Stabilisation frontière du système élastodynamique en présence de singularités

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ÉCOLE CENTRALE DE LYON

Année 2004 numéro d'ordre 2004-26

THÈSE

Présentée et soutenue publiquement le 30 novembre 2004

Pour l'obtention du

Doctorat de l'École Centrale de Lyon

Discipline : Mathématiques et informatique fondamentale (MATHIF)

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Sujet :

Stabilisation frontière du système élastodynamique en présence de singularités

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Résumé : Nous considérons le cas d'un corps faiblement élastique dont une partie de la frontière est encastrée. Notre problème est de déterminer un contrôle sur la partie de la frontière laissée libre (non-encastrée), de telle sorte que le système, quelque soit son état d'origine, s'amortisse le plus rapidement possible.

En d'autres termes, nous considérons un système élastodynamique, amorti au moyen d'une rétroaction définie par une condition de type Neumann sur une partie de la frontière, l'autre partie de la frontière étant munie des conditions de Dirichlet homogène. Nous obtenons des résultats de stabilisation frontière linéaire et non-linéaire, ainsi qu'un résultat de contrôlabilité. Nous démontrons pour cela des relations ad-hoc, dites de Rellich, puis nous utilisons la méthode des multiplicateurs.

L'originalité de ce travail réside dans la présence d'une interface entre la partie Dirichlet et la partie Neumann, qui génère des singularités.

Mots clés : Système linéaire de l'élasticité, Singularité, Régularité, Condition aux limites Dirichlet-Neumann, Stabilisation frontière du système élastodynamique, Contrôlabilité frontière du système élastodynamique, Relation de Rellich, Méthode des multiplicateurs.

Abstract : We consider the case of a weakly elastic body, with a sinked part along its boundary. Our problem is to build a control along the free part of the boundary, such that the energy function vanishes as quick as possible.

In other words, we consider an elastodynamic system, damped by a Neumann feedback on the free part of the boundary. We assume that homogen Dirichlet condition holds on the other part. We obtain linear and nonlinear boundary stabilization results, and a controllability result. We need so-called Rellich relations, and we use multipliers method.

The originality of this work lies in the existence of a non-empty interface between Dirichlet part and Neumann part, which generates singularities.

Key words : Linear elasticity system, Singularity, Regularity, Dirichlet-Neumann boundary condition, Boundary stabilization of elastodynamic system, Boundary controllability of elastodynamic system, Rellich relation, Multipliers method.

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Introduction

Nous considérons le cas d'un corps faiblement élastique dont une partie de la frontière est encastrée. Notre problème est de déterminer une rétroaction (un feedback) sur la partie de la frontière laissée libre (non-encastrée), de telle sorte que le système, quelque soit son état d'origine, s'amortisse le plus rapidement possible.

0.1 Le problème considéré

Soit Ω un ouvert borné de $\mathbb{R}^n$ suffisamment régulier. On suppose que sa frontière $\partial\Omega$ vérifie :

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N, \text{ avec } \begin{cases} \partial\Omega_D \cap \partial\Omega_N = \emptyset, \\ \text{mes}(\partial\Omega_D) \neq 0, \\ \text{mes}(\partial\Omega_N) \neq 0. \end{cases} \tag{1}$$

On note $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$, l'interface entre la partie encastrée $\partial\Omega_D$ et la partie libre $\partial\Omega_N$ (cf figure 1).

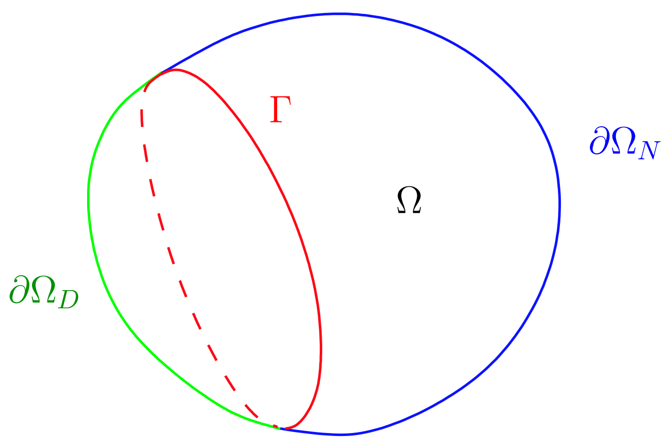

Fig. 1 – Un exemple de domaine Ω.

Soit $\mathbf{v}$ un champ de vecteurs suffisamment régulier. On définit pour ce champ le tenseur des déformations $\epsilon(\mathbf{v}) \in \mathcal{M}_n(\mathbb{R})$ par :

$$\epsilon(\mathbf{v}) = \text{Jac}(\mathbf{v}) +^t \text{Jac}(\mathbf{v}).$$

Nous allons considérer dans la suite un corps élastique homogène et isotrope. Soient $\lambda > 0$ et $\mu > 0$, les coefficients de Lamé, tels que, en vertu de la loi de Hooke, le tenseur des contraintes $\sigma(\mathbf{v}) \in \mathcal{M}_n(\mathbb{R})$ s'écrit :

$$\sigma(\mathbf{v}) = 2\mu\epsilon(\mathbf{v}) + \lambda \text{tr}(\epsilon(\mathbf{v}))I_n.$$

Le système élastodynamique linéaire isotrope, contrôlé sur la partie $\partial\Omega_N$ par le feedback $\mathbf{F}$, s'écrit :

$$\begin{cases} \mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) = 0 & \text{dans } \Omega \times \mathbb{R}_+ \ ; \\ \mathbf{u} = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+ \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{F}(\mathbf{x}, \mathbf{u}') & \text{sur } \partial\Omega_N \times \mathbb{R}_+ \ ; \\ \mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{dans } \Omega \ ; \end{cases} \tag{2}$$

où

- $\mathbf{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ est le champ des déplacements, $\mathbf{u}'$ et $\mathbf{u}''$ les dérivées en temps,
- $\boldsymbol{\nu}$ est le vecteur normal sortant sur la frontière,
- $\mathbf{F}$ est la fonction feedback.
- $(\mathbf{u}_0, \mathbf{u}_1)$ sont les conditions initiales.

Pour ce système, on définit l'énergie :

$$E(\mathbf{u}, t) = \frac{1}{2}\int_\Omega (|\mathbf{u}'|^2 + \sigma(\mathbf{u}) : \epsilon(\mathbf{u}))\, \mathrm{d}\mathbf{x} \ ;$$

où $\sigma(\mathbf{u}) : \epsilon(\mathbf{u}) = \sum_{i,j=1}^n \sigma_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{u})$.

Le problème est de donner des conditions sur Ω, Γ et $\mathbf{F}$ afin que cette énergie soit exponentiellement décroissante en temps, au sens suivant :

Pour tout $C > 1$, il existe une constante $\varpi > 0$ telle que pour toute condition initiale $(\mathbf{u}_0, \mathbf{u}_1)$ suffisamment régulière, la solution $\mathbf{u}$ du problème (2) vérifie

$$\forall t \in \mathbb{R}_+,\ E(\mathbf{u}, t) \leq Ce^{-\varpi t}E(\mathbf{u}, 0).$$

Quelques notations pour la suite

On introduit les espaces suivants :

- $\mathbb{L}^2(\Omega) = (L^2(\Omega))^n, \quad \mathbb{H}^s(\Omega) = (H^s(\Omega))^n, \quad$ pour $s > 0$;
- $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega) \ / \ \mathbf{v} = 0 \text{ sur } \partial\Omega_D\}$;
- $\mathcal{H} = \mathbb{H}^1_D(\Omega) \times \mathbb{L}^2(\Omega)$, avec la norme suivante :

$$\forall (\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}, \quad \|(\mathbf{v}_0, \mathbf{v}_1)\|_\mathcal{H} = \int_\Omega \left(|\mathbf{v}_1|^2 + \sigma(\mathbf{v}_0) : \epsilon(\mathbf{v}_0)\right)\, \mathrm{d}\mathbf{x}.$$

- Pour un champ de vecteurs $\mathbf{v}$ régulier, on définit

$$\nabla\mathbf{v} = (\partial_j v_i)_{1\leq i,j\leq n} = \begin{pmatrix} \partial_1 v_1 & \ldots & \partial_n v_1 \\ \vdots & \ddots & \vdots \\ \partial_1 v_n & \ldots & \partial_n v_n \end{pmatrix}.$$

0.2 Les résultats obtenus précédemment pour le système élastodynamique

L'étude de la stabilisation frontière du système élastodynamique a débuté par Lagnese ([22]), en 1983. Il a pour cela introduit le feedback dit “naturel”. Le système élastodynamique linéaire isotrope, avec ce feedback naturel, s'écrit :

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{dans } \Omega \times \mathbb{R}_+ \ ; \\ \mathbf{u} = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+ \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}') & \text{sur } \partial\Omega_N \times \mathbb{R}_+ \ ; \\ \mathbf{u}(0) = \mathbf{u}_0, \ \mathbf{u}'(0) = \mathbf{u}_1 & \text{dans } \Omega \ ; \end{cases} \tag{3}$$

où $\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}') = -a(\mathbf{x})\mathbf{u} - b(\mathbf{x})\mathbf{u}'$, avec $a, b \in C^1(\overline{\partial\Omega_N})$.

Par une méthode classique de semi-groupe, on montre que le problème (3) admet une unique solution, sous la condition suivante :

$$(\mathbf{u}_0, \mathbf{u}_1) \in \mathcal{H} \tag{4}$$

Pour le problème (3), l'énergie s'écrit :

$$E(\mathbf{u}, t) = \frac{1}{2}\left(\int_\Omega (|\mathbf{u}'|^2 + \sigma(\mathbf{u}) : \epsilon(\mathbf{u}))\, \mathrm{d}\mathbf{x} + \int_{\partial\Omega_N} a|\mathbf{u}|^2\, \mathrm{d}\Gamma\right) \ ;$$

Les résultats obtenus dans [22] ne s'appliquent pas aux systèmes isotropes. Dans [23], Lagnese montre un résultat de stabilisation pour les systèmes isotropes en dimension 2, en introduisant un feedback artificiel. Dans [20], Komornik passe à la dimension 3 pour $\partial\Omega_N$ sphérique, toujours pour ce feedback artificiel.

C'est en 1997, dans [1], qu'Alabau et Komornik obtiennent un résultat de stabilisation avec le feedback naturel pour des systèmes isotropes et anisotropes, à l'aide d'une nouvelle identité. Ces résultats ont été par la suite étendus dans [15, 17, 2, 3], pour des géométries moins restrictives. Cependant, dans tous ces travaux, les différents auteurs supposent que $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} = \emptyset$, ce qui permet d'avoir des solutions fortes dans $C(\mathbb{R}^+, \mathbb{H}^2(\Omega))$. En revanche, lorsque $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} \neq \emptyset$, le changement de conditions au bord génère pour les solutions fortes des solutions singulières qui n'ont plus la régularité $\mathbb{H}^2$, comme dans le cas des ondes que nous décrirons plus loin.

Le résultat le plus général pour le système de Lamé, toujours sans singularité, a été obtenu par M. A. Horn [18], à l'aide de techniques micro-locales, mais ces techniques ne conduisent pas à un taux de décroissance explicite, dans ce cas.

Notre objectif était donc d'obtenir un résultat de stabilisation frontière du système élastodynamique dans le cas où il existe une interface Γ non vide entre la partie Dirichlet $\partial\Omega_D$ et la partie Neumann $\partial\Omega_N$, et si possible avec des techniques permettant d'avoir des taux explicites de décroissance.

0.3 Le cas de l'équation des ondes

Nous nous sommes donc intéressés d'abord à l'équation des ondes (cf [19] et les références de ce livre), pour laquelle cet objectif avait été précédemment atteint. Des conditions géométriques induisant des singularités ont d'abord été étudiées dans [13]. Ce travail a ensuite été étendu dans [21], où a été introduit le feedback décrit ci-dessous, puis dans [4], où le cas de la présence d'une interface Dirichlet-Neumann a été étudié. Nous donnons ici un rappel des résultats obtenus dans cette dernière référence.

Soit $n \geq 3$. Soit Ω un ouvert borné connexe de $\mathbb{R}^n$ tel que, au sens de Nečas ([31]), $\partial\Omega$ est de classe $\mathcal{C}^2$. En tout point $\mathbf{x}$ de $\partial\Omega$, on définit $\boldsymbol{\nu}(\mathbf{x})$ le vecteur normal unitaire sortant (cf figure 2).

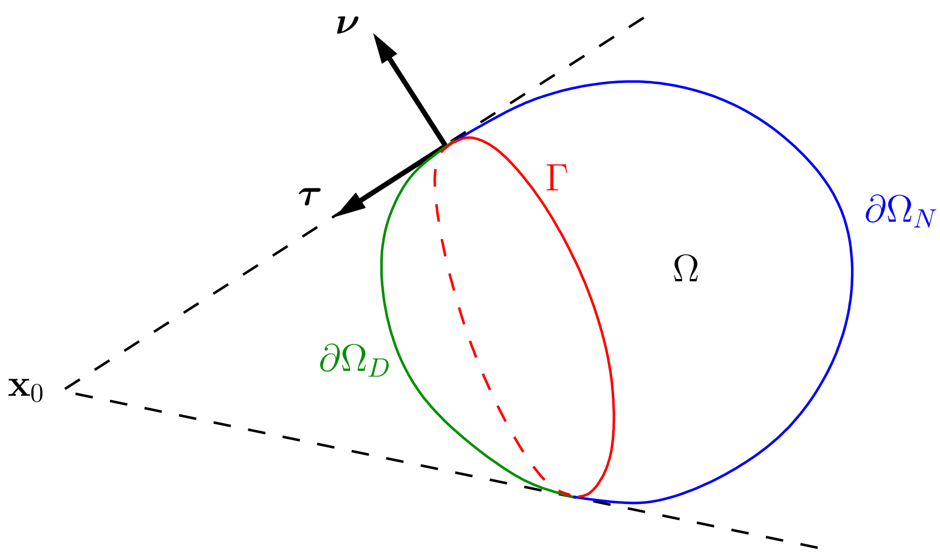

Fig. 2 – Un exemple de domaine Ω.

On fixe $\mathbf{x}_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ et on définit la partition de la frontière de Ω de la façon suivante :

$$\begin{aligned} &\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0, \quad \forall \mathbf{x} \in \mathbb{R}^n \ ; \\ &\partial\Omega_N = \{\mathbf{x} \in \partial\Omega \ / \ \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) > 0\} \ ; \\ &\partial\Omega_D = \{\mathbf{x} \in \partial\Omega \ / \ \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) \leq 0\}. \end{aligned} \tag{5}$$

On note $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$. On suppose que

$$\begin{gathered} \Gamma \text{ est une sous-variété de codimension 2 et de classe } C^3 \text{ telle} \\ \text{qu'il existe un voisinage } \Omega'\text{de } \Gamma \text{ tel que } \partial\Omega \cap \Omega'\text{est une sous-variété} \\ \text{de codimension 1 et de classe } C^3.\text{(cf figure 2)} \end{gathered} \tag{6}$$

Nous pouvons considérer $\partial\Omega_N$ comme un ouvert régulier de $\partial\Omega$, et ainsi définir en tout point $\mathbf{s}$ de Γ, $\boldsymbol{\tau}(\mathbf{s})$ le vecteur unitaire normal à $\partial\Omega_N$, pointant vers l'extérieur de $\partial\Omega_N$ et tangent à $\partial\Omega$. On suppose alors que, pour tout $\mathbf{s} \in \Gamma$, on a

$$\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}) \leq 0, \tag{7}$$

ce qui est toujours vrai si Ω est convexe.

On regarde alors le problème suivant :

$$\begin{cases} u'' - \Delta u = 0 & \text{dans } \Omega \times \mathbb{R}_+ \text{ ;} \\ u = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+ \text{ ;} \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = -(\mathbf{m}\cdot\boldsymbol{\nu})u' & \text{sur } \partial\Omega_N \times \mathbb{R}_+ \text{ ;} \\ u(0) = u_0,\ u'(0) = u_1 & \text{dans } \Omega. \end{cases} \tag{8}$$

On suppose que $(u_0, u_1) \in H^1_D(\Omega) \times L^2(\Omega)$. Dans ce cas, le problème (8) admet une unique solution :

$$u \in C^0(\mathbb{R}^+, H^1_D(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega)).$$

On définit alors son énergie :

$$E(u,t) = \frac{1}{2}\int_\Omega (|u'|^2 + |\nabla u|^2)\,\mathrm{d}\mathbf{x}.$$

Nous considérons u une solution forte. Une singularité apparaît au voisinage de Γ. Par réflexion, on montre qu'en fait, cette singularité est du même type que la singularité qui apparaît dans le cas d'une fissure avec condition de Dirichlet.

On regarde la solution u à $t > 0$ fixé. Au voisinage de tout point de $\overline{\Omega} \setminus \Gamma$, la solution est H^2.

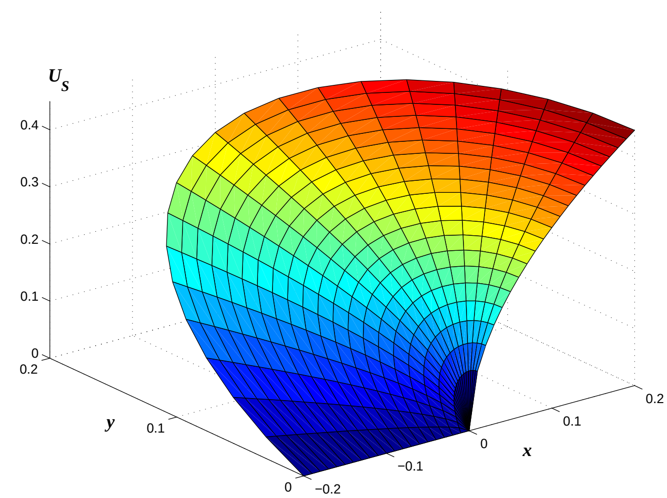

FIG. 3 – Comportement local de la fonction de Shamir.

Au voisinage d'un point $\mathbf{s}$ de Γ, on considère le repère orthonormé formé de $\boldsymbol{\nu}(\mathbf{s})$ et $\boldsymbol{\tau}(\mathbf{s})$, et complété par $n-2$ vecteurs $(\mathbf{z}_1, \dots, \mathbf{z}_{n-2})$. Les dérivées partielles de u par rapport aux $\mathbf{z}_i$ sont H^1. Et surtout, u se décompose en la somme d'une fonction H^2 et d'une fonction singulière, produit tensoriel d'une fonction $H^{\frac{1}{2}}$ sur Γ et de la transformée isomorphe de la fonction u_S, définie sur un demi-disque en coordonnées polaires par :

$$u_S(r,\theta) = \sqrt{r}\sin\left(\frac{\theta}{2}\right)\varsigma(r), \tag{9}$$

où $\varsigma(r)$ est une fonctions de troncature, C^∞, valant 1 au voisinage de 0 et 0 à partir d'un certain r_0. Cette fonction a été introduite par Shamir [36]. (cf figure 3)

Ce résultat permet d'obtenir une relation, dite de Rellich (cf. [34]) :

Soit $u \in H^1(\Omega)$ *tel que :*
- $\Delta u \in L^2(\Omega)$;
- $u_{|\partial\Omega_D} \in H^{3/2}(\partial\Omega_D)$;
- $\partial_\nu u_{|\partial\Omega_N} \in H^{1/2}(\partial\Omega_N)$.

Alors, $2(\mathbf{m}\cdot\nabla u)\partial_{\boldsymbol{\nu}} u - (\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u|^2$ *est intégrable sur* $\partial\Omega$, *et il existe* $\zeta \in H^{1/2}(\Gamma)$ *tel que*

$$\begin{aligned} 2\int_\Omega (\mathbf{m}\cdot\nabla u)\Delta u \,\mathrm{d}\mathbf{x} &= (n-2)\int_\Omega |\nabla u|^2 \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \left(2(\mathbf{m}\cdot\nabla u)\partial_\nu u - (\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u|^2\right) \mathrm{d}\boldsymbol{\gamma} \\ &\quad + \int_\Gamma (\mathbf{m}\cdot\boldsymbol{\tau})|\zeta|^2 \,\mathrm{d}\mathbf{s}. \end{aligned} \tag{10}$$

Ce résultat est en fait utilisé sous la forme suivante, grâce à (7) :

$$2\int_\Omega (\mathbf{m}\cdot\nabla u)\Delta u \,\mathrm{d}\mathbf{x} \leq (n-2)\int_\Omega |\nabla u|^2 \,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \left(2(\mathbf{m}\cdot\nabla u)\partial_\nu u - (\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u|^2\right) \mathrm{d}\boldsymbol{\gamma}.$$

On utilise alors la méthode des multiplicateurs (cf [19]), pour obtenir le résultat de stabilisation frontière suivant :

Il existe $C > 1$ *et* $\varpi > 0$, *indépendants de* (u_0, u_1) *tels que*

$$\forall t \in \mathbb{R}_+,\ E(u,t) \leq Ce^{-\varpi t}E(u,0).$$

0.4 Les résultats obtenus

0.4.1 La piste que nous avons suivie

Nous nous sommes inspirés du cas de l'équation des ondes pour obtenir la stabilisation frontière du système élastodynamique (2). Nous nous sommes d'abord demandés quel feedback $\mathbf{F}$ adopter. La question qui se posait était de savoir si nous pouvions garder le feedback naturel général, ou bien si un feedback plus proche de celui utilisé pour les ondes pouvait mieux convenir. Afin de se donner une première idée de la réponse à cette question, nous avons cherché quel feedback adopter pour stabiliser l'équation suivante, qui est une équation des ondes plus générale :

$$\begin{cases} u'' - \mathrm{div}(A\nabla u) = 0 & \text{dans } \Omega \times \mathbb{R}_+ \ ; \\ u = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+ \ ; \\ \frac{\partial u}{\partial \boldsymbol{\nu}_A} = F(\mathbf{x}, u') & \text{sur } \partial\Omega_N \times \mathbb{R}_+ \ ; \\ u(0) = u_0,\ u'(0) = u_1 & \text{dans } \Omega \ ; \end{cases} \tag{11}$$

où A est une matrice $n \times n$, définie positive, et où $\frac{\partial}{\partial \nu_A}$ est la dérivée conormale associée à la matrice A.

Nous avons raisonné dans le cas simple où $\Gamma = \emptyset$, et on montre qu'alors, le même feedback, $F = -(\mathbf{m}\cdot\boldsymbol{\nu})u'$, stabilise encore. On peut donc voir ici que la matrice A n'intervient pas dans le choix du feedback.

Nous avons donc voulu vérifier si nous pouvions étendre ce résultat au système élastodynamique. En d'autres termes, nous avons voulu savoir si le feedback $\mathbf{F} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'$, qui est en fait un feedback naturel particulier, permettait de stabiliser le système élastodynamique, en présence de singularités.

En étudiant la preuve de la stabilisation de l'équation des ondes, on s'aperçoit qu'un point essentiel du succès du feedback de type $F = -(\mathbf{m}\cdot\boldsymbol{\nu})u'$ est qu'il tend vers 0 lorsqu'on s'approche de Γ. Plus précisément, $(\mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})) \sim d(\mathbf{x}, \Gamma)$, où d est la distance. Ainsi, la forme du feedback doit dépendre de la géométrie du domaine, et non de la forme de l'équation étudiée, ce qui renforce l'idée que $\mathbf{F} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'$ est un feedback susceptible de stabiliser le système élastodynamique.

Nous avons donc d'abord vérifié directement que ce feedback stabilisait lorsque $\Gamma = \emptyset$. Ici, nous avons procédé comme dans le cas des ondes (cf [21]). La difficulté essentielle a été d'obtenir une relation de type "Korn frontière", ce que nous avons fait en nous inspirant de [3].

Puis, nous nous sommes attaqués au cas où $\Gamma \neq \emptyset$, et donc au cas où des singularités apparaissent. Nous avons procédé par étapes, traitant tout d'abord le cas où Ω est un polygone plan, respectant là la chronologie des travaux sur l'équation des ondes (cf [13]). Γ est alors réduit à deux points, et, au voisinage de ces points, Ω est une portion de disque. Puis, nous avons étendu ce résultat à un ouvert plan plus général. Enfin, nous nous sommes intéressés au cas de la dimension supérieure, la seule difficulté résidant dans le passage à la dimension 3, le passage à la dimension $n \geq 4$ se faisant de la même façon.

Comme dans le cas des ondes, la difficulté essentielle est d'obtenir une relation de type Rellich, et pour cela, d'étudier la régularité des solutions fortes à temps fixé. Ensuite, cette relation étant obtenue, la preuve de la stabilisation est identique à tous les cas.

C'est pourquoi nous avons organisé nos articles présentant ces résultats de la façon suivante :

Nous avons groupé les relations de type Rellich pour les différents cas dans [7], ce qui constitue le premier chapitre de cette thèse.

Nous avons ensuite détaillé la preuve de la stabilisation frontière dans [8], ce qui constitue le deuxième chapitre.

Ces résultats sont complétés par un résultat de contrôlabilité et un résultat de stabilisation pour un feedback non linéaire dans [9], ce qui constitue le troisième chapitre.

Nous avions rédigé une note aux Comptes Rendus de l'Académie des Sciences sur la stabilisation de l'équation élastodynamique dans un polygone. Cette note est placée dans la première annexe.

Enfin, un proceeding [10] du *Fifth European Conference on Elliptic and Parabolic Problems : A special tribute to the work of Haim Brezis (30/05-03/06/2004)*, résumant les relations de Rellich pour différentes équations elliptiques, consitue la deuxième annexe.

0.4.2 La relation de type Rellich pour l'équation de l'élasticité dans les différents cas

Nous voulons donc tout d'abord obtenir une relation de type Rellich pour l'élasticité, analogue à (10). Nous reprenons les notations du paragraphe 0.1. On considère l'équation de l'élasticité isotrope avec les conditions mélées suivant :

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{dans } \Omega \ ; \\ \mathbf{u} = \mathbf{g} & \text{sur } \partial\Omega_D \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{h} & \text{sur } \partial\Omega_N \ ; \end{cases} \tag{12}$$

où $\mathbf{f} \in \mathbb{L}^2(\Omega)$, $\mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D)$ et $\mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N)$.

Il est bien connu que ce problème admet une unique solution dans $\mathbb{H}^1(\Omega)$.

Pour $\mathbf{v}_1$ et $\mathbf{v}_2$ deux champs de vecteurs suffisamment réguliers, on introduit la notation suivante :

$$\Theta(\mathbf{v}_1, \mathbf{v}_2) = 2(\sigma(\mathbf{v}_1)\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{v}_2 - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{v}_1) : \epsilon(\mathbf{v}_2).$$

La relation de type Rellich que nous voulons obtenir consiste à estimer l'expression suivante, lorsque $\Theta(\mathbf{u}, \mathbf{u})$ est intégrable sur $\partial\Omega$:

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} - (n-2)\int_\Omega \sigma(\mathbf{u}) : \epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Nous considérons donc successivement les cas suivants :
- $\Omega \subset \mathbb{R}^n$ et $\Gamma = \emptyset$.
- $\Omega \subset \mathbb{R}^2$ est un polygone et Γ est réduit à deux points.
- $\Omega \subset \mathbb{R}^2$ est un domaine régulier de classe $\mathcal{C}^2$ et Γ est réduit à deux points.
- $\Omega \subset \mathbb{R}^n$ est un domaine régulier de classe $\mathcal{C}^2$, Γ une sous-variété de codimension 2.

Le cas d'un domaine de dimension n sans interface

Soit $n \geq 3$. Soit Ω un ouvert borné connexe de $\mathbb{R}^n$ tel que, au sens de Nečas ([31]), $\partial\Omega$ est de classe $\mathcal{C}^2$. En tout point $\mathbf{x}$ de $\partial\Omega$, on définit $\boldsymbol{\nu}(\mathbf{x})$ le vecteur normal unitaire sortant (cf figure 4).

On fixe $\mathbf{x}_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ et on définit la partition de la frontière de Ω comme en (5). On suppose :

$$\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} = \emptyset. \tag{13}$$

On obtient la relation de type Rellich suivante :

Théorème 0.4.1. *Soit $n \geq 2$. Soit $\Omega \subset \mathbb{R}^n$ un domaine borné de classe $\mathcal{C}^2$ dont la frontière vérifie (1) et (13). Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ solution du problème (12) avec*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N).$$

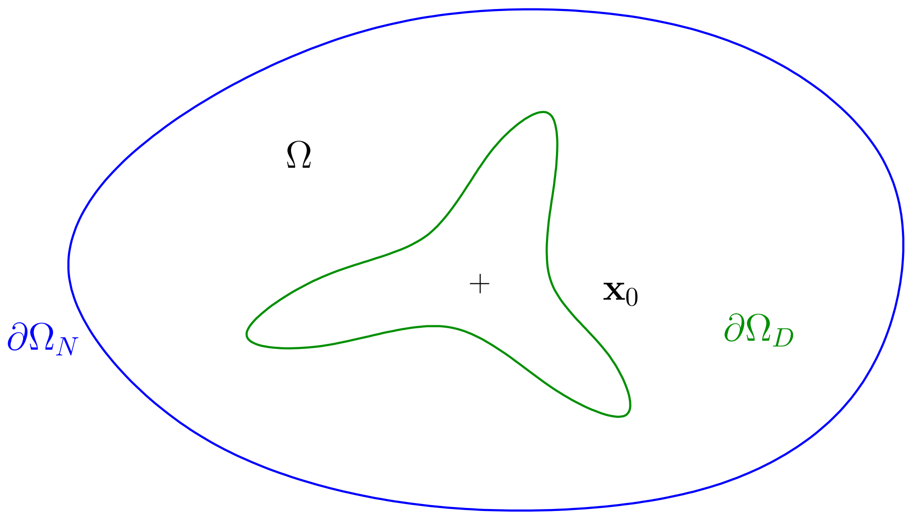

FIG. 4 – Un exemple de domaine Ω sans interface.

$\Theta(\mathbf{u},\mathbf{u})$ *appartient à* $\mathbb{L}^1(\partial\Omega)$ *et*

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = d\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Preuve. Pour prouver cette relation, on commence par prouver que $\mathbf{u}\in\mathbb{H}^2(\Omega)$. Pour la régularité intérieure, on se ramène par troncature à une équation de l'élasticité dans $\mathbb{R}^n$, dont les solutions sont $\mathbb{H}^2$. Pour la régularité à la frontière, on utilise une méthode classique de quotients différentiels.

On applique alors deux fois la formule de Green, pour obtenir le résultat.

Le cas d'un polygone plan

Soit $\Omega\subset\mathbb{R}^2$ un polygone plan convexe, dont la frontière vérifie (1) et (5), (cf figure 5). On a alors

$$\Gamma = \{\mathbf{s}_1,\mathbf{s}_2\}, \tag{14}$$

où $\mathbf{s}_1$ et $\mathbf{s}_2$ sont considérés comme des sommets de $\partial\Omega$. C'est donc en $\mathbf{s}_1$ et $\mathbf{s}_2$ que des singularités apparaissent, et on va noter $c(\mathbf{s})$ le coefficient de singularité associé au point $\mathbf{s}$.

En chaque point $\mathbf{s}_i$, on définit $\omega(\mathbf{s}_i)$ l'angle de Ω, et si $\omega(\mathbf{s}_i)=\pi$, on note $\boldsymbol{\tau}(\mathbf{s}_i)$ le vecteur tangent unitaire à $\partial\Omega$ dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$. La condition (5) implique :

$$\omega(\mathbf{s}_i)=\pi \Rightarrow \mathbf{m}(\mathbf{s}_i)\cdot\boldsymbol{\nu}(\mathbf{s}_i)=0. \tag{15}$$

On va poser

$$\Upsilon = 8\frac{(2\mu+\lambda)(3\mu+\lambda)}{\pi\mu}\left(\pi^2+\ln^2\left(\frac{3\mu+\lambda}{\mu+\lambda}\right)\right), \tag{16}$$

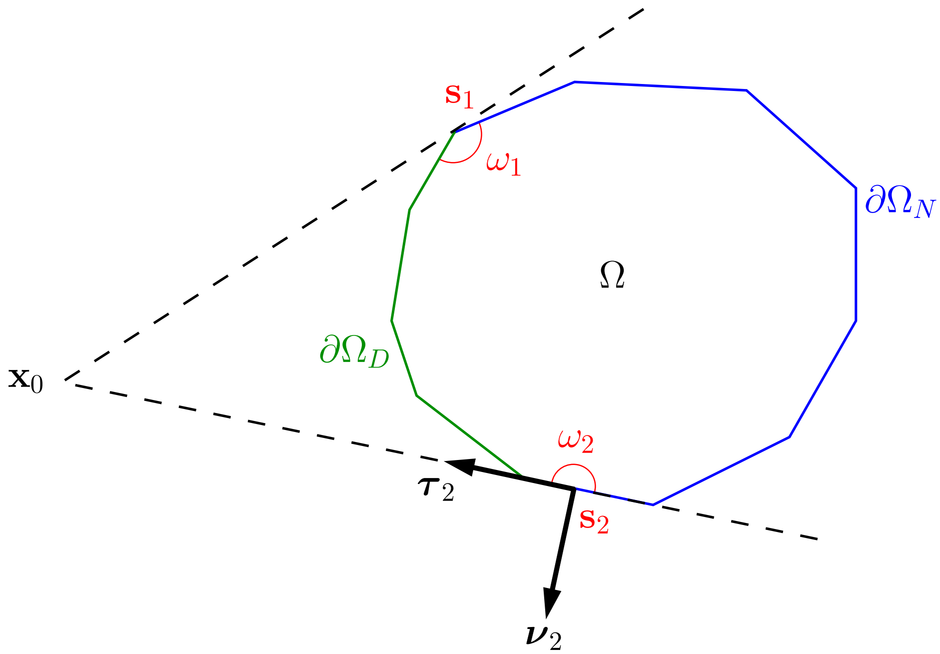

FIG. 5 – Un exemple de domaine polygonal.

et on obtient la relation de type Rellich suivante :

Théorème 0.4.2. *Soit $\Omega \subset \mathbb{R}^2$ un polygone plan convexe, dont la frontière vérifie (1), (14) et (15). Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ la solution du problème (12) avec*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N).$$

$\Theta(\mathbf{u},\mathbf{u})$ *appartient à* $\mathbb{L}^1(\partial\Omega)$ *et*

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} + \Upsilon \sum_{\substack{\mathbf{s}\in\{\mathbf{s}_1,\mathbf{s}_2\}\\ \omega(\mathbf{s})=\pi}} c(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})). \tag{17}$$

Pour prouver ce théorème, on utilise les résultats de P. Grisvard [12], étendus par B. Merouani [27] et on procède comme dans [13].

Preuve. La solution $\mathbf{u}$ est $\mathbb{H}^2$ au voisinage de tout point de $\overline{\Omega}\setminus\Gamma$. Pour un point de $\overline{\Omega}$ qui n'est pas un sommet, on procède comme dans le cas précédent, et pour un sommet qui n'est pas $\mathbf{s}_1$ ou $\mathbf{s}_2$, on utilise [12, 27, 14].

On considère donc $\mathbf{s} \in \Gamma$, et on se ramène par troncature au problème suivant :

$$\begin{cases} -\operatorname{div}(\sigma(\tilde{\mathbf{u}})) = \tilde{\mathbf{f}} & \text{dans } \mathcal{H} \ ; \\ \tilde{\mathbf{u}} = 0 & \text{sur } \partial\mathcal{H}_D \ ; \\ \sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{sur } \partial\mathcal{H}_N \ ; \end{cases} \tag{18}$$

où $\mathcal{H} = \{(r,\theta) \in \mathbb{R}^{*+} \times (0,\omega)\}$, $\partial\mathcal{H}_D = \{(r,0) \ / \ r > 0\}$, $\partial\mathcal{H}_N = \{(r,\omega) \ / \ r > 0\}$ en coordonnées polaires, et où $\tilde{\mathbf{f}} \in \mathbb{L}^2(\mathcal{H})$.

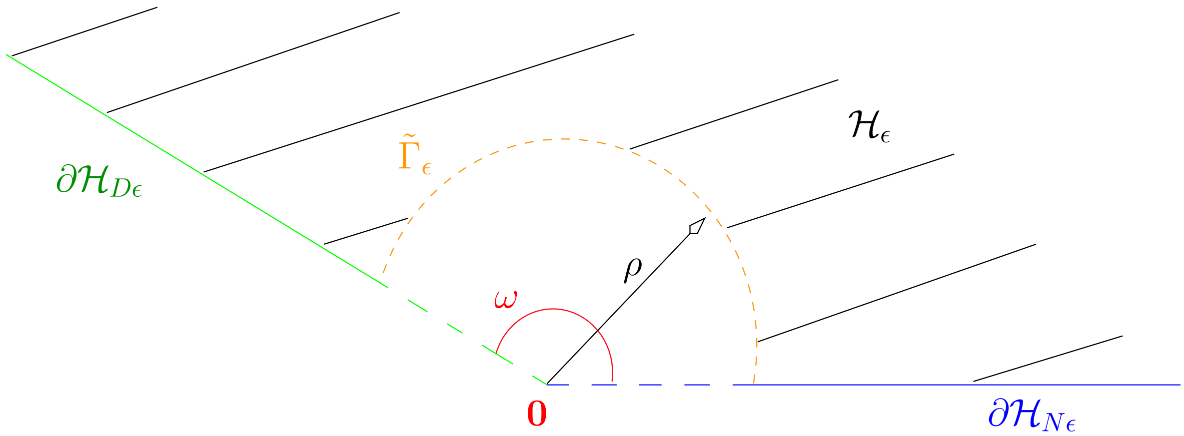

FIG. 6 – Exemple de $\mathcal{H}_\epsilon$ pour $\omega < \pi$.

Pour $\varepsilon > 0$, on définit (cf figure 6) :

$$\left\{\begin{array}{l} \mathcal{H}_\varepsilon = \{(r,\theta) \in (\varepsilon,+\infty)\times(0,\omega)\}, \\ \partial\mathcal{H}_{D\varepsilon} = \{(r,0)\ /\ r > \varepsilon\}, \\ \partial\mathcal{H}_{N\varepsilon} = \{(r,\omega)\ /\ r > \varepsilon\}, \\ \tilde{\Gamma}_\varepsilon = \{(\varepsilon,\theta)\ /\ \theta \in (0,\omega)\}, \end{array}\right.$$

tels que $\partial\mathcal{H}_\varepsilon = \partial\mathcal{H}_{D\varepsilon} \cup \partial\mathcal{H}_{N\varepsilon} \cup \tilde{\Gamma}_\varepsilon$.

Comme $\tilde{\mathbf{u}} \in \mathbb{H}^2(\mathcal{H}_\varepsilon)$, le théorème 0.4.1 donne

$$2\int_{\mathcal{H}_\varepsilon} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla\tilde{\mathbf{u}})\,\mathrm{d}\mathbf{x} = \int_{\partial\mathcal{H}_\varepsilon} \Theta(\tilde{\mathbf{u}},\tilde{\mathbf{u}})\,\mathrm{d}\boldsymbol{\gamma}. \tag{19}$$

Grâce au théorème de Lebesgue, on a

$$\int_{\mathcal{H}_\varepsilon} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{H}} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.$$

Pour la convergence de l'intégrale frontière dans (19), on utilise la structure de la solution de (18) qui est fourni par [12, 27, 14]. On considère alors deux cas.

Premier cas : $\omega < \pi$. On note $\nu = \dfrac{1}{2}\dfrac{\lambda}{\lambda+\mu}$ le coefficient de Poisson du système. On considère alors l'équation en α suivante :

$$\sin^2(\alpha\omega) = \frac{4(1-\nu)^2 - \alpha^2\sin^2\omega}{3-4\nu}. \tag{20}$$

Soit $(\alpha_i)_{i=1,K}$ les racines complexes de (20) telles que $\Re\alpha \in]0,1]$. Grâce à [12, 27, 14], on obtient

$$\begin{aligned} &\exists \mathbf{u}_R \in \mathbb{H}^2(\mathcal{H}),\ \forall i \in [1,K], \exists \mathbf{v}_i^1, \mathbf{v}_i^2 \in (C^\infty([0,\omega],\mathbb{C}))^2 : \\ &\tilde{\mathbf{u}} = \mathbf{u}_R + \textstyle\sum_{i=1}^K \Re[r^{\alpha_i}(\mathbf{v}_i^1(\theta) + \ln(r)\mathbf{v}_i^2(\theta))]. \end{aligned} \tag{21}$$

Or, nous obtenons la propriété suivante sur les racines (cf [37, 33]) :

Lemme 0.4.1. $\forall i \in [1,K],\ \Re\alpha_i > \dfrac{1}{2}$.

Dans ce cas, la régularité est suffisante pour qu'aucun terme correctif n'apparaisse dans la relation de type Rellich. On obtient :

$$\int_{\partial\mathcal{H}_\varepsilon} \Theta(\tilde{\mathbf{u}},\tilde{\mathbf{u}})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\mathcal{H}} \Theta(\tilde{\mathbf{u}},\tilde{\mathbf{u}})\,\mathrm{d}\boldsymbol{\gamma}. \tag{22}$$

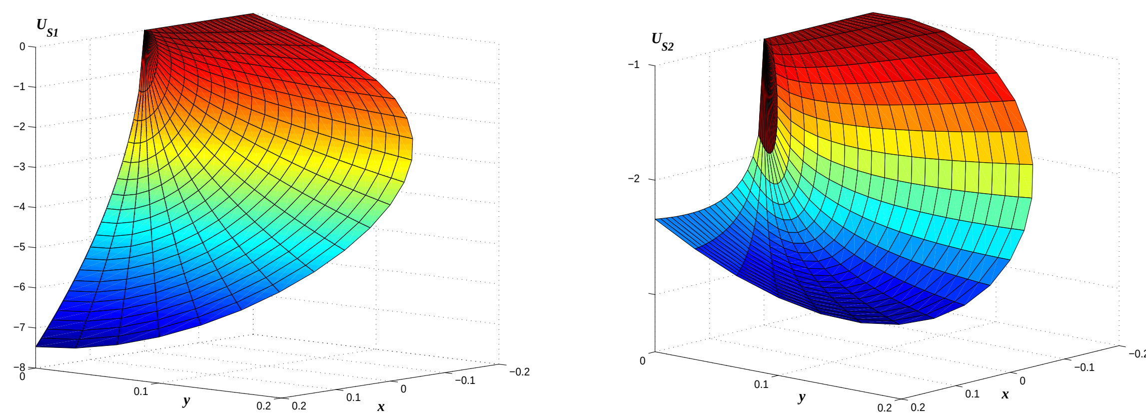

FIG. 7 – Comportement local de u_{S1} et u_{S2}, où $\mathbf{u}_S = (u_{S1}, u_{S2})$ (coefficients de Lamé : $\lambda = \mu = 1$).

Second cas : $\omega = \pi$. Á nouveau, nous utilisons [12, 27, 14]. Dans ce cas, l'équation (20) devient

$$\sin^2(\alpha\pi) = \frac{4(1-\nu)^2 - \alpha^2}{3 - 4\nu}. \tag{23}$$

Les racines de (23) telles que $\Re\alpha \in]0,1]$ sont

$$\alpha = \frac{1}{2} + ik \text{ et } \overline{\alpha} = \frac{1}{2} - ik, \text{ où } k = \frac{\ln(3-4\nu)}{2\pi}.$$

La solution singulière de (18) est donc

$$\mathbf{u}_S(r,\theta) = \Re\left(r^\alpha \mathbf{v}(\theta)\right), \tag{24}$$

où $\mathbf{v}$ est une fonction $\mathcal{C}^\infty$ connue. (cf figure 7) On obtient alors le résultat suivant :

$$\exists! \mathbf{u}_R \in \mathbb{H}^2(\mathcal{H}),\ \exists! c_S \in \mathbb{R} \text{ tels que } \tilde{\mathbf{u}} = \mathbf{u}_R + c_S \mathbf{u}_S. \tag{25}$$

Dans ce cas précis, $\Theta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0$ sur $\partial\mathcal{H}_N$ et sur $\partial\mathcal{H}_D$. Il reste donc l'intégrale sur $\tilde{\Gamma}_\varepsilon$, et on obtient :

$$\lim_{\varepsilon\to 0} \int_{\tilde{\Gamma}_\varepsilon} \Theta(\tilde{\mathbf{u}}, \tilde{\mathbf{u}})\, \mathrm{d}\boldsymbol{\gamma} = c_S^2 \lim_{\varepsilon\to 0} \int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}_S, \mathbf{u}_S)\, \mathrm{d}\boldsymbol{\gamma} = \Upsilon c_S^2 (\mathbf{m}(0)\cdot\boldsymbol{\tau}(0)),$$

ce qui donne le résultat. □

Le cas d'un domaine de dimension 2.

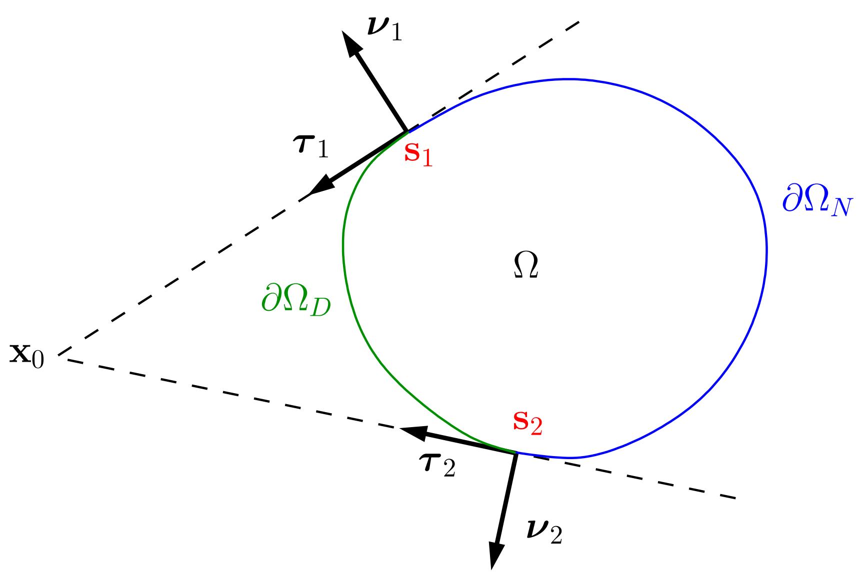

FIG. 8 – Un exemple de domaine Ω en dimension 2.

Nous considérons ici un domaine connexe $\Omega \subset \mathbb{R}^2$, dont la frontière $\partial\Omega$ est de classe C^2 et vérifie (1) et (14) (Dans (14), $\mathbf{s}_1$ et $\mathbf{s}_2$ sont deux points de $\partial\Omega$ (cf figure 8)). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^2$ tel que (5) est vérifié. En tout point $\mathbf{s}_i$, on note $\boldsymbol{\tau}(\mathbf{s}_i)$ le vecteur tangent à $\partial\Omega$, dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$.

On observe que :

$$\mathbf{m}(\mathbf{s}_1)\cdot\boldsymbol{\nu}(\mathbf{s}_1) = \mathbf{m}(\mathbf{s}_2)\cdot\boldsymbol{\nu}(\mathbf{s}_2) = 0. \tag{26}$$

Dans ce cas, la relation de type Rellich s'écrit :

Théorème 0.4.3. *Soit $\Omega \subset \mathbb{R}^2$ un domaine borné connexe de classe $\mathcal{C}^2$ qui vérifie (1), (14) et (26). Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ une solution du problème (12) avec*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{27}$$

$\Theta(\mathbf{u},\mathbf{u})$ appartient à $\mathbb{L}^1(\partial\Omega)$ et

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} + \Upsilon \sum_{\mathbf{s}\in\{\mathbf{s}_1,\mathbf{s}_2\}} c(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))$$

où $c(\mathbf{s})$ est le coefficient de singularité de $\mathbf{u}$ en $\mathbf{s}$, et où Υ est une constante positive définie en (16).

Á nouveau, nous avons besoin de connaître la structure de la solution. Nous allons d'abord regarder le cas d'un demi-disque, puis passer au cas général.

Le cas du demi-disque Pour $\rho > 0$, on définit en coordonnées polaires : $D^+(\rho) = (0,\rho)\times(0,\pi)$; $\partial D_N^+(\rho) = (0,\rho)\times\{\pi\}$ et $\partial D_D^+(\rho) = (0,\rho)\times\{0\}\cup\{\rho\}\times(0,\pi)$ (cf figure 9). On considère alors le problème suivant :

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{dans } D^+(\rho) \text{ ;} \\ \mathbf{u} = 0 & \text{sur } \partial D_D^+(\rho) \text{ ;} \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{sur } \partial D_N^+(\rho) \text{ ;} \end{cases} \tag{28}$$

où $\mathbf{f} \in \mathbb{L}^2(\Omega)$.

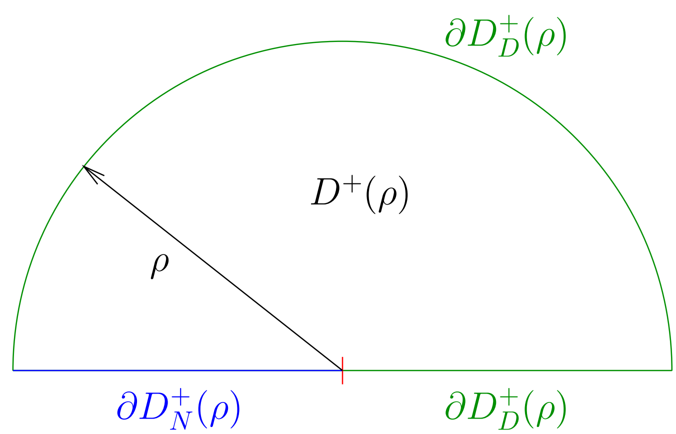

FIG. 9 – Le demi-disque $D^+(\rho)$.

On note $\tilde{\mathbf{u}}_S$ la fonction définie en (24) dans tout le demi-plan. On définit alors

$$\mathbf{u}_S(r,\theta) = \tilde{\mathbf{u}}_S(r,\theta)\varsigma(r), \tag{29}$$

où $\varsigma(r)$ est une fonctions de troncature, C^∞, valant 1 au voisinage de 0 et 0 à partir d'un certain r_0.

Grâce aux résultats du cas polygonal, on a :

Théorème 0.4.4. *Soit* $\mathbf{u}$ *le solution du problème (28), où* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$*. On a :*

$$\exists! \mathbf{u}_R \in \mathbb{H}^2(D^+(\rho)),\ \exists! c_S \in \mathbb{R} \text{ tels que } \mathbf{u} = \mathbf{u}_R + c_S \mathbf{u}_S,$$

où $\mathbf{u}_S$ *est définie en (29).*

En d'autres termes, si on définit l'opérateur $\mathcal{A}_\rho$ par

$$\begin{cases} D(\mathcal{A}_\rho) = \{\mathbf{v} \in \mathbb{H}^1_D(D^+(\rho)) / \mathcal{A}_\rho \mathbf{v} \in \mathbb{L}^2(D^+(\rho)),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ sur } \partial D^+_N(\rho)\} \\ \mathcal{A}_\rho \mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})), \end{cases}$$

on a :

$$D(\mathcal{A}_\rho) \subset \mathbb{H}^2(D^+(\rho)) \oplus \mathbb{R}\mathbf{u}_S.$$

Nous aurons de plus besoin dans la suite de deux théorèmes, permettant d'estimer respectivement le coefficient de singularité c_S et la partie régulière $\mathbf{u}_R$.

Théorème 0.4.5. *Pour* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$*, soit* $\mathbf{u}$ *la solution du problème (28) et soit* c_S *le coefficient de singularité de* $\mathbf{u}$*. Il existe* $\varkappa \in \mathbb{R}^*$*, indépendant de* $\mathbf{f}$*, tel que*

$$c_S = \frac{1}{\varkappa} \int_{D^+(\rho)} \mathbf{S}^* \cdot \mathbf{f} \, \mathrm{d}\mathbf{x}.$$

Preuve. On s'inspire de ce qui a été fait pour l'équation des ondes dans [11], en utilisant une solution $\mathbb{L}^2$ du problème (28), avec $\mathbf{f} = \mathbf{0}$.

Théorème 0.4.6. *Pour* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$*, soit* $\mathbf{u} \in D(\mathcal{A}_\rho)$ *la solution du problème (28). Soit* $\mathbf{u}_R \in \mathbb{H}^2(D^+(\rho))$ *la partie régulière de* $\mathbf{u}$*. il existe* $C > 0$ *indépendant de* $\mathbf{f}$ *tel que*

$$\|\mathbf{u}_R\|_{\mathbb{H}^2(D^+(\rho))} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))}.$$

Preuve. On applique essentiellement le théorème de l'application ouverte.

Le cas général. On considère maintenant le cas d'un domaine Ω vérifiant les conditions géométriques données en début de paragraphe. Comme dans le cas polygonal, la solution du problème (12) est $\mathbb{H}^2$ au voisinage de tout point de $\overline{\Omega} \setminus \Gamma$.

Soit maintenant $\mathbf{s}$ un des deux points de l'interface, et soit $W \subset \mathbb{R}^2$ un voisinage de $\mathbf{s}$. On note (cf figure 10) :

$$w = W \cap \Omega, \quad \partial w_N = W \cap \partial\Omega_N, \quad \partial w_D = (W \cap \partial\Omega_D) \cup (\partial w \cap \Omega). \tag{30}$$

Grâce à un procédé de localisation, on se ramène au problème suivant :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{g} & \text{dans } w \text{ ;} \\ \mathbf{u} = 0 & \text{sur } \partial w_D \text{ ;} \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{sur } \partial w_N \text{ ;} \end{cases} \tag{31}$$

où $\mathbf{g} \in \mathbb{L}^2(w)$.

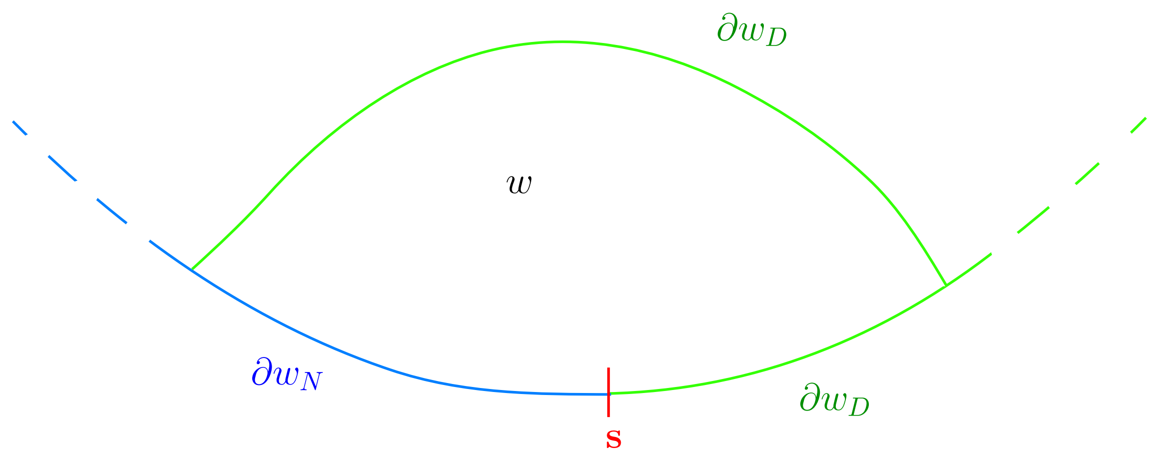

FIG. 10 – Exemple de w.

Pour ce problème, on définit l'opérateur $\mathcal{B}$ par

$$\begin{cases} D(\mathcal{B}) = \{\mathbf{v} \in \mathbb{H}^1_D(w)/\mathcal{B}\mathbf{v} \in \mathbb{L}^2(w),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ sur } \partial w_N\} \\ \mathcal{B}\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})). \end{cases}$$

On peut supposer que w est tel qu'il existe un $\mathcal{C}^2$-difféomorphisme $\boldsymbol{\phi}$ de w sur le demi-disque $D^+(\rho)$, pour un $\rho > 0$, tel que

$$\boldsymbol{\phi}(\mathbf{s}) = 0, \quad \boldsymbol{\phi}(\partial w_D) = \partial D^+_D(\rho), \quad \boldsymbol{\phi}(\partial w_N) = \partial D^+_N(\rho). \tag{32}$$

Grâce à ce difféomorphisme, on se ramène au système perturbé suivant :

$$\begin{cases} -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{u}})) = \tilde{\mathbf{g}} & \text{dans } D^+(\rho)\ ; \\ \tilde{\mathbf{u}} = 0 & \text{sur } \partial D^+_D(\rho)\ ; \\ \tilde{\sigma}(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{sur } \partial D^+_N(\rho)\ ; \end{cases} \tag{33}$$

où $\tilde{\mathbf{g}} \in \mathbb{L}^2(D^+(\rho))$, et où $\tilde{\sigma}(\tilde{\mathbf{u}}) = \mu\nabla\boldsymbol{\phi}(\nabla\tilde{\mathbf{u}}\nabla\boldsymbol{\phi} +^t \nabla\boldsymbol{\phi}^t\nabla\tilde{\mathbf{u}}) + \lambda(\nabla\boldsymbol{\phi} : \nabla\tilde{\mathbf{u}})\nabla\boldsymbol{\phi}$.

Comme dans [4], on regarde le problème (33) comme une petite perturbation du problème (28), et on obtient le résultat de régularité suivant :

Théorème 0.4.7. *Soit w un ouvert de $\mathbb{R}^2$ défini en (30). Il existe $\rho > 0$ et un $\mathcal{C}^2$-difféomorphisme $\boldsymbol{\phi}$ de w sur $D^+(\rho)$ tel que (32) est vérifié et*

$$D(\mathcal{B}) \subset \mathbb{H}^2(w) \oplus \mathbb{R}(\mathbf{u}_S \circ \boldsymbol{\phi}),$$

où $\mathbf{u}_S$ est défini en (29).

Preuve. On prend ρ et $\boldsymbol{\phi}$ comme dans le théorème. Soit $\tilde{\mathcal{A}}_\rho$ l'opérateur défini par

$$\begin{cases} D(\tilde{\mathcal{A}}_\rho) = \{\tilde{\mathbf{v}} \in \mathbb{H}^1_D(D^+(\rho))/\tilde{\mathcal{A}}_\rho\tilde{\mathbf{v}} \in \mathbb{L}^2(D^+(\rho)),\ \tilde{\sigma}(\tilde{\mathbf{v}})\boldsymbol{\nu} = 0 \text{ sur } \partial D^+_N(\rho)\} \\ \tilde{\mathcal{A}}_\rho\tilde{\mathbf{v}} = -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{v}})). \end{cases}$$

La preuve repose alors essentiellement sur le lemme suivant, qui exprime la perturbation :

Lemme 0.4.2. *Pour $\rho > 0$ et $\mathbf{v} \in D(\mathcal{A}_\rho)$, $\tilde{\beta}(\mathbf{v})$ appartient à $H^1(D^+(\rho))^4$ et il existe $C > 0$, indépendant de ρ, tel que*

$$\forall \mathbf{v} \in D(\mathcal{A}_\rho), \quad \|\tilde{\beta}(\mathbf{v})\|_{H^1(D^+(\rho))^4} \leq C\sqrt{\rho}\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Preuve de la relation de type Rellich. On peut maintenant prouver le théorème 0.4.3. Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ une solution du problème (12), satifaisant les conditions (27). On procède comme dans le cas polygonal.

On se ramène par un résultat de trace au problème suivant :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{U})) = \mathbf{F} & \text{dans } \Omega \ ; \\ \mathbf{U} = 0 & \text{sur } \partial\Omega_D \ ; \\ \sigma(\mathbf{U})\boldsymbol{\nu} = 0 & \text{sur } \partial\Omega_N \ ; \end{cases} \tag{34}$$

où $\mathbf{F} \in \mathbb{L}^2(\Omega)$.

On prend un $\varepsilon > 0$ suffisamment petit, et on note

$$\begin{cases} \Omega_\varepsilon = \Omega \setminus (D(\mathbf{s}_1, \varepsilon) \cup D(\mathbf{s}_2, \varepsilon)) \ ; \\ \partial\Omega_{D\varepsilon} = \partial\Omega_D \setminus (D(\mathbf{s}_1, \varepsilon) \cup D(\mathbf{s}_2, \varepsilon)) \ ; \\ \partial\Omega_{N\varepsilon} = \partial\Omega_N \setminus (D(\mathbf{s}_1, \varepsilon) \cup D(\mathbf{s}_2, \varepsilon)) \ ; \\ \tilde{\Gamma}_\varepsilon = (\partial D(\mathbf{s}_1, \varepsilon) \cup \partial D(\mathbf{s}_2, \varepsilon)) \cap \Omega \ ; \end{cases}$$

de telle sorte que $\partial\Omega_\varepsilon = \partial\Omega_{D\varepsilon} \cup \partial\Omega_{N\varepsilon} \cup \tilde{\Gamma}_\varepsilon$.

Pour $\mathbf{s} \in \Gamma$, on applique le théorème 0.4.7 à $\mathbf{U}$ sur $D(\mathbf{s}, \varepsilon) \cap \Omega$. On a alors $\mathbf{u}_R \in \mathbb{H}^2(D(\mathbf{s}, \varepsilon) \cap \Omega)$ et $c_S \in \mathbb{R}$ tels que $\mathbf{u} = \mathbf{u}_R + c_S(\mathbf{u}_S \circ \boldsymbol{\phi})$.

On injecte cette décomposition dans $\Theta(\mathbf{u}, \mathbf{u})$ pour prouver que $\Theta(\mathbf{u}, \mathbf{u})$ appartient à $\mathbb{L}^1(\partial\Omega)$.

Puis, comme $\mathbf{u} \in \mathbb{H}^2(\Omega_\varepsilon)$, on applique deux formules de Green pour obtenir

$$\int_{\Omega_\varepsilon} \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega_\varepsilon} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Grâce au théorème de Lebesgue, on obtient

$$\int_{\Omega_\varepsilon} \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega} \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x},$$

$$\int_{\partial\Omega_{D\varepsilon}\cup\partial\Omega_{N\varepsilon}} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Puis, comme dans le cas polygonal, on a :

$$\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \Upsilon c_S^2(\mathbf{m}(0)\cdot\boldsymbol{\tau}(0)).$$

□

Le cas d'un domaine de dimension n

Nous considérons maintenant le cas d'un domaine $\Omega \subset \mathbb{R}^n$, avec $n \geq 3$, dont la frontière $\partial\Omega$ est de classe C^2 et vérifie (1). On suppose que $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ vérifie (6). (cf figure 11). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) est vérifié, et on observe que :

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0, \quad \text{sur } \Gamma. \tag{35}$$

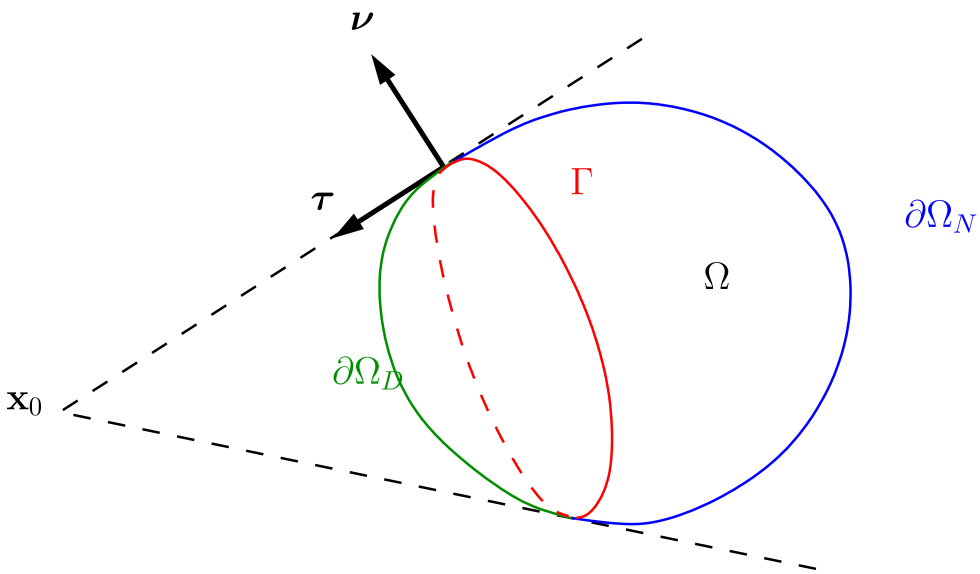

Fig. 11 – Un exemple de domaine Ω.

En tout point $\mathbf{s}$ de Γ, on considère Γ comme une sous-variété de $\partial\Omega$ de co-dimension 1, et on note $\boldsymbol{\tau}(\mathbf{s})$ le vecteur normal unitaire à Γ dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$.

On pose $d = n - 2$. La relation de type Rellich s'écrit :

Théorème 0.4.8. *Soit $\Omega \subset \mathbb{R}^n$ un domaine borné connexe de classe $\mathcal{C}^2$ qui vérifie (1), (6) et (35). Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ une solution du problème (12) avec*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{36}$$

$\Theta(\mathbf{u},\mathbf{u})$ appartient à $\mathbb{L}^1(\partial\Omega)$, et on peut calculer $c_S^e, c_{S1}, \ldots, c_{Sd}$ les coefficients de singularités de $\mathbf{u}$ appartenant à $L^2(\Gamma)$ tels que

$$\begin{aligned} 2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \;&=\; (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \\ &\quad + \int_\Gamma \left(\Upsilon c_S^e(\mathbf{s})^2 + \frac{\mu\pi}{4}\sum_{i=1}^d c_{Si}(\mathbf{s})^2\right)\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})\,\mathrm{d}\mathbf{s}, \end{aligned}$$

où Υ est la constante positive définie en (16).

Encore une fois, nous devons d'abord connaître la structure de la solution. Nous allons d'abord commencer par le cas d'un demi-cylindre, puis nous passerons au cas général.

Le cas du demi-cylindre On considère le cas $\Omega = C^+(\rho) = D^+(\rho) \times \mathbb{R}^d$, (cf figure 12). Les coordonnées de $\mathbf{x} \in C^+(\rho)$ seront notées $(x_1, x_2, \mathbf{z})$, où $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$.

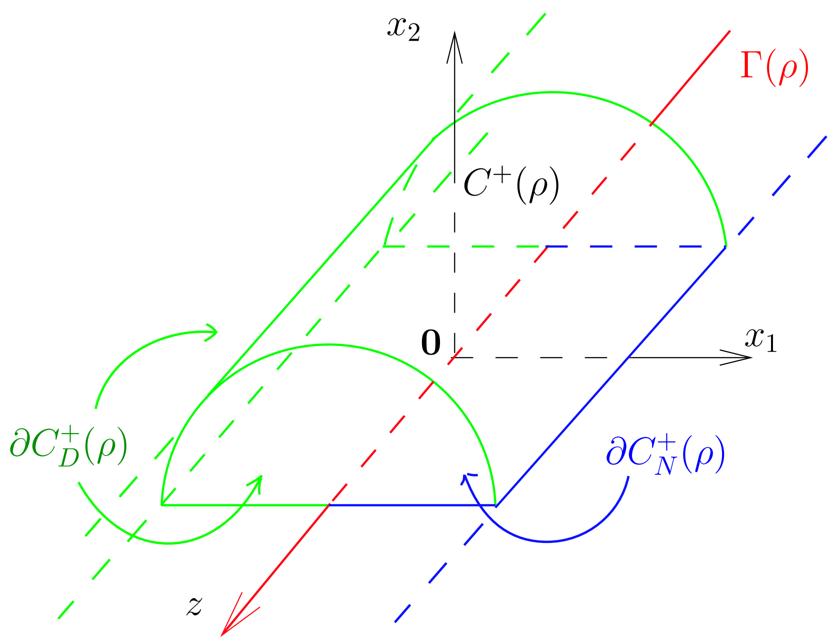

FIG. 12 – Le demi-cylindre $C^+(\rho)$.

On définit $\partial C_N^+(\rho) = \partial D_N^+(\rho) \times \mathbb{R}^d$, $\partial C_D^+(\rho) = \partial D_D^+(\rho) \times \mathbb{R}^d$, $\Gamma(\rho) = \{(0,0)\} \times \mathbb{R}^d$. On considère alors le problème suivant :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{dans } C^+(\rho) \ ; \\ \mathbf{u} = 0 & \text{sur } \partial C_D^+(\rho) \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{sur } \partial C_N^+(\rho). \end{cases} \tag{37}$$

On veut généraliser les résultats obtenus pour la dimension 2. Pour cela, on note $\mathbf{u}_S^e \in (H^1(D^+(\rho)))^2$ la solution singulière définie en (29) pour l'équation de l'élasticité en dimension 2. On note $u_S^l \in H^1(D^+(\rho))$ la fonction de Shamir définie en (9) pour l'équation de Laplace. On définit alors $\mathbf{u}_S \in \mathbb{H}^1(D^+(\rho))$ par :

$$\mathbf{u}_S = \begin{pmatrix} \mathbf{u}_S^e \\ u_S^l \\ \vdots \\ u_S^l \end{pmatrix}. \tag{38}$$

On obtient alors :

Théorème 0.4.9. *Soit* $\mathbf{u}$ *la solution du problème (37), avec* $\mathbf{f} \in \mathbb{L}^2(C^+(\rho))$. *On a :*

1) $\forall i \in \{1, \ldots, d\}$, $\dfrac{\partial \mathbf{u}}{\partial z_i} \in \mathbb{H}^1(C^+(\rho))$, *et il existe* $C > 0$, *indépendant de* ρ *tel que* $\left\| \dfrac{\partial \mathbf{u}}{\partial z_i} \right\|_{\mathbb{H}^1(C^+(\rho))} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$;

2) $\exists! \mathbf{u}_R \in L^2(\mathbb{R}^d, \mathbb{H}^2(D^+(\rho)))$ *et* $\exists! \mathbf{c}_S \in \mathbb{L}^2(\mathbb{R}^d)$ *tels que* $\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S \otimes \mathbf{c}_S$, *où* $\mathbf{u}_S$ *est défini en (38), et tels qu'il existe* $C > 0$, *indépendant de* ρ *tel que*

$$\|\mathbf{u}_R\|_{L^2(\mathbb{R}^d, \mathbb{H}^2(D^+(\rho)))} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))} \text{ et } \|\mathbf{c}_S\|_{\mathbb{L}^2(\mathbb{R}^d)} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

Preuve. Le premier point du théorème s'obtient grâce à la méthode des quotients différentiels.

Pour le deuxième point, on note $\mathbf{u} = {}^t(u_1, \ldots, u_n)$ la solution du problème (37). Nous allons montrer que la solution singulière du système de l'élasticité en dimension n est composée, pour (u_1, u_2), de la solution singulière du système de l'élasticité en dimension 2, et dans les autres directions, de la solution singulière de l'équation de Laplace.

On note

$$\mathbf{u}^e = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in (H^1(C^+(\rho)))^2.$$

On note de plus div_2 l'opérateur de divergence dans $\mathbb{R}^2$, $\epsilon_2(\mathbf{u}^e)$ et $\sigma_2(\mathbf{u}^e)$ le tenseur des déformations et le tenseur des contraintes en dimension 2. $\mathbf{u}^e$ vérifie alors :

$$\left\{ \begin{array}{ll} -\mathrm{div}_2(\sigma_2(\mathbf{u}^e)) = \mathbf{g}^e & \text{dans } C^+(\rho) \; ; \\ \mathbf{u}^e = 0 & \text{sur } \partial C^+_D(\rho) \; ; \\ \sigma_2(\mathbf{u}^e)\boldsymbol{\nu} = \mathbf{h}^e & \text{sur } \partial C^+_N(\rho) \; ; \end{array} \right. \tag{39}$$

où $\mathbf{g}^e \in (L^2(C^+(\rho)))^2$ et $\mathbf{h}^e \in (H^{1/2}(C^+(\rho)))^2$.

Grâce à un résultat de trace, on se ramène à $\mathbf{h}^e = 0$. Comme il n'y a aucune dérivée par rapport à $\mathbf{z}$ dans cette équation, nous pouvons travailler à $\mathbf{z}$ fixé. On se ramène alors à une équation de l'élasticité en dimension 2. Grâce aux théorèmes 0.4.4, 0.4.5 et 0.4.6, on peut écrire $\mathbf{u}^e(., \mathbf{z}) = \mathbf{u}^e_R(., \mathbf{z}) + c^e_S(\mathbf{z})\mathbf{u}^e_S$, où $c^e_S \in L^2(\mathbb{R}^d)$ avec $\|c^e_S\|_{L^2(\mathbb{R}^d)} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$ et $\mathbf{u}^e_R \in (L^2(\mathbb{R}^d, H^2(D^+(\rho))))^2$ avec $\|\mathbf{u}^e_R\|_{(L^2(\mathbb{R}^d, H^2(D^+(\rho))))^2} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$.

On écrit $\Delta_2 = \partial_{11} + \partial_{22}$, et pour $i \in \{3, \ldots, n\}$, on a :

$$\left\{ \begin{array}{ll} -\Delta_2 u_i = g_i & \text{dans } C^+(\rho) \; ; \\ u_i = 0 & \text{sur } \partial C^+_D(\rho) \; ; \\ \partial_2 u_i = h_i & \text{sur } \partial C^+_N(\rho). \end{array} \right. \tag{40}$$

où $g_i \in L^2(C^+(\rho))$ et $h_i \in H^{1/2}(C^+(\rho))$.

Grâce à [4], il existe $u_{Ri} \in L^2(\mathbb{R}^d, H^2(D^+(\rho)))$ et $c_{Si} \in L^2(\mathbb{R}^d)$ tel que

$$u_i = u_{Ri} + u^l_S \otimes c_{Si},$$

et tel qu'il existe $C > 0$ indépendant de ρ tel que

$$\|u_{Ri}\|_{L^2(\mathbb{R}^d, H^2(D^+(\rho)))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}, \quad \|c_{Si}\|_{L^2(\mathbb{R}^d)} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

On note donc

$$\mathbf{u}_R = \begin{pmatrix} \mathbf{u}_R^e \\ u_{R3} \\ \vdots \\ u_{Rn} \end{pmatrix} \text{ et } \mathbf{c}_S = \begin{pmatrix} c_S^e \\ c_S^e \\ c_{S3} \\ \vdots \\ c_{Sn} \end{pmatrix}.$$

et on obtient le résultat. □

Remarque. Nous allons utiliser le théorème 0.4.9 sous une forme différente. Pour $\rho > 0$, soit $B_d(\rho)$ la boule de $\mathbb{R}^d$ de centre $\mathbf{0}$ et de rayon ρ. On définit alors

$$\begin{cases} \tilde{C}^+(\rho) = D^+(\rho) \times B_d(\rho), \\ \partial\tilde{C}_N^+(\rho) = \partial D_N^+(\rho) \times B_d(\rho), \\ \partial\tilde{C}_D^+(\rho) = \partial\tilde{C}^+(\rho) \setminus \partial\tilde{C}_N^+(\rho), \\ \tilde{\Gamma}(\rho) = \{(0,0)\} \times B_d(\rho), \end{cases} \tag{41}$$

et on considère le problème suivant :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{dans } \tilde{C}^+(\rho) \ ; \\ \mathbf{u} = 0 & \text{sur } \partial\tilde{C}_D^+(\rho) \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{sur } \partial\tilde{C}_N^+(\rho). \end{cases} \tag{42}$$

où $\mathbf{f} \in \mathbb{L}^2(\tilde{C}^+(\rho))$.

Si on définit l'opérateur $\mathcal{A}_\rho$ par

$$\begin{cases} D(\mathcal{A}_\rho) = \{\mathbf{v} \in \mathbb{H}_D^1(\tilde{C}^+(\rho))/\mathcal{A}_\rho\mathbf{v} \in \mathbb{L}^2(\tilde{C}^+(\rho)),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ sur } \partial\tilde{C}_N^+(\rho)\} \\ \mathcal{A}_\rho\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})), \end{cases}$$

on obtient

$$D(\mathcal{A}_\rho) \subset \mathbb{H}^2(\tilde{C}^+(\rho)) \oplus \left(\mathbf{u}_S \otimes \mathbb{L}^2(B_d(\rho))\right).$$

Il reste maintenant à estimer la norme $\mathbb{H}^2$ de la partie régulière, et comme dans le cas de la dimension 2, on obtient :

Théorème 0.4.10. *Pour $\mathbf{f} \in \mathbb{L}^2(\tilde{C}^+(\rho))$, soit $\mathbf{u} \in D(\mathcal{A}_\rho)$ la solution du problème (42). Soit $\mathbf{u}_R \in \mathbb{H}^2(\tilde{C}^+(\rho))$ la partie régulière de $\mathbf{u}$. Il existe $C > 0$ indépendant de $\mathbf{f}$ tel que*

$$\|\mathbf{u}_R\|_{\mathbb{H}^2(\tilde{C}^+(\rho))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(\tilde{C}^+(\rho))}.$$

Le cas général. On considère maintenant le cas d'un domaine connexe borné Ω de classe C^2 et vérifiant (1), (6) et (35). Comme dans le cas de la dimension 2, la solution du problème (12) est $\mathbb{H}^2$ au voisinage de tout point de $\overline{\Omega} \setminus \Gamma$.

Soit maintenant $\mathbf{s} \in \Gamma$, et soit $W \subset \mathbb{R}^n$ un voisinage de $\mathbf{s}$. On définit (cf figure 13) :

$$w = W \cap \Omega \ ; \quad \partial w_N = W \cap \partial\Omega_N \ ; \quad \partial w_D = (W \cap \partial\Omega_D) \cup (\partial w \cap \Omega) \ ; \quad \gamma = W \cap \Gamma. \tag{43}$$

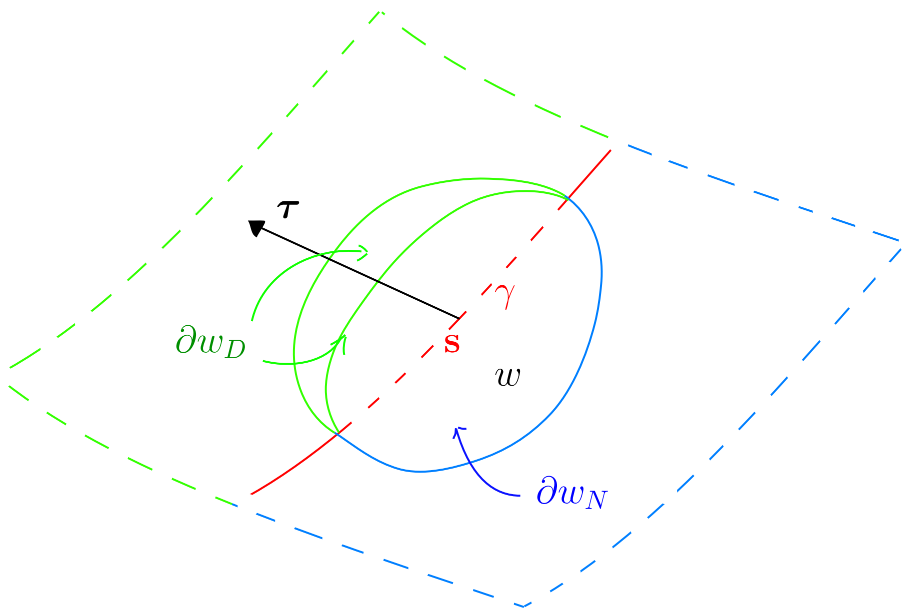

FIG. 13 – Exemple de w.

Par un procédé de localisation, on se ramène au problème suivant :

$$\left\{\begin{array}{ll} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{g} & \text{dans } w \ ; \\ \mathbf{u} = 0 & \text{sur } \partial w_D \ ; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{sur } \partial w_N \ ; \end{array}\right. \tag{44}$$

où $\mathbf{g} \in \mathbb{L}^2(w)$.

Pour ce problème, on définit l'opérateur $\mathcal{B}$ par

$$\left\{\begin{array}{l} D(\mathcal{B}) = \{\mathbf{v} \in \mathbb{H}^1_D(w)/\mathcal{B}\mathbf{v} \in \mathbb{L}^2(w),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ sur } \partial w_N\} \\ \mathcal{B}\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})). \end{array}\right.$$

On peut supposer que w est tel qu'il existe un $\mathcal{C}^2$-difféomorphisme $\mathbf{\Phi}$ de w sur $\tilde{C}^+(\rho)$, pour un $\rho > 0$, qui satisfait :

$$\mathbf{\Phi}(\mathbf{s}) = \mathbf{0}, \quad \mathbf{\Phi}(\partial w_D) = \partial\tilde{C}^+_D(\rho), \quad \mathbf{\Phi}(\partial w_N) = \partial\tilde{C}^+_N(\rho), \quad \mathbf{\Phi}(\gamma) = \tilde{\Gamma}(\rho). \tag{45}$$

De la même façon que dans le cas de la dimension 2, on obtient le résultat de régularité suivant :

Théorème 0.4.11. *Soit w un ouvert de $\mathbb{R}^n$ défini comme en (43). Il existe $\rho > 0$ et un $\mathcal{C}^2$-difféomorphisme $\mathbf{\Phi}$ de w sur $\tilde{C}^+(\rho)$ tel que (45) est vérifié et*

$$D(\mathcal{B}) \subset \mathbb{H}^2(w) \oplus \left(\left(\mathbf{u}_S \otimes \mathbb{L}^2(B_d(\rho))\right) \circ \mathbf{\Phi}\right).$$

où $\mathbf{u}_S$ est donné en (38).

Preuve de la relation de type Rellich. On peut prouver à présent le théorème 0.4.8. Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ une solution du problème (12), vérifiant les conditions (36). On se ramène grâce à un résultat de trace au problème suivant :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{U})) = \mathbf{F} & \text{dans } \Omega \ ; \\ \mathbf{U} = 0 & \text{sur } \partial\Omega_D \ ; \\ \sigma(\mathbf{U})\boldsymbol{\nu} = 0 & \text{sur } \partial\Omega_N \ ; \end{cases} \tag{46}$$

où $\mathbf{F} \in \mathbb{L}^2(\Omega)$.

On prend un $\varepsilon > 0$ suffisamment petit et on note

$$\Omega_\varepsilon = \Omega \backslash D(\Gamma,\varepsilon) \ ; \quad \partial\Omega_{D\varepsilon} = \partial\Omega_D \backslash D(\Gamma,\varepsilon) \ ; \quad \partial\Omega_{N\varepsilon} = \partial\Omega_N \backslash D(\Gamma,\varepsilon) \ ; \quad \tilde{\Gamma}_\varepsilon = \partial D(\Gamma,\varepsilon) \cap \Omega \ ;$$

où $D(\Gamma,\varepsilon) = \{\mathbf{x} \in \mathbb{R}^n / d(\mathbf{x},\Gamma) \leq \varepsilon\}$ et de telle sorte que $\partial\Omega_\varepsilon = \partial\Omega_{D\varepsilon} \cup \partial\Omega_{N\varepsilon} \cup \tilde{\Gamma}_\varepsilon$ (cf figure 14).

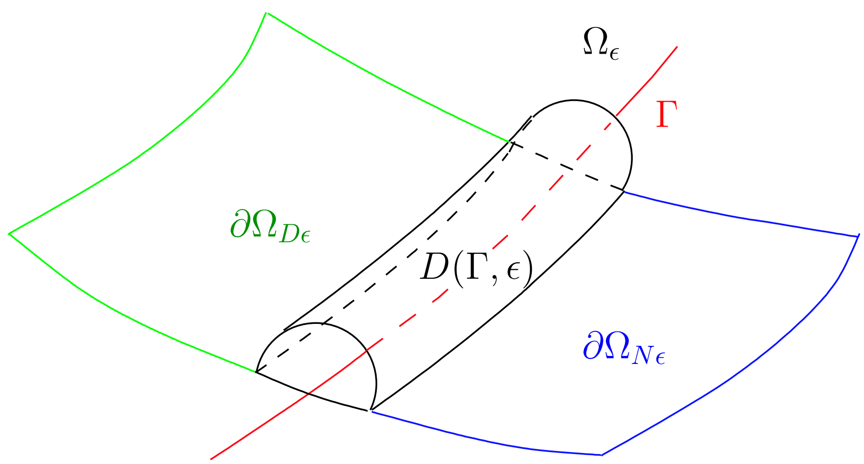

FIG. 14 – Ω_ϵ.

Comme dans le cas de la dimension 2, $\Theta(\mathbf{u},\mathbf{u})$ appartient à $\mathbb{L}^1(\partial\Omega)$. Et comme dans le cas de la dimension 2, on applique la formule de Green pour obtenir :

$$\int_{\Omega_\varepsilon} \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = (n-2)\int_{\Omega_\varepsilon} \sigma(\mathbf{u}) : \epsilon(\mathbf{u})\, d\mathbf{x} + \int_{\partial\Omega_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Toutes les intégrales sur Ω_ε, $\partial\Omega_{N\varepsilon}$ et $\partial\Omega_{D\varepsilon}$ convergent.

Il reste à étudier la convergence de l'intégrale sur $\tilde{\Gamma}_\varepsilon$. Soit $\mathbf{s} \in \Gamma$. Nous supposons que $\mathbf{s}$ est à l'origine, et que $\boldsymbol{\tau}(\mathbf{s})$ et $\boldsymbol{\nu}(\mathbf{s})$ sont les deux premiers axes de coordonnées. On écrit $\tilde{\Gamma}_\varepsilon(\mathbf{s}) = \tilde{\Gamma}_\varepsilon \cap (\mathbf{s},\boldsymbol{\tau},\boldsymbol{\nu})$, où $(\mathbf{s},\boldsymbol{\tau},\boldsymbol{\nu})$ est le plan contenant $\mathbf{s}$ et généré par $\boldsymbol{\tau}$ et $\boldsymbol{\nu}$.

On applique alors le théorème 0.4.11 pour avoir $\mathbf{u} = \mathbf{u}_R + (\mathbf{u}_S \circ \boldsymbol{\Phi}) \otimes \mathbf{c}_S(\mathbf{s})$ sur un voisinage $w(\mathbf{s})$ de $\mathbf{s}$. On a $\boldsymbol{\Phi}(\tilde{\Gamma}_\varepsilon) = \{\varepsilon\} \times (0,\pi)$ en coordonnées polaires.

On obtient :

$$\int_{\tilde{\Gamma}_\varepsilon(\mathbf{s})} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} = c_S^e(\mathbf{s})^2 \int_{\theta=0}^{\pi} \Theta(\mathbf{u}_S^e,\mathbf{u}_S^e)\varepsilon\,\mathrm{d}\theta$$

$$+\sum_{i=1}^{d} c_{Si}(\mathbf{s})^2 \int_{\theta=0}^{\pi} \mu[(\boldsymbol{\nu}\cdot\nabla u_S^l)\cdot(\mathbf{m}\cdot\nabla u_S^l) - (\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u_S^l|^2]\varepsilon\,\mathrm{d}\theta + \mathcal{O}(\varepsilon).$$

En utilisant les résultats de [11, 4], on obtient :

$$\int_{\theta=0}^{\pi} [(\boldsymbol{\nu}\cdot\nabla u_S^l)\cdot(\mathbf{m}\cdot\nabla u_S^l) - (\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u_S^l|^2]\varepsilon\,\mathrm{d}\theta = \frac{\pi}{4}(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})) + \mathcal{O}(\varepsilon).$$

De plus, on a obtenu dans le cas de la dimension 2 :

$$\int_{\theta=0}^{\pi} \Theta(\mathbf{u}_S^e,\mathbf{u}_S^e)\varepsilon\,\mathrm{d}\theta = \Upsilon(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})) + \mathcal{O}(\varepsilon).$$

On obtient donc :

$$\int_{\tilde{\Gamma}_\varepsilon(\mathbf{s})} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} = c_S^e(\mathbf{s})^2\Upsilon(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})) + \sum_{i=1}^{d} c_{Si}(\mathbf{s})^2\mu\frac{\pi}{4}(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})) + \mathcal{O}(\varepsilon).$$

On intègre le résultat précédent sur $w(\mathbf{s})\cap\Gamma$. Par compacité de Γ on peut introduire un recouvrement fini de Γ par les $\{w(\mathbf{s})\}_{\mathbf{s}\in\Gamma}$ et les fonctions de troncatures associées qui constituent une partition de l'unité de Γ. On somme alors un nombre fini de relations pour obtenir :

$$\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} = \int_{\Gamma} \left(\Upsilon c_S^e(\mathbf{s})^2 + \frac{\mu\pi}{4}\sum_{i=1}^{d} c_{Si}(\mathbf{s})^2\right)(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))\,\mathrm{d}\mathbf{s} + \mathcal{O}(\varepsilon).$$

On obtient la relation de type Rellich en passant à la limite quand ε tend vers 0. □

Remarque : on a obtenu au théorème 0.4.11 un résultat de regularité locale. Nous pouvons obtenir un résultat de régularité globale.

Grâce à la régularité de Ω, on peut construire un voisinage Ω' de Γ de telle sorte que les fonctions suivantes soient bien définies :

$$\begin{array}{rccc}(\mathcal{L}_{\mathbf{s}},\mathcal{L}_r,\mathcal{L}_\theta): & \Omega' & \longrightarrow & \Gamma\times\mathbb{R}^+\times[0,\pi] \\ & \mathbf{x} & \longmapsto & (\mathbf{s},r,\theta).\end{array}$$

où $\mathcal{L}_{\mathbf{s}}(\mathbf{x})$ est la projection de $\mathbf{x}$ sur Γ. $\mathbf{x}$ est donc dans le plan contenant $\mathcal{L}_{\mathbf{s}}(\mathbf{x})$ et généré par $\boldsymbol{\nu}(\mathcal{L}_{\mathbf{s}}(\mathbf{x}))$ et $\boldsymbol{\tau}(\mathcal{L}_{\mathbf{s}}(\mathbf{x}))$. $\mathcal{L}_r(\mathbf{x})$ et $\mathcal{L}_\theta(\mathbf{x})$ sont alors les coordonnées polaires de $\mathbf{x}$ dans ce plan (cf figure 15).

Soit ζ une fonction de troncature C^∞ telle que $\zeta\equiv 0$ hors de Ω' et $\zeta\equiv 1$ sur un voisinage de Γ. Il existe alors $\mathbf{u}_R\in\mathbb{H}^2(\Omega)$ et $\mathbf{c}_S\in\mathbb{L}^2(\Gamma)$ tels que

$$\mathbf{u} = \mathbf{u}_R + \zeta\mathbf{c}_S(\mathcal{L}_{\mathbf{s}})\mathbf{u}_S(\mathcal{L}_r,\mathcal{L}_\theta) \quad \text{dans } \Omega,$$

où $\mathbf{u}_S$ est la fonction singulière définie en (38).

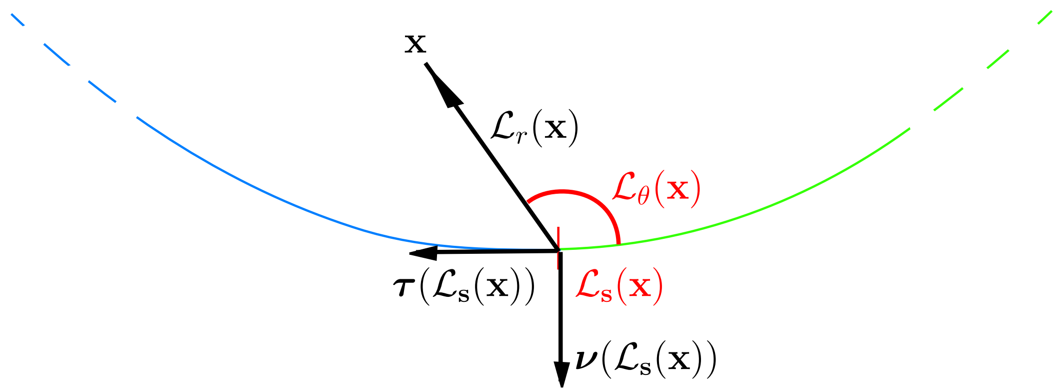

FIG. 15 – Les fonctions $\mathcal{L}$.

0.4.3 La stabilisation frontière du système élastodynamique

On peut à présent montrer le résultat de stabilisation.

Nous considérons le cas d'un domaine $\Omega \subset \mathbb{R}^n$, dont la frontière $\partial\Omega$ est de classe C^2 et vérifie (1). On suppose que $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ vérifie (6). (cf figure 11). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) est vérifié, et, à nouveau, on a (35).

En tout point $\mathbf{s}$ de Γ, on considère Γ comme une sous-variété de $\partial\Omega$ de co-dimension 1, et on note $\boldsymbol{\tau}(\mathbf{s})$ le vecteur unitaire normal à Γ dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$. On suppose alors que (7) est vérifié, ce qui est en particulier toujours le cas quand Ω est convexe.

On considère le système élastodynamique isotrope suivant :

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{dans } \Omega \times \mathbb{R}_+ \text{ ;} \\ \mathbf{u} = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+ \text{ ;} \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}' & \text{sur } \partial\Omega_N \times \mathbb{R}_+ \text{ ;} \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{dans } \Omega \text{ ;} \\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{dans } \Omega. \end{cases} \tag{47}$$

Par la méthode des semi-groupes, on montre que le problème (47) est bien posé, sous la condition (4).

L'énergie du système est donnée par :

$$E(\mathbf{u}, t) = \frac{1}{2}\int_\Omega (|\mathbf{u}'|^2 + \sigma(\mathbf{u}) : \epsilon(\mathbf{u}))\, \mathrm{d}\mathbf{x}. \tag{48}$$

Théorème 0.4.12. *Soit Ω un domaine connexe borné de $\mathbb{R}^n$. On suppose que sa frontière $\partial\Omega$ est de classe C^2 et vérifie (1) et (6). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) et (7) sont vérifiés.*

Il existe $C > 0$ et $\varpi > 0$ tel que pour tout $(\mathbf{u}_0, \mathbf{u}_1)$ vérifiant (4), la solution $\mathbf{u}$ du problème (47) vérifie :

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq Ce^{-\varpi t} E(\mathbf{u}, 0).$$

Preuve. On va suivre la méthode des multiplicateurs, utilisée dans [21] et décrite dans [19].

On prouve d'abord le théorème pour les solutions fortes, le résultat pour les solutions faibles sera ensuite obtenu par densité.

L'outil principal de la preuve est le lemme suivant, montré dans [19] :

Lemme 0.4.3. *Soit $E : \mathbb{R}^+ \to \mathbb{R}^+$ une fonction décroissante telle qu'il existe $C > 0$ indépendant de t tel que*

$$\int_t^\infty E(s)\, ds \leq CE(t), \quad \forall t \geq 0 \; ; \tag{49}$$

on a alors

$$E(t) \leq E(0) e^{1-\frac{t}{C}}, \quad \forall t \geq C.$$

On vérifie facilement que l'énergie décroît avec le temps. En effet, grâce à la formule de Green, on a, pour tout $t > 0$:

$$E'(\mathbf{u}, t) = \int_\Omega (\mathbf{u} \cdot \mathbf{u}' + \sigma(\mathbf{u}) : \epsilon(\mathbf{u}'))\, \mathrm{d}\mathbf{x} = -\int_{\partial\Omega_N} (\mathbf{m} \cdot \boldsymbol{\nu}) |\mathbf{u}'|^2 \, \mathrm{d}\boldsymbol{\gamma}. \tag{50}$$

Soient $T > S > 0$. On introduit le multiplicateur $M\mathbf{u} = 2(\mathbf{m} \cdot \nabla)\mathbf{u} + (n-1)\mathbf{u}$, et on écrit :

$$\int_S^T \int_\Omega \mathbf{u}'' \cdot M\mathbf{u} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = \int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u})) \cdot M\mathbf{u} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \tag{51}$$

On développe alors chacune des deux parties de cette égalité, par des intégrations par parties et des formules de Green.

On est amené à utiliser le théorème 0.4.8 (la relation de Rellich) et la condition (7) pour obtenir :

$$\begin{aligned} 2\int_\Omega \mathrm{div}(\sigma(\mathbf{u})) \cdot (\mathbf{m} \cdot \nabla)\mathbf{u} \, \mathrm{d}\mathbf{x} \quad \leq \quad & (n-2) \int_\Omega \sigma(\mathbf{u}) : \epsilon(\mathbf{u}) \, \mathrm{d}\mathbf{x} \\ & - \int_{\partial\Omega_N} (\mathbf{m} \cdot \boldsymbol{\nu}) [2\mathbf{u}' \cdot (\mathbf{m} \cdot \nabla)\mathbf{u} + \sigma(\mathbf{u}) : \epsilon(\mathbf{u})] \, \mathrm{d}\boldsymbol{\gamma}. \end{aligned}$$

Nous avons alors besoin du lemme suivant, inspiré de [3] :

Lemme 0.4.4. *Pour tout $\theta > 0$ assez petit, il existe $C > 0$ indépendant de S et T tel que*

$$-\int_S^T \int_{\partial\Omega_N} (\mathbf{m} \cdot \boldsymbol{\nu}) [2\mathbf{u}' \cdot (\mathbf{m} \cdot \nabla)\mathbf{u} + \sigma(\mathbf{u}) : \epsilon(\mathbf{u})] \, d\boldsymbol{\gamma} \, dt \leq \theta \int_S^T E(\mathbf{u}, t) \, dt + CE(\mathbf{u}, S).$$

Nous obtenons en fin de compte que, pour tout $\theta > 0$ suffisamment petit, il existe $C > 0$ indépendant de S et de T tel que

$$(2-\theta)\int_S^T E(t)\,\mathrm{d}t \leq C_3 E(\mathbf{u}, S).$$

On prend θ assez petit, et le passage à la limite lorsque T tend vers l'infini permet de conclure. Le résultat est facilement étendu aux solutions faibles par densité, puisque les constantes qui apparaissent ne dépendent pas de la solution forte considérée. □

Remarque : toutes les constantes qui apparaissent dans la preuve sont explicites, et nous pouvons donc obtenir le taux de décroissance en fonction des données (géométrie, coefficients de Lamé).

0.5 Contrôlabilité frontière

Grâce au résultat de stabilisation, nous sommes en mesure d'obtenir la contrôlabilité frontière du système élastodynamique. On utilise pour cela le principe de Russell (cf [35]). Ce problème a été étudié en particulier par Lions [25], et il est classique de déduire la contrôlabilité de la stabilisation. (Voir par exemple [20, 16, 32] pour le système élastodynamique) À noter que, dans [1], la contrôlabilité est démontrée directement par la méthode HUM de Lions.

Soit $\Omega \subset \mathbb{R}^n$ un domaine connexe borné, dont la frontière $\partial\Omega$ est de classe C^2 et vérifie (1). On note $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$, et on suppose que Γ vérifie (6) (cf figure 16).

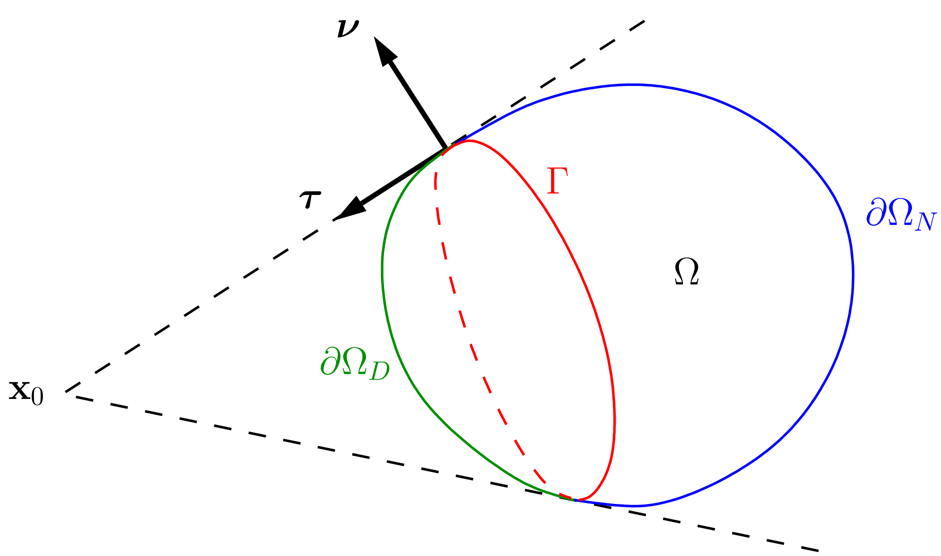

FIG. 16 – Un exemple de domaine Ω.

On note $\boldsymbol{\nu}(\mathbf{x})$ le vecteur normal unitaire en $\mathbf{x} \in \partial\Omega$, pointant hors de Ω. On suppose qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que, si on note $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, on a (5).

En tout point $\mathbf{s} \in \Gamma$, on considère Γ comme une sous variété de $\partial\Omega$ de codimension 1, et on note $\boldsymbol{\tau}(\mathbf{s})$ le vecteur normal unitaire à Γ, dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$. On suppose alors que l'on a (7).

On considère le problème suivant :

On fixe $T > 0$. Pour tout $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, on cherche $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ telle que la solution de problème suivant :

$$\left\{\begin{array}{ll} \mathbf{y}'' - \operatorname{div}(\sigma(\mathbf{y})) = 0 & \text{dans } \Omega \times [0,T]; \\ \mathbf{y} = 0 & \text{sur } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{y})\boldsymbol{\nu} = \boldsymbol{\phi} & \text{sur } \partial\Omega_N \times [0,T]; \\ \mathbf{y}(0) = 0 & \text{dans } \Omega; \\ \mathbf{y}'(0) = 0 & \text{dans } \Omega; \end{array}\right. \tag{52}$$

vérifie

$$\mathbf{y}(T) = \mathbf{y}_0; \quad \mathbf{y}'(T) = \mathbf{y}_1 \quad \text{sur } \Omega. \tag{53}$$

Afin de trouver cette fonction $\boldsymbol{\phi}$, nous considérons deux problèmes intermédiaires. Soit $(\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}$. On note $\mathbf{v}$ la solution du problème :

$$\left\{\begin{array}{ll} \mathbf{v}'' - \operatorname{div}(\sigma(\mathbf{v})) = 0 & \text{dans } \Omega \times [0,T]; \\ \mathbf{v} = 0 & \text{sur } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{v})\boldsymbol{\nu} = (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{v}' & \text{sur } \partial\Omega_N \times [0,T]; \\ \mathbf{v}(T) = \mathbf{v}_0 & \text{dans } \Omega; \\ \mathbf{v}'(T) = \mathbf{v}_1 & \text{dans } \Omega. \end{array}\right. \tag{54}$$

Grâce au théorème 0.4.12, on a :

$$\forall t \in [0,T], \quad E(\mathbf{v}, t) \leq C e^{-\varpi(T-t)} E(\mathbf{v}, T). \tag{55}$$

Soit maintenant $\mathbf{w}$ la solution du problème :

$$\left\{\begin{array}{ll} \mathbf{w}'' - \operatorname{div}(\sigma(\mathbf{w})) = 0 & \text{dans } \Omega \times [0,T]; \\ \mathbf{w} = 0 & \text{sur } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{w})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{w}' & \text{sur } \partial\Omega_N \times [0,T]; \\ \mathbf{w}(0) = \mathbf{v}(0) & \text{dans } \Omega; \\ \mathbf{w}'(0) = \mathbf{v}'(0) & \text{dans } \Omega. \end{array}\right. \tag{56}$$

Á nouveau, on a :

$$\forall t \in [0,T], \quad E(\mathbf{w}, t) \leq C e^{-\varpi t} E(\mathbf{v}, 0). \tag{57}$$

On pose $\mathbf{y} = \mathbf{w} - \mathbf{v}$. $\mathbf{y}$ vérifie (52) avec $\boldsymbol{\phi} = -(\mathbf{m}\cdot\boldsymbol{\nu})(\mathbf{v}' + \mathbf{w}') \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$.

On définit l'opérateur Λ par :

$$\begin{array}{rccc} \Lambda : & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & (\mathbf{v}_0, \mathbf{v}_1) & \longmapsto & (\mathbf{w}(T), \mathbf{w}'(T)). \end{array}$$

et on montre grâce à (55) et (57) qu'il existe $T > 0$ tel que l'opérateur $\Lambda - I_{\mathcal{H}}$ est un isomorphisme. Donc, pour tout $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, on prend $(\mathbf{v}_0, \mathbf{v}_1) = (\Lambda - I_{\mathcal{H}})^{-1}(\mathbf{y}_0, \mathbf{y}_1)$ et $\mathbf{y}$ vérifie alors (53).

On a donc obtenu le théorème de contrôlabilité suivant :

Théorème 0.5.1. *Soit Ω un domaine connexe borné de $\mathbb{R}^n$. On suppose que sa frontière $\partial\Omega$ est de classe C^2 et vérifie (1) et (6). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) et (7) sont vérifiés.*

Il existe alors $T_0 > 0$ tel que, pour tout $T > T_0$ et pour tout $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, il existe $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ tel que la solution $\mathbf{y}$ du problème (52) vérifie $\mathbf{y}(T) = \mathbf{y}_0$ et $\mathbf{y}'(T) = \mathbf{y}_1$ sur Ω.

Remarque : de la preuve, on déduit, ce qui sera utile dans la suite, que

$$\|\tilde{\boldsymbol{\phi}}\|_{\mathbb{L}^2(\partial\Omega_N \times [0,T])} \leq \frac{\sqrt{\|\mathbf{m}\|_\infty}}{1 - Ce^{-\varpi T}} \|(\mathbf{y}_0, \mathbf{y}_1)\|_{\mathcal{H}}.$$

Remarque : Le théoreme 0.5.1 est habituellement exprimé sous la forme suivante :

Théorème 0.5.2. *Soit Ω un domaine connexe borné de $\mathbb{R}^n$. On suppose que sa frontière $\partial\Omega$ est de classe C^2 et vérifie (1) et (6). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) et (7) sont vérifiés.*

Il existe alors $T_0 > 0$ tel que, pour tout $T > T_0$ et pour tout $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, il existe $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ tel que la solution $\mathbf{y}$ du problème suivant :

$$\begin{cases} \mathbf{y}'' - \operatorname{div}(\sigma(\mathbf{y})) = 0 & \textit{dans } \Omega \times [0,T]; \\ \mathbf{y} = 0 & \textit{sur } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{y})\boldsymbol{\nu} = \boldsymbol{\phi} & \textit{sur } \partial\Omega_N \times [0,T]; \\ \mathbf{y}(0) = \mathbf{y}_0 & \textit{dans } \Omega; \\ \mathbf{y}'(0) = \mathbf{y}_1 & \textit{dans } \Omega; \end{cases}$$

vérifie $\mathbf{y}(T) = 0$ et $\mathbf{y}'(T) = 0$ sur Ω.

0.6 Stabilisation frontière non-linéaire

Nous voulons maintenant, à l'aide du résultat de contrôlabilité précédent, élargir le résultat de stabilisation, en considérant un feedback non-linéaire, c'est-à-dire de la forme $\mathbf{F} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{g}(\mathbf{u}')$. En fonction des propriétés de $\mathbf{g}$, nous n'obtiendrons plus forcément une décroissance exponentielle de l'énergie, mais parfois une décroissance polynomiale. On utilise pour cela le principe de Liu (cf [32]).

Soit $\Omega \subset \mathbb{R}^n$ un domaine connexe borné, dont la frontière $\partial\Omega$ est de classe C^2 et vérifie (1). On note $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$, et on suppose que Γ vérifie (6) (cf figure 16).

On note $\boldsymbol{\nu}(\mathbf{x})$ le vecteur normal unitaire en $\mathbf{x} \in \partial\Omega$, pointant hors de Ω. On suppose qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que, si on note $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, on a (5).

En tout point $\mathbf{s} \in \Gamma$, on considère Γ comme une sous variété de $\partial\Omega$ de codimension 1, et on note $\boldsymbol{\tau}(\mathbf{s})$ le vecteur normal unitaire à Γ, dirigé de $\partial\Omega_N$ vers $\partial\Omega_D$. On suppose alors que l'on a (7).

Soit $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ une fonction continue telle que $g(\mathbf{0}) = \mathbf{0}$. On suppose que

$$\mathbf{g}(\mathbf{x})\cdot\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{58}$$

On suppose de plus qu'il existe des constantes $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$, $p \geq 1$ et $q \in]0,1]$ tels que

$$c_1|\mathbf{x}|^p \leq |\mathbf{g}(\mathbf{x})| \leq c_2|\mathbf{x}|^q \quad \forall |\mathbf{x}| \leq 1\ ; \tag{59}$$

$$c_3|\mathbf{x}| \leq |\mathbf{g}(\mathbf{x})| \leq c_4|\mathbf{x}| \quad \forall |\mathbf{x}| > 1\ ; \tag{60}$$

où $|.|$ est ici la norme euclidienne sur $\mathbb{R}^n$.

On considère alors le système élastodynamique suivant :

$$\begin{cases} \mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) = 0 & \text{dans } \Omega \times \mathbb{R}_+; \\ \mathbf{u} = 0 & \text{sur } \partial\Omega_D \times \mathbb{R}_+; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{g}(\mathbf{u}') & \text{sur } \partial\Omega_N \times \mathbb{R}_+; \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{dans } \Omega; \\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{dans } \Omega. \end{cases} \tag{61}$$

Par la méthode des semi-groupes, on montre que le problème (61) est bien posé, sous les conditions (59), (60) et (4).

On veut prouver alors le théorème suivant :

Théorème 0.6.1. *Soit Ω un domaine connexe borné de $\mathbb{R}^n$. On suppose que sa frontière $\partial\Omega$ est de classe C^2 et vérifie (1) et (6). On suppose de plus qu'il existe $\mathbf{x}_0 \in \mathbb{R}^n$ tel que (5) et (7) sont vérifiés.*

Soit $\mathbf{g} \in C(\mathbb{R}^n, \mathbb{R}^n)$ une fonction continue telle que $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ et qui vérifie (58), (59) et (60).

Pour tout $(\mathbf{u}_0, \mathbf{u}_1)$ satisfaisant (4), la solution $\mathbf{u}$ de (61) vérifie

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C_0 e^{-\varpi_0 t} E(\mathbf{u}, 0), \quad \text{si } p = q = 1; \tag{62}$$

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C_1 (1+t)^{-\frac{r}{1-r}}, \quad \text{si } (p,q) \neq (1,1); \tag{63}$$

où $r = min(\frac{2}{p+1}, \frac{2q}{q+1})$, où $\varpi_0 > 0$ et $C_0 > 0$ sont des constantes indépendantes de $(\mathbf{u}_0, \mathbf{u}_1)$, et où $C_1 > 0$ est une constante dépendante de $(\mathbf{u}_0, \mathbf{u}_1)$.

Ce résultat a été obtenu par différents auteurs ([23, 1, 16]), pour des conditions géométriques plus restrictives. En particulier, les travaux précédents supposent que $\Gamma = \emptyset$.

Le résultat le plus général obtenant un résultat de stabilisation non linéaire à partir d'un résultat de stabilisation linéaire a été obtenu par Nicaise [32], pour une equation d'évolution hyperbolique abstraite.

Nous considérons ici le feedback, introduit dans [21], et utilisé dans [5], pour l'équation des ondes. Malgré tout, nous nous sommes inspirés de [16] pour de larges parts de notre preuve. Cependant, notre résultat peut être aussi vu comme une conséquence directe de [32].

Preuve : soit $\mathbf{u}$ la solution du problème (61). On considère ici les solutions fortes. Grâce au théorème 0.5.1, on prend $T > T_0$ et $\boldsymbol{\phi}$ tels que $\mathbf{y}$ solution du problème (52) vérifie $(\mathbf{y}(T), \mathbf{y}'(T)) = (\mathbf{u}(T), \mathbf{u}'(T))$.

On rappelle qu'on a :

$$E'(\mathbf{u},t) = -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{g}(\mathbf{u}')\cdot\mathbf{u}'\,\mathrm{d}\mathbf{x} \tag{64}$$

On multiplie la première équation de (61) et (52) par $\mathbf{y}'$ et $\mathbf{u}'$ respectivement, et on développe. On obtient :

$$E(\mathbf{u},T) = \frac{1}{2}\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(\mathbf{u}'\cdot\tilde{\boldsymbol{\phi}} - \mathbf{y}'\cdot\mathbf{g}(\mathbf{u}')\right)\,\mathrm{d}\Gamma\mathrm{d}t. \tag{65}$$

On distingue alors deux cas.

Premier cas : $p = q = 1$. Grâce à (59), (60) et (64), on obtient

$$E(\mathbf{u},T) \leq k\,(E(\mathbf{u},0) - E(\mathbf{u},T))^{\frac{1}{2}}\,E(\mathbf{u},T)^{\frac{1}{2}},$$

où k est une constante positive indépendante de $\mathbf{u}$ et t.

On a donc

$$E(\mathbf{u},T) \leq \frac{k^2}{1+k^2}E(\mathbf{u},0),$$

et donc, pour tout $m \in \mathbb{N}$, on a

$$E(\mathbf{u},mT) \leq \frac{1}{(1+k^2)^m}E(\mathbf{u},0).$$

On en déduit qu'il existe $C_0 > 0$ et $\varpi > 0$ tels que

$$E(\mathbf{u},t) \leq C_0 e^{-\varpi t}E(\mathbf{u},0),$$

ce qui donne (62). Dans ce cas, ce résultat permet de passer aux solutions faibles, par un argument de densité.

Second cas : $(p,q) \neq (1,1)$. Pour tout $t \in \mathbb{R}^+$, on coupe $\partial\Omega_N$:

$$\partial\Omega_N^+ = \{\mathbf{x} \in \partial\Omega_N : |\mathbf{u}'| > 1\}, \quad \partial\Omega_N^- = \{\mathbf{x} \in \partial\Omega_N : |\mathbf{u}'| \leq 1\}.$$

Puis, grâce à (59), (60) et (64), on a

$$E(\mathbf{u},0) \leq k\left((E(\mathbf{u},0) - E(\mathbf{u},T)) + (E(\mathbf{u},0) - E(\mathbf{u},T))^{\frac{2q}{q+1}} + (E(\mathbf{u},0) - E(\mathbf{u},T))^{\frac{2}{p+1}}\right).$$

De même, pour tout $t > 0$, on a

$$\begin{aligned} E(\mathbf{u},t) \quad &\leq \quad k\left((E(\mathbf{u},t) - E(\mathbf{u},t+T)) + (E(\mathbf{u},t) - E(\mathbf{u},t+T))^{\frac{2q}{q+1}}\right.\\ &\qquad \left. + (E(\mathbf{u},t) - E(\mathbf{u},t+T))^{\frac{2}{p+1}}\right), \end{aligned} \tag{66}$$

où k ne dépend pas de t.

On sait que $E(\mathbf{u},.)$ est une fonction positive décroissante. Elle admet donc une limite à l'infini. Or, le membre de droite de (66) tend vers 0 à l'infini. On obtient alors que

$$E(\mathbf{u},t) \underset{t\to\infty}{\longrightarrow} 0.$$

On prend un T suffisamment large pour que $E(\mathbf{u},T) < 1$. On a alors

$$(E(\mathbf{u},t))^{\frac{1}{r}} \leq k\left(E(\mathbf{u},t) - E(\mathbf{u},t+T)\right), \quad \forall t \geq T,$$

où $r = \min\{\frac{2q}{q+1}, \frac{2}{p+1}\}$, ce qui donne (63) (cf [16, 30]).

0.7 Perturbation de m

Aussi bien dans la définition de la partition de la frontière que dans le feedback, la fonction $\mathbf{m}$ joue un rôle central. Or, par exemple dans l'optique d'une application concrète, il peut être intéressant de s'affranchir de la définition très stricte de $\mathbf{m}$.

Comme dans nombre de travaux précédents ([22, 23, 3] par exemple), nos résultats s'étendent à des fonctions $\mathbf{m} \in \left(\mathcal{C}^1(\overline{\Omega})\right)^n$ telles qu'il existe $\alpha > 0$ tel que :

- $\sigma(\mathbf{v}):(\nabla\mathbf{v}\cdot\nabla\mathbf{m}) \geq \alpha\sigma(\mathbf{v}):\epsilon(\mathbf{v}), \quad \forall \mathbf{v} \in \left(\mathcal{C}^1(\overline{\Omega})\right)^n$;
- $\max\limits_{\overline{\Omega}}(\mathrm{div}(\mathbf{m})) - \min\limits_{\overline{\Omega}}(\mathrm{div}(\mathbf{m})) < 2\alpha$.

Chapitre 1

Stabilisation frontière du système élastodynamique
Partie I : Relations de type Rellich pour un problème de l'élasticité aux conditions frontière mélées

Boundary stabilization of elastodynamic systems. Part I : Rellich-type relations for a mixed boundary problem in elasticity.

Résumé en français Pour le système de Lamé, les conditions mélées génèrent des singularités dans la solution, en particulier lorsque la frontière du domaine est connexe. Nous prouvons ici des relations de type Rellich prenant en compte ces singularités.

Ces relations interviennent dans le problème de stabilisation frontière du système élastodynamique lorsqu'on utilise la méthode des multiplicateurs. Ce problème sera étudié dans la deuxième partie.

Ce chapitre est constitué d'une prépublication soumise à *SIAM, J. Control Optim.*

(n ° 383, http ://maply.univ-lyon1.fr/publis/publiv/2004/publis.html)

Les résultats obtenus dans ce travail ont été repris dans un proceeding [10]. De plus, le cas polygonal a fait l'objet d'une note aux CRAS [6]. Le proceeding et la note aux CRAS ont été placés en annexe.

English abstract For the Lamé's system, mixed boundary conditions generate singularities in the solution, mainly when the boundary of the domain is connected. We here prove Rellich relations involving these singularities.

These relations are useful in the problem of boundary stabilization of the elastodynamic system when using the multiplier method. This problem is studied in Part II.

Introduction

In this work, we present a detailed proof of a result which has been announced in [6] and some extension of this result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set such that its boundary satisfies

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N, \text{ with } \begin{cases} \partial\Omega_D \cap \partial\Omega_N = \emptyset, \\ \text{meas}(\partial\Omega_D) \neq 0, \\ \text{meas}(\partial\Omega_N) \neq 0. \end{cases} \tag{1.1}$$

We denote the boundary interface by $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$. At a given $\mathbf{x} \in \partial\Omega$, we can consider $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outward of Ω. For a regular vector field $\mathbf{v}$, we define the strain tensor and the stress tensor by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i \mathbf{v}_j + \partial_j \mathbf{v}_i), \qquad \sigma(\mathbf{v}) = 2\mu\epsilon(\mathbf{v}) + \lambda \text{div}(\mathbf{v}) I_n.$$

where λ and μ are the Lamé's coefficients and I_n is the identity matrix of $\mathbb{R}^n$.

We introduce following Sobolev spaces : $\mathbb{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbb{H}^s(\Omega) = (H^s(\Omega))^n$, $\forall s > 0$, and $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega) / \mathbf{v} = 0 \text{ on } \partial\Omega_D\}$.

We here consider the following mixed boundary problem :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{in } \Omega; \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega_D; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{h} & \text{on } \partial\Omega_N; \end{cases} \tag{1.2}$$

where $\mathbf{f} \in \mathbb{L}^2(\Omega)$, $\mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D)$ and $\mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N)$.

It is well known that this problem admits a unique solution in $\mathbb{H}^1(\Omega)$.

Furthermore, singularities are generated in the solution when the boundary interface Γ is non-empty. These singularities are described in [12, 27, 14] in the bi-dimensional case. Similar situations appear in Laplace problems and have been addressed by many authors (see for example [13, 4]).

We prove here integral relations for problem (1.2), of the type of those introduced in [34] by Rellich.

For a given point $\mathbf{x}_0$ in $\mathbb{R}^n$, we denote by $\mathbf{m}$ the following function : $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$.

For $\mathbf{v}_1$ and $\mathbf{v}_2$ two vector fields, let us define

$$\Theta(\mathbf{v}_1, \mathbf{v}_2) = 2(\sigma(\mathbf{v}_1)\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{v}_2 - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{v}_1):\epsilon(\mathbf{v}_2).$$

Rellich-type relations are estimate of following expression :

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\, d\mathbf{x} - (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\, d\mathbf{x} + \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\, d\boldsymbol{\gamma}.$$

We obtain in this paper Rellich-type relations in following cases :

- $\Omega \subset \mathbb{R}^n$ and $\Gamma = \emptyset$.
- $\Omega \subset \mathbb{R}^2$ is a polygonal domain and Γ contains two points. We there use results of [12, 27, 14].
- $\Omega \subset \mathbb{R}^2$ is a smooth domain of class $\mathcal{C}^2$ and Γ contains two points.
- $\Omega \subset \mathbb{R}^n$ is a smooth domain of class $\mathcal{C}^2$ and Γ is a $(n-2)$-submanifold.

We emphasize that these Rellich relations are useful for proving boundary stabilization results for the elastodynamic system by using the multiplier method as well as in [13, 21, 4]. This work will be published in a second part.

1.1 The case of the n-dimensional domain without interface

In this section, we assume (1.1) and that $\partial\Omega_N$ and $\partial\Omega_D$ are defined by means of above function $\mathbf{m}$,

$$\partial\Omega_N = \{\mathbf{x} \in \partial\Omega \ / \ \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) > 0\}\,; \quad \partial\Omega_D = \{\mathbf{x} \in \partial\Omega \ / \ \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) < 0\}\,; \tag{1.3}$$

so that (see figure 1.1)

$$\Gamma = \emptyset. \tag{1.4}$$

We obtain the following Rellich-type relation.

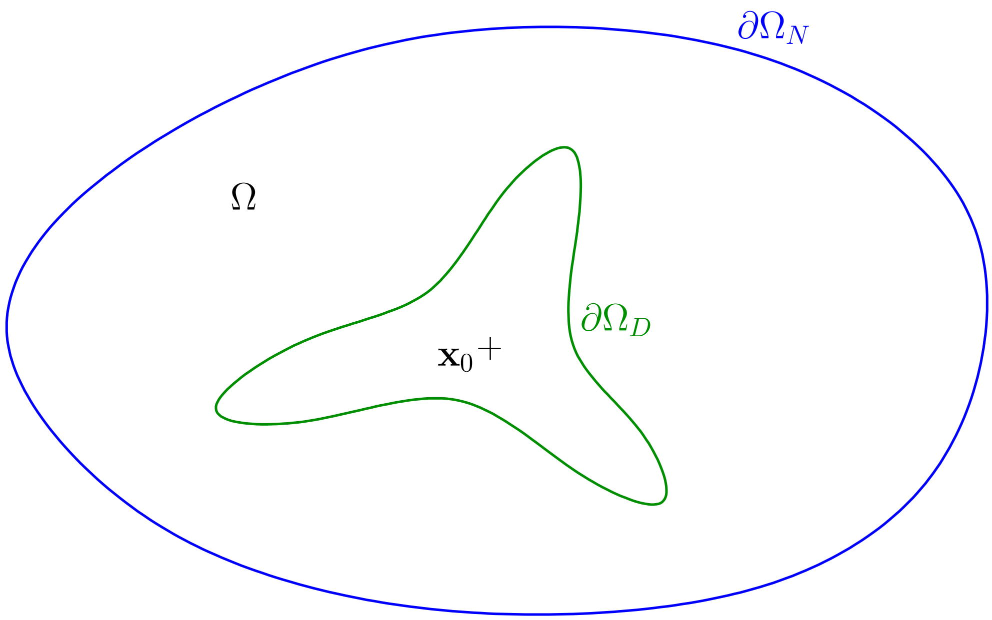

FIG. 1.1 – An example of a domain Ω without interface.

Théorème 1.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set, where $n \geq 2$, such that its boundary is of class $\mathcal{C}^2$ and satisfies (1.1) and (1.4). Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be the solution of problem (1.2) with*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N).$$

Then $\Theta(\mathbf{u}, \mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Proof. Our proof is composed of two steps : first we prove a hidden regularity result, secondly we apply Green's formula.

First step. Let us prove that $\mathbf{u} \in \mathbb{H}^2(\Omega)$.

Interior regularity. For every open subset $\mathcal{V}$ such that $\overline{\mathcal{V}} \subset \Omega$, multiplying by a cut-off function leads us to the following problem

$$-\operatorname{div}(\tilde{\sigma}(\tilde{\mathbf{u}})) = \mathbf{f}, \quad \text{in } \mathbb{R}^n.$$

We know that for this problem, $\mathbf{f} \in \mathbb{L}^2(\mathbb{R}^n)$ implies $\tilde{\mathbf{u}} \in \mathbb{H}^2(\mathbb{R}^n)$. So we have $\mathbf{u} \in \mathbb{H}^2(\mathcal{V})$.

Boundary regularity. let $\mathbf{x}$ be a point of $\partial\Omega_D$. We introduce a neighborhood $\mathcal{V}$ of $\mathbf{x}$ in $\mathbb{R}^n$ such that $\mathcal{V} \cap \partial\Omega = \mathcal{V} \cap \partial\Omega_D$. Multiplying by a cut-off function and changing the coordinates leads us to some problem in the following form

$$\begin{cases} -\operatorname{div}(\tilde{\sigma}(\tilde{\mathbf{u}})) = \bar{\mathbf{f}} & \text{in } \mathbb{R}^{n+}; \\ \tilde{\mathbf{u}} = \bar{\mathbf{g}} & \text{on } \mathbb{R}^{n-1} \times \{0\}; \end{cases}$$

where $\mathbb{R}^{n+} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \ / \ x_n > 0\}$, $\bar{\mathbf{f}} \in \mathbb{L}^2(\mathbb{R}^{n+})$ and $\bar{\mathbf{g}} \in \mathbb{H}^{3/2}(\mathbb{R}^{n-1})$.

Using a trace result, we can build $\tilde{\mathbf{U}} \in \mathbb{H}^2(\mathbb{R}^{n+})$ such that $\tilde{\mathbf{U}} = \bar{\mathbf{g}}$, on $\mathbb{R}^{n-1} \times \{0\}$.

We write $\tilde{\mathbf{f}} = \mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{U}})) - \mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{u}}))$. $\tilde{\mathbf{f}} \in \mathbb{L}^2(\mathbb{R}^{n+})$ and $\mathbf{U} = \tilde{\mathbf{u}} - \tilde{\mathbf{U}}$ is a solution of

$$\begin{cases} -\mathrm{div}(\tilde{\sigma}(\mathbf{U})) = \tilde{\mathbf{f}} & \text{in } \mathbb{R}^{n+}; \\ \mathbf{U} = 0 & \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{cases}$$

Using the differential quotients method, we prove that $\tilde{\mathbf{u}} \in \mathbb{H}^2(\mathbb{R}^{n+})$. So we have $\mathbf{u} \in \mathbb{H}^2(\Omega \cap \mathcal{V})$.

And now, let $\mathbf{x}$ be a point of $\partial\Omega_N$. By a similar method, we prove that $\mathbf{u}$ is $\mathbb{H}^2$ in some neighborhood of $\mathbf{x}$.

Finally, since $\overline{\Omega}$ is compact, we can conclude that $\mathbf{u} \in \mathbb{H}^2(\Omega)$.

Second step. Green's formula.

We can now apply Green's formula a first time,

$$\begin{aligned} 2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} &= 2\int_{\partial\Omega} (\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma} \\ &\quad -2\int_\Omega \sigma(\mathbf{u}):\nabla((\mathbf{m}\cdot\nabla)\mathbf{u})\,\mathrm{d}\mathbf{x} \\ &= 2\int_{\partial\Omega} (\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma} - 2\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} \\ &\quad - \int_\Omega \nabla(\sigma(\mathbf{u}):\epsilon(\mathbf{u}))\cdot\mathbf{m}\,\mathrm{d}\mathbf{x} \end{aligned}$$

and a second time,

$$\begin{aligned} 2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} &= 2\int_{\partial\Omega} (\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma} - 2\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} \\ &\quad - \int_{\partial\Omega} (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} + n\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Hence, we get the required results. □

1.2 The case of a bi-dimensional polygonal domain

We here study the case of a convex polygonal domain $\Omega \subset \mathbb{R}^2$. We assume that its boundary $\partial\Omega$ satisfies (1.1) and furthermore,

$$\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}, \tag{1.5}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ will be considered as two vertices of $\partial\Omega$, (see figure 1.2).

We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that

$$(\mathbf{m}\cdot\boldsymbol{\nu}) \geq 0, \text{ on } \partial\Omega_N; \qquad (\mathbf{m}\cdot\boldsymbol{\nu}) \leq 0, \text{ on } \partial\Omega_D. \tag{1.6}$$

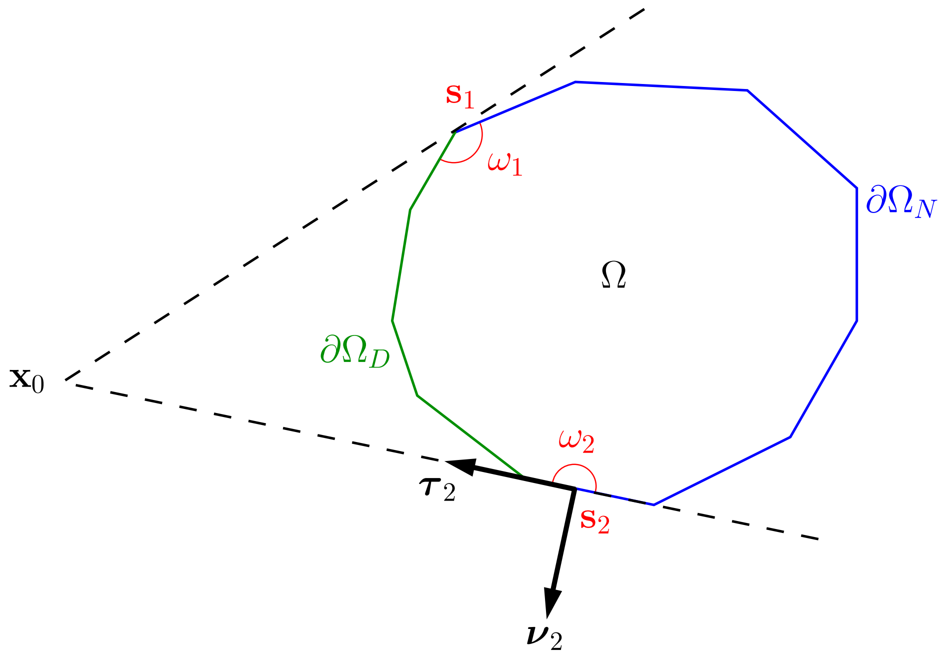

FIG. 1.2 – A polygonal domain Ω with a non-empty interface $\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}$.

Let us recall that changing boundary conditions generates singularities in the solution of our problem. We will denote by $c(\mathbf{s})$ the coefficient of the singularity associated to point $\mathbf{s}$.

At each point $\mathbf{s}_i$, we define $\omega(\mathbf{s}_i)$ the angle of Ω. If $\omega(\mathbf{s}_i) = \pi$, we denote $\boldsymbol{\tau}(\mathbf{s}_i)$ the unit tangent vector to $\partial\Omega$ pointing toward $\partial\Omega_D$, (see figure 1.2). Let us observe that condition (1.6) leads to

$$\omega(\mathbf{s}_i) = \pi \Rightarrow \mathbf{m}(\mathbf{s}_i)\cdot\boldsymbol{\nu}(\mathbf{s}_i) = 0. \tag{1.7}$$

Let us define

$$\Upsilon = 8\frac{(2\mu+\lambda)(3\mu+\lambda)}{\pi\mu}\left(\pi^2 + \ln^2\left(\frac{3\mu+\lambda}{\mu+\lambda}\right)\right).$$

We here obtain a Rellich-type relation in the following form :

Théorème 1.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal open set such that its boundary satisfies (1.1), (1.5) and (1.7). Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be the solution of problem (1.2) with*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N).$$

Then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} + \Upsilon\sum_{\substack{\mathbf{s}\in\{\mathbf{s}_1,\mathbf{s}_2\}\\ \omega(\mathbf{s})=\pi}} c(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})) \tag{1.8}$$

where $c(\mathbf{s})$ is the singularity coefficient of $\mathbf{u}$ at point $\mathbf{s}$.

We emphasize that in (1.8), singular term $c(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))$ appears only if $\omega(\mathbf{s}) = \pi$. In order to prove this theorem, we use results of P. Grisvard and B. Merouani [12, 27, 14] and we proceed as well as in [13].

Proof. As well as for theorem 1.1.1, we can prove that for every open set $\mathcal{V}$ such that $\overline{\mathcal{V}} \subset \Omega$, $\mathbf{u} \in \mathbb{H}^2(\mathcal{V})$. Similarly, for every $\mathbf{x} \in \partial\Omega \setminus \Gamma$ such that $\mathbf{x}$ is not a vertex, there exists a neighborhood $\mathcal{V}$ of $\mathbf{x}$ such that $\mathbf{u} \in \mathbb{H}^2(\mathcal{V} \cap \Omega)$.

For a vertex $\tilde{\mathbf{s}} \in \partial\Omega \setminus \Gamma$, we use [12, 27, 14] to prove that there exists a neighborhood $\mathcal{V}$ of $\tilde{\mathbf{s}}$ such that $\mathbf{u} \in \mathbb{H}^2(\mathcal{V} \cap \Omega)$.

Now consider $\mathbf{s} \in \Gamma$. Let γ_0 and γ_1 be the adjacent edges to $\mathbf{s}$. Using a cut-off function, we can suppose that $\mathbf{u}$ vanishes outside a disk $D(\mathbf{s}, \delta)$. We can suppose that δ is small enough so that $D(\mathbf{s}, \delta) \cap \partial\Omega \subset \gamma_0 \cup \gamma_1$.

We take some $\varepsilon > 0$ and define (see figure 1.3)

$$\begin{cases} \mathcal{H}_\varepsilon = \{(r, \theta) \in (\varepsilon, +\infty) \times (0, \omega)\}, \\ \partial\mathcal{H}_{D\varepsilon} = \{(r, 0) \ / \ r > \varepsilon\}, \\ \partial\mathcal{H}_{N\varepsilon} = \{(r, \omega) \ / \ r > \varepsilon\}, \\ \tilde{\Gamma}_\varepsilon = \{(\varepsilon, \theta) \ / \ \theta \in (0, \omega)\}, \end{cases}$$

so that $\partial\mathcal{H}_\varepsilon = \partial\mathcal{H}_{D\varepsilon} \cup \partial\mathcal{H}_{N\varepsilon} \cup \tilde{\Gamma}_\varepsilon$.

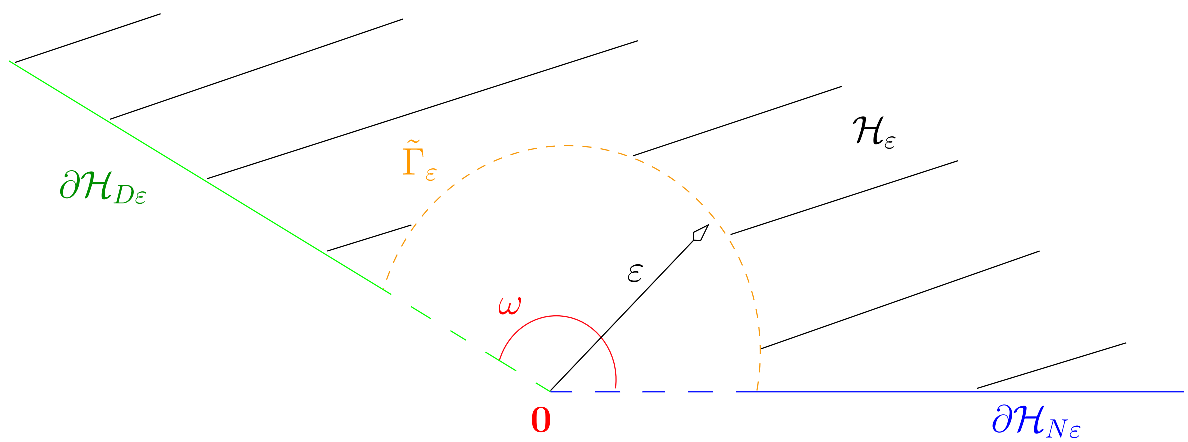

FIG. 1.3 – An example of $\mathcal{H}_\varepsilon$ for $\omega < \pi$.

Then $\mathbf{u} \in \mathbb{H}^2(\mathcal{H}_\varepsilon)$ and theorem 1.1.1 gives

$$2\int_{\mathcal{H}_\varepsilon} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla\mathbf{u})\,\mathrm{d}\mathbf{x} = \int_{\partial\mathcal{H}_\varepsilon} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}. \tag{1.9}$$

Using Lebesgue's theorem, we get easily

$$\int_{\mathcal{H}_\varepsilon} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{H}} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x}.$$

We now consider the convergence of the right-hand side of (1.9). Projecting method and changing coordinates leads us to consider the following problem

$$\begin{cases} -\mathrm{div}(\sigma(\tilde{\mathbf{u}})) = \tilde{\mathbf{f}} & \text{in } \mathcal{H}; \\ \tilde{\mathbf{u}} = 0 & \text{on } \partial\mathcal{H}_D; \\ \sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{on } \partial\mathcal{H}_N; \end{cases} \tag{1.10}$$

where $\mathcal{H} = \{(r,\theta) \in \mathbb{R}^{*+} \times (0,\omega)\}$, $\partial\mathcal{H}_D = \{(r,0) \ / \ r > 0\}$, $\partial\mathcal{H}_N = \{(r,\omega) \ / \ r > 0\}$ in polar coordinates, and where $\tilde{\mathbf{f}} \in \mathbb{L}^2(\mathcal{H})$.

Hence, we can write $\mathbf{u} = \mathbf{U} + \tilde{\mathbf{u}}$ where $\mathbf{U} \in \mathbb{H}^2(\mathcal{H})$. We use the structure of the solution of (1.10) which is given in [12, 27, 14]. We will now consider two cases.

1.2.1 First case : $\omega < \pi$

Let us denote by ν the Poisson's coefficient of the system, i.e. $\nu = \dfrac{1}{2}\dfrac{\lambda}{\lambda+\mu}$.

Following [12, 27, 14], we have to consider this equation in α,

$$\sin^2(\alpha\omega) = \frac{4(1-\nu)^2 - \alpha^2 \sin^2\omega}{3-4\nu}. \tag{1.11}$$

Let $(\alpha_i)_{i=1,K}$ be the sequence of complex roots of (1.11) such that $\Re\alpha \in]0,1]$. From [12, 27, 14], we get

$$\begin{gathered} \exists \tilde{\mathbf{u}}_R \in \mathbb{H}^2(\mathcal{H}),\ \forall i \in [1,K], \exists \mathbf{v}_i^1, \mathbf{v}_i^2 \in (C^\infty([0,\omega],\mathbb{C}))^2 : \\ \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_R + \textstyle\sum_{i=1}^K \Re[r^{\alpha_i}(\mathbf{v}_i^1(\theta) + \ln(r)\mathbf{v}_i^2(\theta))]. \end{gathered}$$

We write $\mathbf{u}_R = \mathbf{U} + \tilde{\mathbf{u}}_R$, and we have :

$$\mathbf{u} = \mathbf{u}_R + \sum_{i=1}^K \Re[r^{\alpha_i}(\mathbf{v}_i^1(\theta) + \ln(r)\mathbf{v}_i^2(\theta))]. \tag{1.12}$$

The main point of our proof lies in the following lemma which will be proved at the end of this section.

Lemme 1.2.1. $\forall i \in [1,K],\ \Re\alpha_i > \dfrac{1}{2}$.

Hence, for the Rellich-type relation, there is no corrective term. We then prove

$$\int_{\partial\mathcal{H}_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\mathcal{H}} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}. \tag{1.13}$$

We know that $\mathbf{u}$ can be written in the form (1.12). Using this form, we expand the left-hand side in (1.13) and we have to consider $\mathbf{u}_1$ and $\mathbf{u}_2$ two functions taken among the following ones : $\mathbf{u}_R$; $\Re[r^{\alpha_i}\mathbf{v}_i^1(\theta)]$; $\Re[r^{\alpha_i}\ln(r)\mathbf{v}_i^2(\theta)]$.

We have, $\exists C > 0 :\ |\Theta(\mathbf{u}_1,\mathbf{u}_2)| \leq C|\nabla\mathbf{u}_1||\nabla\mathbf{u}_2|$.

First case : $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_R$

In this case, $\mathbf{u}_R \in \mathbb{H}^2(\mathcal{H})$, we can then use Lebesgue's theorem to get the result.

Second case : $\mathbf{u}_1 = \mathbf{u}_R$ and $\mathbf{u}_2 = \Re[r^{\alpha_i}\mathbf{v}_i^1(\theta)]$

We have, $\exists C > 0 : \ |\nabla \mathbf{u}_2| \leq C r^{\Re\alpha_i - 1}$.
Hence, $\exists C > 0 : \ |\Theta(\mathbf{u}_1, \mathbf{u}_2)| \leq C |\nabla \mathbf{u}_1| r^{\Re\alpha_i - 1}$.
Now, with lemma 1.2.1, $\Re\alpha_i - 1 > -1/2$. Then we use Lebesgue's theorem and we get

$$\int_{\partial\mathcal{H}_{D\varepsilon}} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \xrightarrow[\varepsilon \to 0]{} \int_{\partial\mathcal{H}_D} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma};$$

$$\int_{\partial\mathcal{H}_{N\varepsilon}} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \xrightarrow[\varepsilon \to 0]{} \int_{\partial\mathcal{H}_N} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma}.$$

Moreover, there exists $C > 0$ such that

$$\left| \int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \right| = \varepsilon \left| \int_0^\omega \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\theta \right| \leq C \varepsilon^{\Re\alpha_i} \xrightarrow[\varepsilon \to 0]{} 0,$$

since $\Re\alpha_i > 1/2$.
Then (1.13) is satisfied.
Remark. We get a similar result with $\mathbf{u}_2 = \Re[r^{\alpha_i} \ln(r) \mathbf{v}_i^2(\theta)]$. We get also the same result, reversing the roles of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Third case : $\mathbf{u}_1 = \Re[r^{\alpha_i}\mathbf{v}_i^1(\theta)]$ and $\mathbf{u}_2 = \Re[r^{\alpha_j}\mathbf{v}_j^1(\theta)]$

We have, $\exists C > 0 : \ |\nabla \mathbf{u}_1| \leq C r^{\Re\alpha_i - 1}, \ |\nabla \mathbf{u}_2| \leq C r^{\Re\alpha_j - 1}$.
Then, $\exists C > 0 : \ |\Theta(\mathbf{u}_1, \mathbf{u}_2)| \leq C r^{\Re\alpha_i + \Re\alpha_j - 2}$.
Now, with lemma 1.2.1, $\Re\alpha_i + \Re\alpha_j - 2 > -1$. Then we use Lebesgue's theorem and we get

$$\int_{\partial\mathcal{H}_{D\varepsilon}} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \xrightarrow[\varepsilon \to 0]{} \int_{\partial\mathcal{H}_D} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma};$$

$$\int_{\partial\mathcal{H}_{N\varepsilon}} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \xrightarrow[\varepsilon \to 0]{} \int_{\partial\mathcal{H}_N} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma}.$$

Moreover, there exists $C > 0$ such that

$$\left| \int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\boldsymbol{\gamma} \right| = \varepsilon \left| \int_0^\omega \Theta(\mathbf{u}_1, \mathbf{u}_2)\, \mathrm{d}\theta \right| \leq C \varepsilon^{\Re\alpha_i + \Re\alpha_j - 1} \xrightarrow[\varepsilon \to 0]{} 0,$$

since $\Re\alpha_i + \Re\alpha_j - 1 > 0$.
Then (1.13) is satisfied.
Remark. we get a similar result with $\mathbf{u}_1$, $\mathbf{u}_2$ or both in the form $\Re[r^{\alpha_i} \ln(r) \mathbf{v}_i^2(\theta)]$. □

Proof of lemma 1.2.1

This lemma has been proved in a different form especially in [37, 33].
Assume that a solution of (1.11) $\alpha = \gamma + i\eta$ is such that $\gamma \in]0, 1/2[$.

We have

$$\begin{aligned}\sin^2(\alpha\omega) &= \frac{1-\cos(2\alpha\omega)}{2}\\ &= \frac{1}{2}(1-\cos(2\gamma\omega)\cosh(2\eta\omega)+i\sin(2\gamma\omega)\sinh(2\eta\omega)).\end{aligned}$$

Moreover,

$$\frac{4(1-\nu)^2-\alpha^2\sin^2\omega}{3-4\nu}=\frac{4(1-\nu)^2-(\gamma^2-\eta^2+2i\gamma\eta)\sin^2\omega}{3-4\nu}.$$

We then get

$$\left\{\begin{aligned}(3-4\nu)(1-\cos(2\gamma\omega)\cosh(2\eta\omega)) &= 2(4(1-\nu)^2+(\eta^2-\gamma^2)\sin^2\omega),\\ (3-4\nu)\sin(2\gamma\omega)\sinh(2\eta\omega) &= -4\gamma\eta\sin^2\omega.\end{aligned}\right. \tag{1.14}$$

The first equation of (1.14) is even with respect to η, the second is odd. We conclude that, if α is a solution of (1.11), then $\overline{\alpha}$ also is.

We then assume in the following $0<\gamma\leq 1/2$ and $\eta\geq 0$.

We rewrite the second equation of (1.14)

$$(3-4\nu)\sin(2\gamma\omega)\sinh(2\eta\omega)+4\gamma\eta\sin^2\omega=0. \tag{1.15}$$

Let us remind that $0<\nu<1/2,\ 0<\omega<\pi$.

We then have in (1.15)

$$\left\{\begin{aligned}(3-4\nu)\sin(2\gamma\omega)\sinh(2\eta\omega) &\geq 0,\\ \gamma\eta\sin^2\omega &\geq 0.\end{aligned}\right.$$

And

$$\left\{\begin{aligned}(3-4\nu)\sin(2\gamma\omega)\sinh(2\eta\omega) &= 0,\\ \gamma\eta\sin^2\omega &= 0.\end{aligned}\right.$$

Hence $\eta=0$, and the first equation of (1.14) gives

$$(3-4\nu)\cos(2\gamma\omega)+8(1-\nu)^2-(3-4\nu)-2\gamma^2\sin^2\omega=0.$$

Since $3-4\nu>0$, for every $\gamma\in]0,1/2]$, we have

$$\begin{aligned}0 &\geq (3-4\nu)\cos\omega+8(1-\nu)^2-(3-4\nu)-\frac{1}{2}\sin^2\omega\\ &= \frac{1}{2}(\cos^2\omega+2(3-4\nu)\cos\omega+16\nu^2-24\nu+9\\ &= \frac{1}{2}(\cos\omega+(3-4\nu))^2\\ &> 0.\end{aligned}$$

which is impossible. □

1.2.2 Second case : $\omega = \pi$

Structure of the solution

Once again, we use [12, 27, 14].
In this case, equation (1.11) becomes

$$\sin^2(\alpha\pi) = \frac{4(1-\nu)^2 - \alpha^2}{3-4\nu}. \tag{1.16}$$

The roots of (1.16) such that $\Re\alpha \in]0,1]$ are

$$\alpha = \frac{1}{2} + ik \text{ and } \overline{\alpha} = \frac{1}{2} - ik, \text{ where } k = \frac{\ln(3-4\nu)}{2\pi}.$$

Then the singular solution of (1.10) is $\mathbf{u}_S(r,\theta) = \Re\left(r^\alpha \mathbf{v}(\theta)\right)$.
We need to know exactly $\mathbf{u}_S$. To this end, we introduce the following intermediate functions

$$\begin{cases} a(\theta) = \nu_0\left[\sin\left(-\frac{3}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right) - 2k\left(\cos\left(-\frac{3}{2}\theta\right) - \cos\left(\frac{1}{2}\theta\right)\right)\right]e^{k\theta}; \\ b(\theta) = \nu_0\left[\cos\left(-\frac{3}{2}\theta\right) - \cos\left(\frac{1}{2}\theta\right) + 2k\left(\sin\left(-\frac{3}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)\right)\right]e^{k\theta}; \\ m(\theta) = 4(\nu_0+2)\sin\left(\frac{1}{2}\theta\right)\cosh(k\theta); \\ n(\theta) = 4(\nu_0+2)\cos\left(\frac{1}{2}\theta\right)\sinh(k\theta); \end{cases}$$

where $\nu_0 = \dfrac{\lambda+\mu}{\mu}$.
We have

$$\mathbf{u}_S(r,\theta) = \sqrt{r}\left[\cos(k\ln(r))\begin{pmatrix} a(\theta) - m(\theta) \\ b(\theta) + n(\theta)\end{pmatrix} + \sin(k\ln(r))\begin{pmatrix} -b(\theta) + n(\theta) \\ a(\theta) + m(\theta)\end{pmatrix}\right]. \tag{1.17}$$

Then we get the following result

$$\exists! \tilde{\mathbf{u}}_R \in \mathbb{H}^2(\mathcal{H}),\ \exists! c_S \in \mathbb{R}:\ \tilde{\mathbf{u}} = \tilde{\mathbf{u}}_R + c_S\mathbf{u}_S.$$

and then, if we write $\mathbf{u}_R = \mathbf{U} + \tilde{\mathbf{u}}_R$, we have

$$\mathbf{u} = \mathbf{u}_R + c_S\mathbf{u}_S. \tag{1.18}$$

End of the proof of theorem 1.2.1 in the second case

As well as in paragraph 1.2.1, we use (1.18) and expand $\Theta(\mathbf{u},\mathbf{u})$. Similarly, terms where $\mathbf{u}_R$ appears converge to 0.
We now consider the remaining term, quadratic with respect to $\mathbf{u}_S$.
Since $\omega = \pi$, we have $(\mathbf{m}\cdot\boldsymbol{\nu}) = 0$, on $\partial\mathcal{H}$.
Hence, $\sigma(\mathbf{u}_S)\boldsymbol{\nu} = 0$, on $\partial\mathcal{H}_N$. Then $\Theta(\mathbf{u}_S,\mathbf{u}_S) = 0$, on $\partial\mathcal{H}_N$.
Moreover, $\mathbf{u}_S = 0$, on $\partial\mathcal{H}_D$. This implies that $(\mathbf{m}\cdot\nabla)\mathbf{u}_S = (\mathbf{m}\cdot\boldsymbol{\nu})(\boldsymbol{\nu}\cdot\nabla)\mathbf{u}_S$. Then $\Theta(\mathbf{u}_S,\mathbf{u}_S) = (\mathbf{m}\cdot\boldsymbol{\nu})(2(\sigma(\mathbf{u}_S)\boldsymbol{\nu})\cdot(\boldsymbol{\nu}\cdot\nabla)\mathbf{u}_S - \sigma(\mathbf{u}_S):\epsilon(\mathbf{u}_S)) = 0$, on $\partial\mathcal{H}_D$.
It only remains to compute the integral on $\tilde{\Gamma}_\varepsilon$.

For a given $\varepsilon > 0$, Maple computation gives

$$\int_{\partial\Gamma_\varepsilon} \Theta(\mathbf{u}_S, \mathbf{u}_S)\,\mathrm{d}\boldsymbol{\gamma} = \Upsilon(\mathbf{m}(0)\cdot\boldsymbol{\tau}(0)) + \mathcal{O}(\varepsilon).$$

We then have

$$\lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}, \mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} = c_S^2 \lim_{\varepsilon\to 0}\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u}_S, \mathbf{u}_S)\,\mathrm{d}\boldsymbol{\gamma} = \Upsilon c_S^2(\mathbf{m}(0)\cdot\boldsymbol{\tau}(0)).$$

This achieves the proof of theorem 1.2.1. □

1.3 The case of a bi-dimensional domain

We here study the case of a bounded connected domain $\Omega \subset \mathbb{R}^2$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (1.1) and (1.5) (In (1.5), $\mathbf{s}_1$ and $\mathbf{s}_2$ are two points of $\partial\Omega$ (see figure 1.4).).

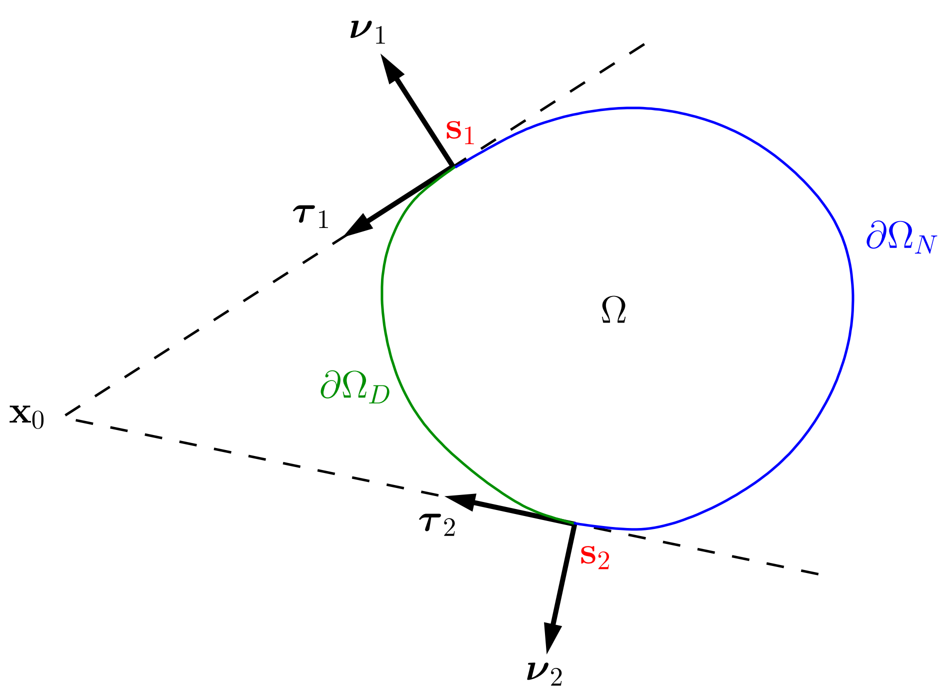

FIG. 1.4 – A smooth bi-dimensional domain Ω with a non-empty interface.

We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that (1.6) is satisfied. At each point $\mathbf{s}_i$, we denote $\boldsymbol{\tau}(\mathbf{s}_i)$ the unit tangent vector to $\partial\Omega$ pointing toward $\partial\Omega_D$.

It can be observed that

$$\mathbf{m}(\mathbf{s}_1)\cdot\boldsymbol{\nu}(\mathbf{s}_1) = \mathbf{m}(\mathbf{s}_2)\cdot\boldsymbol{\nu}(\mathbf{s}_2) = 0. \tag{1.19}$$

In this case, the Rellich-type relation can be written as follows :

Théorème 1.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain of class $\mathcal{C}^2$ which satisfies (1.1), (1.5) and (1.19). Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be a solution of problem (1.2) with*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{1.20}$$

Then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} + \Upsilon \sum_{\mathbf{s}\in\{\mathbf{s}_1,\mathbf{s}_2\}} c(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))$$

where $c(\mathbf{s})$ is the singularity coefficient of $\mathbf{u}$ at $\mathbf{s}$.

Once again, we need to know the structure of the studied functions.

1.3.1 The case of the semi-disk

We first study the case $\Omega = D^+(\rho)$, where $\rho > 0$ so that in polar coordinates, we may define $D^+(\rho) = (0,\rho)\times(0,\pi)$, $\partial D^+_N(\rho) = (0,\rho)\times\{\pi\}$ and $\partial D^+_D(\rho) = (0,\rho)\times\{0\}\cup\{\rho\}\times(0,\pi)$, (see figure 1.5) and we consider the problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{in } D^+(\rho); \\ \mathbf{u} = 0 & \text{on } \partial D^+_D(\rho); \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial D^+_N(\rho). \end{cases} \tag{1.21}$$

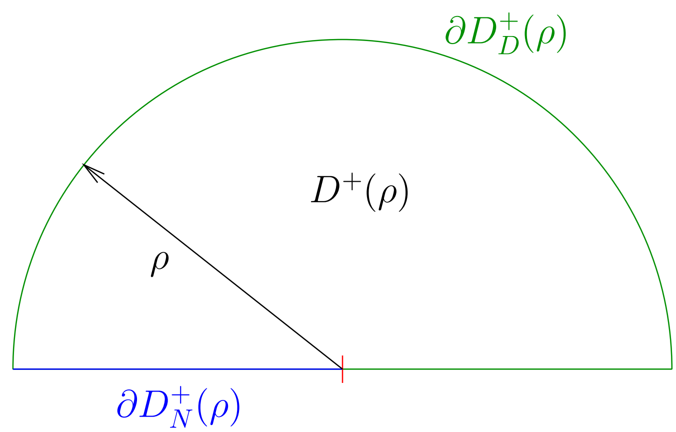

Fig. 1.5 – The considered domain is a semi-disk.

We denote by $\tilde{\mathbf{u}}_S$ the function defined by (1.17) in the whole half-plane. Let ς be a cut-off function belonging to $C^\infty(\mathbb{R}^+,[0,1])$ such that $\operatorname{supp}(\varsigma)\subset[0,\rho_2]$ and $\varsigma\equiv 1$ on $[0,\rho_1]$, with $0<\rho_1<\rho_2<\rho$. We can define

$$\mathbf{u}_S(r,\theta) = \tilde{\mathbf{u}}_S(r,\theta)\varsigma(r). \tag{1.22}$$

Using results of paragraph 1.2.2, we get

Théorème 1.3.2. *Let* $\mathbf{u}$ *be the solution of problem (1.21) where* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$. *We have*

$$\exists! \mathbf{u}_R \in \mathbb{H}^2(D^+(\rho)),\ \exists! c_S \in \mathbb{R} :\ \mathbf{u} = \mathbf{u}_R + c_S \mathbf{u}_S,$$

where $\mathbf{u}_S$ *is defined by (1.22).*

This theorem means that, if we define the operator $\mathcal{A}_\rho$ by

$$\begin{cases} D(\mathcal{A}_\rho) = \{\mathbf{v} \in \mathbb{H}^1_D(D^+(\rho)) / \mathcal{A}_\rho \mathbf{v} \in \mathbb{L}^2(D^+(\rho)),\, \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ on } \partial D^+_N(\rho)\} \\ \mathcal{A}_\rho \mathbf{v} = -\operatorname{div}(\sigma(\mathbf{v})), \end{cases}$$

we get

$$D(\mathcal{A}_\rho) \subset \mathbb{H}^2(D^+(\rho)) \oplus \mathbb{R}\mathbf{u}_S.$$

We want now to characterize the singular coefficient c_S according to $\mathbf{f}$. To this end, we introduce the following intermediate functions

$$\begin{cases} \tilde{a}(\theta) = \nu_0 \left[\sin\left(-\frac{5}{2}\theta\right) - \sin\left(-\frac{1}{2}\theta\right) + 2k\left(\cos\left(-\frac{5}{2}\theta\right) - \cos\left(-\frac{1}{2}\theta\right)\right)\right] e^{k\theta}; \\ \tilde{b}(\theta) = \nu_0 \left[\cos\left(-\frac{5}{2}\theta\right) - \cos\left(-\frac{1}{2}\theta\right) - 2k\left(\sin\left(-\frac{5}{2}\theta\right) - \sin\left(-\frac{1}{2}\theta\right)\right)\right] e^{k\theta}; \\ \tilde{m}(\theta) = 4(\nu_0 + 2)\sin\left(-\frac{1}{2}\theta\right)\cosh(k\theta); \\ \tilde{n}(\theta) = 4(\nu_0 + 2)\cos\left(-\frac{1}{2}\theta\right)\sinh(k\theta); \end{cases}$$

where k and ν_0 have been introduced in subsection 1.2.2.

We then define

$$\boldsymbol{\Sigma}^*(r,\theta) = \frac{\cos(k\ln(r))}{\sqrt{r}} \begin{pmatrix} \tilde{a}(\theta) + \tilde{m}(\theta) \\ \tilde{b}(\theta) - \tilde{n}(\theta) \end{pmatrix} + \frac{\sin(k\ln(r))}{\sqrt{r}} \begin{pmatrix} -\tilde{b}(\theta) - \tilde{n}(\theta) \\ \tilde{a}(\theta) - \tilde{m}(\theta) \end{pmatrix}.$$

In the distributions sense, $\boldsymbol{\Sigma}^*$ is a $\mathbb{L}^2$ solution of the following problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } D^+(\rho); \\ \mathbf{u} = 0 & \text{on } (0,\rho) \times \{0\}; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial D^+_N(\rho). \end{cases}$$

Let us now consider $\mathbf{g}(\theta) = \boldsymbol{\Sigma}^*(\rho,\theta)$, for every $\theta \in]0,\pi[$. We have $\mathbf{g} \in C^\infty([0,\pi])$. Let $\boldsymbol{\Psi}^* \in \mathbb{H}^1(D^+(\rho))$ be a weak solution of the following problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } D^+(\rho); \\ \mathbf{u} = 0 & \text{on } (0,\rho) \times \{0\}; \\ \mathbf{u} = \mathbf{g} & \text{on } \{\rho\} \times (0,\pi); \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial D^+_N(\rho). \end{cases}$$

We define $\mathbf{S}^* = \boldsymbol{\Sigma}^* - \boldsymbol{\Psi}^*$. $\mathbf{S}^*$ is then, in sense of distributions, a $\mathbb{L}^2$ solution of the following problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } D^+(\rho); \\ \mathbf{u} = 0 & \text{on } \partial D^+_D(\rho); \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial D^+_N(\rho). \end{cases}$$

We get the following result

Théorème 1.3.3. *For* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$*, let* $\mathbf{u}$ *be the solution of problem (1.21) and let* c_S *be the singular coefficient of this solution. There exists* $\varkappa \in \mathbb{R}^*$*, independent of* $\mathbf{f}$*, such that*

$$c_S = \frac{1}{\varkappa} \int_{D^+(\rho)} \mathbf{S}^* \cdot \mathbf{f} \, \mathrm{d}\mathbf{x}.$$

Proof. Let $(\mathbf{v}_n)_{n\in\mathbb{N}} \in \mathbb{H}^1_D(D^+(\rho))^{\mathbb{N}}$ be such that $(\mathbf{v}_n)$ tends to $\mathbf{S}^*$ in $\mathbb{L}^2(D^+(\rho))$.
For all $n \in \mathbb{N}$, we have

$$\int_{D^+(\rho)} \sigma(\mathbf{u}_R) : \epsilon(\mathbf{v}_n) \, \mathrm{d}\mathbf{x} + c_S \int_{D^+(\rho)} \sigma(\mathbf{u}_S) : \epsilon(\mathbf{v}_n) \, \mathrm{d}\mathbf{x} = \int_{D^+(\rho)} \mathbf{f} \cdot \mathbf{v}_n \, \mathrm{d}\mathbf{x}.$$

Now, we have of course

$$\int_{D^+(\rho)} \mathbf{f} \cdot \mathbf{v}_n \, \mathrm{d}\mathbf{x} \underset{n\to\infty}{\longrightarrow} \int_{D^+(\rho)} \mathbf{f} \cdot \mathbf{S}^* \, \mathrm{d}\mathbf{x}.$$

Moreover,

$$\begin{aligned} \int_{D^+(\rho)} \sigma(\mathbf{u}_R) : \epsilon(\mathbf{v}_n) \, \mathrm{d}\mathbf{x} &= -\int_{D^+(\rho)} \mathrm{div}(\sigma(\mathbf{u}_R)) \cdot \mathbf{v}_n \, \mathrm{d}\mathbf{x} \\ &\underset{n\to\infty}{\longrightarrow} -\int_{D^+(\rho)} \mathrm{div}(\sigma(\mathbf{u}_R)) \cdot \mathbf{S}^* \, \mathrm{d}\mathbf{x} \;=\; 0, \end{aligned}$$

since $\mathbf{u}_R = \mathbf{u} - c_S \mathbf{u}_S$ satisfies the homogeneous Neumann condition on $\partial D^+_N(\rho)$.
Then, there exists $\varkappa \in \mathbb{R}$ such that

$$\int_{D^+(\rho)} \sigma(\mathbf{u}_S) : \epsilon(\mathbf{v}_n) \, \mathrm{d}\mathbf{x} \underset{n\to\infty}{\longrightarrow} \varkappa.$$

We then get

$$\varkappa c_S = \int_{D^+(\rho)} \mathbf{S}^* \cdot \mathbf{f} \, \mathrm{d}\mathbf{x}.$$

Now, $\varkappa$ is independent of $\mathbf{f}$. Thus $\varkappa \in \mathbb{R}^*$ and the theorem is proved. □

We want now to get an estimate of the $\mathbb{H}^2$-norm of the regular part according to $\mathbf{f}$.

Théorème 1.3.4. *For* $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$*, let* $\mathbf{u} \in D(\mathcal{A}_\rho)$ *be the solution of problem (1.21). Let* $\mathbf{u}_R \in \mathbb{H}^2(D^+(\rho))$ *be the regular part of* $\mathbf{u}$*. There exists* $C > 0$ *independent of* $\mathbf{f}$ *such that*

$$\|\mathbf{u}_R\|_{\mathbb{H}^2(D^+(\rho))} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))}.$$

Proof. We define $\mathbb{L}^2_R(D^+(\rho)) = \{\mathbf{f} \in \mathbb{L}^2(D^+(\rho)) \ / \ \mathbf{u} \in \mathbb{H}^2(D^+(\rho))\}$. Using theorem 1.3.3, we may consider c_S as a continuous linear form of $\mathbf{f}$. We then have $\mathbb{L}^2_R(D^+(\rho)) = \ker(c_S)$ and $\mathbb{L}^2_R(D^+(\rho))$ is closed in $\mathbb{L}^2(D^+(\rho))$.

Now, let us define

$$\begin{aligned} T : \quad \mathbb{L}^2_R(D^+(\rho)) &\longrightarrow D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho)) \\ \mathbf{f}_R &\longmapsto \mathbf{u}_R. \end{aligned}$$

T is continuous linear and one-to-one. Moreover, we have $T^{-1}(\mathbf{u}_R) = -\mathrm{div}(\sigma(\mathbf{u}_R))$, for every $\mathbf{u}_R$ in $D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho))$. We can then apply the open-mapping theorem :

$$\begin{aligned} &\exists C > 0 : \quad \forall \mathbf{u}_R \in D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho)), \\ &\|\mathrm{div}(\sigma(\mathbf{u}_R))\|_{\mathbb{L}^2_R(D^+(\rho))} \leq C \|\mathbf{u}_R\|_{\mathbb{H}^1(D^+(\rho))}. \end{aligned} \tag{1.23}$$

Consider now

$$\begin{array}{rrcl} \tilde{T} : & (D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho)), \|.\|_{\mathbb{H}^2(D^+(\rho))}) & \longrightarrow & \mathbb{L}^2_R(D^+(\rho)) \\ & \mathbf{u}_R & \longmapsto & -\mathrm{div}(\sigma(\mathbf{u}_R)). \end{array}$$

$\tilde{T}$ is linear, one-to-one, and obviously continuous. Moreover, we can easily see that $(D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho)), \|.\|_{\mathbb{H}^2(D^+(\rho))})$ is a Banach space.

We apply once again the open-mapping theorem :

$$\begin{aligned} &\exists C > 0 : \quad \forall \mathbf{u}_R \in D(\mathcal{A}_\rho) \cap \mathbb{H}^2(D^+(\rho)), \\ &\|\mathbf{u}_R\|_{\mathbb{H}^2(D^+(\rho))} \leq C \|\mathrm{div}(\sigma(\mathbf{u}_R))\|_{\mathbb{L}^2_R(D^+(\rho))}. \end{aligned} \tag{1.24}$$

Consider $\mathbf{f} \in \mathbb{L}^2(D^+(\rho))$ and $\mathbf{u} \in D(\mathcal{A}_\rho)$, be the solution of problem (1.21) for $\mathbf{f}$. Let $\mathbf{u}_R \in \mathbb{H}^2(D^+(\rho))$ and $c_S \in \mathbb{R}$ be such that $\mathbf{u} = \mathbf{u}_R + c_S \mathbf{u}_S$. Writing the variational formulation with $\mathbf{v} = \mathbf{u}_R$, we get

$$\int_{D^+(\rho)} \sigma(\mathbf{u}_R) : \epsilon(\mathbf{u}_R)\,\mathrm{d}\mathbf{x} + c_S \int_{D^+(\rho)} \sigma(\mathbf{u}_S) : \epsilon(\mathbf{u}_R)\,\mathrm{d}\mathbf{x} = \int_{D^+(\rho)} \mathbf{f} \cdot \mathbf{u}_R\,\mathrm{d}\mathbf{x}.$$

Using theorem 1.3.3, there exists some constant $C > 0$ independent of $\mathbf{f}$ such that

$$|c_S| \leq C \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))}.$$

Then there exists $C > 0$ independent of $\mathbf{f}$ such that

$$\left| c_S \int_{D^+(\rho)} \sigma(\mathbf{u}_S) : \epsilon(\mathbf{u}_R)\,\mathrm{d}\mathbf{x} \right| \leq C \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))} \|\mathbf{u}_R\|_{\mathbb{H}^1(D^+(\rho))}.$$

Moreover, we get easily the existence of $c > 0$ independent of $\mathbf{f}$ such that

$$\int_{D^+(\rho)} \sigma(\mathbf{u}_R) : \epsilon(\mathbf{u}_R)\,\mathrm{d}\mathbf{x} \geq c \|\mathbf{u}_R\|^2_{\mathbb{H}^1(D^+(\rho))}.$$

At last, we have

$$\left| \int_{D^+(\rho)} \mathbf{f} \cdot \mathbf{u}_R\,\mathrm{d}\mathbf{x} \right| \leq \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))} \|\mathbf{u}_R\|_{\mathbb{H}^1(D^+(\rho))}.$$

Then, there exists $C > 0$ independent of $\mathbf{f}$ such that

$$\|\mathbf{u}_R\|_{\mathbb{H}^1(D^+(\rho))} \leq C \|\mathbf{f}\|_{\mathbb{L}^2(D^+(\rho))}. \tag{1.25}$$

We deduce theorem (1.3.4) from (1.23), (1.24) and (1.25). $\square$

1.3.2 The general case

We now consider the case of a bounded connected domain Ω satisfying geometrical assumptions given in the introduction of this paragraph. As well as in the case of the polygon, the solution of problem (1.2) will be locally $\mathbb{H}^2$ in the neighborhood of every point of Ω and of every point of the boundary, except the points of Γ. We will then work at these points.

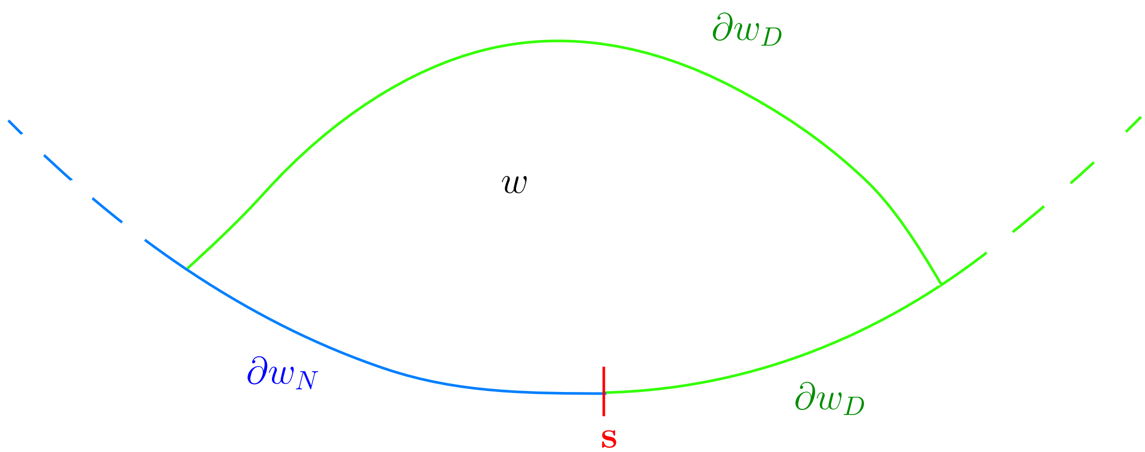

FIG. 1.6 – An example of w.

Let then $\mathbf{s}$ be one of the two points of the interface, and let $W \subset \mathbb{R}^2$ be a neighborhood of $\mathbf{s}$. We denote (see figure 1.6)

$$w = W \cap \Omega, \quad \partial w_N = W \cap \partial\Omega_N, \quad \partial w_D = (W \cap \partial\Omega_D) \cup (\partial w \cap \Omega). \tag{1.26}$$

Using a localization process leads us to consider the following problem

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{g} & \text{in } w; \\ \mathbf{u} = 0 & \text{on } \partial w_D; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial w_N; \end{cases} \tag{1.27}$$

where $\mathbf{g} \in \mathbb{L}^2(w)$.

For this problem, we define the operator $\mathcal{B}$ by

$$\begin{cases} D(\mathcal{B}) = \{\mathbf{v} \in \mathbb{H}^1_D(w) / \mathcal{B}\mathbf{v} \in \mathbb{L}^2(w),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ on } \partial w_N\} \\ \mathcal{B}\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})). \end{cases}$$

We assume that w is such that there exists a $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\phi}$ from w onto $D^+(\rho)$ for some $\rho > 0$, such that

$$\boldsymbol{\phi}(\mathbf{s}) = 0, \quad \boldsymbol{\phi}(\partial w_D) = \partial D^+_D(\rho), \quad \boldsymbol{\phi}(\partial w_N) = \partial D^+_N(\rho). \tag{1.28}$$

Without any restriction, we suppose that $\mathbf{s}$ is located at origin and that x_1-axis is tangent to w at this point. Reducing w if necessary, there exists $\psi \in C^2(\mathbb{R}, \mathbb{R})$ such that

$\partial\Omega \cap W \subset \{(x, \psi(x))/x \in \mathbb{R}\}$. Let us consider

$$\begin{cases} \phi_1(x_1, x_2) = x_1, \\ \phi_2(x_1, x_2) = x_2 - \psi(x_1). \end{cases}$$

Using these local coordinates, (1.27) can be written as follows

$$\begin{cases} -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{u}})) = \tilde{\mathbf{g}} & \text{in } D^+(\rho); \\ \tilde{\mathbf{u}} = 0 & \text{on } \partial D_D^+(\rho); \\ \tilde{\sigma}(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{on } \partial D_N^+(\rho); \end{cases} \tag{1.29}$$

where $\tilde{\mathbf{g}} \in \mathbb{L}^2(D^+(\rho))$ and where we have (with $\tilde{\beta} = \tilde{\sigma} - \sigma$),

$$\begin{cases} \tilde{\beta}_{11}(\tilde{\mathbf{u}}) = -(2\mu + \lambda)\psi'(x_1)\partial_2\tilde{u}_1, \\ \tilde{\beta}_{12}(\tilde{\mathbf{u}}) = -\mu\psi'(x_1)\partial_2\tilde{u}_2, \\ \tilde{\beta}_{21}(\tilde{\mathbf{u}}) = (2\mu + \lambda)\psi'(x_1)^2\partial_2\tilde{u}_1 - (2\mu + \lambda)\psi'(x_1)\partial_1\tilde{u}_1 - (\mu + \lambda)\psi'(x_1)\partial_2\tilde{u}_2, \\ \tilde{\beta}_{22}(\tilde{\mathbf{u}}) = \mu\psi'(x_1)^2\partial_2\tilde{u}_2 - \mu\psi'(x_1)\partial_1\tilde{u}_2 - (\mu + \lambda)\psi'(x_1)\partial_2\tilde{u}_1. \end{cases} \tag{1.30}$$

As well as in [4], we consider problem (1.29) as a perturbation of problem (1.21) in order to get the following regularity result.

Théorème 1.3.5. *Let w be an open subset of $\mathbb{R}^2$ defined in (1.26), There exist $\rho > 0$ and a $\mathcal{C}^2$-diffeomorphism $\boldsymbol{\phi}$ from w onto $D^+(\rho)$ such that (1.28) is satisfied and*

$$D(\mathcal{B}) \subset \mathbb{H}^2(w) \oplus \mathbb{R}(\mathbf{u}_S \circ \boldsymbol{\phi}),$$

where $\mathbf{u}_S$ is given by (1.22).

Proof. We take ρ and $\boldsymbol{\phi}$ as above.

Let $\tilde{\mathcal{A}}_\rho$ be the operator defined by

$$\begin{cases} D(\tilde{\mathcal{A}}_\rho) = \{\tilde{\mathbf{v}} \in \mathbb{H}_D^1(D^+(\rho))/\tilde{\mathcal{A}}_\rho\tilde{\mathbf{v}} \in \mathbb{L}^2(D^+(\rho)),\ \tilde{\sigma}(\tilde{\mathbf{v}})\boldsymbol{\nu} = 0 \text{ on } \partial D_N^+(\rho)\} \\ \tilde{\mathcal{A}}_\rho\tilde{\mathbf{v}} = -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{v}})). \end{cases}$$

We need the following lemma, which gives the perturbation.

Lemme 1.3.1. *For $\rho > 0$ and $\mathbf{v} \in D(\mathcal{A}_\rho)$, $\tilde{\beta}(\mathbf{v})$ belongs to $H^1(D^+(\rho))^4$ and there exists $C > 0$, independent of ρ, such that*

$$\forall \mathbf{v} \in D(\mathcal{A}_\rho), \quad \|\tilde{\beta}(\mathbf{v})\|_{H^1(D^+(\rho))^4} \leq C\sqrt{\rho}\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

We first assume that this lemma is proved. We then take a solution $\tilde{\mathbf{u}}$ of (1.29) belonging to $D(\mathcal{A}_\rho)$. Using lemma 1.3.1, we get that $\tilde{\beta}(\tilde{\mathbf{u}})$ belongs to $H^1(D^+(\rho))^4$. Using a classical trace result, we can build $\tilde{\mathbf{u}}^r \in \mathbb{H}^2(D^+(\rho))$ such that

$$\tilde{\mathbf{u}}^r = 0 \text{ on } \partial D_D^+(\rho), \quad \sigma(\tilde{\mathbf{u}}^r)\boldsymbol{\nu} = -\tilde{\beta}(\tilde{\mathbf{u}}) \text{ on } \partial D_N^+(\rho)$$

and such that there exists $C > 0$, independent of ρ and $\tilde{\mathbf{u}}$, so that

$$\|\tilde{\mathbf{u}}^r\|_{\mathbb{H}^2(D^+(\rho))} \leq C\|\tilde{\beta}(\tilde{\mathbf{u}})\|_{H^1(D^+(\rho))^4}.$$

We can write problem (1.29) in the following form :

$$\begin{cases} -\mathrm{div}(\sigma(\tilde{\mathbf{u}})) - \mathrm{div}(\tilde{\beta}(\tilde{\mathbf{u}})) - \mathrm{div}(\sigma(\tilde{\mathbf{u}}^r)) = \tilde{\mathbf{g}} & \text{in } D^+(\rho); \\ \tilde{\mathbf{u}} = 0 & \text{on } \partial D_D^+(\rho); \\ \sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{on } \partial D_N^+(\rho). \end{cases}$$

We now define the operator $\mathcal{P}_\rho$ by

$$\begin{array}{rcl} \mathcal{P}_\rho : \ D(\mathcal{A}_\rho) & \longrightarrow & \mathbb{L}^2(D^+(\rho)) \\ \tilde{\mathbf{v}} & \longmapsto & -\mathrm{div}(\tilde{\beta}(\tilde{\mathbf{v}})) - \mathrm{div}(\sigma(\tilde{\mathbf{v}}^r)). \end{array}$$

Using lemma (1.3.1), we get that $\mathcal{P}_\rho$ is continuous and that its norm is bounded by $C\sqrt{\rho}$, where C is independent of ρ. Furthermore, $\mathcal{A}_\rho$ is an isomorphism from $D(\mathcal{A}_\rho)$ onto $\mathbb{L}^2(D^+(\rho))$. Hence, for $\rho > 0$ small enough, $\tilde{\mathcal{A}}_\rho = \mathcal{A}_\rho + \mathcal{P}_\rho$ is an isomorphism from $D(\mathcal{A}_\rho)$ onto $\mathbb{L}^2(D^+(\rho))$. Theorem 1.3.5 is then proved. □

Proof of lemma 1.3.1

For $\rho > 0$ and $\mathbf{v} \in D(\mathcal{A}_\rho)$, we will prove that $\tilde{\beta}_{11}(\mathbf{v})$ belongs to $H^1(D^+(\rho))$, the proof will be similar for any $\tilde{\beta}_{ij}(\mathbf{v})$.

Since $\mathbf{v} \in D(\mathcal{A}_\rho)$, there exists $c_S \in \mathbb{R}$ and $\mathbf{v}_R \in \mathbb{H}^2(D^+(\rho))$ such that $\mathbf{v} = \mathbf{v}_R + c_S\mathbf{u}_S$.

For $i \in \{1, 2\}$, we have

$$\partial_i\tilde{\beta}_{11}(\mathbf{v}) = \partial_i\tilde{\beta}_{11}(\mathbf{v}_R) + c_S\partial_i\tilde{\beta}_{11}(\mathbf{u}_S).$$

The term depending on $\mathbf{v}_R$ satisfies

$$\partial_i\tilde{\beta}_{11}(\mathbf{v}_R) = -(2\mu+\lambda)\partial_i\psi'(x_1)\partial_2\mathbf{v}_{R1} - (2\mu+\lambda)\psi'(x_1)\partial_{i2}\mathbf{v}_{R1}.$$

On one hand, there exists $C > 0$ independent of ρ such that

$$\|(2\mu+\lambda)\psi'(x_1)\partial_{i2}\mathbf{v}_{R1}\|_{L^2} \leq C\rho\|\mathbf{v}_R\|_{\mathbb{H}^2}$$

since $\psi'(0) = 0$ and there exists $C > 0$ independent of ρ such that $|\psi'(x_1)| \leq C\rho$.

Thanks to theorem 1.3.4, there exists $C > 0$ independent of ρ such that

$$\|(2\mu+\lambda)\psi'(x_1)\partial_{i2}\mathbf{v}_{R1}\|_{L^2} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

On the other hand, since $\mathbf{v}$ and $\mathbf{u}_S$ satisfy boundary conditions, $\mathbf{v}_R$ satisfy them. Then $\partial_1\mathbf{v}_R$ satisfy a Dirichlet condition on $\partial D_D^+(\rho)$. Thanks to theorem 1.3.4, there exists then $C > 0$ independent of ρ such that

$$\|\partial_1\mathbf{v}_R\|_{\mathbb{L}^2} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)},$$

since Poincaré's constant is proportional to ρ. Moreover, $\sigma_{12}(\mathbf{v}_R)$ and $\sigma_{22}(\mathbf{v}_R)$ satisfy a Dirichlet condition on $\partial D_N^+(\rho)$. Finally we get $C > 0$ independent of ρ such that

$$\|\partial_2\mathbf{v}_R\|_{\mathbb{L}^2} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}$$

and there exists $C > 0$ independent of ρ such that

$$\|(2\mu+\lambda)\partial_i\psi'(x_1)\partial_2\mathbf{v}_{R1}\|_{L^2} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Consider the remaining term depending on $\mathbf{u}_S$. Again, we have

$$\partial_i\tilde{\beta}_{11}(c_S\mathbf{u}_S) = -(2\mu+\lambda)c_S\partial_i\psi'(x_1)\partial_2\mathbf{u}_{S1} - (2\mu+\lambda)c_S\psi'(x_1)\partial_{i2}\mathbf{u}_{S1}.$$

For this terms, we know $\mathbf{u}_S$ explicitly. We then get $C > 0$ independent of ρ such that

$$\|(2\mu+\lambda)c_S\partial_i\psi'(x_1)\partial_2\mathbf{u}_{S1}\|_{L^2} \leq C\sqrt{\rho}|c_S|$$

and there exists $C > 0$ independent of ρ such that

$$\|(2\mu+\lambda)c_S\psi'(x_1)\partial_{i2}\mathbf{u}_{S1}\|_{L^2} \leq C\sqrt{\rho}|c_S|.$$

Now, by theorem 1.3.3, we know that

$$c_S = \frac{1}{\varkappa}\int_{D^+(\rho)} \mathbf{S}^*\mathrm{div}(\sigma(\mathbf{v}))\,\mathrm{d}\mathbf{x}.$$

Then, there exists $C > 0$ independent of ρ such that

$$|c_S| \leq C\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Finally, we get that $\tilde{\beta}_{11}(\mathbf{v})$ belong to $H^1(D^+(\rho))$, and that there exists $C > 0$ independent of ρ such that

$$\|\tilde{\beta}_{11}(\mathbf{v})\|_{H^1(D^+(\rho))} \leq C\sqrt{\rho}\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

We proceed similarly for $\tilde{\beta}_{ij}(\mathbf{v})$ and we obtain the required result. □

1.3.3 Proof of the Rellich-type relation

We can now prove theorem 1.3.1.

Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be a solution of problem (1.2) satisfying conditions (1.20). We proceed as in the polygonal case.

Using a trace theorem, we build $\tilde{\mathbf{u}} \in \mathbb{H}^2(\Omega)$ such that $\tilde{\mathbf{u}} = \mathbf{u}$ and $\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = \sigma(\mathbf{u})\boldsymbol{\nu}$ on $\partial\Omega$. $\mathbf{U} = \tilde{\mathbf{u}} - \mathbf{u}$ is a solution of

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{U})) = \mathbf{F} & \text{in } \Omega; \\ \mathbf{U} = 0 & \text{on } \partial\Omega_D; \\ \sigma(\mathbf{U})\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_N; \end{cases} \tag{1.31}$$

where $\mathbf{F} = \mathrm{div}(\sigma(\tilde{\mathbf{u}})) - \mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega)$.

We take $\varepsilon > 0$ small enough and we define

$$\begin{cases} \Omega_\varepsilon = \Omega \setminus (D(\mathbf{s}_1,\varepsilon) \cup D(\mathbf{s}_2,\varepsilon)), \\ \partial\Omega_{D\varepsilon} = \partial\Omega_D \setminus (D(\mathbf{s}_1,\varepsilon) \cup D(\mathbf{s}_2,\varepsilon)), \\ \partial\Omega_{N\varepsilon} = \partial\Omega_N \setminus (D(\mathbf{s}_1,\varepsilon) \cup D(\mathbf{s}_2,\varepsilon)), \\ \tilde{\Gamma}_\varepsilon = (\partial D(\mathbf{s}_1,\varepsilon) \cup \partial D(\mathbf{s}_2,\varepsilon)) \cap \Omega, \end{cases}$$

so that we have $\partial\Omega_\varepsilon = \partial\Omega_{D\varepsilon} \cup \partial\Omega_{N\varepsilon} \cup \tilde{\Gamma}_\varepsilon$.

For $\mathbf{s} \in \Gamma$, we apply theorem 1.3.5 to $\mathbf{U}$ in $D(\mathbf{s},\varepsilon)\cap\Omega$. We obtain $\mathbf{U}_R \in \mathbb{H}^2(D(\mathbf{s},\varepsilon)\cap\Omega)$ and $c_S \in \mathbb{R}$ such that $\mathbf{U} = \mathbf{U}_R + c_S \mathbf{u}_S \circ \boldsymbol{\phi}$. Writing $\mathbf{u}_R = \mathbf{U}_R + \tilde{\mathbf{u}}$, we get $\mathbf{u} = \mathbf{u}_R + c_S(\mathbf{u}_S \circ \boldsymbol{\phi})$, where $\mathbf{u}_R \in \mathbb{H}^2(D(\mathbf{s},\varepsilon)\cap\Omega)$.

Using this form, we expand $\Theta(\mathbf{u},\mathbf{u})$. Terms with $\mathbf{u}_R$ are obviously integrable on $\partial\Omega \cap D(\mathbf{s},\varepsilon)$. Let us consider terms depending on $\mathbf{u}_S$.

Observe that on $\partial\Omega \cap D(\mathbf{s},\varepsilon)$, $|(\mathbf{m}\cdot\boldsymbol{\nu})| \leq Cd(.,\mathbf{s})$. By a computation, we get that $|(\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}_S \circ \boldsymbol{\phi}) : \epsilon(\mathbf{u}_S \circ \boldsymbol{\phi})|$ is bounded on $\partial\Omega \cap D(\mathbf{s},\varepsilon)$.

Now, $(\sigma(\mathbf{u}_S \circ \boldsymbol{\phi})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}_S \circ \boldsymbol{\phi}$ is of course integrable on $\partial\Omega_N \cap D(\mathbf{s},\varepsilon)$, since it vanishes. On $\partial\Omega_D \cap D(\mathbf{s},\varepsilon)$, since $\mathbf{u}_S \circ \phi = 0$, we have $(\mathbf{m}\cdot\nabla)\mathbf{u}_S \circ \phi = (\mathbf{m}\cdot\boldsymbol{\nu})(\boldsymbol{\nu}\cdot\nabla)\mathbf{u}_S \circ \phi$. Hence, $|(\sigma(\mathbf{u}_S \circ \boldsymbol{\phi})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}_S \circ \boldsymbol{\phi}|$ is bounded on $\partial\Omega_D \cap D(\mathbf{s},\varepsilon)$.

We then obtain that $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega \cap D(\mathbf{s},\varepsilon))$. Hence, it belongs to $\mathbb{L}^1(\partial\Omega)$.

Now, since $\mathbf{u} \in \mathbb{H}^2(\Omega_\varepsilon)$, we apply twice Green's formula and we obtain

$$\int_{\Omega_\varepsilon} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Using Lebesgue's theorem, we get

$$\int_{\Omega_\varepsilon} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\Omega} \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x},$$

$$\int_{\partial\Omega_{D\varepsilon}\cup\partial\Omega_{N\varepsilon}} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

For the remaining integral on $\tilde{\Gamma}_\varepsilon$, we observe that $|J(\boldsymbol{\phi}) - I_2| = \mathcal{O}(\varepsilon)$. With a similar computation as in section 1.2.2, we get

$$\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \underset{\varepsilon\to 0}{\longrightarrow} \Upsilon c_S^2(\mathbf{m}(0)\cdot\boldsymbol{\tau}(0)).$$

The Rellich-type relation is then proved. □

1.4 The case of a n-dimensional domain

We here study the case of a bounded connected domain $\Omega \subset \mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (1.1). We assume furthermore that Γ is a $(n-2)$-dimensional submanifold of class C^3 such that there exists a neighborhood Ω' of Γ such that $\partial\Omega \cap \Omega'$ is a $(n-1)$-submanifold of class C^3, (see figure 1.7).

We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (1.6) is satisfied.

It can be observed that

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0, \quad \text{on } \Gamma. \tag{1.32}$$

At each point $\mathbf{s}$ of Γ, we consider Γ as a submanifold of $\partial\Omega$ of co-dimension 1 and we denote by $\boldsymbol{\tau}(\mathbf{s})$ the unit normal vector to Γ pointing outward of $\partial\Omega_N$.

In this case, the Rellich-type relation can be written as follows :

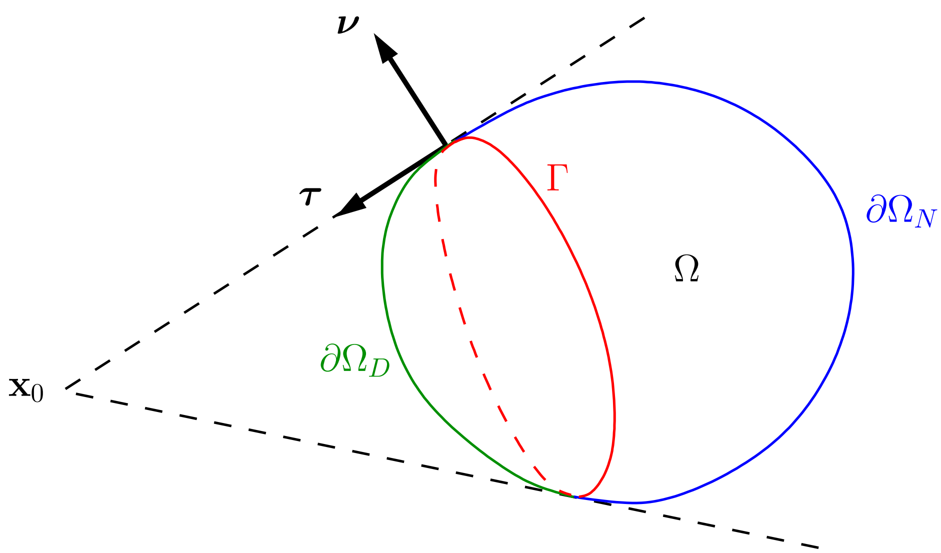

FIG. 1.7 – A general smooth domain Ω with a non-empty interface.

Théorème 1.4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain of class $\mathcal{C}^2$ which satisfies (1.1) and above geometrical assumptions. We assume moreover that the boundary satisfies (1.32). Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be a solution of problem (1.2) with*

$$\mathbf{f} \in \mathbb{L}^2(\Omega), \quad \mathbf{g} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \mathbf{h} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{1.33}$$

Then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and we can compute on Γ singularity coefficients of $\mathbf{u}$ belonging to $L^2(\Gamma)$ such that

$$\begin{aligned} 2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} &= (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \\ &+ \int_\Gamma \left(\Upsilon c_S^e(\mathbf{s})^2 + \frac{\mu\pi}{4}\sum_{i=1}^{n-2} c_{Si}(\mathbf{s})^2\right)\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s})\,\mathrm{d}\mathbf{s}. \end{aligned}$$

We need to know the structure of the studied function.

1.4.1 the case of the semi-cylinder

We use notations of section 1.3.1. $D^+(\rho)$ is the semi-disk with radius ρ in dimension 2. For $d = n-2$, we here consider the case $\Omega = C^+(\rho)$ where $C^+(\rho) = D^+(\rho) \times \mathbb{R}^d$, (see figure 1.8).

Coordinates of $\mathbf{x} \in C^+(\rho)$ will be denoted by $(x_1, x_2, \mathbf{z})$, where $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d$.

Let us define $\partial C_N^+(\rho) = \partial D_N^+(\rho) \times \mathbb{R}^d$, $\partial C_D^+(\rho) = \partial D_D^+(\rho) \times \mathbb{R}^d$.

We consider the problem

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{in } C^+(\rho); \\ \mathbf{u} = 0 & \text{on } \partial C_D^+(\rho); \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial C_N^+(\rho). \end{cases} \tag{1.34}$$

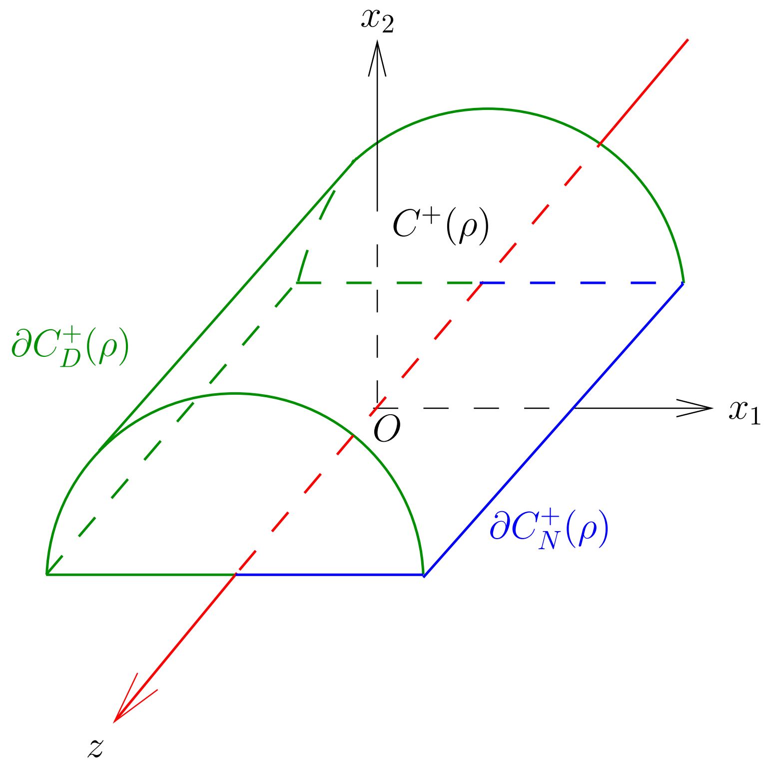

FIG. 1.8 – The considered domain is a semi-cylinder.

We want to generalize regularity results got in section 1.3.1.
Let $\mathbf{u}_S^e \in (H^1(D^+(\rho)))^2$ be the singular solution defined by (1.22).
Let $u_S^l \in H^1(D^+(\rho))$ be the function defined in polar coordinates by

$$u_S^l(r,\theta) = \varsigma(r)\sqrt{r}\sin\left(\frac{\theta}{2}\right),$$

where ς is a cut-off function belonging to $C^\infty(\mathbb{R}^+,[0,1])$ such that $\mathrm{supp}(\varsigma) \subset [0,\rho_2]$ and $\varsigma \equiv 1$ on $[0,\rho_1]$, with $0 < \rho_1 < \rho_2 < \rho$. Roughly speaking, u_S^l is the singular solution of a Laplace equation with mixed boundary conditions (see [4]). This function has been introduced by Shamir in [36].

We define $\mathbf{u}_S \in \mathbb{H}^1(D^+(\rho))$ by

$$\mathbf{u}_S = \begin{pmatrix} \mathbf{u}_S^e \\ u_S^l \\ \vdots \\ u_S^l \end{pmatrix}. \tag{1.35}$$

We then obtain

Théorème 1.4.2. *Let $\mathbf{u}$ be the solution of problem (1.34). We then have*

1) for every i in $\{1,\dots,d\}$, $\dfrac{\partial \mathbf{u}}{\partial z_i}$ belongs to $\mathbb{H}^1(C^+(\rho))$ and there exists $C>0$, independent of ρ such that $\left\|\dfrac{\partial \mathbf{u}}{\partial z_i}\right\|_{\mathbb{H}^1(C^+(\rho))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$;

2) $\exists! \mathbf{u}_R \in L^2(\mathbb{R}^d, \mathbb{H}^2(D^+(\rho)))$ and $\exists! \mathbf{c}_S \in \mathbb{L}^2(\mathbb{R}^d)$ such that $\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S \otimes \mathbf{c}_S$, where $\mathbf{u}_S$ is defined by (1.35) and such that there exists $C>0$, independent of ρ such that

$$\|\mathbf{u}_R\|_{L^2(\mathbb{R}^d,\mathbb{H}^2(D^+(\rho)))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))} \text{ and } \|\mathbf{c}_S\|_{\mathbb{L}^2(\mathbb{R}^d)} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

Proof. The first point of above theorem is easily obtained by using differential quotients. Hence we only develop the proof of the second one.

We denote the solution of problem (1.34) by $\mathbf{u} = {}^t(u_1,\dots,u_n)$. Let us define

$$\mathbf{u}^e = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in (H^1(C^+(\rho)))^2.$$

Using $\Delta_2 = \partial_{11} + \partial_{22}$, we rewrite (1.34) in the following form :

$$\forall i \in \{3,\dots,n\}, \quad \begin{cases} -\Delta_2 u_i = g_i & \text{in } C^+(\rho); \\ u_i = 0 & \text{on } \partial C^+_D(\rho); \\ \partial_2 u_i = h_i & \text{on } \partial C^+_N(\rho). \end{cases} \tag{1.36}$$

With the first point, we have $g_i \in L^2(C^+(\rho))$ and $h_i \in H^{1/2}(C^+(\rho))$.

Moreover, we get that there exists $C>0$ independent of ρ such that

$$\|g_i\|_{L^2(C^+(\rho))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}, \quad \|h_i\|_{H^{1/2}(C^+(\rho))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

Then, using [4], there exist $u_{Ri} \in L^2(\mathbb{R}^d, H^2(D^+(\rho)))$ and $c_{Si} \in L^2(\mathbb{R}^d)$ such that

$$u_i = u_{Ri} + u^l_S \otimes c_{Si},$$

and such that there exists $C>0$ independent of ρ such that

$$\|u_{Ri}\|_{L^2(\mathbb{R}^d,H^2(D^+(\rho)))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}, \quad \|c_{Si}\|_{L^2(\mathbb{R}^d)} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

It remains to obtain the result for $\mathbf{u}^e$. We denote by div_2 the divergence operator in $\mathbb{R}^2$. We also denote by $\epsilon_2(\mathbf{u}^e)$ and $\sigma_2(\mathbf{u}^e)$ the strain tensor and the stress tensor in dimension 2. Hence, $\mathbf{u}^e$ verifies

$$\begin{cases} -\mathrm{div}_2(\sigma_2(\mathbf{u}^e)) = \mathbf{g}^e & \text{in } C^+(\rho); \\ \mathbf{u}^e = 0 & \text{on } \partial C^+_D(\rho); \\ \sigma_2(\mathbf{u}^e)\boldsymbol{\nu} = \mathbf{h}^e & \text{on } \partial C^+_N(\rho); \end{cases} \tag{1.37}$$

where $\mathbf{g}^e \in (L^2(C^+(\rho)))^2$, $\mathbf{h}^e \in (H^{1/2}(C^+(\rho)))^2$, such that there exists $C>0$ independent of ρ such that

$$\|\mathbf{g}^e\|_{(L^2(C^+(\rho)))^2} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}, \quad \|\mathbf{h}^e\|_{(H^{1/2}(C^+(\rho)))^2} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}.$$

Using a trace result, we can build $\tilde{\mathbf{u}}^e \in (H^2(C^+(\rho)))^2$ such that there exists $C > 0$ independent of ρ such that $\|\tilde{\mathbf{u}}^e\|_{(H^2(C^+(\rho)))^2} \le C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$ and such that $\mathbf{U}^e = \mathbf{u}^e - \tilde{\mathbf{u}}^e$ satisfies an equation in the form (1.37) with $\mathbf{h}^e = 0$. In this equation, there are no derivative with respect to $\mathbf{z}$. We then can work for a given $\mathbf{z}$, and we get that, for almost every $\mathbf{z}$, $\mathbf{U}^e$ satisfies

$$\left\{\begin{array}{ll} -\mathrm{div}_2(\sigma_2(\mathbf{U}^e(.,\mathbf{z}))) = \mathbf{G}^e(.,\mathbf{z}) & \text{in } D^+(\rho); \\ \mathbf{U}^e(.,\mathbf{z}) = 0 & \text{on } \partial D_D^+(\rho); \\ \sigma_2(\mathbf{U}^e(.,\mathbf{z}))\boldsymbol{\nu} = 0 & \text{on } \partial C_N^+(\rho); \end{array}\right. \tag{1.38}$$

where $\mathbf{G}^e(.,\mathbf{z}) \in (L^2(D^+(\rho)))^2$, such that there exists $C > 0$ independent of ρ such that $\|\mathbf{G}^e(.,\mathbf{z})\|_{(L^2(D^+(\rho)))^2} \le C\|\mathbf{f}(.,\mathbf{z})\|_{\mathbb{L}^2(D^+(\rho))}$.

With theorems 1.3.2, 1.3.3 and 1.3.4 we can write $\mathbf{u}^e(.,\mathbf{z}) = \mathbf{u}_R^e(.,\mathbf{z}) + c_S^e(\mathbf{z})\mathbf{u}_S^e$ where $\mathbf{u}_R^e(.,\mathbf{z}) \in (H^2(D^+(\rho)))^2$ such that there exists $C > 0$ independent of ρ such that $\|\mathbf{u}_R^e(.,\mathbf{z})\|_{(H^2(D^+(\rho)))^2} \le C\|\mathbf{f}(.,\mathbf{z})\|_{\mathbb{L}^2(D^+(\rho))}$, and

$$c_S^e(\mathbf{z}) = \int_{D^+(\rho)} \mathbf{S}^*(\mathbf{x})\mathbf{G}^e(\mathbf{x},\mathbf{z})\,\mathrm{d}\mathbf{x}.$$

We easily get that $c_S^e \in L^2(\mathbb{R}^d)$ with $\|c_S^e\|_{L^2(\mathbb{R}^d)} \le C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$. We also get that $\mathbf{u}_R^e \in (L^2(\mathbb{R}^d, H^2(D^+(\rho))))^2$ with $\|\mathbf{u}_R^e\|_{(L^2(\mathbb{R}^d,H^2(D^+(\rho))))^2} \le C\|\mathbf{f}\|_{\mathbb{L}^2(C^+(\rho))}$.

We now write

$$\mathbf{u}_R = \begin{pmatrix} \mathbf{u}_R^e \\ u_{R3} \\ \vdots \\ u_{Rn} \end{pmatrix} \text{ and } \mathbf{c}_S = \begin{pmatrix} c_S^e \\ c_S^e \\ c_{S3} \\ \vdots \\ c_{Sn} \end{pmatrix}.$$

and we obtain the required result. □

Remark. Theorem 1.4.2 may be expressed in other words.

For $\rho > 0$, let $B_d(\rho)$ be the ball of $\mathbb{R}^d$ centered at the origin of radius ρ. We define

$$\left\{\begin{array}{l} \tilde{C}^+(\rho) = D^+(\rho) \times B_d(\rho), \\ \partial\tilde{C}_N^+(\rho) = \partial D_N^+(\rho) \times B_d(\rho), \\ \partial\tilde{C}_D^+(\rho) = \partial\tilde{C}^+(\rho) \setminus \partial\tilde{C}_N^+(\rho), \\ \tilde{\Gamma}(\rho) = \{(0,0)\} \times B_d(\rho), \end{array}\right. \tag{1.39}$$

and consider the following problem

$$\left\{\begin{array}{ll} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f} & \text{in } \tilde{C}^+(\rho); \\ \mathbf{u} = 0 & \text{on } \partial\tilde{C}_D^+(\rho); \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial\tilde{C}_N^+(\rho). \end{array}\right. \tag{1.40}$$

Using a cut-off function, theorem 1.4.2 allows us to prove that, if we define the operator $\mathcal{A}_\rho$ by

$$\left\{\begin{array}{l} D(\mathcal{A}_\rho) = \{\mathbf{v} \in \mathbb{H}_D^1(\tilde{C}^+(\rho))/\mathcal{A}_\rho\mathbf{v} \in \mathbb{L}^2(\tilde{C}^+(\rho)),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ on } \partial\tilde{C}_N^+(\rho)\} \\ \mathcal{A}_\rho\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})), \end{array}\right.$$

we get

$$D(\mathcal{A}_\rho) \subset \mathbb{H}^2(\tilde{C}^+(\rho)) \oplus \left(\mathbf{u}_S \otimes \mathbb{L}^2(B_d(\rho))\right).$$

We now can estimate of the $\mathbb{H}^2$-norm of the regular part according to $\mathbf{f}$.

Théorème 1.4.3. *For $\mathbf{f} \in \mathbb{L}^2(\tilde{C}^+(\rho))$, let $\mathbf{u} \in D(\mathcal{A}_\rho)$ be the solution of problem (1.40). Let $\mathbf{u}_R \in \mathbb{H}^2(\tilde{C}^+(\rho))$ be the regular part of $\mathbf{u}$. There exists $C > 0$ independent of $\mathbf{f}$ such that*

$$\|\mathbf{u}_R\|_{\mathbb{H}^2(\tilde{C}^+(\rho))} \leq C\|\mathbf{f}\|_{\mathbb{L}^2(\tilde{C}^+(\rho))}.$$

Proof. We only follow the proof of theorem 1.3.4. □

1.4.2 The general case

We now consider the case of a bounded connected domain Ω satisfying geometrical assumptions given in the introduction of this paragraph. As well as in the 2-dimensional case, the solution of problem (1.2) is locally $\mathbb{H}^2$ in the neighborhood of every point of Ω and of every point of $\partial\Omega \setminus \Gamma$.

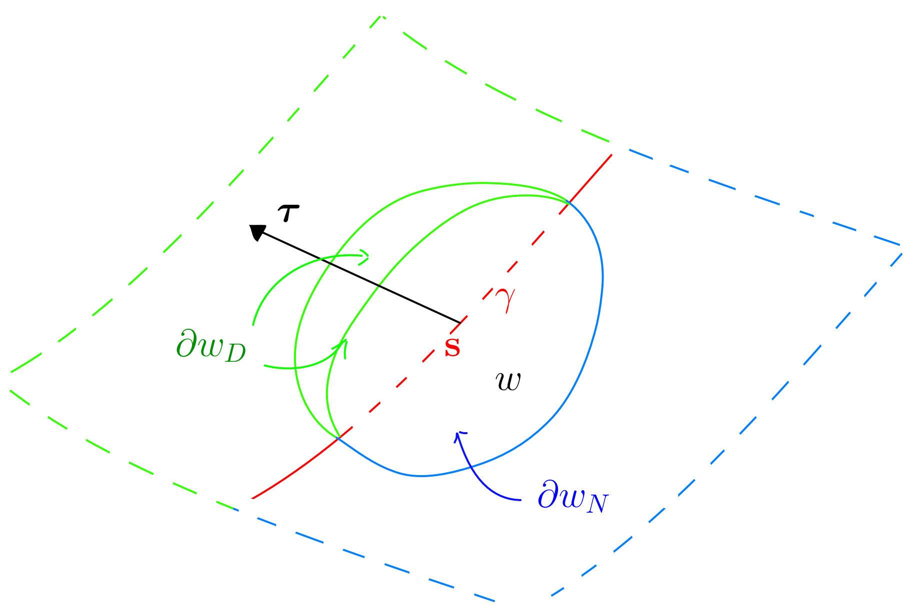

FIG. 1.9 – An example of w.

For $\mathbf{s} \in \Gamma$, let $W \subset \mathbb{R}^n$ be a neighborhood of $\mathbf{s}$. We define (see figure 1.9)

$$\left\{\begin{array}{l} w = W \cap \Omega, \\ \partial w_N = W \cap \partial\Omega_N, \\ \partial w_D = (W \cap \partial\Omega_D) \cup (\partial w \cap \Omega), \\ \gamma = W \cap \Gamma. \end{array}\right. \tag{1.41}$$

A localization process (see [4]) leads us to consider the following problem :

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{g} & \text{in } w; \\ \mathbf{u} = 0 & \text{on } \partial w_D; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = 0 & \text{on } \partial w_N; \end{cases} \tag{1.42}$$

where $\mathbf{g} \in \mathbb{L}^2(w)$.

For this problem, we define the operator $\mathcal{B}$ by

$$\begin{cases} D(\mathcal{B}) = \{\mathbf{v} \in \mathbb{H}^1_D(w)/\mathcal{B}\mathbf{v} \in \mathbb{L}^2(w),\ \sigma(\mathbf{v})\boldsymbol{\nu} = 0 \text{ on } \partial w_N\} \\ \mathcal{B}\mathbf{v} = -\mathrm{div}(\sigma(\mathbf{v})). \end{cases}$$

We assume that w is such that for any $\rho > 0$, there exists a $\mathcal{C}^2$-diffeomorphism $\mathbf{\Phi}$ from w onto $\tilde{C}^+(\rho)$ which satisfies

$$\mathbf{\Phi}(\mathbf{s}) = \mathbf{0}, \quad \mathbf{\Phi}(\partial w_D) = \partial\tilde{C}^+_D(\rho), \quad \mathbf{\Phi}(\partial w_N) = \partial\tilde{C}^+_N(\rho), \quad \mathbf{\Phi}(\gamma) = \tilde{\Gamma}(\rho). \tag{1.43}$$

Modifying w if necessary, and for simplicity, we will consider a particular $\mathbf{\Phi}$.

As well as in subsection 1.3.2, without any restriction, we suppose that the considered point $\mathbf{s}$ of Γ is located at the origin, that ∂w is tangent to x_2-axis and γ to the subspace $\{x_1 = x_2 = 0\}$ at this point.

Reducing w if necessary, there exists $\boldsymbol{\phi} \in C^2(\mathbb{R}^{n-1}, \mathbb{R})$ and $\boldsymbol{\psi} \in C^2(\mathbb{R}^d, \mathbb{R})$ such that

$$\forall \mathbf{x} = (x_1, x_2, \mathbf{z}) \in w, \quad \mathbf{\Phi}(\mathbf{x}) = (x_1 - \boldsymbol{\psi}(\mathbf{z}), x_2 - \boldsymbol{\phi}(x_1, \mathbf{z}), \mathbf{z}) \tag{1.44}$$

and

$$\nabla_{\mathbf{z}}\boldsymbol{\phi}(0, \mathbf{0}) = \nabla_{\mathbf{z}}\boldsymbol{\psi}(\mathbf{0}) = 0, \quad \frac{\partial\boldsymbol{\phi}}{\partial x_1}(0, \mathbf{0}) = 0. \tag{1.45}$$

Hence, the Jacobian matrix of $\mathbf{\Phi}$ is given by :

$$D\mathbf{\Phi} = \begin{pmatrix} 1 & 0 & -{}^t\nabla_{\mathbf{z}}\boldsymbol{\psi} \\ -\frac{\partial\phi}{\partial x_1} & 1 & -{}^t\nabla_{\mathbf{z}}\boldsymbol{\phi} \\ \mathbf{0} & \mathbf{0} & I_d \end{pmatrix}.$$

Especially, $D\mathbf{\Phi}(\mathbf{0}) = I_n$.

Using these local coordinates, (1.42) can be written as follows

$$\begin{cases} -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{u}})) = \tilde{\mathbf{g}} & \text{in } \tilde{C}^+(\rho); \\ \tilde{\mathbf{u}} = 0 & \text{on } \partial\tilde{C}^+_D(\rho); \\ \tilde{\sigma}(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{on } \partial\tilde{C}^+_N(\rho); \end{cases} \tag{1.46}$$

where $\tilde{\mathbf{g}} \in \mathbb{L}^2(\tilde{C}^+(\rho))$ and where

$$\tilde{\sigma}(\tilde{\mathbf{u}}) = \mu\nabla\mathbf{\Phi}(\nabla\tilde{\mathbf{u}}\nabla\mathbf{\Phi} +^t \nabla\mathbf{\Phi}^t\nabla\tilde{\mathbf{u}}) + \lambda(\nabla\mathbf{\Phi} : \nabla\tilde{\mathbf{u}})\nabla\mathbf{\Phi}. \tag{1.47}$$

Once again, we write $\tilde{\beta}(\tilde{\mathbf{u}}) = \tilde{\sigma}(\tilde{\mathbf{u}}) - \sigma(\tilde{\mathbf{u}})$ and we consider $\tilde{\beta}(\tilde{\mathbf{u}})$ as a perturbation of problem (1.40) in order to get the following regularity result.

Théorème 1.4.4. *Let w be an open subset of $\mathbb{R}^n$ defined as in (1.41). There exist $\rho > 0$ and a $\mathcal{C}^2$-diffeomorphism $\mathbf{\Phi}$ from w onto $\tilde{C}^+(\rho)$ such that (1.43) is satisfied and*

$$D(\mathcal{B}) \subset \mathbb{H}^2(w) \oplus \left(\left(\mathbf{u}_S \otimes \mathbb{L}^2(B_d(\rho))\right) \circ \mathbf{\Phi}\right).$$

where $\mathbf{u}_S$ is given by (1.35).

Proof. We take ρ and $\mathbf{\Phi}$ as above.

Let $\tilde{\mathcal{A}}_\rho$ be the operator defined by

$$\begin{cases} D(\tilde{\mathcal{A}}_\rho) = \{\tilde{\mathbf{v}} \in \mathbb{H}^1_D(C^+(\rho))/\tilde{\mathcal{A}}_\rho \tilde{\mathbf{v}} \in \mathbb{L}^2(C^+(\rho)),\ \tilde{\sigma}(\tilde{\mathbf{v}})\boldsymbol{\nu} = 0 \text{ on } \partial C^+_N(\rho)\} \\ \tilde{\mathcal{A}}_\rho \tilde{\mathbf{v}} = -\mathrm{div}(\tilde{\sigma}(\tilde{\mathbf{v}})). \end{cases}$$

We need the following lemma, which gives the perturbation :

Lemme 1.4.1. *For $\rho > 0$ and $\mathbf{v} \in D(\mathcal{A}_\rho)$, $\tilde{\beta}(\mathbf{v})$ belongs to $H^1(\tilde{C}^+(\rho))^{n^2}$ and there exists $C > 0$, independent of ρ, such that*

$$\forall \mathbf{v} \in D(\mathcal{A}_\rho), \quad \|\tilde{\beta}(\mathbf{v})\|_{H^1(\tilde{C}^+(\rho))^{n^2}} \leq C\sqrt{\rho}\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Assuming that lemma 1.4.1 holds, we follow the proof of theorem 1.3.5. □

Proof of lemma 1.4.1

Using (1.47), we can write, for $(i,j) \in \{1,\dots,n\}^2$ and $\mathbf{v} \in D(\mathcal{A}_\rho)$, above functions $\tilde{\beta}_{ij}$ in the form :

$$\tilde{\beta}_{ij}(\mathbf{v}) = \sum_{k,l=1}^{n} \kappa^{ij}_{kl} \partial_k v_l$$

where $\forall (i,j,k,l) \in \{1,\dots,n\}^4$, $\kappa^{ij}_{kl} \in C^1(\tilde{C}^+(\rho))$ and $\kappa^{ij}_{kl}(\mathbf{0}) = 0$.

For $\mathbf{v} \in D(\mathcal{A}_\rho)$ and $t \in \{3,\dots,n\}$, we have

$$\partial_t \tilde{\beta}_{ij}(\mathbf{v}) = \sum_{k,l=1}^{n} \partial_t \kappa^{ij}_{kl} \partial_k v_l + \sum_{k,l=1}^{n} \kappa^{ij}_{kl} \partial_{kt} v_l.$$

On one hand, we use the first point of theorem 1.4.2 : there exists $C > 0$ independent of ρ such that

$$\|\partial_{kt} v_l\|_{L^2(\tilde{C}^+(\rho))} \leq C\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Since $\kappa^{ij}_{kl}(\mathbf{0}) = 0$, there exists $C > 0$ independent of ρ such that

$$\left\| \sum_{k,l=1}^{n} \kappa^{ij}_{kl} \partial_{kt} v_l \right\|_{L^2(\tilde{C}^+(\rho))} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

On the other hand, since $\mathbf{v}$ satisfies a Dirichlet condition on $\partial\tilde{C}^+_D(\rho)$, $\partial_k \mathbf{v}$ satisfies also a Dirichlet condition on a part of the boundary, for $k \in \{3,\dots,n\}$. Then, since the Poincaré's constant is proportional to ρ, there exists $C > 0$ independent of ρ such that

$$\|\partial_k \mathbf{v}\|_{\mathbb{L}^2(\tilde{C}^+(\rho))} \leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Moreover, since $\sigma_{12}(\mathbf{v})$ and $\sigma_{22}(\mathbf{v})$ satisfy a Dirichlet condition on $\partial\tilde{C}_N^+(\rho)$, we get that there exists $C>0$ independent of ρ such that

$$\|\partial_2\mathbf{v}\|_{\mathbb{L}^2(\tilde{C}^+(\rho))}\leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

We get then $C>0$ independent of ρ such that

$$\left\|\sum_{k,l=1}^{n}\partial_t\kappa_{kl}^{ij}\partial_k v_l\right\|_{L^2(\tilde{C}^+(\rho))}\leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

We then have got $C>0$ independent of ρ such that

$$\|\partial_t\tilde{\beta}_{ij}(\mathbf{v})\|_{L^2(\tilde{C}^+(\rho))}\leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

We now assume that $t\in\{1,2\}$. Similarly, there exists $C>0$ independent of ρ such that

$$\left\|\partial_t\left(\sum_{k=3}^{n}\sum_{l=1}^{n}\kappa_{kl}^{ij}\partial_k v_l\right)\right\|_{L^2(\tilde{C}^+(\rho))}\leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Since $\mathbf{v}\in D(\mathcal{A}_\rho)$, there exists $\mathbf{v}_R\in L^2(\mathbb{R}^d,\mathbb{H}^2(D^+(\rho)))$ and $\mathbf{c}_S\in\mathbb{L}^2(\mathbb{R}^d)$ such that $\mathbf{v}=\mathbf{v}_R+\mathbf{u}_S\otimes\mathbf{c}_S$.

Once again, as above, we get $C>0$ independent of ρ such that

$$\left\|\partial_t\left(\sum_{k=1}^{2}\sum_{l=1}^{n}\kappa_{kl}^{ij}\partial_k v_{Rl}\right)\right\|_{L^2(\tilde{C}^+(\rho))}\leq C\rho\|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

For the remaining term with $\mathbf{u}_S$, we have

$$\partial_t\left(\sum_{k=1}^{2}\sum_{l=1}^{n}\kappa_{kl}^{ij}\partial_k(\mathbf{v}_S\otimes\mathbf{c}_S)_l\right)=\sum_{k=1}^{2}\sum_{l=1}^{n}\partial_t\kappa_{kl}^{ij}(\partial_k\mathbf{u}_S\otimes\mathbf{c}_S)_l+\sum_{k=1}^{2}\sum_{l=1}^{n}\kappa_{kl}^{ij}(\partial_{kt}\mathbf{u}_S\otimes\mathbf{c}_S)_l.$$

Observe that $\mathbf{u}_S$ is explicitly known. A technical computation gives $C>0$ independent of ρ such that

$$\left\|\sum_{k=1}^{2}\sum_{l=1}^{n}\partial_t\kappa_{kl}^{ij}(\partial_k\mathbf{u}_S\otimes\mathbf{c}_S)_l\right\|_{L^2(\tilde{C}^+(\rho))}\leq C\sqrt{\rho}\|\mathbf{c}_S\|_{\mathbb{L}^2}$$

and

$$\left\|\sum_{k=1}^{2}\sum_{l=1}^{n}\kappa_{kl}^{ij}(\partial_{kt}\mathbf{u}_S\otimes\mathbf{c}_S)_l\right\|_{L^2(\tilde{C}^+(\rho))}\leq C\sqrt{\rho}\|\mathbf{c}_S\|_{\mathbb{L}^2}.$$

Since there also exists $C>0$ independent of ρ such that

$$\|\mathbf{c}_S\|_{\mathbb{L}^2}\leq C\|\mathbf{v}\|_{D(\mathcal{A}_\rho)},$$

we get $C > 0$ independent of ρ such that

$$\left\| \partial_t \left(\sum_{k=1}^{2} \sum_{l=1}^{n} \kappa_{kl}^{ij} \partial_k (\mathbf{v}_S \otimes \mathbf{c}_S)_l \right) \right\|_{L^2} \leq C\sqrt{\rho} |\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Finally, for every (i,j) in $\{1,\dots,n\}^2$, we get : $\tilde{\beta}_{ij}(\mathbf{v}) \in H^1(\tilde{C}^+(\rho))$, and there exists $C > 0$ independent of ρ such that

$$\|\tilde{\beta}_{ij}\|_{H^1(\tilde{C}^+(\rho))} \leq C\sqrt{\rho}|\mathbf{v}\|_{D(\mathcal{A}_\rho)}.$$

Lemma 1.4.1 is then proved. □

1.4.3 Proof of the Rellich-type relation

We now can prove theorem 1.4.1.

Proof. Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ be a solution of problem (1.2) satisfying conditions (1.33).

Using a trace result, we build $\tilde{\mathbf{u}} \in \mathbb{H}^2(\Omega)$ such that $\tilde{\mathbf{u}} = \mathbf{u}$ and $\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = \sigma(\mathbf{u})\boldsymbol{\nu}$ on $\partial\Omega$. $\mathbf{U} = \mathbf{u} - \tilde{\mathbf{u}}$ satisfies

$$\begin{cases} -\mathrm{div}(\sigma(\mathbf{U})) = \mathbf{F} & \text{in } \Omega; \\ \mathbf{U} = 0 & \text{on } \partial\Omega_D; \\ \sigma(\mathbf{U})\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_N; \end{cases} \tag{1.48}$$

where $\mathbf{F} = \mathrm{div}(\sigma(\tilde{\mathbf{u}})) - \mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega)$.

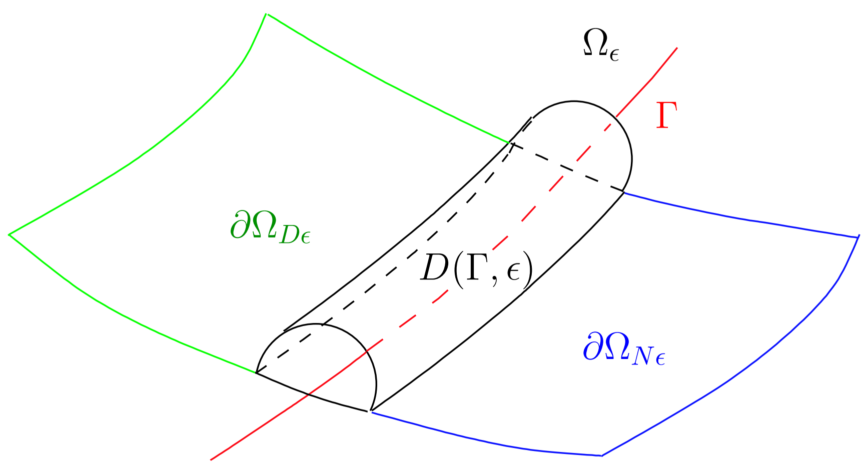

FIG. 1.10 – Ω_ϵ.

Let $\varepsilon > 0$ be small enough and define (see figure 1.10)

$$\begin{cases} \Omega_\varepsilon = \Omega \setminus D(\Gamma,\varepsilon), \\ \partial\Omega_{D\varepsilon} = \partial\Omega_D \setminus D(\Gamma,\varepsilon), \\ \partial\Omega_{N\varepsilon} = \partial\Omega_N \setminus D(\Gamma,\varepsilon), \\ \tilde{\Gamma}_\varepsilon = \partial D(\Gamma,\varepsilon) \cap \Omega, \end{cases}$$

where $D(\Gamma,\varepsilon)=\{\mathbf{x}\in\mathbb{R}^n/d(\mathbf{x},\Gamma)\leq\varepsilon\}$ and so that we have $\partial\Omega_\varepsilon=\partial\Omega_{D\varepsilon}\cup\partial\Omega_{N\varepsilon}\cup\tilde{\Gamma}_\varepsilon$.

As well as for theorem 1.3.1, $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$.

Observe that $\mathbf{u}\in\mathbb{H}^2(\Omega_\varepsilon)$. We apply twice Green's formula and obtain

$$\int_{\Omega_\varepsilon}\operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x}=(n-2)\int_{\Omega_\varepsilon}\sigma(\mathbf{u}):\epsilon(\mathbf{u})\,d\mathbf{x}+\int_{\partial\Omega_\varepsilon}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

Using Lebesgue's theorem, we get

$$\int_{\Omega_\varepsilon}\operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x}\xrightarrow[\varepsilon\to0]{}\int_{\Omega}\operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x},$$

$$\int_{\Omega_\varepsilon}\sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x}\xrightarrow[\varepsilon\to0]{}\int_{\Omega}\sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x},$$

$$\int_{\partial\Omega_{D\varepsilon}\cup\partial\Omega_{N\varepsilon}}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}\xrightarrow[\varepsilon\to0]{}\int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}.$$

It remains to study the convergence of the integral on $\tilde{\Gamma}_\varepsilon$. We consider $\mathbf{s}\in\Gamma$. After convenient rotation and translation, we assume that this point $\mathbf{s}$ is the origin, and that $\boldsymbol{\tau}(\mathbf{s})$ and $\boldsymbol{\nu}(\mathbf{s})$ give the first and the second axises of coordinates. We write $\tilde{\Gamma}_\varepsilon(\mathbf{s})=\tilde{\Gamma}_\varepsilon\cap(\mathbf{s},\boldsymbol{\tau},\boldsymbol{\nu})$ where $(\mathbf{s},\boldsymbol{\tau},\boldsymbol{\nu})$ is the plane containing $\mathbf{s}$ generated by $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$.

Applying theorem 1.4.4 we get $\mathbf{u}=\mathbf{u}_R+(\mathbf{u}_S\circ\boldsymbol{\Phi})\otimes\mathbf{c}_S(\mathbf{s})$ on some neighborhood $w(\mathbf{s})$ of $\mathbf{s}$. We have $\boldsymbol{\Phi}(\tilde{\Gamma}_\varepsilon)=\{\varepsilon\}\times(0,\pi)$ in polar coordinates.

We obtain

$$\int_{\tilde{\Gamma}_\varepsilon(\mathbf{s})}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}=c_S^e(\mathbf{s})^2\int_{\theta=0}^{\pi}\Theta(\mathbf{u}_S^e,\mathbf{u}_S^e)\varepsilon\,\mathrm{d}\theta$$
$$+\sum_{i=1}^{d}c_{Si}(\mathbf{s})^2\int_{\theta=0}^{\pi}\mu[(\boldsymbol{\nu}\cdot\nabla u_S^l)\cdot(\mathbf{m}\cdot\nabla u_S^l)-(\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u_S^l|^2]\varepsilon\,\mathrm{d}\theta+\mathcal{O}(\varepsilon).$$

Using results of [13, 4], we get

$$\int_{\theta=0}^{\pi}[(\boldsymbol{\nu}\cdot\nabla u_S^l)\cdot(\mathbf{m}\cdot\nabla u_S^l)-(\mathbf{m}\cdot\boldsymbol{\nu})|\nabla u_S^l|^2]\varepsilon\,\mathrm{d}\theta=\frac{\pi}{4}(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))+\mathcal{O}(\varepsilon).$$

Moreover, we have obtained in section 1.2.2

$$\int_{\theta=0}^{\pi}\Theta(\mathbf{u}_S^e,\mathbf{u}_S^e)\varepsilon\,\mathrm{d}\theta=\Upsilon(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))+\mathcal{O}(\varepsilon).$$

Hence, we obtain

$$\int_{\tilde{\Gamma}_\varepsilon(\mathbf{s})}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma}=c_S^e(\mathbf{s})^2\Upsilon(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))+\sum_{i=1}^{d}c_{Si}(\mathbf{s})^2\mu\frac{\pi}{4}(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))+\mathcal{O}(\varepsilon).$$

We integrate the previous result on $w(\mathbf{s})\cap\Gamma$. Since Γ is compact, we can introduce a finite covering of Γ by $\{w(\mathbf{s})\}_{\mathbf{s}\in\Gamma}$ and the associated cut-off functions which constitute a partition of the unity of Γ. We then combine a finite number of equalities and we finally obtain

$$\int_{\tilde{\Gamma}_\varepsilon} \Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} = \int_\Gamma \left(\Upsilon c_S^e(\mathbf{s})^2 + \frac{\mu\pi}{4}\sum_{i=1}^{d} c_{Si}(\mathbf{s})^2 \right) (\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))\,\mathrm{d}\mathbf{s} + \mathcal{O}(\varepsilon).$$

The Rellich-type relation is obtained by passing to the limit when ε tends to 0. □

Chapitre 2

Stabilisation frontière du système élastodynamique
Partie II : Stabilisation frontière d'un système élastodynamique en présence de singularités

Boundary stabilization of elastodynamic systems. Part II : Boundary stabilization of an elastodynamic system involving singularities.

Résumé en français Nous considérons un système élastodynamique muni d'un feed-back de type Neumann. Nous prouvons des résultats de stabilisation en utilisant la méthode des multiplicateurs et les relations de Rellich obtenues dans la première partie. Nous prenons en compte les singularités qui apparaissent lorsque les conditions frontière changent.

Ce chapitre est constitué d'une prépublication soumise à *SIAM, J. Control Optim.*

(n ° 384, http ://maply.univ-lyon1.fr/publis/publiv/2004/publis.html)

Les résultats obtenus dans le cas polygonal ont fait l'objet d'une note aux CRAS [6], placée en annexe.

English abstract We consider an elastodynamic system damped by a boundary feedback of Neumann-type. We prove stabilization results by using the multipliers method and Rellich-type relations given in Part I. Especially, we take in account singularities which appear when changing boundary conditions.

Introduction

This work follows Part I, where we have proved some Rellich-type relations for the Lamé's system. We give the detailed proof of some extensions of a result which has been announced in [6].

We here study the problem of boundary stabilization involving singularities. These singularities can appear when the feedback is defined on a strict subset of the boundary.

We have chosen to study our problem by the multipliers method because of its robustness. Furthermore, this method leads to explicit decay rates of the energy function.

Similar problems of boundary stabilization have been addressed by many authors. We especially mention :

– the case of the waves equation (see [19] and references of this book). Geometrical assumptions involving singularities have been firstly studied in [13]. This work has been later extended in [21] and [4].
– the case of elastodynamic systems :

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } \Omega \times \mathbb{R}_+; \\ \mathbf{u} = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = \mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}') & \text{on } \partial\Omega_N \times \mathbb{R}_+; \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega; \\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases} \tag{2.1}$$

In [22], Lagnese has introduced a "natural" feedback : $\mathbf{F}(\mathbf{x}, \mathbf{u}, \mathbf{u}') = -a\mathbf{u} - b\mathbf{u}'$. In [23, 20], another feedback is introduced, in order to obtain a stabilization result. In [1], Alabau and Komornik obtained a stabilization result with the natural feedback. This work has been extended in [15, 16, 2, 3]. In all these works, geometrical restrictions are such that singularities do not appear. The most general result concerning Lamé system has been obtained by M. A. Horn [18], using micro-local analysis techniques. Here, as well as in the case of waves equation, our generalized Rellich relations obtained in Part I leads us to give boundary stabilization results for the elastodynamic system under weak geometrical restrictions.

We use a particular natural feedback which conveniently behaves with respect to Rellich relations. The case of the general natural feedback when singularities appear is still open.

Let us now introduce notations and main assumptions.

We here consider an elastic body which satisfies Lamé's laws. As usual, we define the strain tensor and the stress tensor for a regular vector field $\mathbf{v}$ by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i \mathbf{v}_j + \partial_j \mathbf{v}_i), \qquad \sigma(\mathbf{v}) = 2\mu\epsilon(\mathbf{v}) + \lambda \mathrm{div}(\mathbf{v}) I_n.$$

where λ and μ are the Lamé's coefficients and I_n is the identity matrix of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set such that its boundary satisfies

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N, \text{ with} \begin{cases} \partial\Omega_D \cap \partial\Omega_N = \emptyset, \\ \mathrm{meas}(\partial\Omega_D) \neq 0, \\ \mathrm{meas}(\partial\Omega_N) \neq 0. \end{cases} \tag{2.2}$$

We denote the boundary interface by $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$. We assume that Ω is smooth enough so that for almost every $\mathbf{x} \in \partial\Omega$, we can consider $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outward of Ω.

We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that, setting $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, we have

$$(\mathbf{m}\cdot\boldsymbol{\nu}) \geq 0, \text{ on } \partial\Omega_N, \qquad (\mathbf{m}\cdot\boldsymbol{\nu}) \leq 0, \text{ on } \partial\Omega_D. \tag{2.3}$$

It can be observed that, if $\Gamma \neq \emptyset$, and if $\partial\Omega$ is smooth enough,

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0, \quad \text{on } \Gamma. \tag{2.4}$$

We here consider the linear isotropic elastodynamic system

$$\begin{cases} \mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) = 0 & \text{in } \Omega \times \mathbb{R}_+; \\ \mathbf{u} = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}' & \text{on } \partial\Omega_N \times \mathbb{R}_+; \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega; \\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases} \tag{2.5}$$

As well as in the case of waves equation, because of the change of boundary conditions, singularities appear along Γ. Nevertheless, we have seen in Part I that, since the feedback vanishes on Γ (cf (2.4)), we can obtain Rellich type relations.

We introduce following Sobolev spaces : $\mathbb{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbb{H}^s(\Omega) = (H^s(\Omega))^n$, $\forall s > 0$ and $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega) / \mathbf{v} = 0 \text{ on } \partial\Omega_D\}$.

The semi-group method leads to well-posedness of problem (2.5) under the following assumption

$$(\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{H}^1_D(\Omega) \times \mathbb{L}^2(\Omega). \tag{2.6}$$

Energy function is given by

$$E(\mathbf{u}, t) = \frac{1}{2} \int_\Omega (|\mathbf{u}'|^2 + \sigma(\mathbf{u}) : \epsilon(\mathbf{u}))\, \mathrm{d}\mathbf{x}. \tag{2.7}$$

We will prove, under convenient geometric assumptions, results in the following form

There exist constants $C > 0$ and $\varpi > 0$ such that for all $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (2.6), the solution $\mathbf{u}$ of (2.5) satisfies

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C e^{-\varpi t} E(\mathbf{u}, 0).$$

This paper is organized as follow :

– section 2.1 : we give boundary stabilization results when Ω is a bi-dimensional polygonal domain (Theorem 2.1.1).
– section 2.2 : we give this result when Ω is a n-dimensional smooth domain (Theorem 2.2.1).
– section 2.3 : we give the proof of theorem 2.2.1. This proof can be easily adapted for theorem 2.1.1, following [6].

2.1 Stabilization of the elastodynamic system for a polygonal domain

We here express the result in the case of a polygonal domain $\Omega \subset \mathbb{R}^2$, as in section 2.1. We assume that its boundary $\partial\Omega$ satisfied (2.2) and furthermore,

$$\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}, \tag{2.8}$$

where $\mathbf{s}_1$ and $\mathbf{s}_2$ will be considered as two vertices of $\partial\Omega$ (see figure 2.1).

The following theorem has been announced in [6]. The proof is similar to the proof given in section 2.3.

Théorème 2.1.1. *Let Ω be a polygonal domain of $\mathbb{R}^2$ satisfying (2.2), (2.3) and (2.8). There exist $C > 0$ and $\varpi > 0$ such that for every $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (2.6), the solution $\mathbf{u}$ of (2.5) satisfies*

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C e^{-\varpi t} E(\mathbf{u}, 0).$$

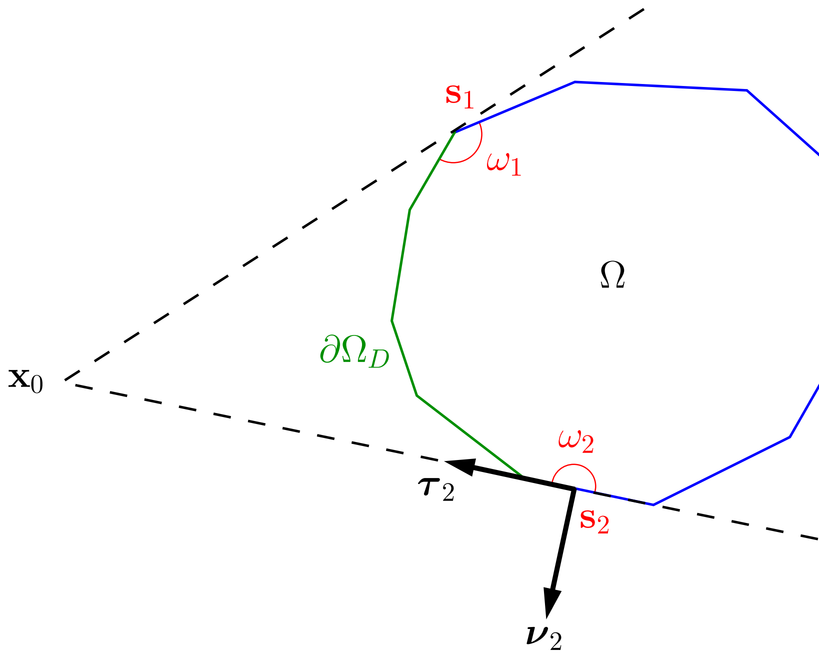

FIG. 2.1 – A polygonal domain Ω with a non-empty interface $\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}$.

2.2 Stabilization of the elastodynamic system for a n-dimensional domain

2.2.1 Introduction

We consider a bounded connected domain $\Omega \subset \mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (2.2). We assume furthermore that :

$$\begin{gathered}\Gamma \text{ is a } (n-2)\text{-dimensional submanifold of class } C^3 \text{ such that}\\ \text{there exists a neighborhood } \Omega' \text{ of } \Gamma \text{ such that } \partial\Omega \cap \Omega'\\ \text{is a } (n-1)\text{-submanifold of class } C^3.\text{(see figure 2.2)}\end{gathered} \tag{2.9}$$

We assume that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (2.3) is satisfied. At each point $\mathbf{s}$ of Γ, we consider Γ as a submanifold of $\partial\Omega$ of co-dimension 1 and we can denote by $\boldsymbol{\tau}(\mathbf{s})$ the unit normal vector to Γ pointing outward of $\partial\Omega_N$. We assume that :

$$\mathbf{m}\cdot\boldsymbol{\tau} \leq 0, \quad \text{on } \Gamma. \tag{2.10}$$

We emphasize that such a condition appears in the case of waves equation (see [4]). When using multipliers method, we shall take in account this condition in Rellich-type relation and this will give the behavior of the energy function.

Condition (2.10) is especially satisfied when Ω is convex, and $\mathbf{x}_0$ outside of $\overline{\Omega}$.

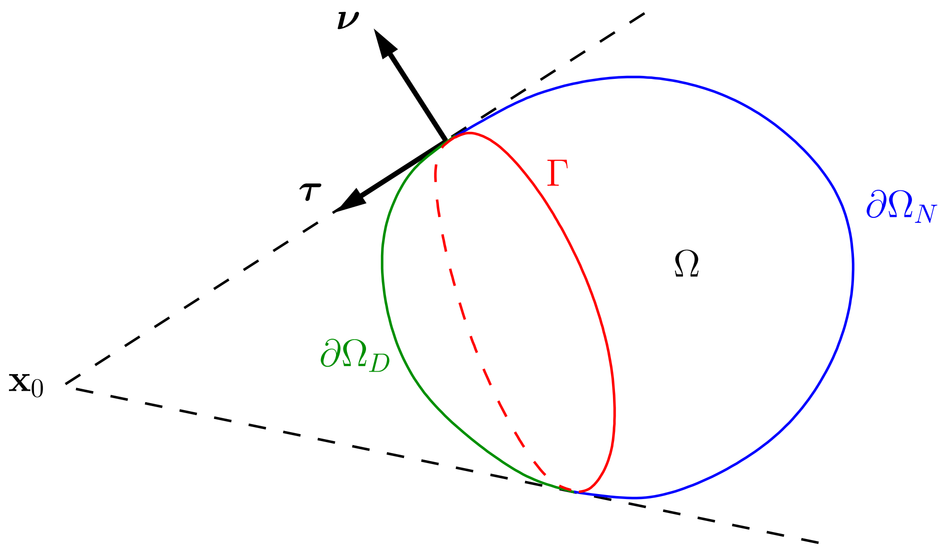

FIG. 2.2 – A general smooth domain Ω with a non-empty interface.

Théorème 2.2.1. *Let Ω be a bounded connected domain of $\mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (2.2) and (2.9). We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (2.3) and (2.10) hold.*

There exist $C > 0$ and $\varpi > 0$ such that for every $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (2.6), the solution $\mathbf{u}$ of (2.5) satisfies

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq Ce^{-\varpi t} E(\mathbf{u}, 0).$$

In order to prove theorem 2.2.1 in this case, we will follow the method used in [21] and described in [19], so-called the multipliers method.

Some notations

For a regular vector field $\mathbf{u}$, we define

$$\nabla \mathbf{u} = (\partial_j u_i)_{1\leq i,j\leq n} = \begin{pmatrix} \partial_1 u_1 & \dots & \partial_n u_1 \\ \vdots & \ddots & \vdots \\ \partial_1 u_n & \dots & \partial_n u_n \end{pmatrix}.$$

As well as in Part I, we need to introduce

$$\Upsilon = 8\frac{(2\mu+\lambda)(3\mu+\lambda)}{\pi\mu}\left(\pi^2 + \ln^2\left(\frac{3\mu+\lambda}{\mu+\lambda}\right)\right).$$

and for two regular vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$, we define

$$\Theta(\mathbf{v}_1, \mathbf{v}_2) = 2(\sigma(\mathbf{v}_1)\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{v}_2 - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{v}_1):\epsilon(\mathbf{v}_2).$$

We first prove theorem 2.2.1 for strong solutions. The result for weak solutions follows thanks to a density result.

2.2.2 The Rellich-type relation

Let us recall results obtained in the first part of this work.

Théorème 2.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected domain of class $\mathcal{C}^2$ which satisfies (2.2)-(2.4). Let $\mathbf{u} \in \mathbb{H}^1(\Omega)$ such that*

$$\operatorname{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega), \quad \mathbf{u} \in \mathbb{H}^{3/2}(\partial\Omega_D), \quad \sigma(\mathbf{u})\cdot\boldsymbol{\nu} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{2.11}$$

Then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathbb{L}^1(\partial\Omega)$ and we can compute on Γ singularity coefficients of $\mathbf{u}$ belonging to $L^2(\Gamma)$ such that

$$\begin{aligned} 2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} &= (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \\ &\quad + \int_\Gamma \left(\Upsilon c_S^e(\mathbf{s})^2 + \frac{\mu\pi}{4}\sum_{i=1}^d c_{Si}(\mathbf{s})^2\right)(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))\,\mathrm{d}\mathbf{s}. \end{aligned}$$

2.3 The proof of theorem 2.2.1

The main tool of our proof is the following lemma proved in [19] :

Lemme 2.3.1. *Let $E:\ \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that there exists $C > 0$ independent of t such that*

$$\int_t^\infty E(s)\,ds \le CE(t), \quad \forall t \ge 0; \tag{2.12}$$

then we have

$$E(t) \le E(0)e^{1-\frac{t}{C}}, \quad \forall t \ge C.$$

We can easily verify that the energy function is non-increasing with respect to time. Indeed, using Green's formula, we get, for every $t > 0$,

$$E'(\mathbf{u},t) = \int_\Omega (\mathbf{u}\cdot\mathbf{u}' + \sigma(\mathbf{u}):\epsilon(\mathbf{u}'))\,\mathrm{d}\mathbf{x} = -\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma}. \tag{2.13}$$

In order to prove that the energy function satisfies (2.12), we use the multipliers method (see for example [19] and [25]).

Let $T > S > 0$ be two constants. We now introduce $M\mathbf{u} = 2(\mathbf{m}\cdot\nabla)\mathbf{u} + (n-1)\mathbf{u}$. We have

$$\int_S^T\int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_S^T\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{2.14}$$

Integrating the left-hand side by parts with respect to t, we get

$$\int_S^T\int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T - \int_S^T\int_\Omega \mathbf{u}'\cdot M\mathbf{u}'\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{2.15}$$

But we can write :

$$\begin{aligned} 2\int_\Omega \mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}'\,\mathrm{d}\mathbf{x} &= \int_\Omega (\mathbf{m}\cdot\nabla)|\mathbf{u}'|^2\,\mathrm{d}\mathbf{x} \\ &= \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma} - \int_\Omega \mathrm{div}(\mathbf{m})|\mathbf{u}'|^2\,\mathrm{d}\mathbf{x} \\ &= \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma} - n\int_\Omega |\mathbf{u}'|^2\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_S^T\int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= \left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T \\ &+\int_S^T\int_\Omega |\mathbf{u}'|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_S^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \end{aligned} \tag{2.16}$$

The right-hand side of (2.14) can be written as follows :

$$\begin{aligned} \int_S^T\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &= 2\int_S^T\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &+(n-1)\int_S^T\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned}$$

Green's formula gives

$$\int_S^T\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = -\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t - \int_S^T\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Now, since $\mathbf{u}$ is a strong solution of (2.5), $\mathbf{u}(t)$ satisfies conditions of theorem 2.2.2 for almost every t. Thanks to (2.10), the integral on Γ is non-positive. Hence, we get

$$\begin{aligned} 2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} &\le (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \\ &\le (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} \\ &\quad -\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}+\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\boldsymbol{\gamma} \\ &\quad +\int_{\partial\Omega_D}[2(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}-(\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\boldsymbol{\gamma}. \end{aligned}$$

We have $\mathbf{u}=0$ on $\partial\Omega_D$. Then, $\forall i$, $\nabla u_i = (\nabla u_i\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ on $\partial\Omega_D$. We observe that $(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} = (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})$ on $\partial\Omega_D$. Thanks to (2.3), we get

$$2\int_{\partial\Omega_D}(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma} - \int_{\partial\Omega_D}(\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\boldsymbol{\gamma} \le 0$$

and

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \;\le\; (n-2)\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} - \int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}+\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\boldsymbol{\gamma}.$$

Hence

$$\begin{aligned}\int_S^T\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;\le\; & -\int_S^T\int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ & -\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}+\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \\ & -(n-1)\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \qquad (2.17)\end{aligned}$$

From (2.16) and (2.17), we obtain

$$\begin{aligned}2\int_S^T E(\mathbf{u},t)\,\mathrm{d}t \;\le\; & -\left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T + \int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t \\ & -(n-1)\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t \\ & -\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}+\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \qquad (2.18)\end{aligned}$$

One can easily prove that there exists $C_1 > 0$ such that for all $t > 0$,

$$\left|\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right| \le C_1E(\mathbf{u},t).$$

Then, since energy function is non-increasing, we get

$$\left|\left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T\right| \le 2C_1E(\mathbf{u},S). \qquad (2.19)$$

Moreover, for a given $\theta > 0$, we get $C_2 > 0$ independent of S and T such that

$$\left|\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t\right| \le \theta\int_S^T E(\mathbf{u},t)\,\mathrm{d}t + C_2E(\mathbf{u},S). \qquad (2.20)$$

At last, we use the following lemma which is proved at the end of this paper.

Lemme 2.3.2. *For all $\theta > 0$ small enough, there exists $C > 0$ independent of S and T such that*

$$-\int_S^T\int_{\partial\Omega_N}(\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}+\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,d\boldsymbol{\gamma}\,dt \le \theta\int_S^T E(\mathbf{u},t)\,dt + CE(\mathbf{u},S). \qquad (2.21)$$

Hence, from (2.18), (2.19), (2.20) and (2.21), we obtain that for every $\theta > 0$ small enough, there exists $C_3 > 0$ independent of S and T such that

$$(2-\theta)\int_S^T E(t)\,\mathrm{d}t \le C_3 E(\mathbf{u}, S). \tag{2.22}$$

It is convenient to take θ small enough. The limit when T tends to infinity gives the result. This result can be extended to weak solutions by a density argument, since decay rate does not depend on the considered strong solution. □

Remark : all constants that appear in the above proof are explicit, so we can get the exponential decay rate with respect to data.

2.3.1 Proof of lemma 2.3.2

For the proof of this lemma, we follow [3].

Since $\partial\Omega$ is of class C^2, for all $\mathbf{x} \in \partial\Omega$, we can build a local C^2-diffeomorphism $\boldsymbol{\phi}$ from an open subset $V_{\mathbf{x}} \subset \mathbb{R}^{n-1}$ onto an open neighborhood $\mathcal{V}_{\mathbf{x}} \subset \partial\Omega$. Then vectors

$$\mathbf{a}_i(\mathbf{x}) = \frac{\partial \boldsymbol{\phi}}{\partial \xi_i}(\boldsymbol{\phi}^{-1}(\mathbf{x})), \quad \forall i \in \{1, \dots, n-1\},$$

are independent and generate $T_{\mathbf{x}}(\partial\Omega)$, the tangent space at a $\mathbf{x}$. Moreover, we denote by $T(\partial\Omega)$ the tangent bundle (see [24] and [38]).

We then denote by g the metric tensor related to $\boldsymbol{\phi}$

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j, \quad \forall (i,j) \in \{1, \dots, n-1\}^2,$$

and by $(g^{ij})_{(i,j)\in\{1,\dots,n-1\}^2}$ its inverse.

We denote by $\pi(\mathbf{x})$ the orthogonal projection on $T_{\mathbf{x}}(\partial\Omega)$. Then, for every vector field $\mathbf{v}$ defined on $\overline{\Omega}$, we have for almost every $\mathbf{x}$ in $\partial\Omega$,

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}_T(\mathbf{x}) + v_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}),$$

where $\mathbf{v}_T(\mathbf{x}) = \pi(\mathbf{x})\mathbf{v}(\mathbf{x}),\ v_{\boldsymbol{\nu}}(\mathbf{x}) = \mathbf{v}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x})$.

Especially for vector $\mathbf{m}$, we have for almost every $\mathbf{x}$ in $\partial\Omega$,

$$\mathbf{m}(\mathbf{x}) = \mathbf{m}_T(\mathbf{x}) + m_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\nu}(\mathbf{x}).$$

We will denote by ∂_T the tangential derivative, by $\partial_{\boldsymbol{\nu}}$ the normal derivative and by ∇_T the tangential gradient. For every smooth enough vector field $\mathbf{v}$, we have

$$\mathrm{d}\mathbf{v} = \pi(\partial_T \mathbf{v}_T)\pi + v_{\boldsymbol{\nu}}(\partial_T \boldsymbol{\nu}) + (\partial_{\boldsymbol{\nu}} \mathbf{v}_T)^t\boldsymbol{\nu} + \boldsymbol{\nu}(\partial_T v_{\boldsymbol{\nu}} - {}^t\mathbf{v}_T(\partial_T \boldsymbol{\nu}) + (\partial_{\boldsymbol{\nu}} v_{\boldsymbol{\nu}})^t\boldsymbol{\nu}), \text{ on } \partial\Omega. \tag{2.23}$$

We then have

$$\epsilon(\mathbf{v}) = \epsilon_T(\mathbf{v}) + \boldsymbol{\nu}^t\epsilon_S(\mathbf{v}) + \epsilon_S(\mathbf{v})^t\boldsymbol{\nu} + \epsilon_{\boldsymbol{\nu}}(\mathbf{v})\boldsymbol{\nu}^t\boldsymbol{\nu}, \quad \text{on } \partial\Omega, \tag{2.24}$$

$$\sigma(\mathbf{v}) = \sigma_T(\mathbf{v}) + \boldsymbol{\nu}^t\sigma_S(\mathbf{v}) + \sigma_S(\mathbf{v})^t\boldsymbol{\nu} + \sigma_{\boldsymbol{\nu}}(\mathbf{v})\boldsymbol{\nu}^t\boldsymbol{\nu}, \quad \text{on } \partial\Omega, \tag{2.25}$$

with

$$\begin{cases} 2\epsilon_T(\mathbf{v}) = \pi(\partial_T \mathbf{v}_T)\pi + \pi{}^t\partial_T \mathbf{v}_T\pi + 2v_{\boldsymbol{\nu}}\partial_T\boldsymbol{\nu}, \\ 2\epsilon_S(\mathbf{v}) = \partial_{\boldsymbol{\nu}}\mathbf{v}_T + \nabla_T v_{\boldsymbol{\nu}} - (\partial_T\boldsymbol{\nu})\mathbf{v}_T, \\ \epsilon_{\boldsymbol{\nu}}(\mathbf{v}) = \partial_{\boldsymbol{\nu}} v_{\boldsymbol{\nu}}, \end{cases}$$

$$\begin{cases} \sigma_T(\mathbf{v}) = 2\mu\epsilon_T(\mathbf{v}) + \lambda(\mathrm{tr}(\epsilon_T(\mathbf{v}) + \epsilon_{\boldsymbol{\nu}}(\mathbf{v}))I_2, \\ \sigma_S(\mathbf{v}) = 2\mu\epsilon_S(\mathbf{v}), \\ \sigma_{\boldsymbol{\nu}}(\mathbf{v}) = 2\mu\epsilon_{\boldsymbol{\nu}}(\mathbf{v}) + \lambda(\mathrm{tr}(\epsilon_T(\mathbf{v}) + \epsilon_{\boldsymbol{\nu}}(\mathbf{v})). \end{cases}$$

Remark : It can be observed that $\epsilon_T(\mathbf{v})$ and $\sigma_T(\mathbf{v})$ correspond to some symmetric $(n-1)\times(n-1)$-matrices, and that $\epsilon_S(\mathbf{v})$ and $\sigma_S(\mathbf{v})$ correspond to some vectors of dimension $n-1$, such that in some orthogonal basis $(\mathbf{t}_1, \dots, \mathbf{t}_{n-1}, \boldsymbol{\nu})$, where $\mathbf{t}_1, \dots, \mathbf{t}_{n-1}$ belong to the tangent space, tensors $\epsilon(\mathbf{v})$ and $\sigma(\mathbf{v})$ are represented by matrices

$$\begin{pmatrix} \epsilon_T(\mathbf{v}) & \epsilon_S(\mathbf{v}) \\ {}^t\epsilon_S(\mathbf{v}) & \epsilon_{\boldsymbol{\nu}}(\mathbf{v}) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sigma_T(\mathbf{v}) & \sigma_S(\mathbf{v}) \\ {}^t\sigma_S(\mathbf{v}) & \sigma_{\boldsymbol{\nu}}(\mathbf{v}) \end{pmatrix}.$$

We first estimate

$$I = \int_S^T \int_{\partial\Omega_N} 2(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t.$$

>From (2.23), we deduce

$$\begin{aligned} \mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} &= \mathbf{u}'_T\cdot(\partial_T\mathbf{u}_T)\mathbf{m}_T + (u_{\boldsymbol{\nu}}\mathbf{u}'_T - u'_{\boldsymbol{\nu}}\mathbf{u}_T)\cdot(\partial_T\boldsymbol{\nu})\mathbf{m}_T \\ &\quad + u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T + m_{\boldsymbol{\nu}}(\mathbf{u}'_T\partial_{\boldsymbol{\nu}}\mathbf{u}_T + u'_{\boldsymbol{\nu}}\partial_{\boldsymbol{\nu}}u_{\boldsymbol{\nu}}). \end{aligned} \tag{2.26}$$

Using this form, we expand I and we study each term.

• Estimate of $I_1 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T\,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t.$

$\partial\Omega$ is a compact manifold of dimension $n-1$. Then there exists a finite number of local maps associated to a partition of unity $(\theta_1, \dots, \theta_k)$. We denote $U_j = \mathrm{supp}(\theta_j)$. We then have

$$\begin{aligned} \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T\,\mathrm{d}\boldsymbol{\gamma} &= \int_{\partial\Omega} 2m_{\boldsymbol{\nu}}u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T\,\mathrm{d}\boldsymbol{\gamma} \\ &= \sum_{j=1}^k \int_{U_j} 2m_{\boldsymbol{\nu}}\theta_j u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T\,\mathrm{d}\boldsymbol{\gamma}. \end{aligned}$$

For $j \in \{1, \dots, k\}$, we consider the j-th term in the above sum. For simplicity, we have denoted θ_j and U_j by θ and U, respectively. We have

$$\mathbf{m}_T = \sum_{i=1}^{n-1} m^i\mathbf{a}_i.$$

We write $|g| = |\det(g)|$ and $W = \boldsymbol{\phi}^{-1}(U)$ and we get

$$\begin{aligned} &\int_U 2m_{\boldsymbol{\nu}}\theta u'_{\boldsymbol{\nu}}\nabla_T u_{\boldsymbol{\nu}}\cdot\mathbf{m}_T\,\mathrm{d}\boldsymbol{\gamma} \\ &= \int_W 2(m_{\boldsymbol{\nu}}\circ\boldsymbol{\phi})(\theta\circ\boldsymbol{\phi})(u'_{\boldsymbol{\nu}}\circ\boldsymbol{\phi})\left(\sum_{i=1}^{n-1}\frac{\partial(u_{\boldsymbol{\nu}}\circ\boldsymbol{\phi})}{\partial\xi_i}m^i\right)|g|^{\frac{1}{2}}\,\mathrm{d}\boldsymbol{\gamma}. \end{aligned} \tag{2.27}$$

We write $v_{\boldsymbol{\nu}} = u_{\boldsymbol{\nu}} \circ \boldsymbol{\phi} \in H^{\frac{1}{2}}(W)$. There exists $C > 0$ such that

$$\|v_{\boldsymbol{\nu}}\|_{H^{\frac{1}{2}}(W)} \leq C \|u_{\boldsymbol{\nu}}\|_{H^{\frac{1}{2}}(U)}.$$

We now introduce a partition of W :

$$\begin{cases} W^+ = \{(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \ / \ m^1(\xi_1, \dots, \xi_{n-1}) > 0\} \cap W, \\ W^- = \{(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \ / \ m^1(\xi_1, \dots, \xi_{n-1}) < 0\} \cap W. \end{cases}$$

and we look at the term with m^1 on W^+ in the right-hand side of (2.27).

We write $\psi = \Big((m_{\boldsymbol{\nu}} \circ \boldsymbol{\phi})(\theta \circ \boldsymbol{\phi}) m^1 |g|^{\frac{1}{2}}\Big)^{\frac{1}{2}}$. So we have

$$\begin{aligned} \int_{W^+} 2(m_{\boldsymbol{\nu}} \circ \boldsymbol{\phi})(\theta \circ \boldsymbol{\phi}) v_{\boldsymbol{\nu}}' \frac{\partial v_{\boldsymbol{\nu}}}{\partial \xi_1} m^1 |g|^{\frac{1}{2}} \, \mathrm{d}\boldsymbol{\gamma} &= \int_{W^+} 2\psi^2 v_{\boldsymbol{\nu}}' \frac{\partial v_{\boldsymbol{\nu}}}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} \\ &= \int_{W^+} 2\psi v_{\boldsymbol{\nu}}' \frac{\partial (\psi v_{\boldsymbol{\nu}})}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} - \int_{W^+} \psi (v_{\boldsymbol{\nu}}^2)' \frac{\partial \psi}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma}. \end{aligned} \tag{2.28}$$

Consider the right-hand side of (2.28). We have

$$\int_S^T \int_{W^+} \psi (v_{\boldsymbol{\nu}}^2)' \frac{\partial \psi}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} \, \mathrm{d}t = \left[\int_{W^+} \psi v_{\boldsymbol{\nu}}^2 \frac{\partial \psi}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} \right]_S^T.$$

There exists $C > 0$ independent of $\mathbf{u}$ and of t such that

$$\begin{aligned} \left| \int_{W^+} \psi v_{\boldsymbol{\nu}}^2 \frac{\partial \psi}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} \right| &\leq C \int_{\partial \Omega_N} |\mathbf{u}|^2 \, \mathrm{d}\boldsymbol{\gamma} \\ &\leq C \int_{\partial \Omega} |\mathbf{u}|^2 \, \mathrm{d}\boldsymbol{\gamma}. \end{aligned}$$

Using a trace result, there exists $C > 0$, independent of $\mathbf{u}$ and t such that

$$\int_{\partial \Omega} |\mathbf{u}|^2 \, \mathrm{d}\boldsymbol{\gamma} \leq C \|\mathbf{u}\|_{\mathbb{H}^1(\Omega)}^2.$$

Now, using Poincaré's inequality and Korn's inequality, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\|\mathbf{u}\|_{\mathbb{H}^1(\Omega)}^2 \leq C E(\mathbf{u}, t).$$

Then, since $E(\mathbf{u}, .)$ is non-increasing, we get $C > 0$, independent of $\mathbf{u}$, such that

$$\left| \int_S^T \int_{W^+} \psi (v_{\boldsymbol{\nu}}^2)' \frac{\partial \psi}{\partial \xi_1} \, \mathrm{d}\boldsymbol{\gamma} \, \mathrm{d}t \right| \leq C E(\mathbf{u}, S). \tag{2.29}$$

Now, for the first term of (2.28), we define

$$G = \begin{cases} \psi v_{\boldsymbol{\nu}} & \text{in } W^+ \times \mathbb{R}, \\ 0 & \text{in } (\mathbb{R}^{n-1} \setminus W^+) \times \mathbb{R}. \end{cases}$$

We have

$$\int_{W^+} 2\psi v'_{\boldsymbol{\nu}} \frac{\partial(\psi v_{\boldsymbol{\nu}})}{\partial \xi_1}\, \mathrm{d}\boldsymbol{\gamma} = \int_{\mathbb{R}^{n-1}} 2G' \frac{\partial G}{\partial \xi_1}\, \mathrm{d}\boldsymbol{\gamma}.$$

We denote by $\hat{G}$ the Fourier transform of G with respect to ξ_1, and we get

$$\int_{W^+} 2\psi v'_{\boldsymbol{\nu}} \frac{\partial(\psi v_{\boldsymbol{\nu}})}{\partial \xi_1}\, \mathrm{d}\boldsymbol{\gamma} = \int_{\mathbb{R}^{n-1}} 2i\pi\eta_1 (\hat{G}^2)'\, \mathrm{d}\boldsymbol{\gamma}.$$

Now, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left| \int_{\mathbb{R}^{n-1}} \eta_1 \hat{G}^2\, \mathrm{d}\boldsymbol{\gamma} \right| \leq C \|G\|^2_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})}.$$

Moreover, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\|G\|^2_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \|u_{\boldsymbol{\nu}}\|^2_{H^{\frac{1}{2}}(\partial\Omega_N)},$$

and thanks to a trace result, we get $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\|G\|^2_{H^{\frac{1}{2}}(\mathbb{R}^{n-1})} \leq C \|\mathbf{u}\|^2_{H^1(\Omega)}.$$

Thanks to (2.13), $E(\mathbf{u}, .)$ is non-increasing. We then can find $C > 0$, independent of $\mathbf{u}$, such that

$$\left| \int_S^T \int_{W^+} 2\psi v'_{\boldsymbol{\nu}} \frac{\partial(\psi v_{\boldsymbol{\nu}})}{\partial \xi_1}\, \mathrm{d}\boldsymbol{\gamma}\, \mathrm{d}t \right| \leq C E(\mathbf{u}, S). \tag{2.30}$$

>From (2.29) and (2.30), we build $C > 0$, independent of $\mathbf{u}$, such that

$$\left| \int_{W^+} 2\psi^2 v'_{\boldsymbol{\nu}} \frac{\partial v_{\boldsymbol{\nu}}}{\partial \xi_1}\, \mathrm{d}\boldsymbol{\gamma} \right| \leq C E(\mathbf{u}, S).$$

For the integral in W^-, we replace a_1 by $-a_1$, m^1 by $-m^1$, respectively and proceed as above. We can also get similar results concerning the integral terms containing m^i, for $i \in \{2, \dots, n-1\}$. Finally, we get $C > 0$, independent of $\mathbf{u}$, such that

$$|I_1| \leq C E(\mathbf{u}, S). \tag{2.31}$$

- Estimate of $I_2 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}} \mathbf{u}'_T \cdot \partial_T \mathbf{u}_T \mathbf{m}_T\, \mathrm{d}\boldsymbol{\gamma}\, \mathrm{d}t$.

We write

$$\mathbf{u}_T = \begin{pmatrix} u_T^1 \\ \vdots \\ u_T^{n-1} \end{pmatrix}.$$

We have

$$2m_{\boldsymbol{\nu}} \mathbf{u}'_T \cdot \partial_T \mathbf{u}_T \mathbf{m}_T = \sum_{i=1}^{n-1} 2m_{\boldsymbol{\nu}} u_T^{i\prime} \nabla_T u_T^i \cdot \mathbf{m}_T.$$

As well as for I_1, we get that, for all $i \in \{1, \dots, n-1\}$, there exists $C > 0$, independent of $\mathbf{u}$, such that

$$\left|\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}} u_T^{i\prime} \nabla_T u_T^i \cdot \mathbf{m}_T \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t\right| \leq CE(\mathbf{u}, S).$$

And then, there exists $C > 0$, independent of $\mathbf{u}$, such that

$$|I_2| \leq CE(\mathbf{u}, S). \tag{2.32}$$

• Estimate of $I_3 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot \partial_{\boldsymbol{\nu}} \mathbf{u}_T \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t$.

We have

$$\partial_{\boldsymbol{\nu}} \mathbf{u}_T = 2\epsilon_S(\mathbf{u}) - \nabla_T \mathbf{u}_{\boldsymbol{\nu}} + \partial_T \boldsymbol{\nu}\, \mathbf{u}_T.$$

Now, for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot (\partial_T \boldsymbol{\nu}\, \mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma}\right| \leq \frac{C}{\theta} \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}_T'|^2 \,\mathrm{d}\boldsymbol{\gamma} + \theta \int_{\partial\Omega_N} |\mathbf{u}|^2 \,\mathrm{d}\boldsymbol{\gamma}.$$

Therefore, as in (2.29), for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot (\partial_T \boldsymbol{\nu}\, \mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma}\right| \leq \frac{C}{\theta} \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma} + \theta E(\mathbf{u}, t). \tag{2.33}$$

Moreover, for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 4m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot \epsilon_S(\mathbf{u}) \,\mathrm{d}\boldsymbol{\gamma}\right| \leq \frac{C}{\theta} \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma} + \theta \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\epsilon_S(\mathbf{u})|^2 \,\mathrm{d}\boldsymbol{\gamma}. \tag{2.34}$$

Let us now estimate the remaining term $\displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot \nabla_T \mathbf{u}_{\boldsymbol{\nu}} \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t$. We have

$$\begin{aligned}\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T' \cdot \nabla_T \mathbf{u}_{\boldsymbol{\nu}} \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t &= \left[\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T \cdot \nabla_T \mathbf{u}_{\boldsymbol{\nu}} \,\mathrm{d}\boldsymbol{\gamma}\right]_S^T \\ &\quad - \int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T \cdot \nabla_T \mathbf{u}_{\boldsymbol{\nu}}' \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \end{aligned} \tag{2.35}$$

We have

$$\begin{aligned}\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T \cdot \nabla_T \mathbf{u}_{\boldsymbol{\nu}}' \,\mathrm{d}\boldsymbol{\gamma} &= -\int_{\partial\Omega_N} 2\mathbf{u}_{\boldsymbol{\nu}}' \mathrm{div}_T(m_{\boldsymbol{\nu}}^2 \mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma} + \int_\Gamma 2\mathbf{u}_{\boldsymbol{\nu}}' m_{\boldsymbol{\nu}}^2 \mathbf{u}_T \cdot \boldsymbol{\tau} \,\mathrm{d}\boldsymbol{\gamma} \\ &= -\int_{\partial\Omega_N} 2\mathbf{u}_{\boldsymbol{\nu}}' \mathrm{div}_T(m_{\boldsymbol{\nu}}^2 \mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma}.\end{aligned}$$

Observe that $\mathrm{div}_T(m_{\boldsymbol{\nu}}^2 \mathbf{u}_T) = m_{\boldsymbol{\nu}}^2 \mathrm{div}_T(\mathbf{u}_T) + 2m_{\boldsymbol{\nu}} \nabla_T(m_{\boldsymbol{\nu}}) \cdot \mathbf{u}_T$. We then get

$$\begin{aligned}\left|\int_{\partial\Omega_N} 2\mathbf{u}_{\boldsymbol{\nu}}' \mathrm{div}_T(m_{\boldsymbol{\nu}}^2 \mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma}\right| &\leq \left|\int_{\partial\Omega_N} 2\mathbf{u}_{\boldsymbol{\nu}}' m_{\boldsymbol{\nu}}^2 \mathrm{div}_T(\mathbf{u}_T) \,\mathrm{d}\boldsymbol{\gamma}\right| \\ &\quad + \left|\int_{\partial\Omega_N} 4\mathbf{u}_{\boldsymbol{\nu}}' m_{\boldsymbol{\nu}} \nabla_T(m_{\boldsymbol{\nu}}) \cdot \mathbf{u}_T \,\mathrm{d}\boldsymbol{\gamma}\right|.\end{aligned}$$

Therefore, for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\begin{aligned}\left|\int_{\partial\Omega_N} 2\mathbf{u}'_{\boldsymbol{\nu}}\mathrm{div}_T(m_{\boldsymbol{\nu}}^2\mathbf{u}_T)\,\mathrm{d}\boldsymbol{\gamma}\right| \;&\le\; \theta\int_{\partial\Omega_N}(|\mathbf{u}|^2+m_{\boldsymbol{\nu}}|\mathrm{div}_T(\mathbf{u}_T)|^2)\,\mathrm{d}\boldsymbol{\gamma}\\ &\quad+\frac{C}{\theta}\int_{\partial\Omega_N} m_{\boldsymbol{\nu}}|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma}.\end{aligned}$$

But $|\mathrm{div}_T(\mathbf{u}_T)|^2 \le 2\epsilon_T(\mathbf{u}_T):\epsilon_T(\mathbf{u}_T)$.
Therefore, for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\begin{aligned}\left|\int_{\partial\Omega_N} 2\mathbf{u}'_{\boldsymbol{\nu}}\mathrm{div}_T(m_{\boldsymbol{\nu}}^2\mathbf{u}_T)\,\mathrm{d}\boldsymbol{\gamma}\right| \;&\le\; \theta E(\mathbf{u},t)+\theta\int_{\partial\Omega_N} m_{\mathbf{v}}\epsilon_T(\mathbf{u}_T):\epsilon_T(\mathbf{u}_T)\,\mathrm{d}\boldsymbol{\gamma}\\ &\quad+\frac{C}{\theta}\int_{\partial\Omega_N} m_{\boldsymbol{\nu}}|\mathbf{u}'|^2\,\mathrm{d}\boldsymbol{\gamma}.\end{aligned}\tag{2.36}$$

Let us now study the remaining term in (2.35), $\displaystyle\left[\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right]_S^T$.

$$\begin{aligned}\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2\mathbf{u}_T\cdot\nabla_T\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma} \;&=\; -\int_{\partial\Omega_N} 2\mathrm{div}_T(m_{\boldsymbol{\nu}}^2\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\\ &=\; -\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2\mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}-\int_{\partial\Omega_N} 4m_{\boldsymbol{\nu}}(\nabla_T m_{\boldsymbol{\nu}}\cdot\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}.\end{aligned}$$

There exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 4m_{\boldsymbol{\nu}}(\nabla_T m_{\boldsymbol{\nu}}\cdot\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right| \le C\int_{\partial\Omega_N}|\mathbf{u}|^2\,\mathrm{d}\boldsymbol{\gamma}.$$

Hence, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 4m_{\boldsymbol{\nu}}(\nabla_T m_{\boldsymbol{\nu}}\cdot\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right| \le CE(\mathbf{u},t).\tag{2.37}$$

It remains $\displaystyle\left[\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2\mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right]_S^T$.
For a given $t > 0$, let us define $\xi \in H^1(\partial\Omega_N)$ such that

$$\begin{cases}\xi-\Delta_T\xi=\mathrm{div}_T(\mathbf{u}_T)(t), & \text{in } \partial\Omega_N;\\ \xi=0 & \text{on } \Gamma.\end{cases}$$

We have $\mathrm{div}_T(\mathbf{u}_T)(t) \in H^{-\frac{1}{2}}(\partial\Omega_N)$, then ξ satisfies

$$\begin{cases}\|\xi\|_{H^1(\partial\Omega_N)} \le C\|\mathbf{u}_T\|_{L^2(\partial\Omega_N,T(\partial\Omega_N))};\\ \xi \in H^{\frac{3}{2}}(\partial\Omega_N) \quad \text{and} \quad \|\xi\|_{H^{\frac{3}{2}}(\partial\Omega_N)} \le C\|\mathbf{u}_T\|_{H^{\frac{1}{2}}(\partial\Omega_N,T(\partial\Omega_N))}.\end{cases}$$

Using ξ, we get

$$\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathrm{div}_T(\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma} = \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\xi\,\mathrm{d}\boldsymbol{\gamma} - \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\Delta_T\xi\,\mathrm{d}\boldsymbol{\gamma}.$$

There exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq C\int_{\partial\Omega_N} (|\mathbf{u}_{\boldsymbol{\nu}}|^2 + |\xi|^2)\,\mathrm{d}\boldsymbol{\gamma}.$$

There exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq C\int_{\partial\Omega_N} |\mathbf{u}|^2\,\mathrm{d}\boldsymbol{\gamma}.$$

And then, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq CE(\mathbf{u}, t).$$

Moreover,

$$\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\Delta_T\xi\,\mathrm{d}\boldsymbol{\gamma} = \int_{\partial\Omega_N} (-\Delta_T)^{\frac{1}{4}}(m_{\boldsymbol{\nu}}^2\mathbf{u}_{\boldsymbol{\nu}})(-\Delta_T)^{\frac{3}{4}}(\xi)\,\mathrm{d}\boldsymbol{\gamma}.$$

Then, we get $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\Delta_T\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq C\|\mathbf{u}_\nu\|_{H^{\frac{1}{2}}(\partial\Omega_N)}\|\xi\|_{H^{\frac{3}{2}}(\partial\Omega_N)},$$

and there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\Delta_T\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq C\|\mathbf{u}\|^2_{\mathbb{H}^1(\Omega)}.$$

Hence, there exists $C > 0$, independent of $\mathbf{u}$ and t, such that

$$\left|\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_{\boldsymbol{\nu}}\Delta_T\xi\,\mathrm{d}\boldsymbol{\gamma}\right| \leq CE(\mathbf{u}, t).$$

We finally get $C > 0$, independent of $\mathbf{u}$, such that

$$\left|\left[\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathrm{div}(\mathbf{u}_T)\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right]_S^T\right| \leq CE(\mathbf{u}, S). \tag{2.38}$$

With (2.37) and (2.38), we get $C > 0$, independent of $\mathbf{u}$, such that

$$\left|\left[\int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 \mathbf{u}_T\cdot\nabla_T\mathbf{u}_{\boldsymbol{\nu}}\,\mathrm{d}\boldsymbol{\gamma}\right]_S^T\right| \leq CE(\mathbf{u}, S). \tag{2.39}$$

Now, thanks to (2.33), (2.34), (2.36) and (2.39), we get that for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$, such that

$$\begin{aligned} |I_3| \leq\ & \theta \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}}(|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T)) \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t + \theta \int_S^T E(\mathbf{u}, t)\,\mathrm{d}t \\ & + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t + CE(\mathbf{u}, S). \end{aligned} \tag{2.40}$$

• Estimate of $I_4 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}(u_{\boldsymbol{\nu}}\mathbf{u}'_T - u'_{\boldsymbol{\nu}}\mathbf{u}_T) \cdot (\partial_T \boldsymbol{\nu})\mathbf{m}_T \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t$.

Using Cauchy-Schwarz inequality, we can prove that for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$, such that

$$|I_4| \leq \theta \int_S^T \int_{\partial\Omega_N} |\mathbf{u}|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t.$$

Then, for some $\theta > 0$, there exists $C > 0$, independent of $\mathbf{u}$, such that

$$|I_4| \leq \theta \int_S^T E(\mathbf{u}, t)\,\mathrm{d}t + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \tag{2.41}$$

• Estimate of $I_5 = \displaystyle\int_S^T \int_{\partial\Omega_N} 2m_{\boldsymbol{\nu}}^2 u'_{\boldsymbol{\nu}} \partial_{\nu} u_{\boldsymbol{\nu}} \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t$.

Similarly, for some $\theta > 0$, we get $C > 0$, independent of $\mathbf{u}$, such that

$$|I_5| \leq \theta \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\partial_{\boldsymbol{\nu}} \mathbf{u}_{\boldsymbol{\nu}}|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t. \tag{2.42}$$

• End of the proof.

We now use (2.31), (2.32), (2.40), (2.41) and (2.42) and, for some $\theta > 0$, we get $C > 0$, independent of $\mathbf{u}$, such that

$$\begin{aligned} |I| \leq\ & \theta \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}}(|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_{\boldsymbol{\nu}} u_{\boldsymbol{\nu}}|^2) \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t \\ & + \theta \int_S^T E(\mathbf{u}, t)\,\mathrm{d}t + \frac{C}{\theta} \int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} |\mathbf{u}'|^2 \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t + CE(\mathbf{u}, S). \end{aligned} \tag{2.43}$$

Now, using (2.24) and (2.25), we get

$$\sigma(\mathbf{u}) : \epsilon(\mathbf{u}) \geq 2\mu(|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_{\boldsymbol{\nu}} u_{\boldsymbol{\nu}}|^2).$$

Therefore, for every $\theta > 0$ small enough,

$$\int_S^T \int_{\partial\Omega_N} m_{\boldsymbol{\nu}} \left(-\sigma(\mathbf{u}) : \epsilon(\mathbf{u}) + \theta(|\epsilon_S(\mathbf{u})|^2 + \epsilon_T(\mathbf{u}_T) : \epsilon_T(\mathbf{u}_T) + |\partial_{\boldsymbol{\nu}} u_{\boldsymbol{\nu}}|^2)\right) \,\mathrm{d}\boldsymbol{\gamma}\,\mathrm{d}t \leq 0.$$

This completes the proof. □

Chapitre 3

Contrôlabilité et stabilisation frontière non-linéaire du système élastodynamique en présence de singularités

Controllability and non-linear boundary stabilization of elastodynamic systems involving singularities.

Résumé en français Nous considérons un système élastodynamique avec des conditions frontière mélées. Nous prouvons un résultat de stabilisation à l'aide d'un feed-back non-linéaire de type Neumann. Nous sommes alors amenés à prouver un résultat de contrôlabilité pour ce système.

Ce chapitre est constitué d'une prépublication soumise aux proceedings de *IFIP7.2*, suite à une communication dans le cadre de la conférence *ECCOMAS, juillet 2004, Jyväskylä, Finlande*, lors du workshop *Boundary control, moving boundary, smart material and structural control.*

(n ° 388, http ://maply.univ-lyon1.fr/publis/publiv/2004/publis.html)

English abstract We consider an elastodynamic system with mixed boundary conditions. We prove boundary stabilization results by damping the system by a nonlinear feed-back of Neumann-type. While proving this result, we prove a controllability result for the system.

Introduction

This paper follows our works ([7, 8]), where we have obtained the boundary stabilization of elastodynamic systems, with a linear feed-back of Neumann-type.

We here consider an elastic body which satisfies Lamé's laws. As usual, we define the strain tensor and the stress tensor for a regular vector field $\mathbf{v}$ by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i \mathbf{v}_j + \partial_j \mathbf{v}_i), \qquad \sigma(\mathbf{v}) = 2\mu\epsilon(\mathbf{v}) + \lambda \mathrm{div}(\mathbf{v}) I_n.$$

where λ and μ are the Lamé's coefficients and I_n is the identity matrix of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set such that its boundary $\partial\Omega$ is of class C^2 and satisfies

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_N, \text{ with } \begin{cases} \partial\Omega_D \cap \partial\Omega_N = \emptyset, \\ \mathrm{meas}(\partial\Omega_D) \neq 0, \\ \mathrm{meas}(\partial\Omega_N) \neq 0. \end{cases} \tag{3.1}$$

We denote the boundary interface by $\Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$. We assume that :

$$\begin{gathered} \Gamma \text{ is a } (n-2)\text{-dimensional submanifold of class } C^3 \text{ such that} \\ \text{there exists a neighborhood } \Omega' \text{ of } \Gamma \text{ such that } \partial\Omega \cap \Omega' \\ \text{is a } (n-1)\text{-submanifold of class } C^3. \end{gathered} \tag{3.2}$$

We consider $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outward of Ω. We assume that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that, setting $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, we have

$$(\mathbf{m}\cdot\boldsymbol{\nu}) \geq 0, \text{ on } \partial\Omega_N, \qquad (\mathbf{m}\cdot\boldsymbol{\nu}) \leq 0, \text{ on } \partial\Omega_D. \tag{3.3}$$

At each point $\mathbf{s}$ of Γ, we consider Γ as a submanifold of $\partial\Omega$ of co-dimension 1 and we can denote by $\boldsymbol{\tau}(\mathbf{s})$ the unit normal vector to Γ pointing outward of $\partial\Omega_N$. We assume that :

$$\mathbf{m}\cdot\boldsymbol{\tau} \leq 0, \quad \text{on } \Gamma. \tag{3.4}$$

Let $g \in C(\mathbb{R}^n, \mathbb{R}^n)$ be a continuous function such that $g(\mathbf{0}) = \mathbf{0}$ and

$$\mathbf{g}(\mathbf{x})\cdot\mathbf{x} \geq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.5}$$

We assume moreover that there exist constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$, $p \geq 1$ and $q \in]0,1]$ such that

$$c_1|\mathbf{x}|^p \leq |\mathbf{g}(\mathbf{x})| \leq c_2|\mathbf{x}|^q \quad \forall |\mathbf{x}| \leq 1; \tag{3.6}$$

$$c_3|\mathbf{x}| \leq |\mathbf{g}(\mathbf{x})| \leq c_4|\mathbf{x}| \quad \forall |\mathbf{x}| > 1; \tag{3.7}$$

where $|.|$ is the euclidian norm on $\mathbb{R}^n$.

We here consider the linear isotropic elastodynamic system

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } \Omega \times \mathbb{R}_+; \\ \mathbf{u} = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{g}(\mathbf{u}') & \text{on } \partial\Omega_N \times \mathbb{R}_+; \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega; \\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases} \tag{3.8}$$

We emphasize that, as in the linear case, singularities appear along Γ beacause of the change of boundary conditions.

We introduce following Sobolev spaces : $\mathbb{L}^2(\Omega) = (L^2(\Omega))^n$, $\mathbb{H}^s(\Omega) = (H^s(\Omega))^n$, $\forall s > 0$ and $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega)/\mathbf{v} = 0 \text{ on } \partial\Omega_D\}$. We define an Hilbert space $\mathcal{H} = \mathbb{H}^1_D(\Omega) \times \mathbb{L}^2(\Omega)$, with the following norm :

$$\forall (\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}, \quad \|(\mathbf{v}_0, \mathbf{v}_1)\|_{\mathcal{H}} = \int_\Omega \left(|\mathbf{v}_1|^2 + \sigma(\mathbf{v}_0) : \epsilon(\mathbf{v}_0)\right) \, \mathrm{d}\mathbf{x}.$$

The semi-group method leads to well-posedness of problem (3.8) under assumptions (3.6), (3.7) and the following one :

$$(\mathbf{u}_0, \mathbf{u}_1) \in \mathcal{H}. \tag{3.9}$$

Our main purpose is to prove the following stabilization result :

Théorème 3.0.1. *Let Ω be a bounded connected domain of $\mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (3.1) and (3.2). We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (3.3) and (3.4) hold.*

Let $\mathbf{g} \in C(\mathbb{R}^n, \mathbb{R}^n)$ be a continuous function such that $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ and such that (3.5), (3.6) and (3.7) hold.

For every $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (3.9), the solution $\mathbf{u}$ of (3.8) satisfies

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C_0 e^{-\varpi_0 t} E(\mathbf{u}, 0), \quad \textit{if } p = q = 1; \tag{3.10}$$

$$\forall t \in \mathbb{R}_+, \quad E(\mathbf{u}, t) \leq C_1(1+t)^{-\frac{r}{1-r}}, \quad \textit{if } (p,q) \neq (1,1); \tag{3.11}$$

where $r = min(\frac{2}{p+1}, \frac{2q}{q+1})$*, where* $\varpi_0 > 0$ *and* $C_0 > 0$ *are some constants independent of* $(\mathbf{u}_0, \mathbf{u}_1)$*, and where* $C_1 > 0$ *is a constant depending on* $(\mathbf{u}_0, \mathbf{u}_1)$*.*

This result has been obtained by some authors ([23, 1, 16]), for stronger geometrical assumptions. In particular, all of previous works assume that $\Gamma = \emptyset$.

The most general result obtaining a nonlinear stabilization result using a linear stabilization result has been obtained by Nicaise [32], for an abstract evolution equation of hyperbolic type. The result proved in this chapter is a direct consequence of this paper, and we can as well follow it to prove our result.

We here consider the feed-back, introduced in [21], and used in [5] for the wave equation. Nevertheless, we follow [16] for large parts of our proof. Hence, our proof is made of two steps. We first prove a boundary controllability result (part 1) by using Russel's principle. In the second part, we deduce the boundary stabilization by a non linear feedback, by using a principle called in [32] Liu's principle.

3.1 Boundary controllability

We will first obtain the boundary controllability of elastodynamic systems. We here use Russel's principle (see [35]). This problem has been studied in particular by Lions [25]. Deducing controllability results thanks to stabilization results is classical. (see for example [20, 16] for elastodynamics systems) Let us consider the following problem :

For a given $T > 0$, for all $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, we look for a control function $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ such that the solution of the following problem :

$$\begin{cases} \mathbf{y}'' - \operatorname{div}(\sigma(\mathbf{y})) = 0 & \text{in } \Omega \times [0,T]; \\ \mathbf{y} = 0 & \text{on } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{y})\boldsymbol{\nu} = \boldsymbol{\phi} & \text{on } \partial\Omega_N \times [0,T]; \\ \mathbf{y}(0) = 0 & \text{in } \Omega; \\ \mathbf{y}'(0) = 0 & \text{in } \Omega; \end{cases} \tag{3.12}$$

satisfies

$$\mathbf{y}(T) = \mathbf{y}_0; \quad \mathbf{y}'(T) = \mathbf{y}_1 \quad \text{on } \Omega. \tag{3.13}$$

In order to find the control function $\boldsymbol{\phi}$, we will consider a first intermediate problem. For $(\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}$., find $\mathbf{v}$ such that

$$\begin{cases} \mathbf{v}'' - \operatorname{div}(\sigma(\mathbf{v})) = 0 & \text{in } \Omega \times [0,T]; \\ \mathbf{v} = 0 & \text{on } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{v})\boldsymbol{\nu} = (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{v}' & \text{on } \partial\Omega_N \times [0,T]; \\ \mathbf{v}(T) = \mathbf{v}_0 & \text{in } \Omega; \\ \mathbf{v}'(T) = \mathbf{v}_1 & \text{in } \Omega. \end{cases} \tag{3.14}$$

This problem admits one and only one solution $\mathbf{v} \in C([0,T], \mathbb{H}^1_D(\Omega))$. We denote by $\tilde{\mathbf{v}}$ the function belonging to $([0,T], \mathbb{H}^1_D(\Omega))$, and such that $\tilde{\mathbf{v}}(t) = \mathbf{v}(T-t)$, $\forall t \in [0,T]$. We apply the boundary stabilisation estimate that we obtained in [8] to get constants $C > 0$ and $\varpi > 0$, such that

$$\forall t \in [0,T], \quad E(\tilde{\mathbf{v}}, t) \leq Ce^{-\varpi t} E(\tilde{\mathbf{v}}, 0).$$

We then have

$$\forall t \in [0,T], \quad E(\mathbf{v}, t) \leq Ce^{-\varpi(T-t)} E(\mathbf{v}, T). \tag{3.15}$$

Let us now consider a second intermediate problem. Find $\mathbf{w}$ such that

$$\begin{cases} \mathbf{w}'' - \mathrm{div}(\sigma(\mathbf{w})) = 0 & \text{in } \Omega \times [0,T]; \\ \mathbf{w} = 0 & \text{on } \partial\Omega_D \times [0,T]; \\ \sigma(\mathbf{w})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{w}' & \text{on } \partial\Omega_N \times [0,T]; \\ \mathbf{w}(0) = \mathbf{v}(0) & \text{in } \Omega; \\ \mathbf{w}'(0) = \mathbf{v}'(0) & \text{in } \Omega. \end{cases} \tag{3.16}$$

This problem admits one and only one solution $\mathbf{w} \in C([0,T], \mathbb{H}^1_D(\Omega))$, and, using [8] again, we get

$$\forall t \in [0,T], \quad E(\mathbf{w}, t) \leq Ce^{-\varpi t} E(\mathbf{v}, 0). \tag{3.17}$$

Now, $\mathbf{y} = \mathbf{w} - \mathbf{v}$ satisfies (3.12) with $\boldsymbol{\phi} = -(\mathbf{m}\cdot\boldsymbol{\nu})(\mathbf{v}' + \mathbf{w}') \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$.
Let us define the operator Λ by :

$$\begin{array}{rrcl} \Lambda : & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & (\mathbf{v}_0, \mathbf{v}_1) & \longmapsto & (\mathbf{w}(T), \mathbf{w}'(T)). \end{array}$$

Let $(\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}$. Using (3.15) and (3.17), we have

$$\begin{aligned} \|\Lambda(\mathbf{v}_0, \mathbf{v}_1)\|^2_{\mathcal{H}} &= \|(\mathbf{w}(T), \mathbf{w}'(T))\|^2_{\mathcal{H}} \\ &= 2E(\mathbf{w}, T) \\ &\leq 2CE(\mathbf{v}, 0)e^{-\varpi T} \\ &\leq 2C^2 E(\mathbf{v}, T)e^{-2\varpi T} \\ &= C^2 e^{-2\varpi T} \|(\mathbf{v}_0, \mathbf{v}_1)\|^2_{\mathcal{H}}. \end{aligned}$$

We then have $\|\Lambda\| \leq Ce^{-\varpi T}$, and then

$$\|\Lambda\| < 1, \quad \text{if } T > \frac{\ln C}{\varpi}. \tag{3.18}$$

We suppose that $T > \frac{\ln C}{\varpi}$. The operator $\Lambda - I_{\mathcal{H}}$ is an isomorphism and then, for all $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, we define $(\mathbf{v}_0, \mathbf{v}_1) \in \mathcal{H}$ by $(\mathbf{v}_0, \mathbf{v}_1) = (\Lambda - I_{\mathcal{H}})^{-1}(\mathbf{y}_0, \mathbf{y}_1)$ and we have

$$(\mathbf{y}(T), \mathbf{y}'(T)) = (\mathbf{w}(T), \mathbf{w}'(T)) - (\mathbf{v}_0, \mathbf{v}_1) = (\Lambda - I_{\mathcal{H}})(\mathbf{v}_0, \mathbf{v}_1) = (\mathbf{y}_0, \mathbf{y}_1).$$

Then $\mathbf{y}$ satisfies (3.13), and we have obtained the following theorem :

Théorème 3.1.1. *Let Ω be a bounded connected domain of $\mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (3.1) and (3.2). We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (3.3) and (3.4) hold.*

Then there exists $T_0 > 0$ such that, for all $T > T_0$ and for all $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, there exists $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ such that the solution $\mathbf{y}$ of problem (3.12) satisfies $\mathbf{y}(T) = \mathbf{y}_0$ and $\mathbf{y}'(T) = \mathbf{y}_1$ on Ω.

Remark : We take $T > T_0$ and $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$. We then have a control function $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ which satisfies theorem 3.1.1. Following the proof of this theorem, we write $\boldsymbol{\phi} = (\mathbf{m}\cdot\boldsymbol{\nu})\tilde{\boldsymbol{\phi}}$. We then have $\tilde{\boldsymbol{\phi}} = -(\mathbf{v}' + \mathbf{w}')$. $\tilde{\boldsymbol{\phi}} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ and we have :

$$\begin{aligned}
\|\tilde{\boldsymbol{\phi}}\|^2_{\mathbb{L}^2(\partial\Omega_N\times[0,T])} &\leq 2R\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(|\mathbf{v}'|^2 + |\mathbf{w}'|^2\right) \,\mathrm{d}\Gamma\mathrm{d}t \\
&= 2R\left(E(\mathbf{v},T) - E(\mathbf{v},0) + E(\mathbf{w},0) - E(\mathbf{w},T)\right) \\
&\leq R\|(\mathbf{v}_0,\mathbf{v}_1)\|^2_{\mathcal{H}},
\end{aligned}$$

Where $R = \|\mathbf{m}\|_\infty$.

We then get :

$$\|\tilde{\boldsymbol{\phi}}\|_{\mathbb{L}^2(\partial\Omega_N\times[0,T])} \leq \frac{\sqrt{R}}{1 - Ce^{-\varpi T}}\|(\mathbf{y}_0,\mathbf{y}_1)\|_{\mathcal{H}}. \tag{3.19}$$

Remark : Theorem 3.1.1 is usually expressed on the following form :

Théorème 3.1.2. *Let Ω be a bounded connected domain of $\mathbb{R}^n$. We assume that its boundary $\partial\Omega$ is of class C^2 and satisfies (3.1) and (3.2). We assume moreover that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that (3.3) and (3.4) hold.*

Then there exists $T_0 > 0$ such that, for all $T > T_0$ and for all $(\mathbf{y}_0, \mathbf{y}_1) \in \mathcal{H}$, there exists $\boldsymbol{\phi} \in \mathbb{L}^2(\partial\Omega_N \times [0,T])$ such that the solution $\mathbf{y}$ of the following problem

$$\begin{cases}
\mathbf{y}'' - \operatorname{div}(\sigma(\mathbf{y})) = 0 & \text{in } \Omega\times[0,T]; \\
\mathbf{y} = 0 & \text{on } \partial\Omega_D\times[0,T]; \\
\sigma(\mathbf{y})\boldsymbol{\nu} = \boldsymbol{\phi} & \text{on } \partial\Omega_N\times[0,T]; \\
\mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega; \\
\mathbf{y}'(0) = \mathbf{y}_1 & \text{in } \Omega;
\end{cases}$$

satisfies $\mathbf{y}(T) = 0$ and $\mathbf{y}'(T) = 0$ on Ω.

This result can be easily deduced from theorem 3.1.1 by reversing time-scale.

3.2 Non-linear stabilization

We will now prove theorem 3.0.1. Let $\mathbf{u}$ be the solution of problem (3.8). We take here strong solutions.

According to the previous section, we take $T_0 > 0$ such that theorem 3.1.1 holds. We then fix $T > T_0$. Then, applying theorem 3.1.1 with $(\mathbf{y}_0, \mathbf{y}_1) = (\mathbf{u}(T), \mathbf{u}'(T))$, we get $\mathbf{y}$ solution of problem (3.12) such that $(\mathbf{y}(T), \mathbf{y}'(T)) = (\mathbf{u}(T), \mathbf{u}'(T))$.

Let us recall that

$$E'(\mathbf{u},t) = -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{g}(\mathbf{u}')\cdot\mathbf{u}'\,\mathrm{d}\mathbf{x} \tag{3.20}$$

Multiplying first equation of (3.8) and (3.12) by $\mathbf{y}'$ and $\mathbf{u}'$ respectively, and integrating by parts, we get :

$$\begin{aligned}
0 &= \int_0^T\int_\Omega (\mathbf{y}'\cdot(\mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) + \mathbf{u}'\cdot(\mathbf{y}'' - \mathrm{div}(\sigma(\mathbf{y}))) \,\mathrm{d}\mathbf{x}\mathrm{d}t \\
&= \int_0^T\int_\Omega (\mathbf{u}'\cdot\mathbf{y}' + \sigma(\mathbf{u}):\epsilon(\mathbf{y}))' \,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(\mathbf{y}'\cdot\mathbf{g}(\mathbf{u}') - \mathbf{u}'\cdot\tilde{\boldsymbol{\phi}}\right) \mathrm{d}\Gamma\mathrm{d}t \\
&= \int_\Omega (\mathbf{u}'(T)\cdot\mathbf{y}'(T) + \sigma(\mathbf{u}(T)):\epsilon(\mathbf{y}(T))) \,\mathrm{d}\mathbf{x} + \int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(\mathbf{y}'\cdot\mathbf{g}(\mathbf{u}') - \mathbf{u}'\cdot\tilde{\boldsymbol{\phi}}\right) \mathrm{d}\Gamma\mathrm{d}t \\
&= 2E(\mathbf{u},T) + \int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(\mathbf{y}'\cdot\mathbf{g}(\mathbf{u}') - \mathbf{u}'\cdot\tilde{\boldsymbol{\phi}}\right) \mathrm{d}\Gamma\mathrm{d}t.
\end{aligned}$$

We then have :

$$E(\mathbf{u},T) = \frac{1}{2}\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(\mathbf{u}'\cdot\tilde{\boldsymbol{\phi}} - \mathbf{y}'\cdot\mathbf{g}(\mathbf{u}')\right) \mathrm{d}\Gamma\mathrm{d}t. \tag{3.21}$$

We now consider two cases.

First case : $p = q = 1$.

In the following, $k, k', k'', \ldots$ denotes non-negative constants, independent of $\mathbf{u}$ and t, which can change line by line. Using (3.6), (3.7) and (3.20), we have

$$\begin{aligned}
E(\mathbf{u},T) &\leq \frac{1}{2}\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(|\mathbf{u}'| + |\mathbf{g}(\mathbf{u}')|\right)\left(|\mathbf{y}'| + |\tilde{\boldsymbol{\phi}}|\right) \mathrm{d}\Gamma\mathrm{d}t \\
&\leq k\left(\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(|\mathbf{u}'| + |\mathbf{g}(\mathbf{u}')|\right)^2 \mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad \left(\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(|\mathbf{y}'|^2 + |\tilde{\boldsymbol{\phi}}|^2\right) \mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k'\left(\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{g}(\mathbf{u}')\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\quad \left(\int_0^T\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\left(|\mathbf{v}'|^2 + |\mathbf{w}'|^2 + |\tilde{\boldsymbol{\phi}}|^2\right) \mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k'\left(E(\mathbf{u},0) - E(\mathbf{u},T)\right)^{\frac{1}{2}}\left(E(\mathbf{v},T) - E(\mathbf{w},T) + R\|\tilde{\boldsymbol{\phi}}\|^2_{\mathbb{L}^2(\partial\Omega_N\times[0,T])}\right)^{\frac{1}{2}} \\
&\leq k''\left(E(\mathbf{u},0) - E(\mathbf{u},T)\right)^{\frac{1}{2}} E(\mathbf{u},T)^{\frac{1}{2}}.
\end{aligned}$$

We then have

$$E(\mathbf{u},T) \leq \frac{k''^2}{1+k''^2} E(\mathbf{u},0),$$

and then, for all $m \in \mathbb{N}$, we get

$$E(\mathbf{u},mT) \leq \left(\frac{k''^2}{1+k''^2}\right)^m E(\mathbf{u},0).$$

We deduce that there exists constants $C_0 > 0$ and $\varpi > 0$ such that

$$E(\mathbf{u},t) \leq C_0 e^{-\varpi t} E(\mathbf{u},0),$$

which gives (3.10). In this case, we have the same result with weak solutions, by using a density argument.

Second case : $(p,q) \neq (1,1)$.

For all $t \in \mathbb{R}^+$, we split $\partial\Omega_N$:

$$\partial\Omega_N^+ = \{\mathbf{x} \in \partial\Omega_N : |\mathbf{u}'| > 1\}, \quad \partial\Omega_N^- = \{\mathbf{x} \in \partial\Omega_N : |\mathbf{u}'| \leq 1\}.$$

Using (3.6), (3.7) and (3.20), we have

$$\begin{aligned}
\mathcal{I}_1^- &= \int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'||\boldsymbol{\phi}|\,\mathrm{d}\Gamma\mathrm{d}t \\
&\leq \left(\int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'^2\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T\!\!\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\boldsymbol{\phi}|^2\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k\left(\int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^{p+1}\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{p+1}} \left(\int_0^T\!\!\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})(|\mathbf{v}'|^2+|\mathbf{w}'|^2)\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k'\left(E(\mathbf{u},0)-E(\mathbf{u},T)\right)^{\frac{1}{p+1}} E(\mathbf{u},T)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathcal{I}_2^- &= \int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{y}'||\mathbf{g}(\mathbf{u}')|\,\mathrm{d}\Gamma\mathrm{d}t \\
&\leq \left(\int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{g}(\mathbf{u}')|^2\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T\!\!\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{y}'|^2\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k\left(\int_0^T\!\!\int_{\partial\Omega_N^-} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{g}(\mathbf{u}')|^{\frac{q+1}{q}}\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{q}{q+1}} \left(\int_0^T\!\!\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})(|\mathbf{v}'|^2+|\mathbf{w}'|^2)\,\mathrm{d}\Gamma\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq k'\left(E(\mathbf{u},0)-E(\mathbf{u},T)\right)^{\frac{q}{q+1}} E(\mathbf{u},T)^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, as well as for $p = q = 1$, we obtain

$$\int_0^T\int_{\partial\Omega_N^+} ((\mathbf{m}\cdot\boldsymbol{\nu}))\left(\mathbf{u}'\cdot\tilde{\boldsymbol{\phi}} - \mathbf{y}'\cdot\mathbf{g}(\mathbf{u}')\right)\,\mathrm{d}\Gamma\mathrm{d}t \leq k\,(E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{1}{2}}\,E(\mathbf{u},T)^{\frac{1}{2}}.$$

We then get

$$\begin{aligned} E(\mathbf{u},T)^{\frac{1}{2}} &\leq k\left((E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{1}{2}} + (E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{q}{q+1}}\right.\\ &\quad \left. + (E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{1}{p+1}}\right). \end{aligned}$$

We finally obtain

$$E(\mathbf{u},0) \leq k\left((E(\mathbf{u},0)-E(\mathbf{u},T)) + (E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{2q}{q+1}} + (E(\mathbf{u},0)-E(\mathbf{u},T))^{\frac{2}{p+1}}\right).$$

Applying above reasoning for any $t > 0$, we get

$$\begin{aligned} E(\mathbf{u},t) &\leq k\left((E(\mathbf{u},t)-E(\mathbf{u},t+T)) + (E(\mathbf{u},t)-E(\mathbf{u},t+T))^{\frac{2q}{q+1}}\right.\\ &\quad \left. + (E(\mathbf{u},t)-E(\mathbf{u},t+T))^{\frac{2}{p+1}}\right), \end{aligned} \tag{3.22}$$

where k doesn't depend on t.

We know that $E(\mathbf{u},.)$ is a non-negative and non-increasing function. It then admits a limit at infinity. The right-hand side of (3.22) tends then to 0 as t tends to infinity. We obtain

$$E(\mathbf{u},t) \xrightarrow[t\to\infty]{} 0.$$

We take T large enough such that $E(\mathbf{u},T) < 1$. We then have

$$(E(\mathbf{u},t))^{\frac{1}{r}} \leq k\,(E(\mathbf{u},t)-E(\mathbf{u},t+T)), \quad \forall t \geq T, \tag{3.23}$$

where $r = \min\{\frac{2q}{q+1}, \frac{2}{p+1}\}$. As weel as in [16], this will give (3.11). We follow the proof given in [29], in a more general case. Since the proof is simpler in our case, we write it.

We write $\phi(t) = E(\mathbf{u},t)^{\frac{r-1}{r}}$, for all $t > 0$. For all $t \geq T$, using (3.23), we have

$$\begin{aligned} \phi(t+T)-\phi(t) &= \int_0^T \frac{\partial}{\partial\theta}\left(\frac{\theta}{T}E(\mathbf{u},t+T) + \left(\frac{T-\theta}{T}\right)E(\mathbf{u},t)\right)^{\frac{r-1}{r}}\,\mathrm{d}\theta\\ &= \frac{1}{T}\frac{1-r}{r}\,(E(\mathbf{u},t)-E(\mathbf{u},t+T))\\ &\quad \int_0^T\left(\frac{\theta}{T}E(\mathbf{u},t+T) + \left(\frac{T-\theta}{T}\right)E(\mathbf{u},t)\right)^{-\frac{1}{r}}\,\mathrm{d}\theta\\ &\geq \frac{1}{T}\frac{1-r}{r}\frac{1}{k}E(\mathbf{u},t)^{\frac{1}{r}}\int_0^T E(\mathbf{u},t)^{-\frac{1}{r}}\,\mathrm{d}\theta\\ &\geq \frac{1-r}{kr}. \end{aligned}$$

We now consider $t > T$, and we take $n \in \mathbb{N}$ such that $n+T \leq t \leq n+T+1$. We have

$$\begin{aligned} \phi(t) &\geq \phi(t-n) + n\frac{1-r}{kr} \\ &\geq \phi(T+1) + (t-T-1)\frac{1-r}{kr}. \end{aligned}$$

We then get

$$E(\mathbf{u},t) \leq \left(\phi(T+1) + (t-T-1)\frac{1-r}{kr}\right)^{-\frac{r}{1-r}},$$

which gives (3.11). □

Annexe A

Stabilisation frontière du système élastodynamique dans un polygone plan

In this paper, we study the boundary stabilization of the elastodynamic system in a plane polygonal domain. Here, we take in account singularities which appear when changing boundary conditions. *To cite this article : R. Brossard, J.-P. Lohéac, C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

Résumé

Dans ce travail, nous étudions la stabilisation frontière du système élastodynamique dans un polygone plan. Ici, nous prenons en compte les singularités générées par un changement de conditions au bord. *Pour citer cet article : R. Brossard, J.-P. Lohéac, C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

Abridged English version

Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex polygonal set and let $\partial\Omega$ be its boundary. We assume

$$\partial\Omega = \overline{\partial\Omega_N} \cup \overline{\partial\Omega_D}, \qquad \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} = \{\mathbf{s}_1, \mathbf{s}_2\}, \tag{A.1}$$

where $\partial\Omega_N$ and $\partial\Omega_D$ are two open connected non-empty parts of $\partial\Omega$, $\mathbf{s}_1$ and $\mathbf{s}_2$ are two points of $\partial\Omega$, which we will consider as vertices of $\partial\Omega$. At almost every point $\mathbf{x}$ of $\partial\Omega$, we denote by $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outward of Ω. We assume that there exists $\mathbf{x}_0 \in \mathbb{R}^2$ such that function $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ satisfies (see figure A.1)

$$(\mathbf{m}\cdot\boldsymbol{\nu}) \geq 0, \text{ on } \partial\Omega_N, \qquad (\mathbf{m}\cdot\boldsymbol{\nu}) \leq 0, \text{ on } \partial\Omega_D. \tag{A.2}$$

Let λ and μ be Lamé's coefficients. If a vector function $\mathbf{v}$ is smooth enough, we define the strain tensor and the stress tensor by

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}(\partial_i \mathbf{v}_j + \partial_j \mathbf{v}_i), \qquad \sigma(\mathbf{v}) = 2\mu\epsilon(\mathbf{v}) + \lambda \mathrm{div}(\mathbf{v}) I_n. \tag{A.3}$$

We will set $\sigma(\mathbf{u}):\epsilon(\mathbf{u}) = \mathrm{tr}(\sigma(\mathbf{u})\epsilon(\mathbf{u}))$. We consider the linear isotropic elastodynamic system

$$\begin{cases} \mathbf{u}'' - \mathrm{div}(\sigma(\mathbf{u})) = 0 & \text{in } \Omega \times \mathbb{R}_+; \\ \mathbf{u} = 0 & \text{on } \partial\Omega_D \times \mathbb{R}_+; \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}' & \text{on } \partial\Omega_N \times \mathbb{R}_+; \\ \mathbf{u}(0) = \mathbf{u}_0,\ \mathbf{u}'(0) = \mathbf{u}_1 & \text{in } \Omega. \end{cases} \tag{A.4}$$

We introduce $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$, $\mathbb{H}^1(\Omega) = (H^1(\Omega))^2$ and $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega)/\mathbf{v} = 0 \text{ on } \partial\Omega_D\}$. One can easily prove the well-posedness of this problem by using semi-group method, assuming

$$(\mathbf{u}_0, \mathbf{u}_1) \in \mathbb{H}^1_D(\Omega) \times \mathbb{L}^2(\Omega). \tag{A.5}$$

Energy function is given by

$$E(\mathbf{u}, t) = \frac{1}{2}\int_\Omega (|\mathbf{u}'|^2 + \sigma(\mathbf{u}):\epsilon(\mathbf{u}))\,\mathrm{d}x. \tag{A.6}$$

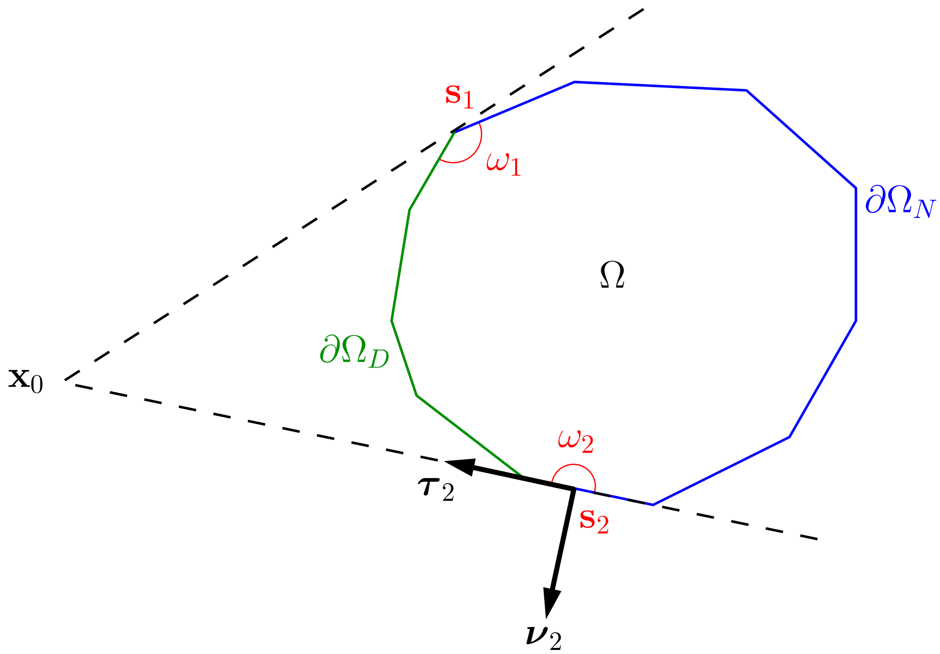

FIG. A.1 – an example of domain Ω.

A similar problem for the wave equation has been studied in [13]. As well as in this work, we prove here stabilization result for problem (A.4).

Theorem A.0.1. *Assume (A.1) and (A.2). Then there exist constants $C > 0$ and $\varpi > 0$ such that for all $(\mathbf{u}_0, \mathbf{u}_1)$ satisfying (A.5), the solution $\mathbf{u}$ of (A.4) satisfies*

$$\forall t \in \mathbb{R}_+,\ \ E(\mathbf{u}, t) \leq Ce^{-\varpi t}E(\mathbf{u}, 0).$$

The case of a disconnected boundary is studied in [22] for some elastodynamic systems with the natural feedback. This work has been extended in [23, 1, 2] for more general systems, including the Lamé system, under less restrictive geometrical assumptions, but always with a boundary such that $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} = \emptyset$.

Here we inspired from [13, 21, 4]. We have chosen the feed-back of the type $(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'$, which appears for the first time in [21]. This is the first step to prove boundary stabilization of the elastodynamic system in a n-dimensional domain, with singularities.

Version française

Soit Ω un ouvert borné polygonal plan convexe et $\partial\Omega$ sa frontière. On va supposer que $\partial\Omega$ est découpée en deux parties connexes ouvertes $\partial\Omega_N$ et $\partial\Omega_D$ telles que (A.1) est vérifié. Pour presque tout point $\mathbf{x}$ de $\partial\Omega$, on note $\boldsymbol{\nu}(\mathbf{x})$ la normale unitaire sortante en $\mathbf{x}$. On suppose qu'il existe $\mathbf{x}_0 \in \mathbb{R}^2$ tel que la fonction $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ satisfait (A.2). (voir figure A.1)

Pour $\mathbf{v}$ un champ de vecteurs suffisamment régulier, le tenseur des déformations et le tenseur
des contraintes sont définis par (A.3).

Par la méthode des semi-groupes, on prouve que le problème (A.4) est bien posé, sous l'hypothèse (A.5). L'énergie est définie par (A.6). Un problème similaire pour l'équation des ondes a été étudié dans [13]. Ici, nous prouvons un résultat analogue pour le problème (A.4).

Théorème A.0.1. *On suppose que (A.1) et (A.2) sont satisfaits. Alors il existe des constantes $C > 0$ et $\varpi > 0$ telles que pour tout $(\mathbf{u}_0, \mathbf{u}_1)$ vérifiant (A.5), la solution $\mathbf{u}$ du problème (A.4) vérifie*

$$\forall t \in \mathbb{R}_+,\ \ E(\mathbf{u}, t) \leq Ce^{-\varpi t}E(\mathbf{u}, 0).$$

Le cas d'une frontière non connexe a été étudié dans [22] pour certains système élastodynamiques, en introduisant un feedback dit naturel. Ce travail a été par la suite étendu dans [23, 1, 2] pour des sytèmes élastodynamiques plus généraux, y compris le système de Lamé, et pour des conditions géométriques moins restrictives, mais toujours pour une frontière qui vérifie $\overline{\partial\Omega_N} \cap \overline{\partial\Omega_D} = \emptyset$.

Nous nous sommes ici inspirés de [13, 21, 4]. Nous avons choisi le feedback du type $(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'$, qui apparait pour la première fois dans [21]. Ce travail constitue une première étape vers la stabilisation du système élastodynamique dans un domaine de $\mathbb{R}^n$, en présence de singularités.

La démonstration de ce théorème repose sur une égalité de type Rellich, énoncée cidessous.

Aux points $\mathbf{s}_i$, on définit ω_i l'angle d'ouverture de Ω. Si $\omega_i = \pi$, on note $\boldsymbol{\tau}(\mathbf{s}_i)$ le vecteur unitaire tangent à $\partial\Omega$ dirigé vers $\partial\Omega_D$ (voir figure A.1). On constate que la condition (A.2) implique

$$\omega_i = \pi \Rightarrow \mathbf{m}(\mathbf{s}_i)\cdot\boldsymbol{\nu}(\mathbf{s}_i) = 0. \tag{A.7}$$

On obtient alors

Théorème A.0.2. *Soit $\Omega \subset \mathbb{R}^2$ un ouvert borné polygonal plan convexe vérifiant (A.1) et soit $\mathbf{x}_0 \in \mathbb{R}^2$ tel que la fonction $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ vérifie (A.7). Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ tel que*

$$\operatorname{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega),\ \mathbf{u} \in \mathbb{H}^{3/2}(\partial\Omega_D)\ \text{ et } \sigma(\mathbf{u})\cdot\boldsymbol{\nu} \in \mathbb{H}^{1/2}(\partial\Omega_N). \tag{A.8}$$

La fonction $2(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})$ est alors intégrable sur $\partial\Omega$ et on a

$$2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} [2(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma$$
$$+8\frac{(2\mu+\lambda)(3\mu+\lambda)}{\pi\mu}\left(\pi^2 + \ln^2\left(\frac{3\mu+\lambda}{\mu+\lambda}\right)\right)\sum_{\substack{\mathbf{s}\in\{\mathbf{s}_1,\mathbf{s}_2\}\\ \omega(\mathbf{s})=\pi}} C(\mathbf{s})^2(\mathbf{m}(\mathbf{s})\cdot\boldsymbol{\tau}(\mathbf{s}))$$

où $C(\mathbf{s})$ est le coefficient de singularité de $\mathbf{u}$ au point $\mathbf{s}$.

Preuve abrégée du théorème A.0.2 : On démontre grâce à la formule de Green que, pour $\mathbf{u} \in \mathbb{H}^2(\Omega)$,

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} [2(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma. \qquad \text{(A.9)}$$

Soit $\mathbf{u} \in \mathbb{H}^1(\Omega)$ vérifiant (A.8). Pour tout ouvert $\mathcal{V}$ tel que $\overline{\mathcal{V}} \subset \Omega$, $\mathbf{u} \in \mathbb{H}^2(\mathcal{V})$.

De plus, pour tout $\mathbf{x} \in \partial\Omega\backslash\{\mathbf{s}_1, \mathbf{s}_2\}$, on peut montrer qu'il existe un voisinage $\mathcal{V}$ de $\mathbf{x}$ tel que

$\mathbf{u} \in \mathbb{H}^2(\mathcal{V} \cap \Omega)$. Pour cela, si $\mathbf{x}$ est un sommet, on utilise les résultats obtenus dans [27]. Sinon, on utilise la méthode des quotients différentiels.

Les problèmes que l'on va rencontrer seront donc en $\mathbf{s}_1$ et $\mathbf{s}_2$. Soit $\mathbf{s}$ l'un des deux $\mathbf{s}_i$. Soit γ_0 et γ_1 les deux côtés attenants à $\mathbf{s}$. Par troncature, on peut supposer que $\mathbf{u}$ est nul en dehors d'un disque $D(\mathbf{s}, \delta)$. On peut supposer que δ est suffisamment petit pour que $D(\mathbf{s}, \delta) \cap \partial\Omega \subset \gamma_0 \cup \gamma_1$. Par un procédé de relèvement et par changement de coordonnées, on se ramène au problème

$$\begin{cases} -\mathrm{div}(\sigma(\tilde{\mathbf{u}})) = \tilde{\mathbf{f}} & \text{dans } \mathcal{H} \\ \tilde{\mathbf{u}} = 0 & \text{sur } \partial\mathcal{H}_D \\ \sigma(\tilde{\mathbf{u}})\boldsymbol{\nu} = 0 & \text{sur } \partial\mathcal{H}_N \end{cases} \qquad \text{(A.10)}$$

où $\mathcal{H} = \{(r,\theta) \in \mathbb{R}^{*+}\times]0,\omega[\}$, $\partial\mathcal{H}_D = \{(r,0)\ /\ r > 0\}$ et $\partial\mathcal{H}_N = \{(r,\omega)\ /\ r > 0\}$ en coordonnées polaires, et où $\tilde{\mathbf{f}} \in \mathbb{L}^2(\mathcal{H})$.

Ainsi, on peut écrire $\mathbf{u}$ comme somme d'une fonction $\mathbf{U} \in \mathbb{H}^2(\mathcal{H})$ et de $\tilde{\mathbf{u}}$ solution de (A.10). En injectant $\mathbf{u} = \mathbf{U} + \tilde{\mathbf{u}}$ dans le terme de gauche de (A.0.2) et en développant, on constate que les termes en $(\mathbf{U}, \mathbf{U})$ et en $(\mathbf{U}, \tilde{\mathbf{u}})$ vérifient une égalité du type (A.9). Il reste alors le terme quadratique en $\tilde{\mathbf{u}}$.

On va prendre $\varepsilon > 0$, et on va poser $\mathcal{H}_\varepsilon = \{(r,\theta) \in]\varepsilon, +\infty[\times]0,\omega[\}$, $\partial\mathcal{H}_{D\varepsilon} = \{(r,0)\ /\ r > \varepsilon\}$, $\partial\mathcal{H}_{N\varepsilon} = \{(r,\omega)\ /\ r > \varepsilon\}$ et $\gamma_\varepsilon = \{(\varepsilon,\theta)\ /\ \theta \in]0,\omega[\}$.

On a donc $\tilde{\mathbf{u}} \in \mathbb{H}^2(\mathcal{H}_\varepsilon)$ et on obtient

$$2\int_{\mathcal{H}_\varepsilon} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla\tilde{\mathbf{u}})\,\mathrm{d}\mathbf{x} = \int_{\partial\mathcal{H}_\varepsilon} [2(\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\tilde{\mathbf{u}}):\epsilon(\tilde{\mathbf{u}})]\,\mathrm{d}\Gamma. \qquad \text{(A.11)}$$

On a facilement, grâce au théorème de Lebesgue

$$\int_{\mathcal{H}_\varepsilon} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}}\,\mathrm{d}\mathbf{x} \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{H}} \mathrm{div}(\sigma(\tilde{\mathbf{u}}))\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}}\,\mathrm{d}\mathbf{x}.$$

Pour la convergence des intégrales frontières de (A.11), on a besoin de la structure de la solution de (A.10), que l'on obtient grâce aux résultats obtenus dans [27]. On va maintenant séparer deux cas.

Premier cas : $\omega < \pi$.

On note ν le coefficient de Poisson du système, $\nu = \dfrac{1}{2}\dfrac{\lambda}{\lambda + \mu}$ et on considère l'équation en α

$$\sin^2(\alpha\omega) = \frac{4(1-\nu)^2 - \alpha^2\sin^2\omega}{3 - 4\nu}. \qquad \text{(A.12)}$$

Soit $(\alpha_i)_{i=1,K}$ les racines complexes de (A.12) telles que $\Re\alpha \in]0,1]$, qui sont en nombre fini. On a alors

$$\exists \mathbf{u}_R \in \mathbb{H}^2(\mathcal{H}),\ \exists (\mathbf{v}_i^1, \mathbf{v}_i^2)_{i=1,K} \in [(C^\infty([0,\omega],\mathbb{C}))^2]^K \text{ et } \exists (\chi_i)_{i=1,K} \in \mathbb{R}^K \text{ tels que}$$
$$\tilde{\mathbf{u}} = \mathbf{u}_R + \textstyle\sum_{i=1}^K \chi_i \Re[r^{\alpha_i}(\mathbf{v}_i^1(\theta) + \ln(r)\mathbf{v}_i^2(\theta))]. \tag{A.13}$$

Or, par des arguments de parité et par des calculs, on peut démontrer le lemme suivant

Lemme A.0.3.

$$\forall i \in [1,K], \Re\alpha_i > \frac{1}{2}.$$

Maintenant, on injecte la forme (A.13) dans les intégrales frontières de (A.11), on développe la double somme et on étudie la convergence terme à terme. On constate que les termes ont soit la bonne régularité, soit la bonne puissance en r pour que toutes les intégrales convergent. On obtient finalement

$$\int_{\partial\mathcal{H}_\varepsilon} [2(\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\tilde{\mathbf{u}}):\epsilon(\tilde{\mathbf{u}})]\, \mathrm{d}\Gamma$$
$$\underset{\varepsilon\to 0}{\longrightarrow} \int_{\partial\mathcal{H}} [2(\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\tilde{\mathbf{u}}):\epsilon(\tilde{\mathbf{u}})]\, \mathrm{d}\Gamma.$$

Deuxième cas : $\omega = \pi$.

On a $(\mathbf{m}\cdot\boldsymbol{\nu}) = 0$ sur $\partial\mathcal{H}$. Donc, grâce aux conditions au bord, on obtient

$$\int_{\partial\mathcal{H}} [2(\sigma(\tilde{\mathbf{u}})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\tilde{\mathbf{u}} - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\tilde{\mathbf{u}}):\epsilon(\tilde{\mathbf{u}})]\, \mathrm{d}\Gamma = 0.$$

Pour la structure de la solution, toujours grâce aux résultats obtenus dans [27], on obtient

$$\exists \mathbf{u}_R \in \mathbb{H}^2(\mathcal{H}),\ \exists C \in \mathbb{R} \text{ tels que } \tilde{\mathbf{u}} = \mathbf{u}_R + C\mathbf{u}_S, \text{ avec } \mathbf{u}_S(r,\theta) = \Re\left(r^{\frac{1}{2}+i\frac{\ln(3-4\nu)}{2\pi}}\mathbf{v}(\theta)\right), \tag{A.14}$$

où $\mathbf{v}$ est une fonction connue de $(C^\infty([0,\omega],\mathbb{C}))^2$. On remarque qu'il n'y a ici qu'une seule singularité.

On injecte la forme (A.14) dans l'intégrale frontière sur γ_ε et on développe les sommes. On constate que toutes les intégrales où $\mathbf{u}_R$ apparaît convergent vers 0. De plus, si on note $\boldsymbol{\tau}(0)$ le vecteur tangent à $\partial\mathcal{H}$ en 0 pointant vers $\partial\mathcal{H}_N$, on obtient par un calcul à ε fixé

$$\int_{\gamma_\varepsilon} [2(\sigma(\mathbf{u}_S)\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}_S - (\mathbf{m}\cdot\boldsymbol{\nu})\sigma(\mathbf{u}_S):\epsilon(\mathbf{u}_S)]\, \mathrm{d}\Gamma$$
$$= 8\frac{(2\mu+\lambda)(3\mu+\lambda)}{\pi\mu}\left(\pi^2 + \ln^2\left(\frac{3\mu+\lambda}{\mu+\lambda}\right)\right)(\mathbf{m}(0).\boldsymbol{\tau}(0)) + \mathcal{O}(\varepsilon).$$

et le théorème A.0.2 est démontré. □

Preuve abrégée du théorème A.0.1 : on va faire la démonstration pour une solution forte $\mathbf{u}$. Le résultat pour les solutions faibles s'en déduira par un argument classique de densité.

Nous allons pour montrer le théorème utiliser un lemme démontré dans [19] :

Lemme A.0.4. *Soit $E : \mathbb{R}^+ \to \mathbb{R}^+$ une fonction décroissante telle qu'il existe $C > 0$ tel que*

$$\int_t^\infty E(s)\,ds \leq CE(t), \quad \forall t \geq 0, \tag{A.15}$$

alors on a

$$E(t) \leq E(0)e^{1-\frac{t}{C}}, \quad \forall t \geq C.$$

Nous voulons donc à présent obtenir une inégalité de la forme (A.15) pour l'énergie. Pour cela, nous allons utiliser la méthode des multiplicateurs (voir par exemple [19]).

Soit $\mathbf{u}$ la solution du problème (A.4). Tout d'abord, on a grâce à la formule de Green, pour tout $t > 0$,

$$E'(\mathbf{u}, t) = -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2 \,\mathrm{d}\Gamma. \tag{A.16}$$

L'énergie est donc une fonction décroissante du temps.

Soient $T > S > 0$. On pose à présent $M\mathbf{u} = 2(\mathbf{m}\cdot\nabla)\mathbf{u} + \mathbf{u}$ et on écrit

$$\int_S^T \int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{A.17}$$

On considère le premier membre de cette égalité. On obtient

$$\int_S^T \int_\Omega \mathbf{u}''\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T + \int_S^T \int_\Omega |\mathbf{u}'|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\Gamma\,\mathrm{d}t. \tag{A.18}$$

Pour le second membre de l'égalité (A.17), l'application du théorème A.0.2 nous donne

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,\mathrm{d}\mathbf{x} \leq -\int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} + \sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma.$$

On a donc

$$\begin{aligned}\int_S^T \int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad &\leq \quad -\int_S^T \int_\Omega \sigma(\mathbf{u}):\epsilon(\mathbf{u})\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\Gamma\,\mathrm{d}t \\ &\quad - \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} + \sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma\,\mathrm{d}t\end{aligned} \tag{A.19}$$

En regroupant (A.18) et (A.19), on obtient

$$\begin{aligned}\int_S^T E(t)\,\mathrm{d}t \quad &\leq \quad -\left[\int_\Omega \mathbf{u}'\cdot M\mathbf{u}\,\mathrm{d}\mathbf{x}\right]_S^T + \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})|\mathbf{u}'|^2\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}'\cdot\mathbf{u}\,\mathrm{d}\Gamma\,\mathrm{d}t \\ &\quad - \int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} + \sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma\,\mathrm{d}t.\end{aligned} \tag{A.20}$$

On montre par un calcul assez compliqué, inspiré de [3], que pour tout $\theta > 0$, il existe $C_1 > 0$ tel que

$$\int_S^T \int_{\partial\Omega_N} (\mathbf{m}\cdot\boldsymbol{\nu})[2\mathbf{u}'\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} + \sigma(\mathbf{u}):\epsilon(\mathbf{u})]\,\mathrm{d}\Gamma\,\mathrm{d}t \leq \theta \int_S^T E(t)\,\mathrm{d}t + C_1 E(\mathbf{u}, S). \quad \text{(A.21)}$$

On montre alors que pour tout $\theta > 0$, il existe $C_2 > 0$ tel que

$$(1-\theta)\int_S^T E(t)\,\mathrm{d}t \leq C_2 E(\mathbf{u}, S). \quad \text{(A.22)}$$

Comme C_2 ne dépend pas de S et de T, il suffit de prendre θ suffisamment petit, puis de faire tendre T vers l'infini pour obtenir le résultat. Ceci conclut la démonstration du théorème. □

Remarque 1 : Le feedback que nous avons choisi ici permet d'obtenir la stabilisation du système élastodynamique dans le cas d'une frontière non-connexe pour un domaine quelconque de $\mathbb{R}^n$.

Remarque 2 : La stabilisation du système élastodynamique en présence de singularités paraît peu accessible par la méthode des multiplicateurs utilisée avec le feedback naturel introduit dans [22].

Remarque 3 : Le cas d'un ouvert à frontière régulière connexe est encore à l'étude et fera l'objet d'une publication ultérieure.

Annexe B

Rellich relations for mixed boundary elliptic problems

For elliptic partial differential equations, mixed boundary conditions generate singularities in the solution, mainly when the boundary of the domain is connected. Following previous works concerning the Laplace equation, we here give Rellich relations involving singularities for the Lamé system.

These relations are useful in the problem of boundary stabilization of the waves equation and the elastodynamic system, respectively, when using the multiplier method.

Introduction

Let Ω be a regular bounded open set of $\mathbb{R}^n$ and consider the following wave problem,

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0, +\infty)\,, \\ u = 0 & \text{on } \partial\Omega_D \times (0, +\infty)\,, \\ \partial_\nu u = F(u') & \text{on } \partial\Omega_N \times (0, +\infty)\,, \\ u(0) = u^0 & \text{in } \Omega\,, \\ u'(0) = u^1 & \text{in } \Omega\,. \end{cases}$$

Here the problem of boundary stabilization is to build some partition $(\partial\Omega_D, \partial\Omega_N)$ of the boundary $\partial\Omega$ and some feedback function F such that the energy of the solution u is (exponentially) decreasing with respect to time.

Many authors have studied this problem by using the multiplier method (see [19] and the references therein). This leads to the choice

$$\begin{aligned} &\partial\Omega_N = \{\mathbf{x} \in \partial\Omega \,/\, \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) > 0\}\,, \\ &\partial\Omega_D = \partial\Omega \setminus \partial\Omega_N = \{\mathbf{x} \in \partial\Omega \,/\, \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) \le 0\}\,, \\ &F(u') = -(\mathbf{m}\cdot\boldsymbol{\nu})u'\,, \end{aligned}$$

where $\boldsymbol{\nu}(\mathbf{x})$ is the normal unit vector pointing outwards of Ω at some point $\mathbf{x} \in \partial\Omega$ and $\mathbf{m}$ is a function depending on a fixed point $\mathbf{x}_0 \in \mathbb{R}^n$: $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$.

The main step of this method is to prove some Gronwall-type inequality concerning the energy of strong solutions (see Theorem 8.1 in [19]). This leads to define $\mathrm{H}^1_D(\Omega) = \{v \in \mathrm{H}^1(\Omega) \,/\, v = 0\,, \text{ on } \partial\Omega_D\}$ and to consider the operator $\mathcal{A}_w$

$$\begin{aligned} &\mathcal{D}(\mathcal{A}_w) = \{(u, \hat{u}) \in \mathrm{H}^1_D(\Omega) \times \mathrm{H}^1_D(\Omega) \,/\, \Delta u \in \mathrm{L}^2(\Omega)\,;\, \partial_\nu u = -(\mathbf{m}\cdot\boldsymbol{\nu})\hat{u} \text{ on } \partial\Omega_N\}\,, \\ &\mathcal{A}_w(u, \hat{u}) = (-\hat{u}, -\Delta u)\,, \quad \forall (u, \hat{u}) \in \mathcal{D}(\mathcal{A}_w)\,. \end{aligned}$$

The crucial point in the proof of above Gronwall-type inequality is to verify that if $(u, \hat{u})$ belongs to $\mathcal{D}(\mathcal{A}_w)$, then u satisfies a Rellich relation [34] in the following form

$$2\int_\Omega \Delta u \; \mathbf{m}\cdot\nabla u \, d\mathbf{x} = (n-2)\int_\Omega |\nabla u|^2 \, d\mathbf{x} + \int_{\partial\Omega} \left(2\,\partial_\nu u \; \mathbf{m}\cdot\nabla u - \mathbf{m}\cdot\boldsymbol{\nu}\; |\nabla u|^2\right) d\Gamma\,. \quad \text{(B.1)}$$

One can easily observe that this relation is satisfied if u is regular enough. A sufficient condition is that u is locally H^2 in $\overline{\Omega}$. For the above problem of boundary stabilization, this holds in the particular case when the interface $\Gamma = \overline{\partial\Omega_N} \cap \overline{\partial\Omega_D}$ is empty (this can be proved by using the method of difference quotients).

On the other hand, when the interface is not empty, some singular part can appear in u and the above "hidden regularity result" is generally false.

Anyway, in all cases, using a classical trace result, we can prove that if $(u, \hat{u})$ belongs to $\mathcal{D}(\mathcal{A}_w)$, then there exists $u_R \in \mathrm{H}^2(\Omega)$ such that $U = u - u_R$ satisfies the following mixed boundary problem for the Laplace equation.

$$\begin{cases} -\Delta U = F & \text{in } \Omega\,, \\ U = 0 & \text{on } \partial\Omega_D\,, \\ \partial_\nu U = 0 & \text{on } \partial\Omega_N\,, \end{cases} \tag{B.2}$$

where the right-hand side F belongs to $\mathrm{L}^2(\Omega)$.

Under reasonable geometrical assumptions about Ω, when $\Gamma = \emptyset$, U is locally H^2 in some neighborhood of any point of $\overline{\Omega}$ and Rellich relation $(B.1)$ is true.

When $\Gamma \neq \emptyset$, U can be singular even if F is very regular. In this case, formula $(B.1)$ must be modified. A further term, which takes into account singularities, appears. This will be presented in the first part of this paper.

In the second part, we consider the case of the Lamé system which is related to the problem of the boundary stabilization of the elastodynamic system.

In the third part, we give a sketch of the proof of the Rellich relation for the Lamé system (detailed proofs can be found in [6, 7]).

B.1 Rellich relation for the Laplace equation

We first introduce the main geometrical assumptions.

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ such that its boundary $\partial\Omega$ satisfies, in the sense of Nečas [31],

$$\partial\Omega \text{ is of class } \mathcal{C}^2\,. \tag{B.3}$$

Given $\mathbf{x}$ a point of $\partial\Omega$, we denote by $\boldsymbol{\nu}(\mathbf{x})$ the normal unit vector pointing outwards of Ω.

We assume that there exists a partition $(\partial\Omega_N, \partial\Omega_D)$ of $\partial\Omega$ such that

$$\begin{gathered} \operatorname{meas}(\partial\Omega_D) \neq 0\,,\ \operatorname{meas}(\partial\Omega_N) \neq 0 \quad \Gamma = \overline{\partial\Omega_D} \cap \overline{\partial\Omega_N} \text{ is a non-empty} \\ \mathcal{C}^3\text{-manifold of dimension } n-2\,, \quad \text{there exists a neighborhood } \omega \text{ of } \Gamma \\ \text{such that } \partial\Omega \cap \omega \text{ is a } \mathcal{C}^3\text{-manifold of dimension } n-1\,. \end{gathered} \tag{B.4}$$

Furthermore, we suppose that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that, setting $\mathbf{m}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$, Γ satisfies

$$\mathbf{m}\cdot\boldsymbol{\nu} = 0 \quad \text{on } \Gamma\,. \tag{B.5}$$

We can consider $\partial\Omega_N$ as a submanifold of $\partial\Omega$, so that at each point $\mathbf{x}$ of its boundary Γ, we can define a normal unit vector $\boldsymbol{\tau}(\mathbf{x})$ pointing outwards of $\partial\Omega_N$. Observe that this vector is tangential with respect to $\partial\Omega$ (see Figure B.1).

Let us now give the extension of Rellich identity.

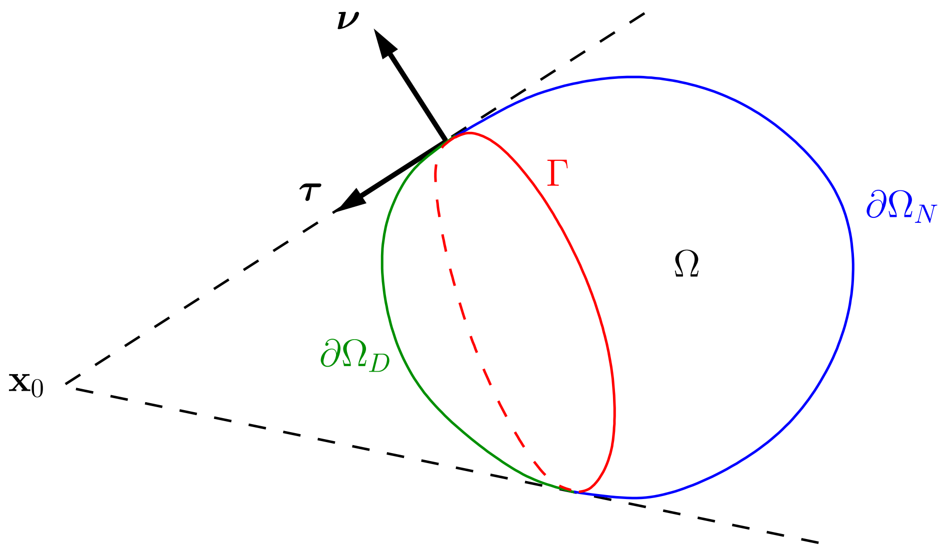

FIG. B.1 – An example of domain Ω with a non-empty interface Γ.

Theorem 1. — *Under assumptions* $(B.3)$-$(B.5)$, *let* $u \in \mathrm{H}^1(\Omega)$ *be such that*

$$\Delta u \in \mathrm{L}^2(\Omega)\,, \qquad u_{/\partial\Omega_D} \in \mathrm{H}^{3/2}(\partial\Omega_D)\,, \qquad \partial_\nu u_{/\partial\Omega_N} \in \mathrm{H}^{1/2}(\partial\Omega_N)\,.$$

Then, $2\,\partial_\nu u\ \mathbf{m}\cdot\nabla u - \mathbf{m}\cdot\boldsymbol{\nu}\ |\nabla u|^2$ *belongs to* $\mathrm{L}^1(\partial\Omega)$ *and there exists* $\zeta \in \mathrm{H}^{1/2}(\Gamma)$ *such that*

$$\begin{aligned} 2\int_\Omega \Delta u\ \mathbf{m}\cdot\nabla u\, d\mathbf{x} &= (n-2)\int_\Omega |\nabla u|^2\, d\mathbf{x} + \int_{\partial\Omega} \left(2\,\partial_\nu u\ \mathbf{m}\cdot\nabla u - \mathbf{m}\cdot\boldsymbol{\nu}\ |\nabla u|^2\right)\, d\Gamma \\ &\quad + \int_\Gamma |\zeta|^2\ \mathbf{m}\cdot\boldsymbol{\tau}\, d\mathbf{s}\,. \end{aligned}$$

The detailed proof of this result can be found in [4].

The first extension has been proved by P. Grisvard [13, 11] who has taken in account singularities generated by vortices of a polygonal domain. Observe in this case, that, if the polygonal domain is convex (with angles lower than π), formula $(B.1)$ holds without any further term.

A further term appears when at some point of the interface, the angle is π. Indeed, this geometrical configuration generates a singularity which behaves locally like the Shamir function [36] given in polar coordinates by

$$U_S(r,\theta) = \varrho(r)\,\sqrt{r}\,\sin\frac{\theta}{2}\,,$$

where ϱ is some cut-off function.

This function satisfies a problem in the form $(B.2)$ and is not locally H^2 in any neighborhood of the origin (see Figure B.2) : observe only that near the origin, the normal derivative satisfies $\partial_r U_S(r,\pi) = O(r^{-1/2})$ (with Landau notations) and is not locally L^2 along the boundary.

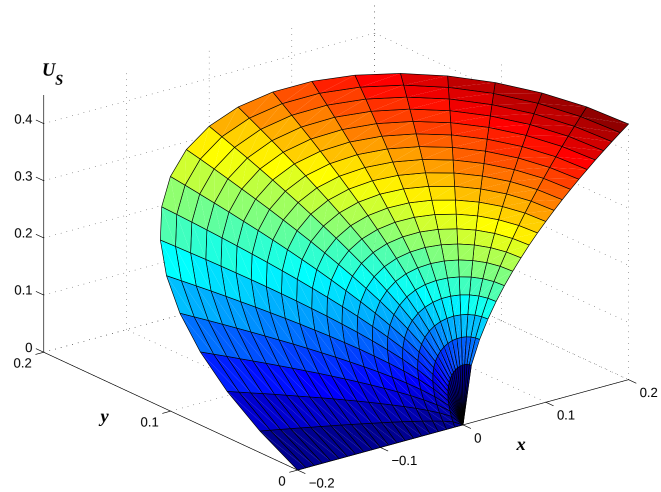

Fig. B.2 – Local behavior of the Shamir function.

The second extension of Rellich formula has been proved by M. Moussaoui [28] when Ω is an infinite semi-cylinder (see Figure B.5 at the end of this paper).

This has been extended for general n-dimensional smooth domains in [4] by using local coordinates at each point of the interface Γ.

B.2 Rellich relation for the Lamé system

We first introduce notations and motivate our work by the study of the boundary stabilization of the elastodynamic system. We end this section by giving main results.

B.2.1 Notations

We will use the following Lamé notations. Assume that $\mathbf{v} = (v_1, v_2, v_3)$ is a regular vector field, we define the strain tensor

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j\right), \quad (i, j) \in \{1, 2, 3\}^2,$$

and the stress tensor

$$\sigma(\mathbf{v}) = 2\mu\,\varepsilon(\mathbf{v}) + \lambda\,\mathrm{div}(\mathbf{v}) I_3\,,$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients and I_3 is the identity matrix of $\mathbb{R}^3$.

We denote the classical inner product by :

$$\sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) = \mathrm{tr}(\sigma(\mathbf{u})\varepsilon(\mathbf{v})) = \sum_i \sum_j \sigma_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v}).$$

We write that $\mathbf{v}$ belongs to $\mathbb{L}^2(\Omega)$ (resp. $\mathbb{H}^s(\Omega)$), if every component of $\mathbf{v}$ belongs to $\mathrm{L}^2(\Omega)$ (resp. $\mathrm{H}^s(\Omega)$).

We also need to define : $\mathbb{H}^1_D(\Omega) = \{\mathbf{v} \in \mathbb{H}^1(\Omega)\,/\,\mathbf{v} = 0\,,\text{ on } \partial\Omega_D\,\}$.

B.2.2 Boundary stabilization of the elastodynamic system

In [7, 8], we have considered the problem of the boundary stabilization of the elastodynamic system

$$\begin{cases} \mathbf{u}'' - \operatorname{div}(\sigma(\mathbf{u})) = 0 & \text{in } \Omega \times (0,+\infty)\,, \\ \mathbf{u} = 0 & \text{on } \partial\Omega_D \times (0,+\infty)\,, \\ \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\mathbf{u}' & \text{on } \partial\Omega_N \times (0,+\infty)\,, \\ \mathbf{u}(0) = \mathbf{u}^0 & \text{in } \Omega\,, \\ \mathbf{u}'(0) = \mathbf{u}^1 & \text{in } \Omega\,. \end{cases}$$

As well as for the wave equation, we can obtain a stabilization result by using multiplier method, provided that some Rellich relation is satisfied.

We follow a similar approach. Especially we have to consider the following operator

$$\mathcal{D}(\mathcal{A}_e) = \{(\mathbf{u},\hat{\mathbf{u}}) \in \mathbb{H}^1_D(\Omega) \times \mathbb{H}^1_D(\Omega) \,/\, \operatorname{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega)\,;\, \sigma(\mathbf{u})\boldsymbol{\nu} = -(\mathbf{m}\cdot\boldsymbol{\nu})\hat{\mathbf{u}}\,,\text{ on } \partial\Omega_N\}\,,$$
$$\mathcal{A}_e(\mathbf{u},\hat{\mathbf{u}}) = (-\hat{\mathbf{u}}, -\operatorname{div}(\sigma(\mathbf{u})))\,, \quad \forall (\mathbf{u},\hat{\mathbf{u}}) \in \mathcal{D}(\mathcal{A}_e)\,.$$

Again, under reasonable geometrical assumptions, we can use a trace result : if $(\mathbf{u},\hat{\mathbf{u}})$ belongs to $\mathcal{D}(\mathcal{A}_e)$, one can build $\mathbf{u}_R \in \mathbb{H}^2(\Omega)$ and $\mathbf{F} \in \mathbb{L}^2(\Omega)$ such that $\mathbf{U} = \mathbf{u} - \mathbf{u}_R$ satisfies the following elasticity system

$$\begin{cases} -\operatorname{div}(\sigma(\mathbf{U})) = \mathbf{F} & \text{in } \Omega\,, \\ \mathbf{U} = 0 & \text{on } \partial\Omega_D\,, \\ \sigma(\mathbf{U})\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_N\,. \end{cases} \tag{B.6}$$

B.2.3 Main results

The regular case

We first give a result which is similar to $(B.1)$ when $\mathbf{u}$ is regular enough.

Proposition 2. — *Assume that the open bounded set* Ω *satisfies* $(B.3)$*. If* $\mathbf{u}$ *belongs to* $\mathbb{H}^2(\Omega)$*, then* $2\,(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} - \mathbf{m}\cdot\boldsymbol{\nu}\;\sigma(\mathbf{u}):\varepsilon(\mathbf{u})$ *belongs to* $\mathrm{L}^1(\partial\Omega)$ *and*

$$\begin{aligned} 2\int_\Omega \operatorname{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,d\mathbf{x} \;&=\; (n-2)\int_\Omega \sigma(\mathbf{u}):\varepsilon(\mathbf{u})\,d\mathbf{x} \\ &\quad + \int_{\partial\Omega}\big(2\,(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{u} - \mathbf{m}\cdot\boldsymbol{\nu}\;\sigma(\mathbf{u}):\varepsilon(\mathbf{u})\big)\,d\Gamma\,. \end{aligned}$$

One can easily prove this result by applying two Green formulæ.

Observe that the above relation holds when $\mathbf{u}$ is the solution of problem $(B.6)$ if $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ is empty.

Let us introduce the useful following notation :

$$\Theta(\mathbf{u},\mathbf{v}) = 2\,(\sigma(\mathbf{u})\boldsymbol{\nu})\cdot(\mathbf{m}\cdot\nabla)\mathbf{v} - \mathbf{m}\cdot\boldsymbol{\nu}\;\sigma(\mathbf{u}):\varepsilon(\mathbf{v}).$$

In order to extend this result, we proceed as well as in the case of Laplace equation.

The case of a plane polygonal domain

Following works of P. Grisvard, we first consider the case of a plane polygonal domain.

We here suppose that Ω is a bounded convex polygonal open subset of $\mathbb{R}^2$ and its boundary is made of two broken lines $\partial\Omega_N$ and $\partial\Omega_D$ defined thanks to some point $\mathbf{x}_0$ belonging to $\mathbb{R}^2 \setminus \Omega$:

$$\partial\Omega_D = cl\big(\{\mathbf{x} \in \partial\Omega \,/\, \mathbf{m}(\mathbf{x})\cdot\boldsymbol{\nu}(\mathbf{x}) \leq 0\}\big)\,, \quad \partial\Omega_N = \partial\Omega \setminus \partial\Omega_D\,, \tag{B.7}$$

so that $\Gamma = \overline{\partial\Omega}_N \cap \overline{\partial\Omega}_D = \{\mathbf{s}_1, \mathbf{s}_2\}$ (see Figure B.3). At $\mathbf{s}_1$ (resp. $\mathbf{s}_2$), let us define angle $\varpi_1 \in (0,\pi]$ (resp. $\varpi_2 \in (0,\pi]$) between $\partial\Omega_N$ and $\partial\Omega_D$. Let us define

$$J(\Omega) = \{\jmath \,/\, \varpi_\jmath = \pi\}\,.$$

For every $\jmath \in J(\Omega)$, we can define as well as above unit vectors $\boldsymbol{\nu}(\mathbf{s}_\jmath)$ and $\boldsymbol{\tau}(\mathbf{s}_\jmath)$ (see an example in Figure B.3).

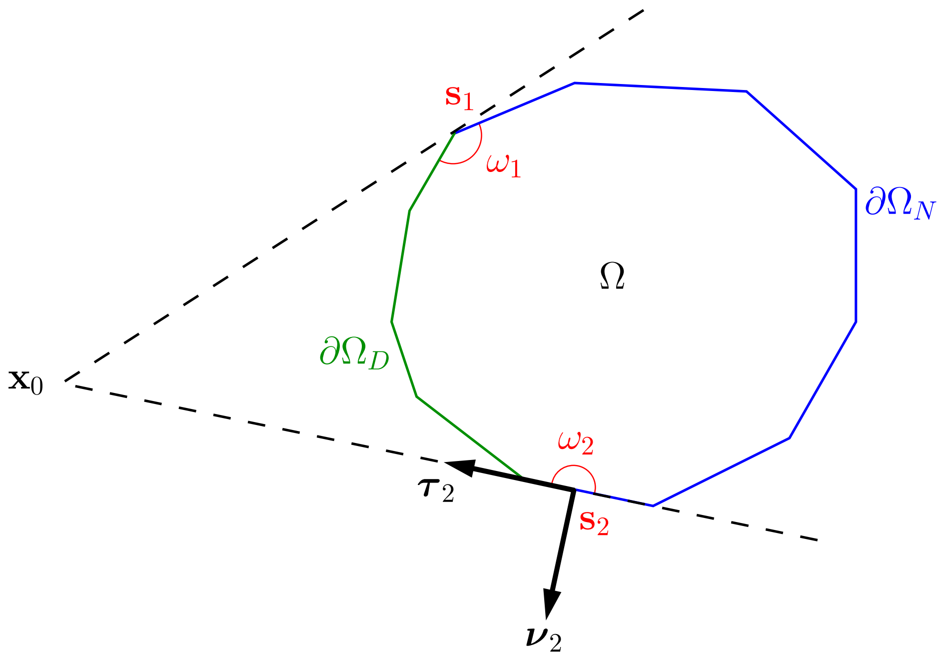

FIG. B.3 – $\Omega \subset \mathbb{R}^2$ is open, bounded, polygonal and convex, $\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}$ and $J(\Omega) = \{2\}$.

Theorem 3. — *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal open set such that its boundary $\partial\Omega$ satisfies (B.7). If $\mathbf{u} \in \mathbb{H}^1(\Omega)$ is such that*

$$\mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega) \quad \mathbf{u}_{/\partial\Omega_D} \in \mathbb{H}^{3/2}(\partial\Omega_D) \quad \sigma(\mathbf{u})\boldsymbol{\nu} \in \mathbb{H}^{1/2}(\partial\Omega_N)\,.$$

then $\Theta(\mathbf{u},\mathbf{u})$ belongs to $\mathrm{L}^1(\partial\Omega)$ and there exist at most two real coefficients $\Upsilon_\jmath$ such that

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,d\mathbf{x} = \int_{\partial\Omega} \Theta(\mathbf{u},\mathbf{u})\,d\Gamma + \sum_{\jmath\in J(\Omega)} \Upsilon_\jmath^2\; \mathbf{m}(\mathbf{s}_\jmath)\cdot\boldsymbol{\tau}(\mathbf{s}_\jmath)\,.$$

This result has been announced in [6]. The reader will find a detailed proof in [7] and we only give main ideas of it.

We first observe that under above assumptions, $\mathbf{u}$ is locally $\mathbb{H}^2$ at every point of $\overline{\Omega}$ which is not a vertex. Hence we can apply Proposition 2 in a subdomain of Ω which does not contain small disks centered at vertices.

The main idea is to compute the limit of each integral term when the rays of these disks tend to 0. To this end, using some localization process, we can write $\mathbf{u}$ as the sum of a $\mathbb{H}^2$-part and a singular part which is given in [27].

At each vertex which does not belong to Γ, the degree of the singularity is convenient and we get the limit without any further term.

A similar process holds at $\mathbf{s}_\imath \in \Gamma$ if $\varpi_\imath < \pi$.

If $\jmath \in J(\Omega)$, then in some neighborhood of $\mathbf{s}_\jmath$, we can write $\mathbf{u}$ as the sum of a $\mathbb{H}^2$-part $\mathbf{u}_R$ and a singular part $\mathbf{u}_S$ which locally behaves like the following function $\mathbf{U}_S$ given in [27] :

$$\mathbf{U}_S(r,\theta) = \Re\left(r^\alpha \mathbf{w}(\theta)\right) ,$$

where $\alpha \in \mathbb{C}$, $\Re\alpha = 1/2$, $\mathbf{w}$ is a complex-valued $\mathcal{C}^\infty$-function (see a detailed formula in [7]). Components of $\mathbf{U}_S$ are represented in Figure B.4 for a particular choice of Lamé coefficients.

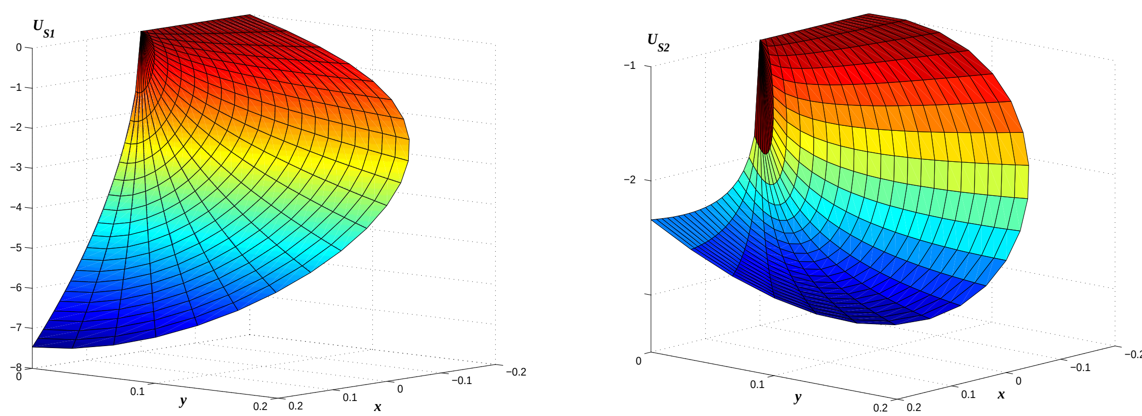

FIG. B.4 – Local behavior of components of $\mathbf{U}_S$ (Lamé coefficients : $\lambda = \mu = 1$).

Hence, we write $\Theta(\mathbf{u},\mathbf{u}) = \Theta(\mathbf{u}_R,\mathbf{u}_R) + \Theta(\mathbf{u}_R,\mathbf{u}_S) + \Theta(\mathbf{u}_S,\mathbf{u}_R) + \Theta(\mathbf{u}_S,\mathbf{u}_S)$ and we carefully compute the limits of corresponding integral terms. The fourth one gives $\Upsilon_\jmath^2\, \mathbf{m}(\mathbf{s}_\jmath)\cdot\boldsymbol{\tau}(\mathbf{s}_\jmath)$ ($\Upsilon_\jmath^2$ depends on the singularity coefficient of $\mathbf{u}$ at $\mathbf{s}_\jmath$), other ones give no further term.

Remark — Assumptions of Theorem 3 can be easily weakened. For a polygonal bounded domain, sufficient conditions are : $\varpi_\imath \in (0,\pi]$ for every $\imath$ and, if $\jmath \in J(\Omega)$, $\mathbf{m}(\mathbf{s}_\jmath)\boldsymbol{\nu}(\mathbf{s}_\jmath) = 0$.

The general case

We here give an extension of Proposition 2 for the geometrical case described in Section 1.

Theorem 4. — *Under assumptions* $(B.3)$-$(B.5)$*, let* $\mathbf{u} \in \mathbb{H}^1(\Omega)$ *be such that*

$$\mathrm{div}(\sigma(\mathbf{u})) \in \mathbb{L}^2(\Omega)\,, \quad \mathbf{u}_{/\partial\Omega_D} \in \mathbb{H}^{3/2}(\partial\Omega_D)\,, \quad \sigma(\mathbf{u})\boldsymbol{\nu} \in \mathbb{H}^{1/2}(\partial\Omega_N)\,.$$

Then $\Theta(\mathbf{u},\mathbf{u})$ *belongs to* $\mathrm{L}^1(\partial\Omega)$ *and there exists* $\Upsilon \in \mathbb{L}^2(\Gamma)$ *such that*

$$2\int_\Omega \mathrm{div}(\sigma(\mathbf{u}))\cdot(\mathbf{m}\cdot\nabla)\mathbf{u}\,d\mathbf{x} = (n-2)\int_\Omega \sigma(\mathbf{u}):\varepsilon(\mathbf{u})\,d\mathbf{x} + \int_{\partial\Omega}\Theta(\mathbf{u},\mathbf{u})\,d\Gamma + \int_\Gamma |\Upsilon|^2\,\mathbf{m}\cdot\boldsymbol{\tau}\,ds\,.$$

The following Section is devoted to a sketch of the proof of this result.

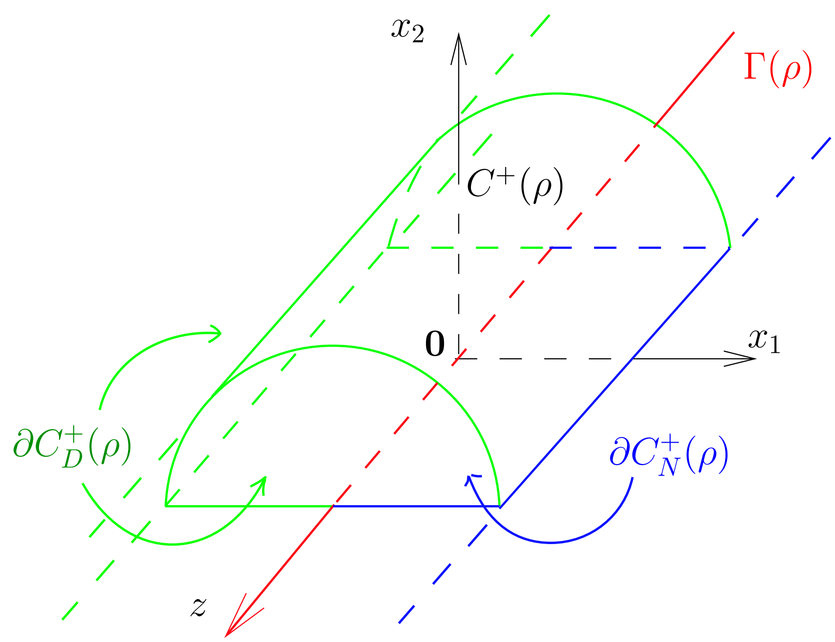

Fig. B.5 – Case of a semi-cylinder.

B.3 Sketch of the proof of Theorem 4

One can find a detailed proof in [7]. This proof is made of three main steps.

First step. We extend theorem 3 for a 2-dimensional open set Ω which satisfies $(B.3)$-$(B.5)$ such that, with notations of Subsection 2.3.2, $\Gamma = \{\mathbf{s}_1, \mathbf{s}_2\}$ and $J(\Omega) = \{1, 2\}$.

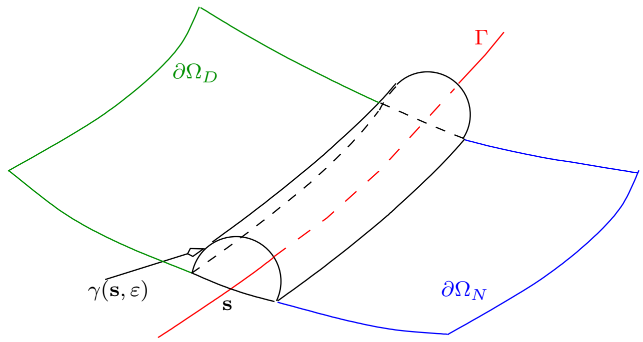

FIG. B.6 – The general case.

At each point $\mathbf{s}_\imath$, we introduce local coordinates and we get a local mixed boundary problem which involves an elasticity operator with non-constant coefficients. We prove that this operator is a small perturbation of Lamé operator and this leads to a similar structure of its solution.

Second step. We study the case considered in [28] : Ω is a semi-cylinder and Γ is its axis. (see figure B.5)

We prove that involved singularities along Γ can be written $\begin{pmatrix} \mathbf{u}_S \\ U_S \end{pmatrix}$ where $\mathbf{u}_S$ is the singular function considered in Subsection 2.3.2 and U_S is the Shamir function (see Section 1).

We then proceed as well as for Theorem 3 : we apply Proposition 2 in a convenient subdomain and we use this particular structure of singularities to get the result.

Third step. We consider a general 3-dimensional open set Ω. As well as in the first step, we use a localization process in a neighborhood of each point of Γ. Similarly, in local coordinates (see Figure B.6), we get a mixed boundary problem which is a small perturbation of previous one.

Thus we can use the above structure of singularities.

We finally build Υ by using a compactness argument.

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