Analyse de régularité locale, quelques applications à l'analyse multifractale

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THÈSE

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Introduction, notions générales

1.1 Régularité locale et analyse multifractale

L'analyse de la régularité locale d'objets tels que des fonctions, des processus stochastiques, des signaux ou des images, constitue une branche jeune des mathématiques. Elle est née avec le perfectionnement des techniques d'analyse numérique, d'expérimentation et de simulation en physique. En particulier, depuis le début des années 1970, la présence de rapides oscillations locales a été mise en évidence dans plusieurs contextes, notamment en mécanique des fluides. Dans ce cadre, les irrégularités mesurées dans la vitesse d'un fluide turbulent reflètent le fait que l'énergie s'y dissipe de façon hétérogène. Ce phénomène de grande variabilité, aujourd'hui dit de "multifractalité", a conduit B. Mandelbrot dans [66] et [69] à introduire ses cascades multiplicatives aléatoires. Ces cascades constituent le premier modèle multifractal, utilisé plus tard par U. Frisch et G. Parisi dans [30] dans le cadre de la turbulence pleinement développée pour illustrer leur conjecture.

Des propriétés d'auto-similarité (c'est-à-dire d'invariance déterministe ou stochastique par certaines similitudes), propriétés qui joueront un rôle important dans ce travail, ont également été mises en évidence dans d'autres phénomènes physiques. Mandelbrot, dans [68] par exemple, a prouvé qu'un grand nombre de phénomènes naturels présentaient des structures auto-similaires fortes. Plus récemment, il a été prouvé que le trafic de données ([61], [60]) peut avoir des propriétés d'auto-similarité (dans le cas d'un réseau local) ou de multifractalité (pour un réseau global du type Internet).

Nous étudions principalement dans cette thèse la caractérisation du comportement local d'un objet autour d'un point donné x_0. Il est fondamental dans les applications citées précédemment mais également d'un point de vue plus théorique, de pouvoir détecter puis classifier les singularités rencontrées. L'analyse de régularité locale se donne pour but de caractériser entièrement, si cela est possible, le comportement local de l'objet (c'est-à-dire autour de chaque point), et donc de compléter de façon pertinente les informations fournies par la simple régularité globale (même si cette dernière présente évidemment un intérêt).

Régularité ponctuelle et ondelettes

La régularité ponctuelle autour d'un point x_0 d'une fonction f est le plus souvent mesurée par l'*exposant ponctuel de Hölder* défini de la manière suivante :

Définition 1.1 *Soit s un réel positif. Une fonction $f : \mathbb{R}^d \to \mathbb{R}$ appartient à $C^s_{x_0}$ si et seulement si on peut trouver un réel $\eta > 0$, un polynôme P, et une constante C tels que*

$$\forall x \in B(x_0, \eta),\ |f(x) - P(x - x_0)| \leq C|x - x_0|^s. \tag{1.1}$$

L'exposant ponctuel de Hölder est défini par $h_f(x_0) = \sup\{s : f \in C^s_{x_0}\}$.

$B(x, \eta)$ désigne la boule ouverte de $\mathbb{R}^d$ de centre x, de rayon η. Dans le cas d'une mesure borélienne positive définie sur $\mathbb{R}$, la régularité locale autour d'un point x_0 est principalement mesurée par un autre exposant de Hölder :

Définition 1.2 *Soit μ une mesure borélienne positive définie sur $\mathbb{R}$. On définit l'exposant de Hölder inférieur de μ au point x_0 par*

$$\underline{\alpha}_\mu(x_0) = \liminf_{r \to 0} \frac{\log \mu(B(x_0, r))}{\log |B(x_0, r)|}. \tag{1.2}$$

Cet exposant est aussi souvent appelé *dimension locale inférieure de μ en x_0*. L'exposant ponctuel de Hölder est une mesure de la régularité locale d'une fonction f. Cet exposant présente pourtant quelques inconvénients, illustrés par l'étude des deux exemples simples suivants.

- la fonction "cusp", $x \to |x|^\gamma$ où γ est un réel positif ($\gamma \notin 2\mathbb{N}$), a un exposant ponctuel de Hölder en 0 égal à γ, mais ne présente pas de phénomènes d'oscillations en ce même point 0. Dans ce cas, l'exposant ponctuel en 0 de toute primitive de cette fonction cusp est égal à $\gamma + 1$.
- la fonction "chirp" $x \to |x|^\gamma \sin(\frac{1}{|x|^\beta})$ où γ et β sont deux réels positifs, a également un exposant ponctuel en 0 égal à γ. En revanche, il est facile de montrer que l'exposant ponctuel en 0 de toute primitive d'un tel chirp est égal à $\gamma + (1 + \beta)$. On obtient donc une régularité supérieure à celle espérée : ceci est dû à la présence de rapides oscillations autour de 0, qui après intégration augmentent l'exposant ponctuel de Hölder.

Le comportement de h_f est donc irrégulier sous l'action des opérateurs de dérivation et d'intégration, et plus généralement sous l'action d'opérateurs pseudo-différentiels (voir [71]). De plus, il ne permet pas de détecter de manière simple les oscillations locales. Dans cette thèse, nous définirons et étudierons d'autres exposants de régularité, qui permettront de compléter la description procurée par h_f.

Tout au long de ce travail nous utiliserons des bases d'ondelettes et la décomposition des fonctions sur de telles bases. Ces bases, qui trouvent également leur motivation pour une grande partie dans le traitement du signal, sont construites par exemple de la manière suivante : partant d'une "ondelette-mère" ψ appartenant à la classe de Schwartz $\mathcal{S}(\mathbb{R}^d)$, on définit, pour $j \in \mathbb{Z}$ et $k \in \mathbb{Z}^d$, les fonctions $\psi_{j,k}(x) = \psi(2^j x - k)$, de telle sorte que sous certaines hypothèses développées dans [71], l'ensemble

$$\{2^{j/2}\psi_{j,k}\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}$$

forme une base orthonormale de $L^2(\mathbb{R}^d)$.

Les coefficients d'ondelettes d'une fonction $f \in L^2(\mathbb{R}^d)$ sont définis par

$$d_{j,k} = 2^j \int_{\mathbb{R}^d} f(t)\psi_{j,k}(t)dt.$$

Remarquons que l'on utilise ici une normalisation L^∞ (et non L^2) pour les coefficients d'ondelettes, et ceci par simple commodité : en effet, prendre comme définition pour les coefficients d'ondelettes $d_{j,k} = 2^{j/2} \int f(t)\psi_{j,k}(t)dt$ nous imposerait un inutile facteur 1/2 dans tous nos résultats.

L'importance et l'utilité des ondelettes dans l'analyse de régularité locale proviennent principalement de la caractérisation suivante de l'exposant ponctuel de Hölder de f en x_0 par coefficients d'ondelettes, trouvée par S. Jaffard dans [41] : si $f \in C^\varepsilon(\mathbb{R}^d)$ pour un certain $\varepsilon > 0$, et si l'ondelette ψ a suffisamment de moments nuls, on a

$$h_f(x_0) = \liminf_{|k2^{-j}-x_0|\to 0} \frac{\log |d_{j,k}|}{\log(2^{-j} + |x_0 - k2^{-j}|)}.$$

Les ondelettes constituent donc un outil naturel pour étudier la régularité locale des objets. De plus, la structure intrinsèquement auto-similaire de la base d'ondelettes est a priori favorable à l'étude des fonctions et processus auto-similaires et multifractals.

Comme mentionné précédemment, plusieurs exposants de régularité ont déjà été introduits afin de détecter et de caractériser la présence et l'intensité des oscillations locales autour d'un point donné. Nous nous sommes particulièrement intéressés à l'un d'entre eux, l'*exposant local de Hölder* noté $\alpha_l(f)(x_0)$. Nous étudierons en détail dans le chapitre 2 les contraintes imposées à l'exposant local de Hölder ainsi que ses relations avec l'exposant ponctuel de Hölder $h_f(x_0)$. En particulier, nous montrerons le théorème suivant

Théorème 1.1 *(Seuret, Lévy Véhel) Soit* $0 < \gamma < 1$, $f : [0,1] \to [\gamma, +\infty)$ *une fonction limite inférieure d'une suite de fonctions continues, avec* $\|f\|_\infty < +\infty$, *et* $g : [0,1] \to [\gamma, +\infty)$ *une fonction semi-continue inférieurement. Supposons que* $\forall t \in [0,1]$, $f(t) \geq g(t)$. *Alors il existe une fonction continue* $F : [0,1] \to \mathbb{R}$ *qui vérifie*

$$\text{pour tout } x,\ \alpha_l(F)(x) = g(x),$$

$$\text{Hors d'un ensemble } D \text{ de dimension de Hausdorff nulle, } h_F(x) = f(x).$$

Ainsi, les deux quantités, pour une fonction f, peuvent différer partout sauf sur un ensemble de dimension de Hausdorff 0. Réciproquement, nous montrerons que ce résultat est optimal, en ce sens que les fonctions d'exposants $x \to h_f(x)$ et $x \to \alpha_l(f)(x)$ doivent coincider au moins sur un ensemble dense non-dénombrable. Ce résultat va à l'encontre de l'idée que ces deux exposants sont le plus souvent égaux, et procure en même temps des fonctions au comportement très erratique. Ces fonctions sont construites à partir de leur décomposition sur une base d'ondelettes.

Ce résultat est comparable à celui obtenu par S. Jaffard dans [46], qui a réussi à prescrire indépendamment l'exposant ponctuel de Hölder et l'*exposant de chirp* (que nous définirons plus tard), sauf sur un ensemble de mesure de Lebesgue nulle.

Vers une caractérisation plus fine de la régularité

Les espaces 2-microlocaux permettent de compléter et de généraliser la notion d'exposant de régularité. Ces espaces fonctionnels, que nous noterons $C_{x_0}^{s,s'}$, ont été introduits par J.M. Bony dans [17], dans le contexte de l'étude du comportement de solutions de certaines EDP.

Ces espaces sont construits à partir de la décomposition de Littlewood-Paley d'une distribution ou d'une fonction f, et sont intimement liés au comportement local de f autour du point x_0. Ils peuvent également être caractérisés par des taux de décroissance des coefficients d'ondelettes. Plus précisément, $f \in C_{x_0}^{s,s'}$ si et seulement si il existe une constante C, telle que pour tous les couples (j,k) vérifiant $|k2^{-j} - x_0| \leq \delta$ (δ est une constante positive), on a

$$|d_{j,k}| \leq C2^{-j(s+s')}(2^{-j} + |k2^{-j} - x_0|)^{-s'}.$$

La propriété suivante est fondamentale : connaissant les espaces 2-microlocaux auxquels appartient une fonction f donnée, on peut retrouver les divers exposants de régularité mentionnés plus haut.

On peut définir la *frontière 2-microlocale* (unique) Γ_{x_0} d'une distribution f au point x_0 par la formule suivante

$$\Gamma_{x_0}(s) = \sup\{s' : f \in C_{x_0}^{s,s'}\}.$$

Cette frontière, notion essentielle dans cette thèse, est (si on la regarde comme fonction $s' = \Gamma_{x_0}(s)$) concave, décroissante, de dérivées à droite et à gauche toujours plus petites que -1 (voir [33], [34] et [72]). Elle constitue une description géométrique de la régularité locale en un point : en effet, on peut maintenant associer à tout point x_0 non plus un ou plusieurs exposants, mais une courbe dans le plan (s,s'). Il est remarquable que dans le cas d'une fonction de $C^\varepsilon(\mathbb{R}^d)$, tous les exposants que nous étudierons peuvent être calculés à partir de Γ. Nous verrons par exemple que si $f \in C^\varepsilon(\mathbb{R}^d)$ pour un $\varepsilon > 0$, alors en chaque point x_0 on a $h_f(x_0) = -\inf\{s' : s(s') + s \geq 0\}$ et $\alpha_l(x_0) = \inf\{s : s' < 0\}$.

Nous montrerons, dans le chapitre 3, qu'il est possible de trouver une caractérisation temporelle des espaces $C_{x_0}^{s,s'}$ dans le cas $s + s' > 0$, $s' < 0$, ce qui correspond aux cas des fonctions ayant une régularité hölderienne uniforme minimale.

Théorème 1.2 *(Seuret, Lévy Véhel) Soit $x_0 \in \mathbb{R}$, et s, s' deux réels tels que $s + s' > 0$, $s + s' \notin \mathbb{N}$, et $s' < 0$. Alors $f \in C_{x_0}^{s,s'}$ si et seulement si il existe $0 < \delta < 1/4$, un polynôme P de degré plus petit que $[s] - [s + s']$, et une constante C telle que*

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}} \right| \leq C|x - y|^{s+s'-m}(|x - y| + |x - x_0|)^{-s'-[s]+m}$$

pour tous x, y tels que $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$.

Remarquons que les espaces de régularité globale (espaces de Sobolev et de Besov) peuvent également être caractérisés spatialement, grâce aux différences finies.

Ce théorème est comparable à certains des nombreux résultats obtenus par Y. Meyer dans [72], qui notamment prouvait que toute fonction $f \in C_{x_0}^{s,s'}$ avec $s + s' > 0$, $s' < 0$, pouvait se décomposer en

$$f(x) = P(x) + |x - x_0|^{-s'} h(x),$$

avec $h \in C^{s+s'}([x_0 - \delta, x_0 + \delta])$ et $|h(x)| \leq C|x - x_0|^{s+s'}$, et P étant un polynôme de degré plus petit que $[s]$. Nous prouverons d'ailleurs l'équivalence entre l'existence de cette décomposition et la validité de notre caractérisation. Y. Meyer démontre des résultats plus généraux sur des distributions (et plus seulement sur des fonctions), dont la preuve fait intervenir la décomposition spectrale de la distribution f.

La démonstration du Théorème 1.2 que nous proposons reste dans le domaine spatial et utilise explicitement l'existence de valeurs ponctuelles de la fonction. L'idée sous-jacente était de rendre accessibles numériquement les espaces 2-microlocaux.

Lorsque $s + s' < 1$ et $s < 1$, l'inégalité ci-dessus prend une forme simple qui avait été trouvée dans [54]

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'}.$$

Cette caractérisation, fondée sur des inégalités faisant intervenir les valeurs ponctuelles de la fonction f, s'avère très utile pour les applications. En effet, si l'on s'intéresse à un signal ou une image f nulle part dérivable (ou de manière équivalente à des exposants plus petits que 1), il est en effet beaucoup plus facile d'estimer pour f des différences du type $|f(x) - f(x_0)|$ que de passer par l'intermédiaire d'une décomposition de Littlewood-Paley ou des ondelettes. Lorsque les exposants sont plus grands que 1, la discussion est différente car un polynôme intervient dans la caractérisation du Théorème 1.2.

À ce titre, un algorithme d'estimation de régularité, et les résultats de cet algorithme appliqué à plusieurs fonctions "types" (un cusp, un chirp, la fonction de Weierstrass, la réalisation d'un mouvemement Brownien fractionnaire), sont présentés à la fin du chapitre 3.

Un nouvel outil d'analyse de régularité : le spectre 2-microlocal

Le coeur de cette thèse se situe dans le chapitre 4. Nous contruisons et développons un outil que nous avons appelé *spectre 2-microlocal*, noté χ_x. Cette fonction est définie sur $[0, 1]$, et ses valeurs $\chi_x(\rho)$ (où $\rho \in [0, 1]$) sont déterminées à partir des coefficients d'ondelettes d'une fonction ou d'une distribution f qui se situent dans un voisinage du point x (i.e. tels que $|k2^{-j} - x| \leq \delta$). Nous démontrons un *formalisme 2-microlocal* (par analogie au formalisme multifractal qui sera défini un peu plus loin dans cette introduction), qui relie via une transformée de Legendre la frontière 2-microlocale Γ au point x et le spectre 2-microlocal χ_x. Plus précisément, on montrera que

Théorème 1.3 *(Lévy Véhel, Seuret) Soit $f \in \mathcal{S}'(\mathbb{R})$ une distribution, et $x \in \mathbb{R}$. Alors*

$$\sigma(s') = \inf_{\rho \in [0,1]} (\rho s' + \chi_x(\rho)),$$

où $s' \to \sigma(s') = s(s') + s'$ est un paramétrage de la frontière Γ.

Le spectre 2-microlocal est indépendant de l'ondelette choisie pour le calculer. On retrouve ainsi que la frontière 2-microlocale (étant l'enveloppe convexe du spectre 2-microlocal) est également indépendante de l'ondelette choisie.

L'étude du spectre et du formalisme 2-microlocal s'avère extrêmement instructive et éclairante. En effet, il sera montré que tous les exposants de régularité usuels se déduisent de χ_x par des formules très simples et naturelles. Nous verrons que si $f \in C^\varepsilon(\mathbb{R})$ pour un certain $\varepsilon > 0$, alors en tout point x on a $h_f(x) = \inf_{\rho \in (0,1]} \frac{\chi_x(\rho)}{\rho}$, et $\alpha_l(x) = \inf_{\rho \in [0,1]} \chi_x(\rho)$.

Le calcul de χ_x se trouvant être souvent aisé, le calcul de la frontière et des exposants peut le devenir aussi.

À notre sens, χ_x apporte un autre point de vue sur l'analyse de régularité locale, et sur les frontières 2-microlocales également. Par exemple, nous trouvons des conditions de compatibilité entre frontières, et sommes capables de prescrire exactement des frontières sur des ensembles dénombrables denses, alors que l'on ne savait jusqu'alors le faire qu'en un seul point (voir [33] et [72] pour ces résultats).

À la fin du chapitre 4, nous faisons le calcul de spectres 2-microlocaux de fonctions usuelles (fonction "non-différentiable" de Riemann, séries lacunaires d'ondelettes), et présentons quelques applications "théoriques".

Le reste de cette thèse traite de l'analyse multifractale des fonctions et des mesures. Tout au long de ces chapitres 5 et 6, le spectre 2-microlocal sera fréquemment utilisé.

Application à l'analyse multifractale

Il est important de comprendre qu'il est souvent impossible, ou pire inintéressant, de calculer exactement ou de tenter d'estimer en chaque point l'exposant ponctuel de Hölder d'une fonction f ou l'exposant de Hölder d'une mesure μ (typiquement, cela est impossible pour la réalisation d'un processus stochastique).

Pour ces raisons on s'intéresse souvent à la taille et à la répartition des ensembles de niveau de h_f, c'est-à-dire aux ensembles E_h^f définis par

$$E_h^f = \{x : h_f(x) = h\}.$$

Pour beaucoup d'exemples de fonctions f, plusieurs ensembles E_h^f se trouvent être des ensembles fractals denses dans l'intervalle de définition de f. Alors, plutôt qu'à la position de chaque point de E_h^f, on s'intéresse à la dimension de Hausdorff de E_h^f, qui donne une idée de la répartition de cet ensemble sur l'intervalle de définition de f.

L'analyse multifractale d'une fonction f correspond à l'étude du *spectre multifractal* de f, c'est-à-dire à la fonction

$$d_f(h) = \dim_H E_h^f,$$

où $\dim_H(E)$ représente la dimension de Hausdorff de l'ensemble E.

De même, l'analyse multifractale des mesures s'occupe du spectre multifractal d'une mesure défini par

$$d_\mu(\alpha) = \dim_H E_\alpha^\mu = \dim_H\{x : \underline{\alpha}_\mu(x) = \alpha\}.$$

Dans les deux compartiments d'analyse multifractale, on étudie la validité d'un *formalisme multifractal*, qui est une formule reliant par une transformée de Legendre le spectre multifractal à une fonction d'énergie libre (généralement notée $\eta_f(p)$ pour les fonctions, et $\tau(q)$ pour les mesures) calculée à partir de la fonction ou de la mesure étudiée. Ces formules, qui sont de la forme

$$d_f(h) = \inf_p(hp + 1 - \eta_f(p)) \text{ ou } d_\mu(\alpha) = \inf_q(\alpha q - \tau(q)),$$

ont été proposées initialement par Frisch et Parisi dans [30], dans le contexte de l'étude de la turbulence pleinement développée. Bien sûr de telles formules ne sont pas valables en toute généralité (il est d'ailleurs facile de construire des contre-exemples, voir notre chapitre 6), mais de larges classes de fonctions, de mesures et de processus satisfont à ce formalisme. Il est intéressant de remarquer que de tels objets obéissent souvent à des lois d'auto-similarité déterministe ou stochastique.

L'analyse multifractale peut être comprise comme l'étude des phénomènes rares. En effet, il arrive souvent que, pour une fonction $f : \mathbb{R} \to \mathbb{R}$ (ou une mesure), on trouve un exposant h_0 *presque sûr*, au sens où presque surement $h_f(x) = h_0$. Ainsi, $d_f(h_0) = 1$. Mais l'analyse multifractale de cette fonction permet de detecter (s'ils existent) des comportements plus rares (i.e. $h_f(x) \neq h_0$), et calculer la dimension de Hausdorff des ensembles de niveau de l'exposant ponctuel de Hölder nous donne à la fois une idée de l'importance de ces phénomènes rares (au sens où ils sont de mesure de Lebesgue 0) et une idée de leur répartition sur le support de la fonction f. Cette étude est importante car il a été prouvé ([66], [67]) que la dissipation d'énergie dans un fluide turbulent pouvait être reliée à la présence et la "densité" de ces événements rares.

Le chapitre 5 est consacré à la définition et à l'étude de fonctions multifractales dont la construction repose une fois de plus sur des bases d'ondelettes. Ces fonctions F_μ sont définies à partir de mesures boréliennes positives μ par la formule suivante

$$F_\mu(x) = \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} \pm 2^{-j(s_0-\frac{1}{p_0})} |\mu\big([k2^{-j},(k+1)2^{-j})\big)|^{\frac{1}{p_0}} \psi_{j,k}(x),$$

Nous prouverons le fait remarquable suivant :

Théorème 1.4 *(Barral, Seuret) Si μ satisfait au formalisme multifractal pour les mesures au point $\alpha \geq 0$, alors F_μ satisfait au formalisme multifractal pour les fonctions au point $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$, et $d_{F_\mu}(h) = d_\mu(\alpha)$.*

Précisons que les formalismes multifractals pour les mesures et les fonctions utilisés dans ce théorème diffèrent légèrement des formalismes usuels.

Ainsi, en plus d'agrandir la classe des fonctions vérifiant le formalisme multifractal, nous établissons un lien satisfaisant entre les théories multifractales des mesures et des fonctions. De surcroit, ce résultat reste stable après perturbations des coefficients d'ondelettes. En effet, sous certaines hypothèses, nous montrerons que l'on conserve au moins une partie du spectre de F_μ après perturbation. En particulier, il est pertinent, notamment dans le cas d'une mesure μ issue d'une construction du type cascade multiplicative de Mandelbrot, de remplacer $\mu\big([k2^{-j},(k+1)2^{-j})\big)$ par $\mu_j\big([k2^{-j},(k+1)2^{-j})\big)$, où $(\mu_j)_j$ est la suite de "pré"-mesures dont la limite faible sera μ. Le spectre de la fonction perturbée pourra se calculer en fonction de la fonction initiale. On apporte ainsi une réponse à A. Arnéodo, E. Bacry et J.F. Muzy qui dans [4] étudiaient le spectre de ces "cascades multiplicatives d'ondelettes".

Il est important de noter que ce résultat s'applique à toutes les classes de mesures connues qui satisfont au formalisme multifractal usuel : les mesures quasi-Bernoulli, les cascades aléatoires de Mandelbrot, les mesures associées aux subordinateurs de Lévy stables ...

Étude des singularités oscillantes

Le chapitre 6 est orienté vers la détection et la création de singularités oscillantes. Il est principalement fondé sur l'étude du spectre 2-microlocal pour des fonctions quelconques.

Nous nous intéresserons particulièrement à l'opérateur de seuillage de coefficients d'ondelettes défini de la manière suivante : l'opérateur de seuillage d'ordre γ associe à une fonction $f = \sum_{j\geq 1}\sum_k d_{j,k}\psi_{j,k}$ de $L^2(\mathbb{R})$ la fonction $f^t = \sum_{j\geq 1}\sum_k d^t_{j,k}\psi_{j,k}$ définie par

$$d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}.$$

Cette opération annule échelle par échelle les coefficients trop petits. Le chapitre 6 est dédié entre autres à l'étude de cet opérateur d'un point de vue multifractal.

Nous verrons qu'un seuillage de ce type *peut* créer des oscillations :

Proposition 1.1 *(Seuret) Soit $f \in C^\varepsilon([0,1])$, et supposons que $E_f^h \neq \emptyset$. Appelons f^t la fonction obtenue après un seuillage de f d'ordre $\gamma < h$. Alors pour tout $x \in E_f^h$, on a deux possibilités : ou bien $h_{f^t}(x) = +\infty$, ou x est une singularité oscillante pour f^t.*

On ne peut bien sûr parler du cas géneral : par exemple appliquer un seuillage de ce type à la fonction f_α dont tous les coefficients d'ondelettes vérifient $d_{j,k} = 2^{-j\alpha}$ ne crée pas d'oscillations (après seuillage, ou bien $\alpha < \gamma$ et f_α est inchangée, ou alors $f^t = 0$ si $\alpha > \gamma$). Nous étudierons donc en détails un cas spécifique où ce seuillage crée des oscillations.

Les mêmes idées nous procureront un résultat intéressant d'analyse multifractale : nous trouvons une condition suffisante sur le spectre multifractal pour la présence d'oscillations. Cette condition est liée au *spectre des grandes déviations* $\tilde{d}_f$ d'une fonction, qui lui aussi est calculé à partir des coefficients d'ondelettes de cette fonction. La valeur $\tilde{d}_f(h)$ (où $h > 0$) est relativement facile à estimer en pratique, et indique heuristiquement qu'à l'échelle j, il y a $2^{j\tilde{d}_f(h)}$ coefficients d'ondelettes d'amplitude 2^{-jh}. Nous montrons

Théorème 1.5 *(Seuret) Soit $f \in C^\varepsilon([0,1])$. Supposons qu'il existe un exposant $h > 0$ tel que $\tilde{d}_f(h) < d_f(h)$. Alors il existe un ensemble E de dimension $d_f(h)$ tel que pour tout x de E, f a une singularité oscillante en x et $h_f(x) = h$.*

En particulier, si le formalisme multifractal pour les fonctions n'est pas satisfait pour une fonction f, f a un comportement oscillatoire sur un large ensemble de points. Plus précisément, il apparaitra clairement dans la démonstration que lorsque le formalisme échoue, "il n'y a pas assez" de coefficients d'ondelettes d'ordre 2^{-jh} pour empêcher les oscillations de survenir. Ce résultat peut s'apparenter au phénomène de Gibbs bien connu en traitement du signal.

Ce théorème doit être mis en parallèle avec le théorème obtenu par C. Melot dans [70], qui démontre que presque toute fonction (au sens de Baire) dans certains espaces fonctionnels (qui sont les mêmes que ceux utilisés par S. Jaffard dans [48] et qui sont rappelés dans le théorème 1.6 plus bas) ne satisfont pas au formalisme multifractal et possèdent des singularités oscillantes.

Évidemment nous pensons aux méthodes de compression de signaux et d'images fondées essentiellement sur le seuillage (différent de celui que nous étudions) des coefficients d'ondelettes. Le seuillage que nous proposons peut sans doute être comparé au seuillages *durs* et *mous*, et par conséquent que de telles méthodes créent génériquement des oscillations visibles. Cela peut apporter quelques indications sur la présence de certains défauts observés après compression de signaux et d'images.

Par les divers objets qu'elle peut traiter, l'analyse de régularité locale peut se revendiquer multidisciplinaire. Si l'objet est un signal ou une image, on la classera en analyse numérique. Lorsque l'on s'intéresse par exemple à la théorie des mesures et ses applications en théorie des nombres, on tombe dans le domaine des mathématiques dites pures. Si on étudie une fonction ou une distribution, elle versera dans les mathématiques appliquées. Si enfin on travaille sur des processus stochastiques ou des mesures, elle sera apparentée aux probabilités.

Mais l'analyse de la régularité locale et l'analyse multifractale représentent en réalité une approche nouvelle et procurent un ensemble de méthodes vouées à l'étude de problèmes mathématiques et physiques.

L'approche multifractale n'est pas anecdotique, en ce sens notamment que les objets multifractals (c'est-à-dire qui ont un spectre multifractal non trivial) apparaissent comme des phénomènes génériques. Rappelons le théorème établi par S. Jaffard dans [48]

Théorème 1.6 *Au sens des catégories de Baire, presque toute fonction de l'espace fonctionnel* $V = \cap_{p>0,\varepsilon>0} B_p^{(\eta_f(p)-\varepsilon)/p,\infty}$ *est multifractale.*

Ce résultat a été amélioré par des travaux d'A. Fraysse et S. Jaffard, en montrant que cette multifractalité reste générique sur des ensembles prévalents dans V.

En quelque sorte, pour une fonction f, être "multifractale" n'est pas une propriété étonnante. Au contraire, ne pas avoir un comportement multifractal est exceptionnel, et relève probablement d'une structure interne spéciale des fonctions considérées.

De même, la proposition suivante, prouvée par exemple dans [51],

Proposition 1.2 *Soit* $f \in B_p^{s,p}(\mathbb{R}^d)$*, avec* $s - \frac{d}{p} > 0$*. L'ensemble des points* x *tels que* $h_f(x) = s - \frac{d}{p}$ *a une dimension de Hausdorff nulle.*

donne un éclairage sur l'optimalité des inclusions dites de Sobolev des espaces de Besov $B_p^{s,p}(\mathbb{R}^d)$ vers les espaces de Hölder uniformes $C^{s-\frac{d}{p}}(\mathbb{R}^d)$.

Pour finir, revenons sur le traitement du signal et d'images. Ce sont des domaines particuliers dans lesquels l'étude de la régularité locale est amenée à jouer un rôle important. En effet, les méthodes classiques d'analyse de signaux et d'images peuvent être améliorées ou complétées grâce à une approche "fractale". Ce n'est d'ailleurs qu'un juste retour des choses, puisque, comme il a été dit précédemment, l'intérêt pour les comportements locaux des objets provient principalement de problèmes issus de l'analyse de signaux réels. Il est également important de comprendre que les méthodes développées dans notre domaine peuvent être appliquées à des signaux qui eux-mêmes ne sont pas "fractals" (au sens où par exemple ils ne présentent pas de structure auto-similaire), à condition toutefois que ces signaux possèdent une certaine irrégularité et que leur comportement local présente un intérêt.

Dans [56], J. Lévy Véhel présente plusieurs outils dédiés à l'analyse de signaux réels et à la synthèse de signaux. Ces derniers sont fondés sur des méthodes d'analyse de régularité locale et multifractale (entre autres en utilisant les systèmes de fonctions itérées). Ils permettent de résoudre certains problèmes spécifiques en traitement du signal, parfois avec des résultats meilleurs que les algorithmes dits classiques. Un autre exemple d'outils est le débruitage multifractal de signaux irréguliers, développé dans [58]. Il est montré qu'appliquer ce type de débruitage à un signal que l'on suppose initialement suffisamment irrégulier (typiquement une fonction de Weierstrass) et auquel un bruit a été ajouté, permet de retrouver la forme irrégulière du signal de départ.

De même, des méthodes de segmentation et de débruitage d'images, ainsi que de détection de contours, sont présentées dans [57] (et également dans [58]). Il apparait que dans certains cas (notamment celui de certaines images satellites), l'approche "fractale" donne là encore de meilleurs résultats que les approches dites classiques.

1.2 Ondelettes, exposants et frontières 2-microlocales

Dans cette section, nous rappelons les définitions des principaux outils qui nous serviront par la suite : la décomposition d'une fonction sur une base d'ondelettes, qui sera fondamentale car elle permet de construire explicitement des objets ayant de bonnes propriétés, les exposants de régularité complétant l'information fournie par l'exposant ponctuel de Hölder, et les espaces 2-microlocaux $C_{x_0}^{s,s'}$.

1.2.1 Les bases d'ondelettes

La décomposition d'une fonction sur des bases d'ondelettes sera à la base de la plupart des travaux exposés plus tard.

Soit ψ une fonction de la classe de Schwartz $\mathcal{S}(\mathbb{R}^d)$. Définissons les versions translatées et dilatées de ψ, $\psi_{j,k}(x) = \psi(2^j x - k)$. Si la fonction ψ satisfaisait les propriétés de décroissance et de reconstruction décrites dans [55], alors l'ensemble de fonctions

$$\{2^{j/2}\psi_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$$

forme une base orthonormale de $L^2(\mathbb{R}^d)$. L'*ondelette* ψ est une fonction à décroissance rapide, comme ses dérivées. Elle est également supposée de moyenne nulle, et on supposera qu'elle possède des moments nuls, i.e.

$$\int_{\mathbb{R}^d} x^k \psi(x)dx = 0 \text{ pour } k = 0, \ldots, N.$$

Il est possible de construire des ondelettes ayant une infinité de moments nuls, mais elle est alors de support infini. Si l'on préfère des ondelettes à support compact (typiquement pour étudier des fonctions à support compact), elles ont nécessairement un nombre fini de moments nuls N, et la taille du support de l'ondelette est relié à ce nombre N (cf [22] pour la construction d'ondelettes à support compact).

Les coefficients d'ondelettes, coordonnées d'une fonction f dans la base *orthogonale* $\{\psi_{j,k}\}$, sont définis par

$$d_{j,k} = 2^j \int_{\mathbb{R}^d} f(t)\psi_{j,k}(t)dt.$$

Les espaces $C^s(x_0)$ peuvent être reliés au taux de décroissance des coefficients d'ondelettes qui se trouvent autour de x_0. Plus précisément, le théorème suivant est démontré par S. Jaffard dans [51]

Théorème 1.7 *Soit $f \in C^s(x_0)$. Si $|k2^{-j} - x_0| \leq 1/2$, alors*

$$|d_{j,k}| \leq C2^{-sj}(1 + 2^j|k2^{-j} - x_0|)^s. \tag{1.3}$$

Réciproquement, si (1.3) est vérifiée pour tous les couples (j, k) tels que $|k2^{-j} - x_0| \leq \delta$ (où δ est une constante positive), et si $f \in C^\varepsilon(\mathbb{R}^d)$, alors on peut trouver une constante C et un polynome P de degré inférieur à $[s]$, tels que

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s \log\left(\frac{2}{|x - x_0|}\right). \tag{1.4}$$

On peut affaiblir la condition $f \in C^\varepsilon$, mais ce ne sera pas utile pour nous par la suite.

Si $f \in C^\varepsilon(\mathbb{R}^d)$, et si l'ondelette ψ a plus que $[h_f(x_0)+1]$ moments nuls, on déduit du Théroème (1.7) la caractérisation de l'exposant ponctuel de Hölder de f en x_0 par coefficients d'ondelettes

$$h_f(x_0) = \liminf_{|k2^{-j}-x_0|\to 0} \frac{\log|d_{j,k}|}{\log(2^{-j}+|x_0-k2^{-j}|)}. \tag{1.5}$$

1.2.2 D'autres exposants

Il existe plusieurs techniques pour pallier au manque d'informations fournies par l'exposant ponctuel de Hölder (et notamment à l'absence de détection d'oscillations). Nous en rappelons quelques-unes.

Exposant Local de Hölder

Cet exposant a été introduit par J. Lévy Vehel et B. Guiheneuf dans [34].

Rappelons tout d'abord la définition des espaces de Hölder globaux $C^\alpha(\mathbb{R})$. Soit $f : \Omega \to \mathbb{R}$ une fonction définie sur un ouvert $\Omega \subset \mathbb{R}$. Pour $0 < s < 1$, $f \in C^s(\Omega)$ si on peut trouver une constante C telle que $\forall(x,y) \in \Omega^2$,

$$|f(x) - f(y)| \leq C|x-y|^s. \tag{1.6}$$

Si $m < s < m+1$ ($m \in \mathbb{N}$), $f \in C^s(\Omega)$ signifie que $\partial^m f \in C^{s-m}(\Omega)$.

On pose alors $\alpha_l(\Omega) = \sup\{s : f \in C_l^s(\Omega)\}$. Remarquons que si $\Omega' \subset \Omega$, $\alpha_l(\Omega') \geq \alpha_l(\Omega)$. Pour définir l'exposant local de Hölder , on utilisera le lemme suivant

Lemme 1.1 *Soit $(O_i)_{i\in I}$ une famille décroissante d'ouverts (i.e. $O_i \subset O_j$ si $i > j$), telle que $\cap_i O_i = \{x_0\}$. Posons*

$$\alpha_l(x_0) = \sup\{\alpha_l(O_i) : i \in I\}. \tag{1.7}$$

$\alpha_l(x_0)$ ne depend pas de la famille $(O_i)_{i\in I}$.

$\alpha_l(x_0)$ peut donc être défini simplement en choisissant une suite d'intervalles contenant x_0.

Définition 1.3 *Soit f une fonction définie au voisinage de x_0. Soit $\{I_n\}_{n\in\mathbb{N}}$ une suite décroissante d'intervalles convergeant vers x_0. L'* exposant Hölder local *de la fonction f au point x_0, noté $\alpha_l(x_0)$, est égal à*

$$\alpha_l(x_0) = \sup_{n\in\mathbb{N}} \alpha_l(I_n) = \lim_{n\to+\infty} \alpha_l(I_n). \tag{1.8}$$

Il est clair que, pour une fonction globalement höldérienne, on a toujours $\alpha_l(x_0) \leq h_f(x_0)$.

Contrairement à l'exposant ponctuel, l'exposant local est stable par intégration et dérivation fractionnaire.

On peut obtenir une caractérisation par ondelettes de l'exposant local de Hölder grâce au théroème suivant [71] :

Théorème 1.8 *Soit $x_0 \in \mathbb{R}$ et $\eta > 0$. Une fonction f appartient à C^α si et seulement si on peut trouver une constante C telle que, pour tous les couples $(j,k) \in \mathbb{Z}^2$ vérifiant $k2^{-j} \in B(x_0,\eta)$, on ait $|d_{j,k}| \leq C2^{-\alpha j}$.*

Ce théorème nous donne directement une caractérisation de l'exposant local de Hölder par coefficients d'ondettes :

Proposition 1.3

$$\alpha_l(x_0) = \lim_{\eta \to 0}(\sup\{s : \exists C,\ k2^{-j} \in B(x_0, \eta) \Rightarrow\ |d_{j,k}| \leq C2^{-sj}\}) \tag{1.9}$$

Il est plus simple algorithmiquement d'estimer la décroissance uniforme d'un ensemble de coefficients d'ondelettes autour d'un point que d'estimer l'exposant ponctuel via l'égalité (1.5). Ainsi, comme $\alpha_l(x_0)$ est relié à un certain taux de décroissance uniforme des coefficients d'ondelettes autour du point x_0, il est souvent calculé à la place de $h_f(x_0)$, et on parle alors également d'exposant ponctuel.

Cet exposant est très utilisé en traitement du signal et d'image (voir [65] et [83] parmi de très nombreuses références), par exemple en détection de contours. En effet, on peut s'attendre à ce que le contour d'un objet dans une image ait un exposant local de Hölder petit, alors que l'intérieur d'une surface uniforme est très régulier (et donc a un exposant grand). L'estimation de la régularité locale peut donc servir d'indicateur de la présence de contours.

Nous étudierons en détail dans le chapitre 2 les contraintes imposées à l'exposant local de Hölder ainsi que ses relations avec l'exposant ponctuel de Hölder h_f. En particulier, nous montrerons que ces deux quantités, pour une fonction f, peuvent différer partout sauf sur un ensemble dense non-dénombrable de dimension de Hausdorff 0, ce qui indique que α_l et h_f procurent bien des informations complémentaires.

Exposant de Chirp

β_c a été introduit dans [72].

Définition 1.4 *Soit $f \in L^1_{loc}(\mathbb{R})$, et $f^{(-l)}$ une primitive de f d'ordre l. f est un chirp du type (h, β) au point x_0 si*

$$\forall n \in \mathbb{N},\ \ f^{(-n)} \in C^{h+n(1+\beta)}_{x_0}.$$

Alors $\beta_c(x_0)$ est la plus grande valeur de β pour laquelle f est un chirp du type (h, β). Cet exposant caractérise le comportement asymptotique autour de x_0 après un grand nombre d'intégrations. Cette définition est naturelle et correspond à l'idée intuitive (expliquée plus haut pour le chirp $x \to |x|^\gamma \sin(1/|x|^\beta)$) que si la fonction f oscille rapidement autour du point x_0, la régularité locale de la primitive de f en x_0 sera meilleure que prévue (c'est-à-dire sera strictement supérieure à $h_f(x_0) + 1$).

Malheureusement, comme il a été remarqué dans [51] par exemple, cet exposant n'est pas stable par l'ajout d'une fonction régulière. En effet, si on ajoute au chirp $|x|^\gamma \sin(1/|x|^\beta)$ la fonction $|x|^{\gamma+1}$, le comportement local de la fonction obtenue est bien régi par le chirp, mais après un certain nombre d'intégrations (typiquement $[1/\beta + 2]$ intégrations), c'est le terme régulier $|x|^{\gamma+1}$ qui domine la régularité locale de la primitive choisie.

Exposant d'oscillation

Comme il a été remarqué dans [2], on peut alors décider de regarder ce qui se passe après une intégration fractionnaire infinitésimale

Définition 1.5 *Soit $f \in L^1_{loc}(\mathbb{R})$, et $h_t(x_0)$ l'exposant ponctuel de Hölder de la primitive fractionnaire de f d'ordre $t > 0$ au point x_0. On pose alors*

$$\beta_f(x_0) = \left(\frac{\partial}{\partial t} h_t(x_0) \right)_{t=0^+} - 1.$$

Il est clair que cette définition est consistante au sens qu'elle est stable par ajout d'une composante régulière.

Plus tard, nous ferons référence à un *comportement oscillatoire*, ou à une singularité *de type chirp* d'une fonction f en un point x : cela signifiera que $\beta_f(x) > 0$ pour la fonction f.

Exposant "weak scaling"

Cet exposant, que nous noterons β_w, a été proposé par Y. Meyer dans [72]

Définition 1.6 *Soit $f \in L^1_{loc}(\mathbb{R})$, et $f^{(-n)}$ une primitive de f d'ordre n. On pose*

$$\beta_w(x_0) = \sup\{s : \exists n, \;\; f^{(-n)} \in C^{s+n}_{x_0}\}.$$

Ces exposants peuvent être caractérisés par des propriétés de décroissance des coefficients d'ondelettes qui se trouvent autour de x_0. Nous ne les développons pas ici, car nous les relierons (par des formules simples) au spectre 2-microlocal dans le chapitre 4.

1.2.3 Espaces 2-microlocaux

Les espaces 2-microlocaux sont initialement définis à partir de la décomposition de Littlewood-Paley d'une distribution tempérée [17].

Considérons une fonction ϕ appartenant à l'espace de Schwartz $\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \forall(\gamma,\delta) \in \mathbb{N}^2, \sup_x |x^\gamma \partial^\delta f(x)| < \infty\}$, dont la transformée de Fourier vérifie

$$\begin{aligned} \hat{\phi}(\xi) &= 1 \quad \text{si} \quad |\xi| \leq 1/2, \\ \hat{\phi}(\xi) &= 0 \quad \text{si} \quad |\xi| \geq 1. \end{aligned}$$

On définit les versions dilatées de ϕ par $\phi_j(x) = 2^j\phi(2^j x)$, et $\psi_j = \phi_{j+1} - \phi_j$. Soit $f \in \mathcal{S}'(\mathbb{R})$ une distribution tempérée, rappelons que

$$\mathcal{S}'(\mathbb{R}) = \{f : \exists C, \; \exists q \in \mathbb{N}, \;\; \forall g \in \mathcal{S}(\mathbb{R}), \; |\langle f, g\rangle| \leq C\pi_q(g)\},$$

(où $\pi_q(g) = \sup\{(1+|x|)^q \, |\partial^\delta g(x)| : |\delta| \leq q, x \in \mathbb{R}\}$). L'analyse de Littlewood-Paley de f consiste en l'ensemble de distributions $\{S_0 f, \Delta_j f\}_{j\geq 0}$, où

$$S_0 f = \phi * f \text{ et } \Delta_j f = \psi_j * f.$$

f peut se décomposer en (voir [72] par exemple pour de plus amples détails)

$$f = S_0 f + \sum_{j=0}^{+\infty} \Delta_j f. \tag{1.10}$$

Les espaces 2-microlocaux $C^{s,s'}_{x_0}$ sont définis de la manière suivante

Définition 1.7 *Soit $x_0 \in \mathbb{R}$ et (s, s') deux réels. Une distribution $f \in \mathcal{S}'(\mathbb{R})$ appartient à $C_{x_0}^{s,s'}$ si l'on peut trouver une constante C telle que*

$$\begin{aligned} |S_0 f(x)| &\leq C(1 + |x - x_0|)^{-s'}, \\ |\Delta_j f(x)| &\leq C 2^{-js}(1 + 2^j |x - x_0|)^{-s'}. \end{aligned}$$

Par la suite, nous nous intéressons principalement au comportement local des objets (distributions entre autres). La définition 1.7 n'est pas adaptée à notre étude, car elle prend en compte le comportement de f loin de x_0, alors que l'on s'intéresse au comportement local autour du point x_0. Nous utiliserons donc une version *locale* des espaces 2-microlocaux, définie par

Définition 1.8 *[72] Soit V un voisinage ouvert de x_0, et $f \in \mathcal{D}'(V)$ une distribution. f appartient à $C_{x_0}^{s,s'}$* localement *si on peut trouver V_0, un voisinage de x_0 inclus dans V, et une distribution g qui appartient à $C_{x_0}^{s,s'}$ (globalement) telle que $f = g$ sur V_0.*

Dorénavant, par convention, $C_{x_0}^{s,s'}$ désignera un espace 2-microlocal *local*.

Nous nous servirons beaucoup par la suite de ces espaces 2-microlocaux, mais principalement de leur caractérisation par coefficients d'ondelettes, ou par tranformée en ondelettes continues.

Nous avons déjà vu les bases d'ondelettes $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}^2}$ engendrées à partir d'une même ondelette-mère ψ. Pour définir la transformée en ondelettes continues [32], nous avons besoin d'une ondelette Ψ analysante dite "admissible" (i.e. satisfaisant des conditions décrites dans [71]). La formule suivante définit la version continue de la transformée en ondelettes

$$W_f(a, b) = \frac{1}{a} \int f(x) \Psi(\frac{x - b}{a}) dx. \tag{1.11}$$

La caractérisation des espaces $C_{x_0}^{s,s'}$ est la suivante [51]

Théorème 1.9 *Soient s, s' deux nombres réels. Supposons que ψ et Ψ ont leurs N premiers moments nuls, avec $N > \max(s, s+s')$ (des conditions précises sur la régularité des ondelettes peuvent être trouvées dans [72]). Les trois assertions suivantes sont équivalentes*

1. $f \in C_{x_0}^{s,s'}$,
2. *Pour tous les couples (j, k) vérifiant $|x_0 - k2^{-j}| \leq 1$, on a*

$$|d_{j,k}| \leq C 2^{-js}(1 + |k - 2^j x_0|)^{-s'},$$

3. $\forall a > 0$, $\forall b$ *tel que* $|b - x_0| < 1$,

$$|W_f(a, b)| \leq C a^s (1 + \frac{|b - x_0|}{a})^{-s'}.$$

Dans le reste de cette thèse nous supposerons toujours que les ondelettes utilisées auront un nombre suffisant de moments nuls pour pouvoir appliquer le Théorème 1.9.

Rappelons quelques propriétés élémentaires des espaces 2-microlocaux

Proposition 1.4 *Pour tout* $x_0 \in \mathbb{R}^d$,

– $t \leq s$ *et* $t + t' \leq s + s' \Rightarrow C_{x_0}^{s,s'} \subset C_{x_0}^{t,t'}$.
– $\forall s > 0,\ C_{x_0}^{s} \subset C_{x_0}^{s,-s}$.
– $\forall (s, s')$ *avec* $s + s' > 0,\ C_{x_0}^{s,s'} \subset C_{x_0}^{s}$.

Une propriété importante des espaces $C_{x_0}^{s,s'}$ est leur stabilité par intégration ou dérivation

Proposition 1.5 *Soit* $n \in \mathbb{N}$. *Pour toute distribution* $f \in \mathcal{S}'(\mathbb{R})$, *pour tous les couples* $(s, s') \in \mathbb{R}$, *on a*

$$f \in C_{x_0}^{s,s'} \Longleftrightarrow f^{(n)} \in C_{x_0}^{s-n,s'}.$$

En réalité, ces espaces sont de manière plus générale stables sous l'action d'opérateurs pseudo-différentiels tels qu'ils sont définis dans [72], et notamment par intégration ou dérivation fractionnaires.

Il est commode pour la suite d'introduire la quantité

$$\sigma = s + s'.$$

1.2.4 Frontière 2-microlocale

Venons-en à l'objet qui nous intéressera beaucoup par la suite.
Pour cela, introduisons le *domaine 2-microlocal* d'une distribution f au point x_0 : $E(f, x_0) = \{(s, s') : f \in C_{x_0}^{s,s'}\}$. Grâce à la Proposition 1.4 et à l'inégalité de Hölder, on voit facilement que $E(f, x_0)$ forme un ensemble convexe du plan (s, s'). Sa fontière, appelée *frontière 2-microlocale*, est une courbe convexe notée $\Gamma(f, x_0)$. Cette frontière peut être définie par

$$\begin{aligned} \Gamma(f, x_0) : \quad & \mathbb{R} \to \mathbb{R} \\ & s' \to s(s') = \sup\{r : f \in C_{x_0}^{r,s'}\}, \end{aligned}$$

mais $\Gamma(f, x_0)$ peut être paramétrisée de différentes manières en fonction des trois quantités s, s' et $\sigma = s + s'$.

Choisir s comme paramètre permet de voir que là où elle n'est pas infinie, la frontière 2-microlocale a des dérivées à gauche et à droite toujours comprises entre $-\infty$ et -1 (toujours à cause des mêmes propriétés des espaces 2-microlocaux $C_{x_0}^{s,s'}$). Mais ce choix n'est pas judicieux, car on ne peut alors atteindre les portions de courbe verticales dans le plan (s, s').

Pour des raisons que l'on explicitera, il est plus pertinent de considérer la paramétrisation $\sigma(s')$. On a alors

Proposition 1.6 *La frontière 2-microlocale d'une distribution* f *au point* x_0, *vue comme fonction* $s' \to \sigma(s')$, *vérifie (si elle n'est pas partout infinie)*

– $\sigma(s')$ *est une fonction concave croissante définie sur* $\mathbb{R}$,
– $\sigma(s')$ *possède des dérivées à gauche et à droite comprises entre 0 et 1.*

Remarquons que l'on a $\forall \varepsilon > 0,\ f \in C_{x_0}^{\sigma(s')-s',s'-\varepsilon}$, mais qu'en toute généralité il est faux que $f \in C_{x_0}^{\sigma(s')-s',s'}$.

Nous affirmons que la frontière 2-microlocale d'une fonction f au point x_0 donne une description géométrique du comportement de f autour de x_0. En effet, à tout point x_0 on peut associer sa frontière, courbe dans $\mathbb{R}^2$. Le point important est que de la connaissance de cette courbe peuvent être déduits tous les exposants de régularité que l'on a présentés plus tôt :

Proposition 1.7 *Soit $s' \to \sigma(s')$ la fontière de f au point x_0. Supposons $f \in C^\varepsilon(\mathbb{R}^d)$ pour un $\varepsilon > 0$. Alors*
- $h_f(x_0) = -\inf\{s' : \sigma(s') \geq 0\}$, *où par convention on pose $h_f(x_0) = +\infty$ si $\sigma(s') > 0$ pour tout $s' \in \mathbb{R}$,*
- $\alpha_l(x_0) = \sigma(0)$,
- $\beta_c(x_0) = \lim_{s' \to -\infty} \frac{s' - \sigma(s')}{\sigma(s')}$,
- $\beta_w(x_0) = \lim_{s' \to -\infty}(\sigma(s') - s')$,
- $\beta_o(x_0) = \lim_{s' \to -h_f^-}(\frac{\sigma(s')}{s' + h_f})^{-1} - 1 = \left((\frac{\partial \sigma}{\partial s'})_-(-h_f)\right)^{-1} - 1.$

Le premier point est prouvé et discuté dans [72]. Les autres points proviennent simplement de l'application de la stabilité sous l'action des opérateurs pseudo-différentiels des espaces $C_{x_0}^{s,s'}$ à la définition des exposants présentés plus haut. En particulier, $\beta_w(x_0)$ et $\beta_c(x_0)$ sont reliés au comportement de la frontière lorsque $s' \to -\infty$, et $\beta_o(x_0)$ est relié au comportement de la frontière autour du point $(-h_f(x_0), 0)$ (dans le plan (s', σ)).

Pour rendre la notion de frontière plus concrète, calculons-la dans quelques cas :

- pour un cusp $x \to |x|^\gamma$, seuls les coefficients qui se trouvent dans le cône d'influence de 0 ont une influence sur la frontière, car les autres ont une décroissance rapide (i.e. $C(a, b)$ décroit plus vite que a^n pour tout n). Ces coefficients se comportent comme a^γ. Cela nous donne une frontière au point $x = 0$ qui est verticale et passe par le point $(\gamma, 0)$ dans le plan (s, s') (voir la figure 1.1). Il parait clair que cette fonction cusp appartient à tous les espaces $C_x^{s,s'}$ quand $x \neq 0$ (elle est C^∞ hors de 0).

- pour un chirp $x \to |x|^\gamma \sin(\frac{1}{|x|^\beta})$, les plus gros coefficients d'ondelettes se trouvent autour de la courbe $b = a^{\frac{1}{1+\beta}}$ (*resp.* de manière équivalente pour les coefficients d'ondelettes autour de la courbe $k2^{-j} = 2^{-j\frac{1}{1+\beta}}$), et ils ont un amplitude comparable $a^{\frac{\gamma}{1+\beta}}$ (*resp.* à $2^{-j\frac{\gamma}{1+\beta}}$). Cela correspond à une frontière 2-microlocale au point 0 qui est une ligne droite dont l'équation est $\sigma(s') = \frac{1}{\beta+1}s' + \frac{\gamma}{\beta+1}$ (voir la figure 1.1). Une fois de plus, on a une singularité isolée et les frontières aux points $x \neq 0$ sont triviales.

- la fonction de Weierstrass

$$W_s(x) = \sum_{n=1}^{+\infty} \lambda^{-ns} \sin(\lambda^n x),$$

où $\lambda > 1$, $0 < s < 1$, vérifie en chaque point x $\alpha_l(x) = h_f(x) = s$. Grâce à la Proposition 4.13 que nous démontrerons, les frontières 2-microlocales de cette fonction sont toutes identiques et ont, dans le plan (s, s'), une partie verticale $s = h_f = \alpha_l$ pour s' négatif et une partie parallèle à la seconde bissectrice et passant par le point $(\gamma, 0)$ pour s' positif (figure 1.1).

Nous rencontrerons par la suite des frontières plus "exotiques". En effet, toute fonction concave, décroissante et de dérivées à gauche et à droite plus petites que -1 est la frontière 2-microlocale d'une fonction f au point 0 (voir [72] et [33]). Nous redémontrerons ce résultat grâce au spectre 2-microlocal dans le chapitre 4.

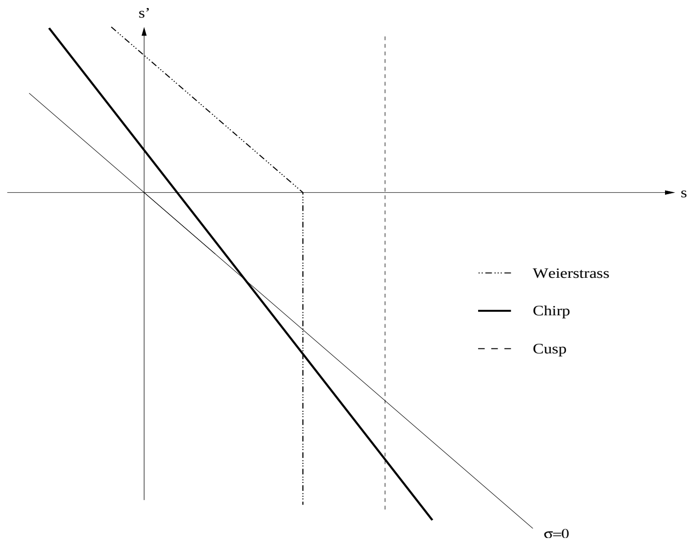

FIG. 1.1 – Exemples de frontières 2-microlocales : cusp, chirp et fonction de Weierstrass.

1.3 Espaces de Besov et formalisme multifractal

L'étude de la validité du formalisme multifractal pour les fonctions nous occupera à la fin de cette thèse, notamment dans les chapitres 5 et 6.

Comme nous l'avons déjà remarqué, il est difficile voire impossible de calculer en chaque point x_0 l'exposant ponctuel de Hölder d'une fonction f en x_0, $h_f(x_0)$. Pour cette raison on s'intéresse souvent aux ensembles de niveau de h_f

Définition 1.9 *Soit Ω un ouvert de $\mathbb{R}^d$ et $f : \Omega \to \mathbb{R}$. On définit, pour tout réel $h \geq 0$, les ensembles E_h^f par*

$$E_h^f = \{x \in \Omega : \ h_f(x) = h\}.$$

L'application $d_f : \mathbb{R}^+ \to \mathbb{R}^+$, qui à $h > 0$ associe $d_f(h) = \dim_H(E_h^f)$, est appelée le spectre multifractal *de f.*

$\dim_H(E)$ est la dimension de Hausdorff d'un ensemble E (pour une definition de la dimension de Hausdorff, voir par exemple [25]). Si $E_h^f = \emptyset$ on pose par convention $d_f(h) = -\infty$. L'analyse multifractale de f consiste en l'étude du spectre multifractal de f.

Nous allons avoir besoin, afin de définir le formalisme multifractal pour les fonctions, des espaces de Besov. L'espace $B_q^{s,p}(\mathbb{R}^d)$ est caractérisable pour tout s et p réels positifs grâce aux coefficients d'ondelettes (contrairement aux espaces de Sobolev). $B_q^{s,p}(\mathbb{R}^d)$ est l'ensemble des fonctions f qui vérifient

$$f \in B_q^{s,p}(\mathbb{R}^d) \Leftrightarrow \left(\sum_k |d_{j,k} 2^{j(s-1/p)}|^p \right)^{1/p} = \varepsilon_j \text{ avec } \varepsilon_j \in l^q. \tag{1.12}$$

Cette définition fait donc intervenir la norme l^p des coefficients $d_{j,k}$. À chaque fonction f on peut associer sa *fonction d'échelle* $\eta_f(p)$ définie pour $p > 0$ par

$$\eta_f(p) = \sup \left\{ u : f \in B_p^{\frac{u}{p},\infty}(I_0) \right\}. \tag{1.13}$$

Cette fonction η_f est concave et croissante, et est donc reliée à l'appartenance de f à certains espaces de Besov. La fonction d'échelle est étudiée en détails dans [48].

Nous avons expliqué qu'il est naturel d'essayer de calculer, d'estimer ou de majorer le spectre multifractal par des méthodes autres que le calcul en chaque point de l'exposant h_f. Frisch et Parisi dans [30] ont proposé une formule liant le spectre multifractal à certaines quantités "moyennes" (suffisamment stables) calculées à partir de la fonction f. La reformulation grâce aux bases d'ondelettes et la généralisation de cette formule peut s'écrire (voir [3], [44], et [48])

$$d_f(h) = \inf_{p>0} (ph - \eta_f(p) + 1). \tag{1.14}$$

(1.14) n'est pas toujours vérifiée : les fonctions f^t et f^{it} du chapitre 6 en donneront des contre-exemples. S. Jaffard a pourtant montré que l'on pouvait en tirer une majoration générale du spectre de toute fonction f ayant un minimum de régularité höldérienne globale

$$\forall\ h \geq 0,\ \ d_f(h) \leq \inf_{p \geq p_c} (ph - \eta_f(p) + 1), \tag{1.15}$$

où $p_c = \inf\{p > 0 :\ \ \eta_f(p) \geq 1\}$. Dans le même travail, il montre également qu'on a égalité dans (1.15) pour quasi-toutes fonctions dans certains espaces fonctionnels.

De même, le formalisme s'est avéré vrai pour de larges classes de fonctions, pour la plupart vérifiant des propriétés d'autosimilarité et ou quasi-autosimilarité ([44], [14] et [1]).

La formule donnant le formalisme multifractal a été modifiée par S. Jaffard dans [50], en s'intéressant aux espaces d'oscillations et en remplacant la fonction d'échelle η_f par une autre fonction de coupure η_f^0 reliée à ces espaces.

1.4 Quelques détails sur les travaux

Afin d'éviter les répétitions, nous avons concentré dans ce chapitre la présentation et les définitions des objets que nous étudierons dans les chapites suivants, par exemple les exposants de régularité, les espaces 2-microlocaux. Nous avons également rappelé les principaux théorèmes que nous utiliserons.

Les chapitres 2, 3 et 4 sont chacuns extraits d'un article (respectivement [80], [82] et [62]) écrit en collaboration avec Jacques Lévy Véhel, dont nous avons donc enlevé la partie introductive.

Le chapitre 5, recopié de [12] et écrit en collaboration avec Julien Barral, ainsi que le chapitre 6, tiré de [79] et écrit seul, sont reproduits dans leur intégralité. Ils traitent de sujets plus spécifiques liés à l'analyse multifractale des fonctions.

Chapitre 2

Étude de la fonction d'Hölder locale d'une fonction continue

Abstract

This work focuses on the local Hölder exponent as a measure the regularity of a function around a given point. We investigate in detail the structure and the main properties of the local Hölder function (i.e. the function that associates to each point its local Hölder exponent). We prove that it is possible to construct a continuous function with prescribed local *and* pointwise Hölder functions outside a set of Hausdorff dimension 0.

2.1 The structure of Hölder functions

One can associate to each x its pointwise Hölder exponent $h_f(x)$. This defines a function $x \to h_f(x)$, called the pointwise Hölder function of f. A natural question is to investigate the structure of the functions $h_f(x)$ when f spans the set of continuous functions. The answer is given by the following theorem ([21]).

Theorem 2.1 *Let $g : \mathbb{R} \to \mathbb{R}^+$ be a function. The two following properties are equivalent :*
- *g is a liminf of a sequence of continuous functions,*
- *There exists a continuous function f such that the pointwise Hölder function of f $h_f(x)$ satisfies $h_f(x) = g(x)$, $\forall x$.*

As in the case of the pointwise exponent, one can associate to each x the local exponent of f at x. This defines a local Hölder function $x \to \alpha_l(x)$. The structure of local Hölder functions is more constrained than the one of pointwise Hölder functions, since the former must be lower semi-continuous functions ([34]). More precisely, we have :

Theorem 2.2 *Let $g : \mathbb{R} \to \mathbb{R}^+$ be a function. The two following properties are equivalent :*
- *g is a non-negative lower semi-continuous (lsc) function.*
- *There exists a continuous function f such that the local Hölder function of f, $\alpha_l(x)$, satisfies $\alpha_l(x) = g(x)$, $\forall x$.*

Proof : From the definition of $\alpha_l(x_0)$, for all $\varepsilon > 0$, there exists an interval I_ε containing x_0 such that

$$\alpha_l(I_\varepsilon) > \alpha_l(x_0) - \varepsilon.$$

Then, using the definition of $\alpha_l(y)$ for every $y \in I_\varepsilon$, one concludes that

$$\forall y \in I_\varepsilon,\ \alpha_l(y) \geq \alpha_l(I_\varepsilon) \geq \alpha_l(x_0) - \varepsilon.$$

This exactly shows that $x \to \alpha_l(x)$ is an lsc function. Obviously, the continuity of f entails $\alpha_l \geq 0$.

That the converse property holds, i.e. any non-negative lsc function is the local Hölder function of a continuous function $f : \mathbb{R} \to \mathbb{R}$, will be a consequence of theorem 2.3 below. ∎

Now that we have discussed the structures of both α_l and h_f, we proceed to examine the relation between them.

2.2 Relations between α_l and h_f

We start with two simple general bounds.

Proposition 2.1 *Let $f : I \to \mathbb{R}$ be a continuous function (I is an interval of $\mathbb{R}$). Let h_f and α_l be respectively its pointwise and local Hölder functions. Then, $\forall x \in I$,*

$$\alpha_l(x) \leq \min(h_f(x), \liminf_{t \to x} h_f(t)). \tag{2.1}$$

Proof : We give the proof in the case $h_f < 1$.
By definition, $\forall \varepsilon$, there exists a constant C such that, for t close enough to x, $|f(t) - f(x)| \leq C|t-x|^{h_f(x)-\varepsilon}$. Comparing this to the definition of $\alpha_l(x)$, one deduces that $\alpha_l(x) \leq h_f(x) - \varepsilon$, $\forall \varepsilon$, hence $\alpha_l(x) \leq h_f(x)$.

On the other hand, for every $\eta > 0$, $\forall y \in B(x, \eta)$, one has $\alpha_l(B(x, \eta)) \leq h_f(y)$. Combining this with the fact that $\alpha_l(x) = \lim_{\eta \to 0} \alpha_l(B(x, \eta))$, one obtains that $\alpha_l(x) \leq \liminf_{t \to x} h_f(t)$. ∎

Proposition 2.2 *Let $f : I \to \mathbb{R}$ be a continuous function (I is an interval of $\mathbb{R}$). If there exists α such that $\{x : h_f(x) = \alpha\}$ is dense around x_0, then $\alpha_l(x_0) \leq \alpha$.*

Proof : The proof is straightforward using Proposition 2.1. ∎

This proposition has an important consequence in multifractal analysis : "multifractal" functions, as IFS (see below and [21]) or repartition functions of multinomial measures [25], usually have the property that, for all α, $E_\alpha = \{x : h_f(x) = \alpha\}$ is either dense on the support of the function or empty. For functions of this kind, α_l is constant. A consequence is that it is not interesting in general to base a multifractal analysis on the local Hölder exponent, since the corresponding spectrum would be degenerate.

Let us now make a few remarks that go against some common thoughts about the relation between local and pointwise Hölder exponents.

– $x \to h_f(x)$ is a continuous function does not imply that $\alpha_l(x) = h_f(x)$ for every x. For a counter-example, consider the sum of a Weierstrass function with pointwise exponent α and a chirp (α, β) at 0, where $\beta < \alpha$. Then $\alpha_l(x) = h_f(x) = \alpha$ for all $x \neq 0$, and $h_f(0) = \alpha$ while $\alpha_l(0) = \beta < \alpha$.
– The converse proposition is also false : $x \to \alpha_l(x)$ is a continuous function does not imply that $\alpha_l(x) = h_f(x)$ for every x : Any well-chosen IFS has a constant local Hölder exponent while $x \to h_f(x)$ is everywhere discontinuous.

We now move to a different kind of relation between h_f and α_l. The following proposition assesses that the two exponents can not differ everywhere :

Proposition 2.3 *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval of $\mathbb{R}$. Assume that there exists $\gamma > 0$ such that $f \in C^\gamma(I)$. Then there exists a subset D of I such that :*
– D is dense, uncountable and has Hausdorff dimension 0.
– $\forall x \in D$, $h_f(x) = \alpha_l(x)$.
Furthermore, this result is optimal, i.e. there exist functions with global Hölder regularity $\gamma > 0$ such that $h_f(x) \neq \alpha_l(x)$ for all x outside a set of Hausdorff dimension 0.

Proof : We give the proof of the last Proposition in the case $\forall x$, $h_f(x) \leq 1$. The general result follows with similar arguments.

Let us consider a ball $B(x_0, \eta_0) \subset I$. We construct three sequences of points $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$ by the following method.

Let $\{\varepsilon_n\}_n$ be a positive sequence converging to 0 when $n \to +\infty$. Let us denote by β_0 the real number $\alpha_l(B(x_0, \eta_0/2))$. By definition of α_l, there exist two real number y_1 and z_1 such that

$$y_1 \in B(x_0, \eta_0/2), z_1 \in B(x_0, \eta_0/2),$$
$$y_1 < z_1 \text{ and } |f(y_1) - f(z_1)| > |y_1 - z_1|^{\beta_0 + \varepsilon_0}.$$

Let us now denote by x_1 the middle point of $[y_1, z_1]$, and by η_1 the number $\min(2^{-1}, |y_1 - z_1|/2)$.

Now consider the smaller ball $B(x_1, \eta_1/2)$, and its associated Hölder exponent $\beta_1 = \alpha_l(B(x_1, \eta_1/2))$. There exist two real numbers y_2 and z_2 such that

$$y_2 \in B(x_1, \eta_1/2), z_2 \in B(x_1, \eta_1/2),$$
$$y_2 < z_2 \text{ and } |f(y_2) - f(z_2)| > |y_2 - z_2|^{\beta_1 + \varepsilon_1}.$$

We denote by x_2 the middle point of $[y_2, z_2]$, and by η_2 the real number $\min(2^{-2}, |y_2 - z_2|/2)$.

We iterate this construction scheme, and thus obtain the desired three sequences $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$.

Now one easily proves that
– The sequence $\{x_n\}_n$ converges to a real number x.
– The sequences $\{y_n\}_n$ and $\{z_n\}_n$ also converge to x.

– For all n, one has the inequalities

$$\begin{array}{rcl} \frac{|y_n - z_n|}{4} \leq & |x - y_n| & \leq |y_n - z_n|, \\ \frac{|y_n - z_n|}{4} \leq & |x - z_n| & \leq |y_n - z_n|. \end{array}$$

One can sum up these inequalities by writing

$$\forall n,\ |x - y_n| \sim |x - z_n| \sim |y_n - z_n|. \tag{2.2}$$

Let us now study the local and pointwise Hölder exponents of the limit point x, respectively denoted by β_x and α_x. Since $f \in C^\gamma([0,1])$, one has $\gamma \leq \beta_x \leq \alpha_x$.

Remark that the sequence $\{\beta_n\}_n$ is non-decreasing, since the intervals $B(x_n, \eta_n/2)$ are embedded. By Proposition 2.2, one has $\beta_x = \lim_n \beta_n$. Indeed, since one can choose any decreasing sequence of open sets converging to x, one specifically chooses the interval $B(x_n, \eta_n/2)$ (the converge of β_n is ensured by the fact than one always has $\beta_n \leq \alpha_x$).

Let us now turn to the pointwise Hölder exponent. For every $\varepsilon > 0$, there exist $\eta > 0$ and a constant C such that, $\forall y \in B(x, \eta)$, one has $|f(x) - f(y)| \leq C|x - y|^{\alpha_x - \varepsilon}$. On the other hand, there exists an infinite number of couples (y_n, z_n) such that $y_n \in B(x, \eta)$ and $z_n \in B(x, \eta)$. For those couples, one can write

$$|f(y_n) - f(z_n)| \geq |y_n - z_n|^{\beta_n + \varepsilon_n}$$

and, on the other side

$$\begin{aligned} |f(y_n) - f(z_n)| &\leq |f(y_n) - f(x)| + |f(x) - f(z_n)| \\ &\leq C|y_n - x|^{\alpha_x - \varepsilon} + C|x - z_n|^{\alpha_x - \varepsilon} \\ &\leq C|y_n - z_n|^{\alpha_x - \varepsilon}, \end{aligned}$$

where one has used (2.2).

Assume now that $\beta_x < \alpha_x$, and let us take $\varepsilon < \frac{\alpha_x - \beta_x}{4}$. Since $\lim_n \beta_n + \varepsilon_n = \beta_x$, there exists N such that $n \geq N$ implies $\beta_n + \varepsilon_n \leq \alpha_x - 2\varepsilon$. For such n's, one has

$$\begin{aligned} \forall n \geq N,\ C|y_n - z_n|^{\alpha_x - 2\varepsilon} \leq C|y_n - z_n|^{\beta_n + \varepsilon_n} &\leq |f(y_n) - f(z_n)| \\ \text{and } |f(y_n) - f(z_n)| &\leq C|y_n - z_n|^{\alpha_x - \varepsilon}, \end{aligned}$$

which gives

$$\forall n \geq N,\ C|y_n - z_n|^{\alpha_x - 2\varepsilon} \leq C|y_n - z_n|^{\alpha_x - \varepsilon}.$$

Since $|y_n - z_n| \to 0$ when n goes to infinity, this is absurd.

One concludes $\alpha_x = \beta_x$ for the x we have found.

A simple modification of the above construction shows that the set $\{x : h_f(x) = \alpha_l(x)\}$ is uncountable. Indeed, starting from the interval $I_0 = [y_0, z_0]$, one can split it into 5 equal parts.

Focus now on the second and the forth subintervals, and apply the construction we have described above. One thus obtains two subintervals I_1^1 (the "left" one) and I_1^2 (the "right" one). Iterating this scheme, at each stage n, one obtains 2^n distinct intervals I_n^i, $i \in \{1, 2, \ldots, 2^n\}$. Using this method one constructs a Cantor set C_f. It is easy to see that it is uncountable, and that each point $x \in C_f$ still satisfies $h_f(x) = \alpha_l(x)$.

Finally, both the optimality and the fact that the set where the exponents coincide has Hausdorff dimension 0 are a consequence of Theorem 2.3 below. Alternatively, one may consider the case of an IFS, for which one has $\alpha_l(x) = h_f(x)$ exactly on a dense uncountable set of dimension 0. More precisely, consider an (attractor of an) IFS defined on $[0, 1]$, verifying the functional identity :

$$f(x) = c_1 f(2x) + c_2 f(2x - 1) \tag{2.3}$$

where $0.5 < |c_1| < |c_2| < 1$. It is known that for such a function, $\alpha_l(t) = -log_2(|c_2|)$ for all t. Furthermore (see [21]), $h_f(t)$ is everywhere discontinuous, and ranges in the interval $[-log_2(|c_2|), -log_2(|c_1|)]$. Finally, for all α in this interval, the set of t for which $h_f(t) = \alpha$ is dense in $[0, 1]$. This is thus an example where the local and pointwise exponents have drastically different behaviors, with a constant α_l and a wildly varying h_f. It is easy to show that the set D on which $h_f(t) = \alpha_l(t) = -log_2(|c_2|)$ is exactly the set of points for which the proportion of 0 in the dyadic expansion is 1. That this set D is dense, uncountable, and of Hausdorff dimension 0 is a classical result in number theory. ■

So far, we have proved that α_l must be not larger than h_f in the sense made precise by proposition 2.1, and that both exponents must coincide at least on a subset of a certain "size". Are there other constraints that rule the relations between α_l and h_f ? The following theorem essentially answers in the negative :

Theorem 2.3 *Let $\gamma > 0$, $f : [0, 1] \to [\gamma, +\infty)$ a liminf of continuous functions, with $\|f\|_\infty < +\infty$, and $g : [0, 1] \to [\gamma, +\infty)$ a lower semi-continuous function. Assume the* compatibility *condition, i.e. $\forall t \in [0, 1]$, $f(t) \geq g(t)$. Then there exists a continuous function $F : [0, 1] \to \mathbb{R}$ such that :*
- *for all x, $\alpha_l(x) = g(x)$,*
- *for all x outside a set D of Hausdorff dimension 0, $h_F(x) = f(x)$*

We prove this theorem in the next section, by explicitly constructing F.

2.3 Joint prescription of the Hölder functions

2.3.1 The case where α_l is constant

We are going in this section to present a function whose local Hölder function is constant, and whose pointwise Hölder function is everywhere constant (and thus equal to the local Hölder exponent) except at 0, where $h_f(0) > h_f(x)$, $x \neq 0$. This is the "inverse" case of a cusp or a chirp, where the regularity at a single point is lower than at all the other points.

This construction is paving the way to the more general result we will prove in the next section.

Proposition 2.4 *Let $0 < \beta < \alpha$ be two real numbers. Then there exists a function $f :]-1,1[\to \mathbb{R}$ such that $\forall x \neq 0, h_f(x) = \beta$ and $h_f(0) = \alpha$. Moreover, one has $\alpha_l(x) = \beta$, $\forall x \in]-1,1[$.*

Proof : The existence of such a function is obvious : take for example the function

$$F_W : \ x \to |x|^{\alpha-\beta} W_\beta(x),$$

where W_β is the Weierstrass function

$$W_\beta(x) = \sum_{n=1}^{+\infty} 2^{-n\beta} \sin(2\pi 2^n x). \tag{2.4}$$

We will exhibit another function f with the same property. This function is built using a wavelet method that can be generalized to prescribe arbitrary Hölder functions.

First we are going to select some particular couples (j,k) among the whole set of indices $\{(j,k)\}_{j\in\mathbb{N},k\in\mathbb{Z}}$. To achieve this, consider the function g defined by

$$g : x \to \begin{array}{ll} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{array}$$

It is known that this function is infinitely differentiable at 0, and that one has $\forall k \in \mathbb{N}, g^{(k)}(0) = 0$.

For all $n \in \mathbb{N}^*$, choose one integer $i \in \{1,\dots,2^n\}$, and define

$$p_{i,n} = \frac{g(i2^{-n})}{2^n}. \tag{2.5}$$

Consider the unique integer j such that $1 \leq 2^j p_{i,n} < 2$, and define another (unique) integer $k = i2^{j-n}$.

We have thus built a function, which associates with each couple (n,i) (where $n \geq 1$ and $i \in \{1,\dots,2^n\}$) another couple of indices (j,k). Let us denote by Γ this set of selected indices.

Let us define the following set of wavelet coefficients :

$$\begin{aligned} d_{j,0} &= 2^{-j\alpha}, \forall j, \\ d_{j,k} &= 2^{-j\beta}, \text{ if } (j,k) \in \Gamma, \\ d_{j,k} &= 0 \text{ everywhere else.} \end{aligned}$$

We add, in a uniform manner, some larger coefficients along exponential curves in the time-frequency domain..

We can define a function f by the reconstruction formula

$$f = \sum_j \sum_k d_{j,k} \psi_{j,k}. \tag{2.6}$$

Let us now prove that this function satisfies the desired properties.

First this function is well defined, since, $\forall (j,k), |d_{j,k}| \leq 2^{-j\beta}$. By the theorem of Jaffard, f is at least $C^\beta(x)$, for all $x \in]-1,1[$.

Case $x \neq 0$.
$\forall j, \forall k$, one has $|d_{j,k}| \leq 2^{-j\beta}$. Thus $h_f(x) \geq \beta$.

The proof of $h_f(x) \leq \beta$ is more delicate. For each integer n, define the unique integer i_n verifying $i_n 2^{-n} \leq x < (i_n+1)2^{-n}$. When $n \to +\infty$, $i_n 2^{-n} \to x$, and, since g is continuous, $g(i_n 2^{-n}) \sim g(x)$. The associated couple (j,k) satisfies

$$\begin{aligned} k2^{-j} &= i_n 2^{-n} \\ 1 \leq \frac{g(i_n 2^{-n})}{2^n} 2^j &< 2 \end{aligned}$$

One can rewrite the last inequality in

$$g(i_n 2^{-n}) 2^{-n-1} \leq 2^{-j} \leq g(i_n 2^{-n}) 2^{-n},$$

or equivalently, using that $g(i_n 2^{-n}) \sim g(x)$ when n goes to infinity, and taking the logarithm,

$$n + C_x \leq j \leq (n+1) + C_x,$$

where C_x is a constant depending only on x.
Now, for the associated couple (j,k), one has

$$\begin{aligned} 2^j |x - k2^{-j}| &\leq C 2^{n+1} |x - k2^{-j}| \\ &\leq C 2^{n+1} |x - i_n 2^{-n}| \\ &\leq C2, \end{aligned}$$

since by construction $|x - i_n 2^{-n}| \leq 2^{-n}$. Thus for such couples (j,k), one has exactly

$$d_{j,k} = 2^{-j\beta} \sim 2^{-j\beta}(1 + 2^j |x - k2^{-j}|)^\beta. \tag{2.7}$$

Hence the inequality $\forall j,k, |d_{j,k}| \leq C 2^{-j\beta}(1 + 2^j |x - k2^{-j}|)^\beta$ is optimal, and $h_f(x) \leq \beta$. One concludes $h_f(x) = \beta$, since we already showed $h_f(x) \geq \beta$.

Case $x = 0$.
One notices first that, by construction, for $k = 0$, $d_{j,0} = 2^{-j\alpha}$, thus $h_f(0) \leq \alpha$.
If $k \neq 0$, $d_{j,k} = 0$, except if there exists an integer $n \geq 1$, and an integer $i \in \{1, \dots, 2^n\}$, such that

$$\begin{aligned} k2^{-j} &= i2^{-n}, \\ 1 \leq 2^j \tfrac{g(i2^{-n})}{2^n} &< 2. \end{aligned}$$

Then, for this kind of indices (j,k),

$$\begin{aligned} |d_{j,k}| &= 2^{-j\beta} \leq (2^{-n} g(i2^{-n}))^\beta \\ &\leq (i2^{-n})^\beta (g(i2^{-n}))^\beta. \end{aligned}$$

But, using the structure of the function g, there exists a constant C (independent of x) such that, $\forall x > 0,\ g(x) \leq C|x|^{\frac{\alpha+1}{\beta}}$.

Thus

$$\begin{aligned} |d_{j,k}| &\leq C(|i2^{-n}|)^{\beta}(|i2^{-n}|^{\frac{\alpha+1}{\beta}})^{\beta} \\ &\leq C|i2^{-n}|^{\alpha+\beta+1} \\ &\leq C|k2^{-j}|^{\alpha+\beta+1} \\ &\leq C2^{-j(\alpha+\beta+1)}(1+|k|)^{\alpha+\beta+1}. \end{aligned}$$

This proves that these coefficients, which are larger than $2^{-j\alpha}$, are nevertheless seen as very regular ones from the point 0. The main contribution to the pointwise regularity is thus given by the wavelet coefficients that are located at 0, the $d_{j,0}$. One concludes $h_f(0) = \alpha$.

To end the proof, we need to prove that $\alpha_l(x) = \beta$, $\forall x \in]-1, 1[$. This is easily done. Indeed, using the characterization given by (1.9), one obtains that $\forall x \neq 0$, $\alpha_l(x) = \beta$. At 0, one can still write $\alpha_l(0) \geq \beta$, but on the other hand one uses (2.1) and concludes that $\alpha_l(0) \leq \liminf_{x\to 0} \alpha_l(x) = \beta$. This concludes the proof. ■

2.3.2 The general case

In the last section, we have built a function whose pointwise exponent at 0 was larger than all the other ones. In particular, at 0, we have forced the local exponent to be equal to a given value β, while at the same time the pointwise exponent was forced to be larger than β. The next step is to be able to do this uniformly, on a set of x as large as possible. The purpose of this subsection is to prove the theorem stated in section 2.2 that we recall here for convenience :

Theorem 4.1

Let $0 < \gamma < 1$, $f : [0,1] \to [\gamma, +\infty)$ a liminf of continuous functions, with $\|f\|_\infty < +\infty$, and $g : [0,1] \to [\gamma, +\infty)$ a lower semi-continuous function. Assume the compatibility *condition, i.e. $\forall t \in [0,1]$, $f(t) \geq g(t)$. Then there exists a continuous function $F : [0,1] \to \mathbb{R}$ such that :*

$$\text{for all } x,\ \alpha_l(x) = g(x), \tag{2.8}$$

$$\text{Outside a set } D \text{ of Hausdorff dimension } 0,\ h_F(x) = f(x). \tag{2.9}$$

Let us make a few remarks.

- The proof is a kind of generalization of the method used in Proposition 2.4. We are going to enlarge some coefficients, but this time we are going to do this "uniformly" and not only around a single point.
- Our construction introduces an asymmetry between the local and the pointwise exponent : one can prescribe *everywhere* the local exponent, while one can not do the same thing at the same time (with this construction) for the pointwise exponent. We believe that this restriction is not intrinsic, and is only a consequence of the approach we have taken.

– Eventually, we will see that, applying the method we introduce, one can prescribe the pointwise exponent everywhere except on a set of Hausdorff dimension 0. This restriction is weaker that the one which occurs when one wants to prescribe at the same time the *chirp* and the pointwise Hölder exponent : S. Jaffard has proved in [46] that, in this frame, the excluded set is of Lebesgue measure 0. Working with the local Hölder exponent thus allows more flexibility.

Proof : We shall one more time construct the function by a wavelet method.

First we are going to construct some specific approximations sequences of continuous functions that will approximate the functions f and g.

By definition, one knows that there exist two sequences of continuous functions $\{f_n^0\}_n$ and $\{g_n^0\}_n$ such that

$$\liminf_n f_n^0 = f, \tag{2.10}$$

$$\sup_n g_n^0 = g. \tag{2.11}$$

We will use the two following lemmas, that roughly say that one can *slow down* the speed of convergence of these two sequences.

Lemma 2.1 *Let f be a liminf of continuous functions. Then there exists a sequence of polynomials f_n^1 that verifies*

$$\begin{aligned} f(t) &= \liminf_n f_n^1(t), \ \forall t \in [0,1], \\ \|(f_n^1)'(t)\|_{L^\infty} &\leq \log n, \ \forall n \geq 1 \text{ and } t \in [0,1]. \end{aligned}$$

The proof of this fact can be found in [42] or [21].

Lemma 2.2 *Let g be an lsc function. Then there exists a sequence of polynomials g_n^1 that verifies*

$$\begin{aligned} g(t) &= \sup_n g_n^1(t), \ \forall t \in [0,1], \\ \|(g_n^1)'(t)\|_{L^\infty} &\leq \log n, \ \forall n \geq 1 \text{ and } t \in [0,1]. \end{aligned}$$

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Proof : This is a little bit more complicated. First let us define, for all n and x, $g_n^2(x) = \max_{p\leq n}\{g_p(x)\}$. One still has $g(x) = \sup_n g_n^2(x)$. One also has $g(x) = \sup_n g_n^3(x)$ with $g_n^3(x) = g_n^2(x) - 1/n$.

For each $n > 0$, there exists a polynomial P_n such that $\|g_n^3 - P_n\|_{L^\infty} \leq 2^{-n}$. One has thus built a sequence of polynomials such that $g = \sup_n P_n$.

One can now, by the same method as in Lemma 2.1, slow down the sequence $\{P_n\}_n$ such that it will satisfy the desired conditions. ∎

We now set the desired sequences $\{f_n\}_n$ and $\{g_n\}_n$ by

$$\begin{aligned} g_n(t) &= \max_{p\leq n}(g_p^1(t), \gamma/2) \\ f_n(t) &= \max(f_n^1(t), g_n(t) + \frac{1}{n}). \end{aligned}$$

They verify the following properties

- They still respectively satisfy (2.10) and (2.11).
- For each n, the right and left derivatives of g_n and f_n at each point $x \in [0,1]$ are lower than $\log n$, since they are maxima of a finite number of polynomials of derivative lower than $\log n$.
- g_n is non-decreasing, i.e. $\forall t \in [0,1]$, $\{g_n(t)\}_n$ is an non-decreasing sequence of real numbers.

- One has the inequality $f_n \geq g_n$ for all $n \in \mathbb{N}^*$.

We are now going to select some couples of indices, which will be the basis of our construction of a function F satisfying (2.8) and (2.9).

For $n \in \{1,2,3,\ldots\}$, and $i \in \{1,2,3,\ldots,2^{n-1}\}$, let us define the two integers j_n and $k_{n,i}$ by

$$\begin{aligned} j_n &= 2^n \\ k_{n,i} &= 2^{j_n}\frac{2i-1}{j_n}. \end{aligned}$$

At each n, one obtains 2^{n-1} couples, which are uniformly distributed on $[0,1]$ in the sense that the $x_{n,i} = k_{n,i}2^{-j_n} = \frac{2i-1}{j_n}$ are uniformly distributed on $[0,1]$. We denote by Λ the set of these selected couples $(j_n, k_{n,i})$.

We are now ready to construct the wavelet coefficients of F. We define

$$\begin{aligned} d_{j,k} &= 2^{-jg_j(x_{n,i})} = 2^{-jg_j(k_{n,i}2^{-j_n})} \text{ if } (j,k) \in \Lambda, \\ d_{j,k} &= 2^{-jf_j(x_{n,i})} \text{ everywhere else.} \end{aligned}$$

The operation we are doing is a re-scaling of some coefficients, according to the local regularity.

Remark that for all (j,k), $|d_{j,k}| \leq 2^{-j\gamma/2}$, thus

$$F(x) = \sum_j \sum_k d_{j,k}\psi_{j,k}(x)$$

is well defined and is $C^{\gamma/2}([0,1])$.

Local Hölder exponent

Let $x_0 \in [0,1]$, and $\varepsilon > 0$. One has $g(x_0) = \sup_n g_n(x_0)$, thus there exists an integer N_1 such that $n \geq N_1 \Rightarrow g_n(x_0) > g(x_0) - \varepsilon/2$. Let N_2 be an integer such that $\log(N_2)2^{-N_2} \leq \varepsilon/2$. Define $N = \max(N_1, N_2)$. Then, using the boundedness of the derivatives of g_N, if $\eta = 2^{-N}$, one obtains $\forall y \in B(x_0, \eta)$,

$$|g_N(y) - g_N(x_0)| \leq (\log N)|y - x_0| \leq (\log N)2^{-N} \leq \varepsilon/2,$$

and thus $\forall y \in B(x_0, \eta)$,

$$g_N(y) \geq g_N(x_0) - \varepsilon/2.$$

One thus has $g_N(y) \geq g_N(x_0) - \varepsilon/2 \geq g(x_0) - \varepsilon$, and since the sequence g_n is non-decreasing, the last property is still true for any g_n, $n \geq N$. One obtains the key property :

$$\forall y \in B(x_0, \eta),\ \forall n \geq N,\ g_n(y) \geq g(x_0) - \varepsilon, \tag{2.12}$$

Consider now the wavelet coefficients $d_{j,k}$ such that their support is included in $B(x_0, \eta)$ (these coefficients are the ones one shall focus on to compute $\alpha_l(B(x_0, \eta))$). There are two sorts of such coefficients

– the "normal" ones, those which do not belong to Λ. One can write for them

$$|d_{j,k}| \leq 2^{-jf_j(k2^{-j})} \leq 2^{-jg_j(k2^{-j})} \leq 2^{-j(g(x_0)-\varepsilon)}.$$

– those which belong to Λ. For them,

$$|d_{j,k}| \leq 2^{-jg_n(x_{n,i})} \leq 2^{-j(g(x_0)-\varepsilon)}.$$

Eventually, for all the interesting couples of coefficients (j, k), $|d_{j,k}| \leq 2^{-j(g(x_0)-\varepsilon)}$. One concludes $\alpha_l(B(x_0, \eta)) \geq g(x_0) - \varepsilon$. The result is clearly still true on every ball $B(x_0, \eta_1)$ with $\eta_1 \leq \eta$, thus one has $\alpha_l(x_0) \geq g(x_0) - \varepsilon$.

On the other hand, $\forall n > 0$, consider the unique integer i that verifies $x_{n,i} = k_{n,i}2^{j_n} \in [x_0 - j_n^{-1}, x_0 + j_n^{-1}]$. Then, using the boundedness of the derivatives of g_n, one can write

$$|g_{j_n}(x_{n,i}) - g_{j_n}(x_0)| \leq \log(j_n)j_n^{-1} \leq n2^{-n}.$$

Let N_3 be such that $N_3 2^{-N_3} \leq \varepsilon/2$. For $n \geq \max(N_3, N)$ (where N has been above defined), one has

$$g_{j_n}(x_{n,i}) \leq g_{j_n}(x_0) + \varepsilon/2 \leq g(x_0) + \varepsilon \tag{2.13}$$

There is an infinite number of such couples (n, i), whose associated wavelet coefficients satisfy

$$|d_{j,k}| = |d_{j_n,k_{n,i}}| = 2^{-j_n g_{j_n}(x_{n,i})} \geq 2^{-j_n(g(x_0)+\varepsilon)}. \tag{2.14}$$

Now, by Proposition 1.8, $\alpha_l(B(x_0, \eta)) \leq g(x_0) + \varepsilon$. Since, one more time, this is also true for any $\eta_1 \leq \eta$, one has $\alpha_l(x_0) \leq g(x_0) + \varepsilon$.

Eventually, $\alpha_l(x_0) = g(x_0)$.

Pointwise Hölder exponent

The estimation of this exponent is more complicated. Let $x_0 \in [0, 1]$, and $\varepsilon > 0$.

Without the rescaled coefficients (i.e. if the $d_{j_n,k_{n,i}}$ were all equal to $2^{-j_n f_{j_n}(x_{n,i})}$), it has been proved in [21] that $\forall x$, $h_F(x) = f(x)$. The question is : do we change something when we modify the values of these specific coefficients ? The modifications may have big influence on regularity, because the new coefficients are larger than the "normal" ones (indeed, remember that $f(x) \geq g(x)$).

We will show that in fact, the rescaled coefficients are not seen by most of the points x. Thus, for such points, one still has $h_F(x) = f(x)$.

Let us define the set E_M by

$$E_M = \{x : \exists C, \exists N_x, \forall n \geq N_x, \forall i, |x - \frac{2i-1}{2^n}| \geq C2^{-2^n \frac{\gamma}{M}}\}, \tag{2.15}$$

where M verifies $M \geq \|f\|_\infty$. Let x_0 be in E_M. Since $x_{n,i} = \frac{2i-1}{2^n}$, one has, for every i and $n \geq N_x$,

$$2^{-2^n \frac{\gamma}{M}} \leq C|x_0 - x_{n,i}|, \tag{2.16}$$

or equivalently, replacing j_n and $k_{n,i}$ by their values,

$$2^{-j_n \frac{\gamma}{M}} \leq C|x_0 - k_{n,i}2^{-j_n}|.$$

We know that $\gamma \leq g_{j_n}$ and $f(x_0) < M$ by construction, thus $\forall y \in [0,1]$, $\frac{g_{j_n}(y)}{f(x_0)} \geq \frac{\gamma}{M}$, and for every i and n,

$$2^{-j_n \frac{g_{j_n}(y)}{f(x_0)}} \leq C|x_0 - k_{n,i}2^{-j_n}|.$$

This is equivalent to

$$2^{-j_n g_{j_n}(x_{n,i})} \leq C|x_0 - k_{n,i}2^{-j_n}|^{f(x_0)},$$

which implies

$$\begin{aligned} 2^{-j_n g_{j_n}(x_{n,i})} &\leq C2^{-j_n f(x_0)}(2^{j_n}|x_0 - k_{n,i}2^{-j_n}|)^{f(x_0)}, \\ &\leq C2^{-j_n f(x_0)}(1 + 2^{j_n}|x_0 - k_{n,i}2^{-j_n}|)^{f(x_0)}. \end{aligned}$$

But $d_{j_n,k_{n,i}} = 2^{-j_n g_{j_n}(x_{n,i})}$, hence, for any $x_0 \in E_M$, there exists a constant C such that

$$|d_{j_n,k_{n,i}}| \leq C2^{-f(x_0)j_n}(1 + 2^{j_n}|x_0 - k_{n,i}2^{-j_n}|)^{f(x_0)}. \tag{2.17}$$

This shows that, if $x_0 \in E_M \cap [0,1], \forall n \geq N_x, \forall p$, one has (2.17), which ensures $h_F(x_0) = f(x_0)$. The large coefficients, those which are rescaled, are not "seen" by the pointwise Hölder exponent at x_0.

To end the proof, it is sufficient to measure the size of E_M. We prove in Section 2.4 that the complementary set D_M of the set E_M has Hausdorff dimension 0. Moreover, any rational number $x = p/q$ belongs to E_M.

■

Remark : One cannot say anything about the x's that are in $D_M = [0,1] \backslash E_M$, except that for such points x, $g(x) = \alpha_l(x) \leq h_F(x)$. Nevertheless some of them must satisfy $h_F(x) = \alpha_l(x)$ even if the functions f and g satisfy $f(y) > g(y)$ for all y in $[0,1]$.

Remark : Combining the construction we used with the construction due to S. Jaffard in [46], one can certainly prescribe, outside a set of Hausdorff dimension 1 but of Lebesgue measure 0, three different regularity exponents at the same time : the local Hölder exponent, the pointwise Hölder exponent, and the *chirp* exponent (cf [71]). This is a first step towards a more complete prescription of the regularity of a function. See [62] for more on this topic.

2.4 Study of the set E_M

We begin by computing the Hausdorff dimension of the complementary set of E_M

Proposition 2.5 *For all $M > 0$, the Hausdorff dimension of the set D_M defined by*

$$D_M = [0,1] \backslash E_M \tag{2.18}$$

is 0.

Proof : Let $M > 0$, $C > 0$, and define E_M^C by

$$E_M^C = \{x \in [0,1] : \exists N_x, \forall n \geq N_x, \forall i, |x - \frac{2i-1}{2^n}| \geq C2^{-2^n \frac{\gamma}{M}}\}, \tag{2.19}$$

or equivalently,

$$E_M^C = \{x \in [0,1] : \exists N_x \in \mathbb{N}, \;\; x \notin \cup_{n \geq N_x} F_n^C\}, \tag{2.20}$$

where

$$F_n^C = \cup_{i=1}^{2^{n-1}} B_{n,i}^C$$

and

$$B_{n,i}^C = \left] \frac{2i-1}{2^n} - C2^{-2^n \frac{\gamma}{M}}, \frac{2i-1}{2^n} + C2^{-2^n \frac{\gamma}{M}} \right[.$$

Let $D_M^C = [0,1] \backslash E_M^C$. D_M^C obviously satisfies

$$D_M^C = \cap_{N \in \mathbb{N}} \cup_{n \geq N} F_n^C.$$

Let $\varepsilon > 0$. One has

$$\begin{aligned} \sum_{n \geq N} \sum_{i=1}^{2^{n-1}} |B_{n,i}^C|^\varepsilon &\leq \sum_{n \geq N} 2^{n-1} |2C2^{-2^n \frac{\gamma}{M}}|^\varepsilon \\ &\leq C' 2^{-2^N \frac{\gamma}{M} \varepsilon + N - 1}, \end{aligned}$$

which goes to zero when N goes to infinity (C' is a constant independent of N). Since for all N, $\cup_{n \geq N} F_n^C$ is obviously a cover of D_M^C by balls of size $2^{-2^N \frac{\gamma}{M}}$, one has exactly shown that the ε-dimensional Hausdorff measure of D_M^C is 0, $\forall \varepsilon > 0$. We conclude that the Hausdorff dimension of D_M^C is 0.

Remark now that $D_M \subset \cap_{n \in \mathbb{N}^*} D_M^{1/n}$. D_M is thus also of Hausdorff dimension 0. ∎

In Theorem 4.1, one may choose, for all x, $f(x) = M > \gamma = g(x) > 0$. Using Proposition 2.3, we deduce that $D_M = [0,1] \backslash E_M$ must be dense and uncountable, otherwise α_l would be different from h_F on a too large set. This implies

Corollary 2.1 *D_M is uncountable and dense in* $[0,1]$.

We remark finally that our construction also allows to prescribe the pointwise Hölder exponent at any rational point (even at dyadic ones). Indeed,

Proposition 2.6 $\mathbb{Q} \cap [0, 1] \subset E_M$.

Proof : Let $x = \frac{p}{q}$ be a rational number.
For every $n \in \mathbb{N}$,

$$|x - \frac{2p-1}{2^n}| = |\frac{p}{q} - \frac{2p-1}{2^n}| = |\frac{2^n p - (2p-1)q}{q2^n}|.$$

Let us decompose the integer q as $q = 2^{n_x} q_1$, where q_1 is an odd integer. Thus, for $n \geq n_x + 1$,

$$2^n p - (2p-1)q = 2^{n_x}(2^{n-n_x} p - (2p-1)q_1) \neq 0,$$

since $2^{n-n_x} p$ is an even integer and $(2p-1)q_1$ is an odd integer. Consequently, $\forall n$ such that $2^n \geq q$,

$$|x - \frac{2p-1}{2^n}| = |\frac{2^n p - (2p-1)q}{q2^n}| \geq \frac{1}{q2^n} \geq (2^{-n})^2.$$

Thus $x \in E_M$ and Proposition 2.6 is proved. ∎

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Chapitre 3

Caractérisation temporelle des espaces 2-microlocaux

Abstract

In [54], new functional spaces, denoted $K_{x_0}^{s,s'}$, were introduced. These spaces characterize the fine local regularity of functions, much in the spirit of 2-microlocal spaces $C_{x_0}^{s,s'}$. In contrast with $C_{x_0}^{s,s'}$ spaces, however, $K_{x_0}^{s,s'}$ spaces are defined through simple estimations on the pointwise values of the functions. In this work, we generalize the definition of $K_{x_0}^{s,s'}$ spaces and prove the equality $C_{x_0}^{s,s'} = K_{x_0}^{s,s'}$ for $s + s' > 0$, $s > 0$.

Using this result, we propose an algorithm able to estimate a part of the 2-microlocal frontier. Experiments on sampled data show that reasonnable accuracy is achieved even for "difficult" functions such as continuous but nowhere differentiable ones. As a by-product, robust estimators of both the pointwise and the local exponents are obtained.

3.1 The Functional Spaces $K_{x_0}^{s,s'}$

3.1.1 Definition of $K_{x_0}^{s,s'}$ Spaces

$K_{x_0}^{s,s'}$ spaces were defined in [54] for nowhere differentiable functions. We extend here this definition to a wider range of exponents

Definition 3.1 *Let $x_0 \in \mathbb{R}$, and s, s' be two real numbers satisfying $s' \leq 0$ and $s + s' \geq 0$ (and thus $s \geq 0$).*

Let $m = [s+s']$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to belong to $K_{x_0}^{s,s'}$ if there exist $0 < \delta < 1/4$, a polynomial P of degree smaller than $[s] - m$, and a constant C, such that

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}} \right| \leq C|x - y|^{s+s'-m}(|x - y| + |x - x_0|)^{-s'-[s]+m} \tag{3.1}$$

for all x, y such that $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$.

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Let us make a few remarks on this definition.

– If $s+s'<1$ and $s<1$ (i.e. $m=[s]=0$), then the original definition of [54] is recovered

$$|f(x)-f(y)| \leq C|x-y|^{s+s'}(|x-y|+|x-x_0|)^{-s'}. \tag{3.2}$$

– If $m<s+s'<m+1$ and $s<m+1$ (i.e. $[s]=m$), one obtains a simpler formulation of the definition

$$|\partial^m f(x)-\partial^m f(y)| \leq C|x-y|^{s-m+s'}(|x-y|+|x-x_0|)^{-s'}. \tag{3.3}$$

– The right term in the above inequality seems to be asymmetric, but it is not. Remarking that $(|x-y|+|x-x_0|) \leq 2(|x-y|+|y-x_0|)$, this right term of (3.1) can be re-written as one of the two following expressions (the last one being symmetric in x and y)

$$|x-y|^{s-m+s'}(|x-y|+|y-x_0|)^{-s'-[s]+m},$$
$$|x-y|^{s-m+s'}\left((|x-y|+|x-x_0|)(|x-y|+|y-x_0|)\right)^{(-s'-[s]+m)/2}$$

– In the following, we will most of the time avoid the critical cases $s+s' \in \mathbb{N}$ and $s \in \mathbb{N}$. Indeed, they would require the use of Zygmund spaces instead of the usual homogeneous Hölder spaces $C^{\alpha}(\mathbb{R})$.

– If $g_m(x)$ denotes the function $\frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}}$, one easily sees that g_m is at least continuous at each point, especially at x_0. Indeed, if $-s'-[s]+m>0$, one writes

$$|g_m(x)-g_m(y)| \leq C|x-y|^{s+s'-m}$$

with $0<s+s'-m<1$. Thus $g_m \in C^{s+s'-m}$ around x_0. On the other hand, if $-s'-[s]+m \leq 0$, one has

$$|g_m(x)-g_m(y)| \quad \leq \quad C|x-y|^{s-[s]}\left(\frac{|x-y|}{|x-y|+|x-x_0|}\right)^{s'+[s]-m} \leq |x-y|^{s-[s]}$$

with $0<s-[s]<1$, thus $g_m \in C^{s-[s]}$ around x_0.

It is thus possible to take $y=x_0$ in (3.1) or in (3.2), and also to consider the real number $g_m(x_0)$.

– The left hand-side of (3.1) and the exponents in use may seem complex. The necessity of the different terms is however easily understood : one tries to reduce the study of f to the one of a new function derived from f that will belong to some $K_{x_0}^{t,t'}$ with $0 \leq t+t'<1$ and $0 \leq t<1$.

Roughly speaking, the subset of $\mathbb{R}^2$ $\{(s,s') : s+s' \geq 0,\ s' \leq 0 \text{ and } s>0\}$ is partitioned into tiles of same size, and the problem is translated to the "initial" tile $\{(s,s') : 0< s+s'<1 \text{ and } s<1\}$ (see figure 1).

For example, if a function f belongs to $K_{x_0}^{s,s'}$ with $m \leq s+s' < m+1$, f admits around x_0 a derivative of order m. The formula simply states that $f \in K_{x_0}^{s,s'}$ means that $\partial^m f \in K_{x_0}^{s-m,s'}$. Then, if $f \in K_{x_0}^{s,s'}$ with $0<s+s'<1$ with $s>1$, f is replaced in formula (3.2) by $\frac{f(x)-P(x)}{|x-x_0|^{[s]}}$, and this new function will belong to $K_{x_0}^{s-[s],s'+[s]}$.

These thoughts can be summarized by saying that the operator

$$f \to \frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}}$$

(for the correct polynomial P) maps $K_{x_0}^{s,s'}$ into $K_{x_0}^{s-[s],s'+[s]-m}$.

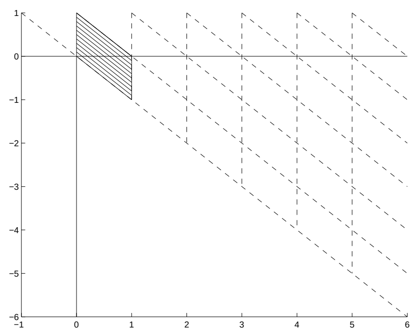

FIG. 3.1 – Paving of the half plane $s > 0$.

3.1.2 First Properties

We stress several interesting properties of these spaces $K^{s,s'}_{x_0}$. Propositions (3.1) to (3.4) are just extensions of the corresponding ones in [54] to the general case.

The first proposition is an embedding property between the spaces $K^{s,s'}_{x_0}$.

Proposition 3.1 *Let $x_0 \in \mathbb{R}$, and s, s', t, t' be four real numbers such that $t \le s$ and $t + t' \le s + s'$. Then $K^{s,s'}_{x_0} \subset K^{t,t'}_{x_0}$.*

The proof of Proposition 3.1 is splitted into two simpler lemmas

Lemma 3.1 *Let $x_0 \in \mathbb{R}$, and s, s', t, t' be four real numbers such that $t \le s$ and $t+t' = s+s'$. Then $K^{s,s'}_{x_0} \subset K^{t,t'}_{x_0}$.*

Proof : Let us treat the case $s + s' < 1$ (i.e. $m = 0$), the general case is then deduced by replacing f by $\partial^m f$ and s by $s - m$.

If $[t] = [s]$, the result is obvious. Let us treat the case $[t] = [s] - 1$, the general result will then easily follow by iteration.

Let us assume that $f \in K^{s,s'}_{x_0}$, $x_0 = 0$ and that $|y| \le |x|$, without loss of generality. One also assumes that $x > 0$, by replacing $z \to f(z)$ by $z \to f(-z)$. There exists a polynomial P such that (3.1) holds. One is now looking for a polynomial P_t that satisfies

$$\left| \frac{f(x) - P_t(x)}{|x|^{[s]-1}} - \frac{f(y) - P_t(y)}{|y|^{[s]-1}} \right| \le C|x-y|^{t+t'}(|x-y| + |x|)^{-t'-[t]} \tag{3.4}$$

Let us denote by P_t the polynomial of degree $[t]=[s]-1$ with the same coefficients as P up to degree $[t]$. To simplify the notations, let us define $g(x)=\frac{f(x)-P(x)}{|x|^{[s]}}$ and $g_t(x)=\frac{f(x)-P_t(x)}{|x|^{[s]-1}}$. Then,

$$\forall x,\ \ g_t(x)=|x|\,g(x)+\frac{P(x)-P_t(x)}{|x|^{[s]-1}},$$

Now,

$$|g_t(x)-g_t(y)|\leq\left|\left(|x|g(x)+\tfrac{P(x)-P_t(x)}{|x|^{[s]-1}}\right)-\left(|y|g(y)+\tfrac{P(y)-P_t(y)}{|y|^{[s]-1}}\right)\right|$$

By construction, $P(x)-P_t(x)$ is a polynomial with only one non-zero coefficient. Thus $P(x)-P_t(x)=ax^{[s]}$, and

$$\begin{aligned}|g_t(x)-g_t(y)| &\leq ||x|g(x)-|y|g(y)|+a\left|\frac{x^{[s]}}{|x|^{[s]-1}}-\frac{y^{[s]}}{|y|^{[s]-1}}\right|\\ &\leq ||x|(g(x)-g(0))-|y|(g(y)-g(0))|+(|a|+|g(0)|)|x-y|\end{aligned}$$

A useful remark is that

$$\begin{aligned}|x-y| &= |x-y|^{t+t'}|x-y|^{-t'+1-t}\\ &\leq |x-y|^{t+t'}(|x-y|+|x|)^{-t'+1-t}\\ &\leq |x-y|^{t+t'}(|x-y|+|x|)^{-t'-[t]}.\end{aligned}$$

(note that $1-t>-[t]$). A direct upper bound for g is obtained by taking $y=0$ in (3.1)

$$\forall x,\ \ |g(x)-g(0)|\leq C|x|^{s-[s]}. \tag{3.5}$$

Applying (3.5) and (3.1), the last term $(T)=||x|(g(x)-g(0))-|y|(g(y)-g(0))|$ is treated as follows

$$\begin{aligned}(T) &\leq ||x|-|y||g(x)-g(0)|+||y|(g(x)-g(0))-|y|(g(y)-g(0))|\\ &\leq |x-y||g(x)-g(0)|+|y||g(x)-g(y)|\\ &\leq C|x-y||x|^{s-[s]}+|y||x-y|^{s+s'}(|x|+|x-y|)^{-s'-[s]}\\ &\leq C|x-y|^{t+t'}|x-y|^{1-(t+t')}|x|^{s-[s]}\\ &\quad +|y||x-y|^{s+s'}(|x|+|x-y|)^{-s'-[s]}\end{aligned}$$

Since $s-[s]$ and $1-(t+t')$ are positive, one upper-bounds $|x|^{s-[s]}$, $|x-y|^{1-(t+t')}$ and $|y|$ respectively by $(|x|+|x-y|)^{s-[s]}$, $(|x|+|x-y|)^{1-(t+t')}$ and $(|x|+|x-y|)$. This gives, using $[t]=[s]-1$,

$$\begin{aligned}(T) &\leq C|x-y|^{t+t'}(|x|+|x-y|)^{-t'-[t]+s-t+2}\\ &\quad +|x-y|^{s+s'}(|x|+|x-y|)^{-s'-[t]+2}.\end{aligned}$$

Eventually, since $t+t'=s+s'$, one has $-s'-[t]+2=-t'-[t]+2+s-t$. One concludes, using $2+s-t>0$, that

$$|g_t(x)-g_t(y)|\leq C|x-y|^{t+t'}(|x|+|x-y|)^{-t'-[t]}$$

which gives the required result. ∎

Lemma 3.2 *Let $x_0 \in \mathbb{R}$, and s, s', t' be three real numbers such that $t' \leq s'$. Then $K^{s,s'}_{x_0} \subset K^{s,t'}_{x_0}$.*

Proof : If $[s+s'] = [s+t']$, the result is obvious.
Assume that $m = [s+s'] = [s+t']+1$, the general result will then easily follow. As usual now, we will assume without loss of generality that $x_0 = 0$ and $|y| \leq x$. By assumption, (3.1) holds, and one wants to prove

$$\left| \frac{\partial^{m-1} f(x) - P(x)}{|x|^{[s]-(m-1)}} - \frac{\partial^{m-1} f(y) - P(y)}{|y|^{[s]-(m-1)}} \right| \leq$$
$$C|x-y|^{s-(m-1)+t'}(|x-y|+|x|)^{-t'-[s]+(m-1)}.$$

Taking $y = 0$ in (3.1) yields

$$|\partial^m f(x) - P(x)| \leq C|x|^{s-m}$$

for a certain polynomial P of degree at most $[s]-m$. Integrating first this last inequality between 0 and x, and then between y and x, one obtains

$$|\partial^{m-1} f(x) - P_{t'}(x)| \leq C|x|^{s-m+1} \tag{3.6}$$

and

$$|(\partial^{m-1} f(x) - P_{t'}(x)) - (\partial^{m-1} f(y) - P_{t'}(y))| \leq C|x-y||x|^{s-m} \tag{3.7}$$

for a polynomial $P_{t'}$ of degree at most $[s]-m+1$ (the same one for both (3.6) and (3.7)). One first writes

$$\left| \frac{\partial^{m-1} f(x) - P_{t'}(x)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|y|^{[s]-m+1}} \right| \leq (I) + (II),$$

where

$$(I) = \left| \frac{\partial^{m-1} f(x) - P_{t'}(x)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|x|^{[s]-m+1}} \right|$$

and

$$(II) = \left| \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|x|^{[s]-m+1}} - \frac{\partial^{m-1} f(y) - P_{t'}(y)}{|y|^{[s]-m+1}} \right|.$$

Dividing (3.7) by $|x|^{[s]-m+1}$, (I) is bounded by $C|x-y||x|^{s-[s]-1}$.
On the other hand, using (3.6), one has

$$\begin{aligned}
(II) &\leq |\partial^{m-1} f(y) - P_{t'}(y)| \left| \frac{1}{|x|^{[s]-m+1}} - \frac{1}{|y|^{[s]-m+1}} \right| \\
&\leq |\partial^{m-1} f(y) - P_{t'}(y)| \frac{|x|^{[s]-m+1} - |y|^{[s]-m+1}}{|xy|^{[s]-m+1}} \\
&\leq |\partial^{m-1} f(y) - P_{t'}(y)| \frac{|x-y||x|^{[s]-m}}{|xy|^{[s]-m+1}} \\
&\leq C|y|^{s-m+1} \frac{|x-y||x|^{-1}}{|y|^{[s]-m+1}} \leq C \frac{|y|^{s-[s]}}{|x|} |x-y| \\
&\leq C \left(\frac{|y|}{|x|} \right)^{s-[s]} |x|^{s-[s]-1} |x-y| \leq C|x-y||x|^{s-[s]-1},
\end{aligned}$$

since $\frac{|y|}{|x|}$ is bounded by 1.

The same kind of manipulations of exponents (but easier) as at the end of the previous proposition can be performed. Remarking that $|x|^{-s'} \leq C(|x-y|+|x|)^{-s'}$ and using $0 < s+t'-(m-1) < 1$, it is easily verified that

$$\begin{aligned} |x-y||x|^{s-[s]-1} &= |x-y|^{s+t'-(m-1)}|x-y|^{1-(s+t'-(m-1))}|x|^{s-[s]-1} \\ &\leq |x-y|^{s+t'-(m-1)}(|x-y|+|x|)^{1-(s+t'-(m-1))+s-[s]-1} \\ &\leq |x-y|^{s+t'-(m-1)}(|x-y|+|x|)^{-t'+(m-1)-[s]} \end{aligned}$$

which gives the result. ■

Combining Lemmas 3.1 and 3.1, Proposition 3.1 is proved.

We now compare these $K_{x_0}^{s,s'}$ with the classical pointwise Hölder spaces.

Proposition 3.2 *Let $x_0 \in \mathbb{R}$, and s be a real number such that $s > 0$, $s \notin \mathbb{N}$. Then $C_{x_0}^s = K_{x_0}^{s,-s}$.*

Proof : Let f be a function in $C_{x_0}^s$. One writes, using the approximating polynomial P found in (1.1),

$$\begin{aligned} \left|\frac{f(x)-P(x)}{|x-x_0|^{[s]}} - \frac{f(y)-P(y)}{|y-x_0|^{[s]}}\right| &\leq \left|\frac{f(x)-P(x)}{|x-x_0|^{[s]}}\right| + \left|\frac{f(y)-P(y)}{|y-x_0|^{[s]}}\right| \\ &\leq C(|x-x_0|^{s-[s]} + |y-x_0|^{s-[s]}) \\ &\leq C(|x-y|+|x-x_0|)^{s-[s]}. \end{aligned}$$

This proves $f \in K_{x_0}^{s,-s}$.

On the other hand, let f be a function in $K_{x_0}^{s,-s}$. Taking $y = x_0$ in (3.1) leads to (remember that $g_m(x)$ is defined by $\frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}}$)

$$|g_m(x)-g_m(0)| = \left|\frac{\partial^m f(x)-P(x)}{|x-x_0|^{[s]-m}} - g_m(0)\right| \leq |x-x_0|^{s-[s]},$$

thus $|\partial^m f(x) - P_1(x)| \leq C|x-x_0|^{s-m}$, where P_1 is a polynomial of order smaller than $[s]-m$. The last inequality can be reformulated in

$$|\partial^m (f-P_2)(x)| \leq C|x-x_0|^{s-m},$$

where P_2 is a polynomial of degree less than $[s]$. This reads $\partial^m f \in C_{x_0}^{s-m}$, which implies $f \in C_{x_0}^s$. ■

It is easily verified that $\forall s' \leq 0$ with $s+s' > 0$, $K_{x_0}^{s,s'} \subset C_{x_0}^s$, thus, for all $s' \leq 0$, $K_{x_0}^{s,s'} \subset K_{x_0}^{s,-s}$. As soon as a function belongs to $K_{x_0}^{s,s'}$ for some s', it automatically belongs to $C_{x_0}^s$. That also means, by reciprocity, that a function f whose pointwise Hölder exponent is $s > 0$ cannot belong to any $K_{x_0}^{t,t'}$, for $t > s$, whatever t' is. This is an important property of $K_{x_0}^{s,s'}$ spaces.

3.1.3 Domain of admissible exponents

Definition 3.2 *Let f be a function : $\mathbb{R} \to \mathbb{R}$. $E(f, x_0)$ denotes the set in the half plane $\{(s, s') : s + s' \geq 0, s' \leq 0$ and $s > 0\}$ of all couples (s, s') such that $f \in K_{x_0}^{s,s'}$.*

Let us notice that this set is convex by Proposition 3.1. Moreover, it cannot intersect the open half-space defined by $\{(s, s') : s > \alpha_p(f)\}$, because of the last remark in the previous section. Eventually, if Γ denotes the boundary of $E(f, x_0)$, it is easily shown that Γ is, again by Propositions 3.1 and 3.2, the graph of a function $s = \gamma(s')$. The function γ is concave, decreasing, with slope greater than -1.

The following proposition links $K_{x_0}^{s,s'}$ spaces with the local Hölder exponent defined in the first section

Proposition 3.3 *Let $x_0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ such that $\gamma(0) > 0$. The local Hölder exponent corresponds to the intersection of the frontier of the domain with the x-axis, i.e. $\alpha_l = \gamma(0)$.*

Proof : From the definition of $K_{x_0}^{s,0}$, one obtains, for $|x - x_0| < \delta$, $|y - x_0| < \delta$, (remark that $s' = 0$ implies $[s] = m$)

$$|\partial^m f(x) - \partial^m f(y)| \quad \leq \quad C|x - y|^{s-m}.$$

On the one hand, if $s < \gamma(0)$, then $f \in C_l^s(B(x_0, \delta))$, for any $\delta < 1/4$. Then, taking the limit when $\delta \to 0$, one has $\alpha_l \geq s$ for any $s < \gamma(0)$. Eventually, one concludes $\alpha_l \geq \gamma(0)$.

On the other hand, if $\alpha_l > \gamma(0)$, then there exists s such that $\gamma(0) < s < \alpha_l$ and $\delta > 0$ such that, for $|x - x_0| < \delta$, $|y - x_0| < \delta$,

$$|\partial^m f(x) - \partial^m f(y)| \quad \leq \quad C|x - y|^{s-m}.$$

Thus $f \in K_{x_0}^{s,0}$, and $\gamma(0) \leq s$, which is absurd. ∎

Proposition 3.4 *Let $x_0 \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ such that $\gamma(0) > 0$. The pointwise Hölder exponent of f is the unique $\alpha > 0$ where the frontier intersects the second diagonal, i.e. α_p is the unique α with $\alpha = \gamma(-\alpha)$.*

Proof : The existence and the unicity of this intersection α is clear, since $\gamma(0) > 0$ and since $|\gamma'(s')| < 1$ for $s' > 0$.

First, this intersection is located on the left of the pointwise exponent of f at x_0 since we know that f cannot belong to $K_{x_0}^{s,s'}$ for $s > \alpha_p$. Thus $\alpha \leq \alpha_p$.

On the other hand, if $\alpha < \alpha_p$, then there exists s such that $\alpha < s < \alpha_p$. By definition of α_p, $f \in C_{x_0}^s$, and by Proposition 3.2, $f \in K_{x_0}^{s,-s}$, which is in contradiction with the unicity of α. ∎

Propositions 3.3 and 3.4 show that, as soon as f has a minimal local regularity (i.e. $\gamma(0) > 0$), one can read the pointwise and local Hölder exponents from the 2-microlocal frontier.

To end up with this section, let us notice that all the above propositions are only consequences of simple manipulations of the several exponents s, s', and $s+s'$. In fact the definitions of $K_{x_0}^{s,s'}$ spaces combine in a smart way two notions of regularity : the *global* regularity around x_0 and the *pointwise* regularity at x_0. They provide us with a deep understanding of the behavior of the considered function f around x_0.

3.2 Relation with 2-microlocal Spaces

The main result of the paper is the following theorem, which identifies in the most interesting cases the 2-microlocal spaces $C_{x_0}^{s,s'}$ with our spaces $K_{x_0}^{s,s'}$. The previous Propositions 3.1 to 3.4, are in fact consequences of the following theorem 3.1, since the corresponding properties have been proved to hold for $C_{x_0}^{s,s'}$ spaces. However, we detail them to show how easier propositions 3.3 and 3.4 were to prove in our frame than in the 2-microlocal frame.

3.2.1 Main Result

Theorem 3.1 *Let $x_0 \in \mathbb{R}$, and s, s' be two real numbers such that $s + s' > 0$, $s + s' \notin \mathbb{N}$, and $s' < 0$. Then*

$$(f \in C_{x_0}^{s,s'}) \Leftrightarrow (f \in K_{x_0}^{s,s'}). \tag{3.8}$$

Let us say first a few words about the contraints on s and s'. As shown before, the condition $s + s' > 0$ implies a minimal global regularity for the function in a neighborhood of x_0, and the existence of a sort of Taylor expansion of f at x_0.

Theorem 3.1 does not contain the critical case $s + s' = 0$. Indeed, $C_{x_0}^{s,-s}$ contains distributions, which obviousy do not belong to any $K_{x_0}^{s,-s}$ spaces. $K_{x_0}^{s,-s}$ spaces are thus strictly included in $C_{x_0}^{s,-s}$ spaces.

Theorem 3.1 can be compared with Theorem 1.7 : Theorem 1.7 assumes a minimal global regularity for the considered function f (namely $f \in C^{log}$) to estimate quantities of the type $|f(x) - P(x - x_0)|$. Theorem 3.1 provides an equivalence and allows to estimate differences of the type $|f(x) - f(y)|$, for any couple of points (x, y) in a neighborhood of x_0. This gain of accuracy is due to the fact that we fully use the assumption of local regularity (i.e. $s+s' > 0$).

3.2.2 Proof in a simple case

We focus on the case $0 < s + s' < 1$, $s < 1$. In addition, we assume that the analysis is done using an orthonormal basis of compactly supported wavelets with at least 2 vanishing moments (see for example [22] for the existence and the construction of such a wavelet). This is slightly different and easier than the general case. This restriction is of great interest for practical purposes, as we shall see later. The proof of Theorem 3.1 in the general case is given in [81].

Proof : Without loss of generality, we assume that $x_0 = 0$, and that $s' < 0$, $0 < s < 1$ and $0 < s + s' < 1$. The definition of $K_{x_0}^{s,s'}$ spaces takes here a nice form, i.e.

$$|f(x) - f(y)| \leq C|x - y|^{s+s'}(|x - y| + |x - x_0|)^{-s'}. \tag{3.9}$$

The important case of functions which are continuous but nowhere differentiable is contained in this frame.

For each couple (j, k), denote by $S_{j,k}$ the support of $\psi_{j,k}$, the translated-dilated version of ψ. Namely, one has $\psi_{j,k}(x) = \psi(2^j x - k)$ and $S_{j,k} = [(k - K)2^{-j}, (k + K)2^{-j}]$, where $2K + 1$ is the length of the support of ψ. The corresponding wavelet coefficient is

$$d_{j,k} = 2^j \int f(x)\psi_{j,k}(x)dx.$$

To prove the result, we will use the characterization of the $C^{s,s'}_{x_0}$ spaces by wavelet coefficients, recalled in Theorem 1.9.

1. $K^{s,s'}_0 \subset C^{s,s'}_0$

Assume that $f \in K^{s,s'}_0$. Then (3.9) holds. We want to study the wavelet coefficients $d_{j,k}$. Using the first vanishing moment of the wavelet, one writes

$$\begin{aligned} |d_{j,k}| &= \left| \int f(x) 2^j \psi_{j,k}(x) dx \right| = 2^j \left| \int_{S_{j,k}} (f(x) - f(k2^{-j})) \psi_{j,k}(x) dx \right| \\ &= 2^j \int_{S_{j,k}} |x - k2^{-j}|^{s+s'} (|x - k2^{-j}| + |k2^{-j}|)^{-s'} |\psi_{j,k}(x)| dx \end{aligned}$$

On the interval $S_{j,k}$, $|x - k2^{-j}|$ is bounded by $K2^{-j}$, where K does not depend on x, y, j or k. Then

$$|x - k2^{-j}| + |k2^{-j}| \leq C2^{-j}(1 + |k|),$$

thus $(|x - k2^{-j}| + |x|)^{-s'} \leq C2^{js'}(1 + |k|)^{-s'}$. Moreover, on $S_{j,k}$, one also has $|x - k2^{-j}|^{s+s'} \leq C2^{-j(s+s')}$ on $S_{j,k}$. One deduces that

$$|d_{j,k}| \leq C2^{js'}(1 + |k|)^{-s'} 2^{-j(s+s')} \int_{S_{j,k}} 2^j |\psi_{j,k}(x)| dx \leq C2^{-js}(1 + |k|)^{-s'},$$

since $\int 2^j |\psi_{j,k}(x)| dx$ is a constant. Thus, f indeed belongs to $C^{s,s'}_0$.

2. $C^{s,s'}_0 \subset K^{s,s'}_0$

We suppose that the wavelet coefficients of f verify $|d_{j,k}| \leq C2^{-js}(1 + |k|)^{-s'}$ ($f \in C^{s,s'}_0$). We aim to show that f satisfies (3.9).

Since $s + s' > 0$, $C^{s,s'}_0 \subset C^{s+s'}$ around 0, and we are allowed to use the reconstruction formula

$$f(x) = \sum_j \sum_k d_{j,k} \psi_{j,k}(x). \tag{3.10}$$

As explained before, it is enough to treat the case $|y| \leq x$. We have to to study the difference

$$|f(x) - f(y)| = \left| \sum_j \sum_k d_{j,k} (\psi_{j,k}(x) - \psi_{j,k}(y)) \right|. \tag{3.11}$$

Denote by j_0 be the integer such that

$$2^{-j_0 - 1} \leq |x - y| < 2^{-j_0}. \tag{3.12}$$

The difference (3.11) can be splitted into three different expressions

$$\begin{aligned} |f(x)-f(y)| &\leq \sum_{j\leq j_0-1}\sum_k |d_{j,k}||\psi_{j,k}(x)-\psi_{j,k}(y)| && (I) \\ &+ \sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| && (II) \\ &+ \sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(y)| && (III) \end{aligned}$$

Let us study first the term (II).

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C\sum_{j\geq j_0}\sum_k 2^{-js}(1+|k|)^{-s'}|\psi_{j,k}(x)|.$$

The crucial fact here is the following : if x and j are fixed, only a fixed number of $\psi_{j,k}(x)$ are different from 0, namely $2K+1$, i.e. the length of the support of ψ. This corresponds to the couples of indices (j,k) such that $|x-k2^{-j}|\leq K2^{-j}$, i.e. $(1+|k|)\sim(1+2^j|x|)$.

Then, using that the dilated-translated $\psi_{j,k}$'s are bounded by the same constant M (M and K are independant of s, s', j and k), one has

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C\sum_{j\geq j_0}(2K+1)M2^{-js}(1+2^j|x|)^{-s'}.$$

If $j\geq j_0$, $2^j|x|>1/2$, thus $2^j|x|<2^j|x|+1\leq 4.2^j|x|$. One thus has

$$\begin{aligned} \sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| &\leq C\sum_{j\geq j_0}2^{-j(s+s')}|x|^{-s'}\leq C2^{-j_0(s+s')}|x|^{-s'} \\ &\leq C|x-y|^{s+s'}|x|^{-s'} \end{aligned}$$

since (3.12) holds. Then, using that $|x-y|\leq 2|x|$, the last inequality gives

$$\sum_{j\geq j_0}\sum_k |d_{j,k}||\psi_{j,k}(x)| \leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'},$$

which is the correct bound.

The third term is bounded by the same method as described above.

We now move to the first term, which is a little bit more delicate to study. We will use the derivative of the wavelet ψ. Indeed, one remarks that

$$|\psi_{j,k}(x)-\psi_{j,k}(y)|\leq|x-y|\sup_{z\in[y,x]}|\psi'_{j,k}(z)|.$$

But we know that $\psi'_{j,k}(x)=(\psi(2^jx-k))'=2^j\psi'(2^jx-k)$, which is uniformly bounded by $C2^j$. Thus one has the property

$$|\psi_{j,k}(x)-\psi_{j,k}(y)|\leq C|x-y|2^j. \tag{3.13}$$

The sum in k contains only a fixed number $2K+1$ of non-zero terms, and for these k's, by the same arguments as before, $(1+|k|)^{-s'} \sim (1+2^j|x|)^{-s'}$. Using (3.13), one writes

$$\begin{aligned} (I) &\leq C \sum_{j \leq j_0 - 1} \sum_k |d_{j,k}| 2^j |x-y| \leq C|x-y| \sum_{j \leq j_0-1} (2K+1) 2^{-js} (1+|k|)^{-s'} \\ &\leq C|x-y| \sum_{j \leq j_0 - 1} 2^{j(1-s)} (1+2^j|x|)^{-s'} \end{aligned}$$

Moreover, $(1+2^j|x|)^{-s'} \leq C1 + (2^j|x|)^{-s'}$, and

$$\begin{aligned} (I) &\leq C|x-y| 2^{j_0(1-s)} + 2^{j_0(1-(s+s'))} |x|^{-s'} \\ &\leq C|x-y|(|x-y|^{s-1} + |x-y|^{s+s'-1}|x|^{-s'}) \\ &\leq C|x-y|^{s+s'}(|x-y|^{-s'} + |x|^{-s'}) \leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'} \end{aligned}$$

The key at this point was that $1-(s+s')$ is strictly positive by construction. One now upper-bounds $|x|^{-s'}$ by $(|x-y|+|x|)^{-s'}$.

This ends the proof of Theorem 3.1. ∎

3.2.3 Applications

We give here two applications of Theorem 3.1. We first exhibit some classes of functions that will belong to $C^{s,s'}_{x_0}$. Second we give a decomposition of any function of $C^{s,s'}_{x_0}$ into "simpler" functions. These propositions were already proved in a more general frame in [72], but our approach shows how easier they are to prove with the help of the $K^{s,s'}_{x_0}$ characterization.

Proposition 3.5 *Let us assume that $s' < 0$ and $s+s' > 0$. Let $\{U_j(x)\}_{j \in \mathbb{N}}$ be a sequence of functions that satisfy, for every $|\alpha| \leq N$,*

$$|\partial^\alpha U_j(x)| \leq C(1+|x|)^{-s'}. \tag{3.14}$$

Then the function f defined by

$$f(x) = \sum_{j=0}^{+\infty} 2^{-sj} U_j(2^j(x-x_0)) \tag{3.15}$$

belongs to $C^{s,s'}_{x_0}$.

Proof : We give here the proof of this proposition in the case where $s+s' < 1$, $s < 1$, the general case only needs an easy adaptation of the following.

With f defined by (3.15), let us study the differences $|f(x)-f(y)|$. Let j_0 be the integer such that $2^{-j_0} \leq |x-y| \leq 2^{-j_0-1}$. Then,

$$\begin{aligned} |f(x)-f(y)| &= |\sum_{j=0}^{+\infty} 2^{-sj}(U_j(2^j(x-x_0)) - U_j(2^j(y-x_0)))| \\ &\leq \sum_{j=0}^{j_0} 2^{-sj}|U_j(2^j(x-x_0)) - U_j(2^j(y-x_0))| \qquad (I) \\ &\quad + \sum_{j=j_0+1}^{+\infty} 2^{-sj}|U_j(2^j(x-x_0)) - U_j(2^j(y-x_0))| \qquad (II) \end{aligned}$$

When $j \geq j_0+1$, by (3.14), one obtains

$$|U_j(2^j(x-x_0)) - U_j(2^j(y-x_0))| \leq 2|U_j(2^j(x-x_0))| \leq C(1+2^j|x-x_0|)^{-s'},$$

and thus

$$\begin{aligned} (II) &\leq C \sum_{j=j_0+1}^{+\infty} 2^{-sj}(1+2^j|x-x_0|)^{-s'} \leq C \sum_{j=j_0+1}^{+\infty} 2^{-sj} + 2^{-(s+s')j}|x-x_0|^{-s'} \\ &\leq C2^{-sj_0} + 2^{-(s+s')j_0}|x-x_0|^{-s'} \leq C|x-y|^s, \end{aligned}$$

since $2^{-j_0} \sim |x-x_0|$.

Consider now the other term (I).

$$\begin{aligned} (I) &\leq C\sum_{j=0}^{j_0} 2^{-sj}|2^j(x-x_0) - 2^j(y-x_0)| \sup_{z\in[2^j(x-x_0),2^j(y-x_0)]} |\partial^1 U_j(z)| \\ &\leq C\sum_{j=0}^{j_0} 2^{-sj}2^j|x-y| \max((1+2^j|x-x_0|)^{-s'},(1+2^j|y-x_0|)^{-s'}) \\ &\leq C2^{(1-s)j_0}(1+2^{j_0}|x-x_0|)^{-s'}|x-y| \\ &\leq C|x-y|^s(1+\frac{|x-x_0|}{|x-y|})^{-s'} \leq C|x-y|^{s+s'}(|x-y|+|x-x_0|)^{-s'}, \end{aligned}$$

where one has assumed that $(1+2^j|x-x_0|)^{-s'} \geq (1+2^j|y-x_0|)^{-s'}$. ■

The following proposition gives a decomposition of any function $f \in K_{x_0}^{s,s'}$ into two terms of different behaviors, the first one being regular and the second one containing the "oscillatory" behavior of f around x_0. It has already been proved in a more general case by Y. Meyer in [72].

Proposition 3.6 *Let s, s' be two real numbers such that $s+s'>0$, $s'<0$, and $x_0 \in \mathbb{R}$. Then the following propositions are equivalent*

- $f \in K_{x_0}^{s,s'}$

– *there exist a constant $\delta > 0$, a polynomial P of degree smaller than $[s]$, and a function h which satisfies $h \in C^{s+s'}([x_0-\delta, x_0+\delta])$ and $|h(x)| \leq C|x-x_0|^{s+s'}$, such that*

$$f(x) = P(x) + |x-x_0|^{-s'} h(x). \tag{3.16}$$

Proof : We treat the case $0 < s+s' < 1$ and $x_0 = 0$.

Assume $f \in K_0^{s,s'}$, and $|y| \leq x$. There exists a polynomial P of degree smaller than s such that (3.1) holds. Let us define the functions g and h by

$$g(x) = \frac{f(x)-P(x)}{|x|^{[s]}}, \;\; h(x) = \frac{f(x)-P(x)-g(0)x^{[s]}}{|x|^{-s'}}.$$

One knows that, for all x, y close enough to 0,

$$|g(x)-g(y)| \leq C|x-y|^{s+s'}(|x-y|+|x|)^{-s'-[s]},$$

and that $h(x) = |x|^{s'+[s]}(g(x)-g(0))$. One always has $-1 < s'+[s] < 1$. If $s'+[s] = 0$, the result is obvious, thus we restrict the study to $s'+[s] \neq 0$.

First note that if $y = 0$,

$$|h(x)-h(y)| = |h(x)| = |x|^{s'+[s]}|g(x)-g(0)| \leq C|x|^{s'+[s]}|x|^{s-[s]} = C|x|^{s+s'}.$$

Now one can assume that $x \neq 0$ and $y \neq 0$, and thus, denoting $\tilde{g}(x) = g(x)-g(0)$,

$$|h(x)-h(y)| \leq ||x|^{s'+[s]}\tilde{g}(x) - |y|^{s'+[s]}\tilde{g}(y)|$$

Using that $|\tilde{g}(x)| \leq C|x|^{s-[s]}$ for all x close enough to 0 and (3.1), one obtains

$$\begin{aligned}
|h(x)-h(y)| &\leq |x|^{s'+[s]}|\tilde{g}(x)-\tilde{g}(y)| + ||x|^{s'+[s]}\tilde{g}(y) - |y|^{s'+[s]}\tilde{g}(y)| \\
&\leq |x|^{s'+[s]}|\tilde{g}(x)-\tilde{g}(y)| + |\tilde{g}(y)|||x|^{s'+[s]} - |y|^{s'+[s]}| \\
&\leq C(|x|^{s'+[s]}|x-y|^{s+s'}(|x-y|+|x|)^{-s'-[s]} \\
&\quad +|g(y)|||x|^{s'+[s]} - |y|^{s'+[s]}|,
\end{aligned}$$

Then we make the same kind of manipulations as before. For the first term, one uses $|x| \leq |x-y|+|x| \leq 3|x|$ to get

$$\begin{aligned}
|x-y|^{s+s'}|x|^{s'+[s]}(|x-y|+|x|)^{-s'-[s]} &\leq C|x-y|^{s+s'}|x|^{s'+[s]}|x|^{-s'-[s]} \\
&\leq C|x-y|^{s+s'}.
\end{aligned}$$

The second term is more delicate to study :

– if $|x-y| \leq |y|$, then

$$\begin{aligned}
|\tilde{g}(y)|||x|^{s'+[s]} - |y|^{s'+[s]}| &\leq C|y|^{s-[s]}|x-y||y|^{s'+[s]-1} \\
&\leq C|x-y|^{s+s'}\left(\frac{|x-y|}{|y|}\right)^{1-(s+s')} \leq C|x-y|^{s+s'}.
\end{aligned}$$

– if $|y| \leq |x-y|$ (i.e. $y < x/2$), two cases must be separated
  – if $-1 < s'+[s] < 0$, $||x|^{s'+[s]} - |y|^{s'+[s]}| \leq C|y|^{s'+[s]}$, and

$$|\tilde{g}(y)|||x|^{s'+[s]} - |y|^{s'+[s]}| \;\; \leq \;\; C|y|^{s-[s]}|y|^{s'+[s]} \leq C|y|^{s+s'} \leq C|x-y|^{s+s'}.$$

– if $0 < s' + [s] < 1$, $||x|^{s'+[s]} - |y|^{s'+[s]}| \leq C|x-y|^{s'+[s]}$ (indeed, the function $t \to t^{s'+[s]}$ is concave), and

$$\begin{array}{rcl} |\tilde{g}(y)|||x|^{s'+[s]} - |y|^{s'+[s]}| & \leq & C|y|^{s-[s]}|x-y|^{s'+[s]} \\ & \leq & C|x-y|^{s-[s]}|x-y|^{s'+[s]} \leq C|x-y|^{s+s'}. \end{array}$$

The function h belongs to $C^{s+s'}([x_0-\delta, x_0+\delta])$.

Let us assume now that f satisfies (3.16) for a certain polynomial P and a function h, but does not satisfy (3.1).

Since (3.1) is not verified, one can find two sequences of real numbers $\{x_n\}_n$ and $\{y_n\}_n$, such that, for all n,

$$|g(x_n) - g(y_n)| \geq n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]}. \tag{3.17}$$

Since all the properties are local, around x_0, one can extract from these sequences two subsequences (still denoted by x_n and y_n) that will satisfy $\lim_n x_n = X$ and $\lim_n y_n = Y$, and

$$|g(x_n) - g(y_n)| \geq C_n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]}, \tag{3.18}$$

with $C_n \to +\infty$ when $n \to +\infty$.

First case $X \neq Y$, $X \neq 0$ and $Y \neq 0$. Since g is continuous, using (3.18) and the fact that $|x_n - y_n| \to_n |X-Y|$ and $|x_n| \to_n |X|$, one obtains that $\forall n$ large enough, $|g(X) - g(Y)| \geq n|X-Y|^{s+s'}(|X-Y| + |X|)^{-s'-[s]}$, which is absurd.

Second case $X = 0$ and $Y \neq 0$. This case is treated similarly as the preceding one.

Third case $X = Y = 0$. One can assume that, for all n, $|x_n| \geq |y_n|$.

By definition one has, for all x, $|g(x)| \leq C|x|^{s-[s]}$. Thus, for all n, $|g(x_n) - g(y_n)| \leq C(|x_n|^{s-[s]} + |y_n|^{s-[s]}) \leq 2C|x_n|^{s-[s]}$. On the other hand, using (3.18), one has

$$\begin{array}{rcl} 2C|x_n|^{s-[s]} & \geq & |g(x_n) - g(y_n)| \geq C_n|x_n - y_n|^{s+s'}(|x_n - y_n| + |x_n|)^{-s'-[s]} \\ & \geq & C_n|x_n - y_n|^{s+s'}|x_n|^{-s'-[s]}, \end{array}$$

The lower-bound $(|x_n - y_n| + |x_n|)^{-s'-[s]}$ by $|x_n|^{-s'-[s]}$ holds even when $-s' - [s] < 0$, still because $|x_n| \sim (|x_n - y_n| + |x_n|)$ for all n. One thus obtains $|x_n| \geq C_n^{\frac{1}{s+s'}}|x_n - y_n|$, which can be rewritten as

$$|x_n - y_n| = o(|x_n|). \tag{3.19}$$

(3.19) says that the couples of points where the inequality (3.16) may fail must satisfy some strong properties : both converge to 0, and the differences $|x_n - y_n|$ are small while the differences $|g(x_n) - g(y_n)|$ stay large. Intuitively it corresponds to the case of strong oscillations around 0.

Let us show that this is impossible. One would have

$$|y_n|^{-s'-[s]} = |x_n|^{-s'-[s]} + (-s')|x_n - y_n||z_n|^{-s'-[s]-1},$$

where z_n is a real number between x_n and y_n. Then,

$$\begin{aligned}|g(x_n)-g(y_n)| &= ||x_n|^{-s'-[s]}h(x_n)-|y_n|^{-s'-[s]}h(y_n)| \\ &\leq |x_n|^{-s'-[s]}|h(x_n)-h(y_n)| \\ &\quad +C|h(y_n)||x_n-y_n||z_n|^{-s'-[s]-1} \\ &\leq C|x_n|^{-s'-[s]}|x_n-y_n|^{s+s'} \\ &\quad +C|y_n|^{s+s'}|x_n-y_n||z_n|^{-s'-[s]-1}\end{aligned}$$

since $h \in C^{s+s'}(\mathbb{R})$.

The first term in the last inequality is bounded by $|x_n-y_n|^{s+s'}(|x_n-y_n|+|x_n|)^{-s'-[s]}$. Let us deal with the last term. Using that $|z_n| \sim |x_n| \sim |y_n|$, one verifies that

$$\begin{aligned}|y_n|^{s+s'}|x_n-y_n||z_n|^{-s'-[s]-1} &\leq C|y_n|^{s+s'}|x_n-y_n||y_n|^{-s'-[s]-1} \\ &\leq C|x_n-y_n||y_n|^{s-[s]-1} \\ &\leq C|x_n-y_n|^{s+s'}|x_n-y_n|^{1-(s+s')}|y_n|^{s-[s]-1} \\ &\leq C|x_n-y_n|^{s+s'}(|x_n-y_n|+|y_n|)^{-s'-[s]}\end{aligned}$$

This eventually gives

$$|g(x_n)-g(y_n)| \leq C|x_n-y_n|^{s+s'}(|x_n-y_n|+|y_n|)^{-s'-[s]},$$

in contradiction with (3.18). ■

The main interest of the last proofs is to show that it is possible to check all the properties of the functions belonging to $C_{x_0}^{s,s'}$ spaces (with $s+s' \geq 0$) using only elementary arguments and a time domain analysis.

3.3 Algorithms

3.3.1 Background Ideas

Let us mention that the following algorithm is valid only if the expected exponents are lower than 1. Indeed, formula (3.1) defining the spaces $K_{x_0}^{s,s'}$ involves a polynomial which approximates the data (a kind of Taylor expansion), which is accessible only with the help of finite differences. We then focus on the simpler case $s<1$ and $s+s'<1$, where we have already seen that there is no polynomial in the definition of $K_{x_0}^{s,s'}$.

In this case, there are three major justifications for the use of $K_{x_0}^{s,s'}$ spaces for the characterization of regularity in practical applications.

- $K_{x_0}^{s,s'}$ spaces give a rather rich description of the regularity structure.
- The computation of both exponents (s,s') is performed using directly the values of the function. One does not lose information by integrating or smoothing the data. Moreover one can extract from the frontier the usual information, i.e. the Hölder exponents.
- it may seem harder to estimate a frontier of a domain in $\mathbb{R}^2$ than only one regularity exponent. But this is not the case. The main reason is that we are using more information : in fact, using (3.1) we extract the whole information avalaible in the data. Combined with the fact that a frontier must satisfy a number of constraints such that its general aspect is known, this leads to some fast estimation procedures.

An algorithm for estimating this part of the frontier has been proposed in [54].

We describe here another approach. This kind of algorithm has already been developed in [31], where wavelet coefficients (instead of pointwise values) were used for the estimations. When $s < 1$, the results provided by the following algorithm are far more accurate than the ones provided by this previous wavelet algorithm. Nevertheless, the algorithm of [31] was able to estimate exponents greater than 1.

3.3.2 Implementation

We want to estimate the $K_{x_0}^{s,s'}$-frontier of a function f at a point x_0. We start with formula (3.9)

$$|f(x) - f(y)| \leq C|x-y|^{s+s'}(|x-y| + |x-x_0|)^{-s'}.$$

We assume that we have at our disposal the discrete values $\{f_i\}_{i=1,\dots,N}$ of a function f at the points $\{x_i\}_{i=1,\dots,N}$ (note that we do not need to assume that the $\{x_i\}_{i=1,\dots,N}$ are equidistant). If $f \in K_{x_0}^{s,s'}$, $\forall i, j$,

$$|f_i - f_j| \leq C|x_i - x_j|^{s+s'}(|x_i - x_j| + |x_i - x_0|)^{-s'}. \tag{3.20}$$

Define $x_{i,j}$ by $x_{i,j} = \log(|x_i - x_j|)$ and $y_{i,j,s'}$ by

$$y_{i,j,s'} \quad = \quad \log(|f_i - f_j|) + s' \log(1 + \frac{|x_i - x_0|}{|x_i - x_j|}).$$

Then, (3.20) reads $\forall\ \lambda = (i,j)$, $y_{\lambda,s'} \leq C x_\lambda$.

Now fix an exponent s'. In order to obtain the other exponent s as a function of s', it suffices to make a regression on the maxima of the set of couples (x_λ, y_λ) (where $\lambda \in [1, \dots, N]^2$) to find the corresponding exponent $s = \Gamma(s')$.

The practical implementation follows the rules

- Choose a set of n discretised values of s', $\{s'_1, s'_2, \dots, s'_n\}$, ranging typically in $[-1, 0.5]$.
- For each s'_i, compute the corresponding y_{λ,s'_i} and x_λ.
- Find the largest y_{λ,s'_i}, when λ belongs to $\Lambda_{x_0} = \{(k,l)/\ |x_0 - x_k| < 1/4$ and $|x_0 - x_l| < 1/4\}$. We obtain the y_{μ,s'_i}, where μ belongs to a subset of Λ_{x_0}.
- Make a linear regression on the set of couples $(x_{\mu,s'}, y_{\mu,s'})$.
- Then the slope of the straight line obtained by regression is the approximation of the exponent s corresponding to the s'.

A set of n samples in the frontier of exponents $s = \Gamma(s')$ is eventually obtained. By a simple method of convexification, it can be modified into a convex set of samples. Applying this method we obtain an approximation of the frontier, which satisfies its basic theoretical properties : it is convex, non-decreasing, with a derivative with modulus less than 1.

A last but important remark is the following : since the simple version of the $K_{x_0}^{s,s'}$ spaces (i.e. without polynomials) is considered, the formula we use can only be applied in the triangle $0 < s < 1$, $s' < 0$ and $0 < s + s' < 1$. This implies that, for a function f whose pointwise Hölder exponent is $\alpha < 1$, the algorithm can not detect any regularity larger than $s = \alpha$, even when $s + s' < 0$. This equivalently means that the frontier the algorithm tries to estimate can not intersect the half-plane $\{(s,s')/\ s > \alpha\}$. This provides us with a sharp localization of the pointwise Hölder exponent.

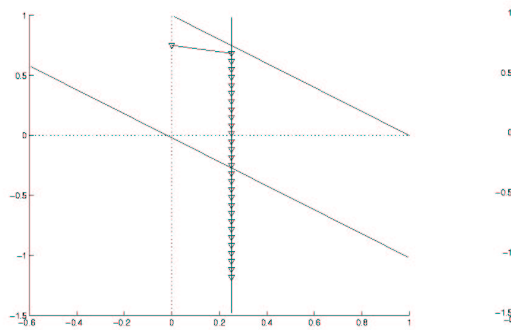

FIG. 3.2 – **Left :** Estimation of the frontier of the cusp $|x|^{0.25}$ at 0. The estimation is plotted with triangles. **Right :** Estimation of the frontier of the chirp $|x|^{0.6}\sin(\frac{1}{|x|^{1.4}})$ at 0.

3.4 Numerical Results

We present the results of the algorithm implemented in different cases. We first treat the case of an isolated singularity, with three examples : a cusp singularity, a chirp singularity, and a sum of two chirps at the same point. Then the more complicated cases of functions which are everywhere continuous, but nowhere differentiable are considered : we deal with the Weierstrass function, the fractionnal Brownian motion, and the generalized Weierstrass function.

In all the figures, we plot the frontier found by the algorithm, and compare it with the theoretical one. We also plot the straight lines $s + s' = 0$, and $s + s' = 1$, which bound the validity of the results (indeed, remember that the formula we are using is only valid for $0 < s + s' < 1$ and $s' < 0$). All the results were obtained using functions sampled on 1024 points.

3.4.1 A cusp

The function considered here is $x \to |x|^{0.25}$. Since there is no oscillation phenomenon, the local and the pointwise Hölder exponents are both equal to $\alpha = 0.25$. The theoretical frontier is a vertical line.

The estimation found for the common value of the two regularity exponents is 0.252, which is extremely precise (see Figure 2).

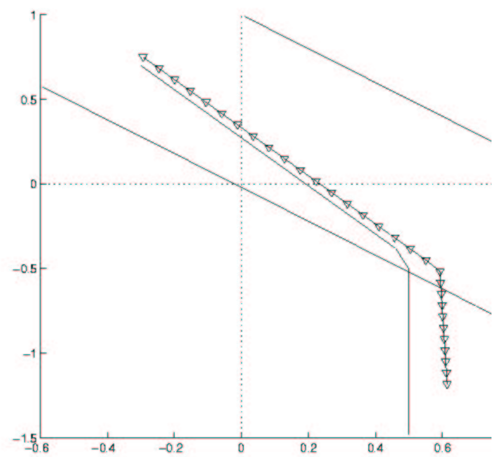

FIG. 3.3 – **Left :** Estimation of the frontier of the function $|x|^{0.8}\sin(\frac{1}{|x|^3}) + |x|^{0.5}\sin(\frac{1}{|x|^{0.25}})$ at 0. **Right :** Estimation of the frontier of the Weierstrass function with $\lambda = 3.23$ and $H = 0.55$ at an arbitrary point.

3.4.2 A chirp

The function we study here is the chirp function, $|x|^{0.6}\sin(\frac{1}{|x|^{1.4}})$. The theoretical Hölder exponents are $\alpha_p = 0.6$, and $\alpha_l = \frac{0.6}{1+1.4} = 0.25$.

The frontier computed by the algorithm (Figure 2) yields the estimations 0.27 and 0.62 for, respectively, the local and the pointwise exponent of f. One notices one more time the precision of the results. The whole fontier is also estimated with good accuracy.

3.4.3 A sum of two chirps

The case of the sum of two chirps located at the same point is delicate. Indeed, it is very hard to distinguish the two behaviors, since there are two types of oscillations (at different frequencies).

We see that, in Figure 3, the two behaviors are identified, since, for example, the theorical pointwise and local exponents (respectively 0.5 and 0.25) are found with a good precision. The phase transition between the two chirps is not well estimated, but, away from it, the approximation of the frontier is accurate.

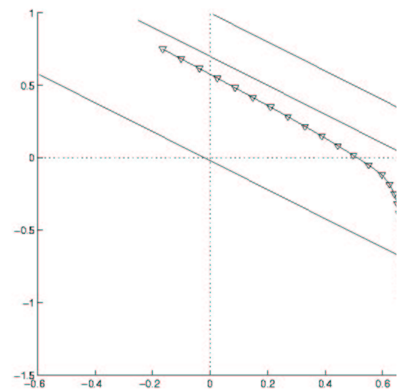

FIG. 3.4 – **Left :** Estimation of the frontier at an arbitrary point of an fBm with $H = 0.7$. The knee in the frontier (at $s' = 0$) is still recovered by the algorithm, but it is smoother in the estimation than in the theoretical frontier. **Right :** Estimation of the pointwise Hölder exponents of a generalized Weierstrass function satisfying $\alpha_p(t) = t/2 + 1/4$, from $t = 0.2$ to $t = 0.8$. The straight line is the theoretical pointwise Hölder function.

3.4.4 A Weierstrass function

The Weierstrass function, defined by

$$W_H(x) = \sum_{j=0}^{+\infty} \lambda^{-jH} \sin(\lambda^j x),$$

where $\lambda \geq 2$ and $0 < H < 1$, belongs to a more complicated type of functions. Indeed, it is well known that W_H is everywhere continuous, nowhere differentiable, and that it has a Hölder exponent equal to H at all points (see for example the original article [84] and [36]).

Remark that, in Figure 3, the knee of the frontier located around the axis $s' = 0$ is found by the algorithm.

3.4.5 An fBm

A path of a Fractional Brownian Motion is another way of obtaining a signal for which the pointwise Hölder function is controlled, and is almost surely everywhere equal to a given exponent H (see [15] for example).

We have tested the algorithm on an fBm with $H = 0.7$ (see Figure 4).

3.4.6 A generalized Weierstrass function

The generalized Weierstrass function [21] is defined by

$$W_H(x) = \sum_{j=0}^{+\infty} \lambda^{-jH(x)} \sin(\lambda^j x),$$

where $\lambda \geq 2$ and $x \to H(x)$ is a C^1-function ranging in $(0, 1]$. It verifies, for every x, $\alpha(x) = \alpha_p(x) = H(x)$. The difference with the classical Weierstrass functions is that one can now prescribe time varying pointwise exponents.

We have run the algorithm on a function generated with $H(x) = x/2+1/4$. The estimated pointwise exponents are plotted on Figure 4 for the range $x \in [0.2, 0.8]$.

The algorithm has been implemented in **FracLab**, a software toolbox avalaible at : http ://www-rocq.inria.fr/fractales/

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Chapitre 4

Formalisme 2-microlocal

Abstract

This paper is devoted to the study of a fine way to measure the local regularity of distributions. Starting from the 2-microlocal analysis introduced by J.M. Bony, we develop a *2-microlocal formalism*, much in the spirit of the multifractal formalism. This allows to define a new regularity function, that we call the *2-microlocal spectrum*. The 2-microlocal spectrum proves to be a powerful tool that we apply in three directions. First, it allows to recover all previously known results on local regularity exponents, as well as to discover new properties about them. Second, the 2-microlocal spectrum provides a deeper understanding of the 2-microlocal frontiers. It yields in particular a natural way of prescribing these frontiers on a countable dense set of points. Finally, we explore the close parallel between the multifractal and 2-microlocal formalisms. These applications are illustrated on examples such as the Weierstrass and the Riemann functions, as well as lacunary wavelet series.

4.1 2-microlocal Formalism

In this section, we elaborate on a fact noticed in [34] : We show that $\sigma(s')$ can be obtained as the Legendre transform of a certain function χ. The function $\chi(\rho)$ roughly measures the exponential rate of decay of the wavelet coefficients on curves of the type $|b| = a^\rho$, $0 < \rho < 1$ in the time-frequency plane (b, a). The relation between χ and σ defines a *2-microlocal formalism*, much in the spirit of the multifractal formalism.

We start with an obvious remark. Remember that, by definition, $f \in C^{s,s'}_{x_0}$ if and only if there exists $b_0 > 0$ and a constant $C_{s,s'}$ such that :

$$\forall a \in (0,1],\ \ \forall b,\ \ |b - x_0| < b_0,\ \ |C(a, x_0 + b)| \leq C_{s,s'} a^\sigma (a + |b|)^{-s'},$$

where $\sigma = s + s'$. This yields the following simple expression for $\sigma(s')$ (we take $x_0 = 0$ to simplify the notations) :

$$\sigma(s') = \liminf_{a \to 0} \inf_{|b| \leq b_0} \frac{s' \log(a + |b|) + \log |C(a, b)|}{\log a} \tag{4.1}$$

or, in a discrete setting :

$$\sigma(s') = \liminf_{j\to\infty} \inf_{|k|=0\ldots 2^j} \frac{s'\log(2^{-j}+|k2^{-j}|)+\log|d_{j,k}|}{-j} \tag{4.2}$$

4.1.1 The 2-microlocal Spectrum

Our aim is to obtain an expression for $\sigma(s')$ more tractable than (4.1) or (4.2). We first need to set some definitions.

Definition 4.1 *Let $f \in \mathcal{S}'(\mathbb{R})$, and denote $C(a,b)$ its wavelet transform using a wavelet of sufficient regularity. For a given $x_0 \in \mathbb{R}$, define :*

– $\theta^0 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$

$$\theta^0(\varepsilon) = \sup\{\gamma : \exists b_o > 0, a^\varepsilon \leq |b-x_0| < b_o \Rightarrow |C(a,x_0+b)| \leq K_\varepsilon a^\gamma\}$$

– $\theta^1 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$

$$\theta^1(\varepsilon) = \sup\{\gamma : \exists b_o > 0, |b-x_0| < \min(b_o, a^\varepsilon) \Rightarrow |C(a,x_0+b)| \leq K_\varepsilon a^\gamma\}$$

Clearly, θ^0 is a non-increasing function of ε, and θ^1 is a non-decreasing function, so that we may define

$$\theta^0(0) = \lim_{\varepsilon\to 0^+} \theta^0(\varepsilon) = \sup\{\theta^0(\varepsilon) : \varepsilon \in (0,1]\}$$

and

$$\theta^1(1) = \lim_{\varepsilon\to 1^-} \theta^1(\varepsilon) = \sup\{\theta^1(\varepsilon) : \varepsilon \in [0,1)\},$$

with $\theta^0(0)$ and $\theta^1(1)$ in $[0,+\infty]$.

Loosely speaking, $\theta^0(0)$ and $\theta^1(1)$ characterize the behaviour of the wavelet coefficients respectively below all curves $|b-x_0| \leq a^\rho$, $\rho > 0$, and in the neighbourhood of the cone of influence $|b-x_0| \leq a$.

Definition 4.2 *For $x_0 \in \mathbb{R}$ and for $\varepsilon > 0$, define the function $\chi^\varepsilon : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup \{+\infty\}$*

$$\begin{aligned}\chi^\varepsilon(\rho) &= \sup\{\gamma : \forall a < b_0, \forall \beta \in [\rho-\varepsilon, \rho+\varepsilon],\ |C(a, x_0 \pm a^\beta)| \leq C_{\rho,\varepsilon} a^\gamma\} \\ &= \liminf_{a\to 0} \inf_{\rho-\varepsilon\leq\beta\leq\rho+\varepsilon} \frac{\log|C(a,x_0\pm a^\beta)|}{\log a}\end{aligned}$$

Note that, for each fixed ρ, $\varepsilon \to \chi^\varepsilon(\rho)$ is a non-increasing function, so that the limit $\lim_{\varepsilon\to 0^+} \chi^\varepsilon(\rho) = \sup\{\chi^\varepsilon(\rho) : \varepsilon > 0\}$ is well defined on $[0,+\infty]$.

Definition 4.3 *Define, for any given x_0, $\chi : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ by*

– $\chi(0) = \theta^0(0)$
– $\rho \in (0,1)$ *:* $\chi(\rho) = \lim_{\varepsilon\to 0^+} \chi^\varepsilon(\rho)$
– $\chi(1) = \theta^1(1)$

χ is called the **2-microlocal spectrum** of f at x_0. The relation between χ and $\sigma(s')$ will be given by Theorem 4.1.

Roughly speaking, $\chi(\rho)$ describes the behaviour of the largest wavelet coefficients that lie around the curve $|b-x_0| = a^\rho$ (see section 4.1.6 for examples of computations of χ).

Let us now investigate some properties of θ^0, θ^1 and χ. Let $\tilde{\chi}$ denote the convex envelop of χ.

Proposition 4.1 *The following relations hold :*

1. $\forall \rho_0$, $\theta^0(\rho_0) \leq \inf_{\rho\in[0,\rho_0)} \chi(\rho)$ *and* $\theta^1(\rho_0) \leq \inf_{\rho\in(\rho_0,1]} \chi(\rho)$
2. $\chi(0) \leq \liminf_{\rho\to 0^+} \chi(\rho)$ *and* $\chi(1) \leq \liminf_{\rho\to 1^-} \chi(\rho)$
3. $\tilde{\chi}(0) = \chi(0)$ *and* $\tilde{\chi}(1) = \chi(1)$

Proof : To simplify notations, assume without loss of generality that $x_0 = 0$. We prove the three assertions for $\chi(0)$, as the other part of the proof (concerning $\chi(1)$) follows the same lines.

(1) Fix ρ_0. For $\rho \in (0, \rho_0)$ and $\varepsilon < \min(\rho, \rho_0 - \rho)$, choose $\beta \in [\rho - \varepsilon, \rho + \varepsilon]$. Set $b = a^\beta$.

First assume $\theta^0(\rho_0) < +\infty$. Then we have $|C(a, a^\beta)| \leq K_{\rho_0,\eta} a^{\theta^0(\rho_0)-\eta}$ for all $\eta > 0$, by definition of θ^0, since $a^{\rho_0} \leq |b|$. Thus, $\chi(\rho) \geq \chi^\varepsilon(\rho) \geq \theta^0(\rho_0) - \eta$, $\forall\eta$.

If $\theta^0(\rho_0) = +\infty$, for every $N > 0$, $|C(a, a^\beta)| \leq K_{\rho_0,\eta} a^N$, and one concludes using the same argument that $\chi(\rho) \geq \chi^\varepsilon(\rho) \geq N$, for all N.

Let us now examine the case $\rho = 0$. By definition, $\chi(0) = \theta^0(0)$ and, since θ^0 is non increasing, $\theta^0(\rho_0) \leq \chi(0)$.

(2) One simply uses the definition of $\theta^0(0)$:

If $\theta^0(0) < +\infty$, $\forall\varepsilon > 0$, $\exists\rho_0$, $\forall\rho \in [0, \rho_0]$, $\theta^0(\rho) \geq \theta^0(0) - \varepsilon$. Using (1), this yields

$$\forall\varepsilon > 0, \exists\rho_0 : \forall\rho \in [0, \rho_0), \chi(\rho) \geq \chi(0) - \varepsilon.$$

If $\theta^0(0) = +\infty$, $\forall N > 0$, $\exists\rho_0$, $\forall\rho \in [0, \rho_0]$, $\theta^0(\rho) \geq N$, and thus $\chi(\rho) \geq N$ for all $N > 0$. This gives the result.

(3) This is a simple consequence of (2), the definition of $\tilde{\chi}$ as the convex envelop of $\tilde{\chi}$, and the fact that 0 is an extremal point of the domain of definition of χ. ■

Remark : The second function of Subsection 4.1.6 provides an example of a function such that, for an exponent ρ_0, $\theta(\rho_0) < \inf_{\rho\in[0,\rho_0)} \chi(\rho)$.

Corollary 4.1 *If* $\chi(0) < +\infty$*, then either there exists* $\varepsilon > 0$ *such that* $\forall\rho \in (0, \varepsilon]$*,* $\tilde{\chi}(\rho) = +\infty$*, or* $\chi(0) = \tilde{\chi}(0) = \lim_{\rho\to 0^+} \tilde{\chi}(\rho)$*.*

The following lemma shows that χ is well-behaved.

Lemma 4.1 *Let* $f \in \mathcal{S}'(\mathbb{R})$ *and* $x_0 \in \mathbb{R}$*. Let* I_{x_0} *denote the interior of* $\{\rho : \chi(\rho) < +\infty\}$*. The function* $\rho \to \chi(\rho)$ *is lower semi-continuous on* I_{x_0}*.*

Proof : Let $\rho \in (0, 1)$, with $\rho \in I_{x_0}$, and $\varepsilon > 0$ small enough such that $[\rho - \varepsilon, \rho + \varepsilon] \subset I_{x_0}$. From the definition of χ one gets

$$\chi^\varepsilon(\rho) \leq \inf_{\beta\in(\rho-\varepsilon,\rho+\varepsilon)} \chi(\beta). \tag{4.3}$$

Indeed the computation of $\chi(\beta)$ for $\beta \in (\rho - \varepsilon, \rho + \varepsilon)$ uses wavelet coefficients that are also taken into account when computing $\chi^\varepsilon(\rho)$. By definition, $\lim_{\varepsilon\to 0^+} \chi^\varepsilon(\rho) = \chi(\rho)$, thus $\forall\eta > 0$, there exists ε_η such that $\varepsilon \le \varepsilon_\eta$ implies $\chi^\varepsilon(\rho) \ge \chi(\rho) - \eta$. Using (4.3), one gets that, $\forall \eta > 0$, there exists ε_η such that $\forall \varepsilon \le \varepsilon_\eta$,

$$\chi(\rho) \le \eta + \inf_{0<\varepsilon\le\varepsilon_\eta} \chi(\rho \pm \varepsilon).$$

Thus $\chi(\rho) \le \liminf_{\varepsilon\to 0^+} \chi(\rho \pm \varepsilon)$.

The same arguments may be used to treat the cases $\rho = 0$ and $\rho = 1$. ∎

Combining Lemma 4.1 with Proposition 4.1 yields that the function $\rho \to \chi(\rho)$ is also lower semi-continuous on the closure of I_{x_0}.

4.1.2 Main Result

For the analysis of the 2-microlocal frontier, it will be useful to define the following partition of $D = (0,1] \times [x - b_0, x + b_0]$ into three regions I_{ρ_0}, II_{ρ_0} and III_{ρ_0} (where $0 < \rho_0 < 1/2$) :

$$\begin{aligned} I_{\rho_0} &= \{(a,b) \in D, |b - x_0| < a^{1-\rho_0}\} \\ II_{\rho_0} &= \{(a,b) \in D, a^{1-\rho_0} \le |b - x_0| \le a^{\rho_0}\} \\ III_{\rho_0} &= \{(a,b) \in D, a^{\rho_0} < |b - x_0|\}. \end{aligned}$$

We first give a result concerning $\chi(0)$.

Proposition 4.2

$$f \in C_x^{s,s'} \Rightarrow \sigma \le \chi(0)$$

Proof : If $\chi(0) = +\infty$, the proposition is obvious.

If $\chi(0) < +\infty$, fix $\eta > 0$. By definition of $\theta^0(\rho)$, there exists a sequence (a_n, b_n) with (a_n) converging to 0 and $a_n^\rho \le |b_n| \le b_0$ for all n, and such that :

$$\forall n, |C(a_n, b_n)| \ge K a_n^{\theta^0(\rho)+\eta}$$

for any fixed K. Indeed, otherwise, there would exist $a_1 > 0$ and $b_1 > 0$ such that, for all $(a,b) \in (0, a_1) \times [-b_1, b_1], |C(a,b)| \le K a^{\theta^0(\rho)+\eta}$, contradicting the definition of θ^0 as a supremum.

Now if $f \in C_x^{s,s'}$, by item 3. of Theorem 1.9, one has, for all n :

$$K a_n^{\theta^0(\rho)+\eta} \le |C(a_n, b_n)| \le C_{\sigma,s'} a_n^\sigma (a_n + |b_n|)^{-s'}$$

This implies

$$K a_n^{\theta^0(\rho)+\eta-\sigma} \le C_{\sigma,s'} (a_n + |b_n|)^{-s'} \tag{4.4}$$

Assume first $s' \le 0$. Then, the right-hand side term of (4.4) remains bounded when n tends to infinity. Since a_n tends to 0 when n tends to infinity, this implies that $\sigma \le \theta^0(\rho) + \eta$. Using $\chi(0) = \sup_\rho \theta^0(\rho)$, we get that, for all positive η, $\sigma \le \chi(0) + \eta$.

If now $s' > 0$, then (4.4) and $a_n + |b_n| \ge a_n^\rho$ imply :

$$K a_n^{\theta^0(\rho)+\eta-\sigma} \le C_{\sigma,s'} a_n^{-\rho s'}$$

And, by letting n go to infinity :

$$\theta^0(\rho) + \eta - \sigma + \rho s' \geq 0$$

Using again $\chi(0) = \sup_\rho \theta^0(\rho)$, we get $\sigma \leq \chi(0) + \rho s' + \eta$ and the result follows be letting ρ go to 0. ∎

We are now ready to state the 2-microlocal formalism. We shall denote by g^* the following Legendre transform of a function g

$$g^*(y) = \inf_{x \in D_g} (xy - g(y)),$$

where D_g is the domain of definition of g.

Theorem 4.1 [1] *Let $f \in \mathcal{S}'(\mathbb{R})$. The 2-microlocal frontier of f at any x_0 is given by :*

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho)). \tag{4.5}$$

Recall that $(-\chi)^*(s') = (-\tilde{\chi})^*(s') = \inf_{\rho \in [0,1]}(\rho s' + \tilde{\chi}(\rho))$, since $\tilde{\chi}$ is the convex envelop of χ. More precisely, since $\tilde{\chi}$ is positive, lower semi-continuous on its support and convex, one has $(-\chi)^{**} = -\tilde{\chi}$.

Proof :

Once again, we take $x_0 = 0$. Let us prove first that one has $\sigma(s') \leq \inf_{\rho \in [0,1]}(\rho s' + \tilde{\chi}(\rho))$.

By Theorem 1.9, if $f \in C_0^{s,s'}$, there exist $b_0 > 0$ and $C_{s,s'}$ such that :

$$\forall a \in (0,1], \forall b, |b| < b_0, |C(a,b)| \leq C_{s,s'} a^\sigma (a + |b|)^{-s'}. \tag{4.6}$$

Applying (4.6) with $b = a^\rho$ leads to

$$f \in C_0^{s,s'} \Rightarrow \forall \rho \in (0,1], \forall a \in (0, b_0], |C(a, \pm a^\rho)| \leq C_{s,s'} a^\sigma (a + a^\rho)^{-s'}.$$

Since $\forall a$, $\forall \rho$, $a^\rho < a + a^\rho \leq 2a^\rho$, one gets :

$$f \in C_0^{s,s'} \Rightarrow \forall \rho \in (0,1], \forall a \in (0, b_0], |C(a, \pm a^\rho)| \leq K_{s,s'} a^{\sigma - \rho s'},$$

where $K_{s,s'} = C_{s,s'}$ if $s' > 0$ and $K_{s,s'} = 2^{-s'} C_{s,s'}$ if $s' < 0$.

Fix $\rho \in (0,1)$ and $\varepsilon < \min(\rho, 1-\rho)$. For all $\beta \in [\rho - \varepsilon, \rho + \varepsilon]$, we have :

$$|C(a, \pm a^\beta)| \leq K_{s,s'} a^{\sigma - \beta s'} \leq K_{s,s'} a^{\sigma - (\rho \pm \varepsilon)s'},$$

where one chooses $\rho + \varepsilon$ if $s' > 0$ and $\rho - \varepsilon$ if $s' < 0$. Thus, from the Definition (4.2) of χ^ε,

$$\chi^\varepsilon(\rho) \geq \sigma - (\rho \pm \varepsilon)s',$$

and letting ε go to 0,

$$\forall \rho \in (0,1), \chi(\rho) \geq \sigma - \rho s'.$$

[1] A version of Theorem 4.1 was already proposed in [34]. However this version was in error, as the examples of Section 4.1.6 show.

The case $\rho = 1$ is treated similarly. Indeed, for $\beta \in [1-\varepsilon, 1]$,

$$|C(a, \pm a^\beta)| \leq K_{s,s'} a^{\sigma - \beta s'} \leq K_{s,s'} a^{\sigma - \max((1-\varepsilon)s', s')},$$

and if $|b| \leq a$,

$$|C(a,b)| \leq K_{s,s'} a^{\sigma - s'}.$$

Then, by definition of θ^1, $\theta^1(1-\varepsilon) \geq \sigma - \max((1-\varepsilon)s', s')$, and thus, letting ε go to 0, $\chi(1) \geq \sigma - s'$.

The case $\rho = 0$ has already been taken care of by proposition 4.2. We end up with :

$$f \in C_0^{s,s'} \Rightarrow \sigma \leq \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho)).$$

Since $-\tilde{\chi}$ is the concave envelop of $-\chi$, both functions have the same Legendre transform, and thus :

$$f \in C_0^{s,s'} \Rightarrow \sigma \leq \inf_{\rho \in [0,1]} (\rho s' + \tilde{\chi}(\rho)).$$

This ends the first part of the proof.

Assume now that $\sigma(s') \neq \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho))$. Then there must exist s'_0 and $\eta > 0$ such that :

$$\sigma(s'_0) < \inf_{\rho \in [0,1]} (\rho s'_0 + \chi(\rho)) - \eta. \tag{4.7}$$

Note that $\sigma(s'_0)$ must verify :

$$\sigma(s'_0) \leq \chi(0). \tag{4.8}$$

By definition of the 2-microlocal frontier, with η defined in (4.7), for all K, there exists a sequence $(a_n, b_n)_{n \in \mathbb{N}}$, such that

$$\lim_{n \to \infty} a_n = 0 \text{ and } |C(a_n, b_n)| \geq K a_n^{\sigma(s'_0)+\eta} (a_n + |b_n|)^{-s'_0}. \tag{4.9}$$

Fix K and define :

$$\tilde{\rho} = \liminf_{n \to \infty} \frac{\log |b_n|}{\log a_n}.$$

There exists a infinite subsequence, still denoted (a_n, b_n) such that

$$\frac{\log |b_n|}{\log a_n} \to_{n \to +\infty} \tilde{\rho}. \tag{4.10}$$

We distinguish three cases, according to the values of $\tilde{\rho}$.

First case : $\tilde{\rho} \in (0,1)$. For n large enough, all the couples (a_n, b_n) belong to the region II_{ρ_0} for some $\rho_0 < \tilde{\rho}$. (4.10) implies

$$\forall \varepsilon > 0, \ \exists N, \ \forall n > N, \ a_n^{\tilde{\rho}+\varepsilon} \leq |b_n| \leq a_n^{\tilde{\rho}-\varepsilon},$$

$$\forall \varepsilon > 0, \ \exists N, \ \forall n > N, \ a_n^{\tilde{\rho}+\varepsilon} \leq a_n + |b_n| \leq 2a_n^{\tilde{\rho}-\varepsilon}.$$

Consider first the case $s'_0 > 0$, for which :

$$(a_n + |b_n|)^{-s'_0} \geq 2^{-s'_0} a_n^{-(\tilde{\rho}-\varepsilon)s'_0}.$$

(4.9) entails :

$$|C(a_n, b_n)| \geq K2^{-s'_0} a_n^{\sigma(s'_0)+\eta-(\tilde{\rho}-\varepsilon)s'_0},$$

and

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \leq \frac{\log K2^{-s'_0}}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0. \tag{4.11}$$

Now, $a_n^{\tilde{\rho}+\varepsilon} \leq |b_n| \leq a_n^{\tilde{\rho}-\varepsilon}$ implies that :

$$\inf_{\beta \in [\tilde{\rho}-\varepsilon, \tilde{\rho}+\varepsilon]} |C(a_n, \pm a_n^{\beta})| \leq |C(a_n, b_n)| \leq \sup_{\beta \in [\tilde{\rho}-\varepsilon, \tilde{\rho}+\varepsilon]} |C(a_n, \pm a_n^{\beta})|,$$

and, for $a_n < 1$:

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \geq \inf_{\beta \in [\tilde{\rho}-\varepsilon, \tilde{\rho}+\varepsilon]} \frac{\log |C(a_n, \pm a_n^{\beta})|}{\log a_n},$$

Together with (4.11), this yields : $\forall \varepsilon > 0$, $\exists N$, $\forall n > N$,

$$\inf_{\beta \in [\tilde{\rho}-\varepsilon, \tilde{\rho}+\varepsilon]} \frac{\log |C(a_n, \pm a_n^{\beta})|}{\log a_n} \leq \frac{\log K2^{-s'_0}}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0.$$

Since $a_n \to 0$ when $n \to \infty$, we get, by definition of χ^{ε} :

$$\chi^{\varepsilon}(\tilde{\rho}) \leq \liminf_{n \to \infty} \inf_{\beta \in [\tilde{\rho}-\varepsilon, \tilde{\rho}+\varepsilon]} \frac{\log |C(a_n, \pm a_n^{\beta})|}{\log a_n} \leq \sigma(s'_0) + \eta - (\tilde{\rho} - \varepsilon)s'_0.$$

Letting ε go to 0, one eventually finds

$$\sigma(s'_0) \geq \tilde{\rho}s'_0 + \chi(\tilde{\rho}) - \eta,$$

in contradiction with (4.7).

If now $s'_0 < 0$, using $(a_n + |b_n|)^{-s'_0} \geq a_n^{-(\tilde{\rho}+\varepsilon)s'_0}$, we get :

$$|C(a_n, b_n)| \geq K a_n^{\sigma(s'_0)+\eta-(\tilde{\rho}+\varepsilon)s'_0},$$

and

$$\frac{\log |C(a_n, b_n)|}{\log a_n} \leq \frac{\log K}{\log a_n} + \sigma(s'_0) + \eta - (\tilde{\rho} + \varepsilon)s'_0,$$

and we end up with the same contradiction.

Second case : $\tilde{\rho} = 0$. For n large enough, all the couples (a_n, b_n) belong to a region I_{ρ_0}, for ρ_o small enough. As a consequence, for n large enough, one has $a_n^{\rho_0} \leq |b_n|$ and

$$|C(a_n, b_n)| \geq K a_n^{\sigma(s'_0)+\eta}(a_n + |b_n|)^{-s'_0}. \tag{4.12}$$

Now, if $\chi(0) = +\infty$, for all $N > 0$, there exist ρ_0 and a constant $K_{\rho_0,N}$ such that, for all (a_n, b_n) that belong to I_{ρ_0} (i.e. for n large enough)

$$|C(a_n, b_n)| \leq K_{\rho_0,N} a_n^N. \tag{4.13}$$

(4.12) and (4.13) imply

$$Ka_n^{\sigma(s_0')+\eta-N} \leq K_{\rho_0,N}(a_n+|b_n|)^{s_0'}.$$

If $s_0' \geq 0$, the right hand side of (4.15) remains bounded when n tends to infinity, which implies that $\sigma(s_0') \geq N-\eta$ for all N.
If $s_0' < 0$, we use $a_n+|b_n| \geq |b_n| \geq a_n^{\rho_0}$ to write :

$$Ka_n^{\sigma(s_0')+\eta-N} \leq K_{\rho_0,N}a_n^{\rho_0 s_0'},$$

which leads to $\sigma(s_0') \geq N$ for all N (when ρ_0 goes to 0). In both cases, $\sigma(s_0') = \chi(0) = +\infty$, which is in contradiction with (4.7).

If $\chi(0) < +\infty$, by definition of θ^0, for all $\varepsilon > 0$ small enough, there exists $K_{\rho_0,\varepsilon}$ such that, for all n large enough :

$$|C(a_n,b_n)| \leq K_{\rho_0,\varepsilon}a_n^{\theta^0(\rho_0)-\varepsilon}. \tag{4.14}$$

(4.12) and (4.14) entail :

$$Ka_n^{\sigma(s_0')+\eta-\theta^0(\rho_0)+\varepsilon} \leq K_{\rho_0,\varepsilon}(a_n+|b_n|)^{s_0'}. \tag{4.15}$$

The same arguments as before allow to conclude. Indeed, if $s_0' \geq 0$, the right hand side of (4.15) remains bounded when n tends to infinity, which implies that :

$$\sigma(s_0') \geq \theta^0(\rho_0)-\eta-\varepsilon.$$

Since this must be valid for all positive ε and η small enough, we get :

$$\sigma(s_0') \geq \theta^0(\rho_0)$$

This inequality holds for all $\rho_0 > 0$ small enough. Letting ρ_0 tend to 0, we get

$$\sigma(s_0') \geq \tilde{\chi}(0).$$

If $s_0' < 0$, we use $a_n+|b_n| \geq |b_n| \geq a_n^{\rho_0}$ to write :

$$Ka_n^{\sigma(s_0')+\eta-\theta^0(\rho_0)+\varepsilon} \leq K_{\rho_0,\varepsilon}a_n^{\rho_0 s_0'},$$

which entails :

$$\sigma(s_0') \geq \theta^0(\rho_0)+\rho_0 s_0'-\eta-\varepsilon,$$

and, again by letting ρ_0 go to 0,

$$\sigma(s_0') \geq \tilde{\chi}(0).$$

Comparing with (4.8), we get that, necessarily

$$\sigma(s_0') = \tilde{\chi}(0).$$

Thus we get that

$$\sigma(s') = \inf_{\rho\in[0,1]}(\rho s'+\tilde{\chi}(\rho)),$$

except maybe for $s' = s'_0$, where s'_0 is such that $\sigma(s'_0) = \tilde{\chi}(0)$, and for which we could have $\sigma(s'_0) < \inf_{\rho\in[0,1]}(\rho s' + \tilde{\chi}(\rho))$. However, this last inequality cannot occur since obviously $\inf_{\rho\in[0,1]}(\rho s' + \tilde{\chi}(\rho)) \leq \tilde{\chi}(0)$.

The case $\tilde{\rho} = 1$ is treated as the case $\tilde{\rho} = 0$. This concludes the proof. ∎

Remark : Note that, since the Legendre transform is invertible for concave functions, Theorem 4.1 also allows to compute $\tilde{\chi}$ from $\sigma(s')$:

$$-\tilde{\chi}(\rho) = \inf_{s'\in\mathbb{R}}\{\rho s' - \sigma(s')\}.$$

Remark : If one parameterizes the frontier using the function $s'(\sigma)$, then the result corresponding to Theorem 4.1 is the following : if the 2-microlocal frontier is nowhere parallel to the second bisector,

$$s'(\sigma) = \sup_{\rho\in[0,1]} \frac{\sigma - \tilde{\chi}(\rho)}{\rho} = \sup_{\rho\in[0,1]} \left(\frac{\sigma}{\rho} - u(1/\rho)\right) \tag{4.16}$$

where $u(\beta) = \beta\tilde{\chi}\left(\frac{1}{\beta}\right)$. It is easy to check that, if a function $x \to g(x)$ is convex, then the function $x \to xg(1/x)$ is also convex. Then (4.16) states that the (convex) function $s'(\sigma)$ is the Legendre transform for convex functions of u.

The proof of this result follows the same lines as the ones of Theorem 4.1, and uses the following additional property, whose proof is omitted :

$$\forall\sigma \leq \chi(0), \quad \sup_{0<\rho\leq 1} \frac{\sigma - \chi(\rho)}{\rho} \geq \sup_{0\leq\rho\leq 1} \frac{\sigma - \tilde{\chi}(\rho)}{\rho}$$

4.1.3 Discrete Setting

It is useful to define the analog of χ in the discrete setting. For every scale j and every $\rho \in [0,1]$, denote by $k_{j,\rho}$ the integer $k_{j,\rho} = [2^{j(1-\rho)}]$.

Definition 4.4 *Let $f \in \mathcal{S}'(\mathbb{R})$ and let $(d_{j,k})$ denote its wavelet coefficients using a wavelet of sufficient regularity. For any given x_0, define*

– $\theta^0 : (0,1) \to \mathbb{R}^+ \cup \ \{+\infty\}$

$$\theta^0(\varepsilon) = \sup\{\gamma : \exists b_0 > 0, \ \forall\beta \leq \varepsilon, \ 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K_\varepsilon 2^{-j\gamma}\}$$

– *for any given $\varepsilon > 0$, $\chi^\varepsilon : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup \ \{+\infty\}$,*

$$\begin{aligned}\chi^\varepsilon(\rho) &= \sup\{\gamma : \exists b_0, \forall\beta \in [\rho-\varepsilon, \rho+\varepsilon], \ 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq C_{\rho,\varepsilon} 2^{-j\gamma}\} \\ &= \liminf_{j\to+\infty} \inf_{\rho-\varepsilon\leq\beta\leq\rho-\varepsilon} \frac{\log|d_{j,[2^j x_0 \pm k_{j,\beta}]}|}{-j}\end{aligned}$$

– $\theta^1 : (0,1) \to \mathbb{R}^+ \cup \ \{+\infty\}$

$$\theta^1(\varepsilon) = \sup\{\gamma : \exists b_0 > 0, \ \forall\beta \geq \varepsilon, \ 2^{-j} \leq b_0 \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K_\varepsilon 2^{-j\gamma}\}$$

Clearly, θ^0 and χ^ε are non-increasing functions, while θ^1 is non-decreasing. We thus define the analog of χ (also denoted χ) in the discrete setting

Definition 4.5 *Define, for any given x_0 : $\chi : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$:*

- $\chi(0) = \lim_{\varepsilon\to 0^+} \theta^0(\varepsilon) = \sup_{\varepsilon>0} \theta^0(\varepsilon)$.
- $\rho \in (0,1)$: $\chi(\rho) = \lim_{\varepsilon\to 0} \chi^\varepsilon(\rho)$
- $\chi(1) = \lim_{\varepsilon\to 1^-} \theta^1(\varepsilon) = \sup_{\varepsilon>0} \theta^1(\varepsilon)$.

As a preparation, let us show that this tool is consistant, in the sense that it is independent of the wavelet ψ that has been chosen.

A critical argument that pleads for the use of the 2-microlocal spectrum is that for any continuous function f, for any point x and for any $\rho \in [0,1]$, the value of $\chi_x(\rho)$ is independent of the wavelet used to compute the wavelet coefficients $d_{j,k}$.

This is achieved by using the notion of robustness developped in [41] or [49] for example. The following definitions are taken from [49], as well as Lemma 4.3.

Definition 4.6 *Let $\gamma > 0$, and define for every couple of dyadic numbers $k2^{-j}$ and $k'2^{-j'}$*

$$\omega_\gamma(k2^{-j}, k'2^{-j'}) = \frac{2^{-|j-j'|(\gamma+2)}}{(1 + 2^{\min(j,j')}|k2^{-j} - k'2^{-j'}|)^{(\gamma+2)}}.$$

An infinite matrix A indexed by the dyadic numbers belongs to $\mathcal{A}^\gamma$ if there exists a constant C such that $\forall k2^{-j}, k'2^{-j'}$, $A(k2^{-j}, k'2^{-j'}) \le C\omega_\gamma(k2^{-j}, k'2^{-j'})$.

A is said to be quasi-diagonal if it is invertible and if A and A^{-1} belong to $\bigcap_{\gamma>0} \mathcal{A}^\gamma$.

The matrix of the operator which maps an orthonormal wavelet basis on another wavelet basis is quasi-diagonal.

Proposition 4.3 *Let $f \in C^\delta(\mathbb{R})$, and $x \in \mathbb{R}$. The 2-microlocal spectrum $\rho \mapsto \chi_x(\rho)$ does not depend on the wavelet basis $\{\psi_{j,k}\}$ that has been chosen in Definitions 4.4 and 4.5.*

We shall use some results established in [49]

Definition 4.7 *Let $\varepsilon' > 0$. The ε'-neighborhood of $k2^{-j}$, denoted by $N_{\varepsilon',k2^{-j}}$, is the set of dyadic numbers $k'2^{-j'}$ that satisfy $|j - j'| \le \varepsilon' j$ and $|k2^{-j} - k'2^{-j'}| \le 2^{-j(1-2\varepsilon')}$.*

Lemma 4.2 *Let $\varepsilon' > 0$ and $(j,k) \in \mathbb{N}^2$. Let j' be such that $|j - j'| \le \varepsilon' j$. The cardinal of the set of dyadic numbers $k'2^{-j'}$ that belong to $N_{\varepsilon',k2^{-j}}$ is bounded by $2^{j'+2}2^{-j(1-2\varepsilon')}$.*

Lemma 4.3 *Let $f \in C^\delta(\mathbb{R})$. Let $\gamma \ge \delta$, and $A \in \mathcal{A}^\gamma$. There exists a constant C such that $\forall j,k$, $|e_{j,k}| \le C2^{-j\delta}$.*
Moreover, if $\gamma \ge \delta + 1/\varepsilon'$, for the same constant C one has

$$\left| \sum_{k'2^{-j'} \notin N_{\varepsilon',k2^{-j}}} A(k2^{-j}, k'2^{-j'}) d_{j,k} \right| \le C2^{-j(\delta+1/\varepsilon')}.$$

Proof : (Proposition 4.3)

As usual, in the following C denotes a constant that does not depend on j and k.

Let $f \in C^{\delta}(\mathbb{R})$. Let $\{\psi_{j,k}\}$ and $\{\psi'_{j,k}\}$ be two orthonormal wavelet bases, and let $\{d_{j,k}\}$ and $\{e_{j,k}\}$ be the corresponding wavelet coefficients of f in these bases. Let us denote by A be the (quasi-diagonal) matrix that maps $\{d_{j,k}\}$ to $\{e_{j,k}\}$. One thus has

$$e_{j,k} = \sum_{k'2^{-j'}} A(k2^{-j}, k'2^{-j'}) d_{j',k'}. \tag{4.17}$$

Without loss of generality, we assume $x_0 = 0$. Let $\rho \in (0,1)$.

For ε small enough, we denote by $\chi^{\psi,\varepsilon}(\rho)$ and $\chi^{\psi',\varepsilon}(\rho)$ the two functions associated with the two wavelet bases (see Definition 4.4).

Let $\varepsilon > 0$, and denote $\alpha = \chi^{\psi,\varepsilon}(\rho)$. Thus, for every $\eta > 0$, there exists a constant C such that for every dyadic number $k2^{-j}$ that satisfies $2^{-j(\rho+\varepsilon)} \leq |k2^{-j}| \leq 2^{-j(\rho-\varepsilon)}$, one has $|d_{j,k}| \leq C2^{-j(\alpha-\eta)}$.

Let $\varepsilon' > 0$ and $\gamma > 1/\varepsilon'$ (their exact values will be precised later) Let us now compute $\chi^{\psi',\varepsilon'}(\rho)$. Let $k2^{-j}$ be a dyadic number that satisfies $2^{-j(\rho+\varepsilon')} \leq |k2^{-j}| \leq 2^{-j(\rho-\varepsilon')}$.

Let us consider the ε'-neighborhood of $k2^{-j}$. If $k'2^{-j'} \in N_{\varepsilon',k2^{-j}}$, one has $|j'-j| \leq \varepsilon' j$ and $|k2^{-j} - k'2^{-j'}| \leq 2^{-j(1-2\varepsilon')}$. As a consequence,

$$|k'2^{-j'}| \leq |k2^{-j}| + 2^{-j(1-2\varepsilon')} \leq 2^{-j(\rho-\varepsilon')} + 2^{-j(1-2\varepsilon')} \leq C2^{-j'\frac{\rho-\varepsilon'}{1-\varepsilon'}}.$$

Similarly,

$$|k'2^{-j'}| \geq |k2^{-j}| - 2^{-j(1-2\varepsilon')} \geq 2^{-j(\rho+\varepsilon')} + 2^{-j(1-2\varepsilon')} \leq C2^{-j'(1+\varepsilon')(\rho+\varepsilon')}.$$

Choose ε' small enough so that $\rho - \varepsilon/2 \leq \frac{\rho-\varepsilon'}{1-\varepsilon'}$ and $(1+\varepsilon')(\rho+\varepsilon') \leq \rho + \varepsilon/2$. Hence the ε'-neighborhood of $k2^{-j}$ $N_{\varepsilon',k2^{-j}}$ is included in $\{k2^{-j} : 2^{-j(\rho+\varepsilon)} \leq |k2^{-j}| \leq 2^{-j(\rho-\varepsilon)}\}$. As a consequence, for every $k2^{-j}$ such that $2^{-j(\rho+\varepsilon')} \leq |k2^{-j}| \leq 2^{-j(\rho-\varepsilon')}$, for every $k'2^{-j'} \in N_{\varepsilon',k2^{-j}}$, $|d_{j',k'}| \leq C2^{-j'(\alpha-\eta)}$.

Finally, choose γ large enough so that $\gamma \geq \max(\alpha, \delta + 1/\varepsilon')$.

We are now able to estimate $\chi^{\psi',\varepsilon'}$. Indeed, let $k2^{-j}$ be such that $2^{-j(\rho+\varepsilon')} \leq |k2^{-j}| \leq 2^{-j(\rho-\varepsilon')}$. One has

$$e_{j,k} = \sum_{k'2^{-j'} \in N_{\varepsilon',k2^{-j}}} A(k2^{-j}, k'2^{-j'}) d_{j,k} + \sum_{k'2^{-j'} \notin N_{\varepsilon',k2^{-j}}} A(k2^{-j}, k'2^{-j'}) d_{j,k} = (1) + (2).$$

Using Lemma 4.2, one gets that the first sum contains at most

$$\sum_{j'=[j(1-\varepsilon')]}^{[j(1+\varepsilon')]} 2^{j'+2} 2^{-j(1-2\varepsilon')} \leq 2^{j(1+\varepsilon')+3}\, 2^{-j(1-2\varepsilon')} = 8 2^{j3\varepsilon'}$$

terms. Each wavelet coefficient in this sum is bounded by $C2^{-j(1-\varepsilon')(\alpha-\eta)}$. Moreover, since $\gamma > \delta + 1/\varepsilon'$, for every $k'2^{-j'} \in N_{\varepsilon',k2^{-j}}$, $\omega_\gamma(k2^{-j}, k'2^{-j'})$ is always lower than the constant C that appears in Lemma 4.3. Thus $|(1)| \leq C2^{-j((\alpha-\eta)(1-\varepsilon')-3\varepsilon')}$.

Since γ has been chosen large enough, by Lemma 4.3, one has $|(2)| \leq C2^{-j\alpha}$.

As a conclusion, $|e_{j,k}| \leq C2^{-j((\alpha-\eta)(1-\varepsilon')-3\varepsilon')}$ for some constant C independant of j and k. This remains true for every couple (j,k) (or equivalently for every dyadic number $k2^{-j}$) such that $2^{-j'(\rho+\varepsilon')} \leq |k2^{-j}| \leq 2^{-j'(\rho-\varepsilon')}$, thus $\chi^{\psi',\varepsilon'}(\rho) \geq (\alpha-\eta)(1-\varepsilon') - 3\varepsilon'$. This also remains true for every $\eta > 0$, hence $\chi^{\psi',\varepsilon'}(\rho) \geq \alpha(1-\varepsilon') - 3\varepsilon' = \chi^{\psi,\varepsilon}(\rho)(1-\varepsilon') - 3\varepsilon'$.

Using that for any wavelet ψ, $\chi^{\psi}(\rho) = \lim_{\varepsilon\to 0}\chi^{\psi,\varepsilon}(\rho)$, one gets $\chi^{\psi'}(\rho) \geq \chi^{\psi}(\rho)$. Since $\{\psi_{j,k}\}$ and $\{\psi'_{j,k}\}$ are generic wavelets, this shows that $\chi(\rho)$ is independent of the orthonormal wavelet basis $\{\psi_{j,k}\}$.

The same technique applies to $\chi(0)$ and $\chi(1)$. ∎

It is thus justified to say that χ contains more information than the 2-microlocal frontier.

Then the following theorem, analogous to Theorem 4.1 in the continuous case, holds :

Theorem 4*(bis)* *Let f be in $\mathcal{S}'(\mathbb{R})$. The 2-microlocal frontier of f at any x_0 is given by :*

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho\in[0,1]}(\rho s' + \chi(\rho)).$$

Theorem 4 *bis* is important since it allows to build functions with explicit formulas for their wavelet coefficients. The proof of Theorem 4 *bis* is an easy adaptation of the one of Theorem 4.1. It also uses Lemma 4.1. This is left to the reader.

4.1.4 Time Domain Version of $\tilde{\chi}$

In the spirit of [54] and [82], a "time domain" equivalent to Theorem 4.1 may be obtained as follows. As recalled in Theorem 3.1, when $(\sigma, s') \in T_0 = \{(\sigma, s') : 0 < \sigma < 1, -1 \leq s' \leq 1, \sigma \geq s'\}$, f belongs to $C_0^{s,s'}$ if and only if there exist a positive real δ and a constant C such that $\forall(x,y)$, $0 < |x-x_0| < \delta$, $0 < |y-x_0| < \delta$,

$$|f(x) - f(y)| \leq C|x-y|^{s+s'}(|x-x_0| + |x-y|)^{-s'}.$$

This yields the following alternative equation for the 2-microlocal frontier :

$$\sigma(s') = \liminf_{x\to x_0}\inf_{y:|y-x_0|<|x-x_0|}\left(\frac{\log|f(x)-f(y)|}{\log|x-y|} + s'\frac{\log(|x-x_0|+|x-y|)}{\log|x-y|}\right)$$

This is formally analogous to (4.1) if we identify a with $|x-y|$, b with $x-x_0$, and $C(a,b)$ with $|f(x)-f(y)|$. Thus, we expect the following relation to hold :

$$(\sigma, s') \in T_0 \Rightarrow \sigma_{x_0}(s') = \inf_{\rho\in[0,1]}(\rho s' + \xi_{x_0}(\rho)) \tag{4.18}$$

where :

$$\begin{aligned}
\xi_{x_0}(0) &= \lim_{\rho\to 0^+}\tilde{\theta}^0_{x_0}(\rho)\\
\xi_{x_0}(\rho) &= \lim_{\varepsilon\to 0^+}\chi^{\tilde{\varepsilon}}_{x_0}(\rho)\\
\xi_{x_0}(1) &= \lim_{\rho\to 1^-}\tilde{\theta}^1_{x_0}(\rho)
\end{aligned}$$

and

$$\begin{array}{rcl}\tilde\theta^0_{x_0}(\rho) & = & \sup\left\{\gamma : \begin{array}{c}\exists b_o>0,\ \ \forall x,y \text{ with } |y-x_0|<|x-x_0|,\\ |x-y|^\rho \le |x-x_0| < b_o \Rightarrow |f(x)-f(y)|\le C|x-y|^\gamma\end{array}\right\},\\ \tilde\theta^1_{x_0}(\rho) & = & \sup\left\{\gamma : \begin{array}{c}\exists b_o>0,\ \ \forall x,y \text{ with } |y-x_0|<|x-x_0|,\\ |x-x_0| \le |x-y|^\rho < b_o \Rightarrow |f(x)-f(y)|\le C|x-y|^\gamma\end{array}\right\},\\ \tilde{\chi^\varepsilon}_{x_0}(\rho) & = & \sup\left\{\gamma : \begin{array}{c}\exists b_o>0, \forall x \text{ with } |x-x_0|\le b_o, \forall\beta\in[\rho-\varepsilon,\rho+\varepsilon],\\ |f(x)-f(x-|x-x_0|^{\frac{1}{\beta}}\mathrm{sgn}(x-x_0))| \le C|x-x_0|^{\frac{\gamma}{\beta}}\end{array}\right\}\end{array}$$

(sgn(x) denotes the sign of x). These quantities should be compared to the ones in Definitions (4.1) and (4.2). Note that they need to be modified when $(\sigma, s') \notin T_0$. This is achieved in the same manner as in [82]. The details are left to the reader.

Let us put formula (4.18) to use in the simplest case of the function $f(x) = |x|^\gamma$, $0 < \gamma < 1$. A straightforward computation yields $\xi_0(\rho) = \rho(\gamma - 1) + 1$ and :

$$\begin{array}{rcl} s' \le 1-\gamma & : & \sigma_0(s') = s' + \gamma \\ s' \ge 1-\gamma & : & \sigma_0(s') = 1 \end{array}$$

This example shows that, as expected, (4.18) gives the right frontier inside T_0, but may yield wrong results outside T_0.

Remark finally that χ_0 and ξ_0 differ : In general, there is no reason why the "time domain" and the "wavelet domain" 2-microlocal spectra should coincide.

4.1.5 Relations between the 2-microlocal Spectrum χ and the Regularity Exponents

To each point x_0 is associated its 2-microlocal spectrum χ_{x_0}. In the same way as the regularity exponents can be deduced from $\sigma_{x_0}(s')$, Theorem 4.1 allows to link them with $\chi_{x_0}(\rho)$. In particular, the following set of relations holds.

Proposition 4.4 *1. If $f \in C^\gamma(\mathbb{R})$ for some $\gamma > 0$, $\alpha_l(x_0) = \inf\{\chi_{x_0}(\rho) : \rho \in [0,1]\}$*

2. If $f \in C^\gamma(\mathbb{R})$ for some $\gamma > 0$, $h_f(x_0) = \inf\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0,1]\}$

3. $\beta_w(x_0) = \chi_{x_0}(1) \ \in [0, +\infty]$

4. If $h_f(x_0) < +\infty$, then $\beta_0(x_0)$ is the smallest real number β that satisfies

$$\chi_{x_0}\left(\frac{1}{\beta+1}\right) = \frac{h_f(x_0)}{\beta+1}.$$

5. $\alpha_l(x_0) = h_f(x_0) \Rightarrow \alpha_l(x_0) = \chi_{x_0}(1)$ and $\beta_o(x_0) = 0$

6. $\alpha_l(x_0) = h_f(x_0) \Rightarrow \chi_{x_0}(\rho) \ge h_f(x_0)$, $\forall\rho$

Proof : We omit the subscript x_0 in the proof to simplify the notations.
1. follows from Theorem 4.1. Indeed, α_l corresponds to $\sigma(0)$, i.e. the intersection between the frontier and the s-axis. Then we use $\sigma(0) = \inf_\rho(0s' + \chi(\rho)) = \inf_\rho \chi(\rho)$.

h_f corresponds to the intersection between the frontier and the second bisector, thus to the s' (if it exists) such that $\sigma(s') = 0 = \inf_{\rho\in[0,1]}(\rho s' + \chi(\rho))$. This leads to 2. If $h_f = +\infty$, this intersection does not exist. By Legendre transform $\chi(\rho) = +\infty$ if $\rho \in (0, 1]$. Thus $h_f = \inf\{\frac{\chi(\rho)}{\rho} : \rho \in (0, 1]\} = +\infty$.

Relation 3. follows simply from :

$$\chi(1) = -\inf_{s'}(s' - \sigma(s')) = \sup_{s'}(\sigma(s') - s') = \lim_{s'\to-\infty}(\sigma(s') - s')$$

Proposition 1.7 gives $\beta_o(x_0) = \left((\frac{\partial\sigma}{\partial s'})_{\text{left}}(-h_f)\right)^{-1} - 1$. To prove 4., remark first that if $h_f < +\infty$, there exists at least one $\rho > 0$ such that $\chi(\rho) = \tilde{\chi}(\rho) = \rho h_f$ (this is due to Lemma 4.1). Assume for simplicity that $s' \to \sigma(s')$ is differentiable at all s'. Then the following parametric form holds for $\tilde{\chi}$:

$$\begin{cases} \rho & = & \frac{d\sigma}{ds'}(s') \\ -\tilde{\chi}(\rho) & = & s'\frac{d\sigma}{ds'}(s') - \sigma(s') \end{cases}$$

When $s' = -h_f$, $\beta_o = \rho^{-1} - 1$ and thus β_o is defined by

$$\begin{aligned} \tilde{\chi}(\rho) &= \tilde{\chi}(\frac{1}{\beta_0 + 1}) \\ &= h_f b + \sigma(-h_f) \\ &= \frac{h_f}{\beta_o + 1}. \end{aligned}$$

since $\sigma(-h_f) = 0$.

Eventually, 5. and 6. are obvious. ∎

Remarks :

- No remarkable relation seems to hold between χ and β_c.
- The values of $\chi(0)$ and h_f are independent.
- $\chi(1)$ controls the shape of the asymptotic branch of the 2-microlocal frontier (when $s' \to -\infty$). Heuristically, for a "multifractal" function for which all the levels sets of h_f are dense, "most of the time", $\chi(1)$ is finite, and thus the 2-microlocal frontier, in the (s, s') plane, has a vertical asymptote $s = \sup\{r : \exists s' \in \mathbb{R},\ f \in C^{r,s'}_{x_0}\}$.
- *5*. shows that, when the local and the pointwise exponents coincide, their common value can be inferred using only the wavelet coefficients "above" the considered point. This case is favorable since it leads to simple estimation procedures.
- The uniform Hölder condition $f \in C^{\gamma}(\mathbb{R})$ is necessary for *2*. to hold in Proposition 4.4. See subsection 4.1.6 for an example of a function with $h_f(x_0) \neq \inf\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0, 1]\}$.

4.1.6 Examples

We provide now examples of computations of σ and χ on various functions, in view of obtaining a more concrete understanding of their relation.

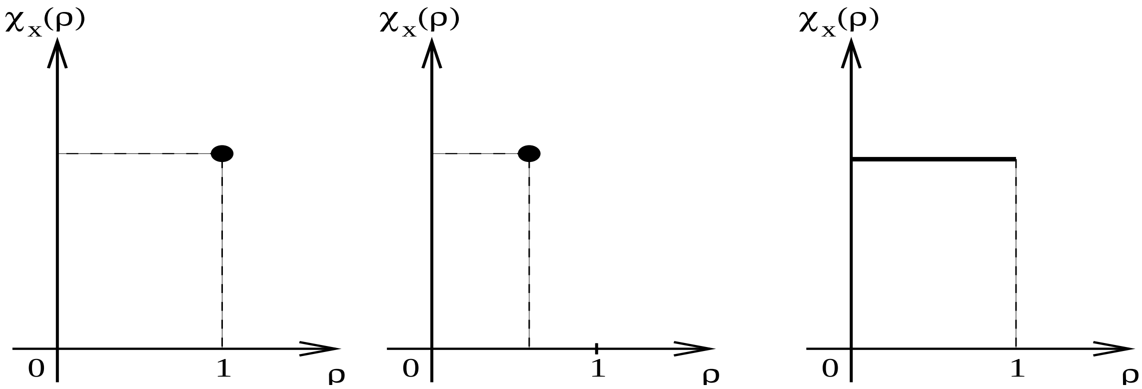

FIG. 4.1 – Typical 2-microlocal spectra : Cusp, Chirp and Weierstrass function.

Simple Functions

Let us start with the simplest function, i.e. the cusp function $x \to |x|^\gamma$. The wavelet coefficients that are not "above" 0 have a fast decay. This implies that $\chi_0(\rho) = +\infty$ for $\rho \in [0,1)$. On the other hand, inside the cone, the largest wavelet coefficients behave like $2^{-j\gamma}$, thus $\chi_0(1) = \gamma$. One easily verifies that $(-\chi_0)^*(s') = \sigma_0(s') = \gamma + s'$.

Let us now apply Theorem 4.1 to the case of a chirp. Recall that, for $f(x) = |x|^\gamma \sin \frac{1}{|x|^\beta}$, $\gamma > 0$, $\beta \geq 0$, one has at 0 :

$$s' = -\frac{\beta+1}{\beta} s + \frac{\gamma}{\beta},$$

which is the same as

$$\sigma(s') = \frac{1}{\beta+1} s' + \frac{\gamma}{\beta+1}.$$

Applying the Legendre transform to $\sigma(s')$ leads to

$$\begin{cases} \tilde{\chi}(\rho) = \infty & \rho \neq \frac{1}{\beta+1} \\ \tilde{\chi}(\frac{1}{\beta+1}) = \frac{\gamma}{\beta+1} & \end{cases}$$

and thus to $\chi = \tilde{\chi}$. We see that we recover the well-known fact that the wavelet coefficients of the chirp have fast decay everywhere except around the curve $a = |b|^{\beta+1}$, on which the largest coefficients verify $C(a, \pm a^{\frac{1}{\beta+1}}) \sim a^{\frac{\gamma}{\beta+1}}$.

Note that, while $\sigma(s')$ is always concave, χ does not have to be convex, and this is why $\tilde{\chi}$ had to be introduced. (Take for instance the sum of two chirps $f(x) = |x|^\gamma \sin \frac{1}{|x|^{\beta_1}} + |x|^\gamma \sin \frac{1}{|x|^{\beta_2}}$, with $\gamma > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 \neq \beta_2$.)

We end this subsection with an example where $\chi(\rho)$ is finite for all ρ : consider the function f_γ, whose wavelet coefficients are $d_{j,k} = 2^{-j\gamma}$ for all j,k, where $\gamma \in (0,1)$. By definition, $\chi(\rho) = \gamma$ for all x and $\rho \in [0,1]$. Taking the Legendre transform of $\chi(\rho)$, we get $\sigma(s') = \gamma$ for $s' \geq 0$ and $\sigma(s') = \gamma + s'$ for $s' \leq \gamma$ (since all points x have the same 2-microlocal features, we drop the subscript x). In particular, $h_f = \alpha_l = \gamma$. In the (s,s') plane, the frontiers are parallel to the second bisector for $s' > 0$ and vertical for $s' \leq 0$. It is easy to see that the Weierstrass function $\sum_{n=1}^{+\infty} \lambda^{-n\gamma} \sin(2\pi\lambda^n x)$ also has $\chi_x(\rho) = \gamma$ for all ρ and all x.

More Elaborate Functions

In [34], a version of Theorem 4.1 was given using, in place of $\chi(\rho)$, the function ξ defined as follows

$$\begin{aligned} \xi(\rho) &= \sup\{\gamma : |C(a, x_0 \pm a^\rho)| \leq C a^\gamma, \ \forall a < b_0\}, \\ \xi(1) &= \sup\{\gamma : |C(a, x_0 \pm b)| \leq C a^\gamma, \ \forall 0 \leq b \leq a < 1\}. \end{aligned}$$

(note that ξ is not defined at 0).

In simple cases, it is true that the 2-microlocal frontier $\sigma(s')$ is given by the Legendre transform of ξ [2]. However, this statement is wrong in general. The heuristic reason is that the function ξ does not consider "enough" wavelet coefficients, as shown by the two following examples.

1 - Let us consider the function f_{β_1,β_2} (for $0 < \beta_2 < \beta_1 < 1$) constructed in [82], Theorem 4.1, with $f(x) = \beta_1$, $g(x) = \beta_2$. This specific function has a local Hölder exponent equal to β_2 everywhere, while the pointwise Hölder exponent equals β_1 everywhere except on a set of Hausdorff dimension 0.

Let us study the 2-microlocal frontier at 0 of f_{β_1,β_2}. With the help of Proposition 6.2 in [82], one easily computes that $\xi(\rho) = \chi(\rho) = \beta_1$ for all $\rho \in (0, 1]$. The Legendre transform $\tilde{\sigma}(s')$ of ξ reads

$$\begin{aligned} \tilde{\sigma}(s') &= \beta_1 + s' \text{ if } s' < 0 \\ \tilde{\sigma}(s') &= \beta_1 \text{ if } s' \geq 0 \end{aligned}$$

In particular, $\sigma = \tilde{\sigma}$ would imply that f_{β_1,β_2} belongs to $C_0^{\beta_1-\varepsilon,0}$ for any $\varepsilon > 0$. This is not true : Indeed, since the local Hölder exponent of f_{β_1,β_2} at 0 is β_2, f_{β_1,β_2} can not belong to any $C_0^{s,0}$ for $s > \beta_2$.

From this first example one sees that considering only curves $b = a^\rho$ for $\rho > 0$ in the time-frequency plane is not enough. It is necessary to consider the case "$\rho = 0$", i.e. to define properly $\chi(0)$.

Indeed, the computation of $\chi(0)$ for the function f_{β_1,β_2} yields $\chi(0) = \beta_2$, leading to

$$\begin{aligned} \sigma(s') &= \beta_2 + s' \text{ if } s' < (\beta_2 - \beta_1) \\ \sigma(s') &= \beta_2 \text{ if } s' \geq (\beta_2 - \beta_1), \end{aligned}$$

which is the correct 2-microlocal frontier of f_{β_1,β_2} at 0 (this is left to the reader). Note that χ is not continuous at 0. ■

[2]In particular, this will occur when $\varepsilon \to \chi^\varepsilon(\rho)$ is continuous at $\varepsilon = 0^+$ and $\rho \to \chi(\rho)$ is continuous at 0 and 1. This is because $\xi(\rho) = \chi^{\varepsilon=0}(\rho)$ for $\rho \in (0, 1)$, and χ and $\tilde{\chi}$ coincide at 0 and 1

2 - A second, and distinct, difficulty occurs when one considers ξ instead of χ : The function ξ is "too focused" on the curves $b = a^\rho$, and this is the reason why a regularization procedure is necessary to properly define χ.

Consider the function f defined by its wavelet coefficients $d_{j,k}$ in an orthonormal wavelet basis $\{\psi_{j,k}\}_{(j,k)}$. The $d_{j,k}$'s are set as follows. For all $n \in \mathbb{N}$, define the integers j_n and k_n by

$$\begin{aligned} j_n &= 2^n, \\ k_n &= 2^{\frac{j_n}{2}} + 2n \end{aligned}$$

Now, we set, with $\beta > \delta > 0$,

$$\begin{aligned} d_{j_n,k_n} &= 2^{-j_n\delta} \ \forall n \in \mathbb{N}, \\ d_{j,k} &= 2^{-j\beta} \text{ for every other couple of indices } (j,k). \end{aligned}$$

This function $f = \sum_{j,k} d_{j,k}\psi_{j,k}$ is clearly well-defined and continuous. Moreover, $f \in C^\delta(\mathbb{R})$, since $\forall (j,k)$, $|d_{j,k}| \leq 2^{-j\delta}$. Let us compute the function ξ associated to this function f at point 0.

For all $\rho > 0$, we denote by $k_{j,\rho}$ the integer $[2^{j(1-\rho)}]$.

- if $\rho = 1/2$, for all $n \in \mathbb{N}$, $k_n = k_{j_n,1/2} + 2n$, thus $dj_n, k_{j_n,1/2} = 2^{-j_n\beta}$. Thus, for all j, $d_{j,k_j,1/2} = 2^{-j\beta}$, and one deduces $\xi(1/2) = \beta$.
- if $\rho \in]0,1[\setminus\{1/2\}$, then there exists N_ρ such that $n \geq N_\rho$ implies $|k_n - k_{j_n,\rho}| \geq 2$. Indeed, $k_n \sim 2^{j_n(1-1/2)}$ while $k_{j_n,\rho} \sim 2^{j_n(1-\rho)}$ when $n \to +\infty$.
  Thus, for $j \geq 2^{N_\rho}$, one has $d_{j,k_{j,\rho}} = 2^{-j\beta}$. One thus concludes that $\xi(\rho) = \beta$.
- One eventually has $\xi(1) = \xi(0) = \beta$, since the "bad" coefficients (those equal to $2^{-j\delta}$) are located around the curve $k2^{-j} = 2^{-j/2}$.

Thus $\xi(\rho) = \beta$ for all $\rho \in [0,1]$. Applying the Legendre transform to ξ yields

$$\begin{aligned} \sigma(s') &= \beta + s' \text{ if } s' < 0 \\ \sigma(s') &= \beta \text{ if } s' \geq 0 \end{aligned}$$

In particular this would imply that f belongs to $C_0^{\beta-\varepsilon,0}$ for all $\varepsilon > 0$. This means that for all (j,k) such that $|k2^{-j}|$ is close enough to 0, $|d_{j,k}| \leq C2^{-j(\beta-\varepsilon)}$. This is obviously wrong, since by construction some of them are equal to $2^{-j\delta}$.

The problem here comes from the fact that the constants $C_{\rho,\gamma}$ used in the definition of $\xi(\rho) = \sup\{\gamma : |d_{j,k_{j,\rho}}| \leq C_{\rho,\gamma}2^{-j\gamma}\}$ are not uniformly bounded in ρ. In particular, in this example, they tend to infinity, and this does not allow to obtain a global bound for the decay of the wavelet coefficients. This leads to a wrong 2-microlocal frontier. On the contrary, if one computes the values of $\chi(\rho)$, one finds

$$\begin{aligned} \chi(\rho) &= \beta \text{ if } \rho \neq 1/2, \\ \chi(1/2) &= \delta, \end{aligned}$$

which gives by Legendre transform the right 2-microlocal frontier (this is left to the reader). ■

A distribution and a function with no positive Hölder regularity

We end this section with two examples where the condition $\alpha_l > 0$ is not verified.

1 - Consider the function $g : x \to \sqrt{|x|}\sin(2\pi \exp(1/|x|)), g(0) = 0$. Following the same lines of computation as in the case of the chirp (see [59]), it is easy to show that, in the neighbourhood of 0, the ratio $|g(x) - g(y)|/|x - y|$ is large only around the sequences $x_k = \frac{1}{\log((4k+1)/4)}$ and $y_k = \frac{1}{\log((4k+3)/4)}$. Now :

$$\log|x_k - y_k| \sim -\log k \text{ when } k \to +\infty,$$

and :

$$\log|g(x_k) - g(y_k)| \sim -1/2 \log\log(k).$$

As a consequence, $\chi_0(\rho) = \infty$ for $\rho \neq 0$, and $\chi_0(0) = \alpha_l = 0 < h_f = 0.5$. Theorem 4.1 yields that $\sigma(s') = 0$ for all s', i.e. the 2-microlocal frontier of g is the second bisector in the (s, s')-plane : In this case of a function with no positive uniform Hölder regularity, h_f is not given by the intersection of the frontier with the s' axis. The formula $h_f(0) = \inf_{\rho\in(0,1]}\left(\frac{\chi_0(\rho)}{\rho}\right)$ does not apply.

Note that in this case the 2-microlocal frontier can be obtained without computations as follows. Any primitive $g^{(-n)}$ of order n of g has infinitely fast oscillations around 0. The oscillating exponent of $g^{(-n)}$ is thus $+\infty$ for all n. The last item of Proposition 1.7 entails that the derivative of the function $s' \to \sigma(s')$ is 0 at infinitely many points. As the frontier is convex, it has to be constant. The fact that $\sigma(0) = 0$ allows to conclude that $\sigma(s') = 0$ for all s'.

The same arguments apply to any function that has the form $g_h : x \to |x|^h w(2\pi \exp(1/|x|))$, where w is an oscillating function in the sense of [51].

2 - On the other hand, the case of the Dirac distribution δ_0 at 0 shows that one may have $h_f(0) = \inf_{\rho\in(0,1]}\left(\frac{\chi_0(\rho)}{\rho}\right)$ even though $\delta_0 \notin \cup_{\varepsilon>0}C^{\varepsilon}(\mathbb{R})$. Assume the wavelet ψ has compact support. Then the wavelet coefficients of δ_0 that are located outside the "cone" above 0 vanish. The largest wavelet coefficients inside the cone grow as 2^j when $j \to +\infty$. As a consequence, $\chi_0(1) = -1$ and $\chi_0(\rho) = +\infty$ for $\rho \in [0, 1)$.

Theorem 4.1 entails that $\sigma(s') = -1 + s'$ for all s'. If one could apply Proposition 4.4, one would get $\alpha_l(0) = h_f(0) = -1$, $\beta_w(0) = -1$, $\beta_o(0) = 0$. Although the definitions we have set for α_l and h_f do not make sense for distributions, the values $\alpha_l(0) = h_f(0) = -1$ are perfectly meaningful : δ_0 is indeed not oscillatory at 0, with regularity exponents equal to -1, in the sense, e.g., that $\delta_0(\lambda.) = \lambda^{-1}\delta_0(.)$.

4.1.7 d-dimensional case

The 2-microlocal spectrum, as well as the other exponents, can be defined in any dimension. The definitions are slightly modified as shown below, but the 2-microlocal formalism still holds.

Definition 4.8 *Let $f \in \mathcal{S}'(\mathbb{R}^d)$, and denote $C(a, b)$ its wavelet transform using a wavelet of sufficient regularity. For a given $x_0 \in \mathbb{R}^d$, define :*

– $\theta^0 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$

$\theta^0(\varepsilon) = \sup\{\gamma : \exists b_o > 0, a^\varepsilon \leq \|b - x_0\| < b_o \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$

– $\chi^\varepsilon : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup \{+\infty\}$

$$\chi^\varepsilon(\rho) = \sup\left\{\gamma : \begin{array}{c} \exists b_o > 0, \forall a < b_0, \forall b \text{ with } a^{\rho+\varepsilon} \leq \|b - x_0\| \leq a^{\rho-\varepsilon}, \\ |C(a, x_0 + b)| \leq C_{\rho,\varepsilon} a^\gamma \end{array}\right\}$$

– $\theta^1 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$

$\theta^1(\varepsilon) = \sup\{\gamma : \exists b_o > 0, \|b - x_0\| < \min(b_o, a^\varepsilon) \Rightarrow |C(a, x_0 + b)| \leq K_\varepsilon a^\gamma\}$

The 2-microlocal spectrum χ in the d-dimensional case is defined as in dimension 1 :

Definition 4.9 *Define, for any given x_0, $\chi : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ by*

– $\chi(0) = \theta^0(0)$

– $\rho \in (0,1)$ *:* $\chi(\rho) = \lim_{\varepsilon \to 0^+} \chi^\varepsilon(\rho)$

– $\chi(1) = \theta^1(1)$

The following analog to Theorem 4.1 holds :

Theorem 4*(ter)* *Let f be a function in $\mathcal{S}'(\mathbb{R}^d)$. The 2-microlocal frontier of f at any $x_0 \in \mathbb{R}^d$ is given by :*

$$\sigma(s') = (-\chi)^*(s') = \inf_{\rho \in [0,1]} (\rho s' + \chi(\rho))$$

4.2 The Neighbourhood Exponent $\chi_x(0)$

The value of $\chi_x(0)$ provides a new regularity exponent, which gives information complementary to α_l and h_f. Since $\chi_x(0)$ is concerned only with what happens in the neighbourhood of x, we shall call it the *neighbourhood exponent*. It appears that $\chi_x(0)$ as well as the function $x \to \chi_x(0)$ hold essential information both from a multifractal and a 2-microlocal point of view. In this section, we describe some of the properties of the neighbourhood exponent.

4.2.1 Relations with other exponents

The following proposition is an obvious consequence of the shape of the frontier :

Proposition 4.5 *Assume $\alpha_l(x) > 0$. Then $\alpha_l(x) \leq \min(h_f(x), \chi_x(0))$.*

Note that there is no general relation between $h_f(x)$ and $\chi_x(0)$, as the following examples show (such a relation is not expected, since, for instance, the wavelet coefficients that contribute to $h_f(x)$ and $\chi_x(0)$ may belong to different regions of the (a, b) plane).

Consider first the function $x \to |x|^\gamma$. In this case, the exponents at 0 are $h_f(0) = \alpha_l(0) = \gamma < \chi_0(0) = +\infty$.

For the reverse inequality $\chi_x(0) < h_f(x)$, recall subsection 3.5.2 and the function f_{β_1,β_2}. We have proved that in this case, $\chi_0(0) = \beta_2$ while $h_f(x) = \beta_1 > \beta_2$. For a more explicit example, consider the function f given by $f(x) = \exp(-1/|x|)\sin(2\pi \exp(1/|x|))$, $f(0) = 0$. Following the same lines of computation as in the case of the chirp [59], one finds that $|f(x)-f(y)|/|x-y|$ is large only around the sequences $x_k = \frac{1}{\log((4k+1)/4)}$, $y_k = \frac{1}{\log((4k+3)/4)}$. Since

$$\log(x_k - y_k) \sim -\log(k), k \to \infty$$

and

$$\log|f(x_k) - f(y_k)| \sim -\log(k), k \to \infty,$$

one gets that $\chi_0(\alpha) = +\infty$ for $\alpha \neq 0$ and $\chi_0(0) = 1$. In particular, $\alpha_l(0) = \chi_0(0) = 1 < \alpha_p(0) = +\infty$. Note that such a function, which has uniform regularity 1, has for frontier the straight line $\sigma(s') = 1$, which never crosses the second bisector. This is consistent with the fact that $h_f = +\infty$ and Proposition 1.6.

Proposition 4.6 *Let $f \in \mathcal{S}'(\mathbb{R})$.*

$$\chi_x(0) = \sup_{s' \in \mathbb{R}} \sigma(s') = \lim_{s' \to +\infty} \sigma(s')$$

The proof is obvious.

The exponent $\chi_x(0)$ shares an important property with α_l. Indeed, the stability of $C_x^{s,s'}$ spaces with respect to fractional integro-differentiation obviously entails the one of $\chi_x(0)$ (remember that this is not the case for h_f). Thus, for instance, for any f and x, $\chi_x^{f'}(0) = \chi_x^f(0) - 1$.

Proposition 4.7 *The exponent $\chi_x(0)$ is stable under the action of fractional integro-differentiation.*

4.2.2 Properties of $x \to \chi_x(0)$, Links with Multifractal Analysis and Compatibility Conditions between 2-microlocal Frontiers

The first two propositions show that, not surprisingly, $\chi_x(0)$ is intimately related to the regularity of the points lying in the neighbourhood of x.

Proposition 4.8 *If $\chi_x(0) < +\infty$, then $\forall \delta > 0$, there exists a neighbourhood V_x^δ of x such that, for all $y \in V_x^\delta$, for all $\rho \in [0,1]$, $\chi_y(\rho) \geq \chi_x(0) - \delta$.*

Moreover, for every $y \in V_x^\delta$, one has $\sigma_y(s') \geq \sigma^{\chi(0),\delta}(s')$, where $\sigma^{\chi(0),\delta}(s')$ is defined by

$$\begin{aligned} \sigma^{\chi(0),\delta}(s') &= (\chi_x(0) - \delta) + s' \text{ if } s' < 0 \\ \sigma^{\chi(0),\delta}(s') &= (\chi_x(0) - \delta) \text{ if } s' \geq 0 \end{aligned}$$

Proof : Let x be such that $\chi_x(0) < +\infty$. By definition, for all $\delta > 0$, there exists $\rho_\delta > 0$ such that $\theta_x^0(\rho_\delta) \geq \chi_x(0) - \delta/2$, where we recall that

$$\theta_x^0(\rho) = \sup \left\{ \gamma : \begin{array}{c} \exists b, C_{\rho,\gamma} \text{ such that } \forall (j,k) \text{ with} \\ 2^{-j\rho} \leq |k2^{-j} - x| \leq b, \text{ one has } |d_{j,k}| \leq C_{\rho,\gamma} 2^{-j\gamma} \end{array} \right\}$$

Thus there exists b_δ and a constant C such that, $\forall (j,k)$ with $2^{-j\rho_\delta} \leq |k2^{-j} - x| \leq b_\delta$, one has $|d_{j,k}| \leq C 2^{-j(\chi_x(0)-\delta)}$. Let us denote by $\Gamma_{x,\delta}$ this set of coefficients.

Let now y be in $(x - b_\delta, x + b_\delta)$, and consider $\eta_y > 0$ such that $[y - \eta_y, y + \eta_y] \subset ((x - b_\delta, x) \cup (x, x + b_\delta))$.

Let $\rho \in [0, 1]$, $\varepsilon > 0$, and let us compute $\chi_y^\varepsilon(\rho)$. One knows that, for all (j, k) such that $|k2^{-j} - y| \leq \eta_y$ and $2^{-j\rho_\delta} \leq |k2^{-j} - x|$,

$$|d_{j,k}| \leq C2^{-j(\chi_x(0)-\delta)}. \tag{4.19}$$

These wavelet coefficients are the ones located around y, with a scale 2^{-j} small enough so that $d_{j,k} \in \Gamma_{x,\delta}$.

Hence, for j large enough, all the coefficients used for the computations of $\chi_y^\varepsilon(\rho)$ verify (4.19). One deduces that $\chi_y^\varepsilon(\rho) \geq \chi_x(0) - \delta$, for all $\varepsilon > 0$.

Letting $\varepsilon \to 0$ leads to the result, i.e. for all $y \in (x - b_\delta, x + b_\delta)$, $\forall \rho \in [0, 1]$, $\chi_y(\rho) \geq \chi_x(0) - \delta$.

The second part of Proposition 4.8 simply follows from the first part, since $\chi_y(\rho) \geq \chi_x(0) - \delta$ $\forall \rho \in [0, 1]$ implies, by applying the Legendre Transform, that $\sigma_y(s') \geq \sigma^{\chi(0),\delta}(s')$ $\forall s' \in \mathbb{R}$. ■

The next proposition, somehow complementary to the previous one, shows that if $\chi_x(0) = +\infty$, then f must be very regular in the (excluded) neighbourhood of x.

Proposition 4.9 *If $\chi_x(0) = +\infty$, then for all $n \in \mathbb{N}$, there exists a neighbourhood V_x^n of x such that $f \in C^n(V_x^n \backslash \{x\})$.*

Proof : The proof is similar to the one of Proposition 4.8. Let x be such that $\chi_x(0) = +\infty$. For all $n > 0$, there exists $\rho_n > 0$ such that $\theta_x^0(\rho_n) \geq n$.

Thus there exists b_n and a constant C such that, $\forall (j, k)$ such that $2^{-j\rho_n} \leq |k2^{-j} - x| \leq b_n$, one has $|d_{j,k}| \leq C2^{-jn}$. Denote by $\Gamma_{x,n}$ this set of coefficients.

Let now y be in $(x - b_n, x + b_n)$, and consider η_y such that $(y - \eta_y, y + \eta_y) \subset ((x - b_n, x) \cup (x, x + b_n))$. Let $\rho \in [0, 1]$, $\varepsilon > 0$, and let us compute $\chi_y^\varepsilon(\rho)$. One knows that, for all (j, k) such that $|k2^{-j} - y| \leq \eta_y$ and $d_{j,k} \in \Gamma_{x,n}$, one has

$$|d_{j,k}| \leq C2^{-jn}.$$

Hence, using the same arguments as in Proposition 4.8, $\chi_y^\varepsilon(\rho) \geq n$, for all $\varepsilon > 0$.

This leads to the following property : for all $y \in (x - b_n, x + b_n)$, $\forall \rho \in [0, 1]$, $\chi_y(\rho) \geq n$.

In particular, for all $y \in (x - b_n, x + b_n)$, the local Hölder exponent at y, which is equal to $\inf_\rho(\chi_y(\rho))$, is larger than n. This concludes the proof. ■

Combining Propositions 4.8 and 4.9 yields in fact the equivalence in both of them.

Rephrasing Proposition 4.9 as follows yields a link between 2-microlocal and multifractal analysis. Recall first that, for a function f, the set E_α is defined by

$$E_\alpha = \{x : h_f(x) = \alpha\},$$

and the Hausdorff spectrum f_h of f is

$$f_h(\alpha) = d_H(E_\alpha),$$

where d_H is the Hausdorff dimension.

Corollary 4.2 *Assume $f \in C^\gamma$ for some $\gamma > 0$. Then :*

$$\sup_{x\in\mathbb{R}} \chi_x(0) = +\infty \Rightarrow \ Supp(f_h) \ unbounded.$$

Proof : The result simply follows from the inequality $h_f(x) \geq \alpha_l(x)$ and Proposition 4.9. ■

In the same spirit, one has

Proposition 4.10 *Let $f \in C^\gamma(\mathbb{R})$ with $\gamma > 0$. Then $E_\infty = \{x : h_f(x) = +\infty\}$ is included in $\{x : \exists!\, \rho$ such that $\chi_x(\rho) < \infty\}$.*

Proof : If $h_f(x) = +\infty$, then the 2-microlocal frontier of f at x (in the (s, s')-plane) never crosses the second bisector. Using that $h_f(x) = \inf_\rho(\frac{\chi_x(\rho)}{\rho})$, this means that $\chi_x(\rho) = +\infty$ whenever $\rho \in (0, 1]$. Only $\chi_x(0)$ may be finite. ■

Proposition 4.11 *Let f_h denote the Hausdorff spectrum of a function $f \in C^\gamma$ for some $\gamma > 0$. If the support of f_h is bounded, for every point x there are at least two exponents ρ such that $\chi_x(\rho) < +\infty$.*

Proof : Assume Supp(f_h) is bounded, and that there exists a point x such that there exists a unique ρ with $\chi_x(\rho) < +\infty$. In this case, the 2-microlocal frontier of f at x is by assumption a straight line, which has a slope less or equal than -1 (in the (s, s')-plane).

Assume the slope is strictly less than -1. Then, using that $\chi_x(0) = \lim_{s'\to+\infty} \sigma(s')$, we get that $\chi_x(0) = +\infty$. Corollary 4.2 then entails that Supp(f_h) is unbounded, hence a contradiction.

Thus the 2-microlocal frontier of f at x must be parallel to the second bisector, which implies $\chi_x(\rho) = +\infty$ for all $\rho \in (0, 1]$, and thus $h_f(x) = +\infty$. This leads to the same contradiction. ■

The following proposition shows that the neighbourhood exponent function satisfies the same regularity constraint as the local Hölder function.

Proposition 4.12 *The function $x \to \chi_x(0)$ (which maps $\mathbb{R}$ to $[0, +\infty]$) is a lower semi-continuous function.*

Proof : This is a consequence of Propositions 4.8 and 4.9. Indeed, if $\chi_x(0) < +\infty$, for all $\delta > 0$, there exists a neighbourhood V_δ of x such that for all $y \in V_\delta$, $\chi_y(0) \geq \chi_x(0) - \delta$.

If $\chi_x(0) = +\infty$, for all $n > 0$, there exists a neighbourhood V_n of x such that for all $y \in V_n$, $\chi_y(0) \geq n$.

These two facts entail that $x \to \chi_x(0)$ is a lower semi-continuous function. ■

A straightforward consequence of Proposition 4.12 is given below

Proposition 4.13 *If $\alpha_l(x) = \chi_x(0)$, then the 2-microlocal frontier at x verifies : $\forall s' > 0$, $\sigma(s') = \alpha_l(x)$. This means that, in the (s, s') plane, the 2-microlocal frontier is a half line parallel to the second bisector for $s' > 0$, passing through the point $(\alpha_l(x), 0)$.*

The proof is obvious and is left to the reader. The last results imply some compatibility conditions on the 2-microlocal frontiers of a function f at neighbouring points.

Proposition 4.14 *Let $f \in C^\gamma$ for some $\gamma > 0$.*

$$\chi_x(0) \leq \liminf_{y \to x} (\inf_{\rho \in [0,1]} (\chi_y(\rho))) = \liminf_{y \to x} \alpha_l(y)$$

Proof : This is a simple consequence of Propositions 4.8 and 4.9.

Indeed, Proposition 4.8 shows that if $\chi_x(0) < +\infty$, $\forall \delta > 0$, there exists a neighbourhood V_δ of x such that for every $y \in V_\delta$, $\inf_{\rho \in [0,1]} \chi_y(\rho) \geq \chi_x(0) - \delta$.

If $\chi_x(0) = +\infty$, there exists a neighbourhood V_n of x such that for every $y \in V_n$, $\inf_{\rho \in [0,1]} \chi_y(\rho) \geq n$. Hence the required result, i.e. $\chi_x(0) \leq \liminf_{y \to x} (\inf_{\rho \in [0,1]} (\chi_y(\rho)))$.

Eventually, Proposition 4.4 gives the relation with the local Hölder exponent α_l. ∎

Combining Proposition 4.5 and Proposition 4.14, one obtains

Proposition 4.15 *For any function f in C^γ, one has*

$$\alpha_l(x) \leq \chi_x(0) \leq \liminf_{y \to x} \alpha_l(y)$$

Using the last Proposition, one recovers a well-known result [34], [80]

Corollary 4.3 *For any function $f \in C^\gamma(\mathbb{R})$ for some $\gamma > 0$, for all x, one has*

$$\alpha_l(x) \leq \liminf_{y \to x} \alpha_l(y),$$

thus $x \to \alpha_l(x)$ is a lower semi-continuous (lsc) function.

We indicate here, just as a parenthesis, that one can also recover the fact that $x \to h_f(x)$ is a $\liminf$ of a sequence of continuous functions ([21], [46]). In fact, one can prove even more :

Proposition 4.16 *Let $f \in C^\gamma(\mathbb{R})$ for some $\gamma > 0$. Then*

- *Let $\rho \in (0,1]$. The function $x \to \chi_x(\rho)$ is a* $\liminf$ *of a sequence of continuous functions.*
- *The function $x \to h_f(x)$ is a* $\liminf$ *of a sequence of continuous functions.*

Proof :

$\forall n \in \mathbb{N}$, define the function g_n by

$$\begin{array}{lll} g_n : & (0,1) \times \mathbb{R} & \to \mathbb{R}^+ \\ & (\rho, x) & \to \inf_{2^{-(n+1)} \leq a < 2^{-n}} \left(\inf_{\rho - 2^{-n} \leq \beta \leq \rho + 2^{-n}} \left(\frac{\log |C(a, x \pm a^\beta)|}{\log a} \right) \right) \end{array}$$

When n is fixed, $(x, \rho) \to g_n(\rho, x)$ is continuous in the variables (x, ρ), since the wavelet transform $(x, a, b) \to C(a, x + b)$ is a continuous function in (x, a, b).

Let $\rho \in (0,1)$. The function $x \to g_n(\rho, x)$ is thus obviously continuous in x. We let the reader check that, for all x, one has $\chi_x(\rho) = \liminf_{n\to+\infty} g_n(\rho, x)$.

If $\rho = 1$, one uses the functions $g_n(1, .)$

$$\begin{array}{rll} g_n(1,.): & \mathbb{R} \to & \mathbb{R}^+ \\ & x \to & \inf_{2^{-(n+1)} \leq a < 2^{-n}} \left(\inf_{-a^{1-2^{-n}} \leq b \leq a^{1-2^{-n}}} \left(\frac{\log |C(a, x+b)|}{\log a} \right) \right), \end{array}$$

and one checks that $\chi_x(1) = \liminf_{n\to+\infty} g_n(1, x)$. This concludes the proof of the first item.

To prove the second item, one uses the continuity of g_n with respect to (x, ρ). Indeed, this continuity implies that, $\forall n \in \mathbb{N}$, the function h_n defined by

$$\begin{array}{rll} h_n: & \mathbb{R} \to & \mathbb{R}^+ \\ & x \to & \inf_{\rho \in [2^{-n}, 1]} \left(\frac{g_n^\rho(x)}{\rho} \right) \end{array}$$

is continuous. It is now easily verified that

$$\liminf_{n\to+\infty} h_n(x) = \inf_{\rho \in (0,1]} \frac{\chi_x(\rho)}{\rho} = h_f(x).$$

■

This leads to a new result on the constraints on the regularity exponents of a function

Corollary 4.4 *The weak scaling exponent function $x \to \beta_w(x)$ is a* lim inf *of a sequence of continuous functions.*

Proof : This is a direct by-product of the last Proposition : Indeed, by Proposition 4.4, $\forall x$, $\chi_x(1) = \beta_w(x)$ and one has proved that $x \to \chi_x(1)$ is a lim inf of a sequence of continuous functions. ■

Proposition 4.13 and 4.15 have in particular a consequence in multifractal analysis. Indeed, for multifractal functions, the local Hölder function $x \to \alpha_l(x)$ is often a continuous function (for IFS or Weierstrass functions, $x \to \alpha_l(x)$ is a constant). This implies that all the frontiers have their upper-part (corresponding to $s' \geq 0$) parallel to the second bisector.

More generally, one has

Proposition 4.17 *For any f defined on an interval I, there exists a residual set* [3] *of I such that, for all $x \in I$, $\sigma(s') = \alpha_l(x)$ for all $s' > 0$.*

Proof : This is a simple consequence of the following fact : If g is a lower semi-continuous function, then g is continuous on a residual set.

¿From Proposition 4.15, $\alpha_l(x) = \chi_x(0)$ at all points x where α_l is continuous. Combining this with the fact that $x \to \alpha_l(x)$ is lower semi-continuous gives the result. ■

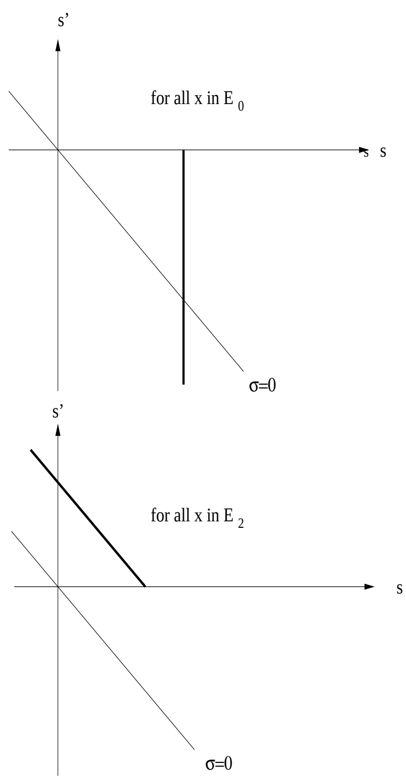

FIG. 4.2 – Typical parts of 2-microlocal frontiers.

Remark : A consequence of Proposition 4.17 is that no function f with $\alpha_l(x) < +\infty \; \forall x$ can behave as a "pure" cusp (i.e. have a vertical frontier) at all points.

In view of the propositions in this section, one has that the generic 2-microlocal frontier has an upper-part parallel to the second bisector, and an vertical asymptote when $s' \to -\infty$. Figure 4.2 illustrates this : The upper-left graph displays the frontier on a set of Hausdorff dimension 0 E_0 (this comes from [80]), the upper-right graph shows the frontier on a set E_1 of measure 0 (where $\beta_c = 0$, see [46]), the lower-left graph shows the frontier of points that belong to a residual set E_2 (Proposition 4.17), and the last graph illustrates Proposition 4.11.

4.3 2-microlocal Frontier Prescription

The problem of prescribing the 2-microlocal frontier of a distribution at one point has been solved in [33] and [72]. We propose here another way of doing so, using the 2-microlocal spectrum.

The advantage of such a method is that it can be extended to any countable dense set of points. More precisely we will be able in Section 4.3.2 to exhibit a function whose 2-microlocal frontiers are prescribed on a countable dense set of points.

We shall use the parameterization $s' \to \sigma(s')$ for the 2-microlocal frontier.

4.3.1 Prescription at one point

Theorem 4.2 *Let $g : \mathbb{R} \to \mathbb{R}$ be a concave, non-decreasing function, with slope between 0 and 1. Assume that $g(0) > 0$. There exists a function f such that the 2-microlocal frontier of f at 0 is $\sigma_0(s') = g(s')$.*

Proof :

The Legendre transform of g, $g^*(\rho) = \inf_{s' \in \mathbb{R}}(\rho s' - g(s'))$, is continuous on its support, and ranges in $[-\infty, -g(0)]$.

Let us first define the functions $\chi_0^j(\rho) = \min(j, -g^*(\rho))$. We are going to build a function f, defined on $[0,1]$, by its wavelet coefficients $\{d_{j,k}\}_{j \geq 1,k}$ on an orthonormal wavelet basis $\{\psi_{j,k}\}_{j,k}$. We will do so in a way such that $\chi_0(\rho) = -g^*(\rho)$ for all $\rho \in [0,1]$.

Let (j,k) be a couple of indices, such that $|k2^{-j}| \leq 1$ and $j \geq 1$. We denote by $E_{j,k}$ the set of exponents ρ such that $0 \leq \rho \leq 1$ and $k = [2^{j(1-\rho)}]$, and by $\beta_{j,k}$ the minimum of χ_0^j on $E_{j,k}$. Then we set, for all (j,k), $d_{j,k} = 2^{-j\beta_{j,k}}$.

By construction, $-g^*(\rho) \geq g(0)$ for all ρ. The function $f = \sum_{j,k} d_{j,k}\psi_{j,k}$ belongs to $C^{g(0)}$ around 0.

For every couple (j,k) and for every exponent $\rho \in [0,1]$, we denote by $k_{j,\rho}$ the integer $[2^{j(1-\rho)}]$. Remark that $\lim_{j \to +\infty} \beta_{j,k_{j,\rho}} = -g^*(\rho)$.

[3] Recall that R is a residual set of I if $R = \cap_{n \in \mathbb{N}} \Omega_n$, where $\{\Omega_n\}_{n \in \mathbb{N}}$ is a sequence of open sets, such that for all n, Ω_n is dense in I.

Let us now compute the function $\rho \to \chi_0(\rho)$ for this function f at 0.

When $0 < \rho < 1$, one must distinguish two cases :

– $-g^*(\rho) < +\infty$:

Let $\varepsilon > 0$. Since $-g^*$ is convex, there exists $\eta > 0$, such that $|\gamma - \rho| \leq \eta$ implies $-g^*(\gamma) \geq -g^*(\rho) - \varepsilon$. Thus, by construction, $|\gamma - \rho| \leq \eta$ also implies $\chi_0^j(\gamma) \geq \chi_0^j(\rho) - \varepsilon$. In particular, $\forall \gamma \in [\rho - \eta/2, \rho + \eta/2]$, and for j such that $2^{-j} \leq \eta/2$, one has $\beta_{j,k_{j,\gamma}} \geq \chi_0^j(\gamma)$. Thus,

$$\begin{aligned} |d_{j,k_{j,\gamma}}| &\leq 2^{-j\beta_{j,k_{j,\gamma}}} \\ &\leq 2^{-j\chi_0^j(\gamma)} \\ &\leq 2^{-j(\chi_0^j(\rho)-\varepsilon)}. \end{aligned}$$

Since $\lim_j \chi_0^j(\rho) = -g^*(\rho)$, one concludes that $\chi_0^\eta(\rho) \geq -g^*(\rho) - \varepsilon$ (remember that χ_0^η takes into account the wavelet coefficients $d_{j,k}$ such that $k \in [[2^{j(1-(\rho+\eta))}], [2^{j(1-(\rho-\eta))}]]$). Reciprocally, let us have a look at the coefficients $d_{j,k_{j,\rho}}$. For all j, one has $d_{j,k_{j,\rho}} = 2^{-j\beta_{j,k_{j,\gamma}}}$, and $\lim_{j\to+\infty} \beta_{j,k_{j,\rho}} = -g^*(\rho)$. Thus $\exists J$, $j \geq J \Rightarrow \beta_{j,k_{j,\rho}} \leq -g^*(\rho) + \varepsilon$. This implies $\chi_0^\eta(\rho) \leq -g^*(\rho) + \varepsilon$. Finally, for all $\varepsilon > 0$, there exists η such that

$$-g^*(\rho) - \varepsilon \leq \chi_0^\eta(\rho) \leq -g^*(\rho) + \varepsilon.$$

Letting ε go to 0 gives $\chi_0(\rho) = -g^*(\rho)$.

– $-g^*(\rho) = +\infty$:

by construction of g^*, for all N, there exists η such that $|\gamma - \rho| \leq \eta$ implies $-g^*(\gamma) \geq N$. Thus $|\gamma - \rho| \leq \eta$ also implies that $\chi_0^j(\gamma) \geq N$ for all $j > N$. In particular, $\forall \gamma \in [\rho - \eta, \rho + \eta]$, for all $j > N$, one has

$$\begin{aligned} |d_{j,k_{j,\gamma}}| &\leq 2^{-j\beta_{j,k_{j,\gamma}}} \\ &\leq 2^{-jN}. \end{aligned}$$

One concludes that $\chi_0^\eta(\rho) \geq N$. This can be done for all $N > 0$, thus $\chi_0(\rho) = +\infty = -g^*(\rho)$.

Let us compute now $\chi_0(0)$. If $-g^*(0) < +\infty$, then, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $\forall \gamma \in [0, \eta]$, $-g^*(\gamma) \geq -g^*(0) - \varepsilon$, and thus $\chi_0^j(\gamma) \geq \chi_0^j(0) - \varepsilon$. This means that, for all coefficients $d_{j,k}$ such that $k \geq [2^{j(1-\eta)}]$, one has

$$\begin{aligned} |d_{j,k}| &\leq 2^{-j\beta_{j,k}} \\ &\leq 2^{-j(\chi_0^j(0)-\varepsilon)}. \end{aligned}$$

Thus $\chi_0^\eta(0) \geq -g^*(0) - \varepsilon$. One concludes that $\chi_0(0) \geq -g^*(0)$.

On the other hand, $d_{j,k_{j,0}} = 2^{-j\beta_{j,0}}$, and $\lim_{j\to+\infty} \beta_{j,0} = -g^*(0)$. Thus for all $\eta > 0$, $\chi_0^\eta(0) \leq -g^*(0)$. This leads to $\chi_0(0) \leq -g^*(0)$, and finally $\chi_0(0) = -g^*(0)$.

The case $-g^*(0) = +\infty$ is treated similarly to the case $-g^*(\rho) = +\infty$ for $0 < \rho < 1$.

Eventually, one computes $\chi_0(1)$ using the same method, and one obtains

$$\forall \rho \in [0,1], \quad \chi_0(\rho) = -g^*(\rho). \tag{4.20}$$

■

Notice that this prescription method chooses the "smoothest"function χ_0 among the ones that lead to the same frontier, because it forces $\chi_0 = \tilde{\chi}_0$.

4.3.2 Prescription on a set

The main advantage of the prescription method used in Theorem 4.2 is that it can be generalized to a countable dense set of points.

Theorem 4.3 *Let $h : (x, \rho) \to h(x, \rho)$ be a function from $[0, 1] \times [0, 1]$ to $[\gamma, +\infty]$ (with $\gamma > 0$), such that*

- *$x \to h(x, 0)$ is an lsc function.*
- *$x \to \inf_{\rho \in [0,1]}(h(x, \rho))$ is an lsc function.*
- *For all x, $h(x, 0) \leq \liminf_{y \to x}(\inf_{\rho \in [0,1]}(h(y, \rho)))$.*

Let $\{x_n\}_n$ be a countable set of points in $[0, 1]$. For each x_n, denote by $\tilde{h}_n$ the convex envelop of $\rho \to h(x_n, \rho)$.

There exists a function f such that the 2-microlocal spectrum $\rho \to \chi_{x_n}(\rho)$ of f at each point x_n is exactly $\rho \to \tilde{h}_n(\rho)$.

Remark : To each continuous function f can be associated a two-variables function $(x, \rho) \to \chi_x(\rho)$. In view of Propositions 4.12 and 4.15, the conditions imposed on the function $h : (x, \rho) \to h(x, \rho)$ make it an admissible candidate to be equal to χ_x.

Proof : For each x_n, let us denote by g_n the Legendre transform of $\rho \to -h(x_n, \rho) : g_n(s') = \inf_{\rho \in [0,1]}(\rho s' + h(x_n, \rho))$.

Theorem 4.2 explains how to force the 2-microlocal frontier to be equal to g_n at one x_n, or equivalently to force χ_{x_n} to be equal to $\tilde{h}_n$. Here we are going to adapt this construction in order to do it simultaneously on all the x_n's.

The construction is iterative : Let Γ denote the set of the wavelet coefficients $\{d_{j,k}\}_{j,k}$ of the function f we are going to build : $\Gamma = \{d_{j,k} : j \geq 1, k \in \{0, \dots, 2^j - 1\}\}$. We only consider the $d_{j,k}$'s such that $j \geq 1$, since they include all the ones that contribute to the local regularity. Then, for any n, we define the "cone" $\Gamma_{x_n} = \{d_{j,k} : |k2^{-j} - x_n| \leq 2^{-j/\log j}\}$. One now proceeds to the following iterative construction :

- one first imposes $d_{j,k} = 2^{-j^2}$, for all $d_{j,k} \in \Gamma$.
- at step 1, one modifies the wavelet coefficients that lie inside the cone $S_1 = \Gamma_{x_1}$, and one prescribes them according to $\tilde{h}_1$, the exponent function expected for x_1, following the construction used in Theorem 4.2 when prescribing the 2-microlocal frontier at one point.
- at step 2, one modifies the wavelet coefficients that lie inside the set $S_2 = \Gamma_{x_2} \backslash \Gamma_{x_1}$, and one prescribes them according to $\tilde{h}_2$.
- at step n, one modifies the wavelet coefficients that lie inside the set S_n defined as $S_n = \Gamma_{x_n} \backslash \{\bigcup_{i=1,\dots,n-1} \Gamma_{x_i}\}$, and one prescribes them according to $\tilde{h}_n$.

Lemma 4.4 $\Gamma = \bigcup_n S_n$*. Moreover, for all $n > 0$, S_n is non-empty, and there exists j_n such that, if $j \geq j_n$ and $|k2^{-j} - x_n| \leq 2^{-j/\log j}$, then $d_{j,k} \in S_n$.*

Proof : Let $d_{j,k} \in \Gamma$. Since $\{x_n\}$ is dense in $[0,1]$, the set $\{i : d_{j,k} \in \Gamma_{x_i}\}$ is non-empty, and it has a smallest element $i_{\min}$. It is easy to verify that $d_{j,k} \in S_{i_{\min}}$.

Let n be an integer greater than 2. For all i such that $1 \leq i \leq n-1$, one has

$$\lim_{j \to +\infty} |(x_i \pm 2^{-j/\log j}) - (x_n \pm 2^{-j/\log j})| = |x_i - x_n|.$$

Thus there exists $\eta_n \leq \frac{1}{2}\min_i(|x_n - x_i|)$, and j_n, such that $\forall\, 1 \leq i \leq n-1$, $j \geq j_n$ implies

$$|(x_i \pm 2^{-j/\log j}) - (x_n \pm 2^{-j/\log j})| \geq \eta_n$$

This equivalently means that, for all (j,k) such that $j \geq j_n$ and $x_n - 2^{-j/\log j} \leq k2^{-j} \leq x_n + 2^{-j/\log j}$, $d_{j,k} \notin S_i$, for $i \leq n-1$. This concludes the proof. ∎

Lemma 4.4 shows that the above construction allows to build a function F by prescribing all its wavelet coefficients. It is obvious that, for any (j,k), $|d_{j,k}| \leq 2^{-j\gamma}$, thus F is well defined and belongs to $C^\gamma([0,1])$.

The crucial point is the following lemma :

Lemma 4.5 *For any point x_n such that $\lim_{\rho \to 0^+} \tilde{h}_n(\rho) < +\infty$, one has $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$ for every $\rho \in [0,1]$.*

Lemma 4.5 says that it is sufficient to prescribe the wavelet coefficients inside the cone Γ_{x_n} to prescribe the whole 2-microlocal frontier. We are going to apply this to effectively compute the regularity of the function.

Proof : (of Lemma 4.5) Using Lemma 4.4, one knows that there exists a scale j_n, such that, for all (j,k) such that $j \geq j_n$ and $|k2^{-j} - x_n| \leq 2^{-j/\log j}$, the wavelet coefficients $d_{j,k}$ are chosen as in Theorem 4.2, i.e. in such a way that $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$ for every $\rho \in [0,1]$. The problems come from the fact that we have only prescribed *a priori* the coefficients that are inside the cone Γ_{x_n}, and Theorem 4.2 does not exactly apply here.

As noticed in Sections 4.1 and 4.2, and also in Corollary 4.1, $\lim_{\rho \to 0^+} \tilde{h}_n(\rho) < +\infty$ implies that

$$\lim_{\rho \to 0^+} \tilde{h}_n(\rho) = \tilde{h}_n(0) < +\infty. \tag{4.21}$$

The only problem is in fact the computation of $\chi_{x_n}(0)$. Indeed, if $\rho > 0$, then the computation of $\chi_{x_n}(\rho)$ uses only the coefficients that are lying inside the cone $\Gamma_{x_n} = \{d_{j,k} : |k2^{-j} - x_n| \leq 2^{-j/\log j}\}$, i.e. those which are correctly scaled.

Let us compute $\theta^0_{x_n}(\eta)$, for $\eta > 0$ small. Two kinds of coefficients need to be taken into account : those that are inside the cone Γ_{x_n}, and those that are outside.

By construction, $\tilde{h}_n(\rho) \leq \tilde{h}_n(0) - \varepsilon$ if η is small enough. Thus, if $d_{j,k} \in \Gamma_{x_n}$, if $|k2^{-j} - x_n| \leq 2^{-j\eta}$ and if $|k2^{-j} - x_n| \leq \eta_1 = 2^{-j_n/\log j_n}$, then

$$d_{j,k} \leq C 2^{-j(\tilde{h}_n(0) - \varepsilon)}. \tag{4.22}$$

Assume for the moment that $\forall\varepsilon > 0$, there exists $\eta_2 > 0$ such that for all the $d_{j,k}$'s with (j,k) satisfying $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta_2$, one has

$$|d_{j,k}| \leq 2^{-j(\tilde{h}_n(0)-\varepsilon)} \tag{4.23}$$

(this concerns all the wavelet coefficients, not only those that belong to Γ_{x_0}). Then one sees that the coefficients which are outside the cone Γ_{x_n} also satisfy (4.22) if $|k2^{-j} - x_n| \leq \eta_2$.

This means that, for $\eta \leq \min(\eta_1, \eta_2)$, if $|k2^{-j} - x_n| \leq \eta$ and $|k2^{-j} - x_n| \leq 2^{-j\eta}$, then $|d_{j,k}| \leq C2^{-j(\tilde{h}_n(0)-\varepsilon)}$. This implies that $\theta^0_{x_n}(\eta) \geq \tilde{h}_n(0) - \varepsilon$ for every $\eta \leq \min(\eta_1, \eta_2)$. This entails $\chi_{x_n}(0) \geq \theta^0_{x_n}(\eta) \geq \tilde{h}_n(0) - \varepsilon$, for all $\varepsilon > 0$. Thus $\chi_{x_n}(0) \geq \tilde{h}_n(0)$.

On the other hand, one always has $\chi_{x_n}(0) \leq \lim_{\rho\to 0^+} \chi_{x_n}(\rho)$ by Proposition 4.1. But, due to (4.21), $\lim_{\rho\to 0^+} \chi_{x_n}(\rho) = \lim_{\rho\to 0^+} \tilde{h}_n(\rho) = \tilde{h}_n(0) < +\infty$.

This proves $\chi_{x_n}(0) \leq \tilde{h}_n(0)$.

The only thing that remains to prove to ensure that one has prescribed the 2-microlocal frontier at each x_n is inequality (4.23). This will be done now in Lemma 4.6, and will be the consequence of the second condition imposed on h in Theorem 4.3.

Lemma 4.6 *If $\tilde{h}_n(0) < +\infty$, then $\forall\varepsilon > 0$, there exists $\eta > 0$ such that, for all the (j,k)'s that satisfy $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$, one has*

$$|d_{j,k}| \leq 2^{-j(\tilde{h}_n(0)-\varepsilon)} \tag{4.24}$$

Proof : (of Lemma 4.6) Let $\varepsilon > 0$. Since $x \to \inf_{\rho\in[0,1]}(h(x,\rho))$ is an lsc function, there exists $\eta_2 > 0$ such that $|y - x_n| \leq \eta_2$ implies

$$\inf_{\rho\in[0,1]}(h(y,\rho)) \geq \inf_{\rho\in[0,1]}(h(x_n,\rho)) - \varepsilon. \tag{4.25}$$

This means that for every point x_i such that $|x_n - x_i| \leq \eta_2$, the infimum $\inf_{\rho\in[0,1]}(h(x_i,\rho))$ is greater than $\inf_{\rho\in[0,1]}(h(x_n,\rho)) - \varepsilon$, or even better, that

$$\alpha_l(x_i) = \inf_{\rho\in[0,1]} \tilde{h}_i(\rho) \geq \inf_{\rho\in[0,1]} \tilde{h}_n(\rho) - \varepsilon.$$

Let us focus on the wavelet coefficients $d_{j,k}$ such that $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta_2$. Such a wavelet coefficient $d_{j,k}$ belong to some S_i (defined during the construction process), it has been modified by the process. One can locate the corresponding x_i. Indeed, one obviously has $|x_i - k2^{-j}| \leq 2^{-j/\log j}$, thus $x_i \in [k2^{-j} - 2^{-j/\log j}, k2^{-j} + 2^{-j/\log j}]$. Using that $|x_n - k2^{-j}| \leq \eta$, one concludes that

$$x_n - 2^{-j/\log j} - \eta \leq x_i \leq x_n + 2^{-j/\log j} + \eta.$$

We now fix j_n and η such that $2^{-j_n/\log j_n} + \eta \leq \min(\eta_2, 2^{-J/\log J})$.

Let us sum up our findings : if (j,k) verifies $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$, then $d_{j,k}$ belongs to one S_i, whose associated point x_i satisfies $|x_i - x_n| \leq \eta \leq \eta_2$. Using (4.25), one knows that the value of $d_{j,k}$ after modification by the process is smaller than $2^{-j\inf_{\rho\in[0,1]}(h(y,\rho))}$, i.e. smaller than $2^{-j(\inf_{\rho\in[0,1]}(h(x_n,\rho))-\varepsilon)}$.

In particular, this shows that, if (j,k) verifies $2^{-j/\log j} \leq |k2^{-j} - x_n| \leq \eta$, then $|d_{j,k}| \leq 2^{-j(h(x_n,0)-\varepsilon)} = 2^{-j(\tilde{h}_n(0)-\varepsilon)}$. This is the result claimed in (4.24). ∎

One can thus conclude that for all $n \in \mathbb{N}$ such that $\lim_{\rho\to 0^+} \tilde{h}_n(\rho) = +\infty$, one has $\chi_{x_n} = \tilde{h}_n$. This ends the proof of Lemma 4.5. ∎

The next lemma, whose proof is omitted, is an easy adaptation of Lemma 4.5 to the case $\lim_{\rho\to 0^+} \tilde{h}_n(\rho) = h_n(0) = +\infty$.

Lemma 4.7 *For any point x_n such that $\lim_{\rho\to 0^+} \tilde{h}_n(\rho) = +\infty$, one has $\chi_{x_n}(\rho) = \tilde{h}_n(\rho)$ for every $\rho \in [0,1]$.*

Eventually, one has exactly prescribed the 2-microlocal frontier simultaneously at all x_n. ∎

One shall remark that a "miracle" is happening here : It is enough to prescribe only a part of the wavelet coefficients that influence the 2-microlocal frontier of a function at some (countable dense) points $\{x_n\}_n$ to recover the whole 2-microlocal frontier at each of these points x_n. The lack of information is compensated by the regularity imposed on the function $x \to \chi_x(0)$.

The loss of information incurred when prescribing the convex envelop $\tilde{h}_n$ of $\rho \to h(x_n,\rho)$ instead of $\rho \to h(x_n,\rho)$ itself, explains why, with the above method, one can not expect to prescribe more than what Theorem 4.3 allows to. Formally the loss of information could be compared to the one in multifractal analysis when considering the Legendre spectrum instead of the Hausdorff or large deviation spectrum.

A natural question is to enquire about the behaviour of the points that do not belong to the $\{x_n\}_n$. Recall that

- the local Hölder function satisfies for every integer n $\alpha_l(x_n) = \inf_{\rho\in[0,1]} \tilde{h}_n(\rho)$ and $\alpha_l(x_n) \leq \liminf_{x_i\to x_n} \alpha_l(x_i)$.
- $\forall n$, $\chi_{x_n}(0) \leq \liminf_{x_i\to x_n} \chi_{x_i}(0)$.

In addition, recall that, since both functions $x \to h(x,0)$ and $x \to \inf_{\rho\in[0,1]}(h(x,\rho))$ are lsc functions, they are completely characterized by their values on a dense set of points (see [63]), respectively $\{y_n\}_n$ and $\{z_n\}_n$. Thus if one prescribes the 2-microlocal frontiers on the three (countable) sets of points $\{x_n\}$, $\{y_n\}_n$ and $\{z_n\}_n$, one obtains the following proposition

Proposition 4.18 *If h satisfies the properties of Theorem 4.3, and if one prescribes the 2-microlocal frontiers on the three sets of points $\{x_n\}$, $\{y_n\}_n$ and $\{z_n\}_n$, then one obtains a function f such that, in addition to the properties of Theorem 4.3, one has for all x,*

$$\begin{aligned} \alpha_l(x) &\leq \inf_{\rho\in[0,1]}(h(x,\rho)), \\ \chi_x(0) &\leq h(x,0). \end{aligned}$$

Proof : We use the following lemma, whose proof is omitted.

Lemma 4.8 *Let v be an lsc function, and $D = \{t_n\}_n$ be a sequence of points in $[0,1]$ that characterizes v in $[0,1]$ (see [63]). Then if w is an lsc function such that $\forall n$, $w(t_n) = v(t_n)$, then $w \leq v$ on $[0,1]$.*

Now, by construction, $x \to \alpha_l(x)$ and $x \to h_{min}(x) = \inf_{\rho\in[0,1]}(h(x,\rho))$ coincide on a dense set of points, namely $\{z_n\}_n$, which entirely characterizes $x \to \alpha_l(x)$. Hence, since they both are lsc, using Lemma 4.8, for all x, one has

$$h_{min}(x) \leq \alpha_l(x).$$

The same argument applies to $x \to \chi_x(0)$. ■

4.4 Riemann's function and lacunary wavelet series from the χ-point of view

We conclude this work with the explicit computation of χ and σ for the celebrated one- and two-dimensional Riemann's function, as well as for lacunary wavelet series.

4.4.1 One-dimensional Riemann function

The function

$$\mathcal{R}(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(\pi n^2 x),$$

was introduced by Riemann and was originally thought to be a continuous but nowhere differentiable function. Since then, deep studies have shown that it is in fact differentiable at rational points $\frac{2p+1}{2q+1}$, $p, q \in \mathbb{Z}^2$ (more precisely $\mathcal{R}$ is $C^{\frac{3}{2}}$ at these points), and that $\mathcal{R}$ is a multifractal function whose spectrum has been calculated in [43].

Our aim here is to compute the 2-microlocal frontier of $\mathcal{R}$ at all points. It is known ([39], [43]) that the behaviour of the continuous wavelet transform of $\mathcal{R}(x)$ can be reduced to the one of the function $T(a,b)$ defined by

$$T(a,b) = a(Im(\Theta(b+ia)) - 1), \tag{4.26}$$

where Θ is the Jacobi theta function defined by

$$\Theta(z) = \sum_{n=1}^{+\infty} e^{i\pi n^2 z}, \text{ for } Im(z) > 0.$$

This is achieved when using the analyzing wavelet $\psi(x) = \frac{1}{(i+x)^2}$, which has only one vanishing moment. One can not thus *a priori* reach exponents greater than 1. However, taking for reconstructing wavelet any function $\tilde{\psi} \in \mathcal{S}(\mathbb{R})$ supported by $[-\varepsilon, \varepsilon]$ such that $\int_{-\varepsilon}^{\varepsilon} \tilde{\psi}(x)dx = 0$, one can verify that

$$f(x) = \int_0^{+\infty} \int_{-\infty}^{+\infty} \tilde{\psi}\left(\frac{x-b}{a}\right) \theta(b+ia) db \frac{da}{a} = \sum_{n=1}^{+\infty} \frac{e^{i\pi n^2 x}}{n^2},$$

whose imaginary part is exactly $\mathcal{R}(x)$. Thus there is no loss of information when using this specific analyzing wavelet.

The function Θ has some invariance properties

$$\Theta(z) = \Theta(z+2) \text{ and } \Theta(z) = \sqrt{\frac{i}{z}}\Theta(-\frac{1}{z}).$$

Thus the rational points can be split into two parts : those which belong to the orbit of 0 under the two transforms above, and those which belong to the orbit of 1. They correspond respectively to the rational points that can be written $\frac{2p}{2q+1}$ or $\frac{2p+1}{2q}$ $(p, q \in \mathbb{Z} \times \mathbb{Z}^*)$ and to whose that can be written $\frac{2p+1}{2q+1}$ $(p, q \in \mathbb{Z}^2)$.

To compute the 2-microlocal spectrum $\chi_x^1(\rho)$ of $\mathcal{R}$ (the 1 in χ_x^1 stands for dimension one), we use the following estimation of the Jacobi theta function found in [39]

Proposition 4.19 **1.** *In a neighbourhood of any point x in the orbit of 0, one has*

$$T(a, x+b) = C_x Im\left(a\sqrt{\frac{i}{b+ia}} + a\phi(b+ia)\right), \quad (4.27)$$

where Im denotes the imaginary part, and ϕ satisfies

$$\begin{aligned} \phi(b+ia) &= \mathcal{O}((b^2+a^2)^{-\frac{1}{4}} e^{-\frac{a}{b^2+a^2}}) \text{ if } \frac{a}{b^2+a^2} > 1, \\ &= \mathcal{O}(a^{-\frac{1}{2}}(b^2+a^2)^{\frac{1}{4}}) \text{ uniformly.} \end{aligned}$$

2. *In a neighbourhood of any point x in the orbit of 1, one has*

$$T(a, x+b) = C_x Im\left(a\phi(b+ia)\right), \quad (4.28)$$

with ϕ satisfying the same properties as before.

Let us define the following sets :

$$\begin{aligned} R_{1/2} &= \{x \in \mathbb{Q} : x \text{ is in the orbit of } 0\} \\ R_{3/2} &= \{x \in \mathbb{Q} : x \text{ is in the orbit of } 1\} \\ \forall \tau \geq 2,\ S_\tau &= \{x \in \mathbb{R}\backslash\mathbb{Q} : \eta(x) = \tau\} \end{aligned}$$

- If $x \in R_{1/2}$:

The first term in (4.27) is easily estimated. We set $b = \pm a^\rho$ and see that

$$\begin{aligned} \left| Im\left(a\sqrt{\frac{i}{\pm a^\rho + ia}}\right)\right| &= \frac{1}{\sqrt{2}} a^{1-\rho/2} \left| Im((1 \pm ia^{1-\rho})^{-\frac{1}{2}})\right| \\ &= \frac{1}{\sqrt{2}} a^{1-\rho/2} \left| Im(1 \mp \frac{i}{2} a^{1-\rho} + \mathcal{O}(a^{2(1-\rho)}))\right| \\ &= \frac{1}{2\sqrt{2}} a^{2-\frac{3}{2}\rho} + \mathcal{O}(a^{\beta+2-\frac{5}{2}\rho}). \end{aligned}$$

It is clear then that the 2-microlocal spectrum corresponding to the first term of (4.27) is $\chi_{x,1}(\rho) = 2 - \frac{3}{2}\rho$.

The second term of (4.27) is more delicate : Set $b = a^\rho$. If $\frac{1}{2} < \rho < 1$, $\lim_{a\to 0} \frac{a}{b^2+a^2} = +\infty$, thus we can use the first bound of ϕ, i.e.

$$
\begin{aligned}
\phi(a^\rho + ia) &= \mathcal{O}((a^{2\rho} + a^2)^{-\frac{1}{4}} e^{-\frac{a}{a^{2\rho}+a^2}}) \\
&= \mathcal{O}(a^{-\frac{1}{2}\rho} e^{-a^{1-2\rho}}),
\end{aligned}
$$

which gives a decay with a faster than any a^N, $N \in \mathbb{N}$. If $0 < \rho < \frac{1}{2}$, $\lim_{a\to 0} \frac{a}{b^2+a^2} = \lim_{a\to 0} \frac{a}{(a^\rho)^2+a^2} = 0$ and we can only use the second bound of ϕ, which leads to

$$
\begin{aligned}
\phi(b + ia) &= \mathcal{O}(a^{-\frac{1}{2}} (a^{2\rho} + a^2)^{\frac{1}{4}}) \\
&= \mathcal{O}(a^{\frac{1}{2}\rho - \frac{1}{2}})
\end{aligned}
$$

Using (4.26), it comes that $\chi_{x,2}(\rho) = \frac{1}{2} + \frac{1}{2}\rho$ if $0 < \rho \leq \frac{1}{2}$, and $\chi_{x,2}(\rho) = +\infty$ otherwise. It is also obvious that $\chi_{x,2}(\frac{1}{2}) = \frac{3}{4}$.

The computations of $\chi_{x.2}(0)$ and $\chi_{x,2}(1)$ easily follow from the above computation : $\chi_{x,2}(0) = \chi_{x,2}(1) = 1/2$.

Eventually, if x is in the orbit of 0, one has

$$\chi_x^1(\rho) = \min(\chi_{x,1}(\rho), \chi_{x,2}(\rho)).$$

$\chi_{x,1}(\rho) \geq \chi_{x,2}(\rho)$ if $0 \leq \rho \leq 1/2$, thus an explicit formula for χ_x^1 is

$$
\begin{aligned}
\chi_x^1(\rho) &= \frac{1}{2} + \frac{1}{2}\rho \text{ if } 0 \leq \rho \leq 1/2 \\
&= 2 - \frac{3}{2}\rho \text{ if } 1/2 < \rho \leq 1
\end{aligned}
$$

It is interesting to remark that $\chi_x^1(0) = \chi_x^1(1) = 1/2$. Thus $\tilde{\chi}_x^1(\rho) = 1/2$ for all $\rho \in [0, 1]$ and the 2-microlocal frontier of $\mathcal{R}$ at x is

$$
\begin{aligned}
\sigma(s') &= 1/2 \text{ if } s' \geq 0, \\
\sigma(s') &= 1 + s' \text{ if } s' < 0,
\end{aligned}
$$

which corresponds to a cusp at each point x in the orbit of 0. Moreover, one recovers that $\alpha_l(x) = \inf_\rho \chi_x^1(\rho) = 1/2$ and $h_f(x) = \inf_\rho \frac{\chi_x^1(\rho)}{\rho} = 1/2$. These points are cusps with regularity 1/2, but one shall remark that the function χ_x^1 at these points contains more information than the single 2-microlocal frontier.

- If $x \in S_\tau$: The case of the irrational points has been studied in [43] : if x_0 is an irrational point, and $\frac{p_n}{q_n}$ its sequence of approximations by continued fractions, then there exists an infinity numbers of integers n such that

$$C(a_n, x_0) \sim C a_n^{1/2 + \frac{1}{2\tau_n}},$$

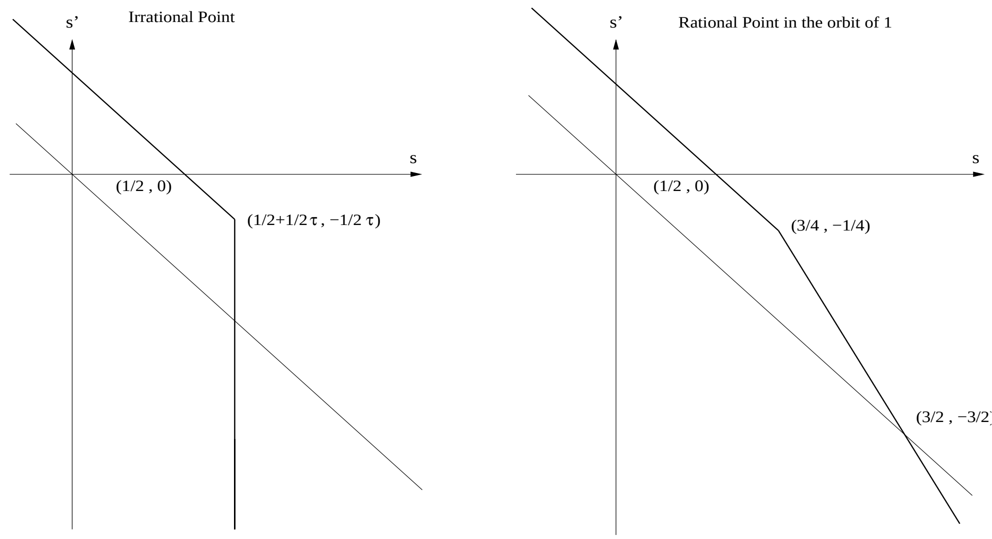

FIG. 4.3 – 2-microlocal Frontiers of the Riemann Function on $R_{1/2}$ (left), $R_{3/2}$ (right).

where $|x_0 - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n}} = a_n$, and at the same time, if $\eta(x_0) = \limsup_n \tau_n$, then $R \in C_{x_0}^{1/2+\frac{1}{2\eta(x_0)}-\varepsilon}$, for every $\varepsilon > 0$. The first point implies that $\chi^1_{x_0}(1) \leq 1/2 + \frac{1}{2\eta(x_0)}$. Together with $\chi^1_{x_0}(0) = 1/2$ (which is in fact true for all $x \in \mathbb{R}$), one obtains $\tilde{\chi}^1_{x_0}(\rho) = 1/2 + \rho\frac{1}{2\eta(x_0)}$ for all $\rho \in [0,1]$. In fact, more can be done, i.e. one can show that

$$\tilde{\chi}^1_{x_0}(\rho) = 1/2 + \frac{\rho}{2\eta(x_0)} \text{ for } \rho \in [0,1]. \tag{4.29}$$

These points have a 2-microlocal frontier equal to $\sigma(s') = 1/2$ if $s' \geq -\frac{1}{2\eta(x_0)}$, and $\sigma(s') = 1/2 + \frac{1}{2\eta(x_0)} + s'$ if $s' < -\frac{1}{2\eta(x_0)}$, which looks like a "cusp" frontier.

Remark : If $x \in S_2$, the wavelet coefficients $T(a, x+b)$ effectively behave like indicated by the corresponding 2-microlocal spectra, i.e. $T(a, x+a^\rho) \sim a^{\chi^1_{x_0}(\rho)}$.

- If $x \in R_{3/2}$ **:**

If x is in the orbit of 1, $\chi^1_x(\rho) = \chi_{x,2}(\rho)$, where $\chi_{x,2}$ has been computed above. Indeed, only the second term needs to be estimated. Thus

$$\begin{aligned}
\tilde{\chi}^1_x(\rho) &= \frac{1}{2} + \frac{1}{2}\rho \text{ if } 0 \leq \rho \leq 1/2 \\
&= +\infty \text{ if } 1/2 < \rho \leq 1,
\end{aligned}$$

which gives a 2-microlocal frontier equal to

$$\begin{aligned}
\sigma(s') &= \beta - 1/2 \text{ if } s' \geq -1/4, \\
\sigma(s') &= \beta - 1/4 + \frac{1}{2}s' \text{ if } s' < -1/4.
\end{aligned}$$

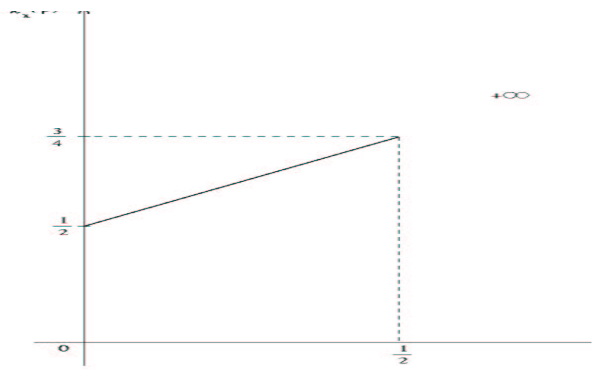

FIG. 4.4 – 2-microlocal spectra of the Riemann Function on $R_{3/2}$ (left), $R_{1/2}$ (middle) and S_τ (right).

One recovers that, at such points, $\alpha_l(x) = \inf_\rho \chi_x^1(\rho) = 1/2$ and $h_f(x) = \inf_\rho \frac{\chi_x^1(\rho)}{\rho} = \frac{3/4}{1/2} = 3/2$.

The complementary information provided by this analysis is that these points are chirps $(3/2, 1)$, i.e. with $h_f = 3/2$, and $\beta_c = \beta_o = 1$.

Remark : Let us insist on the fact that the wavelet coefficients $T(a, x+b)$ effectively behave around the rational points like indicated by the corresponding 2-microlocal spectra, i.e. $T(a, x+a^\rho) \sim a^{\chi_x(\rho)}$. This will be of great importance to treat the two-dimensional case.

4.4.2 Two-dimensional Riemann function

A generalized version of the Riemann function, for $x = (x_1, x_2) \in \mathbb{R}^2$, is (see [76])

$$\mathcal{R}^2(x) = \sum_{(m_1,m_2)\in\mathbb{Z}^2\setminus\{0,0\}} \frac{e^{i\pi(m_1^2 x_1 + m_2^2 x_2)}}{(m_1^2 + m_2^2)^2}.$$

As in the one-dimensional case, the study of the wavelet transform of $\mathcal{R}^2$ can be reduced to the study of the function

$$C(a, b) = a^\beta \Theta(x_1 + b_1)\Theta(x_2 + b_2), \tag{4.30}$$

where $b = (b_1, b_2)$. If $x = (x_1, x_2)$ is fixed, the 2-microlocal spectrum of $\mathcal{R}^2$ at x $\chi_x^2(\rho)$ is related to both $\chi_{x_1}^1(\rho)$ and $\chi_{x_2}^1(\rho)$, where χ_x^1 is the 2-microlocal spectrum of the 1-D Riemann function $\mathcal{R}$. More precisely,

Proposition 4.20 *Let $x = (x_1, x_2) \in \mathbb{R}^2$. Then, for all $\rho \in (0, 1]$,*

$$\chi_x^2(\rho) \geq \min\left(\chi_{x_1}^1(\rho) + \inf_{\rho'\in[\rho,1]} \chi_{x_2}^1(\rho'),\ \inf_{\rho'\in[\rho,1]} \chi_{x_1}^1(\rho') + \chi_{x_2}^1(\rho)\right) \tag{4.31}$$

We will use the following lemma

Lemma 4.9 *Let $f \in C^\gamma$ for $\gamma > 0$, $x \in \mathbb{R}$ and $\rho_0 \in (0,1]$. We denote by $C_f(a,b)$ the continuous wavelet transform of f, and χ_x the 2-microlocal spectrum of f at x. For all $\eta > 0$ small enough, there exists a_0 such that, if $a \le a_0$, for all b with $|x-b| \le a^{\rho_0}$,*

$$|C_f(a,b)| \le a^{\inf_{\rho \in [\rho_0,1]} \chi_x(\rho) - \eta}. \tag{4.32}$$

Proof : (of Lemma 4.9)
For every $\rho \in [\rho_0, 1]$, by definition of $\chi_x(\rho)$, there exists an interval $I_\rho = [\rho - \xi_\rho, \rho + \xi_\rho]$ and a constant a_ρ such that if $a \le a_\rho$, for all $\rho' \in I_\rho$, $|C_f(a, a^\rho)| \le a^{\chi_x(\rho) - \eta}$. The interval $[\rho_0, 1]$ is compact and can be covered by a finite number of such intervals $(I_\rho)_{\rho \in \{\rho_1, \dots, \rho_n\}}$. Now, if $a \le \min_{i=\{1,\dots,n\}} a_i$, for all b such that $|x-b| \le \rho_0$, one has

$$|C_f(a,b)| \le a^{\min_{i=\{1,\dots,n\}} \chi_x(\rho_i) - \eta}.$$

In particular, this implies (4.32). ∎

Proof : (of Proposition 4.20)
We shall work with the $\|.\|_\infty$ norm in $\mathbb{R}^2$. Let $\rho \in (0,1]$, and $\varepsilon > 0$. We need to estimate $\chi_x^{2,\varepsilon}(\rho)$, which can be written

$$\begin{aligned} \chi_x^{2,\varepsilon}(\rho) &= \liminf_{a \to 0 \text{ and } (a,b) \in \Gamma_{x,\rho,\varepsilon}} \frac{\log |a^2 \Theta(x_1+b_1)\Theta(x_2+b_2)|}{\log a} \\ &= \liminf_{a \to 0 \text{ and } (a,b) \in \Gamma_{x,\rho,\varepsilon}} \frac{\log |a \Theta(x_1+b_1)|}{\log a} + \frac{\log |a\Theta(x_2+b_2)|}{\log a} \end{aligned}$$

where $\Gamma_{x,\rho,\varepsilon} = \{(a,b) : a^{\rho+\varepsilon} \le \|x-b\|_\infty \le a^{\rho-\varepsilon}\}$. If $(a,b) \in \Gamma_{x,\rho,\varepsilon}$, then $|x_1 - b_1| \le a^{\rho-\varepsilon}$, $|x_2 - b_2| \le a^{\rho-\varepsilon}$ and $\max(|x_1-b_1|, |x_2-b_2|) \in [a^{\rho+\varepsilon}, a^{\rho-\varepsilon}]$.

Let $\eta > 0$. Assume without loss of generality that $|x_1 - b_1| \in [a^{\rho+\varepsilon}, a^{\rho-\varepsilon}]$. For ε small enough, $\chi_{x_1}^{1,\varepsilon}(\rho) \ge \chi_{x_1}^1(\rho) - \eta/2$, and there exists a_1 such that, for all $a \le a_1$, for all b_1 such that $|x_1 - b_1| \in [a^{\rho+\varepsilon}, a^{\rho-\varepsilon}]$,

$$\frac{\log |a\Theta(x_1+b_1)|}{\log a} \ge \chi_{x_1}^{1,\varepsilon}(\rho) - \eta/2 \ge \chi_{x_1}^1(\rho) - \eta. \tag{4.33}$$

At the same time, Lemma 4.9 applied to $\mathcal{R}$ holds for x_2 and yields that, if $a \le a_2$ and $|x_2 - b_2| \le a^{\rho-\varepsilon}$

$$\frac{\log |a\Theta(x_2+b_2)|}{\log a} \ge \inf_{\rho' \in [\rho-\varepsilon, 1]} \chi_{x_2}^1(\rho') - \eta. \tag{4.34}$$

Both (4.33) and (4.34) are true if $a \le \min(a_1, a_2)$, and together they imply

$$\frac{\log |C(a,b)|}{\log a} \ge \chi_{x_1}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon,1]} \chi_{x_2}^1(\rho') - 2\eta. \tag{4.35}$$

The symmetric case $|x_2 - b_2| \in [a^{\rho+\varepsilon}, a^{\rho-\varepsilon}]$ is treated similarly, and yields for a small enough,

$$\frac{\log |C(a,b)|}{\log a} \ge \chi_{x_2}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon,1]} \chi_{x_1}^1(\rho') - 2\eta. \tag{4.36}$$

Since at least one of (4.35) and (4.36) holds, one has

$$\chi_x^{2,\varepsilon}(\rho) \geq \min\left(\chi_{x_1}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon,1]} \chi_{x_2}^1(\rho'), \chi_{x_2}^1(\rho) + \inf_{\rho' \in [\rho-\varepsilon,1]} \chi_{x_1}^1(\rho')\right) - 2\eta.$$

Now, letting η go to zero, and using that $\varepsilon \to \chi_x^{2,\varepsilon}(\rho)$ is a non-decreasing function yields the announced result. ∎

An important remark is that if for example for x_1, the wavelet transform around x_1 effectively behaves like indicated by the 2-microlocal spectrum χ_x^1 (i.e. if for all $a \leq a_0$, for all $\rho > 0$, $|C(a, a^\rho)| = \mathcal{O}(a^{\chi_x^1(\rho)})$), then one has equality in (4.31).

This is fundamental in our case, since, as noticed in Remarks 1 and 2, this is the case for $\mathcal{R}$ around the rational points and the set S_2 : This will allow to compute exactly the 2-microlocal spectrum of $\mathcal{R}^2$ at some points $x = (x_1, x_2)$.

Theorem 4.4 *Let $x = (x_1, x_2) \in \mathbb{R}^2$.*

1. *if $(x_1, x_2) \in R_{3/2} \times R_{3/2}$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) = 1 + \rho$ if $\rho \in [0, 1/2]$, and $+\infty$ if $\rho \in (1/2, 1]$.*
2. *if $(x_1, x_2) \in R_{1/2} \times R_{3/2} \cup R_{3/2} \times R_{1/2}$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) = 1 + \rho/2$ if $\rho \in [0, 1/2]$, and $+\infty$ if $\rho \in (1/2, 1]$.*
3. *if $(x_1, x_2) \in R_{1/2} \times R_{1/2}$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) = 1 + \rho/2$ if $\rho \in [0, 1/2]$, and $\chi_x^2(\rho) = 4 - 3\rho$ if $\rho \in (1/2, 1]$.*
4. *if $(x_1, x_2) \in R_{3/2} \times S_\tau \cup S_\tau \times R_{3/2}$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) = 1 + \frac{\rho}{2}(1 + 1/\tau)$ if $\rho \in [0, 1/2]$, and $+\infty$ if $\rho \in (1/2, 1]$.*
5. *if $(x_1, x_2) \in R_{1/2} \times S_\tau \cup S_\tau \times R_{1/2}$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) = 1 + \frac{\rho}{2\tau}$ if $\rho \in [0, 1/(1 + 1/3\tau)]$, and $\chi_x^2(\rho) = 5/2 - \frac{3}{2}\rho$ if $\rho \in (1/(1 + 1/3\tau), 1]$.*
6. *if $(x_1, x_2) \in S_\tau \times S_{\tau'} \cup S_{\tau'} \times S_\tau$, the 2-microlocal spectrum of $\mathcal{R}^2$ at x is $\chi_x^2(\rho) \geq 1 + \frac{\rho}{2}(1/\tau + 1/\tau')$. There is equality if either τ or τ' equals 2.*

It is easy to compute the Hausdorff dimensions of the above sets, and the corresponding pointwise Hölder exponents of their points :

1. $d_H(R_{3/2} \times R_{3/2}) = 0$, they correspond to chirps $(3, 1)$.
2. $d_H(R_{1/2} \times R_{1/2} \cup R_{3/2} \times R_{1/2}) = 0$, they correspond to chirps $(5/2, 1)$.
3. $d_H(R_{1/2} \times R_{1/2}) = 0$, they correspond to cusps $(1, 0)$.
4. $d_H(R_{3/2} \times S_\tau \cup S_\tau \times R_{3/2}) = 2/\tau$, they correspond to chirps $(5/2 + 1/2\tau, 1)$.
5. $d_H(R_{3/2} \times S_\tau \cup S_\tau \times R_{31/2}) = 2/\tau$, they correspond to cusps $(1 + 1/2\tau, 0)$.
6. $d_H(S_2 \times S_\tau \cup S_\tau \times S_2) = 1 + 2/\tau$ and they correspond to cusps $(5/4 + 1/2\max(\tau, 2)$.

The other cases, i.e. $x \in S_\tau \times S_{\tau'}$ with τ and τ' strictly greater than 2, are hard to handle since the pointwise Hölder exponent at x can sometimes by higher than simply $1 + 1/2\tau + 1/2\tau'$. Nevertheless one knows that for every $x \in S_\tau \times S_{\tau'}$, $h_f(x) \leq 11/4$.

Parts of the spectrum are drawn on Figure 4.5. The shaded zone corresponds to areas where the spectrum is not known exactly, but where some bounds hold.

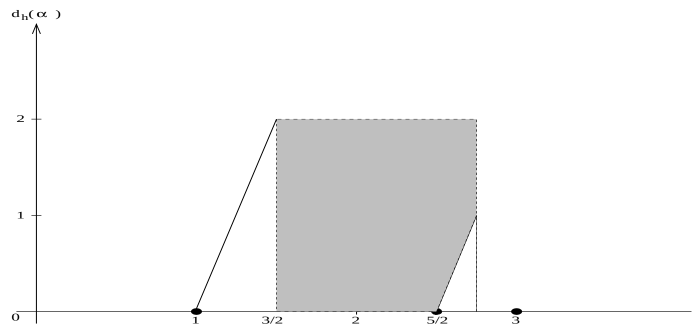

FIG. 4.5 – Multifractal spectrum of the 2-D Riemann Function.

4.4.3 Lacunary wavelet series

Fix $0 < \eta < 1$, and $\alpha > 0$. A lacunary wavelet series [47] is a random process F defined through its wavelet coefficients as follows : Independently at each scale $j \geq 0$, one picks randomly $[2^{\eta j}]$ coefficients among the 2^j ones, according to the uniform probability distribution. These coefficients are attributed the value $2^{-\alpha j}$. The remaining ones are set to 0.

Let us compute $\chi_x(\rho)$ for a sample path of F on $[0,1]$. Remark first that it is obvious that $\forall x$, $\forall \rho \in [0,1]$, $\chi_x(\rho) \in \{\alpha, +\infty\}$.

Define $G_j = \{k : d_{j,k} = 2^{-j\alpha}\}$. For all j, $\text{card}(G_j) = [2^{\eta j}]$.

Proposition 4.21 $\{x : \chi_x(0) = \alpha\} = [0,1]$ *almost surely.*

Proof : Define the set F_j by

$$F_j = \{x : \exists k \in G_j,\ \ 2^{-j\frac{1}{\log j}} \leq |k2^{-j} - x| \leq 2^{-j(\frac{1}{\log j})^2}\}.$$

The following lemma is left to the reader

Lemma 4.10 *If* $x \in \cap_{J\in\mathbb{N}} \cup_{j\geq J} F_j$*, then* $\chi_x(0) = \alpha$.

For any j, let k_i^j, $i \in \{0, \dots, [2^{\eta j}] - 1\}$ be the integers such that $d_{j,k_i^j} = 2^{-j\alpha}$ (there are for $[2^{\eta j}]$ of them). Let $n_j = \sum_{i=1}^{j} [2^{\eta j}]$. For $n \in [n_{j-1}, n_j]$, one sets, for $t_n = k_{n-n_{j-1}}^j 2^{-j}$, $I_n^l = [t_n - 2^{-j(\frac{1}{\log j})^2}, t_n - 2^{-j\frac{1}{\log j}}]$, and $I_n^r = [t_n +, 2^{-j\frac{1}{\log j}} t_n + 2^{-j(\frac{1}{\log j})^2}]$.

F_j is the union of the $2[2^{\eta j}]$ intervals I_n^l and I_n^r for $n \in [n_{j-1}, n_j]$ of size $2^{-j(\frac{1}{\log j})^2} - 2^{-j\frac{1}{\log j}}$.

As explained in [47], one can consider that the t_n are chosen randomly and uniformly in $[0,1]$. Thus the intervals I_n^l (respectively I_n^r) are chosen randomly and uniformly in $[0,1]$, so that one may apply the following Lemma of [47] to I_n^l (and I_n^r) :

Lemma 4.11 *If* $\limsup_{n\to+\infty}(\sum_{j=1}^{n}|I_n^l|-\log n)=+\infty$*, then* $\cap_{N\in\mathbb{N}}\cup_{n\geq N} I_n^l=[0,1]$ *almost surely.*

Lemma 4.11 applies to our intervals I_n^l and I_n^l. Combining this with Lemma 4.10 shows that $\cap_{J\in\mathbb{N}}\cup_{j\geq J} F_j=[0,1]$ a.s., and Proposition 4.21 is proved. ∎

Proposition 4.22 $\{x:\chi_x(\eta)=\alpha\}=[0,1]$ *almost surely.*

Proof : The same arguments as before apply to the sets F_j^η defined by

$$F_j^\eta=\{x:\exists k\in G_j,\ \ 2^{-j(\eta+\frac{1}{\log j})}\leq|k2^{-j}-x|\leq 2^{-j(\eta-\frac{1}{\log j})}\}.$$

Indeed, F_j^η is the union of $2[2^{\eta j}]$ intervals of size $2^{-j\eta}(2^{j\frac{1}{\log j}}-2^{-j\frac{1}{\log j}})$. Combining this with the fact that if $x\in\cap_{J\in\mathbb{N}}\cup_{j\geq J} F_j^\eta$, $\chi_x(\eta)=\alpha$, the result follows at once. ∎

Since $\chi_x(\rho)\in\{\alpha,+\infty\}$ for every ρ and every x, one easily derives the form of the 2-microlocal spectrum at every point x

Proposition 4.23 $\{x:\forall\rho\in[0,\eta],\tilde{\chi}_x(\rho)=\alpha\}=[0,1]$ *almost surely.*

Proposition 4.24 *For almost every sample path of the process F, the Hausdorff dimension of the sets $R_\delta=\{x:\chi_x(\eta+\delta)=\alpha\}$, for $\delta\in(0,1-\eta]$, is $\frac{\eta}{\eta+\delta}$.*

The 2-microlocal frontier of F at an arbitrary point of R_δ, is drawn on Figure 4.6.

Proof : Let us consider the open balls $B_{j,k}^\gamma=(k2^{-j}-2^{-j\gamma},k2^{-j}+2^{-j\gamma})$, for all couples (j,k) such that $d_{j,k}\neq 0$. This set of balls satisfies the conditions of Theorem 2 of [47]. This implies that the Hausdorff dimension of E_γ defined by

$$E_\gamma=\limsup_{j\to+\infty}B_{j,k}^\gamma=\cap_{n\in\mathbb{N}}\cup_{j\geq n}\cup_k B_{j,k}^\gamma \tag{4.37}$$

is $\frac{\eta}{\gamma}$. If $x\in E_\gamma$, then x belongs to an infinite number of balls $B_{j,k}^\gamma$. The following lemmas are obvious consequences of the definition of E_γ

Lemma 4.12 *If $x\notin E_\gamma$, $\chi_x(\rho)=+\infty$ for $\rho>\gamma$.*

Lemma 4.13 *If $x\in E_\gamma\backslash E_{\gamma+\varepsilon}$, there exists $\rho\in[\gamma,\gamma+\varepsilon]$ such that $\chi_x(\rho)=\alpha$.*

Using both Lemma 4.12 and 4.13 yields that if $x\in\cap_{n\in\mathbb{N}}E_{\eta+\delta-1/n}$ but if x does not belong to $\cup_{\varepsilon>\eta+\delta}E_\varepsilon$, then $\chi_x(\eta+\delta)=\alpha$, and $\chi_x(\rho)=+\infty$ if $\rho>\eta+\delta$.

Now, we apply Theorem 2 of [47] : Almost surely the Hausdorff dimension of the set $E_{\eta+\delta-1/n}\backslash\cup_{\varepsilon>\eta+\delta}E_\varepsilon$ is exactly $\frac{\eta}{\eta+\delta}$. This concludes the proof. ∎

The following proposition recapitulates the results obtained above. It sheds interesting light on the interplay between the 2-microlocal and multifractal analysis of lacunary wavelet series

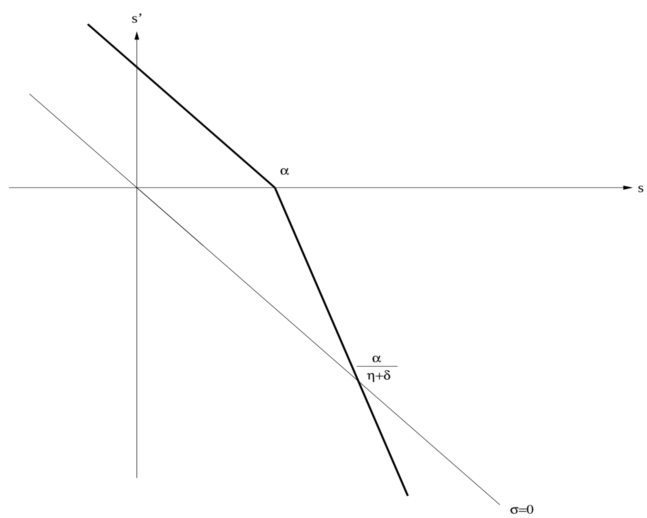

FIG. 4.6 – 2-microlocal frontier for a lacunary wavelet series at $x \in R_\delta$.

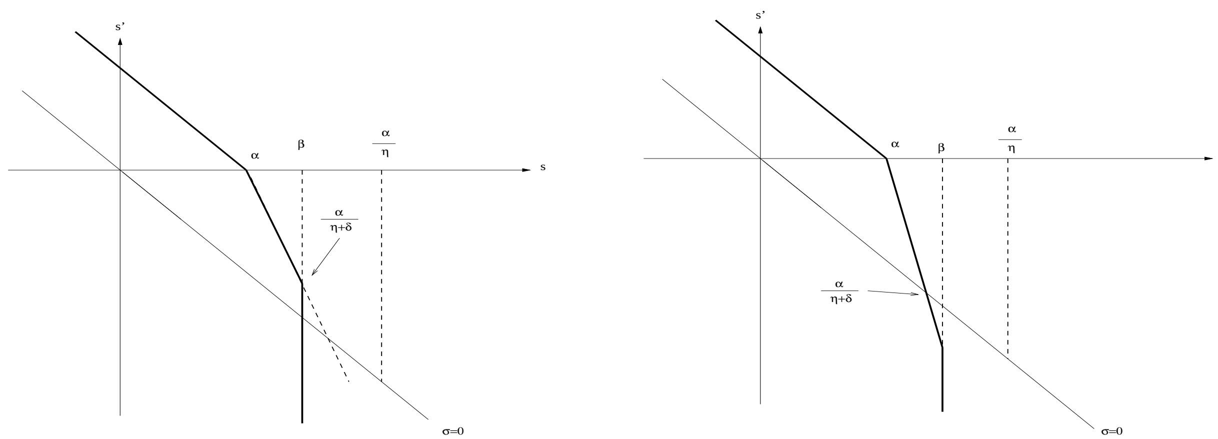

FIG. 4.7 – 2-microlocal frontiers depending on the value of $\frac{\alpha}{\eta+\delta}$ for the modified lacunary wavelet series F_β.

Proposition 4.25 *For almost every sample path of the process F, for all x, there exists $\delta \in [0, 1-\eta]$ such that $\tilde{\chi}_x(\rho) = \alpha$ if $0 \le \rho \le \eta + \delta_x$, and $\tilde{\chi}_x(\rho) = +\infty$ elsewhere. The pointwise exponent at x is $\frac{\alpha}{\eta+\delta_x}$ and the chirp exponent at x is $\frac{1}{\eta+\delta_x} - 1$. For a fixed δ in $[0, 1-\eta]$, the Hausdorff dimension of the points satisfying the above conditions is $\frac{\eta}{\eta+\delta}$.*

Proposition 4.25 allows to recover the fact (proved in [47]) that the corresponding multifractal spectrum is $d(h) = h\frac{\eta}{\alpha}$, for $h \in [\alpha, \frac{\alpha}{\eta}]$.

An interesting generalization of the construction above is to set the "remaining" coefficients to the value $2^{-\beta j}$ instead of 0 (this function can also be understood as a particular case of the random wavelet series introduced in [5]). Let F_β denote this new process. Under the condition that $\beta > \alpha$, the above computations are still valid with one modification : One has now $\chi_x(\rho) = \beta$ instead of $\chi_x(\rho) = +\infty$ when no coefficients equal to $2^{-\alpha j}$ appears in the computation of $\chi_x(\rho)$. The proof of the following proposition is left to the reader.

Proposition 4.26 *If $\beta > \frac{\alpha}{\eta}$, for almost every sample path of the process F_β, the multifractal spectrum is the same as before, i.e. $d(h) = h\frac{\eta}{\alpha}$ if $h \in [\alpha, \frac{\alpha}{\eta}]$. Moreover, one still has that if $h_f(x) = h$, then the chirp exponent at x is $\frac{h}{\alpha} - 1$.*

If $\beta < \frac{\alpha}{\eta}$, for almost every sample path of the process F_β, the multifractal spectrum is $d(h) = h\frac{\eta}{\alpha}$ if $h \in [\alpha, \beta]$, $d(\beta) = 1$, and $d(h) = -\infty$ elsewhere. Moreover, if $h_f(x) = h < \beta$, *then the chirp exponent at x is $\frac{h}{\alpha} - 1$, but if $h_f(x) = \beta$, then the chirp exponent at x is 0 (there are no "oscillations" around x).*

If $\beta = \frac{\alpha}{\eta}$, for almost every sample path of the process F_β, the multifractal spectrum is the same as the one of the usual lacunary series, except that if $h_f(x) = \beta$, then the chirp exponent at x is 0 (there are no more "oscillations" around x).

See Figure 4.7 for examples of 2-microlocal frontiers of F_β. It is interesting to notice that in the second case ($\beta < \frac{\alpha}{\eta}$), one observes a jump in the multifractal spectrum at the critical value β.

Acknowledgements

The authors are grateful to J.M. Bony, Y. Meyer and S. Jaffard for enlightening discussions.

Chapitre 5

Construction de séries multifractales d'ondelettes

Abstract

Given a positive locally finite Borel measure μ on $\mathbb{R}$, a natural way to construct multifractal wavelet series $F_\mu(x) = \sum_{j,k} d_{j,k}\psi_{j,k}(x)$ is to set $|d_{j,k}| = 2^{-j(s_0-1/p_0)}\mu\big([k2^{-j},(k+1)2^{-j})\big)^{1/p_0}$, where $s_0, p_0 \geq 0$. Indeed, under suitable conditions, it is shown that the function F_μ inherits the multifractal properties of μ.

The transposition of multifractal properties works with many classes of statistically self-similar multifractal measures, enlarging the class of processes possessing some self-similarity property and whose multifractal nature is controlled.

Several perturbations of the wavelet coefficients and their impact on the multifractal nature of F_μ are studied. As an application, multifractal gaussian processes associated with F_μ are created. We obtain results for the multifractal spectrum of the so-called $\mathcal{W}$-cascades introduced by Arnéodo *et al.*

5.1 Introduction and motivations

Phenomena exhibiting wild regularity variations are now well identified in many areas. For instance, they occur in fluid mechanics (intermittent turbulence [66, 30]), in traffic analysis (road and Internet traffic [61]), and in finance [69]. Modeling these phenomena is a major issue for further applications. In particular, finding processes, whose local regularity can be controlled is an active domain of research. Among these processes, those which have some properties of statistical self-similarity and of stability under perturbations are of special interest. They are easier to study, since many works have already investigated the subject. They also are better candidates to fit datas from areas listed above, where scaling invariance play important roles.

When they fulfill these conditions, most of the time these processes satisfy some *multifractal formalism*, either for functions [3, 44] or for measures [18, 75]. Multifractal formalisms take their origin for example in [30, 35, 20]. Let us recall that the Hausdorff multifractal spectrum of f is the function

$$d_f : h \mapsto \dim E_h^f = \dim\{x : h_f(x) = h\},$$

where E_h^f is the set of points x where f has an Hölder exponent $h_f(x)$ equal to h. Given such a process f, a multifractal formalism consists in relating, via a Legendre transform, its Hausdorff multifractal spectrum to some kind of free energy function associated with f.

A possible approach to generate such processes consists in using decompositions on wavelet bases [4, 44, 47, 48, 5]. Wavelets are natural tools in multifractal analysis. Indeed, the concept of self-similarity is implicit in the construction of the wavelet basis $\{\psi_{j,k}\}_{j,k}$. Moreover, the pointwise Hölder exponent h_f, which can be viewed as a measure of the local regularity, can be computed through size estimates of the wavelet coefficients $d_{j,k}$.

In this article, we propose a natural construction of functions F_μ based on a measure μ and on a wavelet basis $\{\psi_{j,k}\}$. Namely, given a positive Borel measure μ on $\mathbb{R}$, the function F_μ is defined by

$$F_\mu(x) = \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} \pm 2^{-j(s_0-\frac{1}{p_0})}\mu\big([k2^{-j},(k+1)2^{-j})\big)^{\frac{1}{p_0}}\psi_{j,k}(x),$$

where s_0 and p_0 are two real parameters that rule the regularity of F_μ.

We prove that the control of the Hausdorff multifractal spectrum d_μ of μ yields a control on the Hausdorff multifractal spectrum d_{F_μ} of F_μ. More precisely,

Theorem 5.1 *If μ obeys the multifractal formalism for measures at singularity $\alpha \geq 0$, one has $d_{F_\mu}(h) = d_\mu(\alpha)$, where $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$.*

This result is a simple and satisfactory brigde between multifractal analysis of measures and multifractal analysis of functions. Although the transposition of multifractal properties from μ to F_μ seems natural, the proof involves non-trivial arguments.

The multifractal formalism for measures we are going to use is a slight modification of the one developped in [18] (see Definition 5.7). It is shown that, although built on a dyadic grid, this formalism is satisfied by measures whose construction is not based on the dyadic grid. In particular, Theorem 5.1 can be applied to the the classical families of multifractal measures μ generated by multiplicative procedures, like for example quasi-Bernoulli measures [18] and Mandelbrot b-adic random multiplicative cascades [66]. It also applies to recent compound Poisson cascade measures [10], as well as to stable Lévy measures [16, 45]. When μ is random, we exhibit cases where almost surely the whole multifractal spectrum of F_μ can be computed (and not only each point of this spectrum almost surely). In each case, the verification needs non-obvious arguments, which are developed in a general context in [13]. In this paper, we detail the examples of b-adic random multiplicative cascades and of stable Lévy measures, which illustrate multiplicative and additive chaos.

The list of measures Theorem 5.1 applies to is not exhaustive. For instance, it applies to the recent "Log-infinitely divisible multifractal processes" [6], and also to new examples of random multiplicative measures provided in [11].

An important property of the construction is its stability under perturbations of the wavelet coefficients. Indeed, it is shown that, under reasonable assumptions, a part of the multifractal spectrum remains unchanged. This gives rise to important applications. For example, the famous $\mathcal{W}$-cascades of Arneodo *et al* in [4] can now be seen as a perturbation of a function F_μ associated with a well-chosen random multiplicative cascade measure μ. Using this interpretation, under suitable assumptions, we obtain almost surely the whole multifractal spectrum conjectured in [4] for this class of random wavelet series.

Another application is the following : Given a measure μ satisfying the multifractal formalism, one can explicitely construct Gaussian processes whose multifractal spectra are deduced from the one of μ by affine transformations.

Perturbing the construction is also a way to simplify the simulation of multifractal functions which have the same spectrum as F_μ. Indeed, a multifractal measure μ is often the limit of some simple measure-valued process μ_j. Then a convenient perturbation is often to replace $\mu(I_{j,k})$ by $\mu_j(I_{j,k})$ in the construction of F_μ ($\mathcal{W}$-cascades are obtained like this).

Jaffard [47], Aubry and Jaffard [5], created processes whose wavelet coefficients are mutually independent and identically distributed random variables. They reach non-decreasing Hausdorff multifractal spectra, nowhere strictly concave. Moreover, these processes have oscillating singularities. When working on real datas, due to the use of the Legendre transform, concave spectra with a decreasing part are often encountered. Our construction, as well as the one of [4], reaches functions with theoretical strictly concave Hausdorff spectra (see Section 5.6), with a non-trivial decreasing part (not only in the Legendre spectrum). This certainly comes owing the fact that the wavelet coefficients of F_μ are highly correlated, which could lead to more realistic models. These strong correlations also imply that the function F_μ has no oscillating singularities.

Before proving Theorem 5.1, some work on both multifractal analysis of functions and of measures is needed. Section 5.2 concerns functions : it provides the multifractal formalism for functions well adapted to our construction. The notion of 2-microlocal spectrum introduced in [62] is recalled in Section 5.2.3. This tool is used to provide a short proof of Theorem 5.1.

Section 5.3 introduces a modified version of the multifractal formalism for measures of [18]. Indeed, the single usual Hölder exponent does not provide enough information to control the regularity of the functions we build in Section 5.4, thus the definitions of the usual level sets E^μ_α must be modified. Sufficient conditions for this modified multifractal formalism to hold are given in Theorem 5.2.

The function F_μ is defined and studied in Section 5.4. Perturbations of the wavelet coefficients of F_μ are studied in Section 5.5.

Section 5.6 provides fundamental examples of qualified measures μ and of associated functions F_μ. It also contains the application to $\mathcal{W}$-cascades.

Section 5.7 contains the proofs of Proposition 5.1 and Theorem 5.2. Eventually, Section 5.8 is devoted to the proofs of the results stated in Section 5.6.

5.2 Functions setting

Our aim is to build multifractal functions, i.e. functions with varying regularity. Definitions of the two main tools used to measure local regularity are recalled for reader's convenience : the pointwise Hölder exponent, and the multifractal spectrum. A multifractal formalism adapted to the class of functions we deal with is then defined. An upper bound for the multifractal spectrum is finally obtained.

5.2.1 Regularity exponent, multifractal spectrum

In most cases, one measures the local regularity of a function f around a point x_0 through the pointwise Hölder exponent h_f.

Definition 5.1 *Let I be a nontrivial interval of $\mathbb{R}$, x_0 an interior point of I, and h a positive real number with $h \notin \mathbb{N}$. A function $f : I \to \mathbb{R}$ belongs to $C^h_{x_0}$ if and only if there exist a constant C and a polynomial P of degree smaller than $[h]$ such that*

$$\forall\ x \in \mathbb{R},\ |f(x) - P(x - x_0)| \leq C|x - x_0|^h.$$

The pointwise Hölder exponent *of f at x_0 is then $h_f(x_0) = \sup\{h : f \in C^h_{x_0}\}$.*

As explained before, the decomposition of functions on orthonormal wavelet bases is fundamental in our approach. Let ψ be a function in the Schwartz class, as constructed in [55] or [71]. The set of functions $\{\psi_{j,k} = \psi(2^j \cdot -k)\}$, where $(j,k) \in \mathbb{Z}^2$, forms an orthogonal basis of $L^2(\mathbb{R})$. Thus, any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x),$$

where $d_{j,k}$ is the wavelet coefficient of f defined by

$$d_{j,k} := d_{j,k}(f) = 2^j \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt.$$

The pointwise Hölder exponent $h_f(x_0)$ can be characterized by the rate decay of the wavelet coefficients around x_0. Indeed, if ψ has more than $[h_f(x_0)] + 1$ vanishing moments, it is known (see [41]) that if $f \in C^\varepsilon$ for some $\varepsilon > 0$,

$$h_f(x_0) = \liminf_{k2^{-j}\to x_0} \frac{\log|d_{j,k}|}{\log(2^{-j} + |x_0 - k2^{-j}|)}. \tag{5.1}$$

We mention that ψ can also be chosen with compact support, see [22]. Nevertheless it introduces technical complications unuseful to our purpose. Indeed, if a compactly supported wavelet ψ is used, (5.1) is in concurrence with the regularity of ψ. In particular, even if the wavelet is smooth enough and has enough vanishing moments, outside the support of μ, the regularity of the series F_μ we build will be governed by the one of ψ.

In the sequel, **the wavelet ψ is fixed** and belongs to C^∞. Moreover, all its moments of positive orders are supposed to be null, so that (5.1) holds at every x_0.

Definition 5.2 *Let I be a nontrivial interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$. For every $h \geq 0$, let us define the level sets*

$$E^f_h = \{x \in \mathrm{Int}(I) :\ h_f(x) = h\}.$$

*The mapping $d_f : h \geq 0 \mapsto \dim(E^f_h)$ is called the multifractal spectrum of f (*dim *denotes the Hausdorff dimension).*

If $E_h^f = \emptyset$ by convention one sets $d_f(h) = -\infty$. The multifractal analysis of a function f consists in the study of its multifractal spectrum. This function is rarely reachable when trying to compute the pointwise Hölder exponent point by point. It is thus natural to look for other ways to compute, to estimate or to bound this spectrum. This is the aim of multifractal formalisms.

In Sections 5.4-5.6, if a compactly supported wavelet is used (instead of a C^∞ wavelet), the results below are valid if one replaces E_h^f by $E_h^f \cap \operatorname{supp}(\mu)$, and if the wavelet ψ has a global regularity larger than $s - \frac{1}{p} + \frac{\alpha_{\max}}{p}$, where $\alpha_{\max}$ is the largest Hölder exponent of μ.

5.2.2 Upper bound for $d_f(h)$ and multifractal formalism

It will be seen in Section 5.4 that the multifractal analysis of a function $f \in L^2_{loc}(\mathbb{R})$ can be deduced from those of a set of countable functions f_m derived from f, such that each f_m is C^∞ outside $(0,1)$.

Let $f \in L^2_{loc}(\mathbb{R})$, such that

$$f(x) = \sum_{j \geq 0} \sum_{0 \leq k < 2^j} d_{j,k} \psi_{j,k}(x).$$

For this function, $\dim E_h^f = \dim E_h^f \cap [0,1]$.

For $p, t \in \mathbb{R}$ and $j \geq 0$, let us introduce the quantities

$$S_j(p,t) = \sum_{0 \leq k \leq 2^j - 1}^{*} |d_{j,k}|^p 2^{j(t-1)}$$

where $\sum^*$ means that the sum is taken over those k such that $d_{j,k} \neq 0$. Then, let us define

$$S(p,t) = \limsup_{j \to \infty} S_j(p,t) \quad \text{and} \quad \xi_f(p) = \sup\{t \in \mathbb{R} : \ S(p,t) = 0\}.$$

Alternatively,

$$\xi_f(p) = 1 + \liminf_{j \to \infty} -\frac{1}{j} \log_2 \Big(\sum_{0 \leq k \leq 2^j - 1}^{*} |d_{j,k}|^p \Big). \tag{5.2}$$

Since for each $j \geq 1$ the function $p \mapsto \sum^{*}_{0 \leq k \leq 2^j - 1} |d_{j,k}|^p$ is log-convex, and non-increasing if j is large enough, the function ξ_f is concave and non-decreasing on $\mathbb{R}$ (ξ_f is the limit of the infimum of non-decreasing concave functions). Pay attention to the fact that ξ_f may depend on the wavelet ψ. It is natural to introduce this kind of free energy function in order to formulate a multifractal formalism for functions based on the representation as wavelet series (see [3, 44, 49] for example).

The function ξ_f coincides on $(0,\infty)$ with η_f, the so-called scaling function of f. η_f is related to the fact that f belongs to some Besov spaces $B_p^{s,q}$ (see also [44] where the analogous of ξ_f is defined via a continuous wavelet transform). Besov spaces are an especially relevant frame to work with in the frame of multifractal analysis of functions, and to find natural random wavelet series with concave spectra in such spaces was the initial motivation of this work. Let us recall the characterization of Besov spaces on $\mathbb{R}$ by wavelet coefficients (where any C^∞ wavelet ψ' such that all its moments of positive order equal 0 can be chosen for the decomposition), as well as the definition of η_f : for $p, q, s > 0$,

$$f \in B_p^{s,q}(\mathbb{R}) \Leftrightarrow \Big(\sum_k |d_{j,k} 2^{j(s-1/p)}|^p \Big)^{1/p} = \varepsilon_j \text{ with } \varepsilon_j \in l^q. \tag{5.3}$$

Then, with each function f defined on $\mathbb{R}$ is associated its (unique) scaling function $\eta_f(p)$ defined by

$$\eta_f(p) = \sup \Big\{ u : f \in B_p^{\frac{u}{p},\infty}(\mathbb{R}) \Big\}. \tag{5.4}$$

It is easily deduced from (5.2), (5.3) and (5.4) that $\eta_f(p) = \xi_f(p)$ for every $p > 0$. This function $\eta_f(p)$ does not depend on ψ, so the same holds for $\xi_f(p)$ for $p > 0$. Nevertheless, the function $\xi_f(p)$ may depend on ψ when $p \leq 0$.

Frisch and Parisi proposed in [30] a formula that links the multifractal spectrum of a function f with some averaged quantities derived from f. This formula, generically referred as the *Frisch-Parisi conjecture*, can be generalized and reformulated in (see [30, 44, 48])

$$d_f(h) = \inf_{p>0} \big(ph - \eta_f(p) + 1 \big). \tag{5.5}$$

Of course, (5.5) does not always hold. Nevertheless, Jaffard ([44] for instance) established the following general upper bound for $d_f(h)$ (when f has some minimal uniform regularity)

$$\forall\, h \geq 0,\ d_f(h) \leq \inf_{p \geq p_c} \big(ph - \eta_f(p) + 1 \big), \tag{5.6}$$

where $p_c = \inf\{p > 0 :\ \eta_f(p) \geq 1\}$. Then, Jaffard proved that the upper bound in (5.6) is an equality for quasi-all functions in the sense of Baire's categories in certain function spaces (see [48]).

On the other hand, it turns out that for some classes of functions like self-similar or quasi self-similar functions [44, 14, 1], (5.6) becomes an equality only for a part of the multifractal spectrum. It comes owing the fact that for a self-similar function f, d_f possesses a decreasing part, and (5.6) only reaches non-decreasing spectra, and can even not capture the whole increasing part of d_f. In fact, the general upper bound (5.6) is optimal for $h \leq \eta'_f(p_c^+)$. In [49], in order to recover the decreasing part of the multifractal spectrum, Jaffard introduces a method to extend Besov spaces and scaling function to negative p's, and he derives an upper bound for d_f similar to the case $p \geq p_c$ using another scaling function $\widetilde{\eta}_f(p)$.

Here, we adopt an elementary point of view relative to the fixed wavelet ψ. This gives an upper bound for the spectrum of f by the Legendre transform of ξ_f (Proposition 5.1). In the examples of functions considered in Section 5.6, this upper bound will give the correct multifractal spectrum.

In the sequel, the Legendre transform of a function φ from $\mathbb{R}$ to $\mathbb{R}$ is defined by

$$\varphi^* :\ h \mapsto \inf_{p \in \mathbb{R}} \big(ph - \varphi(p) \big). \tag{5.7}$$

For every $x \in \mathbb{R}$ and $j \geq 0$, let $k_{j,x}$ be the integer such that $k_{j,x} 2^{-j} \leq x < (k_{j,x}+1) 2^{-j}$. Then, one defines the following exponents ($\log(0) := -\infty$)

$$\begin{cases} \underline{\alpha}_d^-(x) = \liminf_{j\to+\infty} \frac{\log |d_{j,k_{j,x}-1}|}{\log 2^{-j}} \\ \underline{\alpha}_d(x) = \liminf_{j\to+\infty} \frac{\log |d_{j,k_{j,x}}|}{\log 2^{-j}} \\ \underline{\alpha}_d^+(x) = \liminf_{j\to+\infty} \frac{\log |d_{j,k_{j,x}+1}|}{\log 2^{-j}} \end{cases}$$

Finally, let us introduce

$$\operatorname{supp}(d) = \left\{ \begin{array}{c} x \in (0,1): \ \forall\, J \geq 0, \ \exists\, j \geq J, \\ \max(|d_{j,k_{j,x}-1}|, |d_{j,k_{j,x}}|, |d_{j,k_{j,x}+1}|) \neq 0 \end{array} \right\}. \tag{5.8}$$

Proposition 5.1 *Let $f \in C^\varepsilon(\mathbb{R})$ be such that $f = \sum_{j=1}^{+\infty} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}$, and let ξ_f be its scaling function (associated with the wavelet ψ).*

1. *If $0 \leq h \leq \xi_f'(p_c^-)$ then $d_f(h) \leq (\xi_f - 1)^*(h) = \inf_{p \geq p_c}(ph - \xi_f(p) + 1)$.*
2. *If $h \in (\xi_f'(p_c^-), \xi_f'(0^+))$ and $E_h^f \subset \{x \in (0,1) : \min(\underline{\alpha}_d^-(x), \underline{\alpha}_d(x), \underline{\alpha}_d^+(x)) = h\}$ then $d_f(h) \leq (\xi_f - 1)^*(h) = \inf_{0 \leq p \leq p_c}(ph - \xi_f(p) + 1)$.*
3. *If $h \in [\xi_f'(0^+), \xi_f'(0^-)]$, $dim(E_h^f \cap supp(d)) \leq (\xi_f - 1)^*(h) = 1 - \xi_f(0)$.*
4. *If $h > \xi_f'(0^-)$, $dim(E_h^f \cap supp(d)) \leq (\xi_f - 1)^*(h) = \inf_{p \leq 0}(ph - \xi_f(p) + 1)$.*
5. *If $(\xi_f - 1)^*(h) < 0$ then $E_h^f \cap supp(d) = \emptyset$.*

Remark one more time that the proposition depends on the wavelet ψ when $p < 0$ (i.e. on the decreasing part of the spectrum).

Proposition 5.1 can be viewed as a consequence of the study of the so-called "oscillation spaces" introduced by S. Jaffard in [49]. Nevertheless, for the sake of completeness, we give a simple proof of Proposition 5.1 adapted to our context in Section 5.7.

Remark that the above results remain valid when $\operatorname{supp}(d) = \{x \in (0,1): \ \forall\, J \geq 0, \ \exists\, j \geq J, \ \exists k \text{ with } |k2^{-j} - x| \leq 2^{-j(1-\varepsilon_{j,x})} \text{ and } d_{j,k} \neq 0\}$, where $\{\varepsilon_{j,x}\}$ is a positive real sequence converging to zero when $j \to +\infty$.

Definition 5.3 *The function f is said to obey the multifractal formalism relatively to ψ at $h \geq 0$ if $d_f(h) = (\xi_f - 1)^*(h)$.*

Remark : The inclusion $E_h^f \subset \{x \in (0,1) : \min(\underline{\alpha}_d^-(x), \underline{\alpha}_d(x), \underline{\alpha}_d^+(x)) = h\}$ is false in general, but one always has $\min(\underline{\alpha}_d^-(x), \underline{\alpha}_d(x), \underline{\alpha}_d^+(x)) \geq h$ if $x \in E_h^f$. For instance, consider the classical chirp function $x \to |x - \frac{1}{2}|^{\frac{1}{4}} \sin(\frac{1}{|x-\frac{1}{2}|})$, for which $h_f(\frac{1}{2}) = \frac{1}{4}$ while $\min(\underline{\alpha}_d^-(\frac{1}{2}), \underline{\alpha}_d(\frac{1}{2}), \underline{\alpha}_d^+(\frac{1}{2})) = +\infty$.

5.2.3 Definition of the 2-microlocal formalism for functions

The notion of 2-microlocal spectrum developped in [62] is essential. It proves to be useful in the proofs of Theorem 1 (Section 5.4) and its extensions (Section 5.5).

Figure 1 helps the understanding of the following definitions. Let us first denote, for every scale j and every exponent $\rho \in [0,1]$, by $k_{j,\rho}$ the integer $k_{j,\rho} = [2^{j(1-\rho)}]$.

Definition 5.4 *Let $f \in L^\infty_{loc}$, and its discrete wavelet decomposition $f = \sum_{j,k} d_{j,k}\psi_{j,k}$. For any given x_0, let us define the functions*

– $\theta^0 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\theta^0(\varepsilon) = \sup\{\gamma : \exists \delta, K, \ \forall \beta \leq \varepsilon, \ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

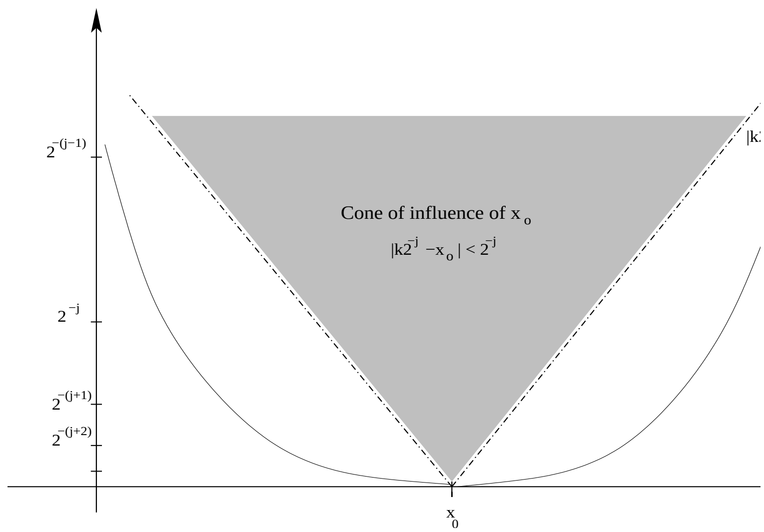

FIG. 5.1 – Time-frequency position of the wavelet coefficients.

– *for any given* $0 < \varepsilon < 1/2$, $\chi^\varepsilon : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\chi^\varepsilon(\rho) = \sup\{\gamma : \exists \delta, K, \forall \beta \in [\rho-\varepsilon, \rho+\varepsilon],\ 2^{-j} \le \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \le K 2^{-j\gamma}\}.$$

– $\theta^1 : (0,1) \to \mathbb{R}^+ \cup \{+\infty\}$,

$$\theta^1(\varepsilon) = \sup\{\gamma : \exists \delta, K,\ \forall \beta \ge \varepsilon,\ 2^{-j} \le \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \le K 2^{-j\gamma}\}.$$

If $k = [2^j x_0 \pm k_{j,\beta}]$, the distance between $k2^{-j}$ and x_0 is approximately $k_{j,\beta}2^{-j} = 2^{-j\beta}$. For $\varepsilon > 0$ small enough, $\theta^0_{x_0}(\varepsilon)$, $\chi^\varepsilon_{x_0}(\rho)$, and $\theta^1_{x_0}(\varepsilon)$ are the maximum rate decay of some selected wavelet coefficients that lie around x_0. $\theta^0_{x_0}(\varepsilon)$ and $\chi^\varepsilon_{x_0}(\rho)$ are non-increasing as functions of ε, and $\theta^1_{x_0}(\varepsilon)$ is a non-decreasing function.

Definition 5.5 *For any $x_0 \in \mathbb{R}$, the 2-microlocal spectrum of f at x_0, $\chi_{x_0} : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ is defined by*

– $\chi_{x_0}(0) = \lim_{\varepsilon\to 0^+} \theta^0_{x_0}(\varepsilon) = \sup\{\theta^0_{x_0}(\varepsilon) : \varepsilon \in (0,1]\}$,
– $\rho \in (0,1)$ *:* $\chi_{x_0}(\rho) = \lim_{\varepsilon\to 0^+} \chi^\varepsilon_{x_0}(\rho) = \sup\{\chi^\varepsilon_{x_0}(\rho) : \varepsilon > 0\}$
– $\chi_{x_0}(1) = \lim_{\varepsilon\to 1^-} \theta^1_{x_0}(\varepsilon) = \sup\{\theta^1_{x_0}(\varepsilon) : \varepsilon \in [0,1)\}$

$\chi_{x_0}(0)$ characterizes the behaviour of the wavelet coefficients that lie in the time-frequency plane below all curves $|k2^{-j} - x_0| = 2^{-j\varepsilon}$, $\varepsilon > 0$, $\chi_{x_0}(1)$ characterizes the behaviour of the wavelet coefficients that lie in the neighbourhood of the cone of influence $|k2^{-j} - x_0| \leq 2^{-j}$, while $\chi_{x_0}(\rho)$ is related to the behaviour of the wavelet coefficients that are located around the curves $|k2^{-j} - x_0| = 2^{-j\rho}$. Moreover, it is proved in [62] that $\rho \mapsto \chi_x(\rho)$ does not depend on the wavelet ψ.

The next proposition [62] relates the 2-microlocal spectrum χ_x to the pointwise Hölder exponent $h_f(x)$. This relation can be compared to (5.1), and it provides a convenient method for computing Hölder exponents.

Proposition 5.2 *Let $f \in C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$, and $x_0 \in \mathbb{R}$. The pointwise exponent of f at x_0 satisfies*

$$h_f(x_0) = \inf\left\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0,1]\right\}.$$

5.3 Multifractal formalism for measures

We consider a slight modification of the multifractal formalism developped in [18]. The main difference is located in the definition of the level sets E^μ_α. For our purpose, we only need the multifractal formalism associated with the dyadic grid of $[0,1]$. Nevertheless, Theorem 5.2 gives sufficient conditions for the validity of this formalism for measures that depend on a b-adic grid with b greater than 2 ; its proof is given in Section 5.7.

5.3.1 Hölder exponent, spectrum of singularity

If $x \in (0,1)$, $\forall j \geq 1$, denote by $I_j(x)$ the closure of the semi-open to the right dyadic interval of length 2^{-j} that contains x. Let us then define $I^+_j(x) = I_j(x) + 2^{-j}$ and $I^-_j(x) = I_j(x) - 2^{-j}$. The convention $\log(0) = -\infty$ is again adopted.

Definition 5.6 *Let μ be a positive Borel measure on $[0,1]$. For $x_0 \in (0,1)$, the lower and upper Hölder exponent of μ at x_0 are respectively defined by*

$$\underline{\alpha}_\mu(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|} \quad \text{and} \quad \overline{\alpha}_\mu(x_0) = \limsup_{j\to+\infty} \frac{\log \mu(I_j(x_0))}{\log |I_j(x_0)|}$$

When $\underline{\alpha}_\mu(x_0) = \overline{\alpha}_\mu(x_0)$, their common value is denoted $\alpha_\mu(x_0)$ and called the Hölder exponent of μ at x_0.

The left and right lower Hölder exponents of μ at x_0 are defined by

$$\underline{\alpha}^-_\mu(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I^-_j(x_0))}{\log |I^-_j(x_0)|} \quad \text{and} \quad \underline{\alpha}^+_\mu(x_0) = \liminf_{j\to+\infty} \frac{\log \mu(I^+_j(x_0))}{\log |I^+_j(x_0)|}.$$

We consider the following level sets for μ, that are necessary in our formalism

Definition 5.7 *For every $\alpha \geq 0$, define*

$$E^\mu_\alpha = \left\{ x \in (0,1) \cap supp(\mu) : \begin{array}{l} \alpha_\mu(x) = \alpha \\ \underline{\alpha}^-_\mu(x) \geq \alpha \\ \underline{\alpha}^+_\mu(x) \geq \alpha \end{array} \right\}. \tag{5.9}$$

The mapping $d_\mu : \alpha \geq 0 \mapsto \dim(E^\mu_\alpha)$ is called the multifractal spectrum of μ.

In the framework of [18], the level sets (5.9) are $\{x \in \mathrm{supp}(\mu) : \alpha_\mu(x) = \alpha\}$. Unfortunately these simpler level sets are not adapted to our construction, since the knowledge of the sole exponent α_μ is not sufficient to guarantee the value of the pointwise Hölder exponent of the wavelet series F_μ.

The optimal choice for the level sets is in fact $\{x : \min(\alpha_\mu(x), \underline{\alpha}_\mu^-(x), \underline{\alpha}_\mu^+(x)) = \alpha\}$. Nevertheless we chose the more restrictive Definition 5.7 to ensure some stability properties after perturbations of wavelet coefficients in (see Section 5.5).

Other multifractal formalisms involve simultaneous information on the quantities $\mu(I_j(x)^-)$, $\mu(I_j(x))$, and $\mu(I_j(x)^+)$, different from the ones we need. In [78] and [85], in order to define a grid-free multifractal formalism for the large deviation spectrum of μ, the function τ is derived from partition functions involving, instead of $\mu(I_{j,k})$, the μ-measure of the boxes $B_{j,k}^+ = I_{j,k}^- \cup I_{j,k} \cup I_{j,k}^+$ such that $\mu(I_{j,k}) \neq 0$.

5.3.2 Multifractal formalism for E_α^μ

Let μ be a positive Borel measure on $[0,1]$. For $j \geq 0$ and $k \in \mathbb{Z}$, let $I_{j,k}$ (*resp.* $I_{j,k}^+$ and $I_{j,k}^-$) denote the interval $[k2^{-j}, (k+1)2^{-j})$ (*resp.* $I_{j,k} + 2^{-j}$ and $I_{j,k} - 2^{-j}$). Following [18], for $q, t \in \mathbb{R}$ and $j \geq 1$, let us define

$$C_j(q,t) = \sum_{0 \leq k \leq 2^j}^{*} \mu^q(I_{j,k}) 2^{kt},$$

where $\sum^*$ means that the sum is taken over those k's such that $\mu(I_{j,k}) > 0$. Then, let us define

$$C(q,t) = \limsup_{j \to +\infty} C_j(q,t) \quad \text{and} \quad \tau(q) = \sup\{t \in \mathbb{R} : C(q,t) = 0\}.$$

The function τ is concave, non-decreasing. An alternative definition of $\tau(q)$ is

$$\tau(q) = \liminf_{j \to +\infty} -\frac{1}{j} \log_2 \Big(\sum_{0 \leq k \leq 2^j}^{*} \mu^q(I_{j,k}) \Big).$$

Noting that E_α^μ is always included in $\{x \in (0,1) : \alpha_\mu(x) = \alpha\}$, it follows from [18] that an upper bound for $\dim(E_\alpha^\mu)$ can be derived from the Legendre transform of τ (see (5.7) for the definition of the Legendre transform)

Proposition 5.3 (Upper bound for $\dim(E_\alpha^\mu)$) *Let $\alpha \geq 0$. One has*

$$\dim(E_\alpha^\mu) \leq \tau^*(\alpha).$$

Moreover, if $\tau^*(\alpha) < 0$ then $E_\alpha^\mu = \emptyset$.

Definition 5.8 *One says that μ obeys the multifractal formalism at $\alpha \geq 0$ if $\dim(E_\alpha^\mu) = \tau^*(\alpha)$.*

Remark : The formalism we use can be improved by considering the sets

$$\widetilde{E}_\alpha^\mu = \{x : \min(\underline{\alpha}_\mu(x), \underline{\alpha}_\mu^-(x), \underline{\alpha}_\mu^+(x)) = \alpha\} \tag{5.10}$$

instead of the E_α^μ. Indeed, Proposition 5.3 also holds for these sets (the proof mimics the one of assertions 2,3,4 and 5 of Proposition 5.1).

5.3.3 A sufficient condition of validity

The following theorem, whose proof is postponed in Section 5.7, gives sufficient conditions for the validity of the multifractal formalism at a given point. It applies in particular on standard classes of statistically self-similar measures that may strongly depend on the b-adic grid with $b \geq 3$. Examples of measures are given in Section 5.6 (a general definition of statistically self-similar measures can be found in [13]).

We need some new definitions.

Let us fix $b \geq 2$. Let $\mathcal{A} = \{0, \dots, b-1\}$. For every $w \in \mathcal{A}^* = \cup_{n \geq 0} \mathcal{A}^n$ ($\mathcal{A}^0 := \{\emptyset\}$), let I_w be the closed b-adic subinterval of $[0,1]$ naturally encoded by w.

If $w \in \mathcal{A}^n$, one can assign to w a unique number $i(w)$ such that the interval I_w can be written $[i(w)b^{-n}, (i(w)+1)b^{-n}]$. Then, if $(v,w) \in \mathcal{A}^n$, $\delta(v,w)$ stands for $|i(v) - i(w)|$.

Let μ be a positive Borel measure on $[0,1]$. For $q, t \in \mathbb{R}$ and $n \geq 0$, let us define

$$C_b(q,t) = \limsup_{n \to \infty} C_{b,n}(q,t) = \limsup_{n \to \infty} \sum_{w \in \mathcal{A}^n}^{*} \mu(I_w)^q b^{nt}.$$

Then let

$$\tau_b(q) = \sup\{t \in \mathbb{R} : \ C_b(q,t) = 0\} = \liminf_{n \to \infty} -\frac{1}{n} \log_b \Big(\sum_{w \in \mathcal{A}^n}^{*} \mu(I_w)^q \Big)$$

Notice that $\tau_2 = \tau$. The following hypothesis (H) on μ is required :

$$(H) \begin{cases} \text{There exists a sequence } (k_j)_{j \geq 1} \text{ such that :} \\ \quad 1.\ \lim_{j \to \infty} \frac{k_j}{j} = 0; \\ \quad 2.\ \text{for every dyadic interval } J \text{ of length } 2^{-j} \text{ such that } \mu(J) \neq 0, \\ \quad\quad \text{there exists a } b\text{-adic interval } J_b \subset J \text{ of length at least } 2^{-j-k_j} \\ \quad\quad \text{such that } \mu(J_b) \neq 0. \end{cases}$$

Hypothesis (H) holds for instance if b is a power of 2 or if $\mathrm{supp}(\mu) = [0,1]$, and in these cases k_j can be taken bounded.

Let us define

$$n_j = \left[\frac{\log 2}{\log b} j \right] \quad \text{and} \quad k'_j = \left[\frac{\log 2}{\log b} k_j \right].$$

Given $q \in \mathbb{R}$, a positive Borel measure μ_q, and a function C_q on $\mathcal{A}^*$ such that

$$\mu_q(I_w) \leq C_q(w) \mu(I_w)^q b^{|w|\tau_b(q)} \ \text{ for all } w \in \mathcal{A}^* \text{ such that } \mu(I_w) > 0 \tag{5.11}$$

holds, if $\tau'_b(q)$ exists, one defines for $\varepsilon, \eta > 0$

$$S_1^{\mu}(q, \varepsilon, \eta) = \sum_{n \geq 1} b^{n\left(\tau_b(q) + (\tau'_b(q) - \varepsilon)\eta\right)} \sum_{v,w \in \mathcal{A}^n : \delta(v,w) \leq b'} \mu(I_v)^{\eta} C_q(w) \mu(I_w)^q,$$

where $b' = 1$ if $b = 2$ and 2 otherwise ;

$$S_2^{\mu}(q,\varepsilon,\eta) = \sum_{j\geq 1} b^{k'_j\tau_b(q)} b^{n_j\left(\tau_b(q)-(\tau'_b(q)+\varepsilon)\eta\right)} \sum_{\substack{v,w\in\mathcal{A}^{n_j+k'_j+2},\ \delta(v,w)\leq b''_j \\ \mu(I_w),\mu(I_v)>0}} \mu(I_v)^{-\eta} C_q(w)\mu(I_w)^q,$$

where $b''_j = 0$ if $b = 2$ and $b^{k'_j+2}+1$ otherwise.

Definition 5.9 *If ν is a positive Borel measure on $[0,1]$, one defines its lower Hausdorff dimension by $\dim(\nu) = \inf\{\dim(B) :\ \nu(B) > 0\}$.*

Theorem 5.2 *Let μ be a positive Borel measure on $[0,1]$, and let us assume that (H) holds for μ. Let $q \in \mathbb{R}$. Suppose that*
(i) there exists a positive Borel measure μ_q on $[0,1]$ and a function C_q on $\mathcal{A}^*$ such that (5.11) holds.
(ii) $\tau'_b(q)$ exists.
(iii) $\dim(\mu_q) \geq \tau_b^*(\tau'_b(q))$.
(iv) for all $\varepsilon > 0$, there exists $\eta > 0$ such that $S_1^{\mu}(q,\varepsilon,\eta) + S_2^{\mu}(q,\varepsilon,\eta) < \infty$.
Then μ obeys the multifractal formalism at $\tau'_b(q)$, i.e. $\dim(E^{\mu}_{\tau'_b(q)}) = \tau^*(\tau'_b(q))$.

Remarks : 1. The assumptions on μ_q in Theorem 5.2 almost imply that μ obeys the multifractal formalism of [18] for a b-adic grid at $\tau'_b(q)$. This is the case for example when C_q is bounded. So this statement is a kind of transfer from the formalism of [18] for a b-adic grid to the formalism of Section 5.3.2 (for a dyadic grid).

2. We do not restrict ourselves to the case $b = 2$. Indeed, one of the main points of this work is the following : Even if the measure μ is not related to the dyadic structure, one can use it to build multifractal wavelet series F_μ (which by construction depend on the dyadic grid). In particular, we explicitely treat the case in Section 5.6 of b-adic Mandelbrot multiplicative cascades. This change of basis is a difficult and interesting issue in multifractal analysis of measures (see [13]).

3. Condition *(iv)*, involving b-adic intervals and their neighbors, is comparable to the one provided in [9] for a measure satisfying the multifractal formalism of [18] for a b-adic grid to also satisfy the Olsen "centered" multifractal formalism [75].

5.4 Building multifractal wavelet series

In all the following sections, two real numbers $s_0 > 0$ and $p_0 > 0$ are fixed such that $s_0 - \frac{1}{p_0} > 0$. These parameters are used to specify the Besov spaces $B_p^{s,\infty}$ the functions belong to.

5.4.1 Explicit construction based on measures

Definition 5.10 *Let μ be a positive measure on $\mathbb{R}$. One defines the wavelet series F_μ, derived from μ, by the following formula*

$$F_\mu(x) = \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} d^\mu_{j,k}\psi_{j,k}(x), \tag{5.12}$$

where the wavelet coefficients $d^\mu_{j,k}$ are

$$d^\mu_{j,k} = 2^{-j(s_0-\frac{1}{p_0})}|\mu(I_{j,k})|^{\frac{1}{p_0}}\,\sigma_{j,k}, \tag{5.13}$$

where $\sigma_{j,k}\in\{-1,1\}$, and by convention $|0|^{\frac{1}{p_0}}=0$.

Any C^∞ wavelet ψ can be used here. As a consequence, several functions are built up here, depending on the choice of ψ. Nevertheless, Theorem 5.1 shows that, under suitable assumptions on μ, these functions have the same multifractal spectrum. One can also consider compactly supported wavelets, but with the restrictions and the modifications we mentioned before (Section 5.2.1).

In order to simplify the multifractal analysis of F_μ, we notice (using Lemma 5.1 below) that

$$\text{for all } h\geq 0,\ d_{F_\mu}(h) = \sup_{m\in\mathbb{Z}} d_{F_{\mu,m}}(h),$$

where $F_{\mu,m}$ is defined on $(0,1)$ by

$$F_{\mu,m}(x) = \sum_{j\geq 0}\ \sum_{\frac{m}{2}\leq k2^{-j}<\frac{m}{2}+1} d^\mu_{j,k}\psi_{j,k}\left(\frac{m}{2}+x\right).$$

Remark that for every point $x\in\mathbb{R}$, there exists an integer m such that $x\in(\frac{m}{2},\frac{m}{2}+1)$. This decomposition holds for any function $f\in L^2_{loc}(\mathbb{R})$. The term $2^{-j(s_0-\frac{1}{p_0})}$ ensures a minimal decay rate of the wavelet coefficients. The reader can verify the following lemma

Lemma 5.1 *If μ is a positive finite measure, then for all $m\in\mathbb{Z}$, $F_{\mu,m}\in B^{s_0,\infty}_{p_0}(\mathbb{R})$. Moreover, $F_{\mu,m}$ is C^∞ outside $[0,1]$.*

5.4.2 Transfer of multifractality theorem

We recall Theorem 5.1. It links the singularity spectrum of F_μ to the one of μ. This result applies on each function $F_{\mu,m}$, $m\in\mathbb{Z}$. Without loss of generality, we redefine μ as its restriction to $(0,1)$, and F_μ is now defined on $(0,1)$ by

$$F_\mu(x) = \sum_{j\geq 0}\ \sum_{0\leq k<2^j}\sigma_{j,k}2^{-j(s_0-\frac{1}{p_0})}|\mu(I_{j,k})|^{\frac{1}{p_0}}\psi_{j,k}(x). \tag{5.14}$$

The functions ξ_{F_μ} and τ_μ (see Sections 5.2.2 and 5.3.2) are simply related :

$$\forall\, p\in\mathbb{R},\ \xi_{F_\mu}(p) = 1+\left(s_0-\frac{1}{p_0}\right)p+\tau_\mu\left(\frac{p}{p_0}\right). \tag{5.15}$$

Theorem 5.1 *Let μ be a positive Borel measure, and let s_0, p_0 be two positive real numbers such that $s_0 - \frac{1}{p_0} > 0$. Let us consider the wavelet series (5.14). If μ obeys the multifractal formalism for measures at $\alpha \geq 0$, then F_μ obeys the multifractal formalism relatively to ψ at $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$, and*

$$d_{F_\mu}(h) = d_\mu(\alpha).$$

Remark : For the function F_μ, supp(d), as defined in (5.8), is obviously equal to supp(μ). Moreover, the function F_μ has the same regularity as the wavelet ψ in the complementary of the support of μ. Indeed, if $x_0 \notin \text{supp}(\mu)$, there exists j_x such that if $j \geq j_x$, $\mu(I_j(x)) = \mu(I_j^+(x)) = \mu(I_j^-(x)) = 0$, thus $d_{j,k} = 0$ for every couple (j,k) with $|k2^{-j} - x| \leq 2^{-j_x}$.

Proof : Assume that μ obeys the multifractal formalism for measures at α.

The inequality $d_{F_\mu}(h) \geq d_\mu(\alpha)$ follows from the following lemma.

Lemma 5.2 *Let $x \in E_\alpha^\mu$. Then $x \in E^{F_\mu}_{s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}}$.*

Proof : For all $\rho \in (0,1)$, let us introduce the sets of wavelet coefficients $B_{\rho,j} = \{d_{j,k} : |k2^{-j} - x| \leq 2^{-j\rho}\}$, and $B_\rho^J = \cup_{j \geq J} B_{j,\rho}$ (B_ρ^J depends on x).

The 2-microlocal spectrum defined in Definitions 5.4 and 5.5 is useful due to the following remark : if there exists a constant C and an integer J such that, for every coefficient $d_{j,k} \in B_\rho^J$, $|d_{j,k}| \leq C 2^{-j\gamma}$ ($\gamma > 0$), then $\forall \rho' > \rho$, $\chi_x(\rho') \geq \gamma$.

Indeed, the computation of $\chi_x(\rho')$ requires the use of wavelet coefficients that all belong to B_ρ^J, i.e. such that $|k2^{-j} - x| \leq 2^{-j\rho}$. Thus a global uniform decay of the coefficients belonging to B_ρ^J gives a lower bound for the values of the 2-microlocal spectrum χ_x at $\rho' > \rho$.

Let now $\rho \in (0,1)$, and $\varepsilon > 0$. $x \in E_\alpha^\mu$ and (5.9) imply that there exists an integer $J > 0$, such that

$$j \geq J \Rightarrow \frac{\log \mu(I_{j,k^*_{j,x}})}{\log 2^{-j}} \geq \alpha - \varepsilon, \tag{5.16}$$

for every $k^*_{j,x} \in \{k^-_{j,x}, k_{j,x}, k^+_{j,x}\}$, where $k^-_{j,x} = k_{j,x} - 1$ and $k^+_{j,x} = k_{j,x} + 1$.

Let now $j \geq \frac{J}{\rho}$, and $d_{j,k} \in B_{\rho,j}$. One sets $j_\rho = [\rho j] \geq J$. $I_{j,k}$ is included in one of the three intervals $I_{j_\rho, k_{j_\rho,x}}$, $I_{j_\rho, k^+_{j_\rho,x}}$, $I_{j_\rho, k^-_{j_\rho,x}}$. Hence

$$\mu(I_{j,k}) \leq \max(\mu(I_{j_\rho,k_{j_\rho,x}}), \mu(I_{j_\rho,k^+_{j_\rho,x}}), \mu(I_{j_\rho,k^-_{j_\rho,x}})) \leq 2^{-j_\rho(\alpha - \varepsilon)}, \tag{5.17}$$

where (5.16) has been used. This translates to the wavelet coefficients into

$$|d_{j,k}| \leq 2^{-j(s_0 - \frac{1}{p_0})} 2^{-(j\rho + 1)\frac{\alpha - \varepsilon}{p_0}} = 2^{\frac{\alpha - \varepsilon}{p_0}} \, 2^{-j(s_0 - \frac{1}{p_0} + \rho\frac{\alpha - \varepsilon}{p_0})}.$$

This decay rate holds for all the wavelet coefficients that belong to $\cup_{j \geq \frac{J}{\rho}} B_{\rho,j} = B_\rho^{[J/\rho]}$. This implies in particular that $\forall \rho' \in (\rho, 1]$, $\chi_x(\rho') \geq s_0 - \frac{1}{p_0} + \rho\frac{\alpha - \varepsilon}{p_0}$. This is true in fact for every $\varepsilon > 0$, and for all $\rho > 0$. Eventually one gets

$$\forall \rho \in (0,1], \chi_x(\rho) \geq s_0 - \frac{1}{p_0} + \rho\frac{\alpha}{p_0} \tag{5.18}$$

Simultaneously, by definition of E^μ_α, $\lim_{j\to+\infty} \frac{\log\mu(I_{j,k_{j,x}})}{\log|I_{j,k_{j,x}}|} = \alpha$. Thus, for a given ρ, the global decay rate of the wavelet coefficients that belong to B^J_ρ is smaller than the one of the special coefficients $d_{j,k_{j,x}}$, which is exactly $s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$. In particular, $\chi_x(1) \leq s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$, and combining this with (5.18) gives

$$\chi_x(1) = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}.$$

Now, Proposition 5.2 allows to conclude

$$h_f(x) = \inf_{\rho\in(0,1]} \frac{\chi_x(\rho)}{\rho} = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}.$$

Moreover, since the largest coefficients are located in the cone of influence of x, there are no chirp-like oscillations around x. One easily shows that the oscillating exponent $\beta_o(x)$ introduced in [51] equals 0.

To be complete about χ_x, let us mention that $\chi_x(0) \geq s_0 - \frac{1}{p_0}$, since $\forall j,k$, $|d_{j,k}| \leq C2^{-j(s_0-\frac{1}{p_0})}$ for some constant C. ■

Lemma 5.2 shows that $E^\mu_\alpha \subset E^{F_\mu}_h$, hence $d_{F_\mu}(h) \geq \dim(E^\mu_\alpha) = d_\mu(\alpha)$.

The inequality $d_{F_\mu}(h) \leq d_\mu(\alpha)$ is obtained as follows.

Using (5.1), for any function f that belongs to C^ε for some $\varepsilon > 0$, if $h_f(x) = h$ then $\min(\underline{\alpha}_d(x), \underline{\alpha}_d^-(x), \underline{\alpha}_d^+(x)) \geq h$. Suppose $x \in E^{F_\mu}_h$ but $\min(\underline{\alpha}_d(x), \underline{\alpha}_d^-(x), \underline{\alpha}_d^+(x)) > h$. This is equivalent, by construction of F_μ, to $\min(\underline{\alpha}_\mu^-(x), \underline{\alpha}_\mu(x), \underline{\alpha}_\mu^+(x)) > \alpha$. It then follows from an easy adaptation of the proof of Lemma 5.2 that $x \in E^{F_\mu}_{h'}$, for some $h' > h$, hence a contradiction with $h_f(x) = h$.

Consequently, $E^{F_\mu}_h \subset \{x \in [0,1] : \min(\underline{\alpha}_d^-(x), \underline{\alpha}_d(x), \underline{\alpha}_d^+(x)) = h\}$. Proposition 5.1 yields $d_{F_\mu}(h) \leq (\xi_{F_\mu} - 1)^*(h)$. Moreover, $(\xi_{F_\mu} - 1)^*(h) = \tau^*(\alpha)$ by (5.15) and $d_\mu(\alpha) = \tau^*(\alpha)$ by assumption. ■

Remarks : 1. Lemma 5.2 is false in general. The special form of the function F_μ we build (i.e. deduced from a positive measure) induces a hierarchy between the wavelet coefficients that makes the lemma hold. In particular, one could consider, instead of a measure μ, more general non decreasing set functions, such as for example Choquet capacities [64], provided that they satisfy some multifractal formalism.

2. As it can be seen in the proof, Theorem 5.1 remains valid if in the multifractal formalism the sets E^μ_α are replaced by the sets $\widetilde{E}^\mu_\alpha$ defined in (5.10). This is used in Section 5.8 to derive the multifractal spectrum of F_μ when μ is a stable Lévy measure. This second formalism is nevertheless hard to manipulate when adding perturbations in the wavelet coefficients.

3. When $p = 1$, $\mu(I_{j,k})$ can be viewed as $\langle\mu, \phi_{j,k}\rangle$, where $\phi(x) = \mathbf{1}_{[0,1]}(x)$. The mapping $\mu \to F_\mu = \sum_{j,k}\langle\mu, \phi_{j,k}\rangle\psi_{j,k}$ is a linear regularization operator.

5.5 Perturbing the construction

A natural question is the stability (from a multifractal point of view) of the construction if a perturbation is introduced in the wavelet coefficients of F_μ.

5.5.1 Principles

The perturbation we consider consists in multiplying the wavelet coefficients by the terms of a real sequence $(\pi(j,k))_{j\geq 0,\ 0\leq k\leq 2^j-1}$. As in Section 5.4.2, without loss of generality, consider the function F_μ restricted to the interval $(0,1)$. Let us define, whenever it exists,

$$F_\mu^{pert}(x) = \sum_{j\geq 0, 0\leq k<2^j} d_{j,k}^{pert}\psi_{j,k}(x),$$

where

$$d_{j,k}^{pert} = 2^{-j(s_0-\frac{1}{p_0})}\mu(I_{j,k})^{\frac{1}{p_0}}\pi(j,k).$$

Let us begin, without proof, with an easy and classical principle of perturbation :

Lemma 5.3 *Assume that $F_\mu \in C^\alpha([0,1])$, and let $\beta \in]-\infty,\alpha)$. We assume that the wavelet ψ has N vanishing moments, with $N > \max\{h_{F_\mu}(x) : x \in (0,1)\} - \min(0,\beta)$.*

The function F_μ^{pert} deduced from F_μ by $d_{j,k}^{pert} = 2^{\beta j} d_{j,k}^\mu$ (i.e. $\pi(j,k) = 2^{\beta j}$) belongs to $C^{\alpha-\beta}([0,1])$ and $d_{F_\mu^{pert}}(h) = d_{F_\mu}(h+\beta)$ for all $h \geq 0$.

We shall need the following properties and definitions.

Property $(\mathcal{P}_1)$: $\displaystyle\limsup_{j\to\infty} \frac{\max_{0\leq k\leq 2^j-1}\log(|\pi(j,k)|)}{j} \leq 0.$

Property $(\mathcal{P}_2)$: $\displaystyle\liminf_{j\to\infty} \frac{\min_{0\leq k\leq 2^j-1}\log(|\pi(j,k)|)}{j} \geq 0.$

Property $(\mathcal{P}_3)$: the set T is empty, where

$$T = \Big\{x : \limsup_{j\to+\infty} \frac{\log|\pi(j,k_{j,x})|}{j} < 0\Big\}.$$

Property $(\mathcal{P}_4(d))$: $0 \leq d < 1$ and $\dim T \leq d$.

Proposition 5.4 *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\geq 0, 0\leq k\leq 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$. Then, the function F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ for every $\varepsilon > 0$, and has the same spectrum as F_μ.*

Proof : For all $x \in [0,1]$, for all $\rho \in [0,1]$, the 2-microlocal spectrum $\chi_x^{pert}(\rho)$ of F_μ^{pert} is equal to the one of F_μ, $\chi_x(\rho)$. Indeed, for every $\varepsilon > 0$, the absolute value of the correction due to the perturbation belongs to $[2^{-j\varepsilon}, 2^{j\varepsilon}]$ for j large enough, thus the modifications induced by the $\pi(j,k)$ are not seen by $\chi_x(\rho)$ (remember that when computing $\chi_x(\rho)$, only decays of order $2^{-j\gamma}$ are taken into account).

Since $\chi_x(\rho)$ is left unchanged, Proposition 5.2 implies that the pointwise Hölder exponent is also left unchanged at each point. ∎

Proposition 5.5 *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\geq 0, 0\leq k\leq 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$. Then, the function F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ for every $\varepsilon > 0$, and has the same spectrum as F_μ.*

Proof : Using the same arguments as in last Proposition, $(\mathcal{P}_1)$ implies that

$$\forall x,\, \forall \rho,\, \chi_x^{pert}(\rho) \geq \chi_x(\rho), \tag{5.19}$$

i.e. that one can only get a better regularity at each point. On the other hand, $(\mathcal{P}_3)$ implies that, for every x such that $\alpha_\mu(x)$ exists, one has

$$\underline{\alpha}_d(x) \geq \underline{\alpha}_d^{pert}(x).$$

Using (5.19), one gets $\chi_x^{pert}(1) = \chi_x(1)$. Proposition 5.2 yields $h_{F_\mu}(x) = h_{F_\mu^{pert}}(x)$. ∎

Proposition 5.6 *Let μ be a positive Borel measure on $[0,1]$. Suppose that the perturbations $(\pi(j,k))_{j\geq 0, 0\leq k\leq 2^j-1}$ satisfy $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ for some $d \in [0,1)$. Then, the function F_μ^{pert} belongs to $B_{p_0}^{s_0-\varepsilon,\infty}$ for every $\varepsilon > 0$, and if μ obeys the multifractal formalism at point $\alpha \geq 0$ such that* $\dim E_\alpha^\mu > d$ *then $d_{F_\mu^{pert}}(h) = d_{F_\mu}(h)$, where $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$.*

Proof : $(\mathcal{P}_4(d))$ replaced $(\mathcal{P}_3)$, so the second argument in the proof of Proposition 5.5 holds at every $x \notin T$. This is enough to conclude. ∎

Remark : Lemma 5.3 and Proposition 5.4 hold in fact for every function $f \in L^2_{loc}(\mathbb{R})$. On the other hand, Propositions 5.5 and 5.6 explicitely use the special structure of the wavelet coefficients of F_μ. Remark that no more hierarchical relation between the wavelet coefficients, such as (5.17), holds after multiplication of $d_{j,k}$ by $\pi(j,k)$. Our analysis shows that the multifractal spectrum of the perturbed function can however be computed.

5.5.2 Random perturbations

We give sufficient conditions for properties $(\mathcal{P}_i)$ to hold almost surely if the sequence $(\pi(j,k))_{j\geq 0, 0\leq k\leq 2^j-1}$ is a sequence of real random variables.

Proposition 5.7 *Sufficient conditions for perturbations :*

- $(\mathcal{P}_1)$ *holds if* $\lim_{q\to\infty} \frac{1}{q} \limsup_{j\to\infty} \frac{\max_{0\leq k\leq 2^j-1} \log \mathbb{E}(|\pi(j,k)|^q)}{j} = 0$.

- $(\mathcal{P}_2)$ *holds if* $\sum_{j\geq 0} 2^j \max_{0\leq k\leq 2^j-1} \mathbb{P}(\pi(j,k) = 0) < \infty$ *and*

$$\lim_{q\to-\infty} \frac{1}{|q|} \limsup_{j\to\infty} \frac{\max_{0\leq k\leq 2^j-1} \log \mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}} |\pi(j,k)|^q)}{j} = 0.$$

- $(\mathcal{P}_3)$ *holds if the random variables $\pi(j,k)$ are independent and if for every $\varepsilon > 0$, one has* $q\lim_{j\to\infty} \max_{0\leq k\leq 2^j-1} \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon}) = 0$.

- $(\mathcal{P}_4(d))$ *holds if the random variables $\pi(j,k)$ are independent and if $\forall\varepsilon > 0$,*

$$\limsup_{j\to+\infty} \max_{0\leq k\leq 2^j-1} \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon}) \leq 2^{d-1}.$$

Proof : We begin by

- $(\mathcal{P}_2)$: Fix $\varepsilon > 0$. For all $q < 0$ and $j \geq 0$,

$$\begin{aligned}
&\mathbb{P}(\exists\, 0 \leq k \leq 2^j - 1 : 0 \leq |\pi(j,k)| \leq 2^{-j\varepsilon}) \\
&\leq \sum_{k=0}^{2^j-1} \mathbb{P}(\pi(j,k) = 0) + \mathbb{P}(\pi(j,k) \neq 0, |\pi(j,k)|^q > 2^{-jq\varepsilon}) \\
&\leq 2^j \Big(\max_{0\leq k\leq 2^j-1} \mathbb{P}(\pi(j,k) = 0) + \max_{0\leq k\leq 2^j-1} 2^{jq\varepsilon}\mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}}|\pi(j,k)|^q) \Big).
\end{aligned}$$

Fix $\varepsilon' > 0$, $j_0 \geq 0$ and $q < 0$ such that $\alpha = 1 + q\varepsilon + |q|\varepsilon' < 0$ and

$$\forall j \geq j_0,\ \frac{1}{j} \max_{0\leq k\leq 2^j-1} \log \mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}}|\pi(j,k)|^q) \leq |q|\varepsilon'.$$

Then, one has

$$\sum_{j\geq j_0} 2^j \max_{0\leq k\leq 2^j-1} 2^{jq\varepsilon}\mathbb{E}(\mathbf{1}_{\{\pi(j,k)\neq 0\}}|\pi(j,k)|^q) \leq \sum_{j\geq j_0} 2^{j\alpha} < +\infty.$$

But, since $\sum_{j\geq 0} 2^j \max_{0\leq k\leq 2^j-1} \mathbb{P}(\pi(j,k) = 0) < +\infty$, one obtains by applying the Borel-Cantelli Lemma that almost surely, there exists an integer J such that, $\forall j \geq J$, $\forall 0 \leq k \leq 2^j-1$, $|\pi(j,k)| > 2^{-j\varepsilon}$. The conclusion follows after letting ε tend to 0 along a countable sequence.

- $(\mathcal{P}_1)$: the same proof as for $\mathcal{P}_2$ holds.

- $(\mathcal{P}_3)$: for every $\varepsilon > 0$, let us define $T_\varepsilon = \{x \in [0,1] : \exists J,\ \forall j \geq J,\ |\pi(j,k_{j,x})| \leq 2^{-j\varepsilon}\}$. One remarks that $T_\varepsilon \subset \bigcup_{J\geq 0} U_J$, where

$$U_J = \bigcap_{j\geq J} \quad \bigcup_{0\leq k\leq 2^j-1:\, |\pi(j,k)|\leq 2^{-j\varepsilon}} I_{j,k}.$$

Each U_J is the boundary of a dyadic branching tree in a random environment with extinction probability $1 - \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon})$ (which tends to 1 uniformly in k when $j \to +\infty$) at node indexed by (j,k). Since the random variables $\pi(j,k)$ are mutually independent, one deduces that T_ε is almost surely empty. Indeed, we see that for j large enough the probability of extinction of one node of the j^{th} generation becomes larger than the one in a subcritical Galton-Watson subtree of $\{0,1\}^*$. Eventually, $T \subset \bigcup_{n\geq 1} T_{1/n}$, thus T is almost surely empty.

- $(\mathcal{P}_4(d))$: In the sense of Fan [27], for every $\varepsilon > 0$, T_ε is included in the set of "bad paths" in $\{0,1\}^{\mathbb{N}}$, where every node is "bad" with probability $P_{j,\varepsilon} = \max_{0\leq k\leq 2^j-1} \mathbb{P}(|\pi(j,k)| \leq 2^{-j\varepsilon})$ and "good" with $1 - P_{j,\varepsilon}$, a node being "good" or "bad" independently of the other nodes.

It follows that almost surely

$$\dim T_\varepsilon \leq 1 - \limsup_{j\to+\infty} \frac{1}{j} \sum_{k=1}^{j} \log_2 \frac{1}{P_{j,\varepsilon}}.$$

One concludes again by writing $T \subset \bigcup_{n\geq 1} T_{1/n}$. ∎

5.5.3 Examples

- Uniform control on $\pi(j,k)$ **:** $(\mathcal{P}_1)$ (resp. $(\mathcal{P}_2)$) holds almost surely if the $\pi(j,k)$ are identically distributed with a random variable with finite moments of every positive (resp. negative) order. This is used in Section 5.6.2.

- Gaussian $\pi(j,k)$ **:** $(\mathcal{P}_1)$ and $(\mathcal{P}_3)$ hold almost surely simultaneously if the $\pi(j,k)$ are independent centered Gaussian random variables with variance $\sigma(j,k)$ such that

$$\lim_{j\to\infty} \frac{\max_{0\le k\le 2^j-1} |\log \sigma(j,k)|}{j} = 0.$$

This makes it possible the construction of Gaussian processes whose spectrum are prescribed via a given measure. This principle works with the examples of Section 5.6 (of course, for a random measure μ, the Gaussian perturbations have to be chosen independently of μ). It also allows to construct very easily "pseudo" Fractional Brownian Motions in the following sense. Let us fix $(s_0,p_0)=(2,1)$, and let us then consider the wavelet series F_ℓ^{pert}), where ℓ is the Lebesque measure, and where the perturbations $\pi(j,k)$ are as above. By construction $\max_{0\le k<2^j} |d_{j,k}(F_\ell^{pert})| = \mathrm{O}(2^{2-\varepsilon})$ for all $\varepsilon>0$. Consequently, for every $H\in(0,1)$, due to Lemma 5.3, the function F_H deduced from F_ℓ^{pert} by

$$d_{j,k}(F_H) = 2^{(2-H)j} d_{j,k}(F_\ell^{pert}) = 2^{-jH}\pi(j,k)$$

is a Gaussian process on $[0,1]$ whose multifractal spectrum is the same as the one of the FBM of exponent H, i.e. $d_{F_H}(H)=1$, and $d_{F_H}(h)=-\infty$ if $h\neq H$ (note that the low frequency part of the FBM is forgotten here). Such a construction has been already considered in [29]. For a construction of the Fractional Brownian Motion of exponent H with wavelet coefficients, see [73].

- Lacunary $\pi(j,k)$ **:** Fix $d\in[0,1]$. $(\mathcal{P}_1)$ and $(\mathcal{P}_4(d))$ hold almost surely if the $\pi(j,k)$ are i.i.d random variables that take the value 0 with probability 2^{d-1} and 1 with probability $1-2^{d-1}$. There, if μ obeys the multifractal formalism on a non trivial interval $[\alpha,\beta]$ such that $d<\sup\{\tau^*(\gamma):\gamma\in[\alpha,\beta]\}$, the perturbation operation produces lacunary wavelet series F_μ^{pert} whose multifractal spectrum agrees with the one of F_μ at every $h=s_0-\frac{1}{p_0}+\frac{\gamma}{p_0}$ such that $\gamma\in[\alpha,\beta]$ and $\tau^*(\gamma)>d$.

Lacunary wavelet series are also considered in [47]. They correspond to perturbations of the function F_ℓ where $s_0=1+\alpha$ and $p_0=1$ for some $\alpha\in(0,1)$. But the way certain wavelet coefficients of generation j are killed is different. Some $\eta\in(0,1)$ is fixed, and at each generation j, $[2^{j\eta}]$ coefficients are selected uniformly and independently among the 2^j, they are kept, and the non selected coefficients are put to 0. Moreover, these selection processes are mutually independent. A major difference with our perturbation process is that there, with probability one, the pointwise Hölder exponent is modified on a set of full Lebesgue measure, and is left unchanged on a set of Hausdorff dimension equal to η. This also gives rise to interesting spectra.

5.6 Wavelet series derived from statistically self-similar measures

In the following examples, when a measure μ is defined on $(I, \mathcal{B}(I))$ with $I \in \{[0,1], \mathbb{R}_+\}$, in order to define F_μ, we implicitely consider on $\mathbb{R}$ the extension of μ by setting $\mu = 0$ outside I. When $\mathrm{supp}(\mu) \in \{\mathbb{R}_+, \mathbb{R}\}$, due to the 1-periodicity of μ or the statistical invariance of μ by positive horizontal translations, without ambiguity we can say that μ (resp. F_μ) obeys the multifractal formalism whenever the redefined measure (resp. function) as defined in Section 5.4.2 obeys it.

Most of the classical families of statistically self-similar measures will satisfy the conditions of Proposition 5.1 and Theorem 5.2. For instance, Quasi-Bernoulli measures [18, 26, 28, 37], Mandelbrot cascades [66], compound Poisson cascade measures [10] and stable Lévy measures [16] belong to this class.

In this work, we detail the example of b-adic Mandelbrot random multiplicative cascades, and we refer the reader to [13] for more details on statistical self-similar measures and for the proof of Proposition 5.1 and Theorem 5.2 in these cases. The wavelet series F_μ associated with Mandelbrot cascades is particularly interesting, since the perturbations of such series allows us to derive the Hausdorff multifractal spectrum of the "random wavelet cascades" of Arnéodo, Bacry and Muzy [4]. We also give some clues of what happens when considering stable Lévy measures.

5.6.1 b-adic random multiplicative cascades

We consider the random "canonical" cascade measures introduced by B. Mandelbrot in [66, 67], and whose analysis led to a large literature [53, 52, 38, 19, 74, 7, 8, 77].

Let us fix an integer $b \geq 2$ and W a non-negative random variable. We assume that W is not a.s. constant and that $\mathbb{E}(W) = \frac{1}{b}$. Let us introduce the function

$$q \in \mathbb{R} \mapsto \widetilde{\tau}(q) = -1 - \log_b \mathbb{E}\big(\mathbf{1}_{\{W>0\}} W^q\big). \tag{5.20}$$

In order to avoid technicalities, unessential to our purpose, we assume in this section that W is positive and that $\widetilde{\tau}(q) > -\infty$ for all $q \in \mathbb{R}$.

Let $(W_w)_{w \in \mathcal{A}^*}$ be a sequence of independent copies of W. For every $n \geq 1$, let us consider the random measure μ_n on $\mathbb{R}$ whose density with respect to the Lebesgue measure is given by

$$b^n W_{w_1} W_{w_1 w_2} \dots W_{w_1 w_2 \dots w_n}$$

on every interval I_w, $w = w_1 w_2 \dots w_n$, and such that $\mu_n = 0$ outside $[0,1]$. With probability one, the sequence μ_n converges vaguely to a measure μ when n goes to infinity. Moreover, if $\widetilde{\tau}'(1) > 0$, one has $\mu \neq 0$ with positive probability (see [53]). Since W is chosen positive, $\mu \neq 0$ a.s. (see [38]). So (H) is satisfied with $k_j = 2$.

Then, let us introduce the set $\mathcal{J} = \{q \in \mathbb{R};\ \widetilde{\tau}^*(\widetilde{\tau}'(q)) > 0\}$. It follows from Theorem 8(iv) in [8] that $\tau_b = \widetilde{\tau}$ on $\mathcal{J}$.

Theorem 5.3 *Let μ be a b-adic random multiplicative cascade. With probability one, for every $q \in \mathcal{J}$, the associated wavelet series F_μ obeys the multifractal formalism relatively to ψ at $h = s_0 - \frac{1}{p_0} + \frac{\widetilde{\tau}'(q)}{p_0}$ and $d_{F_\mu}(h) = \widetilde{\tau}'(q)q - \widetilde{\tau}(q)$.*
Moreover, $E_h^{F_\mu} \cap$ supp$(\mu) = \emptyset$ for all $h \notin \overline{\{s_0 - \frac{1}{p_0} + \frac{\widetilde{\tau}'(q)}{p_0} :\ q \in \mathcal{J}\}}$.

Remarks : 1. If b is a power of 2 and $\mathbb{P}(W=0)>0$, the same kind of conclusion holds, but in certain cases the interval $\mathcal{J}$ has to be reduced. This is due to the non existence of certain moment of negative orders of $\mu([0,1])$ conditionnally to the fact that $\mu\neq 0$ (see [8] Remark 1 and Theorem 8$(i)(b)$ for more details).

2. In certain cases this result can be completed by using some results of [8] on the endpoints of $\mathcal{J}$. We differ the use of these results to the application to [4] below.

5.6.2 The natural perturbation and an application to [4]

Here we fix $b=2$. It turns out from the definition of μ that for every $w\in\mathcal{A}^*$, there exists a copy $Y(w)$ of $\mu([0,1])$ such that

$$\mu(I_w)=Y(w)\mu_{|w|}(I_w).$$

This reflects what we call the statistical self-similarity.

Moreover, if $W\leq 1$ and $\mathbb{P}(W=1)<1/2$, all the moments of $\mu([0,1])$ are finite (see [52, 74, 7] for moments of negative orders and [53] for moments of positive orders). Consequently, we are in the context of the first example of Section 5.2 for the perturbation

$$\pi(j,k)=\left(\frac{\mu_j(I_{j,k})}{\mu(I_{j,k})}\right)^{\frac{1}{p_0}}, \tag{5.21}$$

where we set by convention $\frac{0}{0}:=0$. As a consequence, the conclusions of Theorem 5.3 hold for F_μ^{pert} instead of F_μ.

In [4], a random variable $\mathcal{W}$ is chosen with the following properties : $\mathbb{P}(|\mathcal{W}|>0)=1$, $-\infty<\mathbb{E}(\log|\mathcal{W}|)<0$, and there exists $\eta>0$ such that for every $h\in[0,\eta]$,

$$f(h)=\inf_{q\in\mathbb{R}}\big(hq+1+\log_2\mathbb{E}(|\mathcal{W}|^q)\big)<0. \tag{5.22}$$

Then, a sequence $(\mathcal{W}_w)_{w\in\mathcal{A}^*}$ of independent copies of $\mathcal{W}$ is chosen, and a random wavelet series F is defined by its wavelet coefficients as follows (w is such that $I_{j,k}=I_w$)

$$d_{j,k}(F)=\mathcal{W}_{w_1}\mathcal{W}_{w_1w_2}\dots\mathcal{W}_{w_1w_2\dots w_j}.$$

It can be seen that F converges a.s. in $L^2((0,1))$.

By using large deviations results, the authors show that the pointwise Hölder exponents of F belong to the interval $[h_{\min},h_{\max}]$ where $h_{\min}=\inf\{0<h<-\mathbb{E}(\log_2\mathcal{W}):\ f(h)\geq 0\}$ and $h_{\max}=\inf\{h>-\mathbb{E}(\log\mathcal{W}):\ f(h)<0\}$. Moreover, with probability one, for every $\alpha\in(0,h_{\min})$, F is a Hölder continuous function of exponent α.

We claim that in certain cases the series F can be viewed as a perturbation of one wavelet series F_μ associated with a suitable dyadic random multiplicative cascade measure μ. As a consequence, when this holds, the Hausdorff multifractal spectrum of F can be computed.

Let us assume that all the moments of $\mathcal{W}$ are finite. We consider the function

$$T:\ q\in\mathbb{R}\mapsto -1-\log_2\mathbb{E}(|\mathcal{W}|^q)\quad\text{and}\quad W=\frac{|\mathcal{W}|}{2\mathbb{E}(|\mathcal{W}|)}$$

With W can be associated the scaling function $\widetilde{\tau}$ (5.20) and the interval $\mathcal{J}$. For every $q \in \mathbb{R}$, one easily sees that $T(q) = -q(1 + \log_2 \mathbb{E}(|\mathcal{W}|)) + \widetilde{\tau}(q)$. Hence one has

$$T'(q) = -1 - \log_2 \mathbb{E}(|\mathcal{W}|) + \widetilde{\tau}'(q) \quad \text{and} \quad f(T'(q)) = T^*(T'(q)) = \widetilde{\tau}^*(\widetilde{\tau}'(q)). \tag{5.23}$$

This implies that $(h_{\min}, h_{\max}) = \{T'(q) : \ q \in \mathcal{J}\}$.

Moreover, using (5.22) and (5.23), one gets $\widetilde{\tau}^*(\widetilde{\tau}'(q)) < 0$ for some $q > 0$, which implies that $\widetilde{\tau}$ becomes positive at 1^+ and that $\widetilde{\tau}'(1) > 0$.

As a consequence, let us consider the measure μ constructed as in the previous Section 5.6.1 with the weights $\{W_{w_1\ldots w_n}\}_{w\in\mathcal{A}^*} = \left\{\frac{|\mathcal{W}_{w_1\ldots w_n}|}{2\mathbb{E}(|\mathcal{W}|)}\right\}_{w\in\mathcal{A}^*}$. Let us then introduce the wavelet series F_μ^{pert} associated with μ, with the parameters $s_0 = 2$ and $p_0 = 1$, and with the perturbation (5.21). One can then rewrite the wavelet coefficients of the wavelet series F as ($w = w_1 \ldots w_j$ is chosen so that $I_{j,k} = I_{w_1\ldots w_j}$)

$$\begin{aligned} |d_{j,k}(F)| &= 2^{(s_0 - \frac{1}{p_0})j} \big(2\mathbb{E}(\mathcal{W})\big)^j 2^{-(s_0-\frac{1}{p_0})j} W_{w_1} \ldots W_{w_1\ldots w_j} \\ &= 2^{\left(2+\log_2 \mathbb{E}(|\mathcal{W}|)\right)j} |d_{j,k}(F_\mu^{pert})|. \end{aligned}$$

In order to use the result on F_μ^{pert}, we must assume that W verifies $W \leq 1$ and $\mathbb{P}(W = 1) < 1/2$.

We then apply Theorem 5.3 and Lemma 5.3.

With probability one, for every $q \in \mathcal{J}$, at point

$$h_q = -(2 + \log_2 \left(\mathbb{E}(|\mathcal{W}|)\right) + \left(s_0 - \frac{1}{p_0} + \frac{\widetilde{\tau}'(q)}{p_0}\right) = T'(q),$$

one has $d_F(h_q) = \widetilde{\tau}'(q)q - \widetilde{\tau}(q) = f(h_q)$ ((5.23) has been used here).
Moreover, $E_h^F = \emptyset$ for every $h \notin \overline{\{T'(q) : \ q \in \mathcal{J}\}}$.

Corollary 5.1 *Under the above assumptions on $\mathcal{W}$, with probability one, one has $d_F(h) = f(h)$ for every $h \in (h_{\min}, h_{\max})$.*
Moreover, $E_h^F = \emptyset$ for every $h \notin [h_{\min}, h_{\max}]$.

This result can be completed if one uses the results of [8] concerning the two special values of α, $\alpha_{\min} = \widetilde{\tau}'(\sup(\mathcal{J}))$ and $\alpha_{\max} = \widetilde{\tau}'(\inf(\mathcal{J}))$ (notice that $\alpha_{\min} = \max_{q>0} \frac{\widetilde{\tau}(q)}{q}$ and $\alpha_{\max} = \min_{q<0} \frac{\widetilde{\tau}(q)}{q}$, and that they both belong to $(0, \infty)$). Indeed, in [8], Theorem $8(ii)(b)$ and its proof, establish the following facts :

- If $\mathbb{R}_+ \subset \mathcal{J}$ and $\widetilde{\tau}^*(\alpha_{\min}) > 0$, at $q = \infty$, almost surely there exists a Mandelbrot measure μ_q such that $\dim(\mu_q) \geq \widetilde{\tau}^*(\alpha_{\min})$.
- The same holds if $\mathbb{R}_- \subset \mathcal{J}$ and $\widetilde{\tau}^*(\alpha_{\max}) > 0$.
- In both cases, the analysis done in Section 5.8 to obtain Theorem 5.3 holds at q, and $d_F(h_{\min}) = f(h_{\min})$, in the first case, and $d_F(h_{\max}) = f(h_{\max})$ in the second case.

One also has :
- if $\mathbb{R}_+ \not\subset \mathcal{J}$, $\widetilde{\tau}^*(\alpha_{\min}) = 0$ and $E^\mu_{\alpha_{\min}} \neq \emptyset$, so $d_F(h_{\min}) = f(h_{\min}) = 0$.
- if $\mathbb{R}_- \not\subset \mathcal{J}$, $\widetilde{\tau}^*(\alpha_{\max}) = 0$ and $E^\mu_{\alpha_{\max}} \neq \emptyset$, so $d_F(h_{\max}) = f(h_{\max}) = 0$.

These equalities are obtained by using another kind of statistically self-similar measures. Indeed, in the critical case $\tau^*(\tau'(q)) = 0$, the natural candidate measure for μ_q in Theorem 5.2 is a degenerate Mandelbrot measure, and a substitute is thus needed. It is provided by another martingale construction described in [8].

Using Theorem 3.2 of [77] also allows to compute the function ξ_F on $\mathbb{R}_+$ or $\mathbb{R}_-$ (as well as ξ_{F_μ}). Indeed, due to this result, if $q_{\min} = \inf(\mathcal{J}) > -\infty$ (resp. $q_{\max} = \sup(\mathcal{J}) < \infty$), with probability one, $\tau_2(q) = \alpha_{\max} q$ (resp. $\alpha_{\min} q$) on $(-\infty, q_{\min}]$ (resp. $[q_{\max}, \infty)$).

Corollary 5.2 *Under the assumptions of Corollary 5.1, let us suppose the additional properties on $\mathcal{W}$:*
(i) $\mathbb{R}_+ \subset \mathcal{J}$ and $(-\widetilde{\tau})^*(\alpha_{\min}) > 0$, or $\mathbb{R}_+ \not\subset \mathcal{J}$,
(ii) $\mathbb{R}_- \subset \mathcal{J}$ and $(-\widetilde{\tau})^*(\alpha_{\max}) > 0$, or $\mathbb{R}_- \not\subset \mathcal{J}$.

Then, with probability one,

$$d_F(h) = \begin{cases} f(h) = (\xi_F - 1)^*(h) & \text{if } h \in [h_{\min}, h_{\max}] \\ -\infty & \text{otherwise,} \end{cases}$$

where

$$\xi_F(p) = \begin{cases} h_{\max}\, p & \text{if } -\infty < q_{\min} \text{ and } p \leq q_{\min} \\ -\log_2 \mathbb{E}(|\mathcal{W}|^p) & \text{if } p \in \mathcal{J} \\ h_{\min}\, p & \text{if } q_{\max} < \infty \text{ and } p \geq q_{\max}. \end{cases}$$

5.6.3 Stable Lévy measures

Fix $\beta \in (0,1)$. Let S_β be a Poisson point process in $\mathbb{R}_+ \times \mathbb{R}_+^*$ with intensity $\ell \otimes \nu$, where ℓ is the Lebesgue measure and $\frac{d\nu}{d\ell}(\lambda) = \lambda^{-1-\beta}$. The measure μ defined by

$$\mu = \sum_{(t,\lambda) \in S_\beta} \lambda . \delta_t$$

is almost surely finite and called β-stable Lévy measure. The process $X_\beta(t) = \mu([0,t])$ is a β-stable Lévy subordinator. The measure μ is statistically invariant by positive horizontal translations.

Theorem 5.4 *With probability one, F_μ obeys the multifractal formalism relatively to ψ at every $h \in [s_0 - \frac{1}{p_0}, s_0 - \frac{1}{p_0} + \frac{1}{\beta p_0}]$, with*

$$d_{F_\mu}(h) = \beta p_0 (h - s_0 + \frac{1}{p_0}).$$

Moreover, $E_h^{F_\mu} \cap \mathbb{R}_+ = \emptyset$ for all $h \notin [s_0 - \frac{1}{p_0}, s_0 - \frac{1}{p_0} + \frac{1}{\beta p_0}]$.

5.7 Proofs of Proposition 5.1 and Theorem 5.2

Let us recall the definition of Hausdorff dimension, $\dim(E)$, of a subset E of $\mathbb{R}$: ($|U_i|$ denotes the diameter of the set U_i)

$$\dim(E) = \inf\{d \in \mathbb{R} : \mathcal{H}^d(E) = 0\},$$

where
$$\mathcal{H}^d(E) = \lim_{\delta \downarrow 0} \mathcal{H}^d_\delta(E) = \lim_{\delta \downarrow 0} \inf \Big\{ \sum_{i \in \mathbb{N}} |U_i|^d : \ E \subset \bigcup_{i \in \mathbb{N}} U_i, \ |U_i| \leq \delta \Big\}.$$

Remark : The proof of assertions 2,3, and 4 in Proposition 1 could be deduced, after some efforts, from the one of Theorem 1 in [18]. We give a proof for reader's convenience.

5.7.1 Proof of Proposition 5.1

We begin by an elementary preliminary remark : due to the concavity of ξ_f, if $h \geq \xi_f'(0^+)$ then $(\xi_f - 1)^*(h) = \inf_{p \leq 0}(ph - \xi_f(p) + 1)$, and if $h \in [h_1, h_2]$, where $h_1 \in \{\xi_f'(p_1^-), \xi_f'(p_1^+)\}$ and $h_2 \in \{\xi_f'(p_2^-), \xi_f'(p_2^+)\}$ with $p_1 > p_2 \geq 0$, then $(\xi_f - 1)^*(h) = \inf_{p_2 \leq p \leq p_1}(ph - \xi_f(p) + 1)$.

1. It is the result of Jaffard [44].

2. Fix $h \in (\xi_f'(p_c^-), \xi_f'(0^+))$ and suppose that $E_h^f \subset E_{-1} \cup E_0 \cup E_1$ where

$$\begin{cases} E_{-1} = \{x \in (0,1) : \ \underline{\alpha}_d^-(x) \in [0,h]\} \\ E_0 = \{x \in (0,1) : \ \underline{\alpha}_d(x) \in [0,h]\} \\ E_1 = \{x \in (0,1) : \ \underline{\alpha}_d^+(x) \in [0,h]\}. \end{cases}$$

By the preliminary remark, it is enough to show that for $i \in \{-1, 0, 1\}$ and $p \geq 0$, $\dim E_i \leq ph - \xi_f(p) + 1$. The proof is the same for each E_i, so we only consider the case of E_{-1}.

Fix $p \geq 0$. Then fix $d \geq 0$, $\varepsilon > 0$, $\delta > 0$ and an integer j_δ such that $2^{-j_\delta} < \delta$. By definition, for every $x \in E_{-1}$, one can pick up an integer $j_x \geq j_\delta$ such that

$$\frac{\log |d_{j_x, k_{j_x,x} - 1}|}{\log 2^{-j_x}} \leq h + \varepsilon.$$

Let S stand for $\{[k_{j_x,x} 2^{-j_x}, (k_{j_x,x}+1) 2^{-j_x}] : \ x \in E_{-1}\}$. Let us then define $S' = \{I \in S : \ I \subset J, \ J \in S \Rightarrow J = I\}$. The elements of S' form a δ-covering of E_{-1} by dyadic intervals belonging to the generations greater than or equal to j_δ. Moreover, for every $I = [k2^{-j}, (k+1)2^{-j}] \in S'$, since $p \geq 0$ one has

$$|I|^d \leq |I|^d \, |d_{j,k-1}|^p 2^{jp(h+\varepsilon)}.$$

Consequently,

$$\begin{aligned} \sum_{I \in S'} |I|^d \ &\leq \ \sum_{j \geq j_\delta} \ \sum_{0 \leq k \leq 2^j - 1}^{*} 2^{-jd} |d_{j,k}|^p 2^{jp(h+\varepsilon)} \\ &= \ \sum_{j \geq j_\delta} S_j\big(p, \xi_f(p) + [ph - \xi_f(p) + 1 + p\varepsilon - d]\big). \end{aligned}$$

Suppose $d > ph - \xi_f(p) + 1$. Then, choose ε small enough to have $d - p\varepsilon > ph - \xi_f(p) + 1$ and fix $\varepsilon' \in (0, d - p\varepsilon - [ph - \xi_f(p) + 1])$. One has

$$S_j\big(p, \xi_f(p) + [ph - \xi_f(p) + 1 + p\varepsilon - d]\big) = S_j(p, \xi_f(p) - \varepsilon'') 2^{-j\varepsilon'},$$

where $\varepsilon'' = d - p\varepsilon - [ph - \xi_f(p) + 1] - \varepsilon' > 0$, and by definition of $\xi_f(p)$, $C_p = \sup_{j\geq 0} S_j(p, \xi_f(p) - \varepsilon'') < \infty$. This yields

$$\mathcal{H}^d_\delta(E_{-1}) \leq C_p \sup_{j\geq j_\delta} 2^{-j\varepsilon'},$$

so $\mathcal{H}^d(E_{-1}) = 0$ and $\dim E_{-1} \leq d$. Since this holds for all $d > ph - \xi_f(p) + 1$, the conclusion follows.

3. Notice that $1 - \xi_f(0)$ is an upper bound for the box dimension of supp(d). Then use the preliminary remark.

4. Fix $h > \xi'(0^-)$. By the preliminary remark one only needs to show that $\dim(E^f_h \cap \mathrm{supp}(d)) \leq ph - \xi_f(p) + 1$ for all $p \leq 0$.

Fix $p \leq 0$. It follows from formula (5.1) that for every $x \in E^f_h \cap$ supp(d), one has $\min(\underline{\alpha}^-_d(x), \underline{\alpha}_d(x), \underline{\alpha}^+_d(x)) \geq h$. Hence, for every $\varepsilon > 0$, $E^f_h \cap \mathrm{supp}(d) \subset \cup_{i=-1,0,1} \cup_{n\geq 1} E^i_{h,n,\varepsilon}$ where

$$E^i_{h,n,\varepsilon} = \left\{ x \in (0,1) : \ \forall\, j \geq n,\ \frac{\log |d_{j_x, k_{j_x,x}+i}|}{\log 2^{-j_x}} \geq h - \varepsilon \right\}.$$

The three cases $i = -1, 0, 1$ are similar, so we treat the case $i = -1$.
Fix $d \geq 0$, $n \geq 1$ and $\varepsilon > 0$. For $j \geq 0$, let S^j stand for $\{[k_{j,x}2^{-j}, (k_{j,x}+1)2^{-j}] : x \in E^{-1}_{h,n,\varepsilon} \cap [2^{-j}, 1]\} \cup [0, 2^{-j})$. For every $\delta > 0$ and $j > \max(\log_2 1/\delta, n)$, the elements of S^j form a δ-covering of $E^{-1}_{h,n,\varepsilon}$ and since $p \leq 0$, for every $I = [k2^{-j}, (k+1)2^{-j}] \in S^j$ $(k \neq 0)$

$$|I|^d \leq |I|^d |d_{j,k-1}|^p 2^{jp(h-\varepsilon)}.$$

It follows that

$$\sum_{I\in S^j} |I|^d \leq 2^{-jd} + S_j\big(p, \xi_f(p) + [ph - \xi_f(p) + 1 - p\varepsilon - d]\big). \tag{5.24}$$

Suppose that $d > ph - \xi_f(p) + 1$ and choose ε small enough to have $d + p\varepsilon > ph - \xi_f(p) + 1$. In that case, by definition of $\xi_f(p)$, $\limsup_{j\to\infty} S_j\big(p, \xi_f(p) + [ph - \xi_f(p) + 1 - p\varepsilon - d]\big) = 0$. Consequently, (5.24) yields $\mathcal{H}^d(E^{-1}_{h,n,\varepsilon}) = 0$. This implies that $\dim E^{-1}_{h,n,\varepsilon} \leq d$ for every $n \geq 1$.

Finally, $\dim E^f_h \leq d$ for all $d > ph - \xi_f(p) + 1$, and the conclusion follows.

5. If $0 \leq h < \xi'(0^+)$ then $(\xi - 1)^*(h) < 0$ can occur only if $h \leq \xi'(p_c^+)$, and this case is covered by [48]. If $h \in [\xi'(0^+), \xi'(0^-)]$, we saw that $(\xi - 1)^*(h) = 1 - \xi(0) \geq 0$. If $h > \xi'(0^-)$ and $(\xi - 1)^*(h) < 0$ then $ph - \xi_f(p) + 1 < 0$ for some $p \leq 0$, and if $E^f_h \cap \mathrm{supp}(d) \neq \emptyset$ then using the same computations as in the proof of 4. with $ph - \xi_f(p) + 1 < d < 0$ yields a contradiction, since $\sum_{I\in S^j} |I|^d$ cannot be bounded. ∎

5.7.2 Proof of Theorem 5.2

Theorem 5.2 is a consequence of the next proposition and two lemmas. In all the proofs, the special case $b = 2$, which is immediate, is left to the reader.

Proposition 5.8 1. *For all* $q \geq 0$, $\tau(q) \geq \tau_b(q)$.
2. *Assume (H) holds. For all* $q < 0$, $\tau(q) \geq \tau_b(q)$.

Proof : Remember that, if $(v,w)\in\mathcal{A}^n$, $\delta(v,w)$ stands for $|i(v)-i(w)|$.

1. Fix $q\geq 0$. By definition of n_j, for $j\geq 1$, $b^{-n_j-1}\leq 2^{-j}\leq b^{-n_j}$. So for every $0\leq k\leq 2^j-1$, there exists $v,w\in\mathcal{A}^{n_j}$ such that $\delta(v,w)\leq 1$ and $I_{j,k}\subset I_v\cup I_w$. Moreover, each interval I_v of the n_j^{th} generation covers at most b dyadic intervals of the j^{th} one. It follows that, for $t\in\mathbb{R}$,

$$\sum_{0\leq k\leq 2^j-1}^{*}\mu(I_{j,k})^q 2^{jt} \;\leq\; 2^q b\sum_{w\in\mathcal{A}^{n_j}}^{*}\mu(I_w)^q 2^{jt}\leq 2^q b^{1+|t|}\sum_{w\in\mathcal{A}^{n_j}}^{*}\mu(I_w)^q b^{-n_j t}.$$

Consequently $C(q,t)\leq C_b(q,t)$ and $\tau(q)\geq\tau_b(q)$.

2. Fix $q<0$ and $j\geq 1$. It follows from (H) that every dyadic interval of the j^{th} generation of positive μ-measure contains a b-adic interval of generation $(n_j+k'_j+2)$ with positive μ-measure. It follows that, for $t\in\mathbb{R}$,

$$\sum_{0\leq k\leq 2^j-1}^{*}\mu(I_{j,k})^q 2^{jt} \;\leq\; \sum_{w\in\mathcal{A}^{n_j+k'_j+2}}^{*}\mu(I_w)^q 2^{jt}\leq b^{(3+k'_j)|t|}C_{b,n_j+k'_j+2}(q,t).$$

The conclusion follows from the definitions of $\tau(q)$ and $\tau_b(q)$, together with $\lim_{j\to\infty}k_j/j=0$. ∎

In the sequel, we assume the assumptions of Theorem 5.2.

Lemma 5.4 *$\min(\underline{\alpha}_\mu^-(x),\underline{\alpha}_\mu(x),\underline{\alpha}_\mu^+(x))\geq\tau'_b(q)$ μ_q-almost everywhere.*

Proof : For $j\geq 1$ and $\varepsilon>0$, define

$$\begin{aligned}
\underline{E}^-_{j,\varepsilon} &= \left\{x\in\operatorname{supp}(\mu):\frac{\log\mu(I_j(x)^-)}{\log|I_j(x)^-|}\leq\tau'_b(q)-\varepsilon\right\},\\
\underline{E}_{j,\varepsilon} &= \left\{x\in\operatorname{supp}(\mu):\frac{\log\mu(I_j(x))}{\log|I_j(x)|}\leq\tau'_b(q)-\varepsilon\right\},\\
\underline{E}^+_{j,\varepsilon} &= \left\{x\in\operatorname{supp}(\mu):\frac{\log\mu(I_j(x)^+)}{\log|I_j(x)^+|}\leq\tau'_b(q)-\varepsilon\right\}.
\end{aligned}$$

We shall prove that there exists a function $C(q,\varepsilon,\eta)$ such that for all $\varepsilon,\eta>0$

$$\sum_{j\geq 1}\mu_q\left(\underline{E}^-_{j,\varepsilon}\cup\underline{E}_{j,\varepsilon}\cup\underline{E}^+_{j,\varepsilon}\right)\leq C(q,\varepsilon,\eta)S_1^\mu(q,\varepsilon,\eta).$$

Then, choosing η such that $S_1^\mu(q,\varepsilon,\eta)<\infty$ yields $\sum_{j\geq 1}\mu_q\left(\underline{E}^-_{j,\varepsilon}\cup\underline{E}_{j,\varepsilon}\cup\underline{E}^+_{j,\varepsilon}\right)<\infty$ for all $\epsilon>0$ and the conclusion follows by the Borel-Cantelli Lemma.

Fix $\varepsilon,\eta>0$. It follows from the Tchebitchev inequality that for all $j\geq 1$,

$$\mu_q(\underline{E}^-_{j,\varepsilon})\leq\sum_{0\leq k\leq 2^j-1}\mu_q(I_{j,k})\mu(I^-_{j,k})^\eta 2^{j(\tau'_b(q)-\varepsilon)\eta}.$$

For $0\leq k\leq 2^j-1$, there exists $u,v,w\in\mathcal{A}^{n_j}$ such that $\delta(u,v)=1$, $\delta(v,w)=1$, and $I^-_{j,k}\subset I_u\cup I_v$, $I_{j,k}\subset I_v\cup I_w$. Consequently

$$\begin{aligned}
&\mu_q(I_{j,k})\mu(I^-_{j,k})^\eta 2^{j(\tau'_b(q)-\varepsilon)\eta}\\
\leq\;&(\mu_q(v)+\mu_q(w))2^\eta(\mu(u)^\eta+\mu_q(v)^\eta)2^\eta b^{|\tau'(q)+\varepsilon|\eta}b^{n_j(\tau'_b(q)-\varepsilon)\eta}.
\end{aligned}$$

Using (5.11) gives

$$\mu_q(\underline{E}^-_{j,\varepsilon}) \leq C b^{n_j\left(\tau_b(q)+\eta(\tau_b'(q)-\varepsilon)\right)} \sum_{v,w\in\mathcal{A}^{n_j},\ \delta(v,w)\leq 2} \mu(I_v)^\eta C_q(w)\mu(I_w)^q$$

with $C = b^{|\tau'(q)+\varepsilon|\eta}$. The same upper bound holds for $\mu_q(\underline{E}_{j,\varepsilon})$ and $\mu_q(\underline{E}^+_{j,\varepsilon})$, and the conclusion follows. For the special case $b=2$, $b'=1$ because there is no change of base. ∎

Lemma 5.5 $\overline{\alpha}_\mu(x) \leq \tau_b'(q)$ μ_q*-almost everywhere.*

Proof : For $j\geq 1$ and $\varepsilon>0$, define

$$\overline{E}_{j,\varepsilon} = \left\{x\in\mathrm{supp}(\mu) : \frac{\log\mu(I_j(x))}{\log|I_j(x)|} \geq \tau_b'(q)+\varepsilon\right\}.$$

One shall prove that for all $\varepsilon,\eta>0$

$$\sum_{j\geq 1}\mu_q(\overline{E}_{j,\varepsilon}) \leq b^{2\tau_b(q)} S_2^\mu(q,\varepsilon,\eta).$$

The conclusion follows as in Lemma 5.4.

Fix $\varepsilon,\eta>0$. It follows from the Tchebitchev inequality that for all $j\geq 1$,

$$\mu_q(\overline{E}_{j,\varepsilon}) \leq \sum_{\substack{0\leq k\leq 2^j-1\\ \mu(I_{j,k})>0}} \mu_q(I_{j,k})\mu(I_{j,k})^{-\eta} 2^{-j(\tau_b'(q)+\varepsilon)\eta}.$$

One uses the fact that for $j\geq 1$, every dyadic interval $I_{j,k}$ such that $\mu(I_{j,k})>0$ contains a b-adic interval $I_v(j,k)$ of generation $(n_j+k_j'+2)$ such that $\mu(I_v(j,k))>0$, and is also contained in the union of at most $b^{k_j'+2}+1$ such intervals that we denote by the $I_{w_s}(j,k)$. It follows that

$$\begin{aligned}\mu_q(I_{j,k})\mu(I_{j,k})^{-\eta}2^{-j(\tau_b'(q)+\varepsilon)\eta} &\leq b^{-n_j(\tau_b'(q)+\varepsilon)\eta}\mu(I_v(j,k))^{-\eta}\sum_s \mu_q\big(I_{w_s}(j,k)\big)\\ \text{and } \mu_q(\overline{E}_{j,\varepsilon}) &\leq b^{-n_j(\tau_b'(q)+\varepsilon)\eta} \sum_{\substack{v,w\in\mathcal{A}^{n_j+k_j'+2}\\ \delta(v,w)\leq b_j'',\ \mu(I_v)>0}} \mu(I_v)^{-\eta}\mu_q(I_w)\end{aligned}$$

(we used the inequality $2^{-j(\tau_b'(q)+\varepsilon)\eta} \leq b^{-n_j(\tau_b'(q)+\varepsilon)\eta}$ since $\tau_b'(q)+\varepsilon$ is positive). By using (5.11) this yields

$$\sum_{j\geq 1}\mu_q(\overline{E}_{j,\varepsilon}) \leq b^{2\tau_b(q)} S_2^\mu(q,\varepsilon,\eta).$$

∎

Proof : (of Theorem 5.2) Due to Lemma 5.4 and 5.5, μ_q is carried by $E^\mu_{\tau_b'(q)}$. Consequently, our assumption on $\dim(\mu_q)$ implies that $\dim(E^\mu_{\tau'(q)}) \geq \tau_b^*(\tau_b'(q))$. Moreover, $\dim(E^\mu_{\tau_b'(q)}) \leq \tau^*(\tau_b'(q))$ and $\tau\geq\tau_b$ by Propositions 5.3 and 5.8 respectively. Consequently, $\dim(E^\mu_{\tau_b'(q)}) = \tau^*(\tau_b'(q))$, i.e. the multifractal formalism holds at $\tau_b'(q)$.

5.8 Proofs of Theorems 5.3 and 5.4

5.8.1 b-adic random multiplicative cascades

Some computations use arguments that were already present in [9].

For every $q \in \mathcal{J}$, $v \in \mathcal{A}^*$ and $n \geq 1$, let us define

$$Y_{q,n}(v) = b^{n\widetilde{\tau}(q)} \sum_{w_1 \dots w_n \in \mathcal{A}^n} W^q_{vw_1} W^q_{vw_1w_2} \dots W^q_{vw_1w_2\dots w_n}.$$

It follows from Corollary 5 in [8] that, with probability one, for all $v \in \mathcal{A}^*$ and all $q \in \mathcal{J}$, the limit $Y_q(v) = \lim_{n\to\infty} Y_{q,n}(v)$ exists. Moreover, with probability one, for all $q \in \mathcal{J}$, the mapping μ_q defined on the b-adic intervals by

$$\mu_q(I_v) = b^{|v|\widetilde{\tau}(q)} Y_q(v) \prod_{j=1}^{|v|} W^q_{v_1\cdots v_j} \tag{5.25}$$

extends to a Borel measure (notice that $\mu_1 = \mu$). The measures μ_q all have $[0,1]$ as support and for all $v \in \mathcal{A}^*$ and $q \in \mathcal{J}$,

$$\mu_q(I_v) = C_q(v)\mu(I_v)^q c^{|v|\widetilde{\tau}(q)} \tag{5.26}$$

with $C_q(v) = \frac{Y_q(v)}{Y_1^q(v)}$. Moreover, $\forall\, q \in \mathcal{J}$ one has $\dim(\mu_q) \geq \widetilde{\tau}^*(\widetilde{\tau}'(q)) = \tau_b^*(\tau_b'(q))$.

One knows that $\tau_b = \widetilde{\tau}$ on $\mathcal{J}$ and that (H) holds with $k_j = 2$. Hence, the first part of Theorem 5.3 is a consequence of Theorem 5.1 and 5.2 if the following property holds : For every non trivial compact subinterval K of $\mathcal{J}$, with probability one, for all $q \in K$, for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $\gamma \in \{-1, 1\}$

$$\sum_{n\geq 1} b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} \sum_{\substack{v,w\in\mathcal{A}^n,\ \delta(v,w)\leq b^4+1 \\ \mu(I_w),\mu(I_v)>0}} \mu(I_v)^{\gamma\eta} C_q(w)\mu(I_w)^q < \infty.$$

i.e. $\sum_{n\geq 1} b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} f_{n,\varepsilon,\eta}(q) < \infty$, where

$$f_{n,\varepsilon,\eta}(q) = \sum_{v,w\in\mathcal{A}^n:\, \delta(v,w)\leq b^4+1} Y_1(v)^{\gamma\eta} Y_q(w) \prod_{k=1}^{n} W^{\gamma\eta}_{v_1\dots v_k} W^q_{w_1\dots w_k}$$

(see Section 5.3.3 for the definitions of $\delta(v,w)$ and $i(v)$) Let us fix such a compact K. It turns out that it suffices to show that for every $\varepsilon > 0$, if $\eta > 0$ is small enough,

$$\text{if } \gamma \in \{-1,1\} \begin{cases} \sum_{n\geq 1} \sup_{q\in K} n b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} \mathbb{E}\big(f_{n,\varepsilon,\eta}(q)\big) < \infty \\ \sum_{n\geq 1} \sup_{q\in K} b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} \mathbb{E}\big(|f'_{n,\varepsilon,\eta}(q)|\big) < \infty \end{cases} \tag{5.27}$$

Indeed, if (5.27) holds then, with probabity one,

$$\sum_{n\geq 1} b^{n\left(\tau_b(\inf(K))+\gamma\eta\tau_b'(\inf(K))-\varepsilon\eta\right)} f_{n,\varepsilon,\eta}(\inf K) < \infty,$$

$$\text{and} \qquad \int_K \sum_{n\geq 1} \left| \frac{d}{dq} \left(b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} f_{n,\varepsilon,\eta}(q) \right) \right| dq < \infty.$$

Hence the series $\sum_{n\geq 1} b^{n\left(\tau_b(q)+\gamma\eta\tau_b'(q)-\varepsilon\eta\right)} f_{n,\varepsilon,\eta}(q)$ converges uniformly on K (a similar approach was initially used to get the main result in [8]).

It follows from Lemma 6 in [8] that for η small enough and $\gamma \in \{-1,1\}$

$$C_K(\eta) = \sup_{\substack{q\in K,\ n\geq 1,\\ v,w\in\mathcal{A}^n}} \mathbb{E}\left(\left|\frac{d}{dq}\big(Y_1(v)^{-\gamma\eta}Y_q(w)\big)\right|\right) + \mathbb{E}\left(Y_1(v)^{-\gamma\eta}Y_q(w)\right) < \infty$$

$$\text{and} \qquad C_K'(\eta) = \sup_{\substack{q\in K,\ n\geq 1,\\ v,w\in\mathcal{A}^n}} \frac{\mathbb{E}\left(\left|\frac{d}{dq}W_v^{-\gamma\eta}W_w^q\right|\right)}{\mathbb{E}\left(W_v^{-\gamma\eta}W_w^q\right)} < \infty.$$

By taking into account the fact that the W's are mutually independent, one gets

$$\mathbb{E}\left(|f'_{n,\varepsilon,\eta}(q)|\right) \leq C_K(\eta)\big(1+nC_K'(\eta)\big)g_{n,\varepsilon,\eta}(q),$$

$$\text{and} \qquad g_{n,\varepsilon,\eta}(q) = \sum_{v,w\in\mathcal{A}^n:\,\delta(v,w)\leq b^4+1} \prod_{k=1}^{n} \mathbb{E}\left(W_{v_1\cdots v_k}^{-\gamma\eta} W_{w_1\cdots w_k}^{q}\right).$$

One also has $\mathbb{E}\left(f_{n,\varepsilon,\eta}(q)\right) \leq C_K(\eta) g_{n,\varepsilon,\eta}(q)$.

Let us make an important remark. There exists an integer M with the following property : Given two words v and w of $\mathcal{A}^n$ such that $\delta(v,w) \leq b^4+1$, the two prefixes $\dot{v}$ and $\dot{w}$ of v and w which belong to $\mathcal{A}^{n-M}$ satisfy $\delta(\dot{v},\dot{w}) \leq 1$. Moreover, at most b^{2M} pairs of words $(v',w') \in \mathcal{A}^n \times \mathcal{A}^n$ have $(\dot{v},\dot{w})$ as prefixes of generation $n-M$. Because of the form of the function $g_{n,\varepsilon,\eta}(q)$, one can thus assume without loss of generality that in the sums over $\{v,w\in\mathcal{A}^n;\ \delta(v,w)\leq b^4+1\}$, there are only pairs (v,w) for which $\delta(v,w)\leq 1$.

A common way to represent the pairs of words (v,w) is the following. Let ρ_k be the word consisting of k consecutive zeros and let λ_k be the word consisting of k consecutive $b-1$ (considered as a letter from the alphabet $\{0,1,2,\dots,b-1\}$). A representation of the set of pairs (v,w) in $\mathcal{A}^n$ such that $i(w)=i(v)+1$ is as follows :

$$\bigcup_{k=0}^{n-1} \bigcup_{u\in\mathcal{A}^{n-1-k}} \left\{\big(u.j.\lambda_k, u.(j+1).\rho_k\big) : j\in\{0,\dots,b-2\}\right\}. \tag{5.28}$$

Then, splitting $g_{n,\varepsilon,\eta}(q)$ into the sum over the pairs (w,w) and the sum over the pairs (v,w) such that $\delta(v,w)=1$, and using (5.28), one obtains

$$g_{n,\varepsilon,\eta}(q) = b^{-n\widetilde{\tau}(q+\gamma\eta)} + h_{n,\varepsilon,\eta}(q),$$

where $h_{n,\varepsilon,\eta}(q) = (b-1)\sum_{k=0}^{n-1} b^k \left(\mathbb{E}(W^{q+\gamma\eta})\right)^k \left(\mathbb{E}(W^{\gamma\eta})\mathbb{E}(W^{(q)})\right)^{n-k}$.

One then sees that

$$\begin{aligned} h_{n,\varepsilon,\eta}(q) &= (b-1)\left(\mathbb{E}(W^{\gamma q})\mathbb{E}(W^q)\right)^n \sum_{k=0}^{n-1}\left[\frac{b\mathbb{E}(W^{q+\gamma\eta})}{\mathbb{E}(W^{\gamma q})\mathbb{E}(W^q)}\right]^k \\ &\le b^{1-n\left(2+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q)\right)}\sum_{k=0}^{n-1} b^{k\left(2-\widetilde{\tau}(q+\gamma\eta)+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q)\right)} \\ &\le b^{1-n\left(2+\widetilde{\tau}(\gamma\eta)+\widetilde{\tau}(q)\right)}\sum_{k=0}^{n-1} b^{k\left(2-\gamma\eta\widetilde{\tau}'(q)+\widetilde{\tau}(\gamma\eta)-\eta\varepsilon_q(\eta)\right)} \\ &\le \frac{b^{1-n\left(\widetilde{\tau}(q)+\gamma\eta\widetilde{\tau}'(q)+\eta\varepsilon_q(\eta)\right)}}{b^{\left(2-\gamma\eta\widetilde{\tau}'(q)+\widetilde{\tau}(\gamma\eta)-\eta\varepsilon_q(\eta)\right)}-1}, \end{aligned}$$

with $\varepsilon_q(\eta) \to 0$ uniformly on K when $\eta \to 0$. Then, it is easily seen that (5.27) holds if η is small enough.
The second part of Theorem 5.3 comes from the fact that if α does not belong to $\overline{\{\widetilde{\tau}'(q),\ q \in \mathcal{J}\}}$, then $\widetilde{\tau}^*(\alpha) < 0$. Thus, using that $\tau_b = \widetilde{\tau}$ on $\mathcal{J}$, one has $\tau_b^*(\alpha) < 0$.

5.8.2 Stable Lévy measures

In this section, we use the sets $\widetilde{E}^\mu_\alpha$ defined in (5.10) instead of the E^μ_α. Theorem 5.4 follows if the following property holds : With probability one, μ obeys the multifractal formalism at every $\alpha \in [0, 1/\beta]$ with $\dim(\widetilde{E}^\mu_\alpha) = \beta\alpha$ and $\tau^*(\alpha) < 0$, $\forall \alpha > 1/\beta$.

Let us denote by $\big(X_\beta(t)\big)_{t\in[0,1]}$ the stable subordinator such that, almost surely, $\mu([0, t]) = X_\beta(t)$ for all $t \in [0, 1]$.

We begin by estimating by above the Hausdorff dimension of the sets $\widetilde{E}^\mu_\alpha$. Let us estimate the function τ. One knows that $\mu(I_w)$ is a copy of $2^{-\frac{j}{\beta}}X(1)$ for every $j \ge 1$ and $w \in \mathcal{A}^j$. Moreover, $\mathbb{E}\big(X(1)^q\big) < \infty$ for every $q \in (-\infty, \beta)$. This yields that for every $j \ge 1$ and for every couple $(q, t) \in (-\infty, \beta) \times \mathbb{R}$,

$$\mathbb{E}\big(C_{2,j}(q, t)\big) = 2^{(1+t-q/\beta)j}\mathbb{E}\big(X(1)^q\big). \tag{5.29}$$

For every $q \in \mathbb{R}$, let us introduce the function

$$\widetilde{\tau}(q) = \begin{cases} -1 + q/\beta & \text{if } q \in (-\infty, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (5.29) that for every $q \in]-\infty, \beta[$ and $\varepsilon > 0$, with probability one $\sum_{j\ge1} C_{2,j}(q, \widetilde{\tau}(q)-\varepsilon) < \infty$. Consequently, for every $q \in]-\infty, \beta[$, $\tau(q) \ge \widetilde{\tau}(q)$.

Using that the functions $\tau(q)$ and $\widetilde{\tau}(q)$ are continuous, that τ is non-decreasing, and that $\widetilde{\tau}(\beta) = 0$, one deduces that $\tau \ge \widetilde{\tau}$ on $\mathbb{R}$ almost surely. This implies that almost surely, $\tau^*(\alpha) \le \widetilde{\tau}^*(\alpha) = \alpha\beta$ for all $\alpha \in [0, 1/\beta]$, and that $\tau^*(\alpha) \le \widetilde{\tau}^*(\alpha) = -\infty$ for all $\alpha > 1/\beta$.

Finally, by Proposition 5.3 applied to $\widetilde{E}^\mu_\alpha$, with probability one, $\dim(\widetilde{E}^\mu_\alpha) \le \beta\alpha$ for all $\alpha \in [0, 1/\beta]$, and $\widetilde{E}^\mu_\alpha = \emptyset$ for all $\alpha > 1/\beta$.

It remains to lower bound the dimensions. It is proved in [45] that, almost surely, the set E^X_α is empty if $\alpha > 1/\beta$ and $d_X(\alpha) = \beta\alpha$ if $\alpha \in [0, 1/\beta]$. Moreover, since a stable subordinator is not compensated (see [16]), if $\alpha \in [0, 1/\beta]$, the proof of Proposition 2 in [45] shows that for every $t \in E_\alpha^X$,

$$\forall\, \varepsilon > 0,\ \exists\, C > 0,\ \forall s \in [0,1] \setminus \{t\},\ |X(s) - X(t)| \le C|s-t|^{\alpha-\varepsilon}.$$

Using the triangular inequality, this implies that, with probability one, for every $\alpha \in [0, 1/\beta]$ and $t \in E_\alpha^X$, $\min(\underline{\alpha}_\mu^-(t), \underline{\alpha}_\mu(t), \underline{\alpha}_\mu^+(t)) \ge \alpha$.

On the other hand, because of Proposition 1 of [45], with probability one, for every $\alpha \in [0, 1/\beta]$, if $t \in E_\alpha^X$, for every $\varepsilon > 0$ there exists $(j_n)_{n\ge 1}$, an increasing sequence of integers, and $(t_n)_{n\ge1}$, a sequence of jump points of X such that $X(t_n) - X(t_n^-) \ge 2^{-j_n - 1}$ and $|t - t_n| \le 2^{-j_n/(\alpha+\varepsilon)}$. Let $J_n = [j_n/(\alpha+\varepsilon)] - 1$. It is straightforward to see that if t is not a dyadic point one can assume without loss of generality that $t_n \in I_{J_n, k_{J_n,t}}$. This implies

$$\mu(I_{J_n,k_{J_n,t}}) \ge 2^{-3} 2^{-(\alpha+\varepsilon)J_n} \quad (\forall\, n \ge 1) \tag{5.30}$$

and $\min(\underline{\alpha}_\mu^-(t), \underline{\alpha}_\mu(t), \underline{\alpha}_\mu^+(t)) \le \alpha + \varepsilon$. Since this holds for every $\varepsilon > 0$, and since the set of dyadic points is at most countable, we deduce that with probability one, for all $\alpha \in (0, 1/\beta]$,

$$\dim(\widetilde{E}_\alpha^\mu) \ge \beta\alpha. \tag{5.31}$$

Let us finish with the case $\alpha = 0$. By construction, with probability one, the deterministic set D of dyadic points in $[0,1]$ does not contain any jump point of X. So $E_0^X \setminus D$, which contains the jump points of X, is not empty, and (5.31) also holds for $\alpha = 0$.

Chapitre 6

Détection et création d'oscillations

Abstract

Comparing the multifractal and the large deviation spectra for a given continuous function f, we find sufficients conditions for f to have oscillating singularities.

Using a similar approach, we study the following interesting wavelet threshold operator which associates to any function $f = \sum_j \sum_k d_{j,k}\psi_{j,k}$ the function f^t whose wavelet coefficients are $d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}$, for some fixed $\gamma > 0$. We show that this operator applied to a function f creates a context propitious to have oscillating singularities, and can considerably enlarge the support of the multifractal spectrum of f.

We study in detail an example for which a wavelet threshold effectively leads to the creation of oscillations in the "compressed" function f^{th} which were not present in the inital function f. This will lead to functions with surprising, non concave, multifractal spectra.

6.1 Introduction and motivations

This work is entirely devoted to the investigation of the relations between the failure of the multifractal formalism and the presence of oscillating singularities. Detecting fast oscillations in functions, images, or processes, is a fundamental issue in mathematics and physics. Indeed, from a mathematical point of view, the presence of oscillations in a function is often correlated with a failure of the multifractal formalism (see for example [48], [70]). In physics, and especially in fluid mechanics and in the study of turbulence ([30]), local oscillating behaviors play an important role in the blow-up conditions of solutions of classical PDE's.

Multifractal analysis is concerned with the study of the local regularity of measures, functions or distributions. The local regularity of a function f at a given point x is often measured through the pointwise Hölder exponent $h_f(x)$, which is a precise indicator of the wildness of the behavior of f around x. This exponent is hard to handle with, and one usually studies its level sets $E^f_h = \{x : h_f(x) = h\}$. The multifractal analysis of f consists in the study of the multifractal spectrum defined by

$$h \geq 0 \to \dim_H E^f_h,$$

where $\dim_H$ stands for the Hausdorff dimension.

Wavelets are a natural tool used to study local regularity of functions. In particular, for a sufficiently regular function f, they provide a characterization of the pointwise Hölder exponent of f at a given point x. This exponent is a precise indicator of the wildness of the behavior of f around x. But wavelets also allow to study chirp-like behaviors, i.e. infinitely fast oscillations of the function around any point.

We thus start from the decomposition of a continuous function f on a wavelet basis $\{\psi_{j,k}\}$. The repartition (in time) of the wavelet coefficients is fundamental in the regularity analysis of a function. For example, one easily sees that a same set of wavelet coefficients $\{d_{j,k}\}_{j,k\geq 0}$ can lead to drastically different multifractal spectra. However some *a priori* upper bounds can be found by estimating the histograms of coefficients (see [5]). We show in Section 6.3.2 that the comparison between a large deviation spectrum (based on wavelet coefficients) and the multifractal spectrum can sometimes allow to conclude to the presence of oscillations in the function f. A consequence *a posteriori* is that, when the multifractal formalism is broken in the increasing part of the spectrum, oscillating singularities are present in the function.

A natural way to try to decorrelate the large deviation and the multifractal spectrum is to apply a threshold on the wavelet coefficients. This can be performed by what we call a threshold of order γ, i.e. by putting to 0 all the wavelet coefficients such that $|d_{j,k}| \leq 2^{-j\gamma}$ for some $\gamma > 0$. Threshold is a powerful method used in signal and image compression ([23], [24]) : Indeed it is an easy way to keep the main information of a function (i.e. the large wavelet coefficients) while giving up the less important behaviors. We show that a threshold of order γ, as we define it in Section 6.3, introduces most of the time oscillating singularities in the "compressed" function. For discrete images, *hard* threshold, i.e. by putting to 0 all the wavelet coefficients such that $|d_{j,k}| \leq \varepsilon$ for some $\varepsilon > 0$, can certainly be compared to a threshold of order γ for a certain $\gamma > 0$.

We introduce the *threshold of order* γ, which can be applied to any continuous function $f \in L^2(\mathbb{R})$. if $f = \sum_{j\geq 1}\sum_k d_{j,k}\psi_{j,k}$, then we define $f^t = \sum_{j\geq 1}\sum_k d^t_{j,k}\psi_{j,k}$, where

$$d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}.$$

i.e. one puts to 0 all the wavelet coefficients such that $|d_{j,k}| \leq 2^{-j\gamma}$ for some $\gamma > 0$.

We also introduce the *inverse threshold of order* γ which associates to any continuous function $f \in L^2(\mathbb{R})$ the function $f^{it} = \sum_{j\geq 1}\sum_k d^{it}_{j,k}\psi_{j,k}$, where

$$d^{it}_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\leq 2^{-j\gamma}}.$$

A threshold usually keeps the largest wavelet coefficients, this is the reason why we called this transformation inverse threshold.

The study of these operators appears to be fruitful from a multifractal point of view.

We first show in Section 6.3 that the threshold of order γ can create oscillating singularities for f^t that were not present in the initial function f, and that the support of multifractal spectrum of f^t (i.e. $\{h : E_h^{f^t} \neq \emptyset\}$) can be larger than the one of f.

Using the same ideas, we compare for a given function f the large deviation spectrum (which is based on the histograms of wavelet coefficients, see Section 6.2) with the multifractal sepctrum to find sufficient conditions for the presence of oscillating singularities.

We then study in the rest of this work the action of both thresholds (regular and inverse) on the specific and interesting example, inspired by [12], of a set of wavelet coefficients $\{d_{j,k}\}$

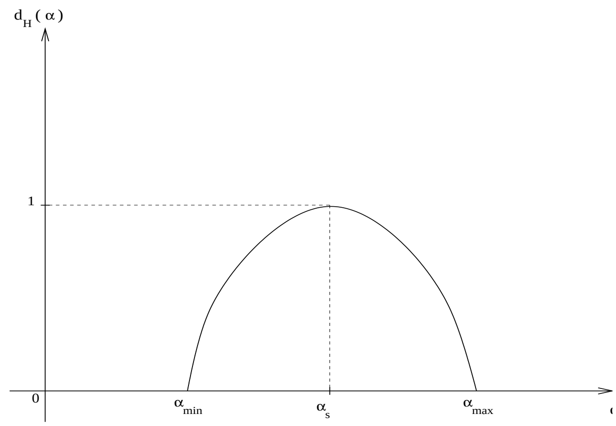

FIG. 6.1 – Multifractal spectrum of μ.

derived from a positive measure μ in the following way. Let $0 < q_0 < 1/2$ and $q_1 = 1 - q_0$ be two real numbers, and let us consider the binomial measure μ with ratios (q_0, q_1) (its construction is recalled in Section 6.4). This measure μ satisfies the multifractal formalism as defined in [18], and its multifractal spectrum is given by the following Legendre transform

$$d_\mu(\alpha) = \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)), \tag{6.1}$$

where the function $\tau(q)$ is defined by

$$\tau(q) = \lim_{j \to +\infty} \frac{-1}{j} \log \sum_{k=0}^{2^j} \left[\mu([k2^{-j}, (k+1)2^{-j}))\right]^q = -\log(q_0^q + q_1^q). \tag{6.2}$$

This spectrum is strictly concave, ranges in $[\alpha_{min}, \alpha_{max}]$, with $\alpha_{min} = -\log_2 q_1$ and $\alpha_{max} = -\log_2 q_0$. $\alpha_s = -\frac{1}{2}\log_2 q_0 q_1$ is the almost-sure exponent of μ (see Figure 1).

Starting from μ, we construct using the same method as in [12] a function f on $(0,1)$ with the formula

$$f = \sum_{j \geq 1} \sum_{k \in \mathbb{Z}} 2^{-j(s_0 - \frac{1}{p_0})} \mu\left([k2^{-j}, (k+1)2^{-j})\right)^{\frac{1}{p_0}} \psi_{j,k}(x). \tag{6.3}$$

It is shown in [12] that the function f has a multifractal spectrum (for functions) which is deduced from the one of μ through the formula

$$d_f(h) = d_\mu(\alpha),$$

where α and h are related by $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$. The function f is also shown to satisfy the multifractal formalism for functions as we define it in Section 6.2. This kind of result on wavelet cascades was actually expected since [4].
In [12], it is also shown that there are no oscillations in f, and more generally that any function built as f (i.e. from any positive Borel measure μ) contains only cusps.

Applying the thresholds defined above to this function f yields two "compressed" functions f^{it} and f^t. These functions f^{it} and f^t will be shown in Section 6.5 to have surprising multifractal spectra. In addition, we show that oscillating singularities arise after applying a threshold on the wavelet coefficients.

Not surprisingly the spectrum of f^{it} vanishes for $h < \gamma$, but Theorem 6.3 also shows that this multifractal spectrum may have a support larger than the one of f. This enlargement of the support of the spectrum is the consequence of the apparition of chirp-like (or oscillating) singularities in f^{it}. The spectrum of f^{it} appears to be the supremum of two functions, and in particular for some values of γ, f^{it} will not satisfy the multifractal formalism for functions any more.

Theorem 6.4 states that the same phenomenon as in Theorem 6.3 happens for f^t. In particular, the intuition that the multifractal spectrum of f^t vanishes for $h > \gamma$ is false. As for f^{it}, infinitely fast oscillations appear.

In both cases, the creation of oscillations after applying a threshold on wavelet coefficients is comparable to a Gibbs phenomenon.

6.2 Local Regularity of Functions

6.2.1 Regularity Exponents

In most cases, the local regularity of a function f around a point x_0 is first measured using the pointwise Hölder exponent h_f.

Definition 6.1 *Let I be an interval of $\mathbb{R}$, x_0 an interior point of I, and h a positive real number with $h \notin \mathbb{N}$. A function $f : I \to \mathbb{R}$ belongs to $C^h_{x_0}$ if and only if there exist a constant C and a polynomial P of degree smaller than $[h]$ such that*

$$\forall\ x \in \mathbb{R},\ |f(x) - P(x - x_0)| \leq C|x - x_0|^h.$$

The pointwise Hölder exponent *of f at x_0, denoted by $h_f(x_0)$, is defined by* $\sup\{h : f \in C^h_{x_0}\}$.

The functions we are interested in are defined through wavelet coefficients. Let ψ be a wavelet in the Schwartz class, as constructed in [55] or [71]. The set of functions $\{\psi_{j,k} = \psi(2^j. - k)\}$, where $(j, k) \in \mathbb{Z}^2$, forms an orthogonal basis of $L^2(\mathbb{R})$. Any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} d_{j,k}\psi_{j,k}(x),$$

where $d_{j,k}$ is the wavelet coefficient of f defined by $d_{j,k} = 2^j \int_{\mathbb{R}} f(t)\psi_{j,k}(t)dt$.

The pointwise Hölder exponent $h_f(x_0)$ can be characterized by the rate decay of the wavelet coefficients around x_0. Indeed, if $f \in C^\varepsilon$ (uniformly) for some $\varepsilon > 0$ in a neighborhood of x_0, and if ψ has more than $[h_f(x_0)] + 1$ vanishing moments, it is known (see [40]) that

$$h_f(x_0) = \liminf_{k2^{-j}\to x_0} \frac{\log|d_{j,k}|}{\log(2^{-j} + |x_0 - k2^{-j}|)}. \tag{6.4}$$

In the sequel, we assume that ψ has enough vanishing moments so that (6.4) holds at every x_0.

$h_f(x)$ is a measure of the behavior of a function f around a point x. But it does not distinguish, for example, at the point 0 the two functions $x \to |x|^\gamma$ and $x \to |x|^\gamma \sin(1/|x|^\beta)$. This second function is called a chirp at 0, and some infinitely fast oscillations occur around 0, in opposition to what happens at 0 for the first function, which is called a cusp.

Several methods ([4], [82]) have been introduced to detect and characterize oscillating singularities, i.e. infinitely fast oscillations around a point. We chose the *oscillating exponent* introduced in [4], and denoted β_f. It is obtained by considering what happens when an infinitesimal integration is performed :

Definition 6.2 *Let $f \in L^\infty_{loc}(\mathbb{R})$. We denote by $h_t(x_0)$ the pointwise Hölder exponent of a fractional primitive of order t of f at x_0. Then*

$$\beta_f(x_0) = \left(\frac{\partial}{\partial t} h_t(x_0)\right)_{t=0^+} - 1.$$

One easily checks that $\beta_f(0) = 0$ for the cusp, and $\beta_f(0) = \beta$ for the chirp. We say that f has a *chirp-like behavior* or an *oscillating singularity* at x if $\beta_f(x) > 0$.

6.2.2 Various Spectra of Singularities for Functions

We define three different spectra of singularity for a function f.

Definition 6.3 *Let I be an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ belonging to $L^\infty_{loc}(I)$. For every $h \geq 0$ define*

$$E_h^f = \{x \in \text{Int}(I) : \ h_f(x) = h\}.$$

$d_f : h \to \dim_H(E_h^f)$ is called the multifractal spectrum of f ($\dim_H$ denotes the Hausdorff dimension). If $E_h^f = \emptyset$ by convention one sets $d_f(h) = -\infty$.

d_f is the classical multifractal spectrum. It measures the size of the level sets of the pointwise Hölder exponent h_f.

The second spectrum we need is the Legendre spectrum of a function $f \in L^1_{loc}$, computed with wavelet coefficients. Let us define

$$\xi_f(p) = 1 + \liminf_{j\to\infty} -\frac{1}{j} \log_2 \left(\sum_{0\leq k\leq 2^j-1}^{*} |d_{j,k}|^p \right).$$

This function ξ_f is concave and non-decreasing. An alternative definition of $\xi_f(p)$ for $p > 0$ is given in terms of Besov spaces. Indeed, one can show that

$$\xi_f(p) = \sup \left\{ u : f \in B_p^{\frac{u}{p},\infty}((0,1)) \right\}.$$

This shows that for $p > 0$, $\xi_f(p)$ is independent of the wavelet that has been chosen to compute the wavelet coefficients. This is not a priori the case when $p < 0$.

Jaffard ([44] for instance) established the following general upper bound for $d_f(h)$ (when f has some minimal uniform regularity)

$$\forall\ h \geq 0,\ d_f(h) \leq \inf_{p \geq p_c} (ph - \xi_f(p) + 1), \tag{6.5}$$

where p_c is the unique real number such that $\xi_f(p_c) = 1$. Since [44], better upper-bounds have been found (essentially by replacing Besov spaces by the more general oscillation spaces, see [49]), but this is not our purpose here. The Legendre spectrum of a function f we focus on is the whole Legendre transform of $\xi_f - 1$ (i.e. taken with $p \in \mathbb{R}$), i.e.

$$(\xi_f - 1)^*(h) = \inf_{p \in \mathbb{R}} (ph - \xi_f(p) + 1).$$

Eventually, the *large deviation* spectrum $\tilde{d}_f$ of a function gives an indication of the repartition of the wavelet coefficients (it is a sort of histogram of wavelet coefficients, see [5] for a complete study of this spectrum). It is also much easier to estimate, in practical cases, than the multifractal spectrum d_f. It is defined as follows.

Definition 6.4 *Let $f \in L^1_{loc}([0,1])$ and $\varepsilon > 0$. For every $j \geq 1$ and $k \in [0, \ldots, , 2^j - 1]$, let $h_{j,k} = \frac{\log_2 |d_{j,k}|}{-j}$ (we set $h_{j,k} = +\infty$ if $d_{j,k} = 0$). We define*

$$N_j^\varepsilon(h) = \#\left\{k \in \{0, \ldots, 2^j\} : |h_{j,k} - h| \leq \varepsilon\right\}. \tag{6.6}$$

and

$$\tilde{d}^\varepsilon(h) = \limsup_{j \to +\infty} \frac{\log_2 N_j^\varepsilon(h)}{j}.$$

The large deviation spectrum $\tilde{d}_f(h)$ is defined as $\tilde{d}_f(h) = \lim_{\varepsilon \to 0} \tilde{d}^\varepsilon(h)$.

While one always has $\tilde{d}_f(h) \leq (\xi_f - 1)^*(h)$, there is no general relationship between $\tilde{d}_f$ and d_f. The examples we will later treat will illustrate this statement.

Definition 6.5 *A function f is said to obey the multifractal formalism at $h \geq 0$ if there exists a wavelet ψ such that $d_f(h) = (\xi_f - 1)^*(h)$.*

6.2.3 2-microlocal analysis of Functions

The notion of 2-microlocal spectrum developped in [62] is essential in the following study of local regularity. For every scale j and every exponent $\rho \in [0,1]$, $k_{j,\rho}$ denotes the integer $k_{j,\rho} = [2^{j(1-\rho)}]$.

Definition 6.6 *Let $f \in L^\infty_{loc}$, and consider its wavelet decomposition $f = \sum_{j,k} d_{j,k} \psi_{j,k}$. For any given x_0, define*

– $\theta^0_{x_0} : (0,1) \to \mathbb{R}^+ \cup\ \{+\infty\}$,

$$\theta^0_{x_0}(\varepsilon) = \sup\{\gamma : \exists \delta,\ K,\ \forall \beta \leq \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

– *for any given $\varepsilon > 0$, $\chi^\varepsilon_{x_0} : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup\ \{+\infty\}$,*

$$\chi^\varepsilon_{x_0}(\rho) = \sup\{\gamma : \exists \delta,\ K,\ \forall \beta \in [\rho - \varepsilon, \rho + \varepsilon],\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K 2^{-j\gamma}\}.$$

– $\theta^1_{x_0} : (0,1) \to \mathbb{R}^+ \cup \ \{+\infty\}$,

$$\theta^1_{x_0}(1-\varepsilon) = \sup\{\gamma : \exists\delta,\, K,\, \forall\beta \geq 1-\varepsilon,\, 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \leq K2^{-j\gamma}\}.$$

For $\varepsilon > 0$ small enough, $\theta^0_{x_0}(\varepsilon)$, $\chi^\varepsilon_{x_0}(\rho)$, and $\theta^1_{x_0}(1-\varepsilon)$ are the maximum decay rate of some selected wavelet coefficients that lie around x_0.

Definition 6.7 *For any $x_0 \in \mathbb{R}$, the 2-microlocal spectrum of f at x_0, $\chi_{x_0} : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ is defined by*

– $\chi_{x_0}(0) = \lim_{\varepsilon\to 0^+} \theta^0_{x_0}(\varepsilon) = \sup\{\theta^0_{x_0}(\varepsilon) : \varepsilon \in (0,1)\}$,
– $\rho \in (0,1)$ *:* $\chi_{x_0}(\rho) = \lim_{\varepsilon\to 0^+} \chi^\varepsilon_{x_0}(\rho) = \sup\{\chi^\varepsilon_{x_0}(\rho) : \varepsilon > 0\}$
– $\chi_{x_0}(1) = \lim_{\varepsilon\to 0^+} \theta^1_{x_0}(1-\varepsilon) = \sup\{\theta^1_{x_0}(1-\varepsilon) : \varepsilon \in (0,1)\}$

$\chi_{x_0}(1)$ characterizes the behaviour of the wavelet coefficients that lie in the neighbourhood of the cone of influence $|k2^{-j} - x_0| \leq 2^{-j}$, while $\chi_{x_0}(\rho)$ is related to the behaviour of the wavelet coefficients that are located in the time-frequency plane around the curves $|k2^{-j} - x_0| = 2^{-j\rho}$. Eventually, $\chi_{x_0}(0)$ characterizes the behaviour of the wavelet coefficients that lie below all curves $|k2^{-j} - x_0| = 2^{-j\varepsilon}$, $\varepsilon > 0$,

The next proposition, proved in [62], relates the 2-microlocal spectrum χ_x to the pointwise Hölder exponent $h_f(x)$ and to the oscillating exponent $\beta_f(x)$.

Proposition 6.1 *Let $f \in C^\varepsilon(\mathbb{R})$ for some $\varepsilon > 0$, and $x_0 \in \mathbb{R}$. One has*

$$h_f(x_0) = \inf\left\{\frac{\chi_{x_0}(\rho)}{\rho} : \rho \in (0,1]\right\} \tag{6.7}$$

and if $h_f(x_0) < +\infty$, $\beta_f(x_0) = \inf\left\{\beta \in [0,+\infty] : \chi_{x_0}\left(\frac{1}{1+\beta}\right) = \frac{h_f(x_0)}{1+\beta}\right\}$.

We will also need the somehow complementary notion of *upper* 2-microlocal spectrum.

Definition 6.8 *Let $f \in L^\infty_{loc}$, and its discrete wavelet decomposition $f = \sum_{j,k} d_{j,k}\psi_{j,k}$. For any given x_0, define*

– $\tilde{\theta}^0_{x_0} : (0,1) \to \mathbb{R}^+ \cup \ \{+\infty\}$,

$$\tilde{\theta}^0_{x_0}(\varepsilon) = \inf\{\gamma : \exists\delta, K,\ \forall\beta \leq \varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K2^{-j\gamma}\}.$$

– *for any given $\varepsilon > 0$, $\chi^{\tilde{\varepsilon}}_{x_0} : (\varepsilon, 1-\varepsilon) \to \mathbb{R}^+ \cup \ \{+\infty\}$,*

$$\chi^{\tilde{\varepsilon}}_{x_0}(\rho) = \inf\{\gamma : \exists\delta, K, \forall\beta \in [\rho-\varepsilon, \rho+\varepsilon],\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K2^{-j\gamma}\}.$$

– $\tilde{\theta}^1_{x_0} : (0,1) \to \mathbb{R}^+ \cup \ \{+\infty\}$

$$\tilde{\theta}^1_{x_0}(1-\varepsilon) = \inf\{\gamma : \exists\delta, K,\ \forall\beta \geq 1-\varepsilon,\ 2^{-j} \leq \delta \Rightarrow |d_{j,[2^j x_0 \pm k_{j,\beta}]}| \geq K2^{-j\gamma}\}.$$

Definition 6.9 *For any $x_0 \in \mathbb{R}$, the upper 2-microlocal spectrum of f at x_0, $\tilde{\chi}_{x_0} : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$ is defined by*

– $\tilde{\chi}_{x_0}(0) = \lim_{\varepsilon\to 0^+} \tilde{\theta}^0_{x_0}(\varepsilon) = \inf\{\theta^0_{x_0}(\varepsilon) : \varepsilon \in (0,1)\}$,
– $\rho \in (0,1)$ *:* $\tilde{\chi}_{x_0}(\rho) = \lim_{\varepsilon\to 0^+} \chi^{\tilde{\varepsilon}}_{x_0}(\rho) = \inf\{\chi^{\tilde{\varepsilon}}_{x_0}(\rho) : \varepsilon > 0\}$
– $\tilde{\chi}_{x_0}(1) = \lim_{\varepsilon\to 0^+} \tilde{\theta}^1_{x_0}(1-\varepsilon) = \inf\{\tilde{\theta}^1_{x_0}(1-\varepsilon) : \varepsilon \in (0,1)\}$

$\tilde{\chi}$ gives an upper bound of the decay rate of the wavelet coefficients around x_0. One always has for every x_0, for every $\rho \in [0,1]$, $\chi_{x_0}(\rho) \leq \tilde{\chi}_{x_0}(\rho)$.

6.3 General Results

6.3.1 An Upper Bound of the multifractal Spectrum

We recall the upper bound of the multifractal spectrum proven in [12]. Define

$$\mathrm{Supp}(d) = \left\{ x \in (0,1) : \begin{array}{c} \forall\ J \geq 0,\ \forall \varepsilon > 0,\ \exists\ j \geq J, \\ \exists k \text{ with } |k2^{-j} - x| \leq 2^{-j(1-\varepsilon)}, \\ \text{such that } d_{j,k} \neq 0 \end{array} \right\}. \tag{6.8}$$

Proposition 6.2 *Let $f \in C^\varepsilon(\mathbb{R})$ be such that $f = \sum_{j=1}^{+\infty} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}$, and let ξ_f be its scaling function (associated with the wavelet ψ).*

1. *If $0 \leq h \leq \xi_f'(p_c^-)$ then $d_f(h) \leq (\xi_f - 1)^*(h) = \inf_{p \geq p_c}(ph - \xi_f(p) + 1)$.*
2. *If $h \in (\xi_f'(p_c^-), \xi_f'(0^+))$ and $E_h^f \subset \{x \in (0,1) : \min(\underline{\alpha}_d^-(x), \underline{\alpha}_d(x), \underline{\alpha}_d^+(x)) = h\}$ then $d_f(h) \leq (\xi_f - 1)^*(h) = \inf_{0 \leq p \leq p_c}(ph - \xi_f(p) + 1)$*
3. *If $h \in [\xi_f'(0^+), \xi_f'(0^-)]$, $dim(E_h^f \cap Supp(d)) \leq (\xi_f - 1)^*(h) = 1 - \xi_f(0)$.*
4. *If $h > \xi_f'(0^-)$, $dim(E_h^f \cap Supp(d)) \leq (\xi_f - 1)^*(h) = \inf_{p \leq 0}(ph - \xi_f(p) + 1)$.*
5. *If $(\xi_f - 1)^*(h) < 0$ then $E_h^f \cap Supp(d) = \emptyset$.*

Remark that Proposition 6.2 depends on the wavelet ψ when $p < 0$ (i.e. on the decreasing part of the spectrum), and that it can be viewed as a consequence of the study of the so-called "oscillation spaces" introduced by S. Jaffard in [49].

Proposition 6.2 is actually a slightly improved version of the proposition stated in [12]. Indeed, the set $\mathrm{Supp}(d)$ is here larger, but the proof follows the same lines as in [12] and is immediate.

Proposition 6.2 gives a sharp upper bound of the multifractal spectrum when $\mathrm{Supp}(d) = [0,1]$. Nevertheless, $\mathrm{Supp}(d)$ is strictly included in $[0,1]$ for f^{it} and f^t in Section 6.5, and a finer study will be needed on $[0,1]\backslash\mathrm{Supp}(d)$.

6.3.2 Detection of Oscillations

When the multifractal formalism for functions does not hold for a function f, the presence of chirp-like oscillations is often detected. For example, Jaffard in [48] showed that almost surely (in the sense of Baire's categories), in some functional spaces V, the multifractal formalism is broken. It has also been proven in [70] that almost surely, the functions of V (and especially the so-called "saturating functions") have oscillating singularities. The following theorem gives a clue on the relation between the failure of the multifractal formalism and the presence of oscillations.

Theorem 6.1 *Let $f \in C^\varepsilon([0,1])$. Assume that there exists an exponent $h > 0$ such that $\tilde{d}_f(h) < d_f(h)$. Then there exists a set E of dimension $d_f(h)$ of oscillating singularities for f such that for every $x \in E$, $h_f(x) = h$.*

Proof : Let $\eta > 0$, such that $\tilde{d}_f(h) + \eta < d_f(h)$. By Definition 6.4 of $\tilde{d}_f(h)$, there exist $\varepsilon > 0$, and a scale J, such that $j \geq J$ implies

$$\frac{\log_2 N_j^\varepsilon(h)}{j} \leq \tilde{d}_f(h) + \eta. \tag{6.9}$$

Let us denote by $S_{j,\varepsilon}$ the set of wavelet coefficents $d_{j,k}$ at scale $j \geq J$, such that $2^{-j(h+\varepsilon)} \leq |d_{j,k}| \leq 2^{-j(h-\varepsilon)}$. In view of Definition 6.6, (6.9) is equivalent to say that, at each scale $j \geq J$, the cardinal of $S_{j,\varepsilon}$ is less than $2^{j(\tilde{d}_f(h)+\eta)}$.

Let us estimate the Hausdorff dimension of the sets S_ε^ρ defined for $0 < \rho < 1$

$$S_\varepsilon^\rho = \bigcap_{J\geq 0} \bigcup_{j \geq J} \bigcup_{k: d_{j,k} \in S_{j,\varepsilon}} \left[k2^{-j} - 2^{-j\rho}, k2^{-j} + 2^{-j\rho}\right]. \tag{6.10}$$

Let $d > \frac{\tilde{d}_f(h)+\eta}{\rho}$. The cardinal of $S_{j,\varepsilon}$ is upper bounded by $2^{j(\tilde{d}_f(h)+\eta)}$, thus

$$\sum_{k: d_{j,k} \in S_{j,\varepsilon}} \left|\left[k2^{-j} - 2^{-j\rho}, k2^{-j} + 2^{-j\rho}\right]\right|^d \leq 2^{j(\tilde{d}_f(h)+\eta)} 2^{d(1-j\rho)} \leq C 2^{j(\tilde{d}_f(h)+\eta-d\rho)},$$

and

$$\sum_{j\geq J} \sum_{k: d_{j,k} \in S_{j,\varepsilon}} \left|\left[k2^{-j} - 2^{-j\rho}, k2^{-j} + 2^{-j\rho}\right]\right|^d < +\infty. \tag{6.11}$$

For every $J \geq 0$, the set S_ε^ρ is obviously covered by the set

$$\bigcup_{j \geq J} \bigcup_{k: d_{j,k} \in S_{j,\varepsilon}} \left[k2^{-j} - 2^{-j\rho}, k2^{-j} + 2^{-j\rho}\right]$$

(remember (6.10)). Hence, because of (6.11), the d-dimensional Hausdorff measure of S_ε^ρ equals zero, for any $d > \frac{\tilde{d}_f(h)+\eta}{\rho}$. As a consequence the Hausdorff dimension of S_ε^ρ is smaller than $\frac{\tilde{d}_f(h)+\eta}{\rho}$.

Let us study the complementary set of S_ε^ρ, that we denote by C_ε^ρ. If $x \in C_\varepsilon^\rho$, there exists a scale J_x such that for every $j \geq J_x$, for every k such that $|k2^{-j} - x| \leq 2^{-j\rho}$, one has $|d_{j,k}| \geq 2^{-j(h-\varepsilon)}$ or $|d_{j,k}| \leq 2^{-j(h+\varepsilon)}$. This means that for every $\rho' \in (\rho, 1]$, $\chi_x(\rho') \in (0, h - \varepsilon) \cup (h + \varepsilon, +\infty]$. In particular, if $h_f(x) = h$, applying Proposition 6.1 gives that $\beta_f(x) > 0$, i.e. there are fast oscillations around x.

Now, let us take $\rho < 1$ such that $\frac{\tilde{d}_f(h)+\eta}{\rho} < d_f(h)$. One has

$$E_h^f = \left(E_h^f \cap S_\varepsilon^\rho\right) \cup \left(E_h^f \cap C_\varepsilon^\rho\right)$$

Let $E = E_h^f \cap C_\varepsilon^\rho$. Since $\dim_H E_h^f \cap S_\varepsilon^\rho \leq \frac{\tilde{d}_f(h)+\eta}{\rho} < d_f(h)$, the set E has a Hausdorff dimension equal to $d_f(h)$. For every $x \in E$, $x \in E_h^f$ thus $h_f(x) = h$, and at the same time, since $E \subset C_\varepsilon^\rho$, $\beta_f(x) \geq 1/\rho - 1 > 0$, i.e. the function f has a chirp-like behavior around x. ∎

It can be guessed that the important sets to consider are in fact the sets $S_{j,\varepsilon}$ and S_ε^ρ. A result derived from the last proof, and slightly more general than Theorem 6.1, (but which is not

directly linked to any of the spectra we defined), is :
If there exist ρ and ε such that $\dim_H S^\rho_\varepsilon < d_f(h)$, then there exists a set E of Hausdorff dimension $d_f(h)$ such that $\forall x \in E$, $h_f(x) = h$ and $\beta_f(x) > 0$.

The condition $\tilde{d}_f(h) < d_f(h)$ allows to upper bound the dimensions of the sets S^ρ_ε. Unfortunately, it is impossible to treat in general the case $\tilde{d}_f(h) > d_f(h)$, since this inequality do not yield neither a sharp upper bound nor a lower bound of $\dim_H S^\rho_\varepsilon$.

If the Legendre transform of $\xi - 1$ at exponent h is strictly less than $d_f(h)$, i.e. if the multifractal formalism (as stated in Definition 6.5) fails at h, then $\tilde{d}_f(h) < d_f(h)$, and Theorem 6.1 can be applied. This brings the following remark : the oscillations do not make the multifractal formalism fail. Theorem 6.1 rather emphasizes that if the multifractal formalism is not satisfied at exponent h, then there are not enough wavelet coefficients to have for every $x \in E^f_h$, $\chi_x(1) = h$ (or equivalently $\beta_f(x) = 0$). Hence there are oscillating singularities in E^f_h. Remark that the property $(\xi - 1)^*(h) < d_f(h)$ can only be satisfied when $h \geq (\xi - 1)'(p_c)$, because of the upper bound (6.5) proved in [44] or [48]. As noticed in [5], this is the reason why (remembering that $\tilde{d}_f(h) \leq (\xi-1)^*(h)$) working on the large deviation spectrum is more general than working on the Legendre spectrum. In particular, the conditions of Theorem 6.1 can occur for every $h \geq 0$.

The counterpart of Theorem 6.1 is that the knowledge of the multifractal spectrum d_f is required to point out oscillations. It can happen that, when estimating this spectrum, one already has had to take into account the presence of oscillations to compute the pointwise Hölder exponents, and thus they are already detected.

6.3.3 Creating Oscillations

We now define and study the wavelet threshold operators

Definition 6.10 *Let $f \in L^2(\mathbb{R})$, $\{d_{j,k}\}_{j,k}$ its wavelet coefficients and $\gamma > 0$. The function f^t obtained after a threshold of order γ is defined by $f^t = \sum_j \sum_k d^t_{j,k}\psi_{j,k}$, where*

$$d^t_{j,k} = d_{j,k}\mathbb{1}_{|d_{j,k}| \geq 2^{-j\gamma}}.$$

We know by Theorem 6.1 now that $\tilde{d}_{f^t}(h) < d_{f^t}(h)$ for some exponent $h > 0$ ensures the existence of oscillations. Such a threshold imposes $\tilde{d}_{f^t}(h) = 0$ for $h > \gamma$. Since a threshold increases (local and global) regularity, there will still exist some points x with pointwise Hölder exponent greater than γ. These points are candidates to be oscillating singularities for f^t.

Theorem 6.2 *Let $f \in C^\varepsilon([0,1])$, and assume that $E^h_f \neq \emptyset$. Let f^t be the function obtained after a threshold of f of order $\gamma < h$. Then for every $x \in E^h_f$, either $h_{f^t}(x) = +\infty$, or x is an oscillating singularity for f^t.*

Proof : Since we cancel wavelet coefficients, the threshold we apply obviously increases the local regularity at each point. Thus $h_{f^t}(x) \geq h_f(x)$ for every x.

Let $x \in E^f_h$. We denote by χ_x the 2-microlocal spectrum of f at x, and by χ^t_x the 2-microlocal spectrum of f^t. One has $h_f(x) = \inf_{\rho\in(0,1]} \frac{\chi_x(\rho)}{\rho} = h$, and

$$h_{f^t}(x) = \inf_{\rho\in(0,1]} \frac{\chi^t_x(\rho)}{\rho} \geq h. \qquad (6.12)$$

Since $\forall(j,k)$, one has either $|d_{j,k}| \geq 2^{-j\gamma}$ or $d_{j,k} = 0$, one gets

$$\forall \rho \in [0,1],\ \chi_x^t(\rho) \in [\varepsilon, \gamma] \cup \{+\infty\}. \tag{6.13}$$

Assume $h_{f^t}(x) < +\infty$. Combining (6.12) and (6.13) implies that there exists $\rho_x \in (0,1)$ such that $\frac{\chi_x^t(\rho_x)}{\rho_x} = h_{f^t}(x)$.
Let $\rho_h < 1$ be the unique real number such that $\frac{\gamma}{\rho_h} = h$. By construction, for every $\rho \in (\rho_h, 1]$, $\frac{\chi_x^t(\rho)}{\rho} \in [\frac{\varepsilon}{\rho}, \frac{\gamma}{\rho}] \cup \{+\infty\}$. In particular, for every $\rho \in (\rho_h, 1]$, $\frac{\chi_x^t(\rho)}{\rho} \neq h_{f^t}(x)$.

In view of Proposition 6.1, this exactly means that $\beta_{f^t}(x) \in [1/\rho_h - 1, 1/\rho_x - 1]$, i.e. that $\beta_{f^t}(x) > 0$ and f^t has an oscillating singularity at x. ∎

Theorem 6.2 can be thought as a sort of Gibbs phenomenom that occurs when forgetting at each scale j in the reconstruction formula the smallest wavelet coefficients.

The existence of oscillating singularities depends on the choice of the parameter γ, and on the function f. Nevertheless, for a multifractal function f, and for γ close enough to h, one can hope that there will be some points x with $h_{f^t}(x) < +\infty$. This will be the case of the function f we focus on in the following sections : The function f^t obtained after thresholding has dense sets of oscillating singularities. In addition the multifractal formalism will fail for this function f^t.

In signal and image processing the *hard* threshold ([23], [24]) is often used to compress datas. It consists in putting to zero all the wavelet coefficients $d_{j,k}$ that are too small, i.e. such that $|d_{j,k}| \leq \varepsilon$ for a given constant $\varepsilon > 0$. The *hard* threshold can certainly be compared to the threshold we define in Definition 6.10 in the following way. Let f be a discrete signal of size $N = 2^J$, and consider its wavelet decomposition $f = \sum_{j=0}^{J} d_{j,k}\psi_{j,k}$. Let ε small enough. Applying a hard threshold amounts to putting to 0, at each scale j, the wavelet coefficients $d_{j,k}$ such that

$$|d_{j,k}| \leq 2^{-j\frac{-\log_2 \varepsilon}{j}}.$$

If one considers only the smallest scales (for example the scales j such that $j \geq [N/2]$), the hard threshold puts to 0 more coefficients than a threshold of order $\gamma^t = 2^{\frac{\log_2 \varepsilon}{J}}$. This means that the points with (discrete) local regularity greater than γ^t will become either C^∞ points, or oscillating singularities. Of course, since real datas are discrete, and every point is in fact C^∞, but this model may give an explanation of some defaults, that look like oscillations, observed around isolated singularities after thresholding in image and signal processing.

6.4 Study of a specific example

Let us first begin with a recall on local regularity of measures. In the following, we will use the notations $h_{\min} = s_0 - \frac{1}{p_0} + \frac{\alpha_{\min}}{p_0}$, $h_s = s_0 - \frac{1}{p_0} + \frac{\alpha_s}{p_0}$, $h_{\max} = s_0 - \frac{1}{p_0} + \frac{\alpha_{\max}}{p_0}$.

6.4.1 Local regularity of Measures

If $x \in (0,1)$, $\forall j \geq 1$, $k_{j,x}$ is the unique integer such that $x \in [k_{j,x}2^{-j}, (k_{j,x}+1)2^{-j})$, and $k_{j,x}^+ = k_{j,x} + 1$, $k_{j,x}^- = k_{j,x} - 1$. By convention, $\log(0) = -\infty$.

Let μ be a positive Borel measure on $[0,1]$. For $x \in (0,1)$, the lower and upper Hölder exponent of μ at x are respectively defined by

$$\underline{\alpha}_\mu(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}})}{\log |I_{j,k_{j,x}})|} \quad \text{and} \quad \overline{\alpha}_\mu(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}})}{\log |I_{j,k_{j,x}}|}$$

When $\underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x)$, their common value is denoted $\alpha_\mu(x)$ and called the Hölder exponent of μ at x.

Similarly, the left and right lower Hölder exponents of μ at x are defined by

$$\underline{\alpha}_\mu^-(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^-})}{\log |I_{j,k_{j,x}^-}|} \quad \text{and} \quad \underline{\alpha}_\mu^+(x) = \liminf_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^+})}{\log |I_{j,k_{j,x}^+}|},$$

and the left and right upper Hölder exponents of μ at x are

$$\overline{\alpha}_\mu^-(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^-})}{\log |I_{j,k_{j,x}^-})|} \quad \text{and} \quad \overline{\alpha}_\mu^+(x) = \limsup_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^+})}{\log |I_{j,k_{j,x}^+}|}.$$

When they exist, The left and right Hölder exponents of μ at x are defined by

$$\alpha_\mu^-(x) = \lim_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^-})}{\log |I_{j,k_{j,x}^-})|} \quad \text{and} \quad \alpha_\mu^+(x) = \lim_{j\to+\infty} \frac{\log \mu(I_{j,k_{j,x}^+})}{\log |I_{j,k_{j,x}^+}|}.$$

We consider the following level sets for μ

Definition 6.11 *For every $\alpha \geq 0$, define*

$$\begin{aligned}
E_\alpha^\mu &= \{x \in (0,1) \cap Supp(\mu) : \alpha_\mu(x) = \alpha\}, \\
F_\alpha^\mu &= \left\{x \in (0,1) \cap Supp(\mu) : \min(\underline{\alpha}_\mu(x), \underline{\alpha}_\mu^+(x), \underline{\alpha}_\mu(x)^-) = \alpha\right\}, \\
G_\alpha^\mu &= \left\{x \in (0,1) \cap Supp(\mu) : \alpha_\mu(x) = \alpha_\mu^+(x) = \alpha_\mu^-(x) = \alpha\right\}, \\
H_\alpha^\mu &= \left\{x \in (0,1) \cap Supp(\mu) : \max(\overline{\alpha}_\mu(x), \overline{\alpha}_\mu^+(x), \overline{\alpha}_\mu(x)^-) = \alpha\right\}.
\end{aligned}$$

One obviously has $G_\alpha^\mu \subset E_\alpha^\mu \cap F_\alpha^\mu \cap H_\alpha^\mu$.

Usually (see [18]), the multifractal spectrum of μ is the mapping $\alpha \geq 0 \to d_\mu(\alpha) = \dim_H E_\alpha^\mu$, i.e. it measures the size of the level sets of the Hölder exponent.

6.4.2 Definition of f

Let f be the wavelet series given by (6.3)

$$f(x) = \sum_{j\geq 1} \sum_{k=0}^{2^j-1} 2^{-j(s_0 - \frac{1}{p_0})} (\mu(I_{j,k}))^{\frac{1}{p_0}} \psi_{j,k}(x),$$

where μ is the binomial measure constructed with ratios (q_0, q_1) with $q_0 < q_1$.

The binomial measure μ associated to the couple (q_0, q_1) is built as follows. Starting from the uniform measure μ^0 on $I_0 = [0,1]$, one splits I_0 into the two dyadic intervals $I_{1,0} = [0, 1/2)$

and $I_{1,1} = [1/2, 1]$, and defines a measure μ^1 by $\mu^1(I_{1,0}) = q_0$ and $\mu^1(I_{1,1}) = q_1$ (uniformly in each interval). Iterating this scheme, one obtains at each scale j a measure μ^j which associates to each interval $I_{j,k} = [k2^{-j}, (k+1)2^{-j})$ the uniform mass $q_0^{\phi(j,k)} q_1^{j-\phi(j,k)}$, where $\phi(j,k)$ is the number of 0 in the dyadic decomposition of $k2^{-j}$. The sequence of measures $\{\mu^j\}$ converges weakly to a probability measure μ, called the binomial measure.

Let us mention that the measure μ_r obtained when the mass is randomly splitted (i.e. one gives with probability 1/2 the mass q_0 to the left sub-interval and the mass q_1 to the right sub-interval, and with probability 1/2 the mass q_1 to the left sub-interval and the mass q_0 to the right sub-interval) is similar to μ, in the sense that it almost surely has the same spectrum as μ (see [18] for example) and all the following propositions and theorems apply to μ_r as well as to μ.

The measure μ is known to satisfy the multifractal formalism as defined in [18], and its multifractal spectrum is given by the Legendre transform (6.1)

$$d_\mu(\alpha) = \inf_{q \in \mathbb{R}} (q\alpha - \tau(q)),$$

where the function $\tau(q)$ is given by (6.2).

Starting from μ (it in fact can be done with any positive Borel measure), one can construct using the same method as in [12] a function f on $(0,1)$ with the formula (6.3)

$$f(x) = \sum_{j \geq 1} \sum_{k \in \mathbb{Z}} 2^{-j(s_0 - \frac{1}{p_0})} \mu \left([k2^{-j}, (k+1)2^{-j}) \right)^{\frac{1}{p_0}} \psi_{j,k}(x).$$

It is shown in [12] that this function f has a multifractal spectrum (for functions) which is deduced from the one of μ through the formula $d_f(h) = d_\mu(\alpha)$, where α and h are related by $h = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$. This is strongly related to the fact that, although they are different, the sets introduced above in (6.11) have the same Hausdorff dimension in the case of a (random and deterministic) binomial measure. Indeed,

Lemma 6.1 *Let μ be the random binomial measure with ratios (q_0, q_1).*

$$\dim_H(E_\alpha^\mu) = \dim_H(F_\alpha^\mu) = \dim_H(G_\alpha^\mu) = \dim_H(H_\alpha^\mu) = \tau^*(\alpha).$$

This lemma is a direct consequence of [18], and proved in [12].

6.4.3 Regularity at $x \in G_\alpha^\mu$

Proposition 6.3 *If $x \in G_\alpha^\mu$, then $\forall \rho \in [0,1]$,*

$$\chi_x(\rho) = s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0}, \tag{6.14}$$

and

$$\tilde{\chi}_x(\rho) = s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_0}{p_0} \tag{6.15}$$

Proof : For all $\rho \in (0,1)$, define the sets of wavelet coefficients $B_{\rho,j}$ and B_ρ^J by $B_{\rho,j} = \{d_{j,k} : |k2^{-j} - x| \leq 2^{-j\rho}\}$, and $B_\rho^J = \cup_{j \geq J} B_{j,\rho}$ (they depend on x).

Remark that if there exist a constant C and an integer J such that, for every coefficient $d_{j,k} \in B_\rho^J$,

$$\frac{1}{C} 2^{-j\gamma_1} \leq |d_{j,k}| \leq C 2^{-j\gamma_2}$$

(where γ_1, $\gamma_2 > 0$), then $\forall \rho' > \rho$, $\gamma_1 \leq \chi_x(\rho') \leq \tilde{\chi}_x(\rho') \leq \gamma_2$. Indeed, remembering Definitions 6.6 to 6.9, the computation of $\chi_x(\rho')$ and $\tilde{\chi}_x(\rho')$ takes into account only wavelet coefficients that lie in B_ρ^J.

Fix $\varepsilon > 0$. $x \in G_\alpha^\mu$ implies that there exists an integer $J > 0$, such that $j \geq J$ implies

$$\alpha - \varepsilon \leq \frac{\log \mu(I_{j,k^*_{j,x}})}{2^{-j}} \leq \alpha + \varepsilon, \tag{6.16}$$

for every $k^*_{j,x} \in \{k^-_{j,x}, k_{j,x}, k^+_{j,x}\}$.

Fix $\rho \in (0,1)$, and $\rho' \in (\rho, 1]$. Let $j \geq (J+1)/\rho$, $k \in \{0, \ldots, 2^j - 1\}$, such that $d_{j,k} \in B^j_{\rho'}$. One has $I_{j,k} \subset [x - 2^{-j\rho}, x + 2^{-j\rho}]$, and simultaneously

$$[x - 2^{-j\rho}, x + 2^{-j\rho}] \subset I_{j_\rho, k_{j_\rho,x}-1} \cup I_{j_\rho, k_{j_\rho,x}} \cup I_{j_\rho, k_{j_\rho,x}+1},$$

where $j_\rho = [j\rho] \geq J$. This implies

$$\mu(I_{j,k}) \leq \max \Big(\mu(I_{j_\rho, k_{j_\rho,x}-1}), \mu(I_{j_\rho, k_{j_\rho,x}}), \mu(I_{j_\rho, k_{j_\rho,x}+1}) \Big)$$

But if $I_{j'',k''}$ is a subinterval of $I_{j',k'}$ with $|I_{j'',k''}| = 1/2 |I_{j',k'}|$, one has either $\mu(I_{j'',k''}) = q_0 \mu(I_{j',k'})$ or $\mu(I_{j'',k''}) = q_1 \mu(I_{j',k'})$. Thus since $I_{j,k}$ is a subinterval of one of the intervals $\{I_{j_\rho, k_{j_\rho,x}-1}, I_{j_\rho, k_{j_\rho,x}}, I_{j_\rho, k_{j_\rho,x}+1}\}$, one obtains

$$\mu(I_{j,k}) \leq \min \Big(\mu(I_{j_\rho, k_{j_\rho,x}-1}), \mu(I_{j_\rho, k_{j_\rho,x}}), \mu(I_{j_\rho, k_{j_\rho,x}+1}) \Big) q_1^{j - j_\rho + 1}$$

and

$$\mu(I_{j,k}) \geq \max \Big(\mu(I_{j_\rho, k_{j_\rho,x}-1}), \mu(I_{j_\rho, k_{j_\rho,x}}), \mu(I_{j_\rho, k_{j_\rho,x}+1}) \Big) q_0^{j - j_\rho + 1}.$$

Combining this with (6.16) yields

$$q_0^{j - j_\rho + 1} 2^{-j_\rho(\alpha + \varepsilon)} \leq \mu(I_{j,k}) \leq q_1^{j - j_\rho + 1} 2^{-j_\rho(\alpha - \varepsilon)},$$

or equivalently

$$\frac{1}{C} 2^{-j(\rho(\alpha+\varepsilon) - (1-\rho)\log_2 q_0)} \leq \mu(I_{j,k}) \leq C 2^{-j(\rho(\alpha-\varepsilon) - (1-\rho)\log_2 q_1)},$$

since j_ρ is equivalent to $j\rho$. Coming back to the wavelet coefficients, one can write for any $d_{j,k} \in B_{j,\rho'}$,

$$\frac{1}{C} 2^{-j\left(s_0 - \frac{1}{p_0} + \frac{\rho(\alpha+\varepsilon) - (1-\rho)\log_2 q_0}{p_0}\right)} \leq d_{j,k} \leq C 2^{-j\left(s_0 - \frac{1}{p_0} + \frac{\rho(\alpha-\varepsilon) - (1-\rho)\log_2 q_1}{p_0}\right)}.$$

This holds for any $j \geq J/\rho$, thus also for any wavelet coefficients $d_{j,k} \in B_{\rho'}^{J/\rho}$.
Applying the first remark leads to

$$s_0 - \frac{1}{p_0} + \frac{\rho(\alpha+\varepsilon) - (1-\rho)\log_2 q_1}{p_0} \leq \chi_x(\rho')$$

and

$$\tilde{\chi}_x(\rho') \leq s_0 - \frac{1}{p_0} + \frac{\rho(\alpha-\varepsilon) - (1-\rho)\log_2 q_0}{p_0}.$$

This is true for any $\rho' \in (\rho, 1]$, but stays also true for any $\rho \in (0,1]$. Hence

$$s_0 - \frac{1}{p_0} + \frac{\rho(\alpha+\varepsilon) - (1-\rho)\log_2 q_1}{p_0} \leq \chi_x(\rho)$$

and

$$\tilde{\chi}_x(\rho) \leq s_0 - \frac{1}{p_0} + \frac{\rho(\alpha-\varepsilon) - (1-\rho)\log_2 q_0}{p_0}.$$

Finally, letting ε go to zero, one obtains

$$s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0} \leq \chi_x(\rho) \tag{6.17}$$

and

$$\tilde{\chi}_x(\rho) \leq s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_0}{p_0}. \tag{6.18}$$

The optimality of (6.17) and (6.18) is a direct consequence of the following argument.
If ρ and j are fixed, $I_{j_\rho,k_{j_\rho,x}}$ is strictly included in $[x - 2^{-j\rho}, x + 2^{-j\rho}]$. The same holds for all the dyadic subintervals of $I_{j_\rho,k_{j_\rho,x}}$. In particular, there exist two dyadic intervals I_{j,k_1} and I_{j,k_2}, subintervals of $I_{j_\rho,k_{j_\rho,x}}$, such that

$$\frac{\mu(I_{j,k_1})}{\mu(I_{j_\rho,k_{j_\rho,x}})} = q_0^{j-j_\rho+1} \text{ and } \frac{\mu(I_{j,k_2})}{\mu(I_{j_\rho,k_{j_\rho,x}})} = q_1^{j-j_\rho+1}.$$

This translates to the wavelet coefficients d_{j,k_2} into

$$|d_{j,k_2}| \geq C 2^{-j(s_0 - \frac{1}{p_0} + \frac{\rho(\alpha+\varepsilon)-(1-\rho)\log_2 q_1}{p_0})}. \tag{6.19}$$

Such coefficients exist in $B_{j,\rho}$ at every scale $j \geq J\rho$. Thus, there exists $\rho_\varepsilon \in [\rho, 1]$ such that

$$\chi_x(\rho_\varepsilon) \leq s_0 - \frac{1}{p_0} + \frac{\rho(\alpha+\varepsilon) - (1-\rho)\log_2 q_1}{p_0}.$$

But combining this with (6.17), one obtains that ρ_ε is close to ρ. In particular, letting ε go to zero, one gets

$$\chi_x(\rho) \leq s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0},$$

and finally (6.14) is proven.

The optimality of (6.18) is proven by applying the same study for the coefficients d_{j,k_1}, and (6.15) is obtained.

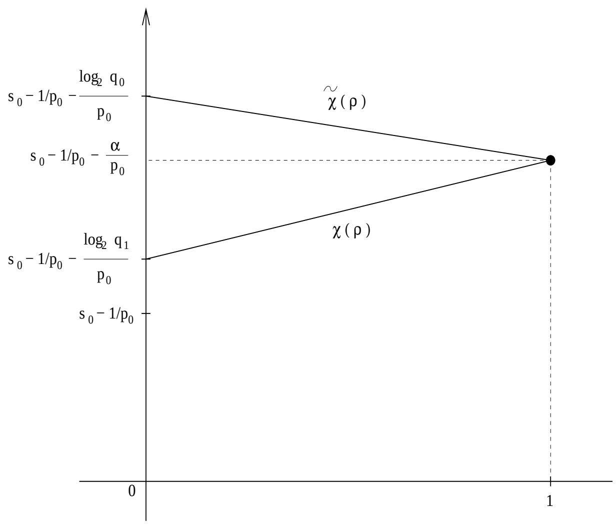

FIG. 6.2 – Plot of the 2-microlocal spectrum and upper 2-microlocal spectrum when $x \in G_\alpha^\mu$.

Remark in particular that $\chi_x(1) = \tilde{\chi}_x(1) = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} = h$.

Eventually, it is obvious that for all (j,k), $d_{j,k} \leq 2^{-j(s_0 - \frac{1}{p_0} + \frac{-\log_2 q_1}{p_0})}$, thus $\chi_x(0) = s_0 - \frac{1}{p_0} + \frac{-\log_2 q_1}{p_0}$. The lower bound $d_{j,k} \geq 2^{-j(s_0 - \frac{1}{p_0} + \frac{-\log_2 q_0}{p_0})}$ yields by the same argument $\tilde{\chi}_x(0) = s_0 - \frac{1}{p_0} + \frac{-\log_2 q_0}{p_0}$. ∎

Remark : (6.14) remains true if $x \in F_\alpha^\mu$, i.e. if one controls the value of the lower Hölder exponents of μ at x. Indeed, for such a point x, a slight modification of the above proof shows that there will still exist a infinite number of j's such that (6.16), (6.19) and then (6.14) hold. Similarly, the value of the upper 2-microlocal spectrum only depends on the upper Hölder exponents of μ at x. This is summarized in the next proposition

Proposition 6.4 *For all $x \in (0,1)$, there exists two real numbers $0 < \alpha \leq \beta$ such that $x \in F_\alpha^\mu$ and $x \in H_\beta^\mu$. Then, for every $\rho \in [0,1]$,*

$$\chi_x(\rho) = s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0}, \tag{6.20}$$

and

$$\tilde{\chi}_x(\rho) = s_0 - \frac{1}{p_0} + \frac{\rho\beta - (1-\rho)\log_2 q_0}{p_0}. \tag{6.21}$$

In particular,

$$h_f(x) = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}, \tag{6.22}$$

and in addition there is no oscillating singularity around x, i.e. $\beta_f(x) = 0$.

Proof : The point that remains to prove is (6.22). One simply applies Proposition 6.1 and gets that $h_f(x) = \inf_\rho \left\{ \frac{\chi_x(\rho)}{\rho} \right\} = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} = h$, and, since $\chi_x(1) = h_f(x)$, $\beta_f(x) = 0$. ∎

Remarks : 1. As proven in [12], (6.22) remains true for any positive Borel measure μ, and not only for the binomial measure.

2. Fix x and $\rho \in (0,1]$. It is easy too see that for every $\varepsilon > 0$, for every $\gamma \in [\chi_x(\rho), \tilde{\chi}_x(\rho)]$, there exists an infinite number of scale j such that

$$\left| \frac{\log_2 |d_{j,k}|}{-j} - \gamma \right| \leq \varepsilon. \tag{6.23}$$

It is now easy to compute the multifractal spectrum of f

Proposition 6.5 *One has*

$$d_f(h) = d_\mu \left(s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} \right)$$

and f obeys the multifractal formalism for functions as we defined it in Definition 6.5.

Proof : From Proposition 6.3, one gets that for every $x \in G^\mu_\alpha$, $h_f(x) = h = s_0 - 1/p_0 + \alpha/p_0$, and from Proposition 6.1, one deduces that $d_f(h) \geq \dim_H(G^\mu_\alpha) = \tau^*(\alpha)$. An easy computation shows that

$$\xi_f(p) = 1 + s_0 - \frac{1}{p_0} + \tau(p). \tag{6.24}$$

Applying Proposition 6.2 for f gives $d_f(h) \leq (\xi_f - 1)^*(h) = \tau^*(\alpha)$, and thus $d_f(h) = d_{\mu_0}(\alpha)$ (remark that for f, Supp(d) as defined in (6.8) equals $[0,1]$). ∎

Remark : Proposition 6.5 is in fact a simple application of the main Theorem of [12], where it is shown that, provided that a measure ν satisfies a certain multifractal formalism for measures (this is the case of the binomial measure μ), the corresponding function f_ν built as in (6.3) satisfies the multifractal formalism for functions we defined.

6.5 Multifractal Spectrum of f^{it} and f^t

Our example f is typically a function which has only cusps (i.e. non-oscillating singularities).

Theorem 6.2 proposes a simple method to create oscillating singularities, and we try it on this specific function f. Before testing this method on f, we begin by the inverse threshold of order γ we mentionned in Section 6.3.

6.5.1 Spectrum of f^{it}

Let $\gamma \in [h_{\min}, h_{\max}]$. We set $f^{it} = \sum_{j,k} d^{it}_{j,k}\psi_{j,k}$, where

$$d^{it}_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\leq 2^{-j\gamma}}. \tag{6.25}$$

Theorem 6.3 *Let $\omega_{it} : [h_{\min}, \gamma] \to (0, +\infty)$ be the decreasing function defined by*

$$u \to \gamma \frac{u - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_0}{p_0}}{\gamma - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_0}{p_0}} \tag{6.26}$$

The multifractal spectrum of f^{it} ranges in $[\gamma, \max(h_{\max}, \omega_{it}^{-1}(h_{\min}))]$, and equals

$$d_{f^{it}}(h) = \max(d_f(h), d_f(\omega_{it}^{-1}(h))). \tag{6.27}$$

ω_{it}^{-1} is the inverse function of ω_{it} : for $h \in (0, +\infty)$, $\omega_{it}^{-1}(h)$ is the unique real number such that $\omega_{it}(\omega_{it}^{-1}(h)) = h$. **Proof :** For every (j,k), $|d_{j,k}| \leq 2^{-j\gamma}$, thus $d_{f^{it}}(h) = -\infty$ if $h < \gamma$. We denote by χ^{it}_x and $\tilde{\chi}^{it}_x$ the 2-microlocal spectra of f^{it} at x.

Let α and β be two real numbers such that $\alpha_{\min} \leq \alpha \leq \beta \leq \alpha_{\max}$. Then $E_{\alpha,\beta}$ is the set defined by

$$E_{\alpha,\beta} = \{x : x \in F^\mu_\alpha \text{ and } x \in H^\mu_\beta\}.$$

From [75] one gets that $\dim_H(E_{\alpha,\beta}) = \min(\tau^*(\alpha), \tau^*(\beta))$ for thebinomial measure μ. Moreover, for a fixed $\beta > 0$, $\cup_{\alpha\leq\beta} E_{\alpha,\beta} = H^\mu_\beta$, thus $\dim_H(\cup_{\alpha\leq\beta} E_{\alpha,\beta}) = \tau^*(\beta)$. Similarly, for a fixed α, $\cup_{\alpha\leq\beta} E_{\alpha,\beta} = F^\mu_\alpha$, and $\dim_H(\cup_{\alpha\leq\beta} E_{\alpha,\beta}) = \tau^*(\alpha)$.

Let $x \in [0,1]$ such that $x \in E_{\alpha,\beta}$ for some α, $\beta > 0$. Proposition 6.4 and equations (6.20) and (6.21) give us the values of the 2-microlocal spectra of f at x. Property (6.23) is here essential : if for a given x and a given ρ, $\chi_x(\rho) \leq \gamma \leq \tilde{\chi}_x(\rho)$, then, after applying (6.25), one gets $\chi^{it}_x(\rho) = \gamma \leq \tilde{\chi}^{it}_x(\rho)$ for f^{it}. Let us distinguish three cases.

- $\gamma \leq s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$ **:** the 2-microlocal spectrum of f^{it} at x becomes

$$\chi^{it}_x(\rho) = \begin{cases} \gamma & \text{if } \rho \in [0, \rho_x] \\ s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0} & \text{if } \rho \in (\rho_x, 1] \end{cases},$$

where ρ_x is the unique ρ such that $s_0 - \frac{1}{p_0} + \frac{\rho_x\alpha - (1-\rho_x)\log_2 q_1}{p_0} = \gamma$. Thus $h_{f^{it}}(x) = h_f(x) = \chi^{it}_x(1) = s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$. The upper 2-microlocal spectrum at x remains unchanged.

- $s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} \leq \gamma \leq s_0 - \frac{1}{p_0} + \frac{\beta}{p_0}$ **:** the 2-microlocal spectrum of f^{it} at x becomes constant and equals γ for every $\rho \in [0,1]$. Hence $h_{f^{it}}(x) = \chi^{it}_x(1) = \gamma > s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} = h_f(x)$. The upper 2-microlocal spectrum at x remains unchanged.

- $s_0 - \frac{1}{p_0} + \frac{\beta}{p_0} < \gamma$ **:** the 2-microlocal spectrum of f^{it} at x becomes

$$\chi^{it}_x(\rho) = \begin{cases} \gamma & \text{if } \rho \in [0, \rho_x] \\ +\infty & \text{if } \rho \in (\rho_x, 1] \end{cases},$$

where ρ_x is the unique ρ such that $s_0 - \frac{1}{p_0} + \frac{\rho_x\beta - (1-\rho_x)\log_2 q_0}{p_0} = \gamma$. Thus

$$h_{f^{it}}(x) = \inf_\rho \frac{\chi^{it}_x(\rho)}{\rho} = \frac{\gamma}{\rho_x}.$$

Remark that ρ_x is explicitely given by

$$\rho_x = \frac{\gamma - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_0}{p_0}}{p_0(\beta + \log_2 q_0)},$$

and thus $h_{f^{it}}(x) = \omega_{it}^{-1}(s_0 - \frac{1}{p_0} + \frac{\beta}{p_0})$, where ω_{it} is the continuous injective function defined in (6.26).
In particular, since $x \notin \text{Supp}(d)$, Proposition 6.1 gives that there are some oscillating singularities around x, with an oscillating exponent equal to

$$\beta_f(x) = \frac{1}{1 + \rho_x}.$$

Now, let us compute the multifractal spectrum of f^{it}.
The range of ρ's when $x \in [0,1]$ is $\left[\frac{\gamma-(s_0-\frac{1}{p_0})+\frac{\log_2 q_0}{p_0}}{\frac{\log_2 q_0 - \log_2 q_1}{p_0}}, 1\right] = [\rho_{\min}, 1]$ (it is minimal for $\beta = \alpha = -\log_2 q_1$, i.e. for $x \in G^{\mu}_{-\log_2 q_1}$). We denote by $h^{it}_{\max}$ the maximal value of the pointwise Hölder exponent $h^{it}_{\max} = \omega_{it}(s_0 - \frac{1}{p_0} + \frac{(-\log_2 q_1)}{p_0}) = \frac{\gamma}{\rho_{\min}}$.

We denote by $E^{f^{it}}_h$ the set of points $x \in [0,1]$ such that $h_{f^{it}}(x) = h$. First note that if $x \in E^{f^{it}}_\gamma$, from the above computations one deduces that $\alpha = \gamma$ or $\beta = \gamma$. Thus $d_{f^{it}}(\gamma) = d_f(\gamma) = \tau^*(\gamma)$.
One must distinguish two cases :
- $h^{it}_{\max} \leq h_{\max}$ **:** If $x \in E^{f^{it}}_h$ with $\gamma < h \leq h^{it}_{\max}$, x either belongs to a set $E_{\alpha,\beta'}$ with $\beta' \geq \alpha$ and $s_0 + 1/p_0 + \alpha/p_0 = h$, or to a set $E_{\alpha',\beta}$ where $\beta = \omega_{it}^{-1}(h) < p_0(\gamma - (s_0 + 1/p_0))$ and $\alpha' \leq \beta$. Reciprocally,

$$E^{f^{it}}_h \subset \left(\cup_{\beta' \geq \alpha} E_{\alpha,\beta'}\right) \cup \left(\cup_{\alpha' \leq \omega_{it}^{-1}(h)} E_{\alpha',\omega_{it}^{-1}(h)}\right).$$

Using the remark of the beginning of this proof, one gets that

$$\dim_h E^{f^{it}}_h = d_{f^{it}}(h) = \max(\tau^*(\alpha), \tau^*(\omega_{it}^{-1}(h))).$$

Eventually, if $h^{it}_{\max} < h \leq h_{\max}$, $E^{f^{it}}_h = E^f_h$ and $d_{f^{it}}(h) = \tau^*(\alpha)$.

- $h_{\max} < h^{it}_{\max}$ **:** If $x \in E^{f^{it}}_h$ with $\gamma < h \leq h_{\max}$, the same argument as above show that $d_{f^{it}}(h) = \max(\tau^*(\alpha), \tau^*(\omega_{it}^{-1}(h)))$.
If $h_{\max} < h \leq h^{it}_{\max}$, $E^{f^{it}}_h = \cup_{\alpha' \leq \omega_{it}^{-1}(h)} E_{\alpha',\omega_{it}^{-1}(h)}$ and $d_{f^{it}}(h) = \tau^*(\omega_{it}^{-1}(h))$.
In both cases, formula (6.27) gives the multifractal spectrum of f^{it}. ■

Since ω_{it} is a linear decreasing function, the graph of the second function $h \to d_f(\omega_{it}^{-1}(h))$ on $[\gamma, h^{it}_{\max}]$ is a symmetric and dilated version of the graph of $h \to d_f(h)$ on $[h_{\min}, \gamma]$. In particular, when $h^{it}_{\max} > h_{\max}$, the multifractal formalism fails, since $d_{f^{it}}(h) > \tilde{d}_{f^{it}}(h) = (\xi - 1)^*(h) = \tau^*(\alpha)$ (with $h = s_0 - 1/p_0 + \alpha/p_0$) (see Figure 6.3 for a plot of the multifractal spectrum of f^{it}).

6.5.2 Multifractal Spectrum of f^t

Let $\gamma \in [h_{\min}, h_{\max}]$. f^t is obtained after applying a threshold of order γ to f as seen in Definition 6.10, i.e. $f^t = \sum_{j,k} d^t_{j,k}\psi_{j,k}$ with $d^t_{j,k} = d_{j,k}\mathbf{1}_{|d_{j,k}|\geq 2^{-j\gamma}}$.

Theorem 6.4 *Let $\omega_t : [\gamma, h_{\max}] \to (0, +\infty)$ be the increasing function*

$$u \to \gamma \frac{u - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_1}{p_0}}{\gamma - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_1}{p_0}}. \tag{6.28}$$

The multifractal spectrum of f^t ranges in $[h_{\min}, \omega_t^{-1}(h_{\max})]$, and equals

$$d_{f^t}(h) = \begin{cases} d_f(h) & \text{if } h \in [h_{\min}, \gamma], \\ d_f(\omega_t^{-1}(h)) & \text{if } h \in (\gamma, \omega_t^{-1}(h_{\max})]. \end{cases}$$

Proof : The proof follows the same lines as the one of Theorem 6.3. We denote by χ^t_x and $\tilde{\chi}^t_x$ the 2-microlocal spectra of f^t at x.Let $x \in [0,1]$ such that $x \in E_{\alpha,\beta}$.
- $\gamma < s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0}$ **:** the 2-microlocal spectrum of f^{it} at x becomes

$$\chi^t_x(\rho) = \begin{cases} \chi_x(\rho) = s_0 - \frac{1}{p_0} + \frac{\rho\alpha - (1-\rho)\log_2 q_1}{p_0} & \text{if } \rho \in [0, \rho_x] \\ +\infty & \text{if } \rho \in (\rho_x, 1] \end{cases},$$

where ρ_x is the unique ρ such that $s_0 - \frac{1}{p_0} + \frac{\rho_x\alpha - (1-\rho_x)\log_2 q_1}{p_0} = \gamma$. ρ_x is explicitely by

$$\rho_x = \frac{\gamma - (s_0 - \frac{1}{p_0}) + \frac{\log_2 q_1}{p_0}}{p_0(\alpha + \log_2 q_1)},$$

Thus $h_{f^t}(x) = \inf_{\rho\in(0,1]} \frac{\chi^t_x(\rho)}{\rho} = \frac{\gamma}{\rho_x} > h_f(x)$. Note that $h_{f^t}(x) = \omega_t^{-1}(s_0 + \frac{1}{p_0} + \frac{\alpha}{p_0}) = \omega_t^{-1}(h_f(x))$ where ω_t is defined by (6.28). Since $\beta_{f^t}(x) = \frac{1}{1+\rho_x} \neq 0$, f^t has an oscillating behavior around x.

- $s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} \leq \gamma \leq s_0 - \frac{1}{p_0} + \frac{\beta}{p_0}$ **:** the 2-microlocal spectrum of f^t at x becomes constant and equals γ for every $\rho \in [0,1]$. Hence $h_{f^t}(x) = \chi^t_x(1) = \gamma \geq s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} = h_f(x)$.

- $s_0 - \frac{1}{p_0} + \frac{\alpha}{p_0} < \gamma$ **:** the 2-microlocal spectrum of f^t at x is the same as the one of f, and $h_{f^t}(x) = h_f(x)$.

The same remarks that were made in the last proof for the computation of the multifractal spectrum of f^{it} apply here for f^t, and the analysis is even simpler in the case of f^t. One can separate three different behaviors :
- if $x \in E^{f^t}_\gamma$, from the above computations one deduces that $x \in E_{\alpha,\beta}$ with $s_0 + 1/p_0 + \alpha/p_0 \leq \gamma \leq s_0 + 1/p_0 + \beta/p_0$, and thus $d_{f^t}(\gamma) = \tau^*(\gamma)$.
- if $x \in E^{f^t}_h$ with $h_{\min} \leq h < \gamma$, x belongs to a set $E_{\alpha,\beta'}$ with $\beta' \geq \alpha$ and $s_0 - 1/p_0 + \alpha/p_0 = h$, thus $d_{f^t}(h) = d_f(h) = \tau^*(\alpha)$.
- if $x \in E^{f^t}_h$ with $\gamma < h$, x belongs to a set $E_{\omega_t^{-1}(h),\beta'}$ with $\beta' \geq \omega_t^{-1}(h)$. Hence $d_{f^t}(h) = d_f(\omega_t^{-1}(h)) = \tau^*(\omega_t^{-1}(h))$. ■

Actually finding functions with strictly decreasing spectra was one of the initial goals of this

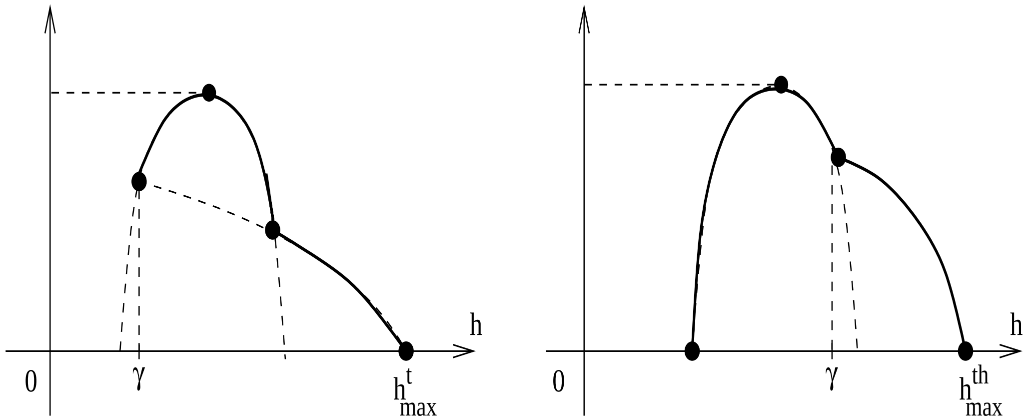

FIG. 6.3 – Multifractal spectra of **Left :** f^{it} and **Right :** f^t

work, and Theorem 6.3 asserts that this is achieved by taking $\gamma = h_s$.

For every value of s_0 and p_0, the multifractal formalism for f^t, as defined in Definition 6.5, is not satisfied when $h \in (\gamma, h^t_{\max}]$ (one has $\tilde{d}_{f^t}(h) = 0$ for $h > \gamma$). Theorem 6.1 confirms *a posteriori* the presence of oscillating singularities in every $E_h^{f^t}$, $h \in (\gamma, h^t_{\max}]$.

Eventually, remark that the multifractal spectra of f^{it} and f^t may be non convex, and that they are homogenous (i.e. they have the same spectrum on any non-trivial subinterval of $[0, 1]$).

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