Rigid geometric structures and hyperbolic dynamical systems

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STRUCTURES GÉOMÉTRIQUES RIGIDES ET SYSTÈMES DYNAMIQUES HYPERBOLIQUES

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Abstract

In this thesis, we are interested in the classification of Anosov flows with smooth decompositions. Under certain supplementary geometric conditions, we obtain a series of classification results and our departing point is an idea called "*go-and-back*". One of the corollaries of our classification results is the following:

Let ϕ_t be an Anosov flow and ψ_t be the geodesic flow of a hyperbolic manifold of dimension at least 3*. If ϕ_t and ψ_t are C^1 orbit equivalent, then they are C^∞ orbit equivalent.*

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Je dédie cette thèse à mes proches.

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Chapter 1

Introduction Générale

Résumé – *Dans le premier chapitre, je donne un résumé détaillé en français de ma thèse. Les chapitres suivants sont rédigés en anglais.*

1.1 Histoire et motivation

L'étude des systèmes dynamiques a plusieurs objectifs, notamment comprendre les propriétés asymptotiques générales des actions des groupes non-compacts, mais aussi classifier certaines classes d'actions spéciales et importantes.

Nous devons admettre le fait que certains systèmes sont fait par la nature, et en conséquence, leurs théories sont harmonieuses et profondes. Les flots géodésiques des variétés à courbure négative sont, à mon avis, parmi ces systèmes bien faits. Rappelons d'abord la définition.

Soit N une variété riemannienne fermée, à courbure négative et SN son fibré unitaire. Pour chaque vecteur $u \in SN$, il existe une unique géodésique tangente à u en $t = 0$, notée γ_u. Donc nous obtenons un flot $\phi : \mathbb{R} \times N \to N$ tel que $\phi(t, u) = \dot{\gamma_u}(t)$, ou $\dot{\gamma_u}(t)$ représent le vecteur tangent de γ_u au temps t.

Comme les courbures sectionnelles sont supposé négatives, les géodésiques différentes divergent souvent l'une de l'autre. Intuitivement, ces flots sont très chaotiques.

Avant l'oeuvre fondamentale de D. V. Anosov, il y avait les travaux de Hedlund, Hopf et Hadamard sur les flots géodésiques des variétés à courbure négative. En particulier, ils ont montré l'ergodicité dans le cas des surfaces. L'un des points essentiels dans leur argument est de montrer l'existence des distributions stable et instable forte, au moins pour les surfaces. Le fait que ces distributions sont souvent très peu différentiables était un obstacle pour montrer l'ergodicité de ces flots dans toute sa généralité.

En 1967, D. V. Anosov publie son oeuvre intitulée "Geodesic flows on Riemann manifolds with negative curvature". Peu après, les systèmes dynamiques axiomatisés pour la première fois dans ce livre sont baptisés des systèmes d'Anosov (ou les systèmes uniformément hyperboliques). Rappellons d'abord la définition.

Soient M une variété fermée avec une métrique riemannienne auxiliaire et ϕ_t un flot C^∞ sur M dont le générateur est noté X. Alors ϕ_t est dit d'*Anosov* s'il existe une décomposition ϕ_t-invariante du fibré tangent de M

$$TM = \mathbb{R}X \oplus E^+ \oplus E^-$$

et deux nombres positifs a et b tel que

$$\| D\phi_{\mp t}(u^\pm) \| \leq ae^{-bt} \| u^\pm \|, \ \forall\ t > 0,\ \forall\ u^\pm \in E^\pm.$$

Ces deux distributions E^+ et E^- s'appellent les distributions instable forte et stable forte de ϕ_t, qui sont uniques et automatiquement continues. Anosov a montré que ces deux distributions s'intègrent en des feuilletages continus dont les feuilles sont C^∞, qui sont notés respectivement $\mathcal{F}^+$ et $\mathcal{F}^-$. Les difféomorphismes d'Anosov sont définis similairement. Rappellons que les flots géodésiques des variétés fermées à courbure négatives sont d'Anosov.

En montrant l'absolue continuité des applications d'holonomie de $\mathcal{F}^+$ et $\mathcal{F}^-$ et en utilisant un argument classique dû à Hopf, Anosov établit l'ergodicité par rapport à la forme volume invariante des flots géodésiques des variétés fermées à courbure négative. Ceci était un triomphe de la théorie des systèmes d'Anosov.

Pendant les vingt ans qui suivent, une étude descriptive de ces systèmes est menée. Elle nous permet de comprendre ces systèmes beaucoup mieux que les autres. Par exemple, nous savons qu'ils sont structurellement stables et ils préservent de nombreuses mesures de probabilité.

Signalons que dans la définition d'un flot d'Anosov, les distributions stable et instable forte sont supposées seulement C^0. Or déja dans [**An**], Anosov a construit des flots d'Anosov analytiques réels dont les distributions ne sont pas C^1. Pour alléger les notations, nous proposons la définition suivante.

Définition 1.1 – Soit ϕ_t un flot d'Anosov. Alors il est dit d'*Anosov-lisse*, si ses distributions instable forte et stable forte sont C^∞.

Le premier résultat concernant les flots d'Anosov-lisses est dû à É. Ghys (voir [**Gh1**]. Il a montré qu'à un changement de temps très spécial et revêtement fini près, un flot d'Anosov-lisse de contact de dimension 3 est C^∞ conjugué au flot géodésique d'une surface hyperbolique. Ce résultat crucial confirme l'intuition générale sur la rareté des flots d'Anosov-lisses.

Ensuite dans [**Ka1**], M. Kanai montre que pour un flot géodésique d'une variété à courbure negative $\frac{4}{9}$-pincée, s'il est Anosov-lisse, alors il est C^∞ conjugué au flot géodésique d'une variété hyperbolique. Puis dans [**FK**], R. Feres et A. Katok ont raffiné l'argument de Kanai pour améliorer la restriction sur le pincement.

Finalement dans [**BFL2**], Y. Benoist, P. Foulon et F. Labourie ont classifié les flots d'Anosov-lisses de contact. Plus précisément, ils ont montré qu'à un changement de temps très spécial et revêtement fini près, chaque flot d'Anosov-lisse de contact est C^∞ conjugué au flot géodésique d'une variété localement symétrique de rang un. Mentionnons que Y. Benoist et F. Labourie ont classifié les difféomorphismes d'Anosov-lisses préservant des connexions linéaires. Avant leur résultat, il y avait les travaux [**Av**] et [**Fl-K**] traitant des cas partculiers.

Maintenant nous expliquons un peu l'idée de la démonstration de ces résultats. Rappellons d'abord que les connexions linéaires sont des structures géométriques rigides, dont les groupes de symétries sont de dimension finie (voir Chapitre 2). En revanche les formes symplectiques ne sont pas rigides, et leurs groupes de symétries sont de dimension infinie.

L'observation cruciale pour les travaux ci-dessus est que l'existence d'une structure géométrique invariante et non-rigide pour un système d'Anosov-lisse entraîne souvent l'existence d'une autre structure géométrique rigide et invariante. Soit par exemple ϕ un difféomorphisme d'Anosov-lisse préservant une forme symplectique ω. Alors ϕ préserve une unique connexion lisse ∇ tel que $\nabla E^{\pm} \subseteq E^{\pm}$ et

$$\nabla\omega = 0, \quad \nabla_{Y^{\pm}} Y^{\mp} = P^{\mp}[Y^{\pm}, Y^{\mp}],$$

où $Y^{\pm}$ représentent des sections de $E^{\pm}$ et $P^{\pm}$ représentent les projections de TM sur $E^{\pm}$.

En effet, pour beaucoup de classes de systèmes d'Anosov-lisses, nous pouvons construire des structures géométriques rigides invariantes et lisses. Et puis, comme il est communément considéré, un système chaotique et une structure géométrique rigide lisse peuvent co-exister uniquement dans des cas très spéciaux (voir [**DG**]).

Dans cette thèse, nous sommes aussi guidés par cette idée. Et en plus, nous avons trouvé une technique d'*aller et retour*, qui est détaillée dans le chapitre 4 et qui nous sert de point de départ pour prouver la rigidité de plusieurs classes de flots d'Anosov-lisses, dont les conséquences sur les flots géodésiques des variétés hyperboliques nous paraîssent frappantes.

1.2 Nos résultats et les idées de démonstration

1.2.1 Chapitre 2: Propriétés des structures géométriques rigides

Dans le chapitre 2, nous établissons quelques propriétés structurelles pour les structures géométriques rigides admettant des pseudo-groupes d'isométries locales topologiquement transitifs, qui nous fournissent la base géométrique pour comprendre les flots d'Anosov-lisses. Notre but est de relier l'étude des structures géométriques rigides localement homogènes, au moins sur un sous-ensemble ouvert-dense, à celle des (G, X)-structures. Sans entrer dans les définitions plus ou moins formelles, nous nous restreignons dans cet introduction aux connexions linéaires, exemples typiques de structures géométriques rigides.

Soient M une variété et ∇ une C^∞ connexion linéaire définie sur M. Chaque difféomorphisme ϕ de M envoie ∇ sur une autre connexion notée $\phi_*\nabla$, et ϕ est appelée une ∇-isométrie si $\phi_*\nabla = \nabla$. Le groupe des ∇-isométries est noté $I(\nabla)$ et l'ensemble des ∇-isométries locales est noté I^{loc}_{∇}. Dans [**Gr1**], M. Gromov a démontré le résultat fondamental suivant.

Théorème 2.1 (M. Gromov) – *Soit* ∇ *une connexion linéaire* C^∞ *sur une variété* M*. Si son pseudo-groupe d'isométries locales* I^{loc}_∇ *admet une orbite dense, alors il admet une unique orbite ouverte et dense.*

Mentionnons que ce théorème a été redémontré avec plus de soin dans [**Be**], [**Fe2**] et [**Ze2**]. Pour nous, ce théorème contitue la première étape pour transformer l'étude des connexion linéaires C^∞ combinées avec des dynamiques chaotiques en celle des (G, X)-structures. La seconde étape consiste à construire un espace modèle. Nous procédons comme suit.

Supposons que le pseudo-groupe des isometries locales de ∇ admet une orbite dense. Alors d'après le théoreme ci-dessus, il admet une unique orbite ouverte et dense, notée Ω. Fixons un point x dans Ω et soit $\mathfrak{g}$ l'ensemble des germes de champs de Killing C^∞ de ∇ au voisinage de x. Nous notons $\mathfrak{h}$ l'ensemble des éléments de $\mathfrak{g}$ qui s'anullent au point x. Alors nous pouvons montrer facilement que $\mathfrak{g}$ et $\mathfrak{h}$ sont des algèbres de Lie de dimension finie. Soit $\bar{G}$ le groupe de Lie connexe et simplement connexe dont l'algèbre de Lie est $\mathfrak{g}$ et soit $\bar{H}$ le sous-groupe de Lie connexe de $\bar{G}$ intégrant $\mathfrak{h}$.

Définition 2.2 – Sous les notations ci-dessus, ∇ est dit *normale* si $\bar{H}$ est fermé dans $\bar{G}$.

Comme $\nabla\mid_\Omega$ est localement homogène, la normalité de ∇ est indépendante du point de base x choisi dans Ω. Nous montrons

Proposition 2.1 – *Soit* ∇ *une connexion linéaire* C^∞*. Si son pseudo-groupe d'isométries locales* I^{loc}_∇ *admet une orbite dense, alors* I^{loc}_∇ *admet une unique orbite ouverte-dense notée* Ω*. Si* ∇ *est en plus normale et* $\bar{G}$ *et* $\bar{H}$ *sont définis comme ci-dessus, alors il existe sur* $\bar{G}/\bar{H}$ *une connexion linéaire* $\bar{\nabla}$*,* $\bar{G}$*-invariante et localement isométrique à* $\nabla\mid_\Omega$*. Si nous prenons l'ensemble des isométries locales de* $(\Omega, \nabla\mid_\Omega)$ *sur* $(\bar{G}/\bar{H}, \bar{\nabla})$ *comme cartes, nous obtenons ainsi sur* Ω *une* $(I(\bar{\nabla}), \bar{G}/\bar{H})$*-structure.*

Rappellons que une connexion linéaire est dit complète si toutes ces géodésiques peuvent être définies sur $\mathbb{R}$. En axiomatisant certaines propriétés des connexions linéaires, nous avons trouvé une notion de complétude pour les structures géométriques rigides générales. En rélevant des géodesiques, nous démontrons la proposition suivante,

Proposition 2.4 – *Soit* M *une variété connexe munie d'une* (G, X)*-structure. Supposons que* G *préserve une* C^∞ *connexion linéaire* $\bar{\nabla}$ *définie sur* X*. Si la structure canoniquement associée* ∇ *sur* M *est complète, alors la* (G, X)*-structure sur* M *est aussi complète.*

Les propositions ci-dessus sont toutes énnoncés pour les connexions linéaires.

Mais modulo des modifications évidentes, elles sont aussi valides pour les structures géométriques rigides générales (voir Chapitre 2). Donc les notions et les propositions prouvées dans ce chapitre nous permettent de ramener l'étude des structures géométriques rigides avec beaucoup d'isométries locales à celle des (G, X)-structures.

1.2.2 Chapitre 3: Propriétés dynamiques et géométriques des systèmes d'Anosov

Dans ce chapitre, nous rappellons et démontrons des propriétés des flots d'Anosov en portant une attention particulière au cas des flots d'Anosov-lisses.

Rappellons d'abord la définition de la mesure de Bowen-Margulis, qui est particulièrement importante tant dans notre idée que dans nos arguments.

Soit ϕ_t un flot d'Anosov topologiquement transitif. Alors il existe une unique mesure de probabilité ϕ_t-invariante μ, dit de Bowen-Margulis, dont l'entropie est égale à l'entropie topologique (voir [**BR**] et [**Ma1**]. Si en plus ϕ_t est topologiquement mélangeant, alors il existe quatre famille de mesures $\mu^{\pm}$ et $\mu^{\pm,0}$ supportées respectivement par les feuilles de $\mathcal{F}^{\pm}$ et $\mathcal{F}^{\pm,0}$ telles que

$$\mu^{\pm} \circ \phi_t = e^{\pm ht}\mu^{\pm}, \quad \mu^{\pm,0} \circ \phi_t = e^{\pm ht}\mu^{\pm,0},$$

où h représente l'entropie topologique de ϕ_t. Ces mesures sont appelées les mesures de Margulis de ϕ_t (voir [**Ma1**]). En plus, nous avons $\mu = \mu^+ \otimes \mu^{-,0} = \mu^- \otimes \mu^{+,0}$ localement.

En utilisant les résultats de cohomologie classiques pour les flots d'Anosov, nous pouvons montrer, en adaptant les arguments de [**Fo1**], que pour un flot d'Anosov-lisse topologiquement mélangeant, si sa mesure de Bowen-Margulis est égale à la mesure de Lebesgue et ses distributions stable et instable fortes sont orientables, alors ses mesures de Margulis sont données par des formes volumes lisses le long des feuilles des distributions stable et instable, qui ne s'annullent nulle part.

A la fin du chapitre, nous détaillons la construction, suggérée par P. Tomter, d'une famille de flots d'Anosov-lisses obtenue à partir de certaines représentations des groupes Spin. Ces flots d'Anosov sont des couplages de flots géodésiques de variétés hyperboliques avec certains automorphismes hyperboliques sur des tores.

1.2.3 Chapitre 4: Les changements de temps spéciaux des flots d'Anosov-lisses

Dans ce chapitre, nous détaillons notre idée d'*aller et retour* dont la force sera démontrée dans les chapitres suivantes. Dans [**Par**], W. Parry a démontré que pour un flot d'Anosov dont la décomposition d'Anosov est C^1, il existe un changement de temps C^1 dont la mesure de Bowen-Margulis est Lebesgue. Nous l'appellons le *changement de temps de Parry.*

Le résultat central de ce chapitre est le théorème suivant, qui constitue le point de départ pour les autres résultats.

Théorème 4.1 – *Soit ϕ_t un flot d'Anosov-lisse préservant une forme de volume. Supposons en plus que E^+ et E^- sont orientables. Alors son changement de temps de Parry est aussi Anosov-lisse.*

Si la mesure de Bowen-Margulis est égale à la mesure de Lebesgue, alors nous avons vu dans le chapitre précédent que les mesures de Margulis sont données par des familles de formes volumes. Nous démontrons le proposition suivante.

Proposition 4.2 – *Soit ϕ_t un flot d'Anosov-lisse topologiquement mélangeant dont la mesure de Bowen-Margulis est égale à la mesure de Lebesgue. Supposons que les dimensions de E^+ et E^- sont respectivement n et m. Si ϕ_t préserve une connexion linéaire, alors il préserve une autre connexion ∇ telle que les fibrés en droite $\wedge^n E^+$ and $\wedge^m E^-$ admettent des sections ∇-parallèlles et non nulles.*

Dans un second temps, nous étudions les changements de temps qui préservent la propriété d'être d'Anosov-lisse. En particulier, nous trouvons tous les changements de temps d'Anosov-lisse des flots géodésiques des variétés localement symétriques ainsi que des suspensions des infra-nilautomorphismes hyperboliques. Notamment, ils sont déterminés par le premier groupe de cohomologie de la variété ambiante.

Notre idée d'*aller et retour* pour prouver la rigidité des flots d'Anosov-lisse est de prendre d'abord le changement de temps de Parry pour renforcer à l'aide de la Proposition 4.2 la structure géométrique sous-jacente. Ensuite nous essayons de classifier ces flots synchronisés. Finalement nous refaisons un changement de temps d'Anosov-lisse pour obtenir des informations sur le flot initial.

Dans les trois chapitres suivants, nous allons utiliser cet idée d'aller et retour pour classifier certaines classes de flots d'Anosov-lisses.

1.2.4 Chapitre 5: Rigidité des flots d'Anosov-lisses transversalement symplectiques

Un flot d'Anosov ϕ_t est dit *transversalement symplectique* s'il préserve une 2-forme fermée ω dont le noyau est $\mathbb{R}X$, où X est le générateur de ϕ_t. Par exemple, comme il est bien connu, les flots géodésiques des variétés riemanniennes sont transversalement symplectiques.

Les flots d'Anosov-lisse transversalement symplectique de dimension 3 sont déja classifiés par É. Ghys. Dans **[FK]**, R. Feres et A. Katok ont étudié les flots d'Anosov-lisse transversalement symplectique de dimension 5. Notamment, ils ont classifié le cas de contact (voir aussi **[BFL]**). Dans ce chapitre, notre résultat principal est une classification complète de ces flots de dimension 5. Plus précisément, nous démontrons

Théorème 5.1 – *Soit ϕ_t un flot d'Anosov-lisse de dimension 5. S'il est transversalement symplectique, alors à un changement de temps spécial et un revêtement* *fini près, ϕ_t est C^∞ conjugué soit au flot géodésique d'une variété hyperbolique de dimension* 3 *soit à la suspension d'un automorphisme hyperbolique et symplectique de* $\mathbb{T}^4$.

Idée de la preuve – Soit λ la 1-forme canonique de ϕ_t, c'est-à-dire,

$$\lambda(X) \equiv 1, \quad \lambda(E^\pm) \equiv 0,$$

où X représente le générateur de ϕ_t. D'après les résultats de [**BFL2**], [**FK**] et [**BL**], il suffit de montrer la non-existence du cas $d\lambda \neq 0$ et $d\lambda \wedge d\lambda = 0$. D'après l'idée d'aller et retour, nous pouvons supposer en plus que la mesure de Bowen-Margulis de ϕ_t est Lebesgue. L'observation cruciale est que dans ce cas là, la décomposition de Liapounov de ϕ_t est forcément lisse. Ensuite nous construisons des connexions invariantes canoniquement associées. Finalement nous éliminons tous les cas possibles en utilisant des arguments de dynamique, d'algèbres de Lie et de groupes discrets. Signalons que la proposition 4.2 est cruciale dans nos arguments.

1.2.5 Chapitre 6 : Sur la classification des systèmes d'Anosov quasiconformes

Un flot d'Anosov est dit *quasiconforme* si ses restrictions sur ses feuilles stables et instables fortes sont toutes quasiconformes, avec des distorsions de quasiconformité uniformément bornées. Par exemple, les flots géodésiques des variétés hyperboliques sont quasiconformes (et même conformes).

Dans [**Yu**], C. Yue a démontré que si le flot géodésique d'une variété de dimension au moins trois et à courbure négative est quasiconforme, alors cette variété est de courbure constante. Avant ce résultat, il y avait des travaux [**Su**] et [**Ka2**]. Très récemment, V. Sadovskaya a classifié les flots d'Anosov quasiconformes de contact dont les distributions stables fortes sont de dimensions au moins 2. En combinant son argument géométrique et notre technique d'aller et retour, nous généralisons tous ces résultats de rigidité en montrant

Théorème 6.1 – *Soit ϕ_t un flot d'Anosov quasiconforme préservant une forme volume. Si $E^+ \oplus E^-$ est C^∞ et les dimensions de E^+ et E^- sont au moins égales à* 2, *alors à changement de temps constant et revêtement fini près, ϕ_t est C^∞ conjugué soit à la suspension d'un automorphisme hyperbolique d'un tore soit à un changement de temps canonique du flot géodésique d'une variété hyperbolique.*

Idée de la preuve – D'aprés l'idée d'*aller et retour*, nous pouvons supposer que la mesure de Bowen-Margulis est égale à la mesure de Lebesgue. Nous montrons que les linéarisations de V. Sadovskaya (voir [**Sa**]) sont C^∞. Nous construisons une connexion invariante, et concluons à l'aide d'arguments géométriques.

Soit ϕ un difféomorphisme d'Anosov quasiconforme. Alors la distribution $E^+ \oplus E^-$ de sa suspension est bien sûr C^∞. Nous déduisons du théorème ci-dessus la classification suivante.

Corollaire 6.1 – *Soit ϕ un difféomorphisme d'Anosov quasiconforme préservant une forme volume. Si E^+ et E^- sont de dimensions au moins égales à 2, alors à un revêtement fini près, ϕ est C^∞ conjugué à un automorphisme hyperbolique d'un tore.*

Dans [**KS**], B. Kalinin et V. Sadovskaya ont classifié des difféomorphismes d'Anosov quasiconfomes et topologiquement transitifs dont les distributions stable et instable forte sont de dimension au moins 3. Leur argument est très élégant, mais rencontre une difficulté essentielle dans les cas où E^+ ou E^- est de dimension 2.

Elaborant sur [**KS**], nous classifions complètement les flots d'Anosov quasiconformes dont les distributions stable et instable fortes sont de dimension assez grandes. Plus précisément, nous démontrons

Théorème 6.2 – *Soit ϕ_t un flot d'Anosov quasiconforme topologiquement transitif. Si E^+ et E^- sont de dimensions au moins égales à 3, alors à un revêtement fini près, ϕ_t est C^∞ orbitalement équivalent soit à la suspension d'un automorphisme hyperbolique d'un tore soit au flot géodésique d'une variété hyperbolique.*

Idée de la preuve – Nous pouvons construire une structure géométrique transverse, invariante par le flot. Ensuite nous montrons la complétude des ces structures. Un argument de diffusion trouvé par É. Ghys est crucial dans la preuve.

Si les dimensions de E^+ et E^- sont au moins égales à 2 et l'une des dimensions de E^+ ou E^- est égale à 2, alors nous pouvons démontrer le résultat partiel suivant.

Proposition 6.1 – *Soit ϕ_t un flot d'Anosov quasiconforme préservant une forme volume. Si E^+ est de dimension 2 et E^- est de dimension au moins égale à 2, et les applications d'holonomie de ϕ_t peuvent être définies globalement, alors à un revêtement fini près, ϕ_t est C^∞ orbitalement équivalent soit à la suspension d'un automorphisme hyperbolique d'un tore soit au flot géodésique d'une variété hyperbolique de dimension* 3.

Les conjugaisons entre les flots d'Anosov ont fait l'objet de beaucoup d'études (voir par exemple [**Ham2**], [**DM**], [**L1**] et [**L2**]). Philosophiquement, il en existe peu, même parmi des conjugaisons C^0.

Or les équivalences orbitales C^0 entre les flots d'Anosov sont abondantes. Par exemple, si deux flots d'Anosov sont suffisament C^1-proches, alors ils sont C^0 orbitalement équivalents d'après la célèbre stabilité structurelle des flots d'Anosov (voir [**HK**]). Donc une question naturelle à se poser est si les équivalences orbitales C^1 entre les flots d'Anosov sont rares.

En montrant que les conditions du théorème 6.2 et de la proposition 6.1 sont préservées par équivalence orbitale C^1, nous déduisons de ces deux résultats la conséquence suivante pour les flots géodésiques des variétés hyperboliques.

Théorème 6.5 – *Soient ϕ_t un flot d'Anosov et ψ_t le flot géodésique d'une variété hyperbolique fermée de dimension au moins* 3. *Si ϕ_t et ψ_t sont C^1 orbitalement équivalents, alors ils sont C^∞ orbitalement équivalents.*

En combinant le théorème 6.5 avec quelques résultats classiques, nous obtenons

Proposition 6.2 – *Soit M une variété connexe et fermée à courbure négative de dimension $m \geq 3$. Alors nous avons les relations suivantes entre la dynamique et la géométrie:*
(1) *Le flot géodésique de M est C^0 orbitalement équivalent à celui d'une variété hyperbolique si et seulement si le revêtement universel $\widetilde{M}$ avec sa métrique relevée est quasi-isométrique à $\mathbb{H}^m$.*
(2) *Le flot géodésique de M est C^1 orbitalement équivalent à celui d'une variété hyperbolique si et seulement si M est à courbure constante négative.*

1.2.6 Chapitre 7 : Sur l'homogénéité des flots d'Anosov-lisses affines

Dans ce chapitre, nous obtenons un résultat d'homogénéité pour les flots d'Anosov-lisses affines. Plus précisément, nous démontrons

Théorème 7.1 – *Soit ϕ_t un flot d'Anosov-lisse affine sur M préservant une forme volume. Noté par $\hat{\phi}_t$ son changement de temps de Parry. Alors nous avons l'alternative suivante :*
(1) *A un changement de temps constant et un revêtement fini près, ϕ est C^∞ conjugué à la suspension d'un nilautomorphisme hyperbolique.*
(2) *$\hat{\phi}_t$ est topologiquement mélangeante et il existe un groupe de Lie G contenant le groupe fondamental Γ de M comme un sous groupe discrete, un sous group fermé H de G et un vecteur α dans l'algèbre de Lie de G tel que $\hat{\phi}_t$ est C^∞ conjugué au flot $\psi_t : \Gamma\diagdown G\diagup H \to \Gamma\diagdown G\diagup H$ donné par $\psi_t(\Gamma g H) = \Gamma(g \cdot exp(t\alpha))H$.*

Idée de la preuve – L'essentiel est de montrer que la structure g est normale. C'est une conséquence du fait que la mesure de Bowen-Margulis de $\hat{\phi}_t$ est égale à la mesure de Lebesgue. Dans la démonstration, nous nous sommes inspirés de [**BFL**].

Rappellons que les flots d'Anosov symétriques sont des exemples de flots d'Anosov-lisses (voir le chapitre 3 pour plus de détail). Nous proposons la conjecture suivante.

Conjecture 7.1 – *Soit ϕ_t un flot d'Anosov-lisse affine préservant une forme volume. Alors ϕ_t est commensurable à un changement de temps très spécial d'un flot d'Anosov symétrique.*

Le théorème 7.1 doit nous fournir le point de départ pour une future classification des flots d'Anosov-lisse affines.

1.3 A short introduction in English

In this thesis, we consider the relationship between rigid geometric structures and hyperbolic dynamical systems. Our goal is to better understand the Anosov flows with C^∞ Anosov distributions. To simplify the notations, we propose the following

Definition 1.1 – Let ϕ_t be a C^∞ Anosov flow. Then it is said to be *Anosov-smooth* if its strong unstable and strong stable distributions are both C^∞.

In chapter 2, we relate the study of rigid geometric structures with topologically transitive pseudo-groups of local isometries with that of (G, X)-structures. Then in chapter 3, we recall and prove certain dynamical and geometrical properties of Anosov-smooth flows. The Bowen-Margulis measures will play a crucial role in our arguments.

In chapter 4, we outline our idea of *go-and-back*, which will serve as the departing point of the following chapters.

In chapter 5, by using the idea of *go-and-back* we classify completely the 5-dimensional transversely symplectic Anosov-smooth flows, which have been studied by R. Feres and A. Katok in [**FK**].

In chapter 6, we classify quasiconformal Anosov systems in most cases. In particular, the volume-preserving quasiconformal Anosov diffeomorphisms are classified. Then we apply our classification results to orbit equivalence rigidity of the geodesic flows of hyperbolic manifold.

In chapter 7, we prove a homogeneity result about affine Anosov-smooth flows, which should furnish the departing point for a possible future classification of such flows.

Part of the material of this thesis has already appeared in journals. Chapter 5 has been announced in [**Fa1**]. The complete version is to appear in *Journal of the Institute of Mathematics of Jussieu* ([**Fa3**]). The first part of chapter 4 has appeared in [**Fa2**]. The second part of chapter 4 combined with the first part of chapter 6 will appear in *Ergodic Theory and Dynamical Systems* ([**Fa4**]).

Chapter 2

Properties of Rigid Geometric Structures

Abstract – *In this chapter, we establish some structural properties for the C^∞ rigid geometric structures admitting topologically transitive pseudogroups of local isometries. These geometric properties are of fundamental importance to understand the Anosov-smooth flows.*

2.1 Introduction

2.1.1 Definition of rigid geometric structures

In this subsection, we fix the notation and recall some elementary facts about geometric structures (see **[CQ]** and **[Gr1]** for more details).

Let M and N be two C^∞ manifolds. Denote by $Diff^\infty_{loc}(M,N)$ the space of C^∞ local diffeomorphisms from M into N. For any $x, y \in M$ and any $k \geq 0$, we define

$$D^k_{(x,y)}(M,N) = \{j^k_x\phi \mid \phi \in Diff^\infty_{loc}(M,N),\ \phi(x) = y\},$$

where $j^k_x\phi$ denotes the k-jet at x of ϕ. We define also

$$G^k(n,\mathbb{R}) = D^k_{(0,0)}(\mathbb{R}^n,\mathbb{R}^n).$$

Then $G^k(n,\mathbb{R})$ is naturally a Lie group with respect to the composition of k-jets. For any $k \geq 1$ we denote by $T^k_0\mathbb{R}^n$ the vector space of $(k-1)$-jets at zero of C^∞ vector fields on $\mathbb{R}^n$. Then $G^k(n,\mathbb{R})$ admits a natural linear representation ρ on $T^k_0\mathbb{R}^n$ such that for any $j^k_0\phi \in G^k(n,\mathbb{R})$ and any $j^{k-1}_0 Y \in T^k_0\mathbb{R}^n$ we have

$$\rho(j^k_0\phi)(j^{k-1}_0 Y) = j^{k-1}_0(D\phi(Y)).$$

It is easily seen that ρ is injective and with respect to this faithful representation, $G^k(n,\mathbb{R})$ becomes a real algebraic group (see **[CQ]** and **[OV]**).

For $l \geq k \geq 0$ we have the natural projection $\pi^l_k : G^l(n,\mathbb{R}) \to G^k(n,\mathbb{R})$, which is in fact a homomorphism of real algebraic groups.

Suppose that M is of dimension n. We define for any $k \geq 1$,

$$F^k M = \cup_{x \in M} D^k_{(0,x)}(\mathbb{R}^n, M), \; D^k M = \cup_{x,y \in M} D^k_{(x,y)}(M, M).$$

Then $G^k(n, \mathbb{R})$ acts naturally on $F^k M$ by right composition and with respect to this action $F^k M$ becomes a $G^k(n, \mathbb{R})$-principal bundle. We call $F^k M$ the *k-th order frame bundle* of M and each element of $F^k M$ a *frame of order k*.

Let Z be a smooth real algebraic variety admitting an algebraic action of $G^k(n, \mathbb{R})$ from left. Then a C^∞ section of the associated fiber bundle $\pi : F^k M \rtimes Z \to M$ is said to be a C^∞ *geometric structure of type* Z *and order* k on M. Recall that these sections are in bijection with C^∞ $G^k(n, \mathbb{R})$-equivariant maps from $F^k M$ into Z (see **[Fe]**).

For a C^∞ diffeomorphism ϕ of M and a C^∞ geometric structure g on M, we denote by $\phi_* g$ the image of the natural action of ϕ on g. Then ϕ is said to be an *isometry* of g if $\phi_* g = g$. The isometry group of g is denoted by $I(g)$.

Now suppose that g is of type Z and order k. For any $i \geq 0$ we denote by $J^i_n Z$ the space of i-jets at zero of smooth maps from $\mathbb{R}^n$ into Z. Then $J^i_n Z$ is also a smooth real algebraic variety admitting a natural algebraic action of $G^{k+i}(n, \mathbb{R})$ (see **[CQ]**). By differentiating g (see **[CQ]** for the details), we get a C^∞ $G^{k+i}(n, \mathbb{R})$-equivariant map $g^i : F^{k+i} M \to J^i_n Z$, i.e. a C^∞ geometric structure of type $J^i_n Z$ and order $(k+i)$. This structure g^i is said to be the *i-th order prolongation* of g.

For any $i \geq 0$ and $\phi \in Diff^\infty_{loc}(M, M)$, ϕ sends g^i to another local geometric structure denoted by $\phi_*(g^i)$. View g^i as a section and define for all $x, y \in M$

$$I^{k+i}_{x,y} = \{ j^{k+i}_x \phi \mid \phi \in Diff^\infty_{loc}(M, M), \; \phi(x) = y \;, (\phi_*(g^i))(y) = g^i(y) \}.$$

If g^i is viewed as a $G^{k+i}(n)$-equivariant map, then

$$I^{k+i}_{x,y} = \{ j^{k+i}_x \phi \mid \phi \in Diff^\infty_{loc}(M, M), \; \phi(x) = y, \; (g^i \cdot J^{k+i} \phi) \mid_{F^{k+i}_x M} = g^i \mid_{F^{k+i}_x M} \}.$$

Similar to $G^{k+i}(n, \mathbb{R})$, $D^{k+i}_{(x,x)}(M, M)$ is naturally a real algebraic group. Since the action of $G^k(n, \mathbb{R})$ on Z is supposed to be algebraic, then it is easily seen that $I^{k+i}_{x,x}$ is a real algebraic subgroup of $D^{k+i}_{(x,x)}(M, M)$ (see **[CQ]**). In addition, for all $j \geq i \geq 0$ the natural projection $\pi^{k+j}_{k+i} : I^{k+j}_{x,x} \to I^{k+i}_{x,x}$ is a homomorphism of real algebraic groups.

Definition 2.1 – Under the notations above, the geometric structure g is said to be *rigid*, if there exists $i \geq 0$ such that for any $x \in M$, $\pi^{k+i+1}_{k+i} : I^{k+i+1}_{x,x} \to I^{k+i}_{x,x}$ is injective.

If g is rigid, then for all $l \geq j \geq i$, $\pi^{k+l}_{k+j} : I^{k+l}_{x,x} \to I^{k+j}_{x,x}$ is also injective (see Corollary 5.5 of **[CQ]**).

We denote by $I^{loc}_{x,y}$ the space of germs at x of g-local isometries sending x to y. Then for any $x \in M$, $I^{loc}_{x,x}$ is a group admitting for any $j \geq 0$ the natural projection $\pi^{loc}_{k+j} : I^{loc}_{x,x} \to I^{k+j}_{x,x}$ such that $\pi^{loc}_{k+j}(\phi) = j^{k+j}_x \phi$. We define

$$I^{k+j} = \cup_{x,y \in M} I^{k+j}_{x,y}, \; I^{loc} = \cup_{x,y \in M} I^{loc}_{x,y}.$$

Each element of I^{k+i} is said to be a $(k+i)$-*th order isometric jet* of g.

I^{k+j} and I^{loc} are both pseudogroups acting naturally on M. For any $x \in M$ we denote respectively by $I^{k+j}(x)$ and $I^{loc}(x)$ their orbits of x. If I^{loc} acts transitively on M, then g is said to be *locally homogeneous.*

Many geometric structures are rigid. For example, pseudo-Riemannian metrics, complete parallelisms and linear connections are all rigid (see [**CQ**]). To illustrate the more or less formal definitions above, let us prove by a simple calculation that Riemannian metrics are rigid.

Let g be a C^∞ Riemannian metric on a n-dimensional manifold M. For any $x \in M$ we want to see the injectivity of $\pi_1^2 : I^2_{x,x} \to I^1_{x,x}$.

Fix a normal coordinate system of an open neighborhood of x. Then $I^{loc}_{x,x}$ is defined by the equations

$$g_{i,j} \circ \phi \cdot \frac{\partial \phi_i}{\partial x_k} \cdot \frac{\partial \phi_j}{\partial x_l} = g_{kl}, \ \forall\ 1 \le k, l \le n.$$

By taking the derivatives of these equations, we get for all $1 \le s, k, l \le n$

$$\partial_r g_{i,j} \circ \phi \frac{\partial \phi_r}{\partial x_s} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_j}{\partial x_l} + g_{i,j} \circ \phi \frac{\partial^2 \phi_i}{\partial x_k \partial x_s} \frac{\partial \phi_j}{\partial x_l} + g_{i,j} \circ \phi \frac{\partial \phi_i}{\partial x_k} \frac{\partial^2 \phi_j}{\partial x_l \partial x_s} = \partial_s g_{k,l}.$$

So $I^2_{x,x}$ is defined by the following algebraic equations

$$g_{i,j}(x) \cdot \frac{\partial \phi_i}{\partial x_k} \cdot \frac{\partial \phi_j}{\partial x_l} = g_{kl}(x),$$

$$\partial_r g_{i,j}(x) \frac{\partial \phi_r}{\partial x_s} \frac{\partial \phi_i}{\partial x_k} \frac{\partial \phi_j}{\partial x_l} + g_{i,j}(x) \frac{\partial^2 \phi_i}{\partial x_k \partial x_s} \frac{\partial \phi_j}{\partial x_l} + g_{i,j}(x) \frac{\partial \phi_i}{\partial x_k} \frac{\partial^2 \phi_j}{\partial x_l \partial x_s} = \partial_s g_{k,l}(x).$$

Since we have taken a normal coordinate system at x, then

$$g_{i,j}(x) = \delta_{i,j}, \ \partial_k g_{i,j}(x) = 0, \ \forall\ 1 \le i, j, k \le n.$$

Suppose that $j^2_x \phi \in I^2_{x,x}$ and $j^1_x \phi = j^1_x Id$. Then we easily get from the previous two algebraic equations

$$\frac{\partial^2 \phi_l}{\partial x_k \partial x_s} = -\frac{\partial^2 \phi_k}{\partial x_l \partial x_s}, \ \forall\ 1 \le s, k, l \le n.$$

So

$$\frac{\partial^2 \phi_l}{\partial x_k \partial x_s} = -\frac{\partial^2 \phi_k}{\partial x_l \partial x_s} = \frac{\partial^2 \phi_s}{\partial x_l \partial x_k} = -\frac{\partial^2 \phi_l}{\partial x_s \partial x_k}.$$

Thus

$$\frac{\partial^2 \phi_l}{\partial x_k \partial x_s} \equiv 0,$$

i.e. $j^2_x \phi = j^2_x Id$. So π_1^2 is injective and g is rigid.

2.1.2 The organization of the chapter

In Section 2.2 we give a criterion for the completeness of a linear connection with parallel torsion and curvature tensors. Then we deduce a criterion for the homogeneity of such a connection possibly enriched with a parallel geometric structure of order one. This result is fundamental for Chapters 5 and 6.

In Section 2.3 the open-dense theorem of Gromov is reproved. Our observation is that everything becomes easy and clear if we consider only the rigid geometric structures admitting topologically transitive pseudogroups of local isometries, which is the only interesting case as far as applications to this thesis are concerned.

In Section 2.4 we study the possibility of constructing a (G, X)-struture from a locally homogeneous rigid geometric structure. We find a sufficient (and almost necessary) condition to carry out such a construction.

In Section 2.5 we find a notion of completeness for locally homogeneous rigid geometric structures which garantes the completeness of (G, X)-structures constructed in the previous section.

So in principle, the results established in this chapter enable us to transfer the study of rigid geometric structures with many local isometries to that of (G, X)-structures.

2.2 Homogeneity of parallel linear connections

Linear connection is one of the most important rigid geometric structures, whose study is both subtle and fundamental. In this section, we prove some properties about a special (but important) class of linear connections.

Let ∇ be a C^∞ linear connection on a C^∞ manifold M. Then ∇ is a C^∞ rigid geometric structure of order two (see **[CQ]**). We denote respectively by T and R the torsion tensor and the curvature tensor of ∇. Then ∇ is said to be *parallel* if $\nabla R = 0$ and $\nabla T = 0$.

Recall that ∇ is said to be *complete* if all its maximal geodesics are defined on $\mathbb{R}$. We can prove the following

Lemma 2.1 – *Let ∇ be a C^∞ parallel linear connection on a connected manifold M. Let $E_1, \cdots, E_l$ be smooth distributions on M. Then ∇ is complete if the following conditions are satisfied:*
(1) *$TM = E_1 \oplus \cdots \oplus E_l$ and $\nabla E_j \subseteq E_j, \forall\ 1 \leq j \leq l$,*
(2) *For any $1 \leq j \leq l$, the geodesics of ∇ tangent to E_j are defined on $\mathbb{R}$.*

Proof – For the terminology below, our reference is **[KN]**. Denote by $\mathcal{F}(M)$ the frame bundle of M which is naturally identified to $F^1(M)$and by π the projection of $\mathcal{F}(M)$ onto M. The linear connection ∇ gives a horizontal distribution $\mathcal{H}$ on $\mathcal{F}M$ and $\mathcal{F}M$ is foliated by holonomy subbundles.

$\mathcal{H}$ is tangent to each holonomy subbundle and then so is any standard horizontal field. For any $u \in \mathcal{F}M$ we denote by $P(u)$ the holonomy subbundle containing

u. The restrictions to $P(u)$ of the standard horizontal fields of $\mathcal{F}M$ are also called standard horizontal. By **[KN]**, ∇ is complete if and only if for any $x \in M$ there exists $u \in \pi^{-1}(x)$ such that the standard horizontal fields of $P(u)$ are all complete.

Take $x \in M$ and $u \in \pi^{-1}(x)$ such that

$$u = (u_1^1, \cdots, u_{i_1}^1, \cdots, u_1^l, \cdots, u_{i_l}^l),$$

where $\{u_1^j, \cdots, u_{i_j}^j\}$ is a basis of $E_j(x)$, $\forall\ 1 \leq j \leq l$. For all $\xi \in \mathbb{R}^n$, We denote by $B^u(\xi)$ the standard horizontal field on $P(u)$ corresponding to ξ and denote by $(e_1, \cdots, e_n)$ the canonical basis of $\mathbb{R}^n$. Take $v \in P(u)$. Because of Assumption (1), v has the same form as u. Since for any $1 \leq m \leq n$, the integral curve of $B^u(e_m)$, begining at v, is just the horizontal lift, beginning at v, of the geodesic tangent to $Pr_m(v)$, then by Assumption (2), this integral curve is defined on $\mathbb{R}$. We deduce that $B^u(e_m)$ is complete.

Fix a basis of the holonomy algebra of ∇ and denote by $\{V_1, \cdots, V_s\}$ the corresponding vertical fields of $P(u)$. Since ∇ is supposed to be parallel, the fields

$$\{V_1, \cdots, V_s, B^u(e_1), \cdots, B^u(e_n)\}$$

generate a Lie algebra (see **[KN]**). Since these fields are all complete, this Lie algebra must be induced by a smooth action on $P(u)$ of a simply connected Lie group. So for all $\xi \in \mathbb{R}^n$, the field $B^u(\xi)$ $(= \sum_{1 \leq i \leq n} \xi_i B^u(e_i))$ is complete. We deduce that ∇ is complete. $\square$

Let ∇ be a C^∞ linear connection on a n-dimensional manifold M. If Z is a smooth algebraic variety admitting an algebraic action from left of $G^1(n, \mathbb{R})$, then ∇ gives rise to an horizontal distribution on the associated bundle $F^1M \rtimes Z$ denoted by $\mathcal{H}_Z$ (see **[KN]**). A smooth curve γ in $F^1M \rtimes Z$ is said to be *parallel* with respect to ∇ if it is everywhere tangent to $\mathcal{H}_Z$. A C^∞ section σ of the bundle $\pi : F^1M \rtimes Z \to M$, i.e. a C^∞ geometric structure of type Z and order one, is said to be *parallel* if $D\sigma(TM) \subseteq \mathcal{H}_Z$. Then it is easily seen that σ is parallel iff it is parallel along any smooth curve in M, i.e. $\sigma \circ \eta$ is parallel for any smooth curve η in M.

Recall that the isometry group of each C^∞ rigid geometric structure g on M denoted by $I(g)$ is a Lie group acting smoothly on M (see **[Gr1]**). The following lemma is of fundamental importance for us.

Lemma 2.2 – *Let ∇ be a C^∞ complete linear connection on a connected and simply-connected manifold M. Let τ be a C^∞ ∇-parallel geometric structure of order one on M. If ∇ is parallel, then $I(\nabla, \tau)$ acts transitively on M.*

Proof – Here $I(\nabla, \tau)$ denotes the isometry group of the combinated geometric structure (∇, τ). Fix $u_0 \in \mathcal{F}M$ and denote by $P(u_0)$ the holonomy subbundle of u_0 (see **[KN]**). Denote by G the group of affine diffeomorphisms of M preserving $P(u_0)$. Then by the assumptions, G is naturally a Lie group and acts transitively on M (see **[KN]**). So we need only prove that g preserves τ for all $g \in G$.

Take $g \in G$ and suppose that τ is of type F. For all $x \in M$, there exists $u \in P(u_0)$ and $a \in F$ such that $[u,\ a] = \tau(x)$, where $[u,\ a]$ denotes an element in the associated bundle $\mathcal{F}M \rtimes F$. Since $g_*(u) \in P(u_0)$, then there exists a piecewise smooth horizontal curve $u(t)$ in $P(u_0)$ such that $u(0) = u$ and $u(1) = g_*(u)$. Project $u(t)$ to M and denote the resulting curve by γ. Then $[u(t),\ a]$ gives a horizontal lift of γ in $\mathcal{F}^1M \rtimes F$ and $[u(0),\ a] = \tau(x)$. Since τ is ∇-parallel, then $\tau \circ \gamma$ is also a horizontal lift of γ. We deduce that $[u(t),\ a] = (\tau \circ \gamma)(t)$ for all $t \in [0,\ 1]$ by the uniqueness of the horizontal lift beginning at a fixed point. Then we have

$$\begin{aligned} \tau(g(x)) &= \tau(\gamma(1)) = [u(1),\ a] \\ &= [g_*(u),\ a] = g_*[u,\ a] = g_*(\tau(x)). \end{aligned}$$

So $G \subseteq I(\nabla, \tau)$. We deduce that $I(\nabla, \tau)$ acts transitively on M. $\Box$

By combining the previous two lemmas we get the following

Corollary 2.1 – *Let M be a C^∞ connected manifold. Let ∇ be a C^∞ parallel linear connection on M and τ be a C^∞ ∇-parallel geometric structure of order one on M. Then $I(\widetilde{\nabla}, \widetilde{\tau})$ acts transitively on $\widetilde{M}$ if there exist smooth distributions $E_1, \cdots, E_l$ on M such that the following conditions are satisfied:*
(1) *$TM = E_1 \oplus \cdots \oplus E_l$ and $\nabla E_j \subseteq E_j, \forall\ 1 \le j \le l$,*
(2) *For any $1 \le j \le l$, the geodesics of ∇ tangent to E_j are defined on $\mathbb{R}$.*

2.3 Open-dense theorem

In the previous section we have seen that a complete parallel connection is locally homogeneous. In spite of its importance, such a connection is not always the relevant geometric structure arising from applications. So in order to understand various phenomenas from the symmetry viewpoint, we need a guiding theory of a more general nature, of which the departing point should be the open-dense theorem discovered by M. Gromov.

In the following sections we shall propose some notions and prove certain properties to furnish such a (though still quite rough) theory. Let us show firstly by a simple example what the open-dense theorem is about.

Lemma 2.3 – *Let A be a C^∞ complete parallelism on a C^∞ manifold M. If its pseudogroup of local isometries I^{loc} admits a dense orbit, then I^{loc} acts transitively on M.*

Proof – Suppose that M is n-dimensional. Recall that a C^∞ complete parallelism on M is by definition a C^∞ section of $\pi : F^1M \to M$. Thus A is identified to n vector fields $\{X_1, \cdots, X_n\}$ such that for all $x \in M$, $\{X_1(x), \cdots, X_n(x)\}$ are independant. So we get C^∞ functions $\{f^k_{i,j}\}_{1 \le i,j,k \le n}$ such that

$$[X_i, X_j] = \sum_{1 \le k \le n} f^k_{i,j} \cdot X_k.$$

Since I^{loc} admits a dense orbit, then each function $f_{i,j}^k$ is constant on a dense subset of M. So these functions are all constant. Thus $\{X_1, \cdots, X_n\}$ generates a Lie algebra denoted by $\mathfrak{g}$.

Denote by G the simply-connected Lie group with $\mathfrak{g}$ as the right-invariant Lie algebra. Then A is induced by a local G-action on M (see **[Fe1]**). Denote by $\{\bar{X}_1, \cdots, \bar{X}_n\}$ the right-invariant fields of G inducing A.

For any $x \in M$ we define $\alpha_x : G \to M$ such that $\alpha_x(g) = gx$. Then in a neighborhood of the unit $e \in G$, α_x is a C^∞ local diffeomorphism sending $\{\bar{X}_1, \cdots, \bar{X}_n\}$ to $\{X_1, \cdots, X_n\}$. So A is locally homogeneous, i.e. I^{loc} acts transitively on M. $\square$

As mentioned above this kind of open-dense phenomena for general rigid geometric structures was first understood by M. Gromov. Inspired by **[Gr1]**, Y. Benoist reobtained this result (see **[Be]**). But his version is not sufficient for some applications. So in the following we retake his arguments to obtain the following application-oriented proposition which should, though weaker than that of M. Gromov, be enough for most (if not all) applications.

Theorem 2.1 (M. Gromov) – *Let M be a C^∞ manifold and g be a C^∞ rigid geometric structure on M such that its pseudogroup of local isometries I^{loc} admits a dense orbit. Then I^{loc} admits a unique open-dense orbit denoted by Ω.*

In addition for any $r \gg 1$, $I^r \cap D^r\Omega$ is a C^∞ submanifold of D^rM such that for any $x, y \in \Omega$ and any $\eta \in I^r_{x,y}$, there exists a unique germ of local isometry integrating η. Furthermore, this germ of local isometry depends smoothly on η.

Remark 2.1 – Under the notations of Theorem 2.1, we suppose that ϕ and ψ are two local g-isometries both defined on a connected open subset $U \subseteq \Omega$. Define for $r \gg 1$

$$\hat{U} = \{x \in U \mid j_x^r\phi = j_x^r\psi\}.$$

By the unique existence of germs of local isometries integrating isometric jets, $\hat{U}$ is seen to be closed and open. So if $\hat{U} \neq \emptyset$, then $\phi\mid_U = \psi\mid_U$.

For all $\zeta \in I^r \cap D^r\Omega$ we denote by $\bar{\zeta}$ the unique germ of local isometry integrating ζ. The C^∞ dependance of germs of local isometries on r-th order isometric jets has the following meaning:

For any $x, y \in \Omega$ and any $\eta \in I^r_{x,y}$, there exists a connected open neighborhood U of x in M and an open neighborhood V of η in $I^r \cap D^r\Omega$ such that for all $\zeta \in V$, $\bar{\zeta}$ can be defined on U and the evaluation map Θ is C^∞, where $\Theta : U \times V \to M$ such that $\Theta(u, \zeta) = \bar{\zeta}(u)$.

If U is a connected open neighborhood of x and V is an open neighborhood of η such that for all $\zeta \in V$, $\bar{\zeta}$ can be defined on U, then it is easily seen that Θ is also C^∞ on $U \times V$.

Theorem 2.1 will be proved via a few lemmas. Suppose that M is of dimension n and g is a C^∞ rigid geometric struture on M satisfying the conditions of Theorem 2.1. For any $A \subseteq M$ we denote by A° its interior and by A^- its closure. Suppose in addition that g is of type Z and order k and fix $x \in M$ such that $\overline{I^{loc}(x)} = M$.

Lemma 2.4 – *Under the notations above, for any $i \geq 0$, $I^{k+i}(x)^{\circ}$ is open-dense in M.*

Proof – Fix $i \geq 0$ and define $W = G^{k+i}(n,\mathbb{R}) \diagdown J_n^i Z$. By the $G^{k+i}(n,\mathbb{R})$-equivariance of g^i there exists a unique C^0 map $\bar{g}^i : M \to W$ such that the following diagram commutes,

$$\begin{array}{ccc} F^{k+i}M & \xrightarrow{g^i} & J_n^i Z \\ \big\downarrow{\scriptstyle pr} & & \big\downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\bar{g}^i} & W \end{array}$$

Then it is easily seen that for all $y, z \in M$, $I^{k+i}(y) = I^{k+i}(z)$ iff $\bar{g}^i(y) = \bar{g}^i(z)$.

Since the action of $G^{k+i}(n,\mathbb{R})$ on $J_n^i Z$ is algebraic, then by a classical result of M. Rosenlicht (see [**Ro**]) there exists a stratification of $J_n^i Z$ into $G^{k+i}(n,\mathbb{R})$-invariant C^∞ submanifolds

$$J_n^i Z = Z_0 \cup \cdots \cup Z_l$$

such that for any $0 \leq j \leq l$, Z_j is open in $Z_0 \cup \cdots \cup Z_j$ and $\pi : Z_j \to G^{k+i}(n,\mathbb{R}) \diagdown Z_j$ is a C^∞ fiber bundle.

Let j be the biggest number such that $g^i(F^{k+i}M) \cap Z_j \neq \emptyset$. Thus $(g^i)^{-1}(Z_j)$ is open in $F^{k+i}M$. Denote by ρ_i the maximal rank of g^i on $(g^i)^{-1}Z_j$ and define

$$\mathcal{S} = \{\eta \in F^{k+i}M \mid g^i(\eta) \in Z_j,\ rank_\eta(g^i) = \rho_i\}.$$

So $\mathcal{S}$ is open in $(g^i)^{-1}Z_j$. We deduce that $V = pr(\mathcal{S})$ is open in M. Again by the $G^{k+i}(n,\mathbb{R})$-invariance of g^i we get $\mathcal{S} = F^{k+i}V$. Define $W_j = G^{k+i}(n,\mathbb{R}) \diagdown Z_j$. Then W_j is a C^∞ manifold and we have the following commutative diagram,

$$\begin{array}{ccc} F^{k+i}V & \xrightarrow{g^i} & Z_j \\ \big\downarrow{\scriptstyle pr} & & \big\downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\bar{g}^i} & W_j \end{array}$$

Note that $\bar{g}^i \mid_V$ is C^∞. Suppose that the fibers of $\pi : Z_j \to W_j$ is of dimension d_i. Then by the definition of $\mathcal{S}$, it is easily seen that $\bar{g}^i \mid_V$ is of constant rank $(\rho_i - d_i)$.

Take $y \in V \cap I^{k+i}(x)$ and define $w = \bar{g}^i(y)$. Since $\bar{g}^i \mid_V$ is of constant rank, then $(\bar{g}^i \mid_V)^{-1}(w)$ is a closed submanifold of V. However

$$(\bar{g}^i \mid_V)^{-1}(w) = (\bar{g}^i)^{-1}(w) \cap V = I^{k+i}(x) \cap V.$$

So $I^{k+i}(x) \cap V$ is a C^∞ closed submanifold of V. Since $\overline{I^{loc}(x)} = M$, then $\overline{I^{k+i}(x)} = M$. So $I^{k+i}(x) \cap V$ is dense in V. We deduce that

$$V \subseteq I^{k+i}(x).$$

Since in addition $I^{loc}(x) \subseteq I^{k+i}(x)$ and $I^{loc}(x)^{-} = M$, then $I^{k+i}(x)$ contains an open-dense subset of M. So $I^{k+i}(x)^{\circ}$ is open-dense in M. $\square$

Denote $I^{k+i}(x)^{\circ}$ by U_{k+i}. Since for all $j \geq i \geq 0$ we have $I^{k+j}(x) \subseteq I^{k+i}(x)$, then $U_{k+j} \subseteq U_{k+i}$. Under the notations above we have $\bar{g}^i(U_{k+i}) \equiv w$. We deduce that $g^i \mid_{F^{k+i}U_{k+i}}$ is of constant rank d_i.

Fix $i \geq 0$ and denote $(k+i)$ by r. Then we get the commutative diagram,

$$\begin{array}{ccc} F^rU_r & \xrightarrow{g^i} & \pi^{-1}(w) \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle \pi} \\ U_r & \xrightarrow{\bar{g}^i} & \{w\} \end{array}$$

Define $q_r : I^r \to M \times M$ such that $q_r(j^r_x\phi) = (x, \phi(x))$. Then we have the following

Lemma 2.5 – *$I^r \cap (D^rU_r)$ is a C^∞ submanifold of constant dimension of D^rU_r and $q_r : I^r \cap D^rU_r \to U_r \times U_r$ is a C^∞ surjective submersion.*

Proof – Suppose that $j^r_x\phi \in I^r \cap (D^rU_r)$ and $\phi(x) = y$. Take a coordinate chart θ of an open neighborhood V of x. Then θ gives canonically a C^∞ local section s of $pr : F^rV \to V$. Take a small open neighborhood V' of y. Then using s, $D^r(V, V')$ is naturally identified with $V \times F^rV'$ in such a way that

$$I^r \cap D^r(V, V') = \{(z, h') \mid (z, h') \in V \times F^rV',\ g^i(s(z)) = g^i(h')\}.$$

Since $g^i \mid_{F^rU_r}$ is of constant rank d_i, then $I^r \cap D^r(V, V')$ is a C^∞ submanifold of codimension d_i of $D^r(V, V')$. So $I^r \cap (D^rU_r)$ is a C^∞ submanifold of codimension d_i of D^rU_r.

Since by definition $U_r = I^r(x)^{\circ}$, then $q_r(I^r \cap (D^rU_r)) = U_r \times U_r$. We want to see that $q_r \mid_{I^r \cap (D^rU_r)}$ is a submersion.

Define $m_r : F^rM \times F^rM \to D^rM$ such that

$$m_r(j^r_0\phi, j^r_0\psi) = j^r_{\phi(0)}(\psi \circ \phi^{-1}).$$

Then m_r is naturally a $G^r(n, \mathbb{R})$-principal bundle. We define

$$L^r = \{(\eta, \zeta) \mid (\eta, \zeta) \in F^rM \times F^rM,\ g^i(\eta) = g^i(\zeta)\}.$$

Then it is easily seen that $L^r = m_r^{-1}(I^r)$. In particular,

$$L^r \cap (F^rU_r \times F^rU_r) = m_r^{-1}(I^r \cup (D^rU_r)).$$

By the definition of L^r, we can see that $L^r \cap (F^rU_r \times F^rU_r)$ is a C^∞ submanifold of $F^rU_r \times F^rU_r$.

Denote by (pr, pr) the projection of $L^r \cap (F^rU_r \times F^rU_r)$ to $U_r \times U_r$. Then we have

$$q_r \circ m_r = (pr, pr).$$

Since m_r is a submersion, then in order to prove that q_r is a submersion, we need only see that (pr, pr) is a submersion.

For any $(\eta, \zeta) \in L^r \cap (F^r U_r \times F^r U_r)$, we define

$$L^r_\eta = \{\zeta' \in F^r U_r \mid g^i(\eta) = g^i(\zeta')\},\ L^r_\zeta = \{\eta' \in F^r U_r \mid g^i(\eta') = g^i(\zeta)\}.$$

Then L^r_η and L^r_ζ are both C^∞ submanifolds of $F^r U_r$. Denote respectively by pr_2 and pr_1 the projections of L^r_η and L^r_ζ onto U_r. Then $D_{(\eta,\zeta)}(pr, pr)$ is surjective if $D_\zeta pr_2$ and $D_\eta pr_1$ are both surjective.

Suppose that $v = \zeta(0)$ and $z = g^i(\zeta)$. Then we get a similar diagram as that in Subsection 2.3 of [**Be**]. Thus by a simple diagram chasing, we can see that $D_\zeta pr_2$ is surjective. Similarly $D_\eta pr_1$ is also surjective. So q_r is a submersion. □

Remark 2.2 – For any $x \in U_r$, we define

$$I^r_{x,\cdot} = q_r^{-1}(\{x\} \times U_r),\ I^r_{\cdot,x} = q_r^{-1}(U_r \times \{x\}).$$

Then by Lemma 2.5, $I^r_{x,\cdot} \cap F^r U_r$ and $I^r_{\cdot,x} \cap F^r U_r$ are both C^∞ submanifolds of $D^r U_r$ and their projections onto U_r are both C^∞ surjective submersions.

Lemma 2.6 – *For any $r \gg 1$, $\pi^{r+1}_r : I^{r+1} \cap (D^{r+1} U_{r+1}) \to I^r \cap (D^r U_r)$ is a C^∞ local diffeomorphism.*

Proof – For any $y, z \in U_r$, $I^r_{y,z} = q_r^{-1}(y \times z)$ is a C^∞ submanifold of $D^r U_r$. Since g is rigid, then for all $x \in M$, $\pi^{r+1}_r \mid_{I^{r+1}_{x,x}}$ is an injective homomorphism. In particular $\pi^{r+1}_r \mid_{I^{r+1}_{x,x}}$ is an immersion. We deduce that $\pi^{r+1}_r : I^{r+1}_{y,z} \to I^r_{y,z}$ is an immersion. Since $q_{r+1} \mid_{I^{r+1} \cap (D^{r+1} U_{r+1})}$ is a submersion by the previous lemma, then it is easy to see that $\pi^{r+1}_r \mid_{I^{r+1} \cap (D^{r+1} U_{r+1})}$ is an immersion. So in order to prove the lemma, we need only see that $I^r \cap (D^r U_r)$ is of constant dimension for $r \gg 1$.

Since $dim(I^r_{x,x})$ is decreasing with r, then for all $r \gg 1$, $dim(I^r_{x,x})$ is constant. Since we have by the previous lemma

$$dim(I^r \cap (D^r U_r)) = dim(I^r_{x,x}) + dim(M \times M),$$

then $dim(I^r \cap (D^r U_r))$ is constant for all $r \gg 1$. Thus the lemma follows. □

Now we can finish the proof of Theorem 2.1 as in [**Be**]. Fix $r \gg 1$ and suppose that $\eta \in I^r \cap D^r U_r$. Take a small open neighborhood V_r of η in $I^r \cap (D^r U_r)$ such that if $V_{r-1} = \pi^r_{r-1}(V_r)$, then $\pi^r_{r-1} \mid_{V_r}$ and $\pi^{r-1}_{r-2} \mid_{V_{r-1}}$ are C^∞ diffeomorphisms onto their images.

View V_{r-1} as a C^∞ differential relation. Then its holonomy solutions correspond to C^∞ local isometries of g. By the same arguments as in Subsection 3.2 of [**Be**], g is seen to be C^∞-*complete* and *consistent* (see [**Be**] for the definitions). Then by the Frobenius theorem (see [**Be**]), there exists a unique germ of solution of V_{r-1} passing by $\pi^r_{r-1}(\eta)$, i.e. there exists a unique germ of C^∞ g-local isometry ϕ integrating

$\pi^r_{r-1}(\eta)$. Since g is rigid, then ϕ integrates also η. The C^∞ dependance on r-th order isometric jets of germs of g-local isometries follows from the proof of the Frobenius theorem (see Subsection 3.2 of [**Be**]).

So each element of $I^r \cap (D^r U_r)$ can be integrated to a local isometry of g. Thus $U_r \subseteq I^{loc}(x)$. Inversely we have $I^{loc}(x) \subseteq U_r$ (see the proof of Lemma 2.4). So we get

$$U_r = I^{loc}(x).$$

In particular $I^{loc}(x)$ is open-dense in M. Since two open-dense orbits must intersect, then $I^{loc}(x)$ is the unique open-dense orbit of I^{loc}. So Theorem 2.1 is proved. We can easily deduce from this proposition the following

Corollary 2.2 – *Under the conditions of Theorem* 2.1*, we have for all* $r \gg 1$*,* $I^{loc}_{x,x}$ *is isomorphic to* $I^r_{x,x}$ *for any* $x \in \Omega$*. In particular* $I^{loc}_{x,x}$ *is naturally a real algebraic group for any* $x \in \Omega$*.*

2.4 Normality and (G, X)-structures.

Throughout this section we denote by g a C^∞ rigid geometric structure on M and suppose that its pseudogroup of local isometries I^{loc} admits a dense orbit. Then by Theorem 2.1, I^{loc} admits a unique open-dense orbit denoted by Ω. In this section we want to construct a (G, X)-structure on Ω. To simplify certain notations, we consider in the following the right-invariant Lie algebras of Lie groups in the place of their left-invariant Lie algebras, i.e. the Lie algebras of left-invariant vector fields.

Let Y be a C^∞ vector field on M. Then Y is said to be a *Killing field* of g if the local flow of Y preserves g. Fix $x \in \Omega$ and denote by $\mathfrak{g}$ the space of germs at x of C^∞ local Killing fields of g. Denote by $\mathfrak{h}$ the subset of $\mathfrak{g}$ consisting of elements vanishing at x.

Lemma 2.7 – *Under the notations above,* $\mathfrak{g}$ *is a Lie algebra of finite dimension and* $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$*. In addition there exists* $r \gg 1$ *such that* $Y = Z$ *in* $\mathfrak{g}$ *iff* $j^r_x Y = j^r_x Z$*.*

Proof – For two germs of local Killing fields Y and Z we define their bracket as the germ at x of the local field $[Y, Z]$. Define their sum as the germ at x of the local field $(Y+Z)$ and for any $a \in \mathbb{R}$, we define $a \cdot Y$ as the germ at x of the local field $a \cdot Y$. We want to see that $\mathfrak{g}$ forms a Lie algebra with respect to these operations.

Suppose that g is of type Z and order k and view g as a map from $F^k M$ into Z. For each vector field X on M, we can lift it naturally to a vector field on $F^k M$, denoted by $X_{(k)}$ (see [**GS**]). Then X is Killing iff $Dg(X_{(k)}) \equiv 0$. Thus we get easily $Y + Z \in \mathfrak{g}$ and $a \cdot Y \in \mathfrak{g}$. Since

$$[Y_{(k)}, Z_{(k)}] = [Y, Z]_{(k)},$$

then $[Y, Z]$ is also contained in $\mathfrak{g}$. So $\mathfrak{g}$ forms a Lie algebra and $\mathfrak{h}$ is easily seen to be a Lie subalgebra of $\mathfrak{g}$.

Fix $r \gg 1$ such that Theorem 2.1 is valid. Suppose that $Y \in \mathfrak{h}$ and $j_x^r Y = 0$. Since we have naturally (see **[CQ]**)

$$exp(t \cdot j_x^r Y) = j_x^r \phi_t^Y, \ \forall \ | t | \ll 1,$$

then $j_x^r \phi_t^Y \equiv j_x^r Id$ for all $| t | \ll 1$, where ϕ_t^Y denotes the local flow of Y. So by Theorem 2.1, there exists a small open neighborhood U of x such that ϕ_t^Y can be defined on U and $\phi_t^Y |_U = Id |_U$ for all $| t | \ll 1$. So $Y = 0$ in $\mathfrak{h}$. Thus for $Y, Z \in \mathfrak{h}$, $Y = Z$ iff $j_x^r Y = j_x^r Z$. We deduce that $\mathfrak{h}$ is of finite dimension. Then so is $\mathfrak{g}$. $\square$

Remark 2.3 – Suppose that U is a connected open subset of Ω. Let Y and Z be two C^∞ Killing fields of g defined on U. If there exists $y \in U$ such that $j_y^r Y = j_y^r Z$, then by Lemma 2.7, $Y = Z$ on U.

Suppose that $\bar{G}$ is the connected and simply-connected Lie group with $\mathfrak{g}$ as its right-invariant Lie algebra. Denote by $\bar{H}$ the connected Lie subgroup of $\bar{G}$ integrating $\mathfrak{h}$.

Definition 2.2 – Under the notations above, g is said to be *normal* if $\bar{H}$ is closed in $\bar{G}$.

Since $g |_\Omega$ is locally homogeneous, then the normality of g is independant of the base point x chosen in Ω. If $\hat{G}$ denotes the connected and simply-connected Lie group with $\mathfrak{g}$ as its left-invariant Lie algebra and $\hat{H}$ denotes the connected Lie subgroup of $\hat{G}$ integrating $\mathfrak{h}$, then it is easily seen that g is normal iff $\hat{H}$ is closed in $\hat{G}$.

Lemma 2.8 – *Under the notations above, if the center of* $\mathfrak{g}$ *is trivial, then* g *is normal.*

Proof – Recall that by Corollary 2.2, $I_{x,x}^{loc}$ is a real algebraic group. Define $\rho : I_{x,x}^{loc} \to Aut(\mathfrak{g})$ such that $\rho(h)(Y) = Dh(Y)$. Since ρ is a homomorphism of real algebraic groups, then $\rho(I_{x,x}^{loc})$ is a closed subgroup of $Aut(\mathfrak{g})$ (see **[OV]** and **[Bo]**).

By the local action of $I_{x,x}^{loc}$ on M, each right-invariant vector field of $I_{x,x}^{loc}$ induces naturally a germ of local Killing field vanishing at x (see Remark 2.1). In this way, we can identify the right-invariant Lie algebra of $I_{x,x}^{loc}$ with $\mathfrak{h}$.

On the other hand, we have the adjoint representation $Ad : \bar{G} \to Aut(\mathfrak{g})$. It is easily seen that $Ad(\bar{H})$ and $\rho(I_{x,x}^{loc})$ have the same Lie algebra in $Aut(\mathfrak{g})$. So

$$Ad(\bar{H}) = (\rho(I_{x,x}^{loc}))_0.$$

Since the center of $\mathfrak{g}$ is trivial, then $\bar{H} = (Ad^{-1}((\rho(I_{x,x}^{loc}))_0)_0$. So $\bar{H}$ is closed in $\bar{G}$, i.e. g is normal. $\square$

From now on we suppose that g is in addition normal. Then we get a C^∞ manifold $\bar{G}/\bar{H}$. In the following we want to contruct on $\bar{G}/\bar{H}$ a $\bar{G}$-invariant geometric structure $\bar{g}$ which is locally isometric to $g\mid_\Omega$.

Fix $r \gg 1$ such that Theorem 2.1 is valid. Then by Remak 2.2, $I^r_{x,\cdot} \cap F^r\Omega$ is a C^∞ submanifold of F^rM and $pr : I^r_{x,\cdot} \cap F^r\Omega \to \Omega$ is a C^∞ surjective submersion. Since $I^r_{x,x}$ acts freely and properly and transitively on the fibers of pr by right composition, then $pr : I^r_{x,\cdot} \cap F^r\Omega \to \Omega$ is a $I^r_{x,x}$-principal bundle. So we get the C^∞ diffeomorphism

$$\bar{pr} : (I^r_{x,\cdot} \cap F^r\Omega)/I^r_{x,x} \xrightarrow{\sim} \Omega.$$

For any $\eta \in I^r_{x,\cdot} \cap F^r\Omega$, we denote by $\bar{\eta}$ the unique germ of local isometry inducing η.

Take an open neighborhood V_r of $j^r_x(Id)$ in $I^r_{x,\cdot} \cap F^r\Omega$. If V_r is small enough, then the following operation is well-defined and C^∞. For any $\eta, \zeta \in V_r$,

$$\eta \cdot \zeta = j^r_x(\bar{\eta} \circ \bar{\zeta}).$$

We call this operation the multiplication of V_r. Recall that $\bar{\eta}$ and $\bar{\zeta}$ denote respectively the local isometries integrating η and ζ of g. Certainly $\eta \cdot \zeta$ is not necessarily contained in V_r. In the following, V_r is supposed to be small enough such that all the expressions below make sense.

Denote by $V_r/I^r_{x,x}$ the image of V_r in $(I^r_{x,\cdot} \cap F^r\Omega)/I^r_{x,x}$. Then $V_r/I^r_{x,x}$ is open and $\bar{pr} : V_r/I^r_{x,x} \xrightarrow{\sim} \bar{pr}(V_r/I^r_{x,x}) = V'_r \subseteq M$.

Denote by $T_e(V_r)$ the tangent space of V_r at $e = j^r_x(Id)$. Then it can be identified to local right-invariant fields on V_r as following:

For any $u \in T_e(V_r)$ and any $\beta \in V_r$ we define a local vector field $\bar{u}$ on V_r such that

$$\bar{u}(\beta) = DR_\beta(u),$$

where R_β denotes the right-multiplication by β. Since the multiplication of V_r is associative, then $\bar{u}$ is right-invariant on V_r. So for any $\alpha \in V_r$ and all $\mid t \mid \ll 1$, we have

$$\phi^{\bar{u}}_t(\alpha) = \phi^{\bar{u}}_t(e) \cdot \alpha.$$

In particular, for small munbers t and s

$$\phi^{\bar{u}}_{t+s}(e) = \phi^{\bar{u}}_t(e) \cdot \phi^{\bar{u}}_s(e).$$

Thus $\{\phi^{\bar{u}}_t(e)\}_{|t|\ll 1}$ gives a local homomorphism of $\mathbb{R}$ into V_r.

Conversely each local right-invariant vector field Won V_r is determined by $W(e)$, i.e. there exists $u \in T_e(V_r)$ such that $W = \bar{u}$. So for any $u, v \in T_e(V_r)$, we can define their bracket $[u, v]$ such that

$$\overline{[u, v]} = [\bar{u}, \bar{v}].$$

In this way $T_e(V_r)$ becomes a Lie algebra. Denote by $\bar{V}$ an open neighborhood of zero in $T_e(V_r)$. If $\bar{V}$ is small enough, then the following map is well-defined and C^∞,

$$exp : \bar{V} \longrightarrow V_r,$$

$$u \to \phi_1^{\bar{u}}(e).$$

In addition for t and s small, we have $exp(tu) \cdot exp(su) = exp((t+s)u)$. It is easy to see that $D_e exp = Id$. So exp is a C^∞ local diffeomorphism near zero. We define

$$Ad : V_r \longrightarrow Aut(T_e(V_r))$$

$$\alpha \to (u \to \frac{\partial}{\partial t} |_{t=0} (\alpha^{-1} \cdot exp(tu) \cdot \alpha)).$$

Then it is easily seen that Ad is a well-defined local homomorphism. Recall that we are using throughout the right-invariant Lie algebras. Define also $ad : T_e(V_r) \to Der(T_e(V_r))$ such that $ad(u)(v) = [u, v]$. Then it is easily checked that $D_e Ad = ad$.

By using these notions and checking through the proof of Theorem 4.29 of chapter one of [**KMS**], we can see that the classical Campbell-Baker-Hausdorff theorem is also valid in our situation, i.e. for any $u, v \in \bar{V} \subseteq T_e(V_r)$ such that $\bar{V}$ is very small, we have

$$exp(u) \cdot exp(v) = exp(u + v - \frac{1}{2}[u, v] + \cdots).$$

The formula is the same as that for Lie groups. Recall that we are using throughout the right-invariant Lie algebras in the place of left-invariant Lie algebras.

Define

$$\rho : T_e(V_r) \longrightarrow \mathfrak{g}$$

$$u \longrightarrow (y \to \frac{\partial}{\partial t} |_{t=0} (\phi_t^{\bar{u}}(e))(y)),$$

where $\phi_t^{\bar{u}}(e)$ denotes the germ extension of the local one-parameter subgroup $\phi_t^{\bar{u}}(e)$ of V_r. Then ρ is easily seen to be a Lie algebra isomorphism by checking through the proof of Proposition 4.1 of chapter one of [**KN**]. Thus we get a local diffeomorphism ϕ sending $\bar{e}$ to e such that the following diagram commutes (when defined)

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho^{-1}} & T_e(V_r) \\ \downarrow{\scriptstyle exp} & & \downarrow{\scriptstyle exp} \\ \bar{G} & \xrightarrow{\phi} & V_r \end{array}$$

Because of the Campbell-Baker-Hausdorff formula, we have for any a and b near $\bar{e}$ in $\bar{G}$, $\phi(a) \cdot \phi(b) = \phi(ab)$. In addition, ϕ sends an open neighborhood of $\bar{e}$ in $\bar{H}$ onto an open neighborhood of e in $I^r_{x,x}$. Denote by π the projection of $\bar{G}$ onto $\bar{G}/\bar{H}$ and take a small neighborhood O of $\bar{e}$ in $\bar{G}$. Then we get a well-defined C^∞ local diffeomorphism $\bar{\phi} : \pi(O) \to V_r / I^r_{x,x}$ by defining for $g \in O$,

$$\bar{\phi}(g\bar{H}) = \phi(g) I^r_{x,x}.$$

Define $\theta = (\bar{pr} \circ \bar{\phi})^{-1}$. Then θ is a C^∞ local diffeomorpism from M to $\bar{G}/\bar{H}$ sending x to $\bar{e}\bar{H}$. For each small Y in $\mathfrak{g}$, it is easy to verify by the definition of θ that $\theta(\phi_1^Y(x)) = exp(Y)\bar{H}$.

Define $\hat{g} = \theta_* g$. Then $\hat{g}$ is a local C^∞ geometric structure on $\bar{G}/\bar{H}$.

Lemma 2.9 – *Under the notations above, the germ of $\hat{g}$ at $\bar{e}\bar{H}$ can be uniquely extended to a $\bar{G}$-invariant C^∞ geometric structure $\bar{g}$ on $\bar{G}/\bar{H}$, which is locally isomorphic to $g\,|_\Omega$.*

Proof – By the definition of θ and $\hat{g}$ we can see that for any a near $\bar{e}$, the left multiplication by a sends the germ of $\hat{g}$ at $\bar{e}\bar{H}$ to that of $\hat{g}$ at $a\bar{H}$. In particular, we deduce that each element of $\bar{H}$ preserves the germ of $\hat{g}$ at $\bar{e}\hat{H}$.

So we can find a small open neighborhood $\bar{U}$ of $\bar{e}\bar{H}$ such that $\hat{g}$ can be defined on $\bar{U}$ and for any $y \in \bar{U}$ there exists $a \in \bar{G}$ such that the left multiplication by a sends the germ of $\hat{g}$ at $\bar{e}\bar{H}$ to that of $\bar{g}$ at y.

For any $b\bar{H} \in \bar{G}/\bar{H}$, we define $\bar{g}$ such that its germ at $b\bar{H}$ is $b_*\hat{g}$. We need see that $\bar{g}$ is well-defined. If $y, z \in \bar{U}$ and $b, c \in \bar{G}$ such that $b(y) = c(z)$, then by the definition of $\bar{U}$, there exist $\alpha, \beta \in \bar{G}$ such that they send the germ of $\hat{g}$ at $\bar{e}\bar{H}$ to the germs of $\hat{g}$ at y and z. Thus $\beta^{-1} \cdot c^{-1} \cdot b \cdot \alpha \in \bar{H}$. Since each element of $\bar{H}$ preserves the germ of $\hat{g}$ at $\bar{e}\bar{H}$, then $b_*\hat{g}$ and $c_*\hat{g}$ have the same germ at $b(y)$. So $\bar{g}$ is well-defined on $\bar{G}/\bar{H}$. Then by its definition $\bar{g}$ is $\bar{G}$-invariant and locally isomorphic to $g\,|_\Omega$. □

Recall that the isometry group of $\bar{g}$ is denoted by $I(\bar{g})$. It is a Lie group acting analytically and transitively on $\bar{G}/\bar{H}$ (see **[Gr1]**).

Lemma 2.10 – *Under the notations above, each C^∞ local $\bar{g}$-isometry defined on a connected open subset can be uniquely extended to a C^∞ global isometry of $\bar{g}$.*

Proof – For any $Y \in \mathfrak{g}$ we denote by Y^R the right-invariant vector field on $\bar{G}$ corresponding to Y. Denote by $\bar{Y}^R$ its quotient field on $\bar{G}/\bar{H}$. Then by the definition of θ, it is easily seen that $\theta_*(Y) = \bar{Y}^R$ as germs of fields at $\bar{e}\bar{H}$.

Since $pr : I^r_{x,\cdot} \cap (F^r\Omega) \to \Omega$ is a surjective submersion, then for any $u \in T_x\Omega$ there exists $u' \in T(I^r_{x,\cdot} \cap (F^r\Omega))$ such that

$$D(pr)(u') = u.$$

Since u' can be integrated to a local 1-parameter subgroup of V_r, then u is tangent to a local Killing field. We deduce that

$$\mathfrak{g}\,|_x = \{Z(x) \mid Z \in \mathfrak{g}\} = T_xM.$$

So we have $(\theta_*(\mathfrak{g}))\,|_{\bar{e}\bar{H}} = T_{\bar{e}\bar{H}}(\bar{G}/\bar{H})$. In addition, $\theta_*(\mathfrak{g})$ is the Lie algebra of germs at $\bar{e}\bar{H}$ of C^∞ local Killing fileds of $\bar{g}$.

Take $h \in I^{loc}_{\bar{e}\bar{H},\bar{e}\bar{H}}(\bar{g})$. Then we get a Lie algebra isomorphism $h_* : \theta_*(\mathfrak{g}) \to \theta_*(\mathfrak{g})$ such that $h_*(W) = Dh(W)$. Thus there exists a Lie algebra isomorphism A of $\mathfrak{g}$ such that $\theta_* \circ A = h_* \circ \theta_*$. We have $A(\mathfrak{h}) = \mathfrak{h}$.

Denote by ψ the Lie group automorphism of $\bar{G}$ integrating A. Then we have $\psi(\bar{H}) = \bar{H}$. Denote by π the projection of $\bar{G}$ onto $\bar{G}/\bar{H}$. then there exists a unique C^∞ diffeomorphism $\bar{\psi}$ of $\bar{G}/\bar{H}$ such that $\pi \circ \psi = \bar{\psi} \circ \pi$.

Recall that for any $Y \in \mathfrak{g}$, $\theta_*(Y) = \bar{Y}^R = \pi_*(Y^R)$ as germs of fields at $\bar{e}\bar{H}$. Then we have the following equalities of germs at $\bar{e}\bar{H}$

$$\bar{\psi}_*(\theta_*(Y)) = \bar{\psi}_*(\pi_*(Y^R)) = \pi_* \circ \psi_*(Y^R) = \pi_*((D_{\bar{e}}\psi(Y))^R)$$

$$= \pi_*(A(Y)^R) = \theta_*(A(Y)) = h_*(\theta_*(Y)).$$

Since $\mathfrak{g}$ is of finite dimension, then we can find a small open neighborhood U of x such that for any $Y \in \mathfrak{g}$, Y can be defined on U and $\bar{\psi}(\theta_*(Y)) = h_*(\theta_*(Y))$ on U. Since $(\theta_*(\mathfrak{g})) \mid_{\bar{e}\bar{H}} = T_{\bar{e}\bar{H}}(\bar{G}/\bar{H})$ and $\theta_*(Y) = \bar{Y}^R$, then we can well choose U so that for all $y \in U$ there exists $Z \in \mathfrak{g}$ such that

$$\phi^{\theta_* Z}_{[0,1]}(\bar{e}\bar{H}) \subseteq \theta(U),\ \phi^{\theta_* Z}_1(\bar{e}\bar{H}) = \theta(y),$$

where $\phi^{\theta_* Z}_t$ denotes the flow of $\theta_* Z$. Since $\bar{\psi}$ and h fix both $\bar{e}\bar{H}$, then $\bar{\psi}(\theta(y)) = h(\theta(y))$. So we get $\bar{\psi} = h$ as germs of diffeomorphisms at $\bar{e}\bar{H}$. Since $\bar{g}$ is $\bar{G}$-invariant and $\bar{\psi}$ is induced by an automorphism of $\bar{G}$, then $\bar{\psi}$ is in fact a global isometry of $\bar{g}$.

Since $\bar{G}$ acts transitively by left-multiplication on $\bar{G}/\bar{H}$, then for each local C^∞ isometry h' of $\bar{g}$ sending y to z, there exists a global C^∞ isometry ψ' of $\bar{g}$ such that $h' = \psi'$ as germs at y of diffeomorphisms.

Suppose that f is a local isometry defined on a connected open subset U of M. For each global $\bar{g}$-isometry ϕ, we define $U_\phi = \{z \in U \mid\ \phi = f \text{ as germs at } z\}$. Then we have

$$U = \cup_{\phi \in I(\bar{g})} U_\phi.$$

In addition, $U_{\phi_1} \cap U_{\phi_2} = \emptyset$ if $\phi_1 \neq \phi_2$. Since U is connected, then there exists a unique global isometry ϕ of $\bar{g}$ such that $\phi \mid_U = f$. □

By combining the lemmas above we get the following

Proposition 2.1 – *Let g be a C^∞ rigid geometric structure. If its pseudogroup of local isometries admits a dense orbit, then it admits a unique open-dense orbit denoted by Ω. If g is in addition normal and $\bar{G}$ and $\bar{H}$ are defined as above, then there exists on $\bar{G}/\bar{H}$ a $\bar{G}$-invariant geometric structure $\bar{g}$ which is locally isometric to $g \mid_\Omega$. In addition by taking the local isometries from $(\Omega, g \mid_\Omega)$ into $(\bar{G}/\bar{H}, \bar{g})$ as charts, we get on Ω a $(I(\bar{g}), \bar{G}/\bar{H})$-structure.*

Lemma 2.11 – *Let g be a real analytic rigid geometric structure on a real analytic manifold M. Suppose that its pseudogroup of local isometries I^{loc} admits a dense orbit. Then I^{loc} has a unique open-dense orbit denoted by Ω. If U is an open subset of Ω, then each C^∞ g-Killing field defined on U is real analytic.*

Proof – Denote by X such a Killing field. By the argument above, for $x \in U$, the germ of X at x is induced by a vector $u \in T_e(V_r)$. Since g and M are both real analytic, then V_r is real analytic and $\bar{u}$ is a real analytic field. So $\{\phi_t^{\bar{u}}(e)\}_{|t|<<1}$ gives a local real analytic homomorphism of $\mathbb{R}$ into V_r. Since the evaluation map Θ (see Remark 2.1) is certainly real analytic, then the germ of X at x is induced by a real analytic local flow. So X is also real analytic. □

Remark 2.4 – Lemma 2.11 is certainly wrong for nonrigid geometric stuctures. For example, $(\frac{\partial}{\partial x},\ \mathbb{R}^2)$ gives a counter-example.

Let g be a C^∞ rigid geometric structure on a connected and simply connected manifold M. If its isometry group $I(g)$ acts transitively on M, then M and g are naturally real analytic. By **[DG]**, $\mathfrak{g}$ is identified to the Lie algebra of real analytic global g-Killing fields, which is by the previous lemma just the Lie algebra of C^∞ global g-Killing fields. If we suppose in addition that each C^∞ global Killing field of g is complete, then $\mathfrak{g}$ is isomorphic to the Lie algebra of $I(g)$ and $\mathfrak{h}$ is isomorphic to the Lie algebra of the isotropy subgroup in $I(g)$ of x. Then we easily deduce that g is normal in this case.

2.5 About completeness

2.5.1 Geodesic structures and associated notion of completeness

Classically, geometry has two points of view, that of Riemann and that of Klein. Rigid geometric structures are natural generalizations of Riemannian metrics and (G, X)-structures are generalizations of homogeneous geometries. Since non-complete (G, X)-structures have little use in the context of our thesis, then we want to find a notion of completeness for rigid geometric structures to garantee the completeness of the (G, X)-structures constructed in Proposition 2.1 from a given locally homogeneous rigid geometric structure.

Recall that for each (G, X)-structure on M there exists a developing map $\mathcal{D} : \widetilde{M} \to X$, which is a local diffeomorphism (see **[Th]**). Then a (G, X)-structure is said to be *complete* if its developing map is a surjective covering map onto X. Our objective is to generalize the following classical proposition (see **[Th]**)

Proposition 2.2 – *Let M be a connected manifold with a (G, X)-structure. Suppose that G preserves a C^∞ Riemannian metric $\bar{g}$ on X. If the canonically associated Riemannian metric g on M is complete, then the (G, X)-structure on M is also complete.*

Let us first generalize it to linear connections.

Lemma 2.12 – *Let ∇_1 and ∇_2 be two C^∞ linear connections on two connected manifolds M_1 and M_2. Suppose that f is a C^∞ local diffeomorphism from M_1 to M_2*

sending ∇_1 to ∇_2. If ∇_1 is complete, then f is a surjective covering map onto M_2 and ∇_2 is also complete.

Proof – Since ∇_1 is complete, then we can lift each piecewise smooth curve consisting of pieces of ∇_2-geodesics. Since any two points of M_2 can be related by such a curve, then f is surjective and ∇_2 is complete.

For all $x \in M_2$ we fix an open neighborhood U of x such that U is a normal neighborhood of each of its point (see [**He**]). Then $f^{-1}(U)$ is decomposed as the disjoint union of its connected components

$$f^{-1}(U) = \cup\, U_i.$$

By lifting ∇_2-geodesics, it is easy to see that $f(U_i) = U$ for each connected component U_i. So in order to prove that $f\,|_{U_i}\colon U_i \to U$ is a C^∞ diffeomorphism, we need only see the injectivity of $f\,|_{U_i}$.

Take $y, z \in U_i$ such that $f(y) = f(z)$. Take a C^∞ curve γ in U_i such that

$$\gamma(0) = y,\ \gamma(1) = z.$$

Define $\bar{\gamma} = f \circ \gamma$ and $\bar{y} = \bar{\gamma}(0)$. Since U is a normal neighborhood of $\bar{y}$ and ∇_1 is complete, then the following curve is well-defined and smooth

$$\hat{\gamma}(t) = exp_y^{\nabla_1}\{(D_y f)^{-1}[(exp_{\bar{y}}^{\nabla_2})^{-1}(\bar{\gamma}(t))]\}.$$

Define $\Lambda = \{t \in [0,1] \mid \hat{\gamma}(t) = \gamma(t)\}$. Then Λ is closed in $[0,1]$. Since f is supposed to be a local diffeomorphism and

$$f \circ \hat{\gamma} = \bar{\gamma} = f \circ \gamma,$$

then Λ is also open in $[0,1]$. However $\hat{\gamma}(0) = y = \gamma(0)$. Thus $\Lambda = [0,1]$. In particular we get $y = \hat{\gamma}(1) = \gamma(1) = z$. So $f\,|_{U_i}$ is injective. We deduce that f is a covering map. □

By considering the developing maps, we can deduce from Lemma 2.12 the following

Proposition 2.3 – *Let M be a connected manifold with a (G, X)-structure. Suppose that G preserves a C^∞ linear connection $\bar{\nabla}$ on X. If the canonically associated linear connection ∇ on M is complete, then the (G, X)-structure is also complete.*

Now we want to abstract certain notions above to give a definition of completeness for more general geometric structures.

Let Z be a smooth real algebraic variety acted upon algebraicly by a certain $G^k(n, \mathbb{R})$. Denote by ρ this action and by $\Theta(\rho)$ the space of C^∞ geometric structures of type Z and order k with respect to ρ. Then $\Theta(\rho)$ forms a category with isometries as morphisms.

Definition 2.3 – A subset Λ of $\Theta(\rho)$ is said to be *localized*, if it satisfies the following conditions:
(1) If $(M, g) \in \Lambda$, then for any open subset V of M, $(V, g\,|_V)$ as well as its isometry class are all contained in Λ.
(2) If $(M, g) \in \Theta(\rho)$ and for any $y \in M$ there exists an open neighborhood V_y of y such that $(V_y, g\,|_{V_y}) \in \Lambda$, then we have $(M, g) \in \Lambda$.

For $(M, g) \in \Theta(\rho)$, the *hull* of g is by definition the intersection of all the localized subsets containing (M, g) of $\Theta(\rho)$, which is denoted by $Hul(g)$. Thus $Hul(g)$ is the smallest localized subset containing (M, g) of $\Theta(\rho)$.

Definition 2.4 – A *geodesic structure* for a localized subset Λ of $\Theta(\rho)$ is a functorial association to each element (M, g) of Λ a family of finitely piecewise-smooth curves on M (with parameterization and defined on connected intervals) which are said to be the geodesics on M, such that the following conditions are satisfied:
(1) Each isometry sends geodesics to geodesics.
(2) For any $x \in M$ there exists an open neighborhood U of x in M such that any two points in U are joined by at least a geodesic contained in U and defined on $[0, 1]$. Denote by G_U the space of such geodesics and denote by π_U the projection of G_U onto $U \times U$ sending each element of G_U to its endpoints. Then there exists a C^0 section defined on $U \times U$ of π_U with respect to the natural uniform topology of G_U.
(3) The restriction of a geodesic to a connected subinterval of definition is still a geodesic, which is called a subgeodesic of the initial one.
(4) If γ_1 and γ_2 are two C^∞ geodesics such that γ_1 coincide with γ_2 on a non-empty open subset of definition, then they coincide on their common connected interval of definition. In addition C^∞ geodesics depend continuously on their subgeodesics.

For each C^∞ geodesic γ there exists by Condition (4) a unique C^∞ geodesic containing γ and defined on a maximal interval, which is denoted by $\bar{\gamma}$. Then a C^∞ geodesic γ is said to be *maximal* if $\bar{\gamma} = \gamma$.

Definition 2.5 – Under the notations above, an element g of Λ is said to be *complete* with respect to a given geodesic structure of Λ, if each maximal C^∞ geodesic of g is defined on $\mathbb{R}$.

Then a geometric structure g is said to be *complete* if there exists a geodesic structure on $Hul(g)$ with respect to which g is complete.

For a localized subset Λ containing (M, g), each geodesic structure on Λ restricts to a geodesic struture on $Hul(g)$. In addition (M, g) is complete with respect to this geodesic structure on Λ iff it is complete with respect to the restricted geodesic structure on $Hul(g)$.

Proposition 2.4 – *Let M be a connected manifold with a (G, X)-structure. Suppose* *that G preserves a C^∞ geometric structure $\bar{g}$ on X. If the canonically associated structure g on M is complete, then the (G, X)-structure is also complete.*

Proof – We just mimic the proof of Proposition 2.3. Since g is complete, then the universal lift $\widetilde{g}$ of g is easily seen to be also complete. So this proposition follows from the following

Sublemma – *Suppose that $f : (M_1, g_1) \to (M_2, g_2)$ is a local isometry and g_1 is complete. Suppose in addition that $g_2 \in Hul(g_1)$. Then f is a surjective covering map onto M_2.*

Proof – Since g_1 is complete, then by Condition (1) we can lift each curve consisting of pieces of g_2-geodesics. Because of Condition (2), any two points of M_2 can be related by such a curve. So f is surjective.

For any $x \in M_2$ we fix an open neighborhood U of x satisfying Condition (2). Then as in the proof of Lemma 2.5.1 we have

$$f^{-1}(U) = \cup\, U_i$$

and $f(U_i) = U$. Take $y, z \in U_i$ such that $f(y) = f(z)$. Take a smooth curve γ in U_i relating y and z and define $\bar{\gamma} = f \circ \gamma$. By Condition (2) we can find a continuous family of g_2-geodesics defined on $[0, 1]$ and jointing $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$ for all $t \in [0, 1]$. Then by Condition (4) and the completeness of g_1 we can lift this family of geodesics to a continuous family of g_1-geodesics. So we get a C^0 curve $\hat{\gamma}$ such that $f \circ \hat{\gamma} = \bar{\gamma}$ and $\hat{\gamma}(0) = \hat{\gamma}(1) = y$. So as in the proof of Lemma 2.12, we get $y = \hat{\gamma}(1) = \gamma(1) = z$, i.e. $f\,|_{U_i}$ is injective. We deduce that f is a covering map. $\square$

2.5.2 Several illustrating examples and propositions

(1) Complete parallelisms.

Suppose that a C^∞ n-dimensional complete parallelism on M is given by $(X_1, \cdots, X_n)$. Define its geodesics as the space of curves of the form $\gamma_1 * \cdots * \gamma_k$, where $1 \le k \le n$ and for each $1 \le j \le k$, γ_j is a piece of orbit of $a \cdot X_{i_j}$ for a certain $a \in \mathbb{R}^+$.

In this way we get a geodesic structure on the category of n-dimensional complete parallelisms. In addition, $(X_1, \cdots, X_n)$ is complete with respect to this geodesic structure, iff X_j is complete for all $1 \le j \le n$.

Suppose that the pseudogroup of local isometries of $A = (X_1, \cdots, X_n)$ admits a dense orbit and X_j is complete for all $1 \le j \le n$. Then by Lemma 2.3.1, $\Omega = M$, i.e. A is locally homogeneous. It is easily seen that $\mathfrak{h}$ is trivial. So A is normal. Then by Proposition 2.1, we can construct a $(I(\bar{A}), \bar{G})$-structure on M. Since each X_j is supposed complete, then by Proposition 2.4 this $(I(\bar{A}), \bar{G})$-structure is also complete. So we get the following (see the proof of Lemma 2.3)

Proposition 2.5 – *Let A be a C^∞ complete parallelism on a connected manifold M. Suppose that each component of A is complete. Then the following conditions*

are equivalent:
(1) *The pseudogroup I^{loc} of A admits a dense orbit,*
(2) *The vector space spanned by the components of A is a Lie algebra,*
(3) (M, A) *is C^∞ isomorphic to $(\Gamma \backslash G, \bar{A})$, where G is a connected and simply connected Lie group, Γ is a discrete subgroup of G and $\bar{A}$ is given by a basis of the left-invariant vector fields of G.*

(2) Linear connections.

By associating to each C^∞ linear connection its usual geodesics we get on the category of n-dimensional affine manifolds (for each fixed n) a geodesic structure.

In this case, Proposition 2.4 specializes to Proposition 2.3. For higher order connections, we have similar geodesic structures by projecting the orbits of the standard horizontal fields onto the base manifold.

Another particularly interesting case is $\Lambda = \{(X, E, F, \nabla)\}$, where X is a vector field , E and F are two distributions and ∇ is a linear connection on M such that

$$TM = \mathbb{R}X \oplus E \oplus F$$

and

$$\nabla X = 0, \ \nabla E \subseteq E, \ \nabla F \subseteq F.$$

Then Λ is a localized set. The geodesics are defined to be the products of the orbits of $a \cdot X$ (for $a \in \mathbb{R}^+$) and the ∇-geodesics tangent to E or F. Then (X, E, F, ∇) is complete with respect to this geodesic structure iff X is complete and the geodesics tangent to E or F are complete.

We can associate to each affine Anosov-smooth flow ϕ_t an element of Λ, i.e. (X, E^+, E^-, ∇). Then by the Anosov property, this structure is easily seen to be complete (see Chapter VII for more details).

(3) Riemannian metrics.

To get a geodesic structure we simply associate to each n-dimensional Riemannian manifold its usual geodesics. Then a Riemannian metric is complete with respect to this geodesic strucure iff it is complete in the usual sense. In this case Proposition 2.4 specializes to Proposition 2.2.

We define a magnetic Riemannian metric to be a couple (g, ω), where g denotes a C^∞ Riemannian metric and ω denotes a C^∞ closed 2-form. Then the space of n-dimensional magnetic Riemannian metrics gives a localized subset of a certain $\Theta(\rho)$, denoted by Λ. We can find a geodesic structure on Λ as following.

Suppose that $(g, \omega) \in \Lambda$ and (g, ω) is defined on M. For each small open set U of M, we have the Lagrangian $\mathcal{L} : TU \to \mathbb{R}$ such that

$$\mathcal{L}(u) = \frac{1}{2} g(u, u) - \theta(u),$$

where $d\theta = \omega$. Thus locally we have the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0.$$

By projecting the integral curves of this equation to M, we get a family of local curves. To see that the integral curves of different equations fit smoothly, we recall simply that they are all solutions of the following equation

$$\nabla_{\partial_t}\dot{\gamma} = Y(\dot{\gamma}),$$

where ∇ denotes the Levi-Civita connection of g and Y denotes the endomorphism associated to ω, i.e. $\omega(\cdot,\cdot) = g(Y(\cdot),\cdot)$ (see **[Pa]**). In this way, we get a geodesic structure on Λ. Then it is well-known that (g,ω) is complete with respect to this geodesic structure if M is compact and ω is exact.

Proposition 2.6 – *Let g be a C^∞ rigid geometric structure on M. If it is normal and locally homogeneous and complete, then $I(\widetilde{g})$ acts transitively on $\widetilde{M}$.*

Proof – Since g is normal and locally homogeneous, then by Proposition 2.1 we can construct on M a $(I(\bar{g}),\bar{G}/\bar{H})$-structure. Since g is complete, then by Proposition 2.4 this structure is also complete. Since $\bar{G}/\bar{H}$ is simply-connected, then $\widetilde{g}$ is isometric to $\bar{g}$ under the developing map of this $(I(\bar{g}),\bar{G}/\bar{H})$-structure. We deduce that $I(\widetilde{g})$ acts transitively on $\widetilde{M}$. $\square$

We can not prove the existence of a geodesic structure for each category of C^∞ rigid geometric structures. For a fixed category, the geodesic structure is rarely unique.

In most (if not all) physical modelizations, the space of states admits a underlying rigid geometric structure, where rigidity corresponds to finite-order observability. To understand the symmetry of the system, experiments can only tell us the existence of local isometries on at most countably many points. Then the open-dense theorem can give us the local homogeneity on a large scale, which in turn justifies that the underlying structure should be rigid. The existence of a geodesic structure should also be postulated because nothing can be deduced without a natural family of curves corresponding to some kind of natural orbits of movement.

2.5.3 Concerning a theorem of Liouville

Let g be a C^∞ geometric structure on M. If $I(g)$ acts transitively on M, then g is said to be *homogeneous*. Furthurmore, g is said to be *completely homogeneous* if it is homogeneous and each C^∞ local g-isometry defined on a connected open subset of M can be extended to a C^∞ global g-isometry. For example, $\mathbb{R}^2$ and $\mathbb{S}^2$ and $\mathbb{H}^2$ are all completely homogeneous with respect to their canonical constantly curved Riemannian metrics.

For all $n \geq 2$, the isometry group of the canonical conformal structure c_n of the sphere S^n is just the Möbius group Möb(n) (see **[Mos]**), which acts transitively on S^n. Then the following classical theorem of Liouville says essentially that c_n is completely homogeneous for $n \geq 3$.

Theorem 2.2 (Liouville) – *For $n \geq 3$, each C^∞ local conformal isometry defined on a connected open subset of S^n is the restriction of a unique element of Möb(n).*

Definition 2.6 – *Let X be a connected and simply-connected manifold and g be a C^∞ rigid geometric structure on X. Then (g, X) is said to be a rigid form if g is complete and $I(g)$ acts transitively on X and each C^∞ global g-Killing field is complete.*

Lemma 2.13 – *If (g, X) is a rigid form, then g is completely homogeneous.*

Proof – Since (g, X) is a rigid form, then by Remark 2.4, g is normal. Then by Proposition 2.1 we can construct on X a $(I(\bar{g}), \bar{G}/\bar{H})$-structure. Since g is complete, then by Proposition 2.4 this $(I(\bar{g}), \bar{G}/\bar{H})$-struture is also complete. So (g, X) is isometric to $(\bar{g}, \bar{G}/\bar{H})$. Then we conclude by Lemma 2.10. □

We can deduce from the previous lemma the following

Corollary 2.3 – *Let M be a connected and simply-connected manifold and g be a C^∞ Riemannian metric on M. Then g is homogeneous iff it is completely homogeneous. If ∇ is a C^∞ complete linear connection on M, then ∇ is homogeneous iff it is completely homogeneous.*

Let us return to the conformal structures. For any $n \in \mathbb{N}$, we associate to each n-dimensional C^∞ conformal manifold its usual conformal geodesics (see [**Fer**]). Then in this way we get on the category of n-dimensional conformal manifolds a geodesic structure. Thus a C^∞ conformal structure is complete iff all its conformal geodesics are defined on $\mathbb{R}$.

The geodesics of c_n on S^n are circles (with proper parametrizations). Then it is easy to see that c_n is complete with respect to this geodesic structure. Since S^n is compact, then each C^∞ global Killing field of c_n is complete. As mentioned above, for any $n \geq 2$ the isometry group of c_n is the Möbius group Möb(n) (see [**Mos**]). So $I(c_n)$ acts transitively on S^n. In addition for any $n \geq 3$ the n-dimensional conformal structures are rigid. So (c_n, S^n) is a rigid form for any $n \geq 3$. Then we can deduce from Lemma 2.13 that c_n is completely homogeneous for any $n \geq 3$. In [**Mos**], G. D. Mostow used some purely geometric arguments to prove that $I(c_n) =$Möb(n) for all $n \geq 2$. So by combining his arguments with the complete homogeneity of c_n, we can reobtain Theorem 2.2.

Let (g, X) be a rigid form and M be a connected C^∞ manifold. Then by Lemma 2.13, it is easy to see that there exists on M a geometric structure g' locally isometric to g iff M admits a $(I(g), X)$-structure.

2.5.4 Decomposition of $\mathfrak{g}$ for parallel linear connections

Let us return to consider the complete parallel linear connections. Denote by ∇ such a connection on a connected and simply connected manifold M and by τ a C^∞ ∇-parallel geometric structure of order one on M. Denote by g the combinated geometric structure (∇, τ). Then by Lemma 2.2 the isometry group $I(g)$ of g is a Lie group acting transitively on M, which is denoted by G. Fix a point x in M and denote by H the isotropy subgroup of x in G.

Since ∇ is complete, each C^∞ global g-Killing field is complete. In particular (g, M) is a rigid form. So by Lemma 2.13, each element of $I^{loc}_{x,x}$ defined on a connected open subset of M extends uniquely to an element of H, i.e. $I^{loc}_{x,x} \cong H$. So by Corollary 2.2 H is naturally a real algebraic group. In particular H has only finitely many connected components.

Since each element of H preserves ∇, the linear isotropy representation $i : H \to GL(T_xM)$ such that $i(h) = D_xh$ is injective. Since i is in addition algebraic, $i(H)$ is a closed Lie subgroup of $GL(T_xM)$ isomorphic to H under i. In the following we identify H with $i(H)$.

Denote by $\mathfrak{g}'$ and $\mathfrak{h}'$ the Lie algebras of G and H. Denote by $\mathfrak{g}^K$ the Lie algebra of C^∞ global g-Killing fields and by $\mathfrak{h}^K$ its subalgebra containing the elements vanishing at x. Then $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}^K$ under the following map

$$r : \mathfrak{g}' \longrightarrow \mathfrak{g}^K$$

$$u \to (a \to \frac{\partial}{\partial t} |_{t=0} \, exp(-tu) \cdot a).$$

We denote by Y^u the corresponding Killing field of u under r. Recall that by **[DG]** (see also Lemma 2.11) each local Killing field of g extends uniquely to a global Killing field of g. So we get the following Lie algebra isomorphisms

$$\mathfrak{g}' \cong \mathfrak{g}^K \cong \mathfrak{g}$$

and

$$\mathfrak{h}' \cong \mathfrak{h}^K \cong \mathfrak{h}.$$

In addition we have the following injective linear map

$$j : \mathfrak{g}' \longrightarrow T_xM \oplus End(T_xM)$$

$$u \to (Y^u_x, (\mathcal{L}_{Y^u} - \nabla_{Y^u}) |_x).$$

By a simple calculation we get $Di = j |_{\mathfrak{h}'}$. In the following we identify $\mathfrak{h}'$ with $Di(\mathfrak{h}')$. We identify also $\mathfrak{g}'$ and $\mathfrak{h}'$ with $\mathfrak{g}$ and $\mathfrak{h}$ as above. Then by some classical arguments (see Theorem 2.8 of **[KN]**) we get the following linear bijection

$$j : \mathfrak{g} \xrightarrow{\sim} T_xM \oplus \mathfrak{h}$$

$$Y \to (Y_x, (\mathcal{L}_Y - \nabla_Y) |_x).$$

The corresponding Lie algebra structure on $T_xM \oplus \mathfrak{h}$ is easily verified to be

$$[(u,A),(v,B)] = (Av - Bu + T(u,v), [A,B] - R(u,v)),$$

where T and R denote respectively the torsion tensor and the curvature tensor of ∇.

Chapter 3

Dynamical and Geometrical Properties of Anosov Systems

Abstract – *Anosov-smooth flows can only be effectively studied in the flexible environment of general Anosov flows. So in this chapter, we recall and prove certain dynamical and ergodic properties for general Anosov flows paying particular attention to the Anosov-smooth case.*

3.1 Plan of the chapter

In Section 3.2 we recall some elementary facts concerning general flows and Anosov(-smooth) flows. In Section 3.3 we give some applications of the Lifschitz cohomology theorem for Anosov flows. In particular, Lemma 3.8 is very important for Chapter 4. In Section 3.4 we recall some known results about symmetric Anosov flows and explain a construction of non-classical symmetric Anosov flows due to P. Tomter. It should be mentioned that symmetric Anosov flows are Anosov-smooth.

3.2 Preliminaries

3.2.1 Multiplicative ergodic theorem

The multiplicative ergodic theorem of Oseledec is a central piece of the general theory of dynamical systems, which is recalled as following (see [**Le**]).

Theorem 3.1 (Oseledec) – *Let μ be a probability measure on a measure space X and θ be an invertible measurable isomorphism of X preserving μ. In addition θ is supposed to be μ-ergodic. If A is a measurable map of X into $GL(d,\mathbb{R})$ for a certain $d \in \mathbb{Z}^+$ such that $\log \parallel A(\cdot) \parallel$ and $\log \parallel A^{-1}(\cdot) \parallel$ are both integrable, then there exist a measurable θ-invariant subset B of X such that $\mu(B) = 1$ and for each $x \in B$ a decomposition $\mathbb{R}^d = \oplus_{1 \le i \le r} W_x^i$ such that*
(1) *dim W_x^i is positive and constant and $x \to W_x^i$ is measurable,*

(2) $A(x)W^i_x = W^i_{\theta(x)}$,
(3) $v \in W^i_x$ *iff* $\frac{1}{n}log \parallel A^{(n)}(x)(v) \parallel \rightarrow \chi_i$ *when* n *goes to positive infinity or negative infinity, where* $A^{(n)}(x) = A(\theta^{n-1}x)\cdots A(x)$ *for* $n > 0$ *and* $A^{(n)}(x) = A^{-1}(\theta^{-n}x)\cdots A^{-1}(\theta^{-1}x)$ *for* $n < 0$.
In addition, if we denote by $det_j(x)$ *the determinant of the restriction of* A *to* W^j_x *, then we have*

$$\text{dim}W^j_x \cdot \chi_j = \int_X log \mid det_j(x) \mid d\mu.$$

Now if ϕ_t is a C^∞ flow on a C^∞ manifold M, then by choosing a measurable vector bundle isomorphism τ of TM onto $M \times \mathbb{R}^d$, we get a measurable map A of M into $GL(d, \mathbb{R})$ such that $A(x) = \tau_{\theta(x)} \circ D_x(\phi_1) \circ \tau_x^{-1}$, where ϕ_1 denotes the time-one map of ϕ_t. So we deduce easily from the previous theorem the following

Theorem 3.2 – *Let* ϕ_t *be a* C^∞ *flow on a* C^∞ *manifold* M. *Let* μ *be a* ϕ_t*-invariant probability measure on* M *such that* ϕ_t *is* μ*-ergodic. Then there exist a measurable* ϕ_t*-invariant subset* B *of* M *such that* $\mu(B) = 1$ *and a measurable* ϕ_t*-invariant decomposition* $TM \mid_B = \oplus_{1\leq i\leq r} L_i$ *such that*
(1) $dimL_i$ *is constant and positive,*
(2) *For any* $u \in TB$ *we have* $u \in L_i$ *iff* $\lim_{t\to\pm\infty} \frac{1}{t}log \parallel D\phi_t(u) \parallel = \chi_i$.

This decomposition is called the *Lyapunov decomposition* of ϕ_t with respect to μ. For any $1 \leq i \leq r$, L_i is called the *Lyapunov subbundle* of ϕ_t with *Lyapunov exponent* χ_i, which is often denoted by L_{χ_i}. If a is not a Lyapunov exponent of ϕ_t, then L_a is defined to be $\{0\}$. In general the Lyapunov decomposition of ϕ_t with respect to an invariant measure μ is only defined on a μ-conull subset whose geometry can be extremely complicated. The following lemma is proved in **[FK]**.

Lemma 3.1 – *Under the conditions of Theorem* 3.2*, if* σ *denotes a bounded measurable* ϕ_t*-invariant tensor of type* $(0,k)$ *and* $\sum_{1\leq j\leq k}\chi_{i_j} \neq 0$*, then we have* $\sigma(L_{\chi_{i_1}}, \cdots, L_{\chi_{i_k}}) \equiv 0$.

Proof – Suppose firstly that $\sum_{1\leq j\leq k}\chi_{i_j} > 0$. Then for all $u_j \in L_{i_j}$ and all $t \gg 0$ and $\epsilon \ll 1$, we have

$$\mid \sigma(u_1, \cdots, u_k) \mid = \mid \sigma(D\phi_{-t}(u_1), \cdots, D\phi_{-t}(u_k)) \mid$$

$$\leq \parallel \sigma \parallel \cdot \parallel D\phi_{-t}(u_1) \parallel \cdots \parallel D\phi_{-t}(u_k) \parallel \leq \parallel \sigma \parallel \cdot e^{-t(\sum_{1\leq j\leq k}\chi_{i_j} - k\epsilon)}.$$

We deduce that $\sigma(u_1, \cdots, u_k) = 0$. If $\sum_{1\leq j\leq k}\chi_{i_j} < 0$, then similar argument works. □

3.2.2 Basic facts concerning Anosov flows

Now let us return to Anosov flows. Let ϕ_t be a C^∞ Anosov flow on a closed manifold M. As mentioned in the Introduction, $E^\pm$ and $E^{\pm,0}$ are all integrable to continuous foliations with C^∞ leaves (see **[An]**) denoted respectively by $\mathcal{F}^\pm$ and $\mathcal{F}^{\pm,0}$. For Anosov-smooth flows this fundamental fact can be easily deduced from the Anosov property.

Lemma 3.2 – *If ϕ_t is Anosov-smooth, then $E^\pm$ and $E^{\pm,0}$ are all integrable to C^∞ foliations.*

Proof – Since ϕ_t is supposed to be Anosov-smooth, then E^+ and E^- are both C^∞ distributions on M. Denote by $P^{-,0}$ the projection of TM onto $E^{-,0}$ with respect to the Anosov splitting. Define for any C^∞ sections Y and Z of E^+, $K(Y,Z) = P^{-,0}[Y,Z]$. Since the Anosov splitting is ϕ_t-invariant, then it is easily seen that K defines a ϕ_t-invariant section of $(E^+)^* \otimes (E^+)^* \otimes E^{-,0}$. Thus we get

$$D\phi_{-t}(K(Y,Z)) = K(D\phi_{-t}(Y), D\phi_{-t}(Z)) \leq \| K \| \cdot \| D\phi_{-t}(Y) \| \cdot \| D\phi_{-t}(Z) \| \to 0,$$

if $t \to +\infty$. So $K(Y,Z) \subseteq E^+$. We deduce that $K(Y,Z) = 0$, i.e. $[Y,Z]$ is tangent to E^+. Then by the Frobenius theorem, E^+ is integrable. Similarly E^- is seen to be integrable. Since E^+ and E^- are both ϕ_t-invariant, then $E^{+,0}$ and $E^{-,0}$ are both involutive, i.e. integrable to C^∞ foliations. $\square$

Now let us recall the following elementary proposition (see **[HK]** and **[M]**).

Proposition 3.1 – *Let ϕ_t be a C^∞ Anosov flow on M. Suppose that E^+ is of dimension k and E^- is of dimension l. Then each leaf of $\mathcal{F}^+$ is C^∞ diffeomorphic to $\mathbb{R}^k$ and each leaf of $\mathcal{F}^-$ is C^∞ diffeomorphic to $\mathbb{R}^l$.*

In addition a leaf of $\mathcal{F}^{+,0}$ is C^∞ diffeomorphic to $\mathbb{R}^{k+1}$ iff it contains no periodic orbit of ϕ_t. Otherwise it is C^∞ diffeomorphic to $S^1 \times \mathbb{R}^k$ and it contains a unique periodic orbit. Similarly a leaf of $\mathcal{F}^{-,0}$ is C^∞ diffeomorphic to $S^1 \times \mathbb{R}^l$ or $\mathbb{R}^{l+1}$ depending on whether it contains a periodic orbit. In addition it contains at most one periodic orbit of ϕ_t.

For any $x \in M$ the leaves containing x of $\mathcal{F}^+$ and $\mathcal{F}^-$ are denoted respectively by W_x^+ and W_x^-. The leaves containing x of $\mathcal{F}^{+,0}$ and $\mathcal{F}^{-,0}$ are denoted by $W_x^{+,0}$ and $W_x^{-,0}$. By the following lemma we can suppose, up to finite covers, that these four foliations and M are all orientable.

Lemma 3.3 – *Let $\bar{M}$ be a finite cover of a closed manifold M and ψ_t be a C^∞ flow on M. Denote by $\bar{\psi}_t$ the lift of ψ_t onto $\bar{M}$. Then ψ_t is Anosov iff $\bar{\psi}_t$ is Anosov. In addition ψ_t is Anosov-smooth iff so is $\bar{\psi}_t$.*

Proof – If ψ_t is Anosov, then $\bar{\psi}_t$ is certainly Anosov. Conversely we suppose that $\bar{\psi}_t$ is Anosov. Fix a C^∞ Riemannian metric on M. Then there exists a C^0-decomposition

$$T\bar{M} = \mathbb{R}\bar{X} \oplus \bar{E}^+ \oplus \bar{E}^-$$

and $a, b > 0$ such that with respect to the lifted metric,

$$\| D\bar{\psi}_{\mp t}(u^\pm) \| \leq a \cdot e^{-bt} \| u^\pm \|, \ \forall\ t > 0, \ \forall\ u^\pm \in \bar{E}^\pm.$$

Denote by π the projection of $\bar{M}$ onto M. Take $x \in M$ and suppose that $\pi(y) = x = \pi(z)$.

If $D\pi(\bar{E}^+_y) \not\subseteq D\pi(\bar{E}^+_z)$, then there exists $u^+ \in \bar{E}^+_y$, $v^+ \in \bar{E}^+_z$, $v^{-,0} \in \bar{E}^{-,0}_z$ such that

$$D\pi(u^+) = D\pi(v^+) + D\pi(v^{-,0}), \ v^{-,0} \neq 0.$$

Thus we get

$$\| D\psi_{-t}(D\pi(u^+)) \| = \| D\bar{\psi}_{-t}(u^+) \| \overset{t\to\infty}{\longrightarrow} 0$$

and

$$\| D\psi_{-t}(D\pi(u^+)) \| = \| D\bar{\psi}_{-t}(v^+ + v^{-,0}) \| \overset{t\to\infty}{\not\longrightarrow} 0,$$

which is a contradiction. So we get $D\pi(\bar{E}^+_y) \subseteq D\pi(\bar{E}^+_z)$. Thus $D\pi(\bar{E}^+_y) = D\pi(\bar{E}^+_z)$. Similarly we have $D\pi(\bar{E}^-_y) = D\pi(\bar{E}^-_z)$. So we can push down by π the Anosov splitting of $\bar{\psi}_t$ to abtain a continuous splitting of TM, which satisfies certainly the Anosov property for ψ_t. So ψ_t is Anosov. It is evident that ψ_t is Anosov-smooth iff so is $\bar{\psi}_t$. $\square$

A general flow ψ_t is said to be *topologically transitive* if it admits a dense orbit. For an Anosov flow ϕ_t on M, topological transitivity is equivalent to each of the following conditions (see **[HK]**):
(1) The set of periodic orbits of ϕ_t is dense in M.
(2) The set of nonwandering points of ϕ_t is M.
Recall that x is said to be a *nonwandering* point of ϕ_t if for any open neighborhood U of x, there exists $T > 0$ such that $\phi_T U \cap U \neq \emptyset$.

If ϕ_t preserves a volume form, i.e. a probability measure in the Lebesgue measure class, then by the recurrence theorem of Poincaré, its nonwandering points are dense in M. However the set of nonwandering points is closed in M by its definition. So we deduce that each volume-preserving Anosov flow is topologically transitive.

For each C^∞ Anosov flow ϕ_t, its *canonical* 1*-form* λ is by definition the C^0 section of T^*M such that

$$\lambda(X) \equiv 1, \ \lambda(E^\pm) \equiv 0.$$

Since E^+ and E^- are ϕ_t-invariant, then λ is also ϕ_t-invariant. If ϕ_t is Anosov-smooth, then λ is a C^∞ 1-form on M.

If ϕ_t is Anosov-smooth, we define its *rank* as the following even number

$$\text{rank}(\phi_t) = 2 \cdot \max\{k \geq 0 \mid \wedge^k d\lambda \not\equiv 0\}.$$

By convention, $\wedge^0 d\lambda = 1$. If ϕ_t is topologically transitive and its rank is $2k$, then $\wedge^k d\lambda$ vanishes nowhere on an open-dense subset of M.

If M is of dimension m, then we get easily $0 \leq \text{rank}(\phi_t) \leq 2 \cdot [\frac{m}{2}]$, where for each $a \in \mathbb{R}$ the symbol $[a]$ denotes the biggest among the integers smaller than a.

A C^∞ Riemannian metric on M is said to be *Lyapunov* for an Anosov flow ϕ_t if there exists $b > 0$ such that

$$\| D\phi_{\mp t}(u^\pm) \| \leq e^{-bt} \| u^\pm \|, \ \forall\ u^\pm \in E^\pm, \ \forall\ t \geq 0.$$

The existence of Lyapunov metrics for general Anosov flows is widely accepted. By mimicing the arguments for diffeomorphisms, we give in the following a proof of this fact for the Anosov-smooth case, which is enough for the need of the next section.

Lemma 3.4 – *If ϕ_t is Anosov-smooth, then it admits a Lyapunov metric.*

Proof – Fix a C^∞ Riemannian metric g on M. Then there exist $a, b > 0$ such that

$$\| D\phi_{\mp t}(u^\pm) \| \leq a \cdot e^{-bt} \cdot \| u^\pm \|, \ \forall\ u^\pm \in E^\pm, \ \forall\ t \geq 0,$$

Fix $T >> 0$ such that $a^2 \cdot e^{-2bT} < 1$ and define

$$g_1 = (\int_0^T \phi_t{}^*(g\,|_{E^-})dt) \oplus (\int_0^T \phi_{-t}{}^*(g\,|_{E^+})dt) \oplus \lambda^2,$$

where λ denotes the canonical 1-form of ϕ_t. Since ϕ_t is Anosov-smooth, then g_1 is a C^∞ Riemannian metric. To simplify the notations, we denote $D\phi_t(u)$ by $\phi_t u$ for all $u \in TM$. For any $u^- \in E^-$ and any $s \in (0, 1]$ we have

$$< u^-, u^- >_1 = \int_0^T < \phi_t u^-, \phi_t u^- > dt$$

$$\leq \int_0^s < \phi_t u^-, \phi_t u^- > dt + \cdots + \int_{[\frac{T}{s}]s}^{([\frac{T}{s}]+1)s} < \phi_t u^-, \phi_t u^- > dt$$

$$\leq (1 + a^2 \cdot e^{-2bs} + \cdots + a^2 \cdot e^{[\frac{T}{s}]s}) \cdot \int_0^s < \phi_t u^-, \phi_t u^- > dt.$$

On the other side, we have for any $s \in (0, 1]$

$$< \phi_s u^-, \phi_s u^- >_1 = \int_0^T < \phi_{s+t} u^-, \phi_{s+t} u^- > dt$$

$$= < u^-, u^- >_1 + \int_T^{T+s} < \phi_t u^-, \phi_t u^- > dt - \int_0^s < \phi_t u^-, \phi_t u^- > dt$$

$$\leq < u^-, u^- >_1 + (a^2 \cdot e^{-2bT} - 1) \cdot \int_0^s < \phi_t u^-, \phi_t u^- > dt.$$

Since $a^2 \cdot e^{-2bT} < 1$, then $< \phi_s u^-, \phi_s u^- >_1 \leq \Delta_s \cdot < u^-, u^- >_1$, where

$$\Delta_s = \frac{a^2 e^{-2bT} + a^2 \cdot e^{-2bs} + \cdots + a^2 \cdot e^{[\frac{T}{s}]s}}{1 + a^2 \cdot e^{-2bs} + \cdots + a^2 \cdot e^{[\frac{T}{s}]s}}.$$

We have $\Delta_s < 1$ for all $s \in (0,1]$. Define

$$\Lambda_s = \frac{1 + e^{-2bT}(1 - e^{-2bs})}{1 + \frac{1}{a^2}(1 - e^{-2bs})}.$$

Then for all $s \in (0,1]$ we easily get $0 < \Delta_s < \Lambda_s < 1$ and

$$\lim_{s \to 0+} \frac{1}{s} log \Lambda_s = 2b(e^{-2bT} - \frac{1}{a^2}) < 0.$$

So there exists $b' > 0$ such that $\Lambda_s < e^{-2b's}$ for all $s \in (0,1]$. Thus

$$\| \phi_s u^- \|_1 \leq e^{-b's} \| u^- \|_1 .$$

Similarly we get $\| \phi_{-s} u^+ \|_1 \leq e^{-b's} \| u^+ \|_1$ for all $u^+ \in E^+$.

For all $t > 0$ we take $N \gg 1$ such that $\frac{t}{N} < 1$. Then

$$\| \phi_{\mp t} u^{\pm} \|_1 = \| \phi_{\frac{\mp t}{N}} \circ \cdots \circ \phi_{\frac{\mp t}{N}} (u^{\pm}) \|_1 \leq e^{-b't} \| u^{\pm} \|_1 .$$

So g_1 is a C^∞ Lyapunov metric for ϕ_t. $\square$

Definition 3.1 – For a C^∞ Anosov flow ϕ_t, we denote by $\widetilde{\Phi}$ the orbit foliation of its lifted flow $\widetilde{\phi}_t$ on $\widetilde{M}$. Then ϕ_t is said to have the *section property* if for each $x \in \widetilde{M}$ there exists a transverse section Σ of $\widetilde{\Phi}$ containing x such that each leaf of $\widetilde{\Phi}$ intersects Σ at most once. Such a transverse section is said to be *fine*.

If ϕ_t admits no homotopically trivial periodic orbit, then $\widetilde{\phi}_t$ admits no periodic orbit, i.e. the orbits of $\widetilde{\phi}_t$ are all diffeomorphic to $\mathbb{R}$.

Lemma 3.5 – *Let ϕ_t be a C^∞ Anosov flow. Then under the notations above, ϕ_t has the section property if and only if it admits no homotopically trivial periodic orbit.*

Proof – Suppose that ϕ_t admits no homotopically trivial periodic orbit. Fix a C^∞ Riemannian metric on M and suppose on the contrary that ϕ_t does not have the section property. Then there exist $x \in \widetilde{M}$ and two sequences $\{x_n\}_{n=1}^\infty \subseteq \widetilde{M}$ and $\{T_n\}_{n=1}^\infty \subseteq \mathbb{R}^+$ such that

$$d(x_n, x) < \frac{1}{n}, \ d(\widetilde{\phi}_{T_n}(x_n), x) < \frac{1}{n}.$$

Since the generator X of ϕ_t vanishes nowhere, then for any $n \gg 1$, T_n is bounded below by a positive munber. If $\{T_n\}_{n=1}^\infty$ is bounded above, then by taking a convergent subsequence, we can produce a $\widetilde{\phi}_t$-periodic orbit, which contradicts the assumption.

So $\{T_n\}_{n=1}^{\infty}$ is not bounded above. Let us recall first the following

Closing lemma ([**An**], Lemma 13.1) – *Let ϕ_t be a C^∞ Anosov flow on a closed manifold M. Then for any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ and $t_0 > 0$ such that if $x \in M$, $T > t_0$ and $d(x, \phi_T(x)) < \delta$, then there exists $x' \in M$ and $T' > 0$ such that $\mid t - T' \mid < \epsilon$ and $\phi_{T'}(x') = x'$ and $d(\phi_s(x), \phi_s(x')) < \epsilon, \ \forall \ s \in [0, T']$.*

Denote by π the projection of $\widetilde{M}$ onto M. Then with respect to the lifted metric, there exists $a > 0$ such that

$$B(y, a) \cap \pi^{-1}(\pi(y)) = \{y\}, \ \forall \ y \in \widetilde{M}.$$

Define $A = sup \parallel X \parallel$ and $\epsilon = min(\frac{a}{3+A}, i(M))$, where $i(M)$ denotes the injectivity radius of M. Then by the Closing lemma above, we get two numbers δ and t_0 with respect to ϵ. Fix $N \gg 1$ such that $\frac{1}{N} < \delta$ and $T_N > t_0$. So we get $x' \in M$ and $T' > 0$ such that $\mid T' - T_N \mid < \epsilon$, $\phi_{T'}(x') = x'$ and

$$d(\phi_s(x'), \phi_s(\pi(x_N)) < \epsilon, \ \forall \ s \in [0, T'].$$

Since $\epsilon < i(M)$, then we can construct a homotopy from $\mathcal{O}_{[0,T']}(x')$ to $\mathcal{O}_{[0,T']}(\pi(x_N))$ by relating the corresponding points by the unique geodesic of length smaller than ϵ, where $\mathcal{O}(x')$ denotes the ϕ_t-orbit of x'.

We can lift this homotopy such that the lift of $\mathcal{O}_{[0,T']}(\pi(x_N))$ is $\widetilde{\mathcal{O}}_{[0,T']}(x_N)$. Denote by $\widetilde{x}'$ the lifted point of x'. Then we get

$$d(\widetilde{x}', \widetilde{\phi}_{T'}(\widetilde{x}')) \leq d(\widetilde{x}', x_N) +$$

$$+ d(x_N, \widetilde{\phi}_{T_N}(x_N)) + d(\widetilde{\phi}_{T_N}(x_N), \widetilde{\phi}_{T'}(x_N)) + d(\widetilde{\phi}_{T'}(x_N), \widetilde{\phi}_{T'}(\widetilde{x}'))$$

$$< 3\epsilon + A \cdot \epsilon \leq a.$$

We deduce that $\widetilde{x}' = \widetilde{\phi}_{T'}(\widetilde{x}')$, which again contradicts the assumption. So ϕ_t has the section property if it has no homotopically trivial periodic orbit.

If ϕ_t admits a homotopically trivial periodic orbit, then $\widetilde{\phi}_t$ admits a periodic point denoted by x. Suppose on the contrary that there exists a fine transverse section containing x. Then by the uniformity of foliations (see [**CM**]), there exists also a fine transverse section Σ containing a piece of $\widetilde{W_x^-}$. Thus by the Anosov property, each $\widetilde{\phi}_t$-orbit passing through a point near x of $\widetilde{W_x^-}$ accumulates to x in the positive direction of $\widetilde{\phi}_t$. So Σ is not fine, which is a contradiction. We deduce that ϕ_t has not the section property if ϕ_t admits a homotopically trivial periodic orbit. $\square$

If ϕ_t has the section property, then by taking small transverse sections of $\widetilde{\Phi}$ as charts, the lifted orbit space $\widetilde{M} / \widetilde{\Phi}$ becomes a first countable C^∞ manifold, which is however not necessarily Hausdoff.

3.3 Cohomological results for Anosov flows

Denote by ϕ_t a C^∞ Anosov flow on M and by G a Lie group. A measurable map $C : M \times \mathbb{R} \to G$ is said to be a *G-cocycle* if

$$C(x, s+t) = C(\phi_t x, s) \cdot C(x, t)$$

for all $x \in M$ and all $s, t \in \mathbb{R}$. Each measurable map $u : M \to G$ gives a G-cocycle, denoted by C_u, such that

$$C_u(x, t) = u(\phi_t x) \cdot u(x)^{-1}$$

for any $x \in M$ and all $t \in \mathbb{R}$.

Let μ be a ϕ_t-invariant measure on M. Two measurable functions f_1 and f_2 on M are said to be *equal with respect to* μ, denoted by $f_1 = f_2,\ a.e.\ \mu$, if we have $\mu\{f_1 \neq f_2\} = 0$. Then two G-cocycles C_1 and C_2 are said to be *equal with respect to* μ, denoted by

$$C_1 = C_2,\ a.e.\ \mu,$$

if for all $t \in \mathbb{R}$, $C_1(\cdot, t) = C_2(\cdot, t),\ a.e.\ \mu$.

The following theorem is a combination of some classical results, which is essential for the chapters below.

Theorem 3.3 (A. N. Livsic, R. de La llave, J. Marco, R. Moriyón) – *Let ϕ_t be a C^∞ Anosov flow on M and let C be a C^∞ G-cocycle, where G is either the additive group $\mathbb{R}$ or the multiplicative group $\mathbb{R}_+^*$. Suppose that ϕ_t preserves a Lebesgue measure μ. If there exists a measurable map $u : M \to G$ such that $C = C_u,\ \mu$ a.e, then there exists a C^∞ map $\bar{u} : M \to G$ such that $\bar{u} = u$ almost everywhere with respect to μ and $C(x,t) = \bar{u}(\phi_t x) \cdot \bar{u}(x)^{-1}$ for all $x \in M$ and all $t \in \mathbb{R}$.*

Proof – Suppose at first $G = \mathbb{R}$. Then by Theorem 9 of **[Li]** there exists a continuous function $\bar{u}$ such that $\bar{u} = u$ almost everywhere with respect to μ. By continuty we get

$$C(x, t) = \bar{u}(\phi_t x) - \bar{u}(x)$$

for all $x \in M$ and all $t \in \mathbb{R}$. So for each ϕ_t-periodic point x of period T we have $C(x, T) = 0$. Since each volume-preserving Anosov flow is topologically transitive, then we can deduce from Theorem 2.1 of **[LMM]** the existence of a C^∞ function $\hat{u}$ such that $C(x, t) = \hat{u}(\phi_t x) - \hat{u}(x)$ for all $x \in M$ and all $t \in \mathbb{R}$. Then we get $\bar{u} = \hat{u}$ by the topological transitivity of ϕ_t.

If C is a $\mathbb{R}_+^*$-cocycle, then $log \circ C$ is a $\mathbb{R}$-cocycle. So we can conclude from the case of $\mathbb{R}$-cocycle above. □

The previous proposition is often useful to prove the smoothness of certain, a priori measurable, geometric objects associated to C^∞ Anosov flows. Here are some applications.

Lemma 3.6 – *Let ϕ_t be a C^∞ Anosov flow on an orientable manifold M. If it preserves a Lebesgue measure μ, then μ is given by a nowhere-vanishing C^∞ volume form on M.*

Proof – Fix a C^∞ volume form ν on M. Then there exists a measurable function f such that $f > 0$ and $\mu = f\nu$. Define a C^∞ map $\theta : M \to \mathbb{R}^+$ such that for any $x \in M$ and any $t \in \mathbb{R}$,

$$(\phi_t{}^*\nu)(x) = \frac{1}{\theta(x,t)} \cdot \nu(x).$$

Then θ is easily seen to be a C^∞ $\mathbb{R}^*_+$-cocycle.

Since μ is ϕ_t-invariant, then for any $t \in \mathbb{R}$, $\phi_t{}^*(f\nu) = f \circ \phi_t \cdot \frac{1}{\theta(\cdot,t)} \cdot \nu = f\nu$. So $\frac{f\circ\phi_t}{f} = \theta(\cdot,t)$, *a.e.* μ. We deduce that

$$C_f = \theta, \ \textit{a.e. } \mu.$$

Then by Theorem 3.3, there exists a C^∞ map $\bar{f} : M \to \mathbb{R}^+$ such that $\bar{f} = f$, *a.e.* μ. So μ is given by the nowhere-vanishing C^∞ volume form $\bar{f}\nu$. □

A general flow ψ_t is said to be *topologically mixing*, if for all non-empty open subsets U and V of M there exists $T > 0$ such that for all $t > T$ we have $\psi_t U \cap V \neq \emptyset$. The following lemma is a simple reformulation of the principal result of **[Pl1]**.

Lemma 3.7 – *Let ϕ_t be a C^∞ topologically transitive Anosov flow. Then we have the following alternative:*
(1) *ϕ_t is topologically mixing,*
(2) *ϕ_t admits a C^∞ closed global section with constant return time.*

Proof – If Case (2) is true, then up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to the suspension of a C^∞ Anosov diffeomorphism. Thus it is not topologically mixing. So the alternative is exclusive.

If there exists $x \in M$ such that W_x^+ is not dense in M, then by Theorem 1.8 of **[Pl1]** $E^+ \oplus E^-$ is the tangent bundle of a C^1 foliation $\mathcal{F}$. In addition the leaves of $\mathcal{F}$ are all compact. So ϕ_t admits a C^1 closed global section with constant return time. Then to realize Case (2) we need only prove that $E^+ \oplus E^-$ is C^∞ in this case.

Denote by λ the canonical 1-form of ϕ_t. Then λ is, a priori, a continuous 1-form on M. For each point $y \in M$ we take a small neighborhood F_y of y in the leaf containing y of $\mathcal{F}$. Then we can construct a local C^1 chart $\theta_y : (-\epsilon,\epsilon) \times F_y \to M$ such that $\theta_y(t,z) = \phi_t(z)$. In this chart we have $\lambda = dt$. We deduce that $\int_\gamma \lambda = 0$ for each piecewise C^1 closed curve γ contained in the image of θ_y. So λ is locally closed (see Section two of **[Pl1]** for the definition). Then by Proposition 2.1 of **[Pl1]**, λ is seen to be closed in a weak sense, i.e. for every C^1 immersed two-disk σ such that $\partial\sigma$ is piecewise C^1,

$$\int_{\partial\sigma} \lambda = 0.$$

So by integrating along the closed curves, λ gives an element in $Hom(\pi_1(M), \mathbb{R})$, i.e. the space of group homomorphisms of $\pi_1(M)$ into $\mathbb{R}$, where $\pi_1(M)$ denotes the fundamental group of M. However we have naturally

$$Hom(\pi_1(M), \mathbb{R}) \cong (H_1(M, \mathbb{R}))^* \cong H^1(M, \mathbb{R}),$$

where $H^1(M, \mathbb{R})$ denotes the first de Rham cohomology group of M. So there exists a C^∞ closed 1-form β such that for each C^∞ closed curve γ,

$$\int_\gamma \lambda = \int_\gamma \beta.$$

So by integrating $(\lambda - \beta)$ along curves, we get on M a well-defined continuous function f. Then for any $y \in M$ and any $t \in \mathbb{R}$ we have

$$(f \circ \phi_t)(y) - f(y) = \int_0^t (1 - \beta(X))(\phi_s(y))ds.$$

Since the right-hand side of this identity is a C^∞ $\mathbb{R}$-cocycle, then by Theorem 3.3, f is smooth. However by the definition of f we have

$$\lambda - \beta = df.$$

So λ is also smooth. We deduce that $E^+ \oplus E^-(= Ker\lambda)$ is C^∞. So Case (2) is realized if W_x^+ is not dense for a certain point $x \in M$.

Now suppose that for any $x \in M$, W_x^+ is dense in M. Fix a Riemannian metric on M. For any $x \in M$ and any $r > 0$ we denote by $M_{x,r}$ and $W_{x,r}^+$ the balls of center x and radius r in M and W_x^+. Take arbitrarily two open subsets U and V in M and a small ball $M_{y,\epsilon}$ in V. Since M is closed and each strong unstable leaf is supposed to be dense in M, then we can find $R < +\infty$ such that

$$W_{x,R}^+ \cap M_{y,\epsilon} \neq \emptyset, \ \forall\, x \in M.$$

Take a small disk $W_{x,\delta}^+$ in U. Then by the Anosov property there exists $T > 0$ such that

$$\phi_t(W_{x,\delta}^+) \supseteq W_{\phi_t(x),R}^+, \ \forall\, t \geq T.$$

So $\phi_t U \cap V \neq \emptyset$, $\forall\, t \geq T$, i.e. ϕ_t is topologically mixing. □

Among the dynamical invariants of an Anosov flow ϕ_t, its topological entropy is the most fundamental one, which is denoted by $h_{top}(\phi_t)$ (see **[HK]** for the definition). Denote by $\mathcal{M}(\phi_t)$ the set of ϕ_t-invariant probability measures and by $h_\mu(\phi_t)$ the metric entropy of each μ in $\mathcal{M}(\phi_t)$ (see **[HK]** for the definition). Then it is well-known that (see **[HK]**)

$$h_{top}(\phi_t) = sup_{\mu \in \mathcal{M}(\phi_t)}\{h_\mu(\phi_t)\}.$$

The following proposition is a special case of Theorem 3.3 of [**BR**].

Proposition 3.2 – *Let ϕ_t be a C^∞ topologically transitive Anosov flow on M. Then there exists a unique ϕ_t-invariant probability measure μ such that $h_\mu(\phi_t) = h_{top}(\phi_t)$. Furthermore, μ is ϕ_t-ergodic and positive on non-empty open subsets of M.*

This measure μ is said to be the *Bowen-Margulis measure* of ϕ_t. If ϕ_t is in addition topologically mixing, then its Bowen-Margulis measure can be decomposed into pieces. Let us recall firstly some notations.

Take a C^∞ Riemannian metric on M. Then for any $x \in M$ and any $\delta > 0$ we denote by $W^+_{x,\delta}$ the sphere in W^+_x of center x and of radius δ with respect to the induced metric on W^+_x. Similarly we define $W^-_{x,\delta}$ and $W^{\pm,0}_{x,\delta}$. Then by [**PS**], there exists $\epsilon > 0$ such that for all $\delta < \epsilon$ and all $x \in M$ the following map is a well-defined local homeomorphism

$$\theta : W^{-,0}_{x,\delta} \times W^+_{x,\delta} \to M,$$

$$(y,\ z) \to W^+_{y,2\delta} \cap W^{-,0}_{z,2\delta}.$$

The image of θ is said to be a *product neighborhood* of x. If ϕ_t is Anosov-smooth, then θ is a C^∞ local diffeomorphism.

Take a curve l tangent to $W^{-,0}_x$ such that $l(0) = x$ and $l(1) = y$. Since $W^+_{x,\delta}$ and $W^+_{y,\delta}$ are both transverse sections of the foliation $\mathcal{F}^{-,0}$, then we get a holonomy map of $\mathcal{F}^{-,0}$ along l with respect to these two transverse sections. Since the curve l is often clear from the context or matters little, then this holonomy map is usually denoted by $H^{-,0}_{x,y}$ and is said to be a *weak stable holonomy map*. We can define and denote the other holonomy maps similarly. If ϕ_t is Anosov-smooth, then all these holonomy maps are C^∞.

The following proposition is proved in [**Ma1**].

Theorem 3.4 (G. A. Margulis) – *Let ϕ_t be a C^∞ topologically mixing Anosov flow. Denote by h its topological entropy. Then there exists a unique (up to scalars) family $\mu^{+,0}$ of measures supported by the leaves of $\mathcal{F}^{+,0}$ such that*

$$\mu^{+,0} \circ \phi_t = e^{ht} \mu^{+,0}$$

and invariant under the strong stable holonomy maps. Similarly there exists a unique (up to scalars) family $\mu^{-,0}$ of measures supported by the leaves of $\mathcal{F}^{-,0}$ such that

$$\mu^{-,0} \circ \phi_t = e^{-ht} \mu^{-,0}$$

and invariant under the strong unstable holonomy maps. There exist also two (unique up to scalars) families of measures μ^+ and μ^- supported respectively by the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$ such that $\mu^\pm \circ \phi_t = e^{\pm ht} \mu^\pm$ and they are absolutely continuous with respect to the weak holonomy maps.

In each product neighborhood, the Bowen-Margulis measure μ is proportional to $\mu^+ \otimes \mu^{-,0}$ and $\mu^- \otimes \mu^{+,0}$ and is equivalent to $\mu^{+,0} \otimes \mu^-$ and $\mu^{-,0} \otimes \mu^+$.

Remark 3.1 – The families of measures above are called the *Margulis measures* of ϕ_t. Note that they are defined only for topologically mixing Anosov flows.

See [**Fo**] or [**Has**] for the definition of $\mu^+ \otimes \mu^{-,0}$. The other products of Margulis measures in a product neighborhood are defined similarly.

Take a C^∞ Riemannian metric on M and denote by $\nu^\pm$ the induced C^∞ volume forms along the leaves of $\mathcal{F}^\pm$. Denote by $\nu^{\pm,0}$ the induced volume forms along the leaves of $\mathcal{F}^{\pm,0}$. Then we can prove the following lemma, which was originally proved for three-dimensional contact Anosov flows in [**Fo1**].

Lemma 3.8 – *Let ϕ_t be a C^∞ topologically mixing Anosov-smooth flow on M. Suppose that its Bowen-Margulis measure is equal to Lebesgue measure and E^+ and E^- are both orientable. Then there exist C^∞ positive functions on M, $f^\pm$ and $f^{\pm,0}$ such that $\mu^\pm = f^\pm\nu^\pm$ and $\mu^{\pm,0} = f^{\pm,0}\nu^{\pm,0}$.*

Proof – Let us prove firstly that along each leaf of $\mathcal{F}^+$, μ^+ is absolutely continuous with respect to ν^+, denoted by $\mu^+ \ll \nu^+$.

Take a small ball $W^+_{x,\delta}$ in W^+_x and $A \subseteq W^+_{x,\delta}$ such that $\nu^+(A) = 0$. Define $\Omega = W^{-,0}_{x,\delta} \times A$ via the local product structure. Since E^+ is C^∞, then for all $y \in W^{-,0}_{x,\delta}$ we have $\nu^+(y \times A) = 0$. Then by Fubini Theorem, Ω is of Lebesgue measure zero. Since μ is supposed to be in the Lebesgue class, then $\mu(A) = 0$. Since μ is locally decomposed as a product of μ^+ and $\mu^{-,0}$, then

$$\mu^+(y \times A) = 0, \ a.e. \ \mu^{-,0}$$

Since μ^+ is absolutely continuous with respect to the weak stable holonomy maps, then $\mu^+(A) = 0$. So $\mu^+ \ll \nu^+$. Similarly we have $\nu^+ \ll \mu^+$. So on each leaf of $\mathcal{F}^+$, μ^+ is equivalent to ν^+. Thus by the definition of μ^+ (see [**HK**]), we can find a measurable function f^+ on M such that $f^+ > 0$ and on each leaf W^+_x of $\mathcal{F}^+$,

$$\mu^+ = f^+ \mid_{W^+_x} \nu^+.$$

For all $(x, \ t) \in M \times \mathbb{R}$ we define

$$f(x, \ t) = \frac{d\nu^+}{d(\nu^+ \circ \phi_t)}(x).$$

Then it is easily seen that f is a C^∞ $\mathbb{R}^*_+$-cocycle. We have for all $t \in \mathbb{R}$,

$$f^+ \circ \phi_t = \frac{d(\mu^+ \circ \phi_t)}{d(\nu^+ \circ \phi_t)}$$

$$= \frac{d(\mu^+ \circ \phi_t)}{d\mu^+} \cdot \frac{d\mu^+}{d\nu^+} \cdot \frac{d\nu^+}{d(\nu^+ \circ \phi_t)} = e^{ht} \cdot f^+ \cdot f(\cdot, \ t), \ a.e. \ \mu.$$

Define $C : M \times \mathbb{R} \to \mathbb{R}^+$ such that $C(x,t) = e^{ht} \cdot f(x,t)$. Then C is also a C^∞ $\mathbb{R}_+^*$-cocycle and

$$C_{f^+} = C, \text{ a.e. } \mu.$$

So by Theorem 3.3, f^+ coincides with a C^∞ positive function almost everywhere with respect to μ. Then by Lemma 20.5.11 of **[HK]** f^+ can be taken to be C^∞ positive such that on each leaf W_x^+ of $\mathcal{F}^+$,

$$\mu^+ = f^+ |_{W_x^+} \nu^+,$$

which is denoted also by $\mu^+ = f^+\nu^+$. Similarly we get the C^∞ positive functions f^- and $f^{\pm,0}$. □

Recall that each topologically transitive Anosov diffeomorphism is topologically mixing (see **[HK]**). Then by similar arguments for flows (see also **[Ma1]** and **[HK]**), we can easily get the following

Lemma 3.9 – *Let ϕ be a C^∞ topologically transitive Anosov diffeomorphism on M. Then there exists a unique ϕ-invariant probability measure μ on M such that $h_\mu(\phi) = h_{top}(\phi)$. Furthermore, μ is ϕ-ergodic and positive on non-empty open subsets of M. This measure μ is called the Bowen-Margulis measure of ϕ.*

There exist two families (unique up to scalars) μ^+ and μ^- of measures supported respectively by the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$ such that

$$\mu^\pm \circ \phi = e^{\pm h} \mu^\pm$$

and invariant under the stable and unstable holonomy maps, where h denotes the topological entropy of ϕ. In each product neighborhood, μ is proportional to $\mu^+ \otimes \mu^-$ and $\mu^- \otimes \mu^+$.

If the Bowen-Margulis measure of ϕ is Lebesgue and E^+ and E^- are both orientable, then μ^+ and μ^- are given respectively by C^∞ nowhere-vanishing volume forms along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$.

3.4 Symmetric Anosov flows

3.4.1 Generalities

Let us recall firstly the following definition (see **[To2]**).

Definition 3.2 – Let ψ_t be a C^∞ flow on a manifold N. Then a Lie transformation group G of N is said to be a *symmetry* group of (N, ψ_t) if G centralizes $\{\psi_t\}$ in Diff(N) and the isotropy subgroups are compact in G.

A C^∞ flow ϕ_t on a closed manifold M is said to be *symmetric*, if there exists a normal covering space $\bar{M}$ of M, such that the group of deck transformations is contained as a uniform lattice in an effective and transitive symmetry group of the lifted flow $\bar{\phi}_t$ on $\bar{M}$.

Compared to **[To2]**, we have added to the definition of symmetric flows the effectiveness of the symmetry group action, which makes no essential difference.

Definition 3.3 – An *infra-homogeneous* manifold is a manifold of the form $\Gamma \diagdown G \diagup K$ such that
(1) G is a Lie group and K is a compact subgroup of G such that G/K is connected.
(2) Γ is a uniform lattice in G acting freely on G/K.

For an infra-homogeneous manifold we denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K. If $\alpha \in \mathfrak{g}$ such that $[\alpha, \mathfrak{k}] \equiv 0$ and $\{exp(t\alpha)\}_{t\in\mathbb{R}}$ normalize K, then we get on $\Gamma \diagdown G \diagup K$ a well-defined flow $\psi_t(\Gamma g K) = \Gamma(g \cdot exp(t\alpha))K$, $\forall\ t \in \mathbb{R}$, which is said to be an *algebraic flow*. A C^∞ flow is said to be *algebraic* if it is C^∞ flow equivalent to an algebraic flow (see **[Ze]**).

The following proposition is proved by P. Tomter (see **[To2]**)

Proposition 3.3 – *Let ψ_t be a C^∞ flow on a closed manifold M. Then ψ_t is symmetric if and only if it is algebraic.*

Let ϕ_t be a symmetric Anosov flow with generator X on a closed manifold M of dimension one. Then $M \cong \mathbb{S}^1 \subseteq \mathbb{C}^*$. Since a symmetry group of a certain lift $\bar{\phi}_t$ of ϕ_t acts transitively on $\bar{M}$, then X is identically zero or vanishes nowhere. Suppose that X vanishes nowhere and denote by T the minimal positive periodic of ϕ_t. Then we get a C^∞ diffeomorphism $\rho : \mathbb{R}\diagup\mathbb{Z} \to M$ such that $\rho(t+\mathbb{Z}) = \phi_{tT}(1)$, which conjugates the flow of $\frac{1}{T}\cdot\partial_t$ to ϕ_t. Thus up to C^∞ flow equivalence, one-dimensional algebraic flows are generated by $\{a \cdot \partial_t\}_{a\in\mathbb{R}}$.

So in the following, we consider only the algebraic flows defined on a manifold of dimension at least two. Denote by ϕ_t such a flow. We suppose that ϕ_t is generated by $\alpha \in \mathfrak{g}$ such that $[\alpha, \mathfrak{k}] \equiv 0$. Then we have the primary decomposition of $\mathfrak{g}$ with respect to $ad\alpha$, $\mathfrak{g} = \oplus_{\lambda\in spec(ad\alpha)}\mathcal{E}_{\lambda,\bar{\lambda}}$, where $spec(ad\alpha)$ denotes the spectrum of $ad\alpha$ on $\mathfrak{g}$. We define

$$\mathcal{E}^+ = \oplus_{Re(\lambda)>0}\mathcal{E}_{\lambda,\bar{\lambda}},\ \mathcal{E}^0 = \oplus_{Re(\lambda)=0}\mathcal{E}_{\lambda,\bar{\lambda}},\ \mathcal{E}^- = \oplus_{Re(\lambda)<0}\mathcal{E}_{\lambda,\bar{\lambda}}.$$

So we get

$$\mathfrak{g} = \mathcal{E}^+ \oplus \mathcal{E}^0 \oplus \mathcal{E}^-.$$

If K is connected, then because of $[\alpha, \mathfrak{k}] \equiv 0$ we can get on $\Gamma \diagdown G \diagup K$ three C^∞ ϕ_t-invariant vector bundles E^+ and E^0 and E^- by translating $D\pi(\mathcal{E}^+)$ and $D\pi(\mathcal{E}^0)$ and $D\pi(\mathcal{E}^-)$ by left multiplications, where π denotes the projection of G onto $G\diagup K$. Then it is easy to see that the vectors in E^+ are contracted exponentially in the positive direction of ϕ_t and those in E^- are dilated exponentially. In addition the vectors in E^0 are neither contracted nor dilated exponentially (see **[To1]** for full details). We deduce that ϕ_t is Anosov iff $\mathcal{E}^0 = \mathbb{R}\alpha + \mathfrak{k}$.

If K is not connected, then there exists a finite lift $\bar{\phi}_t$ of ϕ_t with connected isotropy subgroup. However by Lemma 3.3, ϕ_t is Anosov iff $\bar{\phi}_t$ is Anosov. So we get the following

Lemma 3.10 – *Let ϕ_t be an algebraic flow defined on a manifold of dimension at least two. Fix an element $\alpha \in \mathfrak{g}$ such that $[\alpha, \mathfrak{k}] \equiv 0$ and α generates the flow ϕ_t. Then under the notations above ϕ_t is Anosov iff $\mathcal{E}^0 = \mathbb{R}\alpha + \mathfrak{k}$. In addition algebraic Anosov flows are Anosov-smooth.*

Remark 3.2 – Under the notations above, we define $\mathcal{E}^a = \oplus_{Re(\lambda)=a} \mathcal{E}_{\lambda,\bar{\lambda}}$ for any $a \in \mathbb{R}$. If we denote by $\mathcal{E}^a$ the translated vector bundle on $\Gamma \diagdown G \diagup K$, then by similar arguments as in Lemma 3.3, it is easily seen that $\mathcal{E}^a$ is well-defined and is a Lyapunov bundle with Lyapunov exponent a of ϕ_t (see also [**To1**]).

The classical examples of symmetric Anosov flows are the suspensions of hyperbolic infra-nilautomorphisms and the geodesic flows of closed locally symmetric Riemannian spaces of rank one. Symmetric Riemannian spaces have been classified by É. Cartan (see [**He**] for the details). Let us recall the definition of hyperbolic infra-nilautomorphisms.

Let N be a connected and simply-connected nilpotent Lie group. Let C be a compact subgroup of $Aut(N)$. If Γ is a torsion-free uniform lattice of $N \rtimes C$, then it is said to be an *almost Bieberbach* group. In this case $\Gamma \diagdown N$ is a C^∞ manifold called an *infra-nilmanifold*.

Let ψ be an automorphism of N. If no eigenvalue of the differential $D_e\psi$ is of unit absolute value and $\psi \circ \Gamma \circ \psi^{-1} = \Gamma$, then the induced diffeomorphism $\bar{\psi}$ of $\Gamma \diagdown N$ is said to be a *hyperbolic infra-nilautomorphism* which is easily seen to be Anosov. If $\Gamma \subseteq N$, then $\bar{\psi}$ is said to be a *hyperbolic nilautomorphism*. It is easy to see that the suspensions of such hyperbolic infra-nilautomorphisms are symmetric Anosov flows (see [**To1**] for the details).

3.4.2 A construction using Clifford algebras

In [**To1**] P. Tomter constructed explicitely a seven-dimensional non-classical symmetric Anosov flow. By using his idea we construct here a whole family of new symmetric Anosov flows of dimension 13. The construction is based on some representations of spin groups. Let us recall firstly some notions about Clifford algebras.

Let V be a real (or complex) vector space with a quadratic form q. Then the Clifford algebra of (V, q) is defined to be the quotient algebra of the tensor algebra $\otimes V$ by the idea generated by

$$u \otimes u + q(u) \cdot 1, \ \forall \ u \in V,$$

which is denoted by $Cl(V, q)$. The linear map $\alpha(u) = -u$ on V extends uniquely to an algebra isomorphism of $Cl(V, q)$ denoted also by α. We define for $i = 0$ or 1

$$Cl^i(V, q) = \{\phi \in Cl(V, q) \mid \alpha(\phi) = (-1)^i \phi\}.$$

Then $Cl^0(V,q)$ is said to be the *even part* of $Cl(V,q)$ and $Cl^1(V,q)$ is said to be the *odd part* of $Cl(V,q)$. In addition we have

$$Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$$

as vector spaces. Denote by $Cl^*(V,q)$ the group of invertible elements in $Cl(V,q)$. Then there is a homomorphism

$$Ad : Cl^*(V,q) \to Aut(Cl(V,q))$$

called the *adjoint representation* which is given by

$$Ad_\phi(x) = \phi x \phi^{-1}.$$

The anti-automorphism of $\otimes V$ defined by reversing the order in a simple product, i.e. sending $v_1 \otimes \cdots \otimes v_p$ to $v_p \otimes \cdots \otimes v_1$, determines a unique anti-automorphism of $Cl(V,q)$ which is denoted by $x \to \check{x}$ and refered to as the *check involution.* Then the *reduced Clifford group* is defined as follows

$$spin_0(V,q) = \{\phi \in Cl^*(V,q) \cap Cl^0(V,q) \mid Ad_\phi(V) \subseteq V,\ \phi\check{\phi} = 1\},$$

which is nothing but the identity component of the spin group $spin(V,q)$ (see **[Ha]** or **[LM]**).

Now we begin to construct a new non-classical symmetric Anosov flow. Take an integral quadratic form $q = -n_0 dx_0^2 + n_1 dx_1^2 + n_2 dx_2^2 + n_3 dx_3^2$ such that $\{n_0, \cdots, n_3\}$ are all positive integers and it admits no non-zero integral solutions. Such a quadratic form exists (see **[Di]**, Chapter XI).

Denote by $\{\partial_0, \cdots, \partial_3\}$ the canonical basis of $\mathbb{R}^4 = V$. Then each element a of $Cl^0(V,q)$ is of the following form

$$a = a_0 + a_{01}\partial_0\partial_1 + a_{02}\partial_0\partial_2 + a_{03}\partial_0\partial_3 + a_{12}\partial_1\partial_2 +$$

$$+ a_{13}\partial_1\partial_3 + a_{23}\partial_2\partial_3 + a_{0123}\partial_0\partial_1\partial_2\partial_3.$$

By a simple calculation we can see that $a \in spin_0(V,q)$ iff the following two algebraic equations are satisfied:

$$(1)\quad a_0 a_{0123} - a_{01}a_{23} + a_{02}a_{13} - a_{03}a_{12} = 0,$$

$$(2)\quad a_0^2 - n_0 n_1 a_{01}^2 - n_0 n_2 a_{02}^2 - n_0 n_3 a_{03}^2 + n_1 n_2 a_{12}^2 +$$

$$+ n_1 n_3 a_{13}^2 + n_2 n_3 a_{23}^2 + n_0 n_1 n_2 n_3 a_{0123}^2 = 1.$$

So $spin_0(V,q)$ is naturally a six-dimensional real algebraic group. It is easy to see that its Lie algebra is

$$\mathfrak{g} = \{b_{01}\partial_0\partial_1 + \cdots + b_{23}\partial_2\partial_3 \mid b_{01}, \cdots, b_{23} \in \mathbb{R}\}.$$

By a simple calculation we can see that the spectrum of $ad(\partial_0\partial_1)$ on $\mathfrak{g}$ is

$$\{0,\ 2\sqrt{n_0 n_1},\ -2\sqrt{n_0 n_1}\}$$

and $\mathcal{E}^0$ is generated by $\partial_0\partial_1$ and $\partial_2\partial_3$.

Denote by K the connected subgroup of $spin_0(V,q)$ generated by $\partial_2\partial_3$. Then K is compact. In fact we have for all $t \in \mathbb{R}$

$$e^{t(\partial_2\partial_3)} = cos(\sqrt{n_2 n_3}\cdot t) + \frac{1}{\sqrt{n_2 n_3}} sin(\sqrt{n_2 n_3}\cdot t)\cdot \partial_2\partial_3.$$

The group $spin_0(V,q)$ acts by multiplication on $Cl^0(V,q)$. The spectrum of the linear action of $\partial_0\partial_1$ on $Cl^0(V,q)$ is easily seen to be $\{\sqrt{n_0 n_1}, -\sqrt{n_0 n_1}\}$. Thus in the primary decomposition of $Cl^0(V,q) \rtimes \mathfrak{g}$ with respect to $ad(\partial_0\partial_1)$, we have

$$\mathcal{E}^0 = \mathfrak{g}_K \oplus \mathbb{R}\cdot \partial_0\partial_1.$$

So in order to construct a symmetric Anosov flow we need only find a torsion-free uniform lattice in $Cl^0(V,q) \rtimes spin_0(V,q)$ (see Lemma 3.10). Let us recall firstly some notions.

Let G be a linear algebraic group defined over $\mathbb{Q}$. A subgroup Γ of G is said to be an *arithmetic* subgroup if there exists a certain faithful $\mathbb{Q}$-representation $\rho : G \to GL_n$ such that $\rho(\Gamma)$ is commensurable with $\rho(G) \cap GL_n(\mathbb{Z})$. The same condition is then fulfilled for every faithful $\mathbb{Q}$-representation of G.

Denote by $G_{\mathbb{R}}$ the subgroup of real points of G. Then an arithmetic subgroup Γ is always discrete in $G_{\mathbb{R}}$. By the compactness criterion for arithmetic subgroups, for reductive G, Γ is uniform in $G_{\mathbb{R}}$ iff G is anisotropic over $\mathbb{Q}$, i.e. if it has no $\mathbb{Q}$-split torus $S \neq \{e\}$ (see [**Bor**]).

The complex Clifford algebra $Cl(V^{\mathbb{C}},q)$ is defined similarly as $Cl(V,q)$. We define a linear algebraic group $spin_0(V^{\mathbb{C}},q)$ by the two algebraic equations above. Then we have $(spin_0(V^{\mathbb{C}},q))_{\mathbb{R}} = spin_0(V,q)$. In addition $Ad\ |_{spin_0(V^{\mathbb{C}},q)}$ is defined over $\mathbb{Q}$ and is a finite covering map onto $SO(q)$, which denotes the special q-orthogonal subgroup of $GL(V^{\mathbb{C}})$.

Since q admits no non-zero integral solutions, then $SO(q)$ is anisotropic over $\mathbb{Q}$ (see [**Bor**]). We deduce that $spin_0(V^{\mathbb{C}},q)$ is also anisotropic over $\mathbb{Q}$.

The linear action by multiplication of $spin_0(V^{\mathbb{C}},q)$on $Cl^0(V^{\mathbb{C}},q)$ is defined over $\mathbb{Q}$ and faithful. Take the lattice $\Lambda = \mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\partial_0\partial_1\partial_2\partial_3 \subseteq Cl^0(V,q)$ and denote by Γ the isotropy subgroup of this lattice in $spin_0(V,q)$. Since $spin_0(V^{\mathbb{C}},q)$ is anisotropic over $\mathbb{Q}$, then by the compactness criterion above, Γ is seen to be a uniform lattice in $spin_0(V,q)$. By passing to a finite index torsion-free subgroup of Γ, we get a torsion-free uniform lattice $\Lambda \rtimes \Gamma$ of $Cl^0(V,q) \rtimes spin_0(V,q)$. In particular $\Lambda \rtimes \Gamma$ acts freely on $(Cl^0(V,q) \rtimes spin_0(V,q))\diagup K$.

So by Lemma 3.10, $\partial_0\partial_1$ gives a non-classical algebraic Anosov flows on the infra-homogeneous manifold $\Lambda \rtimes \Gamma\diagdown Cl^0(V,q) \rtimes spin_0(V,q)\diagup K$. By taking the direct product of these kind of representations we can thus get many examples of non-classical symmetric Anosov flows.

Recall that P. Tomter used a three-variable quadratic form $-n_0 dx_0^2 + n_1 dx_1^2 + n_2 dx_2^2$ to construct his seven-dimensional example.

3.4.3 A rough classification of symmetric Anosov flows

Recall that two flows ϕ_t^1 and ϕ_t^2 are said to be *commensurable*, if some finite normal cover of ϕ_t^1 is C^∞ flow equivalent to some finite normal cover of ϕ_t^2.

Since the isotropy subgroups of symmetric Anosov flows are supposed to be compact, then it is essentially a Lie theoretical problem to classify such flows. By Theorems five and six of **[To2]** we have the following

Proposition 3.4 – *Let ϕ_t be a symmetric Anosov flow on a closed m-dimensional manifold. Then we have the following descriptions:*
(1) *If rank$(\phi_t) = 2[\frac{m}{2}]$, then ϕ_t is commensurable to the geodesic flow of a closed locally symmetric Riemannian space of rank one.*
(2) *If rank$(\phi_t) = 0$, then ϕ_t is commensurable to the suspension of a hyperbolic infra-nilautomorphism.*
(3) *If $0 <$rank$(\phi_t) < 2[\frac{m}{2}]$, then ϕ_t is commensurable to a flow ψ_t constructed as following:*

Define $H = Spin(n,1) \times K_1 \times \cdots \times K_p$ where $n \geq 2$ and $spin(n,1)$ denotes the spin group of a quadratic form of index one and $K_1, \cdots, K_p$ are compact, simply connected and almost simple Lie groups. Let $\mathfrak{k}' \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{so}(n,\ 1)$ and α be a non-zero element of $\mathfrak{p}$. Let $\mathfrak{k}$ be the centralizer of α in $\mathfrak{k}'$ and K be the connected Lie subgroup of H with Lie algebra $\mathfrak{k} \oplus \mathfrak{g}_{K_1} \oplus \cdots \oplus \mathfrak{g}_{K_p}$. Let V be a real vector space of positive dimension and $\rho : H \to GL(V)$ be a linear representation such that in the Lie algebra of the semidirect product $G = V \rtimes_\rho H$ we have

$$\mathcal{E}^0 = \mathfrak{g}_K \oplus \mathbb{R}\alpha,$$

where $\mathcal{E}^0$ is defined as above using the primary decomposition of $\mathfrak{g}_G$ with respect to adα. Let Γ be a uniform lattice in G acting freely on G/K. Then we get on the infra-homogeneous manifold $\Gamma\backslash G/K$ the following well-defined flow

$$\psi_t(\Gamma g K) = \Gamma(g \cdot exp(t\alpha))K,\ \forall\ t \in \mathbb{R},\ \forall\ g \in G.$$

Remark 3.3 – For Case (3) in the previous proposition, we can find in **[To2]** an algebraic characterization of the linear representations fulfilling the conditions, i.e. the existence of a uniform lattice and $\mathcal{E}^0 = \mathfrak{g}_K \oplus \mathbb{R}\alpha$.

If the rank of a symmetric Anosov flow ϕ_t is zero or maximal, then it preserves certainly a C^∞ volume form. Otherwise in Case (3) of the previous proposition, since $ad\alpha$ acts on V without pure imaginary eigenvalues, then it is easy to verify that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. So the Haar measure of G is bi-invariant, which passes to the quotient to give a flow-invariant volume form. So we deduce that each symmetric Anosov flow preserves a C^∞ volume form.

Chapter 4

On Special Time Changes of Anosov-smooth Flows

Abstract – *Given an Anosov-smooth flow, we characterize all its Anosov-smooth time changes and we prove the existence of an Anosov-smooth time change whose Bowen-Margulis measure is in the Lebesgue class. In Subsection* 4.5*, we outline our go-and-back idea.*

4.1 Introduction

4.1.1 Anosov-smooth time changes

Let ϕ_t be an Anosov flow on a closed manifold M. Denote by X the generator of ϕ_t. For each C^∞ positive function f the flow of fX is said to be a *smooth time change* of ϕ_t. Since ϕ_t is Anosov, then the flow of fX is also Anosov (see [**HK**]). Denote by E^+_{fX} and E^-_{fX} the strong unstable and strong stable distributions of the flow of fX. Then it is easily seen (see Lemma 1.2 of [**LMM**]) that

$$E^+_{fX} = \{u^+ + \theta^+(u^+) \cdot X \mid \forall\ u^+ \in E^+\},$$

where θ^+ is the unique C^0-section of $(E^+)^*$ such that

$$\mathcal{L}_X(f^{-1}\theta^+) = f^{-2} \cdot df\ |_{E^+}\ .$$

Similarly we have $E^-_{fX} = \{u^- + \theta^-(u^-) \cdot X \mid \forall\ u^- \in E^-\}$, where θ^- is the unique C^0-section of $(E^-)^*$ such that

$$\mathcal{L}_X(f^{-1}\theta^-) = f^{-2} \cdot df\ |_{E^-}\ .$$

In particular we have $E^{+,0}_{fX} = E^{+,0}_X$ and $E^{-,0}_{fX} = E^{-,0}_X$.

Definition 4.1 – Let ϕ_t be a C^∞ Anosov flow on M with generator X. If α is a C^∞ 1-form on M such that $\alpha(X) > 0$ and $\mathcal{L}_X d\alpha = 0$, then the flow of $\frac{X}{\alpha(X)}$ is said to be a *special time change* of ϕ_t.

Lemma 4.1 – *Let ϕ_t be an Anosov-smooth flow and ψ_t be a C^∞ time change of ϕ_t. Then ψ_t is Anosov-smooth iff it is a special time change of ϕ_t.*

Proof – Suppose that the generator of ψ_t is fX and ψ_t is Anosov-smooth. Denote by λ_f its canonical 1-form. Then λ_f is C^∞ and we have $\lambda_f(fX) = 1$, i.e. $f = \frac{1}{\lambda_f(X)} > 0$. Since λ_f is ψ_t-invariant, then by the Anosov property of ψ_t, we have for any $u^+ \in E^+_{fX}$ and any $t > 0$,

$$| d\lambda_f(fX, u^+) |=| d\lambda_f(fX, D\psi_{-t}(u^+)) |\leq\| d\lambda_f \| \cdot \| fX \| \cdot \| D\psi_{-t}(u^+) \|\to 0.$$

So we get $d\lambda_f(fX, E^+_{fX}) \equiv 0$. Similarly $d\lambda_f(fX, E^-_{fX}) \equiv 0$. We deduce that $i_{fX}d\lambda_f = 0$. Thus $i_X d\lambda_f = 0$ and

$$\mathcal{L}_X d\lambda_f = i_X d(d\lambda_f) + di_X(d\lambda_f) = 0.$$

So λ_f fulfills the conditions in the definition of special time changes. Since $fX = \frac{X}{\lambda_f(X)}$, then ψ_t is a special time change of ϕ_t.

Conversely we suppose that ψ_t is a special time change of ϕ_t generated by $\frac{X}{\alpha(X)}$. Since $\mathcal{L}_X d\alpha = 0$, then by the Anosov property we get as above $i_X d\alpha = 0$. Denote $\frac{1}{\alpha(X)}$ by f. Then we have

$$\mathcal{L}_X(f^{-1} \cdot (-\frac{\alpha}{\alpha(X)})) = -\mathcal{L}_X\alpha$$

$$= -(i_X d\alpha + d(\alpha(X))) = -d(\alpha(X)).$$

On the other hand, $f^{-2} \cdot df = \alpha(X)^2 \cdot d(\frac{1}{\alpha(X)}) = -d(\alpha(X))$. Then by the formula above, we get

$$E^\pm_{\frac{X}{\alpha(X)}} = \{u^\pm - \frac{\alpha}{\alpha(X)}(u^\pm) \cdot X \mid \forall\ u^\pm \in E^\pm\}.$$

So ψ_t is Anosov-smooth. □

Definition 4.2 – Let ϕ_t be an Anosov-smooth flow on M with generator X. If $a > 0$ and β is a C^∞ closed 1-form on M such that $a + \beta(X) > 0$, then the flow of $\frac{X}{a+\beta(X)}$ is said to be a *canonical time change* of ϕ_t.

Certainly each canonical time change of ϕ_t is special. The problem is to determine for a certain Anosov-smooth flow whether its special time changes are all canonical.

4.1.2 The organization of the chapter

In Section 4.2 we prove that the special time changes are canonical for either the geodesic flows of closed locally symmetric Riemannian spaces of rank one or the suspensions of hyperbolic infra-nilautomorphisms. In Section 4.3 we prove for each

Anosov-smooth flow the existence of an Anosov-smooth time change whose Bowen-Margulis measure is Lebesgue. Then in Section 4.4 we show that for such a time change the Margulis measures are flat with respect to certain linear connections. Finally in Section 4.5 we outline our *go-and-back* idea to prove the rigidity of Anosov-smooth flows, which will furnish us the departing point of Chapters V and VI and VII.

4.2 Special time changes of certain algebraic Anosov flows

The following proposition is proved in [**Ham1**].

Proposition 4.1 (U. Hamenstadt) – *Let ψ_t be the geodesic flow of a closed negatively curved manifold. If its Anosov splitting is C^1, then for each C^1 1-form α such that $d\alpha$ is ψ_t-invariant, $d\alpha$ is proportional to $d\lambda$, where λ denotes the canonical 1-form of ψ_t.*

We deduce from this proposition the following

Lemma 4.2 – *Let ϕ_t be the geodesic flow of a closed locally symmetric space of rank one. Then each special time change of ϕ_t is canonical.*

Proof – Denote by α a C^∞ 1-form such that $\alpha(X) > 0$ and $\mathcal{L}_X d\alpha = 0$. Then by the previous theorem there exists $a \in \mathbb{R}$ such that

$$d\alpha = a \cdot d\lambda,$$

where λ denotes the canonical 1-form of ϕ_t. So there exists a C^∞ closed 1-form β such that $\alpha = a \cdot \lambda + \beta$. We need only see that $a > 0$.

Suppose on the contrary that $a \leq 0$. Since $\alpha(X) = a + \beta(X) > 0$, then $\beta(X) > 0$. Suppose that the locally symmetric space is of dimension n. Then we get

$$0 = -\int_{\partial M} \beta \wedge \lambda \wedge (\wedge^{n-1} d\lambda) = \int_M \beta \wedge (\wedge^n d\lambda)$$

$$= \int_M \beta(X) \cdot \lambda \wedge (\wedge^n d\lambda) > 0,$$

which is a contradiction. □

Now we want to find out all the special time changes of the suspensions of hyperbolic infra-nilautomorphisms (see Subsection 3.4.1 for definition).

Lemma 4.3 – *Let ϕ be a hyperbolic infra-nilautomorphism on $\Gamma \diagdown N$. If α be a C^1 closed 1-form such that $d\alpha$ is ϕ-invariant, then $d\alpha$ is zero.*

Proof – By a result of L. Auslander (see [**Au**]), each hyperbolic infra-nilautomorphism is normally and finitely covered by a hyperbolic nilautomorphism. So we need only prove this lemma for hyperbolic nilautomorphisms. So in the following we suppose that $\Gamma \subseteq N$. To simplify the notations we denote $\Gamma\diagdown N$ also by M. Remark that any hyperbolic nilautomorphism is topologically transitive and has dense periodic orbits.

Firstly let us recall some notions. Consider the complexified vector bundles $TM \otimes \mathbb{C}$ and $T^*M \otimes \mathbb{C}$. Any diffeomorphism of M acts naturally on these two complex vector bundles and also on their exterior powers by acting respectively on the real and imaginary parts. The brackets of the sections of $TM \otimes \mathbb{C}$ are defined by extending $\mathbb{C}$-linearly the brackets of real vector fields. The action of sections of $TM \otimes \mathbb{C}$ on $\mathbb{C}$-valued C^∞ functions on M are defined by extending $\mathbb{C}$-linearly the action of real vector fields on $\mathbb{R}$-valued C^∞ functions. Then for all $k \in \mathbb{N}$ the exterior derivation of the sections of $\wedge^k(T^*M \otimes \mathbb{C})$ is defined as in the real case and is also denoted by d, i.e. for arbitrary C^∞ sections $\{Y_0, \cdots, Y_k\}$ of $TM \otimes \mathbb{C}$ and arbitrary C^1 section α of $\wedge^k(T^*M \otimes \mathbb{C})$ we have

$$d\alpha(Y_0, \cdots, Y_k) = \sum_{0 \le i \le k} (-1)^i \cdot Y_i(\alpha(Y_0, \cdots, \hat{Y}_i, \cdots, Y_k)) +$$

$$+ \sum_{i<j} (-1)^{i+j} \cdot \alpha([Y_i, Y_j], Y_0, \cdots, \hat{Y}_i, \cdots, \hat{Y}_j, \cdots, Y_k).$$

Any smooth k-form γ can be extended $\mathbb{C}$-linearly to a section of $\wedge^k(T^*M \otimes \mathbb{C})$ denoted by $\gamma^{\mathbb{C}}$. Then γ is ϕ-invariant iff $\gamma^{\mathbb{C}}$ is ϕ-invariant.

A section of $\wedge^k(TM \otimes \mathbb{C})$ or $\wedge^k(T^*M \otimes \mathbb{C})$ (or $\wedge^k T^*M$) is said to be *left-invariant* if its lift to N is left-invariant.

Now let us returm to the proof of the lemma. Take independent left-invariant sections of $TM \otimes \mathbb{C}$, $\{X_1, \cdots, X_n\}$ such that the matrix A of ϕ_* is in Jordan form:

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}; \quad A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}, \ \forall\ 1 \le j \le k.$$

Then for all $m \ge n$ we have

$$(A_j)^m = \begin{pmatrix} (\lambda_j)^m & \binom{m}{1}(\lambda_j)^{m-1} & \dots & \binom{m}{l-1}(\lambda_j)^{m-l+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \binom{m}{1}(\lambda_j)^{m-1} \\ 0 & \dots & 0 & (\lambda_j)^m \end{pmatrix}, \ \forall\ 1 \le j \le k,$$

where l denotes the order of A_j. Denote by $\{X_1^*, \cdots, X_n^*\}$ the dual basis of $\{X_1, \cdots, X_n\}$. Note that the matrix of ϕ^* is just A^T in the basis $\{X_1^*, \cdots, X_n^*\}$. There exist C^0

complex functions $\{f_{ij}\}$ on M such that

$$(d\alpha)^{\mathbb{C}} = \sum_{1\leq i<j\leq n} f_{ij} X_i^* \wedge X_j^*.$$

Denote by I_j the set of indices of A_j in A. Take $i,\ j \in \{1, \cdots, k\}$ such that $i \leq j$ and define an order on $I_i \times I_j$ as following:

For $(s,t),\ (s',t') \in I_i \times I_j$, $(s,t) < (s',t')$ iff $s < s'$ or if $s = s'$, then $t < t'$. Let (k,l) be the smallest element of $I_i \times I_j$. For all $m \geq n$ we have $(\phi^m)^*(d\alpha)^{\mathbb{C}} = (d\alpha)^{\mathbb{C}}$, i.e.

$$(d\alpha)^{\mathbb{C}} = \sum_{1\leq i<j\leq n} f_{ij} \circ \phi^m (\phi^*)^m X_i^* \wedge (\phi^*)^m X_j^*. \qquad (*)$$

By comparing the coefficients of $X_k^* \wedge X_l^*$ of both sides we get

$$f_{kl} = (\lambda_i \lambda_j)^m (f_{kl} \circ \phi^m).$$

So if $\mid \lambda_i\lambda_j \mid \neq 1$, then $f_{kl} \equiv 0$. If $\mid \lambda_i\lambda_j \mid = 1$ and $f_{kl} \not\equiv 0$, then by the density of the periodic orbits of ϕ there exists a point x in M and $p \geq \mathbb{N}$ such that $f_{kl}(x) \neq 0$ and $\phi^p(x) = x$. So we get $(\lambda_i\lambda_j)^p = 1$. We deduce that $f_{kl} \circ \phi^p = f_{kl}$. But ϕ^p is also a hyperbolic nilautomorphism, so topologically transitive. We deduce that f_{kl} is constant.

Take $(s,t) \in I_i \times I_j$ and suppose that $f_{s't'}$ is constant for all $(s',t') < (s,t)$. Then by comparing the coefficients of $X_s^* \wedge X_t^*$ of both sides of $(*)$ and using the special form of A^m, we easily get

$$f_{st} = (\lambda_i\lambda_j)^m (f_{st} \circ \phi^m + P(m)),$$

where $P(m)$ is a polynomial without term of degree zero. Then using similar arguments as above we see that f_{st} is constant. So by induction on $I_i \times I_j$, f_{ab} is seen to be constant for all $(a,b) \in I_i \times I_j$. If in addition $\mid \lambda_i\lambda_j \mid \neq 1$, then we have in fact $f_{ab} \equiv 0$ for all $(a,b) \in I_i \times I_j$. So in order to prove this lemma, we need only see that $f_{ab} = 0$ for all $(a,b) \in I_i \times I_j$ such that $\mid \lambda_i\lambda_j \mid = 1$.

By the arguments above, $(d\alpha)^{\mathbb{C}}$ is seen to be left-invariant. So $d\alpha$ is also left-invariant. In particular we deduce that $d\alpha$ is C^∞. Since the Stokes formula is valid for C^1 forms, then it is easily seen that $d\alpha$ represents the zero cohomology class in the first cohomology group of M. So by **[No]** there exists a left-invariant 1-form β on $\Gamma \diagdown N$ such that $d\beta = d\alpha$. Then we have $(d\alpha)^{\mathbb{C}} = (d\beta)^{\mathbb{C}} = d(\beta^{\mathbb{C}})$. Fix $i \leq j$ such that $\mid \lambda_i\lambda_j \mid = 1$. Let (k,l) be the smallest element of $I_i \times I_j$ then we have

$$\phi_*[X_k, X_l] = \lambda_i\lambda_j [X_k, X_l].$$

Since ϕ is hyperbolic, then $\lambda_i\lambda_j$ can not be an eigenvalue of ϕ_*. So we get $[X_k, X_l] = 0$. Now by induction on $I_i \times I_j$ we easily get $[X_a, X_b] = 0$ for all $(a,b) \in I_i \times I_j$. Since $\beta^{\mathbb{C}}$ is left-invariant, then

$$f_{ab} = d(\beta^{\mathbb{C}})(X_a, X_b) = -\beta^{\mathbb{C}}([X_a, X_b]) = 0.$$

We deduce that $(d\alpha)^{\mathbb{C}} = 0$, i.e. $d\alpha = 0$. □

Lemma 4.4 – *Let ϕ_t be a C^∞ flow on M generated by X. If f is a C^∞ function on M such that $1 + X(f) > 0$, then ϕ_t is C^∞ flow equivalent to the flow of $\frac{X}{1+X(f)}$.*

Proof – We define the following map,

$$\psi^X : M \longrightarrow M$$

$$x \rightarrow \phi_{f(x)}(x).$$

Then by a simple calculation we get

$$D\psi^X\left(\frac{X}{1+X(f)}\right) = X.$$

Since $X = \frac{\frac{X}{1+X(f)}}{1+(\frac{X}{1+X(f)})(-f)}$, then we get a similar map $\psi^{\frac{X}{1+X(f)}}$ such that

$$D\psi^{\frac{X}{1+X(f)}}(X) = \frac{X}{1+X(f)}.$$

Denote by ϕ_t^f the flow of $\frac{X}{1+X(f)}$. Then for all $t \in \mathbb{R}$ and all $x \in M$ we have

$$\phi_t(x) = \phi^f_{\beta(t,x)}(x),$$

where $\beta(t,x) = \int_0^t (1 + X(f))(\phi_s(x))ds$. So we get

$$\beta(t,x) = t + f(\phi_t(x)) - f(x).$$

Then for $x \in M$ we have

$$\psi^{\frac{X}{1+X(f)}} \circ \psi^X(x) = \phi^f_{(-f)(\phi_{f(x)}(x))}(\phi_{f(x)}(x))$$

$$= \phi^f_{-f(\phi_{f(x)}(x))}(\phi^f_{f(\phi_{f(x)}(x))}(x)) = x.$$

So we get $\psi^{\frac{X}{1+X(f)}} \circ \psi^X = Id$. Similarly we have $\psi^X \circ \psi^{\frac{X}{1+X(f)}} = Id$. So ϕ_t is C^∞ flow equivalent to the flow of $\frac{X}{1+X(f)}$. □

Now we can prove the following

Lemma 4.5 – *Let ϕ_t be the suspension of a hyperbolic infra-nilautomorphism. Then each special time change of ϕ_t is homothetic to ϕ_t, i.e. up to a constant change of time scale, each special time change of ϕ_t is C^∞ flow equivalent to ϕ_t.*

Proof – Suppose that ϕ_t is the suspension flow of the hyperbolic infra-nilautomorphism (ϕ, M). Denote by $\bar{M}$ the suspension manifold and by X the generator of ϕ_t.

Let α be a C^∞ 1-form on $\bar{M}$ such that $\alpha(X) > 0$ and $\mathcal{L}_X d\alpha = 0$. Since $d\alpha$ is ϕ_t-invariant, then its restriction to M, $d\alpha \mid_M$, is ϕ-invariant. Then by Lemma 4.3, $d\alpha \mid_M$ is zero, i.e. $d\alpha(E^+, E^-) \equiv 0$. However by the Anosov property, we have $d\alpha(X, E^+) = d\alpha(X, E^-) = 0$. Thus $d\alpha = 0$.

By Theorem 4 of **[Hi]** and Lemma 3.3 of **[Pl1]**, we get $H^1(\bar{M}, \mathbb{R}) \cong \mathbb{R}$. Thus there exists $a \in \mathbb{R}$ such that $[\alpha] = [a \cdot \lambda]$ in $H^1(\bar{M}, \mathbb{R})$, where λ denotes the canonical 1-form of ϕ_t. By integrating along a periodic orbit of ϕ_t we get $a > 0$. Then by Lemma 4.4 the flow of $\frac{X}{\alpha(X)}$ is C^∞ flow equivalent to that of $\frac{1}{a}X$. $\square$

4.3 Parry time change.

Let ϕ_t be a C^∞ volume-preserving Anosov-smooth flow on M. Fix a C^∞ Lyapunov metric g on M. Then there exists $b > 0$ such that

$$\| D\phi_{\mp t}(u^\pm) \| \leq e^{-bt} \| u^\pm \|, \ \forall \ t > 0, \ \forall \ u^\pm \in E^\pm. \tag{$*$}$$

We suppose that E^+ and E^- are both orientable. Denote by n and m the dimensions of E^+ and E^- and by $\nu^\pm$ the induced volume forms of $g \mid_{E^\pm}$ on $E^\pm$. Then ν^+ and ν^- are C^∞ nowhere-vanishing sections of $\wedge^n(E^+)^*$ and $\wedge^m(E^-)^*$. For all $x \in M$ and all $t \in \mathbb{R}$ we define

$$det(D\phi_t \mid_{E_x^+}) = \frac{(\phi_t^*\nu^+)_x}{\nu_x^+}, \ det(D\phi_t \mid_{E_x^-}) = \frac{(\phi_t^*\nu^-)_x}{\nu_x^-}.$$

Then by $(*)$ we get for all $t > 0$,

$$det(D\phi_t \mid_{E_x^+}) \geq e^{nbt}, \ det(D\phi_t \mid_{E_x^-}) \leq e^{-mbt}. \tag{$**$}$$

For all $x \in M$ we define

$$\phi^\pm(x) = \frac{\partial}{\partial t} \mid_{t=0} log(det(D\phi_t \mid_{E_x^\pm})).$$

Since $E^\pm$ are both C^∞, then ϕ^+ and $\phi^-$are both smooth. In addition by $(**)$ we get

$$\phi^+ \geq nb > 0, \ \phi^- \leq -mb < 0.$$

Denote by X the generator of ϕ_t and define a C^∞ time change $X^+ = \frac{X}{\phi^+}$ whose flow is denoted by ϕ_t^+. Then ϕ_t^+ is also a C^∞ volume-preserving Anosov flow. In **[Par]**, W. Parry proved that the Bowen-Margulis measure of ϕ_t^+ is in the Lebesgue measure class. We propose the following

Definition 4.3 – Let ϕ_t be a C^∞ Anosov-smooth flow whose strong stable and strong unstable distributions are both orientable. Under the notations above, the flow generated by the vector field $\frac{X}{\phi^+}$ is said to be the *Parry time change* of ϕ_t.

The purpose of this section is to prove the following

Theorem 4.1 – *Let ϕ_t be a C^∞ volume-preserving Anosov-smooth flow on M. We suppose that E^+ and E^- are both orientable. Then its Parry time change is also Anosov-smooth.*

We shall prove this theorem via several lemmas. Under the notations above, we denote by ϕ_t^+ the Parry time change of ϕ_t and denote by ϕ_t^- the flow of $\frac{X}{-\phi^-}$. We shall prove Theorem 4.1 by finding explicitly the strong stable and strong unstable distributions of ϕ_t^+. Let us prove firstly the following

Lemma 4.6 – Under the notations above, ϕ_t^+ is C^∞ flow equivalent to ϕ_t^-.

Proof – Denote by λ the canonical 1-form of ϕ_t. Since ϕ_t preserves a volume form, then we can find a C^∞ ϕ_t-invariant volume form ν (see Lemma 3.6) and a C^∞ positive function g such that

$$\nu = g \cdot \lambda \wedge \nu^+ \wedge \nu^-.$$

So we get

$$\nu = \phi_t^* \nu = g \circ \phi_t \cdot \lambda \wedge \phi_t^* \nu^+ \wedge \phi_t^* \nu^-$$
$$= \frac{g \circ \phi_t}{g} \cdot (det(D\phi_t \mid_{E^+})) \cdot (det(D\phi_t \mid_{E^-})) \cdot \nu,$$

i.e. $g = g \circ \phi_t \cdot (det(D\phi_t) \mid_{E^+}) \cdot (det(D\phi_t) \mid_{E^-})$. By taking the logarithm of this relation and differentiating at zero with respect to t, we get

$$\phi^+ + \phi^- = -X(log(g)).$$

Recall that $\phi^+ > 0$ and $-\phi^- > 0$. So we have for $H = -log(g)$,

$$\frac{X}{\phi^+} = \frac{X}{-\phi^- + X(H)} = \frac{\frac{X}{-\phi^-}}{1 + (\frac{X}{-\phi^-})(H)}.$$

Then by Lemma 4.4, the flow of $\frac{X}{-\phi^-}$ is C^∞ flow equivalent to that of $\frac{X}{\phi^+}$. □

Let $\mathcal{F}$ be a C^∞ foliation on M whose tangent bundle is denoted by F. Then the leafwise differential forms of $\mathcal{F}$ are just C^∞ sections of the exterior powers of F^*. The exterior differentiation of leafwise differential forms of $\mathcal{F}$ can be defined leafwise and is denoted again by d. For each section Y of F, the classical operations $\mathcal{L}_Y$ and i_Y can be defined naturally and the famous Cartan formula

$$\mathcal{L}_Y = d \circ i_Y + i_Y \circ d$$

is certainly valid in this foliated context. Under these notations, we can prove the following lemma by using the same calculation as that of the proof of Lemma 4.1.

Lemma 4.7 – *Let ϕ_t be a C^∞ Anosov flow with generator X. If E^- is C^∞ and α is a C^∞ section of $(E^{-,0})^*$ such that $\alpha(X) > 0$ and $d\alpha(X, E^-) \equiv 0$, then the strong stable distribution of the flow of $\frac{X}{\alpha(X)}$ is also C^∞ and is given by $\{u^- - \frac{\alpha(u^-)}{\alpha(X)}X \mid u^- \in E^-\}$.*

Similarly if E^+ is C^∞ and α is a C^∞ section of $(E^{+,0})^*$ such that $\alpha(X) > 0$ and $d\alpha(X, E^+) \equiv 0$, then the strong unstable distribution of the flow of $\frac{X}{\alpha(X)}$ is also C^∞ and is given by $\{u^+ - \frac{\alpha(u^+)}{\alpha(X)}X \mid u^+ \in E^+\}$.

Thus in order to prove Theorem 4.1, we need only, by the previous two lemmas, find two sections of $(E^{-,0})^*$ and $(E^{+,0})^*$ to fulfill the conditions of Lemma 4.7 with respect to ϕ^+ and $-\phi^-$. Let us recall firstly some notations.

Definition 4.4 – Let $\mathcal{F}$ be a C^∞ foliation on M and E be a vector bundle on M. Denote by F the tangent bundle of $\mathcal{F}$. Then a bilinear map $\nabla : sec(F) \times sec(E) \to sec(E)$ is called a C^∞ *linear connection on* E *along* $\mathcal{F}$ if for sections Y and Z of F and E and C^∞ function f on M we have

$$\nabla_{fY} Z = f\nabla_Y Z, \quad \nabla_Y fZ = Y(f)Z + f\nabla_Y Z.$$

If in the previous definition, E is supposed in addition to be an orientable line bundle, then by taking a nowhere-vanishing section ω of E, we can define the connection form β and the curvature form Ω of ∇ with respect to ω such that

$$\nabla_Y \omega = \beta(Y)\omega, \ (\nabla_Y \nabla_Z - \nabla_Z \nabla_Y - \nabla_{[Y,Z]})\omega = \Omega(Y, Z)\omega$$

for sections Y and Z of F. Thus we have $\beta \in F^*$ and $\Omega \in \wedge^2 F^*$. By a simple calculation, we get the classical structural equation of Cartan in this situation, i.e. $d\beta = \Omega$. In addition, it is easily seen that Ω is independent on the nowhere-vanishing section chosen.

To our Anosov-smooth flow ϕ_t, two linear connections on vector bundles along foliations are canonically associated. We define $\nabla^+ : sec(E^{-,0}) \times sec(E^+) \to sec(E^+)$ such that

$$(\nabla^+)_{fX+Y^-} Y^+ = f[X, Y^+] + P^+[Y^-, Y^+].$$

Then it is easily verified that ∇^+ is a C^∞ linear connection on E^+ along $\mathcal{F}^{-,0}$. Similarly we have a C^∞ linear connection of E^- along $\mathcal{F}^{+,0}$ denoted by ∇^-.

In a natural way (see **[KN]**, Chapter I), ∇^+ induces a C^∞ connection on $\wedge^n E^+$ along $\mathcal{F}^{-,0}$, which is again denoted by ∇^+. Similarly we get a C^∞ connection on $\wedge^m E^-$ along $\mathcal{F}^{+,0}$ denoted again by ∇^-. Take the dual sections $\omega^\pm$ of $\nu^\pm$ and denote by β^- and β^+ the connection forms of ∇^- and ∇^+ with respect to ω^- and ω^+. Their curvature forms are denoted respectively by Ω^- and Ω^+.

Since ∇^+ is ϕ_t-invariant in a natural sense and Ω^+ is a section of $\wedge^2(E^{-,0})^*$, then by the Anosov property of ϕ_t, we have $\Omega^+ \equiv 0$. Similarly we have $\Omega^- \equiv 0$. Then by the structural equation of Cartan, we get

$$d\beta^+(X, E^-) \equiv 0, \quad d\beta^-(X, E^+) \equiv 0.$$

Lemma 4.8 – *Under the notations above, $-\beta^+(X) = \phi^+$ and $-\beta^-(X) = \phi^-$.*

Proof – Fix $x \in M$ and take a C^∞ diffeomorphism $\psi : \mathbb{R}^n \to W_x^+$. Then we get the following smooth map,

$$\rho : \mathbb{R} \times \mathbb{R}^n \to W_x^{+,0},$$

$$(t, v) \to \phi_t(\psi(v)).$$

If $\epsilon \ll 1$, then $\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}$ is easily seen to be diffeomorphic to $(-\epsilon, \epsilon) \times \mathbb{R}^n$ under ρ^{-1}. In addition, ρ^{-1} sends $\mathcal{F}^+ \mid_{\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}}$ to the foliation $\{t \times \mathbb{R}^n\}_{-\epsilon<t<\epsilon}$ of $(-\epsilon, \epsilon) \times \mathbb{R}^n$.

So on $\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}$ we can find a C^∞ connection along the foliation $\mathcal{F}^+ \mid_{\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}}$, which is denoted by ∇_x^+. Then there exists a C^∞ connection ∇_x on $\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}$ such that

$$(\nabla_x)X = 0, \; (\nabla_x)_{Y^+} Z^+ = (\nabla_x^+)_{Y^+} Z^+,$$

$$(\nabla_x)_X Y^+ = [X, Y^+],$$

where Y^+ and Z^+ denote arbitrary C^∞ sections of $E^+ \mid_{\cup_{-\epsilon<t<\epsilon} W^+_{\phi_t x}}$. Denote by τ_t the ∇_x-parallel transport of E_x^+ along the ϕ_t-orbit of x. Then by the definition of ∇_x we get

$$\tau_t = D\phi_t.$$

If we denote by Δ_t the determinant of τ_t, i.e.

$$\Delta_t = \frac{({\tau_t}^* \nu^+)_x}{\nu_x^+} = \frac{\omega_x^+}{(\tau_{-t})_* \omega^+_{\phi_t(x)}},$$

then we have

$$\Delta_t = det(D\phi_t \mid_{E_x^+}).$$

By differentiating the two sides of this equality with respect to t at zero we get

$$\frac{\partial}{\partial t} \mid_{t=0} \Delta_t = \phi^+(x).$$

By the definitions of ∇_x and β^+, we have $(\nabla_x)_X \omega^+ = \mathcal{L}_X \omega^+ = \beta^+(X)\omega^+$. Since

$$((\nabla_x)_X \omega^+)(x) = \frac{\partial}{\partial t} \mid_{t=0} (\tau_{-t})_* \omega^+_{\phi_t(x)},$$

then we get

$$\beta^+(X)(x) = \frac{\partial}{\partial t} \mid_{t=0} (\frac{1}{\Delta_t}) = -\frac{\partial}{\partial t} \mid_{t=0} \Delta_t.$$

So for any $x \in M$ we have

$$-\beta^+(X)(x) = \phi^+(x),$$

i.e. $-\beta^+(X) = \phi^+$. Similarly we can prove $-\beta^-(X) = \phi^-$. □

Proof of Theorem 4.1 – Since $-\beta^+(X) = \phi^+$ and $d\beta^+(X, E^-) \equiv 0$, then by Lemma 4.7, the strong stable distribution of ϕ_t^+ is C^∞. Since $-\beta^-(X) = \phi^-$ and $d\beta^-(X, E^+) \equiv 0$, then by Lemma 4.7, the strong unstable distribution of ϕ_t^- is C^∞. However by Lemma 4.6, ϕ_t^+ and ϕ_t^- are C^∞ flow equivalent. Thus the Parry time change ϕ_t^+ of ϕ_t is Anosov-smooth. □

4.4 A geometric property of Parry time change

Throughout this subsection we denote by ϕ_t a topologically mixing Anosov-smooth flow preserving a C^∞ linear connection ∇. We suppose that E^+ and E^- are both orientable and the Bowen-Margulis measure of ϕ_t is in the Lebesgue measure class.

Denote by n and m the dimensions of E^+ and E^-. Denote by h the topological entropy of ϕ_t which is equal to its metric entropy by assumption. We can construct another ϕ_t-invariant C^∞ linear connection ∇' such that

$$\nabla' X = 0,\ \nabla' E^\pm \subseteq E^\pm,$$

$$\nabla'_{Y^\pm} Z^\mp = P^\mp[Y^\pm, Z^\mp],\ \nabla'_{Y^\pm} Z^\pm = P^\pm(\nabla_{Y^\pm} Z^\pm),$$

$$\nabla'_X Y^+ = [X, Y^+] + \frac{h}{n} Y^+,\ \nabla'_X Y^- = [X, Y^-] - \frac{h}{m} Y^-,$$

This connection ∇' is said to be the *canonical connection associated to* ∇.

Definition 4.5 – *Under the notations above, ∇ is said to be canonical if it coincides with the canonical connection associated to ∇.*

Remark 4.1 – In order to define the notion of canonical connections, it is not necessary to suppose that ϕ_t is topologically mixing and the Bowen-Margulis measure of ϕ_t is in the Lebesgue class.

Since the Bowen-Margulis measure of ϕ_t is supposed to be in the Lebesgue class and ϕ_t is topologically mixing, then by Lemma 3.8 the Margulis measures $\mu^{+,0}$ and $\mu^{-,0}$ are given by C^∞ nowhere-vanishing volume forms along the leaves of $\mathcal{F}^{+,0}$ and $\mathcal{F}^{-,0}$. So $\mu^{+,0}$ and $\mu^{-,0}$ are C^∞ nowhere-vanishing sections of $\wedge^{n+1}(E^{+,0})^*$ and $\wedge^{m+1}(E^{-,0})^*$.

Lemma 4.9 – *Under the notations above, if ∇ is canonical, then $\mu^{+,0}$ and $\mu^{-,0}$ are both ∇-parallel.*

Proof – Since $\nabla E^{+,0} \subseteq E^{+,0}$, then ∇ induces on $E^{+,0}$ a linear connection which in turn induces a linear connection on the line bundle $\wedge^{n+1}(E^{+,0})^*$. We denote this induced connection also by ∇. Then we want to see that $\nabla\mu^{+,0} = 0$.

Denote by a^+ the number $\frac{h}{n}$. Then by the definition of ∇ the parallel transport of E^+ along an orbit of ϕ_t is given by

$$u^+ \to e^{-a^+t} \cdot D\phi_t(u^+).$$

Since $\nabla X = 0$, then $X \circ \phi_t$ is parallel. Fix $x \in M$ and a basis $\{u_i^+\}_{1\leq i\leq n}$ of E_x^+. We define

$$S(t) = (X \circ \phi_t) \wedge e^{-a^+t} D\phi_t(u_1^+) \wedge \cdots \wedge e^{-a^+t} D\phi_t(u_n^+).$$

Then $S(t)$ is parallel along the ϕ_t-orbit of x, i.e. $\nabla_{\partial_t} S(t) = 0$. Since by Proposition 3.4

$$\mu^{+,0} \circ \phi_t = e^{na^+t} \mu^{+,0},$$

then

$$(\phi_t)^* \mu^{+,0} = e^{na^+t} \mu^{+,0},$$

i.e.

$$\mu^{+,0}(\phi_t(x)) = e^{na^+t} \cdot (\phi_{-t})^*(\mu^{+,0}(x)).$$

So with respect to the natural pairing of $\mu^{+,0}$ and S, we have

$$0 = \partial_t < \mu^{+,0}(\phi_t(x)),\ S(t) >$$

$$=< \nabla_{\partial_t} \mu^{+,0},\ S > + < \mu^{+,0},\ \nabla_{\partial_t} S >$$

$$=< \nabla_{\partial_t} \mu^{+,0},\ S > .$$

So $\nabla_{\partial_t} \mu^{+,0} = 0$, i.e. $\mu^{+,0}$ is parallel along the orbits of ϕ_t. We deduce that $\nabla_X \mu^{+,0} = 0$.

Take a smooth curve γ tangent to E^- and beginning at x. Since $\mu^{+,0}$ is invariant under the stable holonomy maps, then we get

$$\mu^{+,0}(\gamma(t)) = (H^-_{x,\gamma(t)})_*(\mu^{+,0}(x)).$$

By the definition of ∇, for any $u^+ \in E_x^+$, the parallel transport of u^+ along γ is obtained by the differentials of the weak stable holonomy maps,

$$u^+ \to (DH^{-,0}_{x,\gamma(t)})(u^+).$$

Take a small curve l tangent to E^+ and with u^+ as the tangent vector at zero. Fix t, then for all $s \ll 1$, $H^-_{x,\gamma(t)}(l(s))$ and $H^{-,0}_{x,\gamma(t)}(l(s))$ are contained in $W^{+,0}_{\gamma(t)} \cap W^{-,0}_{l(s)}$. More precisely, in a product neighborhood of $\gamma(t)$, they are contained in the same plats of $W^{+,0}_{\gamma(t)}$ and $W^{-,0}_{l(s)}$. So for $\epsilon \ll 1$, we can find a smooth function $b : [0,\ \epsilon] \to \mathbb{R}$ such that $b(0) = 0$ and

$$H^-_{x,\gamma(t)}(l(s)) = \phi_{b(s)}(H^{-,0}_{x,\gamma(t)}(l(s))),\ \forall\ s \in [0,\ \epsilon].$$

By differentiating the relation above with respect to s at zero, we get a number $a(t)$ such that

$$DH^-_{x,\gamma(t)}(u^+) = DH^{-,0}_{x,\gamma(t)}(u^+) + a(t) \cdot X(\gamma(t)).$$

Take a basis of E_x^+ as above, $\{u_1^+, \cdots, u_n^+\}$ and define

$$S(t) = (DH^-_{x,\gamma(t)})(X_x \wedge u_1^+ \wedge \cdots \wedge u_n^+).$$

Since $(DH^-_{x,\gamma(t)})(X_x) = X(\gamma(t))$, then by the relation above we get

$$S(t) = X(\gamma(t)) \wedge (DH^{-,0}_{x,\gamma(t)})(u_1^+) \wedge \cdots \wedge (DH^{-,0}_{x,\gamma(t)})(u_n^+).$$

We deduce that

$$\nabla_{\partial_t} S = 0.$$

Then we have

$$0 = \partial_t < \mu^{+,0}(t),\ S(t) >=< \nabla_{\partial_t}\mu^{+,0},\ S(t) > .$$

So $\mu^{+,0}$ is parallel along γ. We deduce that $\nabla_{Y^-}\mu^{+,0} = 0$ for each section Y^- of E^-.

Since $\nabla E^{+,0} \subseteq E^{+,0}$, then for each point y in M, ∇ induces naturally a linear connection $\nabla |_{W_y^{+,0}}$ on $W_y^{+,0}$ of which the parallel transport of a vector in $TW_y^{+,0}$ along a curve in $W_y^{+,0}$ coincides with that of ∇.

We denote by $\Omega^{+,0}$ the curvature form of the connection ∇ on $\wedge^{n+1}(E^{+,0})^*$. Since $\Omega^{+,0}$ is a ϕ_t-invariant C^∞ 2-form on M, then by the Anosov property we get

$$\Omega^{+,0}(X,\ E^+) = 0,\ \Omega^{+,0}(E^+,\ E^+) = 0.$$

It is easy to see that $\Omega^{+,0} |_{W_x^{+,0}}$ is just the connection form of the induced connection of $\nabla |_{W_x^{+,0}}$ on $\wedge^{n+1}(TW_x^{+,0})^*$. We deduce that the line bundle $\wedge^{n+1}(TW_x^{+,0})^*$ is flat with respect to $\nabla |_{W_x^{+,0}}$. So if two curves are homotopic with fixed endpoints in $W_x^{+,0}$, then their parallel transports are the same.

Along each curve l, we denote by $P^l_{s_1,s_2}$ the parallel transport from $l(s_1)$ to $l(s_2)$ of the line bundle $\wedge^{n+1}(E^{+,0})^*$ with respect to ∇. For any $y \in M$ we denote by $\mathcal{O}_y$ the ϕ_t-orbit of y. Now we take a curve γ tangent to E^+ and beginning at x. For each fixed t and all $s > 0$ we get

$$\begin{aligned} c(t) &= \frac{P^\gamma_{0,t}(\mu^{+,0}(x))}{\mu^{+,0}(\gamma(t))} \\ &= \frac{P^{\mathcal{O}_{\gamma(t)}}_{0,-s} \circ P^\gamma_{0,t}(\mu^{+,0}(x))}{P^{\mathcal{O}_{\gamma(t)}}_{0,-s}(\mu^{+,0}(\gamma(t)))} \\ &= \frac{P^{\phi_{-s}\circ\gamma}_{0,\ t}(\mu^{+,0}(\phi_{-s}(x)))}{\mu^{+,0}(\phi_{-s}(\gamma(t)))}, \end{aligned}$$

where we have used that $\mu^{+,0}$ is parallel along the orbits of ϕ_t. If $s \to +\infty$, then the length of $\phi_{-s} \circ \gamma$ goes to zero. Then by the compactness of M, $c(t)$ goes to 1 if $s \to +\infty$. We deduce that $c(t) = 1$, i.e. $\mu^{+,0}$ is parallel along γ. So we have $\nabla_{Y^+}\mu^{+,0} = 0$ for each section Y^+ of E^+.

In conclusion, we have proved that $\nabla_X\mu^{+,0} = 0$ and $\nabla_{Y^\pm}\mu^{+,0} \equiv 0$ for arbitrary sections of E^+ and E^-. Thus $\mu^{+,0}$ is ∇-parallel. Similarly we can see that $\mu^{-,0}$ is also ∇-parallel. □

We deduce from the previous lemma that $i_X\mu^{+,0}$ and $i_X\mu^{-,0}$ are also ∇-parallel, which are respectively C^∞ nowhere-vanishing sections of $\wedge^n(E^+)^*$ and $\wedge^m(E^-)^*$. So by taking their dual sections, we get the following

Proposition 4.2 – *Let ϕ_t be a topologically mixing Anosov-smooth flow of which the Bowen-Margulis measure is in the Lebesgue class. We suppose in addition that E^+ and E^- are both orientable and of dimension respectively n and m. If ϕ_t preserves a C^∞ canonical linear connection ∇, then the line bundles $\wedge^n E^+$ and $\wedge^m E^-$ admit both C^∞ nowhere-vanishing ∇-parallel sections.*

4.5 The idea

Our idea to prove the rigidity of certain Anosov-smooth flows is to take at first the Parry time change in order to strengthen the geometric information by Proposition 4.2. Then we manage to classify these synchronised Anosov-smooth flows. Finally we go back to find out the initial flows by the lemmas established in Sections 4.1 and 4.2 about special time changes.

The strength of this *go-and-back* idea will be demonstrated in the chapters below (see the proof of Theorem 5.2 for a particularly clear illustration of our idea).

Chapter 5

Smooth Rigidity of Transversely Symplectic Anosov-smooth Flow

Abstract – *In this chapter we study the Anosov-smooth flows preserving a transverse symplectic form. In particular we give a classification of such flows in the case of dimension five.*

5.1 Introduction

5.1.1 Motivation and our main result

Let ϕ_t be a C^∞ Anosov flow on M with generator X. If it preserves a C^∞ closed 2-form ω such that $\text{Ker}\omega = \mathbb{R}X$, then ϕ_t is said to be *transversely symplectic*. This closed 2-form ω is said to be a *transverse symplectic form* of ϕ_t. Recall that the kernel of ω is defined to be

$$\text{Ker}\omega = \{u \in TM \mid i_u\omega = 0\}.$$

The classical examples of transversely symplectic Anosov flows are the geodesic flows of negatively curved closed Riemannian manifolds and the suspensions of symplectic Anosov diffeomorphisms. More precisely, if ϕ_t denotes the geodesic flow of a negatively curved manifold, then its canonical 1-form λ is contact. In addition ϕ_t is the Reeb flow of λ (see **[Pa]**), i.e.

$$\lambda(X) \equiv 0, \quad d\lambda(X, \cdot) \equiv 0.$$

So $d\lambda\mid_{E^+\oplus E^-}$ is non-degenerate, i.e. $\text{Ker}(d\lambda) = \mathbb{R}X$. Thus ϕ_t is transversely symplectic with respect to $d\lambda$.

If ϕ denotes a sympletic Anosov diffeomorphism with symplectic form $\bar{\omega}$, then we can extend $\bar{\omega}$ to a closed 2-form ω on the suspension manifold by defining $\omega(\partial_t, \cdot) \equiv 0$, where ∂_t denotes the generator of the suspension flow. Since $\bar{\omega}$ is non-degenerate, then the suspension of ϕ is transversely symplectic with respect to ω.

So there are plenty of transversely symplectic Anosov flows, which is due to the fact that invariant transverse symplectic forms of Anosov flows are not rigid geometric structures. Roughly speaking, transversely symplectic Anosov flows are rather soft. But the situation changes radically in the case of transversely symplectic Anosov-smooth flows, since we can then use the given transverse symplectic form and the C^∞ Anosov splitting of the flow to construct a C^∞ invariant pseudo-Riemannian metric, or more essentially a C^∞ invariant linear connection (see **[Ka1]** and **[BFL2]** and Subsection 5.2.2 below). These canonically associated C^∞ rigid geometric structures have furnished the departing point for the classification of certain special (but important) classes of transversely symplectic Anosov-smooth flows, notably the contact ones and the suspension ones (see **[BFL2]** and **[BL]** and Theorem 5.2 below).

Here are the known algebraic models of transversely symplectic Anosov-smooth flows. The classical examples are certainly the suspensions of symplectic hyperbolic infra-nilautomorphisms and the geodesic flows of closed locally symmetric spaces of rank one. In addition, it is easy to verify that the spin examples constructed in Subsection 3.4.2 are also transversely symplectic. This richness of algebraic models makes very striking the classification problem of general transversely symplectic Anosov-smooth flows.

In **[Gh1]**, É. Ghys has classified three-dimensional transversely symplectic Anosov-smooth flows (see also Theorem 5.2.2 below). In **[FK]**, R. Feres and A. Katok studied the five-dimensional transversely symplectic Anosov-smooth flows. They established some quite raffined dynamical properties for such flows and they classified the contact case. In this chapter, our main result is the complete classification of such flows of dimension five. More presicely we prove

Theorem 5.1 – *Let ϕ_t be a five-dimensional transversely symplectic Anosov-smooth flow. Then up to a constant change of time scale and finite covers, it is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$.*

5.1.2 Plan of the chapter

In Section 5.2 we prove some properties for general transversely symplectic Anosov-smooth flows. In particular using our idea outlined in Section 4.5, we give a new proof of the classification of three-dimensional volume-preserving Anosov-smooth flows originally obtained by É. Ghys. Then in Section 5.3 we prove Theorem 5.1.

5.2 Preliminaries

5.2.1 Some elementary facts

Let us first prove some properties about transversely symplectic Anosov-smooth flows. Denote by ϕ_t such a flow on M with generator X. Denote by ω its transverse symplectic form and by ϕ_t^β a special time change of ϕ_t given by the C^∞ 1-form β. Then we have $i_X\omega = 0$. Thus

$$\mathcal{L}_{\frac{X}{\beta(X)}}\omega = i_{\frac{X}{\beta(X)}}d\omega + di_{\frac{X}{\beta(X)}}\omega = 0.$$

So we get the following lemma furnishing the basis to use our idea of taking suitable time changes (see Section 4.5).

Lemma 5.1 – *Each special time change of a transversely symplectic Anosov-smooth flow is also transversely symplectic and Anosov-smooth.*

Since $\mathrm{Ker}\omega = \mathbb{R}X$, then $\omega\mid_{E^+\oplus E^-}$ is non-degenerate. For $x \in M$ and $u, v \in E^+_x$ we have by the Anosov property

$$\| \omega(u,v) \| = \| \omega(\phi^*_{-t}u, \phi^*_{-t}v) \| \leq \| \omega \| \cdot \| \phi^*_{-t}u \| \cdot \| \phi^*_{-t}v \|$$

$$\leq a^2 \cdot e^{-2bt} \cdot \| \omega \| \cdot \| u \| \cdot \| v \|, \ \forall\, t > 0.$$

We deduce that $\omega(E^+, E^+) \equiv 0$. Similarly we get $\omega(E^-, E^-) \equiv 0$. So E^+ and E^- are both Lagrangian subspaces of $\omega\mid_{E^+\oplus E^-}$. In particular we deduce that E^+ and E^- have the same dimension and the dimension of M must be odd.

Denote by λ the canonical 1-form of ϕ_t. Then it is easy to see that $\lambda \wedge (\wedge^n\omega)$ is a ϕ_t-invariant volume form if E^+ is n-dimensional. In particular we deduce that ϕ_t is topologically transitive (see Subsection 3.2.2).

5.2.2 Two dynamical lemmas and invariant parallel connections

Let ϕ_t be a C^∞-flow on a closed manifold M and ν be a ϕ_t-invariant probability measure. If ϕ_t is ergodic with respect ν, then by Theorem 3.2, there exists a ν-conull ϕ_t-invariant subset Λ of M and a ϕ_t-invariant measurable Lyapunov decomposition of $TM\mid_\Lambda$,

$$TM\mid_\Lambda = \oplus_{1\leq i\leq k} L_i,$$

such that for any $u_i \in L_i$,

$$\lim_{t\to\pm\infty} t^{-1}\log \| D\phi_t(u_i) \| = \chi_i.$$

Recall that L_i is said to be the Lyapunov subbundle with Lyapunov exponent χ_i, which is also denoted by L_{χ_i}.

The Lyapunov decomposition of ϕ_t, with respect to ν, is said to be *smooth* if there exists a ν-conull ϕ_t-invariant subset Λ_1 of M and a C^∞ decomposition of TM into smooth subbundles

$$TM = \oplus_{1\leq i\leq k} E_i,$$

such that the Lyapunov decomposition is defined on Λ_1 and

$$E_i \mid_{\Lambda_1} = L_i \mid_{\Lambda_1},\ \forall\ 1 \leq i \leq k.$$

If the support of ν is M, then this C^∞ decomposition of TM is certainly unique and ϕ_t-invariant and is said to be the *smooth Lyapunov decomposition* of ϕ_t. The Lyapunov exponent of E_i is by definition that of L_i for any $1 \leq i \leq k$. We define $E_a = L_a = \{0\}$ if a is not a Lyapunov exponent.

The following lemma is proved in [**BFL1**].

Lemma 5.2 – *We suppose that the Lyapunov decomposition of ϕ_t is C^∞ and the support of ν is M. Under the notations above, if K is a C^∞ ϕ_t-invariant tensor of type $(1,\ r)$, then*

$$K(E_{\chi_{i_1}}, \cdots, E_{\chi_{i_r}}) \subseteq E_{\chi_{i_1}+\cdots+\chi_{i_r}}.$$

Let ∇ be a ϕ_t-invariant C^∞ connection on M such that

$$\nabla E_i \subseteq E_i,\ \forall\ 1 \leq i \leq k.$$

Then we have $\nabla K = 0$ iff $\nabla_{E_0} K = 0$, where E_0 denotes the C^∞ Lyapunov subbundle of Lyapunov exponent zero.

Proof – Because of the continuty of K we need only prove that

$$K(L_{\chi_{i_1}}, \cdots, L_{\chi_{i_r}}) \subseteq L_{\chi_{i_1}+\cdots+\chi_{i_r}}$$

on Λ. Take $x \in \Lambda$ and $u_i \in L_{\chi_i}(x)$. Then we get

$$\| D\phi_t(K(u_1, \cdots, u_r)) \| = \| K(D\phi_t(u_1), \cdots, D\phi_t(u_r)) \|$$

$$\leq \| K \| \cdot \| D\phi_t(u_1) \| \cdots \| D\phi_t(u_r) \| .$$

So we have

$$\lim_{t\to+\infty} \frac{1}{t} log(\| D\phi_t(K(u_1, \cdots, u_r)) \|) \leq \lim_{t\to+\infty} \frac{1}{t} log(\| D\phi_t(u_1) \| \cdots \| D\phi_t(u_r) \|)$$

$$= \chi_{i_1} + \cdots + \chi_{i_r}$$

and

$$\lim_{t\to-\infty} \frac{1}{t} log(\| D\phi_t(K(u_1, \cdots, u_r)) \|) \geq \lim_{t\to-\infty} \frac{1}{t} log(\| D\phi_t(u_1) \| \cdots \| D\phi_t(u_r) \|)$$

$$= \chi_{i_1} + \cdots + \chi_{i_r}$$

We deduce that $K(u_1, \cdots, u_r) \in L_{\chi_{i_1}+\cdots+\chi_{i_r}}$. So $K(E_{\chi_{i_1}}, \cdots, E_{\chi_{i_r}}) \subseteq E_{\chi_{i_1}+\cdots+\chi_{i_r}}$. Recall that by convention $L_a = E_a = \{0\}$ if a is not a Lyapunov exponent.

Since K and ∇ are both ϕ_t-invariant, then ∇K is a ϕ_t-invariant tensor of type $(1, r+1)$. So we get

$$(\nabla_{E_b} K)(E_{\chi_{i_1}}, \cdots, E_{\chi_{i_r}}) \subseteq E_{E_b+\chi_{i_1}+\cdots+\chi_{i_r}}.$$

Since ∇ respects the C^∞ Lyapunov decomposition, then

$$(\nabla_{E_b} K)(E_{\chi_{i_1}}, \cdots, E_{\chi_{i_r}}) = \nabla_{E_b}(K(E_{\chi_{i_1}}, \cdots, E_{\chi_{i_r}})) - \cdots \subseteq E_{\chi_{i_1}+\cdots+\chi_{i_r}}.$$

So we get $\nabla_{E_b} K = 0$ if $b \neq 0$. We deduce that $\nabla K = 0$ iff $\nabla_{E_0} K = 0$. □

We can deduce from the previous lemma a special case of Lemma 3.1.

Lemma 5.3 – *Denote by ϕ_t a C^∞ flow with smooth Lyapunov decomposition and suppose that the support of ν is M. Under the notations above, if τ is a C^∞ ϕ_t-invariant tensor field of type $(0, r)$ and $\sum_{1 \leq l \leq r} \chi_{i_l} \neq 0$, then we have $\tau(E_{i_1}, \cdots, E_{i_r}) =$ 0.*

Proof – Denote by X the generator of ϕ_t and define $\bar{\tau} = \tau \otimes Id$. Then $\bar{\tau}$ is a ϕ_t-invariant tensor of type $(1, r+1)$. By the definition of $\bar{\tau}$ we have

$$\bar{\tau}(E_{i_1}, \cdots, E_{i_r}, X) = \tau(E_{i_1}, \cdots, E_{i_r}) \cdot X \subseteq E_0.$$

By Lemma 5.2 we get

$$\bar{\tau}(E_{i_1}, \cdots, E_{i_r}, X) \subseteq E_{\sum_{1 \leq l \leq r} \chi_{i_l}}.$$

Since by assumption $\sum_{1 \leq l \leq r} \chi_{i_l} \neq 0$, then $E_{\sum_{1 \leq l \leq r} \chi_{i_l}} \cap E_0 = \{0\}$. So we have $\tau(E_{i_1}, \cdots, E_{i_r}) = 0$. □

Now we denote by ϕ_t a transversely symplectic Anosov-smooth flow on M. Suppose in addition that it has C^∞ Lyapunov decomposition with respect to its invariant volume form. Then under the notations above we can construct a C^∞ ϕ_t-invariant linear connection ∇ on M such that

$$\nabla X = 0,\ \nabla \omega = 0,\ \nabla E_i^{\pm} \subseteq E_i^{\pm},$$

$$\nabla_{Y_i^{\pm}} Y_j^{\mp} = P_j^{\mp}[Y_i^{\pm}, Y_j^{\mp}],\ \nabla_X Y_i^{\pm} = [X, Y_i^{\pm}] + a_i^{\pm} Y_i^{\pm},$$

where $Y_i^{\pm}$ are smooth sections of $E_i^{\pm}$ with Lyapunov exponent $a_i^{\pm}$.

Let K be a C^∞ ϕ_t-invariant tensor field of type $(1, r)$ and $Z_1, \cdots, Z_r$ be the sections of the C^∞ Lyapunov subbundles, $E_{\chi_1}, \cdots, E_{\chi_r}$. Then by Lemma 5.2, we have

$$(\nabla_X K)(Z_1, \cdots, Z_r)$$

$$= \nabla_X(K(Z_1, \cdots, Z_r)) - \sum_{1\le i\le r} K(Z_1, \cdots, \nabla_X Z_i, \cdots, Z_r)$$

$$= [X, K(Z_1, \cdots, Z_r)] + (\sum_{1\le i\le r} a_i)K(Z_1, \cdots, Z_r) - K([X, Z_1] + a_1 Z_1, \cdots) \cdots$$

$$= [X, K(Z_1, \cdots, Z_r)] - \sum_{1\le i\le r} K(Z_1, \cdots, [X, Z_i], \cdots, Z_r)$$

$$= (\mathcal{L}_X K)(Z_1, \cdots, Z_r) = 0.$$

Since ϕ_t is Anosov, then $E_0 = \mathbb{R}X$. So by Lemma 5.2, we get $\nabla K = 0$. Thus by using this remark to the torsion tensor T and the curvature tensor R of ∇, we get

$$\nabla T = 0, \quad \nabla R = 0$$

and

$$T(E_a, E_b) \subseteq E_{a+b}, \quad R(E_c, E_d) = 0,$$

if $c + d \neq 0$. In particular, ∇ is seen to be parallel.

Lemma 5.4 – *Under the notations above, ∇ is complete.*

Proof – Since ϕ_t is supposed to have C^∞ Lyapunov decomposition, then ϕ_t is Anosov-smooth. So we have the C^∞ decomposition $TM = \mathbb{R}X \oplus E^+ \oplus E^-$. Since $\nabla X = 0$, then each geodesic tangent to $\mathbb{R}X$ is defined on $\mathbb{R}$. So by Lemma 2.1, in order to see the completeness of ∇, we need only prove that the geodesics tangent to E^+ or E^- are defined on $\mathbb{R}$.

Fix a C^∞ Riemannian metric on M. Then there exists $\epsilon > 0$ such that the geodesic γ_u (tangent to u) is defined on $(-1, 1)$ if $\| u \| < \epsilon$. For any $u^+ \in E^+$, we have $\| D\phi_{-t}(u^+) \| < \epsilon$ if $t >> 1$. Since ϕ_t preserves ∇, then $\gamma_{u^+} = \phi_t(\gamma_{\phi_t^* u^+})$. So γ_{u^+} is defined on $(-1, 1)$. We deduce that each geodesic tangent to E^+ is defined on $\mathbb{R}$. Similar we can see that the geodesics tangent to E^- are complete. □

So for each transversely symplectic Anosov-smooth flow with C^∞ Lyapunov decomposition ϕ_t, we have constructed a C^∞ ϕ_t-invariant complete parallel linear connection ∇, which is said to be *canonically associated* to ϕ_t. The idea of constructing such a connection is due to M. Kanai, Y. Benoist, P. Foulon and F. Labourie (see **[Ka1]** and **[BFL1]**).

5.2.3 Characterisation of classical models and the three-dimensional case

By combining some known results, we can prove the following

Proposition 5.1 – *Let ϕ_t be a transversely symplectic Anosov-smooth flow on a m-dimensional manifold M.*

(1) *If rank*$(\phi_t) = 2[\frac{m}{2}]$, *then up to a constant change of time scale and finite covers,* ϕ_t *is* C^∞ *flow equivalent to a canonical time change of the geodesic flow of a closed locally symmetric space of rank one.*
(2) *If rank*$(\phi_t) = 0$, *then up to a constant change of time scale and finite covers,* ϕ_t *is* C^∞ *flow equivalent to the suspension of a symplectic hyperbolic infranilautomorphism.*

Proof – Suppose that E^+ is of dimension n. Then by Subsection 5.2.1 we have $m = 2n + 1$.

If $\mathrm{rank}(\phi_t) = 2[\frac{m}{2}]$, then $\wedge^n d\lambda \not\equiv 0$. So the ϕ_t-invariant C^∞ m-form $\lambda \wedge (\wedge^n d\lambda)$ is not identically zero. Since ϕ_t is topologically transitive, then there exists $c \neq 0$ such that $\lambda \wedge (\wedge^n d\lambda) = c \cdot \lambda \wedge (\wedge^n \omega)$. We deduce that $\lambda \wedge (\wedge^n d\lambda)$ vanishes nowhere, i.e. λ is a contact form. So we conclude in this case by the classification of contact Anosov-smooth flows (see **[BFL2]**).

If $\mathrm{rank}(\phi_t) = 0$, then $d\lambda \equiv 0$. So $E^+ \oplus E^-$ is integrable. By Theorem 3.1 of **[Pl1]**, ϕ_t admits a global section Σ (a global section is by definition a connected closed submanifold of codimension 1 which intersects each orbit transversally). Denote by τ the *first return time* function of Σ. Then the Poincaré map of Σ is by definition $\psi = \phi_{\tau(\cdot)}(\cdot)$. For the sake of completeness, we prove in detail the following.

Sublemma – *The previous Poincaré map* ψ *is a symplectic Anosov-smooth diffeomorphism.*

Proof – Denote by $\mathcal{F}^{+,0}$ and $\mathcal{F}^{-,0}$ respectively the corresponding foliations of $E^{+,0}$ and $E^{-,0}$. Since Σ is transverse to X, then $\mathcal{F}^{+,0} \cap \Sigma$ gives a C^∞ foliation on Σ whose tangent distribution is denote by E^+_Σ. Similarly we have the tangent distribution E^-_Σ of $\mathcal{F}^{-,0} \cap \Sigma$.

Since $\mathcal{F}^{+,0}$ is ϕ_t-invariant, then the foliation $\mathcal{F}^{+,0} \cap \Sigma$ is ψ-invariant. So E^+_Σ is ψ-invariant. Similarly E^-_Σ is also ψ-invariant.

Fix a Riemannian metric on M. Since $E^+ \mid_\Sigma$ and E^+_Σ are both transverse to $\mathbb{R}X$ (along Σ), then we can project E^+_Σ onto $E^+ \mid_\Sigma$ with respect to $\mathbb{R}X$. Denote this projection by P^+. Since Σ is compact, then we can find two positive numbers M_1 and M_2 such that

$$M_1 \parallel u \parallel \leq \parallel P^+ u \parallel \leq M_2 \parallel u \parallel, \ \forall\, u \in E^+_\Sigma.$$

For any $x \in \Sigma$ and any $u \in (E^+_\Sigma)_x$, u splitts uniquely as

$$u = P^+_x(u) + aX_x, \ a \in \mathbb{R}.$$

We have

$$(D_x\psi)(u) = (D_x\tau(u) + a)X_{\psi(x)} + (D_x\phi_{\tau(x)})(P^+_x u).$$

Thus

$$(D_x\psi)(u) = (P^+_{\psi(x)})^{-1}[(D_x\phi_{\tau(x)})(P^+_x u)].$$

So for all $n \in \mathbb{N}$,

$$(D_x\psi^n)(u) = (P^+_{\psi^n(x)})^{-1}(D_x\phi_{\tau(x)+\cdots+\tau(\psi^{n-1}(x))})(P^+_x u).$$

We have a similar formula for E^-_Σ. Now a simple estimation shows that ψ is an Anosov diffeomorphism with C^∞ unstable and stable distributions E^+_Σ and E^-_Σ, i.e. ψ is Anosov-smooth.

Denote by ω the transverse symplectic form of ϕ_t. Then ω restricts to a C^∞ closed 2-form ω_Σ on Σ. Since $\mathrm{Ker}\omega = \mathbb{R}X$, then it is easy to see that ω_Σ is non-degenerate and ψ-invariant. So ψ is symplectic. □

By **[BL]** and the previous sublemma, ψ is seen to be C^∞ conjugate to a hyperbolic infra-nilautomorphism. Then by Corollary 3.5 of **[Pl1]**, the integral manifolds of $E^+ \oplus E^-$ are compact. So we can take a leaf of $E^+ \oplus E^-$ as Σ. With respect to this section, the first return time function is contant. So case two of Proposition 5.1 is true. □

It is easy to see by **[HuK]** that a three-dimensional Anosov-smooth flow is transversely symplectic iff it is volume-preserving. Since such a flow must be either contact or a suspension, then the previous theorem gives a classification of three-dimensional volume-preserving Anosov-smooth flows originally obtained by É. Ghys (see **[Gh1]**). In the following we reobtain this classification by using our *go-and-back* idea (see Subsection 4.5).

Theorem 5.2 (É. Ghys) – *Let ϕ_t be a volume-preserving Anosov-smooth flow on a closed three-dimensional manifold M. Then up to finite covers and a constant change of time scale, ϕ_t is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a closed hyperbolic surface or to the suspension of a hyperbolic automorphism of $\mathbb{T}^2$.*

Proof – By replacing ϕ_t by its Parry time change, we suppose firstly that the Bowen-Margulis measure of ϕ_t is in the Lebesgue measure class. Up to finite covers, we suppose also that E^+ and E^- are both orientable.

By Lemma 3.7, ϕ_t is either topologically mixing or a suspension. These two cases will be treated seperately.

If ϕ_t is topologically mixing, then by Lemma 3.8, the Margulis measures μ^+ and μ^- are given by C^∞ nowhere-vanishing volume forms along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. So we can define two C^∞ fields Y^+ and Y^- tangent to E^+ and E^- such that $\mu^\pm(Y^\pm) \equiv 1$. Since for $t \in \mathbb{R}$

$$\mu^+ \circ \phi_t = e^{ht}\mu^+, \quad \mu^- \circ \phi_t = e^{-ht}\mu^-,$$

where h denotes the topological entropy of ϕ_t, then we have

$$[X, Y^+] = -h \cdot Y^+, \quad [X, Y^-] = h \cdot Y^-.$$

In particular we deduce that $\phi_t^*[Y^+, Y^-] = [Y^+, Y^-]$ for $t \in \mathbb{R}$. So $[Y^+, Y^-]$ is tangent to $\mathbb{R}X$. Since ϕ_t preserves a volume form, then it is topologically transitive. So there exists $a \in \mathbb{R}$ such that $[Y^+, Y^-] = a \cdot X$.

If $a = 0$, then $E^+ \oplus E^-$ is integrable. Since ϕ_t is of codimensional one, then we deduce by **[Pl1]** that the leaves of $E^+ \oplus E^-$ are all compact. So ϕ_t is not topologicaly mixing, which is a contradiction. We deduce that $a \neq 0$.

So we get a C^∞ complete parallelism (Y^+, Y^-, X) on M. Denote by $\mathfrak{g}$ the Lie algebra generated by these three fields. Then we have $\mathfrak{g} \cong \mathfrak{sl}_2(2, \mathbb{R})$. So by Proposition 2.5, ϕ_t is C^∞ flow equivalent to the algebraic Anosov flow on $\Gamma \diagdown \widetilde{SL(2, \mathbb{R})}$ given by the right multiplication of diagonal matrices. Thus up to finite covers, ϕ_t is C^∞ flow equivalent to the geodesic flow of a closed hyperbolic surface.

If ϕ_t is not topologically mixing, then by Lemma 3.7, $E^+ \oplus E^-$ is integrable with closed leaves. Take a leaf Σ of this foliation and denote by ϕ its poincaré map. Since the Bowen-Margulis measure of ϕ_t is Lebesgue, then the Bowen-Margulis measure of ϕ is also Lebesgue. Thus by Lemma 3.9 and the arguments above, we can find two C^∞ nowhere-vanishing vector fields Y^+ and Y^- tangent respectively to E^+ and E^- such that $\phi_* Y^+ = e^h Y^+$ and $\phi_* Y^- = e^{-h} Y^-$, where h denotes the topologically entropy of ϕ. So we get $\phi_*[Y^+, Y^-] = [Y^+, Y^-]$. Thus $[Y^+, Y^-] = 0$.

Denote by ∂_t the generator of ϕ_t. Then the vector fields $Z^\pm = e^{\mp ht} Y^\pm$ are well-defined on M. So we get again a C^∞ complete parallelism (Z^+, Z^-, X) on M. Denote by $\mathfrak{g}$ the Lie algebra generated by these three fields. Then we have $\mathfrak{g} \cong \mathbb{R}^2 \rtimes \mathbb{R}$, where the semidirect product is given by multiplication on $\mathbb{R}^2$ of order two diagonal matrices of trace zero. So by Proposition 2.5, ϕ_t is C^∞ flow equivalent to the algebraic Anosov flow on $\Gamma \diagdown \mathbb{R}^2 \rtimes \mathbb{R}$ given by the right multiplication of the one-parameter subgroup of $\mathbb{R}^2 \rtimes \mathbb{R}$ generated by $((0,0),1)$. Then it is easy to see that up to finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic automorphism of $\mathbb{T}^2$ (see **[To1]**).

Since the initial Anosov-smooth flow ϕ_t is a special time change of its Parry time change, then we finish the proof of this theorem by Lemmas 4.2 and 4.5. $\square$

Remark 5.1 – In **[Gh3]**, a final classification of three-dimensional Anosov flows with smooth weak stable and unstable distributions is achieved, which is related to some very interesting results about group actions on the circle.

5.3 Classification of five-dimensional transversely symplectic Anosov-smooth flows.

In this section, we prove Theorem 5.1.

5.3.1 Some preparations.

Suppose that ϕ_t satisfies the conditions of Theorem 5.1. Denote by X its generator and by ω its transverse symplectic form. Denote by ν the invariant volume form of

ϕ_t. If a is a Lyapunov exponent of ϕ_t with respect to ν, then by Lemma 3.1, $-a$ is also a Lyapunov exponent of ϕ_t. Since M is of dimension five, then there exist only two possibilities concerning the Lyapunov exponents of ϕ_t,

$$(1)\ \ -a < 0 < a,$$

$$(2)\ \ -a < -b < 0 < b < a.$$

Denote by $\hat{\phi}_t$ the Parry time change of ϕ_t. Then by Lemma 5.1, $\hat{\phi}_t$ is also a transversely symplectic Anosov-smooth flow. Since M is of dimension five, then the rank of $\hat{\phi}_t$ can only be zero or two or four.

If $\mathrm{rank}(\hat{\phi}_t) = 4$, then by Proposition 5.1, up to a constant change of time scale, $\hat{\phi}_t$ is finitely covered by a canonical time change of the geodesic flow of a three-dimensional locally symmetric space of rank one. But such a Riemannan space must have contant negative curvature. Then we deduce from Lemma 4.2 that up to finite covers, ϕ_t is C^∞ flow equivalent to a canonical time change of the geodesic flow of a three-dimensional hyperbolic manifold, i.e. Theorem 5.1 is true in this case.

If $\mathrm{rank}(\hat{\phi}_t) = 0$, then by Proposition 5.1, up to a constant change of time scale, $\hat{\phi}_t$ is finitely covered by the suspension of a four-dimensional symplectic hyperbolic nilautomorphism. But in the case of dimension four such a hyperbolic nilautomorphism must be of the form $(\mathbb{T}^4,\ \bar{A})$, where $\bar{A}$ is the induced application of an invertible hyperbolic matrix A in $GL(4,\mathbb{Z})$. Then we deduce from Lemma 4.5 that up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a symplectic hyperbolic automorphism of $\mathbb{T}^4$, i.e.Theorem 5.1 is also true in this case.

Thus in order to prove Theorem 5.1, we need only prove the non-existence of the case of rank two. So throughout the following we suppose on the contrary that ϕ_t is a five-dimensional transversely symplectic Anosov-smooth flow of rank two whose Bowen-Margulis measure is Lebesgue (i.e. we denote $\hat{\phi}_t$ again by ϕ_t). We shall find a contradiction as following:

At first ϕ_t is proved to have C^∞ Lyapunov decomposition. Then we get a C^∞ complete parallel linear connection ∇ associated to ϕ_t, which is constructed in Subsection 5.2.2. So by Lemma 2.2, ϕ_t is homogeneous. Then by some dynamical and Lie theoretical arguments we finish the proof by showing that each possible algebraic model of ϕ_t is absurd.

Now let us procced into the proof. Denote by λ the canonical 1-form of ϕ_t. Since $\mathrm{rank}(\phi_t) = 2$, then

$$d\lambda \not\equiv 0, \quad d\lambda \wedge d\lambda \equiv 0.$$

Define $U = \{x \in M \mid (d\lambda)_x \neq 0\}$. Since ϕ_t is topologically transitive and preserves $d\lambda$, then U is a ϕ_t-invariant open-dense subset of M. Denote by π the projection of TM onto M. We define

$$E_1 = \{u \in E^+ \oplus E^- \mid i_u d\lambda = 0,\ \pi(u) \in U\}$$

and

$$E_1^+ = E_1 \cap E^+, \quad E_1^- = E_1 \cap E^-.$$

Since ϕ_t preserves $d\lambda$ and E^+ and E^-, then E_1 and E_1^+ and E_1^- are all ϕ_t-invariant.

Lemma 5.5 – *Under the notations above, E_1 is a two-dimensional C^∞ subbundle of $TM\mid_U$. E_1^+ and E_1^- are both one-dimensional C^∞ subbundles of $TM\mid_U$ such that $E_1 = E_1^+ \oplus E_1^-$.*

Proof – For all $x \in U$, we have $(d\lambda)_x \neq 0$. So near x, we can find C^∞ local sections of $E^+ \oplus E^-$, V_1 and V_2, such that $d\lambda(V_1,\ V_2) \equiv 1$. Denote by V the C^∞ local distribution spanned by V_1 and V_2 and denote by $V^\perp$ the orthogonal of V with respect to $d\lambda\mid_{E^+\oplus E^-}$.

Since $d\lambda\mid_V$ is non-degenerate, then $V\cap V^\perp = \{0\}$. For all $u \in E^+\oplus E^-$ such that $\pi(u)$ near x, the following vector is contained in $V^\perp$,

$$P(u) = u - d\lambda(u, V_2(\pi(u)))\cdot V_1(\pi(u)) - d\lambda(V_1(\pi(u)), u)\cdot V_2(\pi(u)).$$

So we deduce that locally $E^+\oplus E^- = V\oplus V^\perp$ and the projection of $E^+\oplus E^-$ onto $V^\perp$ with respect to this decomposition is C^∞. In particular $V^\perp$ is C^∞.

Since $d\lambda\mid_V$ is non-degenerate and $d\lambda\wedge d\lambda \equiv 0$, then $d\lambda\mid_{V^\perp}\equiv 0$. Thus locally $E_1 = V^\perp$. In particular, E_1 is C^∞ and two-dimensional.

Since $d\lambda(E^\pm, E^\pm)\equiv 0$ by the Anosov property of ϕ_t, then for all $u\in E_1$, its projections to E^+ and E^- are also contained in E_1. So we have $E_1 = E_1^+\oplus E_1^-$. If for some point x in U, $(E_1^+)_x$ is of dimension two, then $(d\lambda)_x$ will be zero, which contradicts our assumption. Thus E_1^+ and E_1^- are both of dimension one and C^∞. □

Lemma 5.6 – *Under the notations above, the Lyapunov decomposition of ϕ_t with respect to its invariant volume form is smooth.*

Proof – If ϕ_t has only one positive Lyapunov exponent, then its Lyapunov decomposition is just the restriction of that of Anosov onto a ν-conull subset of M. Since ϕ_t is Anosov-smooth, then the lemma is true in this case.

Suppose that ϕ_t has two positive Lyapunov exponents $b < a$. Then there exists a ν-conull subset Λ of M, such that

$$TM\mid_\Lambda = L_1^+\oplus L_1^-\oplus L_2^+\oplus L_2^+\oplus \mathbb{R}X,$$

where $L_1^\pm$ and $L_2^\pm$ are the Lyapunov subbundles with exponents $\pm b$ and $\pm a$ (see Subsection 5.2.2).

Since U is a ϕ_t-invariant open-dense subset and the flow is ν-ergodic (see [**An**]), then U is ν-conull. So $\nu(U\cap\Lambda) = 1$.

Take $x\in U\cap\Lambda$ and non-zero vectors $l_i^\pm \in (L_i^\pm)_x$ for $i = 1$ or 2. By Lemma 3.1 we get

$$d\lambda(l_1^+, l_2^-) = 0,\quad d\lambda(l_1^-, l_2^+) = 0.$$

Since $(d\lambda)_x \neq 0$, then we must have $d\lambda(l_1^+, l_1^-)\neq 0$ or $d\lambda(l_2^+, l_2^-)\neq 0$.

Suppose at first that $d\lambda(l_2^+, l_2^-)\neq 0$. Since $d\lambda\wedge d\lambda\equiv 0$, then we must have $d\lambda(l_1^+, l_1^-) = 0$. So $l_1^+\in (E_1^+)_x$ and $l_1^-\in(E_1^-)_x$, i.e. $(L_1^+)_x = (E_1^+)_x$ and $(L_1^-)_x =$

$(E_1^-)_x$. Since $\omega \mid_{E^+ \oplus E^-}$ is non-degenerate and $\omega(l_1^+, l_2^-) = 0$ by Lemma 3.1, then $\omega(l_1^+, l_1^-) \neq 0$. We deduce that $(d\lambda \wedge \omega)_x \neq 0$. So $\lambda \wedge d\lambda \wedge \omega$ is not identically zero. Then by the topological transitivity of ϕ_t, there exists $c \neq 0$ such that

$$\lambda \wedge d\lambda \wedge \omega = c \cdot \lambda \wedge \omega \wedge \omega.$$

So $\lambda \wedge d\lambda \wedge \omega$ is nowhere zero. We deduce that $d\lambda$ vanishes nowhere and $U = M$. In particular, E_1 and $E_1^{\pm}$ are all C^∞ subbundles of TM.

If $d\lambda(l_1^+, l_1^-) \neq 0$, then by similar arguments, we can see that $(L_2^{\pm})_x = (E_2^{\pm})_x$ and $d\lambda$ vanishes nowhere on M and $E_1^{\pm}$ are both C^∞ subbundles of TM.

So for any $x \in \Lambda$, $(E_1^{\pm})_x = (L_1^{\pm})_x$ or $(L_2^{\pm})_x$. Define

$$\Lambda_i = \{y \in \Lambda \mid E_1^{\pm}(y) = L_i^{\pm}(y)\}, \ i = 1, 2.$$

Then Λ_1 and Λ_2 are both mesurable and ϕ_t-invariant. So one of them is ν-conull. Suppose at first that $\nu(\Lambda_1) = 1$. Then we have $E_1^{\pm} \mid_{\Lambda_1} = L_1^{\pm} \mid_{\Lambda_1}$.

By Lemma 3.1, we have on Λ_1:

$$L_2^{\pm} = [Ker(v \mapsto \omega(L_1^{\mp}, v))] \cap E^{\pm}.$$

Since $d\lambda \wedge \omega$ vanishes nowhere, then we can define two ϕ_t-invariant C^∞ line subbundles of TM as following

$$E_2^{\pm} = [Ker(v \mapsto \omega(E_1^{\mp}, v))] \cap E^{\pm}.$$

Then we have $E_2^{\pm} \mid_{\Lambda_1} = L_2^{\pm} \mid_{\Lambda_1}$. So the Lyapunov decomposition coincides on Λ_1 with a C^∞ decomposition of TM, i.e. it is smooth.

If $\nu(\Lambda_2) = 1$, then similar argument works. □

Remark 5.2 – In the proof of the previous lemma, we have seen that $d\lambda \wedge \omega \neq 0$ if ϕ_t has two positive Lyapunov exponents. If $d\lambda \wedge \omega \neq 0$, then we must have $\omega(E_1^+, E_1^-) \neq 0$. Thus we can define two C^∞ line bundles as in the previous lemma, i.e.

$$E_2^{\pm} = [Ker(v \mapsto \omega(E_1^{\mp}, v))] \cap E^{\pm}.$$

Then we have the following smooth decomposition

$$TM = \mathbb{R}X \oplus E_1^+ \oplus E_1^- \oplus E_2^+ \oplus E_2^-.$$

If ϕ_t has two positive Lyapunov exponents, then this decomposition is just the C^∞ Lyapunov decomposition of ϕ_t. If ϕ_t has only one Lyapunov exponent, then the C^∞ Lyapunov decomposition of ϕ_t is $TM = \mathbb{R}X \oplus E^+ \oplus E^-$.

Since ϕ_t has C^∞ Lyapunov decomposition with respect to its invariant volume form, then by Subsection 5.2.2 (see Lemma 5.4), we get a C^∞ complete parallel linear connection ∇ canonically associated to ϕ_t. Since by the construction of ∇, $(\widetilde{X}, \widetilde{E}_i^{\pm})$ is a C^∞ $\widetilde{\nabla}$-parallel geometric structure of order one on $\widetilde{M}$, then by Lemma 2.2, the isometry group of $(\widetilde{\nabla}, \widetilde{X}, \widetilde{E}_i^{\pm})$ acts transitively on $\widetilde{M}$. In particular we deduce that $d\lambda$ vanishes nowhere. So E_1^+ and E_1^- are two well-defined C^∞ line bundles on M.

Up to finite covers we suppose in the following that $E^{\pm}$ and $E_1^{\pm}$ and M are all orientable.

5.3.2 The construction of a parallel geometric structure

Larger is the geometric structure, smaller is its isometry group. So in this subsection we try to enrich the underlying invariant geometric structure $(\nabla, X, E_i^{\pm})$ of ϕ_t to get a smaller (but transitive) isometry group.

Since the Bowen-Margulis measure of ϕ_t is Lebesgue, then by Pesin's entropy formula, the topological entropy of ϕ_t is equal to the sum of its positive Lyapunov exponents (with multiplicity). If ϕ_t has only one positive Lyapunov exponent, then ∇ is canonical (see Section 4.4 and Subsection 5.2.2). So by Proposition 4.2, $\wedge^2 E^+$ admits a C^∞ nowhere-vanishing ∇-parallel section denoted by ω^+.

However ∇ is not canonical if ϕ_t has two positive Lyapunov exponents. So in this case we need more delicate arguments to find such a section to enrich the geometric structure.

Proposition 5.2 – *Under the notations above, if ϕ_t has two positive Lyapunov exponents, then $\wedge^2 E^+$ admits a C^∞ nowhere-vanishing ∇-parallel section.*

This proposition will be proved via several lemmas. Recall firstly that there exist C^∞ ϕ_t-invariant line bundles E_2^+ and E_2^- in this case (see Remark 5.2).

Since μ^+ is given by C^∞ volume forms along the leaves of $\mathcal{F}^+$ by Lemma 3.8, then it can be viewed as a C^∞ nowhere-vanishing section of $\wedge^2(E^+)^*$. Denote by ω^+ its dual section. ∇ induces a C^∞ linear connection on $\wedge^2 E^+$ again denoted by ∇. Denote by β^+ the connection form of $(\nabla, \wedge^2 E^+)$ with respect to ω^+, i.e.

$$\nabla_\cdot \omega^+ = \beta^+(\cdot)\omega^+.$$

Then $\Omega^+ = d\beta^+$ is the curvature form of $(\nabla, \wedge^2 E^+)$. The following calculation is inspired by **[Fo2]** and **[BFL2]**.

Lemma 5.7 – $d\lambda \wedge \Omega^+ = 0$, $\Omega^+ \wedge \Omega^+ = 0$, $\Omega^+ \wedge \omega = 0$.

Proof – Since Ω^+ is ϕ_t-invariant and the flow is topologically transitive, then there exists a constant c such that

$$\lambda \wedge d\lambda \wedge \Omega^+ = c \cdot \lambda \wedge \omega \wedge \omega.$$

So

$$c \int_M \lambda \wedge \omega \wedge \omega = \int_M \lambda \wedge d\lambda \wedge \Omega^+$$
$$= -\int_M d(\lambda \wedge d\lambda \wedge \beta^+) = \int_{\partial M} \lambda \wedge d\lambda \wedge \beta^+ = 0.$$

So $c = 0$. We deduce that $d\lambda \wedge \Omega^+ = i_X(\lambda \wedge d\lambda \wedge \Omega^+) = 0$. Similarly we get $\Omega^+ \wedge \Omega^+ = 0$ using $d\lambda \wedge \Omega^+ = 0$.

If $\lambda \wedge \Omega^+ \wedge \omega = s \cdot \lambda \wedge \omega \wedge \omega$, then

$$s \int_M \lambda \wedge \omega \wedge \omega = \int_M \beta^+ \wedge d\lambda \wedge \omega.$$

If $\lambda \wedge d\lambda \wedge \omega = \delta \cdot \lambda \wedge \omega \wedge \omega$, then

$$\beta^+ \wedge d\lambda \wedge \omega = \delta \cdot \beta^+ \wedge \omega \wedge \omega$$

$$= \delta \cdot \beta^+(X)\lambda \wedge \omega \wedge \omega.$$

In the following we prove that $\int_M \beta^+(X)\lambda \wedge \omega \wedge \omega = 0$, which will finish the proof.

Since E_1^+ and E_2^+ are both orientable line bundles, then we can find two C^∞ nowhere-vanishing sections ω_1^+ and ω_2^+ of them. Using these two sections we get from ϕ_1 a C^∞ map A of M to $GL(2,\mathbb{R})$. Then with respect to $\omega_1^+ \wedge \omega_2^+$ we get from Theorem 3.1

$$\int_M log(det_{\omega_1^+ \wedge \omega_2^+} D\phi_1 \mid_{E^+}) d\nu = \chi_1 + \chi_2.$$

By similar arguments as in the proof of Theorem 4.3.1, we get with respect to ω^+

$$-\beta^+(X) = \frac{\partial}{\partial t} \mid_{t=0} (log(det_{\omega^+} D\phi_t \mid_{E^+})) - (\chi_1 + \chi_2).$$

Since $log(det_{\omega_1^+ \wedge \omega_2^+} D\phi_1 \mid_{E^+})$ and $log(det_{\omega^+} D\phi_t \mid_{E^+})$ are two cohomologuous $\mathbb{R}$-cocycle of ϕ_t, then their integrals with respect to the invariant volume form coincide. We deduce that $\int_M \beta^+(X)\lambda \wedge \omega \wedge \omega = 0$. $\square$

Lemma 5.8 – *Under the notations above we have* $\Omega^+ = 0$.

Proof – In the direction of X, the situation is always clear. So in the following we consider only the restrictions onto $E^+ \oplus E^-$ of the forms and endomorphisms.

Since $\omega \mid_{E^+ \oplus E^-}$ is non-degenerate, then we can find a section ψ of $End(E^+ \oplus E^-)$ such that $\Omega^+(\cdot,\cdot) = \omega(\psi(\cdot), \cdot)$. For all $y \in M$ we take $l_{1,2}^\pm \in (E_{1,2}^\pm)_y$ such that $(l_2^+, l_1^+, l_2^-, l_1^-)$ forms a dual basis of ω_y, i.e.

$$\omega(l_2^+, l_2^-) = \omega(l_1^+, l_1^-) = 1, \; \omega(l_2^+, l_1^-) = \omega(l_1^+, l_2^-) = 0.$$

If $\psi_y(l_1^+) = 0$, then in this basis we get

$$\psi_y = \begin{pmatrix} A & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\Omega^+ \wedge \omega = 0$ by Lemma 5.7, then $\mathrm{Tr}\psi = 2A = 0$. By Lemma 3.1, we have

$$0 = \Omega_y^+(l_2^+, l_1^-) = \omega(\psi l_2^+, l_1^-) = B \cdot \omega(l_1^+, l_1^-).$$

So $B = 0$. We deduce that $\psi_y = 0$.

Now suppose that $\psi_y(l_1^+) \neq 0$. Since $\Omega^+ \wedge \Omega^+ = 0$, then $\det(\psi_y) = 0$. So there exists $y_1^+ = \alpha l_2^+ + \delta l_1^+$ such that $\alpha \neq 0$ and $\psi_y(y_1^+) = 0$. Then in a dual basis of the form $(y_1^+, l_1^+, y_1^-, z^-)$, we have

$$\psi_y = \begin{pmatrix} 0 & A & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \end{pmatrix}$$

Since $\Omega^+ \wedge \omega = 0$, then $\mathrm{Tr}(\psi_y) = 2B = 0$. Again by Lemma 3.1, we get

$$0 = \Omega_y^+(l_1^+, l_2^-) = \omega(Ay_1^+, l_2^-) = A \cdot \alpha \cdot \omega(l_2^+, l_2^-) = A \cdot \alpha.$$

So $A = 0$. We deduce that $\psi \equiv 0$, i.e. $\Omega^+ \equiv 0$. □

Proof of Proposition 5.2 – By similar arguments as in Lemma 4.9, we can see that $\nabla_X \mu^+ = 0$. We deduce that $\nabla_X \omega^+ = 0$. So $\beta^+(X)\omega^+ = \nabla_X \omega^+ = 0$, i.e. $\beta^+(X) = 0$.

Since $d\beta^+ = \Omega^+ = 0$ by Lemma 5.8, then we have

$$\mathcal{L}_X \beta^+ = d(\beta^+(X)) + i_X d\beta^+ = 0.$$

So there exists $a \in \mathbb{R}$ such that $\beta^+ = a \cdot \lambda$. Thus $0 = \beta^+(X) = a$, i.e. $\beta^+ = 0$. We deduce that $\nabla \omega^+ = 0$. □

So in any case $\wedge^2 E^+$ adimts a C^∞ nowhere-vanishing ∇-parallel section denoted by ω^+. Define $\tau = (X, E_i^\pm, \omega, \omega^+)$. Then τ is a C^∞ ∇-parallel geometric structure of order one. Define $G = I(\widetilde{\nabla}, \widetilde{\tau})$. Then by Lemma 2.2, G acts transitively on $\widetilde{M}$. Fix $x \in \widetilde{M}$ and denote by H the isotropy subgoup of x in G. Denote by Γ the fundamental group of M. Then Γ is contained as a discrete subgroup in G and we have $M \cong \Gamma \diagdown G \diagup H$.

5.3.3 Properties of the isometry group

Denote by $\mathfrak{g}$ the Lie algebras of G which is identified to the Lie algebra of germs of local Killing fields at x (see Subsection 2.5.4). Denote by $\mathfrak{h}$ the Lie algebra of H. At first let us find out the generator in $\mathfrak{g}$ of ϕ_t.

We define $G' = Is(\widetilde{X}, \widetilde{E}_i^\pm, \widetilde{\nabla}, \widetilde{\omega})$. Then $G \subseteq G'$ and G' acts also transitively on $\widetilde{M}$. Denote by H' the isotropy subgroup of x in G'. Their Lie algebras are denoted respectively by $\mathfrak{g}'$ and $\mathfrak{h}'$. Then by Subsection 2.5.4, we have the decompositions

$$\mathfrak{g} \cong T_x\widetilde{M} \oplus \mathfrak{h}, \quad \mathfrak{g}' \cong T_x\widetilde{M} \oplus \mathfrak{h}'.$$

Under the identifications in Subsection 2.5.4, $\widetilde{X}$ is contained in the center of $\mathfrak{g}'$ and there exists an element L_0 in $\mathfrak{h}'$ such that

$$\widetilde{X} = \widetilde{X}_x + L_0.$$

Then on G'/H' the lifted flow $\widetilde{\phi}_t$ is given by

$$\widetilde{\phi}_t(gH') = (g \cdot exp(-t(\widetilde{X}_x + L_0))H' = (g \cdot exp(-t\widetilde{X}_x))H'.$$

So on G/H the lifed flow $\widetilde{\phi}_t$ is given by

$$\widetilde{\phi}_t(gH) = (g \cdot exp(-t\widetilde{X}_x))H.$$

Denote by G_0 the identity comoponent of G. Then G_0 acts also transitively on $\widetilde{M}$. By the long exact sequence of homotopy we get $H_0 = H \cap G_0$. Since by Subsection 2.5.4, G and H have both finitely many connected components, then up to finite covers we suppose that $\Gamma \subseteq G_0$.

Denote respectively by $Q^{\pm}$ and $P^{\pm}$ the stabilizers in G_0 of $\widetilde{W}_x^{\pm}$ and $\widetilde{W}_x^{\pm,0}$. Then we can see as following that they are all connected Lie subgroups of G_0, though not a priori closed:

Since $\pi : G \to G/H$ is a fiber bundle, then each curve in G/H can be lifted to G. So for each element $a \in P^+$, we can find a C^∞ curve l in G such that $l(0) = e$ and $l(1)(x) = a(x)$ and $l \subseteq P^+$. Thus $a^{-1} \cdot l(1) \in H_0$. Since H_0 is connected, then e can be related to $a^{-1} \cdot l(1)$ in H_0. We deduce that a can be joined to e by a piecewise C^∞ curve in G contained completely in P^+. Then by a classical result of Yamabe (see [**KMS**]), P^+ is seen to be a connected Lie subgroup of G. Similarly $Q^{\pm}$ and P^- are seen to be connected Lie subgroups of G.

Their Lie algebras are respectively

$$\mathfrak{q}^{\pm} = \{(u, A) \in \mathfrak{g} \mid u \in E_x^{\pm}\}$$

and

$$\mathfrak{p}^{\pm} = \{(u, A) \in \mathfrak{g} \mid u \in E_x^{\pm,0}\},$$

where we have identified $T_x\widetilde{M}$ with T_xM to simplify the notations.

For $Y = (u, A) \in \mathfrak{g}$ we have

$$0 = \mathcal{L}_Y\widetilde{\omega}^+ = (\mathcal{L}_Y - \nabla_Y)\widetilde{\omega}^+ = Tr(A \mid_{E_x^+}) \cdot \widetilde{\omega}^+.$$

We deduce that $Tr(A \mid_{E_x^+}) = 0$ for all $(u, A) \in \mathfrak{g}$.

Since we have $D_xh(X_x) = X_x$ for all $h \in H_0$, then $\mathfrak{h}(X_x) \equiv 0$. So in the following we identify H_0 and $\mathfrak{h}$ to their restrictions onto $E_x^+ \oplus E_x^-$. A basis (u^+, v^+, u^-, v^-) of $E_x^+ \oplus E_x^-$ is said to be *dual* with respect to ω if $u^{\pm}$ and $v^{\pm}$ are in $E_x^{\pm}$ and

$$\omega(u^+, u^-) = \omega(v^+, v^-) = 1,\ \omega(u^+, v^-) = \omega(v^+, u^-) = 0.$$

Lemma 5.9 – *Under the notations above, G_0 is simply connected and H_0 is isomorphic to either $\{0\}$ or $\mathbb{R}$.*

Proof – If $d\lambda \wedge \omega \neq 0$, then by Remark 5.2, there exist two C^∞ ϕ_t-invariant line bundles E_2^+ and E_2^- such that

$$\omega(E_1^+, E_2^-) \equiv 0,\ \omega(E_1^-, E_2^+) \equiv 0.$$

Take a basis $(l_2^+, l_1^+, l_2^-, l_1^-)$ of $(E^+ \oplus E^-)_x$ such that $l_{1,2}^{\pm} \in (E_{1,2}^{\pm})_x$. Then for each element $h \in H_0$, we have in this basis

$$D_x h = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_1} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_2} \end{pmatrix},$$

since $D_x h$ preserves ω_x and $E_1^{\pm}(x)$ and $E_2^{\pm}(x)$. Since h preserves also $\widetilde{\omega}^+$, then we have $\lambda_1 \cdot \lambda_2 = 1$. So H_0 is isomorphic to 0 or $\mathbb{R}$.

If $d\lambda \wedge \omega = 0$, then we can find a dual basis of $E_x^+ \oplus E_x^-$, (y^+, l_1^+, l_1^-, y^-), such that $l_1^{\pm} \in E_1^{\pm}$ and $d\lambda(y^+, y^-) = 1$. Denote by φ the section of $End(E^+ \oplus E^-)$ such that

$$d\lambda(\cdot,\ \cdot) = \omega(\varphi\ \cdot,\ \cdot).$$

Since $d\lambda \wedge \omega = 0$, then $\mathrm{Tr}\varphi = 0$. Since we have in addition $d\lambda(l_1^{\pm}, \cdot) = 0$ and $d\lambda$ vanishes nowhere, then there exists $B \neq 0$ such that

$$\varphi_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For all $h \in H_0$, $D_x h$ preserves $d\lambda_x$. So in the basis above the matrix of $D_x h$ must have the following form

$$D_x h = \begin{pmatrix} c & 0 & 0 & 0 \\ d & c & 0 & 0 \\ 0 & 0 & \frac{1}{c} & -\frac{d}{c^2} \\ 0 & 0 & 0 & \frac{1}{c} \end{pmatrix}$$

Since h preserves $\widetilde{\omega}^+$, then we have in addition $c^2 = 1$. Thus H_0 is isomorphic to 0 or $\mathbb{R}$.

So in any case H_0 is simply connected. Then by the long exact sequence of homotopy it is easily seen that G_0 is also simply connected. □

We have the following

Lemma 5.10 – *Under the notations above $E_1^+ \oplus E_1^-$ is integrable. If $d\lambda \wedge \omega \neq 0$, then $E_2^+ \oplus E_2^- \oplus \mathbb{R}X$ is also integrable.*

Proof – Let Y, Z be two sections of $E_1^+ \oplus E_1^-$. Then $0 = d\lambda(Y, Z) = -\lambda([Y, Z])$. So $[Y, Z]$ is also a section of $E^+ \oplus E^-$. Since in addition

$$i_{[Y,Z]} d\lambda = (\mathcal{L}_Y i_Z - i_Z \mathcal{L}_Y) d\lambda = -i_Z(di_Y + i_Y d) d\lambda = 0,$$

then $[Y, Z]$ is a section of $E_1^+ \oplus E_1^-$. So $E_1^+ \oplus E_1^-$ is integrable.

Since E_2^+ and E_2^+ are both ϕ_t-invariant, then $[X, E_2^\pm] \subseteq E_2^\pm$. Define two tensor fields $K^\pm$ of type $(1,2)$ on M such that

$$K^\pm(Y,Z) = P_1^\pm[P_2^+(Y), P_2^-(Z)],\ \forall\ Y,Z \subseteq TM.$$

Then $K^\pm$ are both ϕ_t-invariant. So by Lemma 5.2, we have $K^\pm(E_2^+, E_2^-) \subseteq \mathbb{R}X$. We deduce that $[E_2^+, E_2^-] \subseteq E_2^+ \oplus E_2^- \oplus \mathbb{R}X$, i.e. $E_2^+ \oplus E_2^- \oplus \mathbb{R}X$ is integrable. □

In conclusion we need eliminate the following cases in order to prove the non-existence of ϕ_t:
(I0) Two positive Lyapunov exponents and $\dim\mathfrak{h} = 0$.
(I1) Two positive Lyapunov exponents and $\dim\mathfrak{h} = 1$.
(II1) One positive Lyapunov exponent, $\dim\mathfrak{h} = 1$ and $d\lambda \wedge \omega \neq 0$.
(II0) One positive Lyapunov exponent and $\dim\mathfrak{h} = 0$.
(III1) One positive Lyapunov exponent, $\dim\mathfrak{h} = 1$ and $d\lambda \wedge \omega = 0$.
In the subsection below we shall eliminate these cases one by one.

5.3.4 Elimination case by case

(I0) In this case we have $TM = \mathbb{R}X \oplus E_1^+ \oplus E_2^+ \oplus E_1^- \oplus E_2^-$. Up to a constant change of time scale, we suppose that the Lyapunov exponents of E_1^+ and E_2^+ are 1 and a.

As in Subsection 2.5.4, we identify the Lie algebra of G with that of the germs of Killing fields at x. Since by assumption $\dim\mathfrak{h} = 0$, then we have $\mathfrak{g} = T_xM$ (see Subsection 2.5.4). Now we can find explicitly $\mathfrak{g}$ as following.

For u and v in T_xM we have

$$[u,v] = T(u,v) - R(u,v) \in T_xM.$$

So $R(u,v) = 0$ and $[u,v] = T(u,v),\ \forall\ u,v \in T_xM$.

Take a basis of T_xM, $(X_x, l_2^+, l_1^+, l_2^-, l_1^-)$, such that $l_{1,2}^\pm \in (E_{1,2}^\pm)_x$ and $d\lambda(l_2^+, l_2^-) = 1$. Extend $l_{1,2}^\pm$ to local sections of $E_{1,2}^\pm$ denoted by $\bar{l}_{1,2}^\pm$. By the definition of ∇ we get

$$[X_x, l_1^\pm] = T(X_x, l_1^\pm) = \pm l_1^\pm$$

and

$$[X_x, l_2^\pm] = \pm a l_2^\pm.$$

Since $E_1^+ \oplus E_1^-$ is integrable by Lemma 5.10, then we get

$$[l_1^+, l_1^-] = T(l_1^+, l_1^-)$$

$$= (P_1^-[\bar{l}_1^+, \bar{l}_1^-] + P_1^+[\bar{l}_1^+, \bar{l}_1^-] - [\bar{l}_1^+, \bar{l}_1^-]) = 0.$$

Similarly we get $[l_2^+, l_2^-] = X_x$ by Lemma 5.10.

Lemma 5.11 – *Under the notations above, we have $1 < a$.*

Proof – Suppose on the contrary that $1 > a$ (Recall that $a \neq 1$). Then by Subsection 5.2.2, we have

$$T(E_1^+, E_2^-) \subseteq E_{1-a}.$$

If $1 - a \neq a$, then $[l_1^+, l_2^-] = T(l_1^+, l_2^-) = 0$. If $1 - a = a$, then there exists $b \in \mathbb{R}$ such that $[l_1^+, l_2^-] = b \cdot l_2^+$. So in any case there exists $c \in \mathbb{R}$ such that $[l_1^+, l_2^-] = c \cdot l_2^+$.

Since $T(E_1^+, E_2^+) \subseteq E_{1+a} = \{0\}$, then $[l_1^+, l_2^+] = 0$. By the Jacobi identity of l_1^+ and l_2^+ and l_2^-, we get

$$0 = [l_2^+, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, l_2^+]] + [l_2^-, [l_2^+, l_1^+]] = [l_2^+, c \cdot l_2^+] + [l_1^+, -X_x] = l_1^+,$$

which is absurd. $\square$

Since $a > 1$, then we can suppose that $[l_1^+, l_2^-] = c \cdot l_1^-$ and $[l_1^-, l_2^+] = d \cdot l_1^+$. Again by the Jacobi identity of l_1^+ and l_2^+ and l_2^-, we have

$$0 = [l_2^+, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, l_2^+]] = (1 - c \cdot d) l_1^+.$$

So $c \cdot d = 1$. Now replacing l_2^- by $\frac{1}{c} l_2^-$ and l_2^+ by $c \cdot l_2^+$, we get the following bracket relations of $\mathfrak{g}$

$$[X_x, l_1^\pm] = \pm l_1^\pm, \ [X_x, l_2^\pm] = \pm a l_2^\pm,$$
$$[l_1^+, l_1^-] = 0, \ [l_1^+, l_2^-] = l_1^-,$$
$$[l_1^-, l_2^+] = l_1^+, \ [l_2^+, l_2^-] = X_x.$$

The brackets, not appeared above, vanish by Subsection 5.2.2.

Since $[l_1^+, l_2^-] = l_1^-$, then $E_{1-a} \neq \{0\}$. We deduce that $a = 2$. Then by the bracket relations above we get clearly

$$\mathfrak{g} \cong \mathbb{R}^2 \rtimes \mathfrak{sl}(2, \mathbb{R}),$$

where the semi-direct product is given by matrix multiplication.

Since G_0 is simply connected by Lemma 5.9, then we get

$$G_0 \cong \mathbb{R}^2 \rtimes \widetilde{SL(2, \mathbb{R})}.$$

Since H_0 is trivial, then Γ is a uniform lattice of G_0. So case (I0) is eliminated by the following

Lemma 5.12 – $\mathbb{R}^2 \rtimes \widetilde{SL(2, \mathbb{R})}$ *has no cocompact lattice.*

Proof – Suppose that there exists a cocompact lattice, denoted by Δ. Then by **[Ra]**, the projection of Δ to $\widetilde{SL(2, \mathbb{R})}$ must be discrete, denoted by Δ_2. Since Δ_2 is also cocompact in $\widetilde{SL(2, \mathbb{R})}$, then $\mathrm{cd}(\Delta_2) = 3$ (cd denotes the cohomological dimension of a discrete group). We deduce that $\Delta \cap \mathbb{R}^2 \neq 0$. Since the linear action of $\widetilde{SL(2,\mathbb{R})}$ on $\mathbb{R}^2$ is irreducible, then $\Delta \cap \mathbb{R}^2$ is in fact cocompact in $\mathbb{R}^2$. A simple calculation shows that Δ_2 preserves $\Delta \cap \mathbb{R}^2$ for the natural linear action. So its image $\pi(\Delta_2)$ in $SL(2,\mathbb{R})$ (under the natural projection $\pi : \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$) is conjugate to a subgroup of $SL(2,\mathbb{Z})$. Since Δ_2 is cocompact in $\widetilde{SL(2,\mathbb{R})}$, then $\pi(\Delta_2)$ is also cocompact in $SL(2,\mathbb{R})$. We deduce that $SL(2,\mathbb{Z})$ is cocompact in $SL(2,\mathbb{R})$, which is absurd. □

(I1) Define $S = P_2^+ - P_1^+ - P_2^- + P_1^-$. Then $\mathfrak{h}$ is generated by S and we have $\mathfrak{g} = T_xM \oplus \mathbb{R}S$.

As above we suppose that the Lyapunov exponents of E_1^+ and E_2^+ are 1 and a. Take a basis of T_xM, $(X_x, l_2^+, l_1^+, l_2^-, l_1^-)$ such that $l_{1,2}^\pm \in E_{1,2}^\pm$ and $d\lambda(l_2^+, l_2^-) = 1$. Suppose at first that $a > 1$. Then we can find c and d such that

$$[l_1^+, l_2^-] = c \cdot l_1^-,\ [l_1^-, l_2^+] = d \cdot l_1^+.$$

By the Jacobi identity of S and l_1^+ and l_2^-, we get

$$0 = [S, [l_1^+, l_2^-]] + [l_1^+, [l_2^-, S]] + [l_2^-, [S, l_1^+]]$$

$$= c \cdot l_1^- + [l_1^+, l_2^-] + [l_2^-, -l_1^+] = 3c \cdot l_1^-.$$

So $c = 0$. Similarly we get $d = 0$. If $a < 1$, then we can find c' and d' such that

$$[l_1^+, l_2^-] = c' \cdot l_2^+,\ [l_1^-, l_2^+] = d' \cdot l_2^-.$$

Then again by the Jacobi identity of S and l_1^+ and l_2^- we get as above $c' = d' = 0$. We deduce that in any case

$$[l_1^+, l_2^-] = 0,\ [l_1^-, l_2^+] = 0.$$

Now by similar calculations as above we get the following bracket relations,

$$[S, l_1^\pm] = \mp l_1^\pm,\ [S, l_2^\pm] = \pm l_2^\pm,$$

$$[X_x, l_1^\pm] = \pm l_1^\pm,\ [X_x, l_2^\pm] = \pm a l_2^\pm,\ [l_2^+, l_2^-] = X_x + S.$$

The brackets, not appeared above, vanish. Define two elements

$$\alpha = \frac{X_x + S}{a+1}, \quad \beta = \frac{-X_x + aS}{a+1}.$$

Thus $\mathfrak{g}$ is decomposed as a direct product of two ideals

$$\mathfrak{g} \cong (\mathbb{R}l_1^+ \oplus \mathbb{R}l_1^- \oplus \mathbb{R}\beta) \oplus (\mathbb{R}l_2^+ \oplus \mathbb{R}l_2^- \oplus \mathbb{R}\alpha).$$

Then by the bracket relations above, we get

$$\mathfrak{g} \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}),$$

where the semi-direct product $\mathbb{R}^2 \rtimes \mathbb{R}$ is given by the linear action on $\mathbb{R}^2$ of the order two diagonal matrices of trace zero. Since G_0 is simply-connected by Lemma 5.9, then we have

$$G_0 \cong (\mathbb{R}^2 \rtimes \mathbb{R}) \times \widetilde{SL(2,\mathbb{R})}.$$

Before studying the action of Γ on the space of lifted weak unstable leaves, we shall at first prove a lemma about general Anosov flows.

Let ψ_t be a C^∞ Anosov flow on a closed manifold N. Denote by $\widetilde{\psi}_t$ its lifted flow on the universal covering space $\widetilde{N}$. Denote by $\widetilde{\mathcal{F}}^{+,0}$ the lifted foilation of $\mathcal{F}^{+,0}$ and by $\widetilde{N}/\widetilde{\mathcal{F}}^{+,0}$ the space of lifted weak unstable leaves with the quotient topology. Thus the fundamental group $\pi_1(N)$ acts naturally on $\widetilde{N}/\widetilde{\mathcal{F}}^{+,0}$. The following lemma has appeared in some special contexts (see for example [**BFL2**] and [**Ba**]). For the sake of completeness, we prove it in detail.

Lemma 5.13 – *Under the notations above, if* $\gamma \in \pi_1(N)$ *and* $\gamma \neq e$, *then each* γ*-fixed point of* $\widetilde{N}/\widetilde{\mathcal{F}}^{+,0}$ *is either* γ*-contractive or* γ*-repulsive.*

Proof – Recall at first some notions. Suppose that ϕ is a C^∞ diffeomorphism on a manifold M and $a \in M$ such that $\phi(a) = a$. Then ϕ is said to be *contractive* on an open neighborhood U of a, if for each open neighborhood W of a there exists $N > 0$ such that $\phi^n U \subseteq W$ for all $n \geq N$. This fixed point a is said to be ϕ*-contractive* if ϕ is contractive on some open neighborhood of a. Then a is said to be ϕ*-repulsive* if it is ϕ^{-1}*-contractive*.

Now let us return to the proof of the lemma. Suppose that $\widetilde{W}_x^{+,0}$ is fixed by γ. Then there exists $t \in \mathbb{R}$ such that

$$\gamma \widetilde{W}_x^+ = \widetilde{\phi}_t \widetilde{W}_x^+.$$

If $t = 0$, then we can take a curve l in $\widetilde{W}_x^+$, such that $l(0) = x$ and $l(1) = \gamma x$. If $s \ll 0$, then $\phi_s(\pi(l))$ will be tiny, where π denotes the projection of $\widetilde{N}$ onto N. Thus $\phi_s(\pi(l))$ is homotopically trivial. We deduce that $\pi(l)$ is also homotopically trivial, i.e. $\gamma = e$, which is a contradiction. So $t \neq 0$.

By replacing γ by γ^{-1} if necessary, we suppose that $t < 0$. We can see as following that $\widetilde{W}_x^{+,0}$ is γ-contractive.

Fix a C^∞ Riemannian metric g on N. Denote by $\widetilde{g}$ the lifted metric on $\widetilde{N}$. By [**An**] the induced metrics on the leaves of $\mathcal{F}^{+,0}$ are all complete. Then with respect to its induced metric $\widetilde{W}_x^+$ is a complete metric space. Since γ acts isometrically, then $\gamma^{-n} \circ \widetilde{\phi}_{nt}$ is a contraction of $\widetilde{W}_x^+$ if $n \gg 1$. Thus it admits a unique fixed point in $\widetilde{W}_x^+$ denoted again by x. So we get

$$\gamma x = \widetilde{\phi}_t x,$$

i.e. the orbit of x is fixed by γ.

Denote by $\bar{U}$ the saturated set of $\widetilde{W}_x^-$ with respect to $\widetilde{\mathcal{F}}^{+,0}$. Then by the local product structure of $\widetilde{\phi}_t$, $\bar{U}$ is open. So the projection of $\widetilde{W}_x^-$ into $\widetilde{N}/\mathcal{F}^{+,0}$ is an open neighborhood of $\widetilde{W}_x^{+,0}$ denoted by U. For $y \in \widetilde{W}_x^-$ we have

$$\gamma^n \widetilde{W}_y^{+,0} = \widetilde{W}^{+,0}_{(\widetilde{\phi}_{-nt} \circ \gamma^n)(y)}.$$

Since $\gamma x = \widetilde{\phi}_t x$, then $(\widetilde{\phi}_{-nt} \circ \gamma^n)(x) = x$. So $(\widetilde{\phi}_{-nt} \circ \gamma^n)(y) \xrightarrow[n \to +\infty]{} x$. We deduce that $\gamma^n \widetilde{W}_y^{+,0} \xrightarrow[n \to +\infty]{} \widetilde{W}_x^{+,0}$ uniformly. So γ contracts on U. $\square$

Now let us return to our ϕ_t. Denote by P_0 the stabilizer of $\widetilde{W}_x^{+,0}$ in G_0. Then $H_0 \subsetneq P_0$ and P_0 is connected. So $G_0 \diagup P_0$ is identified to the lifted leaf space $\widetilde{M} \diagup \widetilde{\mathcal{F}}^{+,0}$. Define

$$\mathfrak{p}^+ = \mathbb{R}X_x \oplus \mathfrak{h} \oplus \mathbb{R}l_1^+ \oplus \mathbb{R}l_2^+.$$

Then $\mathfrak{p}^+$ is the Lie algebra of P_0 and P_0 is seen to be closed in G_0. Since G_0 is simply connected by Lemma 5.9, then by the long exact sequence of homotopy we get $\pi_1(G_0 \diagup P_0) = 0$.

Define $G_0^1 = (\mathbb{R}^2 \rtimes \mathbb{R}) \times SL(2, \mathbb{R})$ and denote by P_0^1 the connected Lie subgroup of G_0^1 with Lie algebra $\mathfrak{p}^+$. Then $G_0^1 \diagup P_0^1$ is naturally identified to $\mathbb{R}^1 \times \mathbb{S}^1$. Denote by π the projection of G_0 onto G_0^1 and by P_0' the group $\pi^{-1}(P_0^1)$. Then we get

$$G_0 \diagup P_0' \cong G_0^1 \diagup P_0^1 \cong \mathbb{R}^1 \times \mathbb{S}^1.$$

We deduce that $G_0 \diagup P_0 \cong \mathbb{R}^1 \times \mathbb{R}^1$.

Since ϕ_t preserves a volume form, then the set of periodic points of ϕ_t is dense in M. Take $gH_0 \in G_0 \diagup H_0$ such that its projection in M is of period T. If $\widetilde{\phi}_T(gH_0) = gH_0$, then each orbit of $\widetilde{\phi}_t$ is periodic. We deduce that each ϕ_t-orbit is periodic by the homogeneity of $\widetilde{\phi}_t$, which contradicts the topological transitivity of ϕ_t. So $\widetilde{\phi}_T(gH_0) \neq gH_0$.

Now take $\gamma \in \Gamma \subseteq G_0$ such that $\gamma(gH_0) = \widetilde{\phi}_T(gH_0)$. Then we have $\gamma \neq e$ and there exists $h \in H_0$ such that (see the beginning of Subsection 5.3.3)

$$\gamma = g(h \cdot exp(-TX_x))g^{-1}.$$

Since γ fixes the orbit of gH_0, then it fixes also gP_0 and gP_0' for its natural action on $G_0 \diagup P_0$ and $G_0 \diagup P_0'$. So by Lemma 5.13, the γ-action on $G_0 \diagup P_0'$ admits at least a contractive (or repulsive) fixed point, notably gP_0'. Then by some direct but lengthy calculations the corresponding γ-action on $G_0 \diagup P_0' \cong \mathbb{R}^1 \times \mathbb{S}^1$ is seen to be as following:

$$\gamma : \mathbb{R}^1 \times \mathbb{S}^1 \to \mathbb{R}^1 \times \mathbb{S}^1 \qquad (*)$$

$$(y,\ [u]) \to (exp^{-c}y + d,\ [Au]),$$

where $c \neq 0$ and A is matrix with two different positive eigenvalues. Here $\mathbb{S}^1$ is viewed as the set of directions, i.e.

$$\mathbb{S}^1 \cong \{[u] \mid u \in \mathbb{C}^*, u \sim v \Leftrightarrow u = tv, t > 0\}.$$

Then $GL(2,\mathbb{R})$ acts on $\mathbb{S}^1$ by matrix multiplication. It is easy to see (by drawing a picture) that γ has exactly four fixed points on $\mathbb{R}^1\times\mathbb{S}^1$. Two of them are saddles.

Up to an isomorphism of covering spaces, the projection of G_0/P_0 onto G_0/P_0' is as following

$$\mathbb{R}^1\times\mathbb{R}^1 \longmapsto \mathbb{R}^1\times\mathbb{S}^1 \qquad (**)$$
$$(x,\ \theta)\mapsto(x,\ [e^{i\theta}]).$$

Since the γ-action on G_0/P_0 is a lift of the γ-action on G_0/P_0', then by $(*)$ and $(**)$ it is easily seen that on G_0/P_0 γ admits either a saddle or no fixed point. But either of them is absurd.

(II1) Define as above $S=P_2^+-P_1^+-P_2^-+P_1^-$. Then $\mathfrak{h}$ is generated by S and $\mathfrak{g}\cong T_xM\oplus\mathbb{R}S$. Up to a constant change of time scale we suppose that the unique positive Lyapunov exponent of ϕ_t is one. Take a basis $(X_x,l_2^+,l_1^+,l_2^-,l_1^-)$ as in Case (I1). Then by the definition of ∇ we get easily

$$[l_1^+,l_2^-]=0,\quad [l_1^-,l_2^+]=0.$$

Then by some similar calculations as before $\mathfrak{g}$ is seen to be the same as that of Case (I1) except that $a=1$. But in Case (I1) we have found two elements α and β which have eliminated the effect of a on the structure of $\mathfrak{g}$. So in this case we get the same G_0 and H_0 as in Case (I1). So the same argument proves the non-existence of this case.

(II0) Since $\dim\mathfrak{h}=0$, then $\mathfrak{g}=T_xM$. Take a basis $(X_x,l_2^+,l_1^+,l_2^-,l_1^-)$ of T_xM such that $l_1^\pm\in(E_1^\pm)_x$ and $d\lambda(l_2^+,l_2^-)=1$. By similar calculations as in Case (I0) we get directly

$$[X_x,l_{1,2}^\pm]=\pm l_{1,2}^\pm,\ \ [l_2^+,l_2^-]=X_x.$$

The brackets, not appeared above, vanish. We deduce that

$$0=[l_1^+,[l_2^+,l_2^-]]+[l_2^+,[l_2^-,l_1^+]]+[l_2^-,[l_1^+,l_2^+]]=[l_1^+,X_x]=-l_1^+,$$

which is absurd.

(III1) In this case we can find a dual basis $(l_2^+,l_1^+,l_1^-,l_2^-)$ such that $l_1^\pm\in E_1^\pm(x)$ and $d\lambda(l_2^+,l_2^-)=1$. Define an element S in $End(E_x^+\oplus E_x^-)$ such that

$$S(l_2^\pm)=\pm l_1^\pm,\quad S(l_1^\pm)=0.$$

Then we have $S\in\mathfrak{h}$ (see the proof of Lemma 5.9). Since $\dim\mathfrak{h}=1$, then we have $\mathfrak{g}\cong T_xM\oplus\mathbb{R}S$.

By using the Jacobi identities we get the following bracket relations

$$[S,l_2^\pm]=\pm l_1^\pm,\ \ [S,l_1^\pm]=0,$$

$$[X_x,l_{1,2}^\pm]=l_{1,2}^\pm,\ \ [l_1^+,l_1^-]=0,$$

$$[l_1^+, l_2^-] = -[l_1^-, l_2^+] = S, \ [l_2^+, l_2^-] = X_x + a \cdot S,$$

where a is an unknown parameter whose appearence is due to the non-unique choice of l_2^+ and l_2^-.

Since $[l_2^+, l_2^- - a \cdot l_1^-] = X_x$, then by replacing l_2^- by $l_2^- - a \cdot l_1^-$ we get

$$\mathfrak{g} = (\mathbb{R}l_1^+ \oplus \mathbb{R}l_1^- \oplus \mathbb{R}S) \rtimes (\mathbb{R}l_2^+ \oplus \mathbb{R}l_2^- \mathbb{R}X_x).$$

By taking a change of basis of $\mathbb{R}l_1^+ \oplus \mathbb{R}l_1^- \oplus \mathbb{R}S \cong \mathbb{R}^3$ we get

$$\mathfrak{g} \cong \mathbb{R}^3 \rtimes \mathfrak{so}(1,2),$$

where $\mathfrak{so}(1,2)$ denotes the Lie algebra of the isometry group of the quadratic form $q = -x_0^2 + x_1^2 + x_2^2$ and the semidirect product is given by linear multiplication.

Then G_0 and H_0 can be realized as following

$$G_0 \cong \mathbb{R}^3 \rtimes \widetilde{SO_0(1,2)},$$

where $SO_0(1,2)$ is the identity component of the isometry group of the quadratic form q above. The semidirect product is given by the composition of the projection of $\widetilde{SO_0(1,2)}$ onto $SO_0(1,2)$ and the linear action of $SO_0(1,2)$ on $\mathbb{R}^3$. Let $((0,0,1),0) \in \mathbb{R}^3 \rtimes \mathfrak{so}(1,2)$. Then H_0 is the one-parameter subgroup of G_0 generated by this vector.

The fundamental group Γ is contained as a discrete subgroup in G_0 acting freely and properly and cocompactly on $G_0 \diagup H_0$. Now we find a contradiction by showing

Lemma 5.14 – *G_0 admits no discrete subgroup acting properly and freely and cocompactly on $G_0 \diagup H_0$.*

Proof – Suppose on the contrary that there exists Γ satisfying the conditions of the lemma. Denote by π the natural projection of G_0 onto the real algebraic group $G' = \mathbb{R}^3 \rtimes SO_0(1,2)$. Denote by $\overline{\pi(\Gamma)}$ the Zariski closure of $\pi(\Gamma)$ in G'.

If Γ is solvable, then $\pi(\Gamma)$ is solvable. So its Zariski closure $\overline{\pi(\Gamma)}$ is also sovable. Then there exists a closed connected sovable subgroup R of G' containing cocompactly a finite index subgroup of $\pi(\Gamma)$. So by replacing $\pi(\Gamma)$ by a finite index subgroup if necessary, $\pi(\Gamma)$ must be contained in a maximal connected sovable subgroup of G'. However there exist only two such groups in G', i.e. $\mathbb{R}^3 \rtimes AN$ and $\mathbb{R}^3 \rtimes SO(2)$, where KAN is the Iwasawa decomposition of $SO_0(1,2)$. Denote by $cd(\cdot)$ the cohomological dimension of a group. Since Γ acts freely and properly and cocompactly on $G_0 \diagup H_0$ which is contratible, then $cd(\Gamma) = 5$.

If $\pi(\Gamma) \subseteq \mathbb{R}^3 \rtimes SO(2)$, then $\Gamma \subseteq \mathbb{R}^3 \rtimes \widetilde{SO(2)} \cong \mathbb{R}^3 \rtimes \mathbb{R}$. We deduce that $cd(\Gamma) = 4$, which is a contradiction.

If $\pi(\Gamma) \subseteq \mathbb{R}^3 \rtimes AN$, then $\Gamma \subseteq (\mathbb{R}^3 \rtimes AN) \times \mathbb{Z}$, where $\mathbb{Z}$ denotes the center of $\widetilde{SO_0(1,2)}$. Define $\Gamma' = \Gamma \cap (\mathbb{R}^3 \rtimes AN)$. Then we can see that $cd(\Gamma') \geq 4$ (see **[Br]**). So as above, up to finite index, Γ' is contained cocompactly in a closed and connected sovable Lie subgroup R' of $\mathbb{R}^3 \rtimes AN$. Since $\mathbb{R}^3 \rtimes AN$ is not unimodular, then R' is of dimension at most four. We deduce that R' must be either $\mathbb{R}^3 \rtimes A$ or $(\mathbb{R}^2 \rtimes N) \rtimes A$. Since $(\mathbb{R}^2 \rtimes N) \rtimes A$ is neither unimodular, then Γ' can not be contained cocompactly in this group. We deduce that Γ' is a cocompact discrete subgroup of $\mathbb{R}^3 \rtimes A$. Since H_0 is contained in $\mathbb{R}^3$, then Γ' can not act properly on G_0/H_0, which is a contradiction.

We deduce that Γ is not sovable. Then the natural projection of $\pi(\bar{\Gamma})$ to $SO_0(1,2)$ must be surjective. If the kernal of this projection is trivial, then Γ must be contained in $\widetilde{SO_0(1,2)}$. So Γ can not acts cocompactly on G_0/H_0, which is a contradiction. We deduce that the kernal of this projection is not trivial. However since the action of $SO_0(1,2)$ on $\mathbb{R}^3$ is irreducible, then we must have $\pi(\bar{\Gamma}) = G'$, i.e. $\pi(\Gamma)$ is Zariski dense in G'.

Denote by Δ the image of the projection of Γ into $\widetilde{SO_0(1,2)}$ and by $\bar{\Delta}$ its closure in $\widetilde{SO_0(1,2)}$ with respect to the Lie group topology. If the Lie algebra of $\bar{\Delta}$ is denoted by $\mathfrak{l}$, then by Theorem 8.24 of **[Ra]**, $\mathfrak{l}$ is sovable. For all $\gamma \in \Gamma$ we have $Ad(\pi(\gamma))(\mathfrak{l}) \subseteq \mathfrak{l}$. Since this condition is algebraic, then by the Zariski density of $\pi(\Gamma)$, we get $Ad(g)(\mathfrak{l}) \subseteq \mathfrak{l}$ for all $g \in G'$. We deduce that $\mathfrak{l}$ is a sovable ideal in $\mathfrak{so}(1,2)$. So $\mathfrak{l}$ must be zero. We deduce that Δ is discrete in $\widetilde{SO_0(1,2)}$. Thus $cd(\Delta) \leq 3$.

Since $cd(\Gamma) = 5$, then $\Gamma \cap \mathbb{R}^3 \neq \{0\}$. Since the action of $\widetilde{SO_0(1,2)}$ on $\mathbb{R}^3$ is irreducible, then $\Gamma \cap \mathbb{R}^3$ must be cocompact in $\mathbb{R}^3$. We deduce that Γ can not acts properly on G_0/H_0, which is a contradiction. $\square$

Chapter 6

On Quasiconformal Anosov Systems

Abstract – *In this chapter we study the quasiconformal Anosov systems. The striking point is that they are already rigid without any assumptions about the smoothness of the strong stable and unstable distributions.*

6.1 Introduction

6.1.1 Motivations

Conformal geometry is a classically and currently fascinating subject, which is meaningfully mixed together with hyperbolic dynamical systems under the impulsion of the classical [**Su**], [**Ka2**] and [**Yu**] and the recent [**L1**], [**L2**], [**Sa**] and [**KS**]. Here we study the rigidity of such systems by combining quasiconformal techniques with our *go-and-back* idea.

Let us recall first some notions. Let ϕ_t be a C^∞ Anosov flow on a closed manifold M. Define two functions on $M \times \mathbb{R}$ as following

$$K^+(x,t) = \frac{max\{\| D\phi_t(u) \| \mid u \in E_x^+,\ \| u \|= 1\}}{min\{\| D\phi_t(u) \| \mid u \in E_x^+,\ \| u \|= 1\}}$$

and

$$K^-(x,t) = \frac{max\{\| D\phi_t(u) \| \mid u \in E_x^-,\ \| u \|= 1\}}{min\{\| D\phi_t(u) \| \mid u \in E_x^-,\ \| u \|= 1\}}.$$

If K^- (K^+) is bounded, then the Anosov flow ϕ_t is said to be *quasiconformal on the stable (unstable) distribution*. If K^+ and K^- are both bounded, then ϕ_t is said to be *quasiconformal*. If it is the case, then the superior bound of K^+ and K^- is said to be the *distortion* of ϕ_t. The corresponding notions for Anosov diffeomorphisms are defined similarly (see [**Sa**]).

Recall that two C^∞ Anosov flows $\phi_t : M \to M$ and $\psi_t : N \to N$ are said to be C^k *flow equivalent* ($k \geq 0$) if there exists a C^k diffeomorphism $h : M \to N$ such that

$\phi_t = h^{-1} \circ \psi_t \circ h$ for all $t \in \mathbb{R}$. They are said to be C^k *orbit equivalent* ($k \geq 0$) if there exists a C^k diffeomorphism $h : M \to N$ such that the flow $\chi_t = h^{-1} \circ \psi_t \circ h$ is a time change of ϕ_t. By convention, a C^0 diffeomorphism means a homeomorphism. Then we can prove the following

Lemma 6.1 – *Let ϕ_t and ψ_t be two C^∞ Anosov flows. If they are C^1 orbit equivalent and ψ_t is quasiconformal, then ϕ_t is also quasiconformal.*

Proof – Denote by $\phi : M \to N$ the C^1 orbit conjugacy between ϕ_t and ψ_t and by $\bar{\mathcal{F}}^\pm$ and $\bar{\mathcal{F}}^{\pm,0}$ the Anosov foliations of ψ_t. Then we have the following

Sublemma – *Under the notations above, $\phi(\mathcal{F}^{\pm,0}) = \bar{\mathcal{F}}^{\pm,0}$.*

Proof – Define $\hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1}$. Then $\hat{\phi}_t$ is a C^1 flow on N with the same orbits as ψ_t. So there exists a C^1 map $\alpha : \mathbb{R} \times N \to \mathbb{R}$ such that $\hat{\phi}_t(x) = \psi_{\alpha(t,x)}(x)$. Define $\hat{E}^- = (D\phi)(E^-)$. Then it is the C^0 tangent bundle of the C^1 foliation $\hat{\mathcal{F}}^- = \phi(\mathcal{F}^-)$.

Let us prove firstly that $\hat{E}^- \subseteq \bar{E}^{-,0}$. Fix a C^0 Riemannian metric g on N such that $\bar{E}^+$ and $\bar{E}^-$ and $\bar{X}$ are orthogonal to each other. Since ϕ is C^1, then it is bi-Lipschitz. We deduce that for all $x \in N$ and $\hat{u} \in \hat{E}^-_x$, $\| D\hat{\phi}_t(\hat{u}) \| \to 0$ if $t \to +\infty$.

If $\hat{u} = \bar{u}^+ + a\bar{X}_x + \bar{u}^-$ and $\bar{u}^+ \neq 0$, then by a simple calculation we get for a certain function $b_x : \mathbb{R} \to \mathbb{R}$,

$$D\hat{\phi}_t(\hat{u}) = D(\psi_{\alpha(t,x)})(\bar{u}^+) + b_x(t) \cdot \bar{X}_{\hat{\phi}_t(x)} + D(\psi_{\alpha(t,x)})(\bar{u}^-).$$

So we get $\| D\hat{\phi}_t(\hat{u}) \| \geq \| D(\psi_{\alpha(t,x)})(\bar{u}^+) \| \to +\infty$ if $t \to +\infty$, which is a contradiction. We deduce that $\hat{E}^- \subseteq \bar{E}^{-,0}$. So $\phi(\mathcal{F}^0) \subseteq \bar{\mathcal{F}}^{-,0}$. Then it is easy to see that $\phi(\mathcal{F}^{-,0}) = \bar{\mathcal{F}}^{-,0}$,i.e. ϕ sends C^1 diffeomorphically each leaf of $\mathcal{F}^{-,0}$ onto a leaf of $\bar{\mathcal{F}}^{-,0}$. Similarly we have $\phi(\mathcal{F}^{+,0}) = \bar{\mathcal{F}}^{+,0}$. $\square$

In paticular we deduce from the sublemma above that $\hat{E}^\pm \oplus \mathbb{R}\bar{X} = \bar{E}^\pm \oplus \mathbb{R}\bar{X}$. By projecting parallel to the direction of $\bar{X}$, we get a C^0 section P of $End(\hat{E}^+, \bar{E}^+)$ and two positive constants A_1 and A_2 such that

$$A_1 \| u \| \leq \| P(u) \| \leq A_2 \| u \|, \ \forall \ u \in \hat{E}^+.$$

Then it is easy to verify that for all $x \in N$, $P \circ D_x\hat{\phi}_t = D_x\psi_{\alpha(t,x)} \circ P$.

If ψ_t is quasiconformal with distortion K, then we have the following estimation

$$\hat{K}^+(x,t) = \frac{sup\{\| D\hat{\phi}_t(\hat{u}^+) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| D\hat{\phi}_t(\hat{u}^+) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq \frac{A_2}{A_1} \cdot \frac{sup\{\| P(D\hat{\phi}_t(\hat{u}^+)) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| P(D\hat{\phi}_t(\hat{u}^+)) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq (\frac{A_2}{A_1})^2 \cdot \frac{sup\{\| D_x\psi_{\alpha(t,x)}(\frac{P(\hat{u}^+)}{\|P(\hat{u}^+)\|}) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}{inf\{\| D_x\psi_{\alpha(t,x)}(\frac{P(\hat{u}^+)}{\|P(\hat{u}^+)\|}) \| \mid \| \hat{u}^+ \| = 1, \ \hat{u}^+ \in \hat{E}^+_x\}}$$

$$\leq (\frac{A_2}{A_1})^2 \cdot K.$$

Similarly $\hat{K}^-$ is also bounded. So $\hat{\phi}_t$ is quasiconformal. Since ϕ is a C^1 diffeomorphism, then it is bi-Lipschitz. We deduce that ϕ_t is also quasiconformal. $\square$

It is easy to see that the geodesic flow of a closed real hyperbolic manifold is quasiconformal (even conformal). Then by the previous lemma, each C^∞ time change of such a flow is quasiconformal. An Anosov diffeomorphism is quasiconformal iff its suspension is quasiconformal. So if ϕ denotes a semisimple hyperbolic automorphism of a torus with two enginvalues, then its suspension is a quasiconformal Anosov flow.

It seems to be a common phenomena in mathematics that things can only be effectively studied and understood when placed in a suitable and flexible environment. Conformal structures (Anosov flows) are pretty rigid while quasiconformal structures (Anosov flows) seem to be much more flexible. We wish to better understand the classical conformal Anosov flows, notably the geodesic flows of closed hyperbolic manifolds, by using quasiconformal techniques, which is our motivation to study general qusiconformal Anosov systems.

6.1.2 Main theorems

Among quasiconformal Anosov systems, the quasiconformal geodesic flows were classically studied. In [**Ka2**] and [**Yu**] some elegant rigidity results were obtained via the sphere at infinity. For example, in [**Yu**], C. Yue proved that if the geodesic flow of a negatively curved closed manifold of dimension at least three is quasiconformal, then this Riemannian manifold is of constant curvature. Quite recently in [**Sa**], V. Sadovskaya proved that up to a very special time change, a quasiconformal contact Anosov flow of dimension at least five is C^∞ flow equivalent to the geodesic flow of a hyperbolic manifold. Here we improve all these rigidity results by proving

Theorem 6.1 – *Let ϕ_t be a C^∞ volume-preserving quasiconformal Anosov flow on a closed manifold M. If $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least two, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus or to a canonical time change of the geodesic flow of a hyperbolic manifold.*

Under the conditions of the previous theorem, if in addition the Bowen-Margulis measure of ϕ_t is supposed to be in the Lebesgue measure class, then we prove that up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus or to the geodesic flow of a hyperbolic manifold. By considering the suspensions, we can deduce from Theorem 6.1 the following classification.

Corollary 6.1 – *Let ϕ be a C^∞ volume-preserving quasiconformal Anosov diffeomorphism on a closed manifold Σ. If the dimensions of E^+ and E^- are at least two, then up to finite covers ϕ is C^∞ conjugate to a hyperbolic automorphism of a torus.*

In [**KS**], B. Kalinin and V. Sadovskaya classified the topologically transitive quasiconformal Anosov diffeomorphisms such that the dimensions of $E^\pm$ are at least 3. Their argument, though quite elegant, meets some essential difficulties in the case such that the dimension of E^+ or E^- is 2.

Recently P. Foulon proved an entropy rigidity theorem for three-dimensional contact Anosov flows (see [**Fo**]). Since a three-dimensional Anosov flow is certainly quasiconformal, then by combining his result with Theorem 6.1, we get the following

Corollary 6.2 – *Let ϕ_t be a C^∞ quasiconformal contact Anosov flow. If its Bowen-Margulis measure is in the Lebesgue class, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the geodesic flow of a hyperbolic manifold.*

By extending partially our Theorem 6.1 to the case of codimension one and Anosov-smooth, we get the following corollary generalizing the classification result in [**Gh1**] (see Theorem 5.2).

Corollary 6.3 – *Let ϕ_t be a C^∞ volume-preserving quasiconformal Anosov-smooth flow. Then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus or to a canonical time change of the geodesic flow of a hyperbolic manifold.*

Based on the classification result in [**KS**] and Corollary 6.1, we try to classify the quasiconformal Anosov flows up to C^∞ orbit equivalence. If the strong stable and strong unstable distributions have relatively high dimensions, then we get the following final classification.

Theorem 6.2 – *Let ϕ_t be a C^∞ topologically transitive quasiconformal Anosov flow such that E^+ and E^- are of dimension at least three. Then up to finite covers, ϕ_t is C^∞ orbit equivalent either to the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.*

Under the conditions of the previous theorem, if $E^+ \oplus E^-$ is in addition C^1, then we prove that up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

If one of the dimensions of E^+ and E^- is two, then by using Corollary 6.1, we get the following partial result.

Proposition 6.1 – *Let ϕ_t be a C^∞ volume-preserving quasiconformal Anosov flow such that E^+ is of dimension two and E^- is of dimension at least two. If ϕ_t has the sphere-extension property, then up to finite covers, ϕ_t is C^∞ orbit equivalent either to the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.*

If $E^+ \oplus E^-$ is in addition C^1, then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent either to a canonical time change of the geodesic flow of a three-dimensional hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.

The sphere-extension property will be defined in Section 6.3. Let us just mention that this property is invariant under C^1 orbit equivalence.

Now we are in a position to get some concrete applications of our results above. Recall at first that flow conjugacies between Anosov flows have been and being extensively studied. The philosophical conclusion is that they exist rarely, even C^0 ones. Let us just mention two of the most beautiful supporting results (see also **[L1]** and **[L2]**) :

Theorem 6.3 (U. Hamenstädt, **[Ham2]**) – *Let M be a closed negatively curved manifold. If the geodesic flow of M is C^0 flow equivalent to that of a locally symmetric space of rank one N, then M is isometric to N.*

Theorem 6.4 (R. de la Llave and R. Moriyón, **[DM]**) – *Let ϕ_t and ψ_t be two C^∞ three-dimensional volume-preserving Anosov flows. If they are C^1 flow equivalent, then they are C^∞ flow equivalent.*

On the contrary to the rareness of flow conjugacies, there exist plenty of C^0 orbit conjugacies between Anosov flows. For example, if two C^∞ Anosov flows are sufficiently C^1-near, then they are Hölder-continuous orbit equivalent by the celebrated structural stability (see **[An]**). However we can deduce from Theorem 6.2 and Proposition 6.1 the following result showing that C^1 orbit conjugacies are rare in some cases, while Hölder-continuous orbit conjugacies are abundant.

Theorem 6.5 – *Let ϕ_t be a C^∞ Anosov flow and ψ_t be the geodesic flow of a closed hyperbolic manifold of dimension at least three. If ϕ_t and ψ_t are C^1 orbit equivalent, then they are C^∞ orbit equivalent.*

Let us recall firstly some notions for the next corollary.

Definition 6.1 – Let (X, d_X) and (Y, d_Y) be two metric spaces. Then they are said to be *quasi-isometric* if there is a map $f : X \to Y$ and two positive numbers C and D such that the following two conditions are satisfied:
(1) $\frac{1}{C} d_X(x,y) - D \le d_Y(f(x), f(y)) \le C d_X(x,y) + D$, $\forall\, x, y \in X$.
(2) For any $y \in Y$, there exists $x \in X$ such that $d_Y(y, f(x)) \le D$.

Roughly speaking, two metric spaces are quasi-isometric if and only if they are bi-Lipschitz equivalent in the large scale. Recall that for any $n \geq 2$, a n-dimensional Riemannian manifold M is said to be *hyperbolic* if it has constant sectional curvature -1. Then we denote by $\mathbb{H}^n$ the unique simply connected hyperbolic manifold. By combining some classical results with our previous corollary, we get the following

Proposition 6.2 – *Let M be a n-dimensional closed Riemannian manifold of negative curvature such that $n \geq 3$. Then we have the following relations between dynamics and geometry:*
(1) *The geodesic flow of M is Hölder-continuously orbit equivalent to that of a hyperbolic manifold if and only if the universal covering space $\widetilde{M}$ with its lifted metric is quasi-isometric to $\mathbb{H}^n$.*
(2) *The geodesic flow of M is C^1 orbit equivalent to that of a hyperbolic manifold if and only if M has constant negative curvature.*

6.1.3 The organization of the chapter

In Section 6.2 we prove the rigidity of quasiconformal Anosov flows under the assumption that $E^+ \oplus E^-$ is smooth. More precisely we prove in this section Theorem 6.1 and Corollary 6.1. In Section 6.3 we recall and prove some properties of transverse (G,T)-structures of foliations. Then in Section 6.4 we prove Theorem 6.2 and Proposition 6.1. Finally in Section 6.5 we apply all these results to geodesic flows to deduce Theorem 6.5 and Proposition 6.2.

6.2 Rigidity of quasiconformal Anosov flows

6.2.1 Linearizations and smooth conformal structures

In this subsection we review and adapt some results of **[Sa]** to our situation. The starting point is the following elegant proposition from **[Sa]**, which generalizes a one-dimensional result in **[KL]**.

Proposition 6.3 – *Let f be a diffeomorphism of a compact Riemannian manifold M, and let W be a continuous f-invariant foliation with C^∞ leaves. Suppose that $\| Df \mid_{TW} \| < 1$, and there exist $C > 0$ and $\epsilon > 0$ such that for any $x \in M$ and $n \in \mathbb{N}$,*

$$\| (Df^n \mid_{T_xW})^{-1} \| \cdot \| Df^n \mid_{T_xW} \|^2 \leq C(1-\epsilon)^n.$$

Then for any $x \in M$, there exists a C^∞ diffeomorphism $h_x : W_x \to T_xW$ such that
(1) $h_{fx} \circ f = Df_x \circ h_x$,
(2) $h_x(x) = 0$ *and* $(Dh_x)_x$ *is the identity map,*
(3) h_x *depends continuously on* x *in* C^∞ *topology.*
In addition, the family h of maps h_x satisfying (i), (ii) and (iii) is unique.

Now let ϕ_t be a quasiconformal Anosov flow on a closed manifold M. Fix a Lyapunov metric on M. Then for each $s < 0$, $\| D\phi_s |_{E^+} \| < 1$. Denote by K the distortion of ϕ_t. Then for any $x \in M$ and any $n \in \mathbb{N}$ we have

$$\| (D\phi_s{}^n |_{E_x^+})^{-1} \| \cdot \| D\phi_s{}^n |_{E_x^+} \| = K^+(x, ns) \leq K.$$

So by the Anosov property of ϕ_t, $(\phi_s,\ \mathcal{F}^+)$ satisfies the conditions of the previous proposition. So for any $x \in M$ there exists a C^∞ diffeomorphism $h_x^{+,s} : W_x^+ \to E_x^+$ such that
(1) $h_{\phi_s(x)}^{+,s} \circ \phi_s = D_x\phi_s \circ h_x^{+,s}$,
(2) $h_x^{+,s}(x) = 0$ and $(Dh_x^{+,s})_x$ is the identity map,
(3) $h_x^{+,s}$ depends continuously on x in the C^∞ topology.

For any $m \in \mathbb{N}$, we observe easily that $\{h^{+,\frac{s}{m}}\}_{x \in M}$ satisfies also these three conditions with respect to ϕ_s. Then by the uniqueness of this family of maps for ϕ_s, we get

$$h_x^{+,\frac{s}{m}} = h_x^{+,s},\ \forall\ x \in M.$$

We deduce that for all $a \in \mathbb{Q}$ and $a < 0$, $h_x^{+,a} = h_x^{+,-1}$, $\forall\ x \in M$. Then by Condition (3), we get

$$h_{\phi_t(x)}^{+,-1} \circ \phi_t = (D_x\phi_t) \circ h_x^{+,-1},\ \forall\ x \in M,\ \forall\ t < 0.$$

Denote $h_x^{+,-1}$ by h_x^+. Thus we have

$$h_{\phi_t(x)}^{+} \circ \phi_t = (D_x\phi_t) \circ h_x^{+},\ \forall\ x \in M,\ \forall\ t \in \mathbb{R}.$$

This continuous family of C^∞ maps $\{h_x^+\}_{x \in M}$ is said to be the *unstable linearization* of E^+. Similarly we get the *stable linearization* $\{h_x^-\}_{x \in M}$ of E^-.

By similar arguments, we get the stable and unstable linearizations for quasiconformal Anosov diffeomorphisms. Let us recall the following results established in **[Sa]**:

Theorem 6.5 (**[Sa]**, Theorem 1.3) *Let f be a topologically transitive C^∞ Anosov diffeomorphism (ϕ_t be a topologically mixing C^∞ Anosov flow) on a closed manifold M which is quasiconformal on the unstable distribution. Then it is conformal with respect to a Riemannian metric on this distribution which is continuous on M and C^∞ along the leaves of the unstable foliation.*

Theorem 6.6 (**[Sa]**, Theorem 1.4) *Let f (ϕ_t) be a C^∞ Anosov diffeomorphism (flow) on a closed manifold M with $dimE^+ \geq 2$. Suppose that it is conformal with respect to a Riemannian metric on the unstable distribution which is continuous on M and C^∞ along the leaves of the unstable foliation. Then the (weak) stable holonomy maps are conformal and the (weak) stable distribution is C^∞.*

Let us recall briefly the steps to prove these two theorems in the case of flow. Denote by ϕ_t a topologically mixing quasiconformal Anosov flow. Then by some classical arguments (see **[Su]** and **[Tu]**), V. Sadovskaya found two measurable ϕ_t-invariant conformal structures $\bar{\tau}^+$ and $\bar{\tau}^-$ along respectively the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. Then as is usual for Anosov flows, these two conformal structures were pertubated to continuous ϕ_t-invariant ones denoted by τ^+ and τ^-. Using the linearizations, she proved that along each leaf of $\mathcal{F}^+$, τ^+ is isometric to a vector space with its canonical conformal struture, which has permitted her to blow up the smoothness of weak holonomy maps. Then by using a result of J. L. Journé, she proved the smoothness of the weak stable and unstable distributions.

By using Lemma 3.7, We can deduce from the previous two theorems the following

Lemma 6.2 – *Let ϕ_t be a C^∞ topologically transitive quasiconformal Anosov flow such that the dimensions of E^+ and E^- are at least two. Then $E^{+,0}$ and $E^{-,0}$ are both C^∞. If $E^+ \oplus E^-$ is in addition supposed to be C^∞, then ϕ_t is Anosov-smooth.*

Proof – If ϕ_t is topologically mixing, then by Theorems 6.5 and 6.6, $E^{+,0}$ and $E^{-,0}$ are both C^∞.

If ϕ_t is not topologically mixing, then by Lemma 3.7, $E^+ \oplus E^-$ is a C^∞ integrable distribution with C^∞ compact leaves. Take a leaf Σ of the foliation of $E^+ \oplus E^-$ and $T > 0$ such that $\phi_T(\Sigma) = \Sigma$. Then ϕ_T is a C^∞ topologically transitive quasiconformal Anosov diffeomorphism. Again by Theorems 6.5 and 6.6, the unstable and stable distributions of ϕ_T are seen to be C^∞. We deduce that $E^{+,0}$ and $E^{-,0}$ are also C^∞.

If $E^+ \oplus E^-$ is in addition supposed to be C^∞, then $E^\pm = (E^+ \oplus E^-) \cap E^{\pm,0}$ are certainly C^∞. $\square$

If ϕ_t is a topologically transitive quasiconformal Anosov flow, then by Theorem 6.5 and Lemma 3.7, it preserves two continuous conformal strutures τ^+ and τ^- which are C^∞ along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$. Then by Theorem 6.6, these two conformal structures are invariant under the weak holonomy maps. Thus if ϕ_t is Anosov-smooth, then τ^+ and τ^- are both C^∞. So we can deduce from Lemma 6.2 the following

Lemma 6.3 – *Let ϕ_t be a C^∞ topologically transitive quasiconformal Anosov flow such that $E^+ \oplus E^-$ is C^∞ and the dimensions of E^+ and E^- are at least two. Then ϕ_t preserves two C^∞ conformal structures along $\mathcal{F}^+$ and $\mathcal{F}^-$ denoted respectively by τ^+ and τ^-, which are invariant under weak holonomy maps.*

Let ϕ_t be a topologically transitive Anosov flow such that dim$E^\pm \geq 2$. For any $x \in M$ we can extend the conformal structure τ_x^+ at $0 \in E_x^+$ to all other points of E_x^+ via linear translations. If the resulting translation-invariant conformal structure on E_x^+ is denoted by σ_x^+, then (E_x^+, σ_x^+) is isometric to the canonical conformal structure of $\mathbb{R}^n$ if E^+ is n-dimensional.

By Lemma 3.1 of **[Sa]**, h_x^+ sends $\tau^+ \mid_{W_x^+}$ to σ_x^+. So for any $y \in W_x^+$, $h_x^+ \circ (h_y^+)^{-1}$ is a conformal diffeomorphism from $(\sigma_y^+,\ E_y^+)$ onto $(\sigma_x^+,\ E_x^+)$. Since the dimension of E^+ is at least two, then $h_x^+ \circ (h_y^+)^{-1}$ is an affine map, i.e. the composition of a linear map with a translation.

So if we pull back by h_x^+ and h_y^+ the canonical flat linear connections of E_x^+ and E_y^+ onto W_x^+, we get the same C^∞ connection on W_x^+. Thus by pulling back by the unstable linearization the canonical flat connections on the fibers of E^+, we get a well-defined transversely continuous connection along $\mathcal{F}^+$ denoted by ∇^+ (see **[Ka1]** for some details about connections along a foliation). By Condition (1) of the unstable linearization, ∇^+ is easily seen to be ϕ_t-invariant. Similarly we get a transversely continuous ϕ_t-invariant connection along $\mathcal{F}^-$ denoted by ∇^-.

If the linearizations $\{h_x^\pm\}_{x\in M}$ depend smoothly on x, then ∇^+ and ∇^- are certainly C^∞. But in general $\{h_x^\pm\}_{x\in M}$ depend only continuously on x, even though E^+ and E^- are both smooth. However if the Bowen-Margulis measure of ϕ_t is in the Lebesgue measure class, then these two linearizations can be effectively proved to depend smoothly on their base points, which is the key abservation of the following subsection.

6.2.2 Construction of an invariant connection and homogeneity

Denote by ϕ_t an Anosov flow on a closed manifold M satisfying the conditions of Theorem 6.1. In addition, we suppose throughout this section that its Bowen-Margulis measure is Lebesgue. So if we denote by μ the Bowen-Margulis measure of ϕ_t, then μ is given by a C^∞ nowhere-vanishing volume form on M (see Lemma 3.6). Up to finite covers, We suppose that M and E^+ and E^- are all orientable. Under these assumptions, ϕ_t is Anosov-smooth by Lemma 6.2.

Lemma 6.4 – *Under the notations above, if ϕ_t is in addition topologically mixing, then h_x^+ and h_x^- depend smoothly on x. In particular ∇^+ and ∇^- are C^∞ on M.*

Proof – Suppose that $\dim E^+ = n$ (≥ 2). By Lemmas 6.2 and 6.3, E^+ is C^∞ and ϕ_t preserves a C^∞ conformal structure τ^+ along $\mathcal{F}^+$.

Fix a C^∞ Riemannian metric on M. Denote by ν^+ the induced Riemannian volume form along the leaves of $\mathcal{F}^+$ and by μ^+ the Margulis measure of ϕ_t supported by the leaves of $\mathcal{F}^+$. Then by the assumptions and Lemma 3.8, there exist on M a C^∞ positive function f^+ such that $\mu^+ = f^+\nu^+$, i.e. μ^+ is given by a family of C^∞ nowhere-vanishing volume forms along the leaves of $\mathcal{F}^+$.

The volume of a frame of E^+ is by definition the evaluation of μ^+ on this frame. Then by claiming the τ^+-conformal frames of volume one to be orthonormal, we get a well-defined C^∞ Riemannian metric along the leaves of $\mathcal{F}^+$, which is denoted by g^+.

Denote by $\bar{\nabla}^+$ the leafwise Levi-Civita connections of g^+. Since τ^+ is ϕ_t-invariant and $\mu^+ \circ \phi_t = e^{ht}\mu^+$, then

$$\phi_t^* g^+ = e^{\frac{2h}{n}t} g^+, \ \forall\ t \in \mathbb{R}.$$

Thus $\bar{\nabla}^+$ is ϕ_t-invariant. For any $x \in M$, we define $\bar{h}_x^+ : E_x^+ \to W_x^+$ such that $\bar{h}_x^+(u) = exp^{\bar{\nabla}^+}(u)$. Because of the ϕ_t-invariance of $\bar{\nabla}^+$, we get

$$\bar{h}^+_{\phi_t(x)} \circ D\phi_t = \phi_t \circ \bar{h}_x^+, \ \forall\ x \in M, \ \forall\ t \in \mathbb{R}.$$

Evidently, we have $\bar{h}_x^+(0) = x$ and $D_x(\bar{h}_x^+) = Id$ and that $\bar{h}_x^+$ depends smoothly on x.

Since g^+ is certainly complete along each leaf of $\mathcal{F}^+$, then $\bar{h}_x^+$ is surjective. Fix a Riemannian metric on M. Then by the compactness of M, there exists $\epsilon > 0$ such that for any $x \in M$, $\bar{h}_x^+ \mid_{\{u \in E_x^+ \ \mid\ \|u\| < \epsilon\}}$ is a C^∞ diffeomorphism onto its image. If $t \gg 1$, then ϕ_{-t} contracts E^+ exponentially. Since we have in addition

$$\bar{h}_x^+ = \phi_t \circ \bar{h}^+_{\phi_{-t}(x)} \circ D\phi_{-t}, \ \forall\ t > 0,$$

then $\bar{h}_x^+$ is in fact injective and nowhere singular. We deduce that for any $x \in M$, $\bar{h}_x^+$ is a C^∞ diffeomorphism. By the uniqueness of the unstable linearization (see Proposition 6.3), we get

$$h_x^+ = (\bar{h}_x^+)^{-1}, \ \forall\ x \in M.$$

We deduce that h_x^+ depends smoothly on x. Similarly we get the C^∞ dependence of h_x^- on x. Thus the leafwise connections ∇^+ and ∇^- constructed in the previous subsection are C^∞ on M. □

Lemma 6.5 – *Under the notations above, if ϕ_t is not topologically mixing, then Theorem* 6.1 *is true.*

Proof – Since ϕ_t is not topologically mixing, then by Lemma 3.7, $E^+ \oplus E^-$ is integrable with C^∞ compact leaves. Take a leaf Σ of the foliation of $E^+ \oplus E^-$ and $T > 0$ such that $\phi_T(\Sigma) = \Sigma$. Then ϕ_T is a C^∞ topologically transitive quasiconformal Anosov diffeomorphism of Σ. In addition by Lemma 6.2, the stable and unstable distributions of ϕ_T are both C^∞. Since the Bowen-Margulis measure of ϕ_t is Lebesgue, then the Bowen-Margulis measure of ϕ_T is also lebesgue.

After some evident modifications, Lemma 6.4 is also valid for (ϕ_T, Σ) (see Lemma 3.9). So we get as in the case of flow two C^∞ ϕ_T-invariant connections along $\mathcal{F}^\pm_\Sigma$ denoted by $\nabla^\pm_\Sigma$ (see Subsection 6.2.1). Then we can construct on Σ a C^∞ ϕ_T-invariant connection ∇ such that for arbitrary C^∞ sections $Y^\pm$ and $Z^\pm$ of $E^\pm_\Sigma$,

$$\nabla_{Y^\pm} Y^\mp = P^\mp[Y^\pm, \ Y^\mp], \ \nabla_{Y^\pm} Z^\pm = (\nabla^\pm_\Sigma)_{Y^\pm} Z^\pm,$$

where $P^\pm_\Sigma$ denote the projections of $T\Sigma$ onto $E^\pm_\Sigma$.

Then by **[BL]**, ϕ_T is C^∞-conjugate to a hyperbolic infranilautomorphism. Since ϕ_T is in addition quasiconformal, then ϕ_T must be finitely covered by a hyperbolic automorphism of a torus. So Theorem 6.1 is true in this case. □

Throughout the following of this subsection, we suppose that ϕ_t is topologically mixing. Since ϕ_t is quasiconformal, then it has exactly three Lyapunov exponents with respect to μ denoted by $\{a^-, 0, a^+\}$ such that $a^- < 0 < a^+$. Since ϕ_t is Anosov-smooth by Lemma 6.2, then its Lyapunov decomposition is C^∞. In addition, the smooth Lyapunov decomposition of ϕ_t coincides its Anosov decomposition. By using Lemma 6.4, we can construct a C^∞ connection ∇ on M such that

$$\nabla X = 0, \ \nabla E^\pm \subseteq E^\pm,$$

$$\nabla_{Y^\pm} Z^\mp = P^\mp [Y^\pm, \ Z^\mp], \ \nabla_{Y^\pm} Z^\pm = (\nabla^\pm)_{Y^\pm} Z^\pm,$$
$$\nabla_X Y^\pm = [X, \ Y^\pm] + a^\pm Y^\pm,$$

where $P^\pm$ denote the projections of TM onto $E^\pm$ with respect to the Anosov splitting. Then it is easily seen that ∇ is ϕ_t-invariant.

Suppose that $\dim E^+ = n$ and $\dim E^- = m$. Then by Pesin's entropy formula and Lemma 3.1 we get

$$h = n \cdot a^+ \ \text{ and } \ n \cdot a^+ + m \cdot a^- = 0.$$

We deduce that ∇ is canonical (see Section 4.4). Then by the assumptions above and Proposition 4.2, the line bundles $\wedge^n E^+$ and $\wedge^m E^-$ admit both C^∞ nowhere-vanishing ∇-parallel sections denoted respectively by ω^+ and ω^-.

Define $\tau = (X, \ E^\pm, \ \tau^\pm)$. Then τ is a C^∞ ϕ_t-invariant geometric structure of order one on M.

Lemma 6.6 – *Under the notations above, τ is ∇-parallel.*

Proof – τ is in a natural sense the sum of the geometric structures X and $(E^+, \ \tau^+)$ and $(E^-, \ \tau^-)$. Then τ is ∇-parallel iff these structures are parallel respectively. Since $\nabla X = 0$, then X is ∇-parallel.

Let us consider the structure $(E^+, \ \tau^+)$ denoted by σ^+ to simplify the notations. It is easily seen that σ^+ is ∇-parallel iff $\nabla E^+ \subseteq E^+$ and the τ^+-conformal frames are preserved by the parallel transport of E^+ along each piecewise smooth curve of M (see **[KN]**).

By the definition of ∇, the parallel transport of E^+ along the orbits of ϕ_t is given by

$$u^+ \to e^{-a^+ t} \cdot D\phi_t(u^+).$$

Since τ^+ is ϕ_t-invariant, then the τ^+-conformal frames are preserved by the ∇-parallel transport along the orbits of ϕ_t, i.e. $\sigma^+ \circ \phi_t$ is horizontal. So $D\sigma^+(X) \subseteq \mathcal{H}$, where $\mathcal{H}$ denotes the ∇-horizontal distribution of the corresponding fiber bundle.

Take a smooth curve γ tangent to E^+. The restriction of ∇ to the leaves of $\mathcal{F}^+$ is ∇^+. On the leaf containing γ, ∇^+ is isomorphic to the canonical flat connection of a vector space. So the τ^+-conformal frames are certainly preserved by the parallel transport along γ, i.e. $\sigma^+ \circ \gamma$ is horizontal. Thus $D\sigma^+(E^+) \subseteq \mathcal{H}$.

Take another smooth curve γ tangent to E^-. Then by the definition of ∇, the parallel transport along γ is given by the differentials of the weak stable holonomy maps. Since τ^+ is invariant with respect to these maps (see Lemma 6.3), then we deduce that $D\sigma^+(E^-) \subseteq \mathcal{H}$.

So we get $D\sigma^+(TM) = D\sigma^+(E^+ \oplus E^- \oplus \mathbb{R}X) \subseteq \mathcal{H}$, i.e. $(E^+,\ \tau^+)$ is ∇-parallel. Similarly $(E^-,\ \tau^-)$ is seen to be also parallel. we deduce that τ is ∇-parallel. $\square$

Recall that the Bowen-Margulis measure of ϕ_t is supposed to be in the class of Lebesgue and ∇ is canonical, then by Proposition 4.2, $\wedge^n E^+$ and $\wedge^m E^-$ admit both C^∞ nowhere-vanishing ∇-parallel sections denoted respectively by ω^+ and ω^-.

We define $\sigma = (\tau,\ \omega^+,\ \omega^-)$. Then σ is a C^∞ ∇-parallel geometric structure of order one on M. Denote by $\widetilde{M}$ the universal covering space of M and by $\tilde{\sigma}$ and $\tilde{\nabla}$ the lifts of σ and ∇.

By similar argumments as in Subsection 5.2.2, we get $\nabla T = 0$ and $\nabla R = 0$ and the completeness of ∇, where T and R denote respectively the torsion tensor and curvature tensor of ∇. So ∇ is a C^∞ ϕ_t-invariant complete parallel linear connection. Thus by Lemma 2.2, $I(\widetilde{\nabla}, \widetilde{\sigma})$ acts transitively on $\widetilde{M}$.

Denote by G this isometry group. Fix a point $x \in \widetilde{M}$ and denote by H the isotropy subgroup of x. Denote by Γ the fundamental group of M. Then it is contained in G as a discrete subgroup and we have $M \cong \Gamma \diagdown G \diagup H$.

By claiming the τ^+-conformal frames of ω^+-volume one to be orthonormal, we can construct as in Lemma 6.4 a C^∞ fiber metric on E^+ denoted again by g^+. Similarly we can construct a C^∞ fiber metric g^- on E^-. Then we get a C^∞ Riemannian metric g on M such that

$$g = \lambda^2 \oplus g^+ \oplus g^-,$$

where λ denotes the canonical 1-form of ϕ_t. By the definition of σ, each element of G preserves $\tilde{g}$. So G is contained as a closed subgroup in $I(\tilde{g})$. We deduce that H is a compact Lie subgroup of G (see **[Be]** ch.I, 1.78). So ϕ_t turns out to be a symmetric Anosov flow (see Subsection 3.4.1).

By the rough classification of symmetric Anosov flows, i.e. Proposition 3.6, we get the following

Lemma 6.7 – *Suppose that ψ_t is a quasiconformal symmetric Anosov flow on a m-dimensional manifold. Then we must have rank$(\psi_t) = 0$ or $2[\frac{m}{2}]$.*

Proof – Suppose on the contrary that $0 <$rank$(\psi_t) < 2[\frac{m}{2}]$. Then under the notations of Proposition 3.6, we have that up to commensurability, a lift of ψ_t is given by

$$G/K \xrightarrow{\bar{\psi}_t} G/K$$

$$gK \to (g \cdot expt\alpha)K,$$

such that $\mathcal{E}^0 = Ker(ad\alpha) = \mathfrak{g}_K \oplus \mathbb{R}\alpha$.

Since $\alpha \in \mathfrak{p}$ (see Proposition 3.6) and $\mathfrak{so}(n,1)$ is of rank one, then there exists $a > 0$ and $X_\pm \in \mathfrak{so}(n,1)$ such that

$$[a \cdot \alpha,\ X_\pm] = \pm 2X_\pm,\ [X_+,\ X_-] = -a \cdot \alpha.$$

Denote $a \cdot \alpha$ again by α, i.e. considering the flow given by $a \cdot \alpha$. Denote by $\mathfrak{g}_\alpha$ the Lie subalgebra generated by $\{\alpha,\ X_+,\ X_-\}$. Then we have

$$\mathfrak{g}_\alpha \cong \mathfrak{sl}(2, \mathbb{R}).$$

Recall that $G = V \rtimes (Spin(n,1) \times K_1 \times \cdots \times K_p)$, where V is a vector group of positive dimension. By identifying V with its Lie algebra, we get from this semidirect product a linear representation of $\mathfrak{so}(n,1)$ on V. The restriction onto $\mathfrak{g}_\alpha$ of this representation gives a $\mathfrak{sl}(2,\mathbb{R})$-module (see **[Bo1]** 3).

Now by Proposition two of (**[Bo2]** ch.VIII, 1.2), there exists a non-zero vector e in V and $m \in \mathbb{Z}^+ \cup \{0\}$ such that

$$\alpha(e) = m \cdot e.$$

Since $Ker(ad\alpha) = \mathfrak{g}_K \oplus \mathbb{R}\alpha$, then $Ker(ad\alpha) \cap V = \{0\}$. We deduce that $m \neq 0$.

By Remark 3.2 (see **[To1]**), it is easily seen that the Lyapunov exponents of ψ_t are exactly $\Re(spec(ad\alpha))$, i.e. the real part of the spectrum of $ad\alpha$. Since ψ_t is quasiconformal, then $\Re(spec(ad\alpha))$ has only three elements. Since X_+ is taken such that $[\alpha,\ X_+] = 2X_+$, then we must have $m = 2$.

Define $e_1 = -X_-(e)$. Then by Propositions one and two of (**[Bo2]** ch.VIII, 1.2), we get

$$e_1 \neq 0, \quad \alpha(e_1) = 0,$$

which contradicts to $Ker(ad\alpha) \cap V = \{0\}$. □

We deduce by the previous lemma that rank(ϕ_t) is maximal if ϕ_t is topologically mixing. Then again by Proposition 3.6, ϕ_t is commensurable to the geodesic flow of a locally symmetric space of rank one. Since dim$E^\pm \geq 2$, then ϕ_t is in fact finitely covered by the geodesic flow of a locally symmetric space of rank one. Since in addition ϕ_t is quasiconformal, then this locally symmetric space must have constant negative curvature. So by combining the results above, we get the following

Lemma 6.8 – *Let ϕ_t be a C^∞ quasiconformal Anosov flow such that $E^+ \oplus E^-$ is C^∞ and dim$E^\pm \geq 2$. If its Bowen-Margulis measure is in the Lebesgue measure class, then up to finite covers and a constant change of time scale, ϕ_t is C^∞ flow equivalent either to the geodesic flow of a closed hyperbolic manifold or to the suspension of a hyperbolic automorphism of a torus.*

Remark 6.1 – *By* **[Bow]**, *the Bowen-Margulis measure of each symmetric Anosov flow is in the Lebesgue measure class.*

6.2.3 Proofs of Theorem 6.1 and other rigidity results

Proof of Theorem 6.1 – Suppose that ϕ_t satisfies the conditions of Theorem 6.1. Then by Lemma 6.2, ϕ_t is Anosov-smooth. Denote by $\hat{\phi}_t$ its Parry time change. Then $\hat{\phi}_t$ satisfies the conditions of Lemma 6.8. So it is C^∞ flow equivalent either to the suspension of a hyperbolic automorphism of a torus or to the geodesic flow of a closed hyperbolic manifold. Now Theorem 6.1 is a direct consequence of Lemmas 4.2 and 4.5 concerning the special time changes of such flows. □

Proof of Corollary 6.1 – Suppose that ϕ satisfies the conditions of Corollary 6.1 and denote by ϕ_t the suspension of ϕ. Thus the distribution $E^+ \oplus E^-$ of ϕ_t is C^∞. So ϕ_t satisfies the conditions of Theorem 6.1. Since ϕ_t admits a global section, then it can not be the time change of a geodesic flow. So by Theorem 6.1, up to a constant change of time scale, ϕ_t is finitely covered by the suspension of a hyperbolic automorphism of a torus. We deduce that ϕ is finitely covered by a hyperbolic automorphism of a torus. □

Corollary 6.2 is direct conclusion of Theorem 6.1 and Theorem one of **[Fo]**. In order to prove Corollary 6.3, we need only prove the following

Lemma 6.9 – *Let ϕ_t be a C^∞ volume-preserving quasiconformal Anosov-smooth flow such that $dimE^+ = 1$ and $dimE^- \geq 2$. Then up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic automorphism of a torus.*

Proof – Denote by ϕ_t^Y the Parry time change of ϕ_t. Since E_Y^+ and E_Y^- are also C^∞, then by **[Gh2]** $E_Y^+ \oplus E_Y^-$ is integrable with smooth compact leaves. Fix a leaf Σ of the foliation of $E_Y^+ \oplus E_Y^-$ and $T > 0$ such that $\phi_T^Y \Sigma = \Sigma$. Then the Bowen-Margulis measure of ϕ_T^Y is in the Lebesgue measure class. Thus by Lemma 3.9, the Margulis measures $\mu^\pm$ of ϕ_T^Y are given by C^∞ nowhere-vanishing volume forms along the leaves of $\mathcal{F}_\Sigma^\pm$.

Since $\dim E_Y^- \geq 2$, then we can prove as in Lemma 6.4 that the stable linearization $\{h_x^-\}$ (well-defined) depends smoothly on x. So we get as in Subsection 6.2.1 a C^∞ ϕ_T^Y-invariant connection along the leaves of $\mathcal{F}_\Sigma^-$, which is denoted by ∇_Σ^-.

Denote by Y^+ the smooth section of E_Y^+ such that $\mu^+(Y^+) \equiv 1$. Denote by h the topological entropy of ϕ_T^Y. Since $\mu^+ \circ \phi_T^Y = e^h \mu^+$, then

$$(\phi_T^Y)_* Y^+ = e^h Y^+.$$

Thus we get a C^∞ ϕ_T^Y-invariant connection ∇_Σ^+ along the leaves of $\mathcal{F}_\Sigma^+$ such that

$$(\nabla_\Sigma^+)_{Y^+} Y^+ = 0.$$

So ϕ_T^Y preserves a C^∞ connection as in the proof of Lemma 6.5. Thus by **[BL]** ϕ_T^Y is finitely covered by a hyperbolic automorphism of a torus. Since ϕ_t is a special time change of ϕ_t^Y, then we conclude by Lemma 4.5. □

Now Corollary 6.3 is just a combination of the previous lemma and Theorem 6.1 and Theorem 5.2.

6.3 Geometric preparations

In this section we lie down the geometric basis for the proof of Theorem 6.2 and Proposition 6.1.

6.3.1 Transverse (G, T)-structures

In this subsection, we consider the transverse (G, T)-structures of foliations. Let $\mathcal{F}$ be a C^∞ foliation on a connected manifold M. Denote by $Q_{\mathcal{F}}$ the leaf space of $\mathcal{F}$ and by $\mathcal{F}(A)$ the saturation of $\mathcal{F}$ on A for any $A \subseteq M$. We assume that the holonomy maps of $\mathcal{F}$ are defined on connected transverse sections.

Let G be a real Lie group acting effectively and transitively on a connected manifold T. If Σ is a C^∞ transverse section of $\mathcal{F}$ and ϕ is a C^∞ diffeomorphism of Σ onto its open image in T, then (Σ, ϕ) is said to be a *transverse T-chart*. Two transverse T-charts (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are said to be *compatible* if for each holonomy map h of a germ of Σ_1 to a germ of Σ_2, the map $\phi_1 \circ h \circ \phi_2^{-1}$ is locally the restriction of elements of G.

A family of transverse sections is said to be *covering* if each leaf of $\mathcal{F}$ intersects at least one of the sections in this family. By definition, a *transverse (G, T)-structure* on $\mathcal{F}$ is a maximal family of compatible transverse T-charts of which the underlying family of transverse sections is covering.

In order to define a transverse (G, T)-structure, we need just seperate out a family of covering compatible T-charts. Then by considering all the T-charts compatible with this family, we get automatically a transverse (G, T)-structure.

Denote by $\widetilde{\mathcal{F}}$ the lifted foliation on the universal covering space $\widetilde{M}$ of M and denote by π the projection of $\widetilde{M}$ onto M. For each transverse (G, T)-structure on $\mathcal{F}$, we get naturally a lifted transverse (G, T)-structure on $\widetilde{\mathcal{F}}$ by considering the composition of π with the T-charts of the given transverse (G, T)-structure on $\mathcal{F}$. Then by **[Go]** there exists a C^∞ submersion $\mathcal{D} : \widetilde{M} \to T$ and a group homomorphism $\mathcal{H} : \pi_1(M) \to G$ satisfying the following two conditions:
(1) $\mathcal{D}(\gamma x) = \mathcal{H}(\gamma)\mathcal{D}(x)$, $\forall\ x \in \widetilde{M}$, $\forall\ \gamma \in \pi_1(M)$.
(2) The lifted foliation $\widetilde{\mathcal{F}}$ is defined by $\mathcal{D}$.
This submersion $\mathcal{D}$ is said to be the *developing map* of the transverse (G, T)-structure of $\mathcal{F}$ and $\mathcal{H}$ is said to be the *holonomy representation* of $\mathcal{D}$. The transverse (G, T)-structure of $\mathcal{F}$ is said to be *complete* if $\mathcal{D}$ is a C^∞ fibre bundle over $\mathcal{D}(\widetilde{M})$.

If $\mathcal{D}'$ denotes another developing map with holonomy representation $\mathcal{H}'$, then by **[Go]** there exists a unique element $g \in G$ such that $\mathcal{D}' = g \circ \mathcal{D}$ and $\mathcal{H}' = g \cdot \mathcal{H} \cdot g^{-1}$.

Since $\mathcal{D}$ is obtained by analytic continuation along curves (see **[Go]**), then for each transverse section Σ of $\widetilde{\mathcal{F}}$ such that $\mathcal{D}\mid_\Sigma$ is a C^∞ diffeomorphism onto its image, $\mathcal{D}\mid_\Sigma$ is a transverse T-chart of the lifted transverse (G,T)-structure of $\widetilde{\mathcal{F}}$.

Since $\widetilde{\mathcal{F}}$ is defined by the submersion $\mathcal{D}$, then $\mathcal{D}$ sends each leaf of $\widetilde{F}$ to a point of T. Thus for any $x \in \widetilde{M}$ there exists a small C^∞ transverse section Σ containing x of $\widetilde{\mathcal{F}}$ such that each leaf of $\widetilde{\mathcal{F}}$ intersects Σ at most once, i.e. $\mathcal{F}$ has the section property (see Subsection 3.2.2). Then it is easily seen that each leaf of $\widetilde{\mathcal{F}}$ is closed in $\widetilde{M}$.

Denote by $Q_{\widetilde{\mathcal{F}}}$ the leaf space of $\widetilde{\mathcal{F}}$. Then we have the quotient map $\bar{\mathcal{D}} : Q_{\widetilde{\mathcal{F}}} \to \mathcal{D}(\widetilde{M})$. Since each leaf of $\widetilde{\mathcal{F}}$ is closed, then $\bar{\mathcal{D}}$ is bijective iff the $\mathcal{D}$-inverse image of each point of T is connected. If this is the case, then by considering the projections of small transverse T-charts, $Q_{\widetilde{\mathcal{F}}}$ becomes naturally a C^∞ (seperable) manifold such that $\bar{\mathcal{D}}$ is a C^∞ diffeomorphism of $Q_{\widetilde{\mathcal{F}}}$ onto $\mathcal{D}(\widetilde{M})$. In addition, the fundamental group $\pi_1(M)$ of M acts naturally on $Q_{\widetilde{\mathcal{F}}}$.

By **[Hae]** we have the following

Proposition 6.4 – *Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two C^∞ foliations with complete transverse (G,T)-structures. Suppose that their developing maps have both connected fibres and the holonomy covers of their leaves are all contractible. If the $\pi_1(M_1)$-action on $Q_{\widetilde{\mathcal{F}_1}}$ is C^∞ conjugate to the $\pi_1(M_2)$-action on $Q_{\widetilde{\mathcal{F}_2}}$, then there exists a C^∞ map $f : M_1 \to M_2$ such that the following conditions are satisfied:*
(1) f is a surjective homotopy equivalence.
(2) f sends each leaf of $\mathcal{F}_1$ onto a leaf of $\mathcal{F}_2$ and f sends different leaves to different leaves.
(3) f is transversally a local C^∞ diffeomorphism conjugating the two transverse (G,T)-structures.

The lemma below is self-evident and will be used several times in the following.

Lemma 6.10 – *Let $(M, \mathcal{F})$ be a C^∞ foliation with a transverse (G,T)-structure. If T_1 is an open subset of T and G_1 is a closed Lie subgroup of G acting transitively on T_1 such that $\mathcal{D}(\widetilde{M}) \subseteq T_1$ and $\mathcal{H}(\pi_1(M)) \subseteq G_1$, then $\mathcal{F}$ admits a transverse (G_1, T_1)-structure with the same developing map $\mathcal{D}$ and the same holonomy representation $\mathcal{H}$, which is compatible with the initial transverse (G,T)-structure.*

By adapting the arguments in **[Gh3]** we can prove the following

Lemma 6.11 – *Let $(M, \mathcal{F})$ be a C^∞ foliation with a transverse (G,T)-structure. If the closed leaves of $\mathcal{F}$ are dense in M and the $\mathcal{D}$-inverse image of each point of T is connected, then $\mathcal{H}(\pi_1(M))$ is a discrete subgroup of G.*

Proof – Denote by π_1 the projection of $\widetilde{M}$ onto $Q_{\widetilde{\mathcal{F}}}$ and by π_2 the projection of $Q_{\widetilde{\mathcal{F}}}$ onto $Q_{\mathcal{F}}$ the leaf space of $\mathcal{F}$. Take a closed leaf F_x of $\mathcal{F}$ and $\widetilde{x} \in \widetilde{M}$ such that

$\pi(\widetilde{x}) = x$. Since F_x is closed, then we can find a fine transverse section Σ passing through $\widetilde{x}$ such that π sends Σ diffeomorphically onto its image and $F_x \cap \pi(\Sigma) = \{x\}$. So for each $y \in \Sigma$ and $y \neq \widetilde{x}$, $\pi(y)$ is not in F_x. Thus $\pi_2^{-1}(F_x)$ is discrete in $Q_{\widetilde{\mathcal{F}}}$.

Since F_x is closed and $\pi^{-1}(F_x) = \pi_1^{-1}(\pi_2^{-1}(F_x))$, then $\pi_2^{-1}(F_x)$ is closed in $\widetilde{Q}_{\mathcal{F}}$. So the $\pi_1(M)$-orbit of $\widetilde{F}_{\widetilde{x}}$, i.e. $\pi_2^{-1}(F_x)$ is closed and discrete in $Q_{\widetilde{\mathcal{F}}}$.

Since the closed leaves of $\mathcal{F}$ are dense in M i.e. the union of all the closed leaves is dense, then π_2-inverse images of these closed leaves form a dense subset P of $Q_{\widetilde{\mathcal{F}}}$ such that the $\pi_1(M)$-orbit of each point of P is closed and discrete.

Suppose on the contrary that $\mathcal{H}(\pi_1(M))$ is not discrete in G. Since the $\mathcal{D}$-inverse image of each point of T is connected, then $\mathcal{D}$ induces a C^∞ diffeomorphism $\bar{\mathcal{D}} : Q_{\widetilde{\mathcal{F}}} \to \mathcal{D}(\widetilde{M})$. So $\bar{\mathcal{D}}(P)$ is dense in $\mathcal{D}(\widetilde{M})$ and the $\mathcal{H}(\pi_1(M))$-orbit of each point of $\bar{\mathcal{D}}(P)$ is discrete and closed in $\mathcal{D}(\widetilde{M})$.

Take a non-trivial one-parameter subgroup g_t of the closure of $\mathcal{H}(\pi_1(M))$ in G. For each $t \in \mathbb{R}$, g_t preserves the closed complement of $\mathcal{D}(\widetilde{M})$. So we have

$$g_t(\mathcal{D}(\widetilde{M})) = \mathcal{D}(\widetilde{M}).$$

Thus g_t fixes each point in $\bar{\mathcal{D}}(P)$. We deduce that g_t is a trivial one-parameter subgroup, which is a contradiction. □

6.3.2 Sphere-extension property

Let M be a C^∞ manifold. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous foliations with C^1 leaves on M such that

$$T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM.$$

If $\mathcal{F}_1$ is a foliation by planes, i.e. each leaf is C^1 diffeomorphic to a certain $\mathbb{R}^n$, then $(\mathcal{F}_1, \mathcal{F}_2)$ is said to be a *plane foliation couple*. The local leaves of $\mathcal{F}_1$ are natural transverse sections of $\mathcal{F}_2$ and we consider only the holonomy maps of $\mathcal{F}_2$ with respect to these special transverse sections.

For each leaf $F_{1,x}$ of $\mathcal{F}_1$ we denote by $S_{1,x}$ its one-point compactification which is homeomorphic to a standard sphere. The point at infinity of $S_{1,x}$ is denoted by ∞.

Definition 6.2 – Under the notations above, the plane foliation couple $(\mathcal{F}_1, \mathcal{F}_2)$ is said to have the *sphere-extension* property if for each holonomy map θ of $\mathcal{F}_2$ sending x to y there exists a homeomorphism $\Theta : S_{1,x} \to S_{1,y}$ which coincides locally with $\mathcal{F}_2$-holonomy maps on $S_{1,x} \setminus \{\infty, \Theta^{-1}(\infty)\}$ and extends the germ of θ at x.

If $\bar{\phi}_t$ is a lifted flow of an C^∞ Anosov flow ϕ_t, then $\bar{\phi}_t$ is said to have the *sphere-extension* property if $(\bar{\mathcal{F}}^+, \bar{\mathcal{F}}^{-,0})$ and $(\bar{\mathcal{F}}^-, \bar{\mathcal{F}}^{+,0})$ have both the sphere-extension property.

Recall that $\bar{\mathcal{F}}^+$ and $\bar{\mathcal{F}}^-$ are both foliations by planes. The corresponding notion for Anosov diffeomorphisms is defined similarly.

Denote by $(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ the lifted couple on $\widetilde{M}$ of $(\mathcal{F}_1, \mathcal{F}_2)$. Then it is easily seen that $(\mathcal{F}_1, \mathcal{F}_2)$ has the sphere-extension property if $(\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ has this property. So by considering the lifted flows and drawing pictures, we can easily see that the geodesic flows of closed negatively curved manifolds have the sphere-extension property.

It is easily verified that hyperbolic infra-nilautomorphisms have the sphere-extension property. However by **[Man]** each Anosov diffeomorphism defined on a infra-nilmanifold is topologically conjugate to a hyperbolic infra-nilautomorphism. We deduce that the suspensions of Anosov diffeomorphisms on infra-nilmanifolds have the sphere-extension property.

Lemma 6.12 – *Let ϕ_t and ψ_t be two C^1 orbit equivalent C^∞ Anosov flows. Suppose that the strong stable and strong unstable distributions of ψ_t are of dimension at least two. If ψ_t has the sphere-extension property, then ϕ_t has also this property.*

Proof – Denote by ϕ the C^1 orbit equivalence. Define $\hat{\phi}_t = \phi \circ \phi_t \circ \phi^{-1}$ and denote by $\bar{E}^{\pm}$ the strong distributions of ψ_t. Define $\hat{\mathcal{F}}^{\pm} = \phi(\mathcal{F}^{\pm})$ and $\hat{\mathcal{F}}^{\pm,0} = \phi(\mathcal{F}^{\pm,0})$. It is clear that ϕ_t has the sphere-extension property iff $\hat{\phi}_t$ has this property in the natural sense.

Since ϕ_t and ψ_t are C^1 orbit equivalent, then $\hat{\mathcal{F}}^{\pm,0} = \bar{\mathcal{F}}^{\pm,0}$. Take a leaf of $\bar{\mathcal{F}}^{+}$ and take a non-periodic point x in this leaf. We can identify $\hat{W}^+_x$ and $\bar{W}^+_x$ naturally as following.

For all $y \in \hat{W}^+_x$, there exists $t \in \mathbb{R}$ such that $\psi_t(y) \in \bar{W}^+_x$. If $\hat{W}^{+,0}_x$ contains no periodic orbit, this number t is unique for each y in $\hat{W}^+_x$. If $\hat{W}^{+,0}_x$ contains a periodic orbit, then it contains exactly one periodic orbit (see Proposition 3.1). Denote by T its mininal positive period with respect to ψ_t. So if $\psi_t(y) \in W^+_x$, then for all $k \in \mathbb{Z}$, $\psi_{t+k\cdot T}(y) \in \bar{W}^+_x$.

Conversely if $\psi_{t_1}(y) \in \bar{W}^+_x$ and $\psi_{t_2}(y) \in \bar{W}^+_x$, then $\psi_{t_2-t_1}\bar{W}^+_x = \bar{W}^+_x$. Thus $t_2 - t_1 \in T\cdot\mathbb{Z}$. So by associating $t + T\cdot\mathbb{Z}$ to y, we get a well-defined C^∞ map from $\hat{W}^+_x$ to $\mathbb{R}/T\mathbb{Z}$.

Thus by taking a lift if necessary, there exists a unique C^1 map $\theta_x : \hat{W}^+_x \to \mathbb{R}$ such that

$$\theta_x(x) = 0,\ \psi_{\theta_x(y)}(y) \in \bar{W}^+_x,\ \forall\ y \in \hat{W}^+_x.$$

Define a C^1 map $\hat{\eta}_x : \hat{W}^+_x \to \bar{W}^+_x$ such that

$$\hat{\eta}_x(y) = \psi_{\theta_x(y)}(y).$$

Then $\hat{\eta}_x$ is easily seen to be a local C^1 diffeomorphism such that $\hat{\eta}(x) = x$. Similar we get $\bar{\eta} : \bar{W}^+_x \to \hat{W}^+_x$ such that $\bar{\eta}(x) = x$. If $\hat{W}^{+,0}_x$ contains no ψ_t-periodic orbit, then $\hat{\eta}$ and $\bar{\eta}$ are both C^1 diffeomorphisms.

Suppose that $\bar{W}^{+,0}_x$ contains a unique ψ_t-periodic orbit of period T. Denote by z the unique intersection point of this periodic orbit with $\bar{W}^+_x$. For each $k \in \mathbb{Z}$, we define

$$\Lambda_k = \{y \in \bar{W}^+_x \setminus \{z\} \mid \hat{\eta} \circ \bar{\eta}(y) = \psi_{kT}(y)\}.$$

Then $\bar{W}^+_x \setminus \{z\}$ is the disjoint union of $\{\Lambda_k\}_{k\in\mathbb{Z}}$. Each Λ_k is closed in $\bar{W}^+_x \setminus \{z\}$ and $x \in \Lambda_0$. Take $y \in \Lambda_0$ and since ψ_{-T} is a contracting diffeomorphism of $\bar{W}^+_x$, then a small ball containing y intersects with at most finitely many Λ_l non-trivially. We deduce that Λ_0 is open.

Since $\bar{E}^+$ is at least two-dimensional, then $\bar{W}_x^+ \setminus \{z\}$ is connected . We deduce that $\Lambda_0 = \bar{W}_x^+ \setminus \{z\}$, i.e. $\hat{\eta} \circ \bar{\eta} = Id$. Similarly we have $\bar{\eta} \circ \hat{\eta} = Id$. We identify $\hat{W}_x^+$ and $\bar{W}_x^+$ under these two sliding C^1 diffeomorphisms $\bar{\eta}$ and $\hat{\eta}$. We can identify $\hat{W}_x^-$ and $\bar{W}_x^-$ similarly.

Since these identifications conjugate the holonomy maps and ψ_t has the sphere extension property, then $\hat{\phi}_t$ has also this property. We deduce that ϕ_t has the sphere-extension property. □

6.4 On orbit equivalence of quasiconformal Anosov flows

In this section we prove Theorem 6.2 and Proposition 6.1.

6.4.1 Construction of a transverse geometric structure

Denote by ϕ_t a C^∞ topologically transitive quasiconformal Anosov flow such that E^+ and E^- are of dimensions at least three. Then by Theorem 6.5 and Lemma 6.2, $E^{+,0}$ and $E^{-,0}$ are both C^∞ and there exist τ^+ and τ^- two continuous ϕ_t-invariant conformal structures on E^+ and E^- which are C^∞ along the leaves of $\mathcal{F}^+$ and $\mathcal{F}^-$.

Denote by Φ the orbit foliation of ϕ_t. For each transverse section Σ of Φ we get two C^∞ foliations $\mathcal{F}_\Sigma^+$ and $\mathcal{F}_\Sigma^-$ on Σ by intersecting $\mathcal{F}^{\pm,0}$ with Σ. Denote their tangent distributions by E_Σ^+ and E_Σ^- respectively.

We can identify $E_\Sigma^\pm$ and $E^\pm$ by projecting $E^\pm$ onto $E_\Sigma^\pm$ parallel to the direction of the flow. Under this identification, we get two conformal structures τ_Σ^+ and τ_Σ^- on E_Σ^+ and E_Σ^-. Since τ_Σ^- is easily seen to be invariant under the Φ-holonomy maps and the $\mathcal{F}_\Sigma^+$-holonomy maps, then τ_Σ^- is C^∞ on Σ. Similarly we can see that τ_Σ^+ is also C^∞ on Σ. So we get on each transverse section Σ a C^∞ geometric structure $(\mathcal{F}_\Sigma^\pm, \tau_\Sigma^\pm)$ which is invariant under the Φ-holonomy maps.

Denote by c_n the canonical conformal structure on the n-dimensional sphere S^n and by M_n the isometry group of c_n. Then M_n acts transitively on S^n and is called the Möbius group. Suppose that E^+ is of dimension n and E^- is of dimension m. Then we can construct as following a transverse $(M_n \times M_m, S^n \times S^m)$-structure on Φ.

For any $x \in M$ we denote by $\bar{S}_x^+$ and $\bar{S}_x^-$ the one-point compactifications of E_x^+ and E_x^-. Then they admit naturally C^∞ conformal structures extending σ_x^+ and σ_x^-. Since

$$(h_x^+)_*(\tau^+) = \sigma_x^+ \quad \text{and} \quad (h_x^-)_*(\tau^-) = \sigma_x^-,$$

then S_x^+ and S_x^-, i.e. the one-point compactifications of W_x^+ and W_x^- admit also natural conformal structures isometric to those of $\bar{S}_x^+$ and $\bar{S}_x^-$ under the natural extensions of h_x^+ and h_x^-, which are denoted by $\bar{h}_x^+$ and $\bar{h}_x^-$.

By fixing two conformal frames of E_x^+ and E_x^- we get two C^∞ conformal isometries $\phi_x^+ : \bar{S}_x^+ \to S^n$ and $\phi_x^- : \bar{S}_x^- \to S^m$.

Take a C^∞ small transverse section Σ_x containing x and pieces of W_x^+ and W_x^-. Thus for $\delta \ll 1$ we get the local diffeomorphism

$$\theta_x : W_{x,\delta}^+ \times W_{x,\delta}^- \to \Sigma_x$$

$$(y,z) \to W_{\Sigma_x,y,2\delta}^- \cap W_{\Sigma_x,z,2\delta}^+.$$

Then we define $\phi_x : \Sigma_x \to S^n \times S^m$ such that $\phi_x = (\phi_x^+ \times \phi_x^-) \circ (\bar{h}_x^+ \times \bar{h}_x^-) \circ \theta_x^{-1}$.

Since τ_Σ^+ and τ_Σ^- are invariant under respectively the $\mathcal{F}_\Sigma^-$-holonomy maps and the $\mathcal{F}_\Sigma^+$-holonomy maps, then by its definition, ϕ_x is easily seen to be a local isometry of $(\mathcal{F}_{\Sigma_x}^\pm, \tau_{\Sigma_x}^\pm)$ to $(\{S^n \times *\}, \{* \times S^m\}, c_n \times c_m)$.

Let h be any Φ-holonomy map from a germ of Σ_x to a germ of Σ_y. Then it is easy to see that $\theta_y^{-1} \circ h \circ \theta_x$ is given by weak holonomy maps of ϕ_t. We deduce that $\phi_y \circ h \circ \phi_x^{-1} = \phi \times \psi$, where ϕ and ψ are respectively local isomertries of S^n and S^m. Since $n, m \geq 3$, then by the classical theorem of Liouville, i.e. Theorem 2.2, ϕ and ψ can be both extended to global isometries of S^n and S^m. So $\{(\Sigma_x, \phi_x)\}_{x \in M}$ gives a transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ.

6.4.2 Completeness

Fix a developing map $\mathcal{D}$ of the transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ defined in the previous subsection. Denote by $\mathcal{H}$ the associated holonomy representation.

For each $x \in \widetilde{M}$ we construct an open subset U_x of $\widetilde{M}$ such that U_x is the union of the leaves of $\widetilde{\mathcal{F}}^{-,0}$ intersecting $\widetilde{W}_x^{+,0}$.

Lemma 6.13 – *Under the notations above, each leaf of $\widetilde{\mathcal{F}}^+$ intersects each leaf of $\widetilde{\mathcal{F}}^{-,0}$ at most once.*

Proof – By the definition of $\mathcal{D}$, for any $x \in \widetilde{M}$, $\mathcal{D}(\widetilde{W}_x^+)$ must be contained in a certain subset $S^n \times b$. Then for any $y \in \widetilde{W}_x^+$, there exists $\gamma_1 \times \gamma_2 \in M_n \times M_m$ such that

$$(\gamma_1 \times \gamma_2) \circ \mathcal{D} = \widetilde{\phi}_y,$$

where $\widetilde{\phi}_y$ is defined similarly as above. Denote by $\mathcal{D}_1$ the composition $pr_1 \circ \mathcal{D}$. Then there exists an open neighborhood V_y of y in $\widetilde{W}_x^+$ such that

$$\gamma_1 \circ \mathcal{D}_1 \mid_{V_y} = \widetilde{\phi}_y^+ \circ \widetilde{h}_y^+.$$

Since $\widetilde{h}_x^+ \circ \widetilde{h_y^+}^{-1}$ is an affine map (see Subsection 6.2.1), then there exists $\gamma \in M_n$ such that

$$\gamma \circ \mathcal{D}_1 \mid_{\widetilde{W}_x^+} = \widetilde{\phi}_x^+ \circ \widetilde{h}_x^+.$$

So $\mathcal{D}$ sends $\widetilde{W}_x^+$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$.

For any $y \in S^n \setminus a$ such that $\mathcal{D}(z) = y$ and $z \in \widetilde{W}_x^+$, $\mathcal{D}$ sends $\widetilde{W}_z^-$ diffeomorphically onto a set of the form $y \times (S^m \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^m$.

Now suppose that $\widetilde{W}_x^+$ intersects $\widetilde{W}_x^{-,0}$ at a point x' other than x. Then there exist $y, y' \in S^n$ such that $y \neq y'$ and

$$\mathcal{D}(W_x^-) \subseteq y \times S^m, \quad \mathcal{D}(W_{x'}^-) \subseteq y' \times S^m.$$

Denote by x'' the intersection of the $\widetilde{\phi}_t$-orbit of x' with $\widetilde{W}_x^-$. Then we have $\mathcal{D}(x'') \neq \mathcal{D}(x')$. However by the definition of $\mathcal{D}$, $\mathcal{D}(x'') = \mathcal{D}(x')$, which is a contradiction. We deduce that each leaf of $\widetilde{\mathcal{F}}^+$ intersects each leaf of $\widetilde{\mathcal{F}}^{-,0}$ at most once. □

The following lemma is a direct consequence of the previous lemma, which is first observed by T. Barbot in **[Ba]**.

Lemma 6.14 – *Under the notations above, the lifted orbit space $Q_{\widetilde{\Phi}}$ is Hausdoff.*

Proof – Suppose on the contrary that there exist two different orbits $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ such that each $\widetilde{\Phi}$-saturated open neighborhood of $\widetilde{\Phi}_1$ intersects that of $\widetilde{\Phi}_2$. We want to see that these two orbits are contained in the same leaf of $\widetilde{\mathcal{F}}^{-,0}$.

Suppose that it is not the case. Denote by F_1 and F_2 the leaves of $\widetilde{\mathcal{F}}^{-,0}$ containing respectively these two orbits. Then by assumption the $\widetilde{\mathcal{F}}^+$-saturated sets of F_1 and F_2 intersect non-trivially. We deduce that there exists a leaf $\widetilde{W}_x^+$ intersecting F_1 and F_2. Denote by V_1 and V_2 two disjoint open subsets of $\widetilde{W}_x^+$ containing respectively the intersection of $\widetilde{W}_x^+$ with F_1 and that of $\widetilde{W}_x^+$ with F_2. Then by assumption the $\widetilde{\mathcal{F}}^{-,0}$-saturated set of V_1 intersects that of V_2 non-trivially, which contradicts Lemma 6.13.

Thus $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ are contained in the same leaf of $\widetilde{\mathcal{F}}^{-,0}$. Similar we can prove that they are contained in the same leaf of $\widetilde{\mathcal{F}}^{+,0}$. Then by Lemma 6.13, we have $\widetilde{\Phi}_1 = \widetilde{\Phi}_2$, which is a contradiction. □

We can find a sequence $\{x_i\}_{i=1}^{\infty} \subseteq \widetilde{M}$ satisfying the following conditions:
(1) $\cup_{i \geq 1} U_i = \widetilde{M}$.
(2) For each $k \geq 1$, $\Omega_k = \cup_{i=1}^{k} U_i$ is connected,
where U_{x_i} is denoted by U_i. Then we can prove the following lemma. It should be mentioned that we are largely inspired by **[Gh5]**.

Lemma 6.15 – *For each $k \geq 1$, $\mathcal{D} |_{\Omega_k}: \Omega \to \mathcal{D}(\Omega_k)$ is a C^∞ fiber bundle with fiber $\mathbb{R}$ over $\mathcal{D}(\Omega_k)$. In addition $\mathcal{D}(\Omega_k)$ is either the complement in $S^n \times S^m$ of the graph of a continuous map from S^n to S^m or the complement of the union of $\{*\} \times S^m$ and of the graph of a continuous map from $(S^n \setminus \{*\})$ to S^m.*

Proof – We prove this lemma by induction. For $k = 1$ we have $\Omega_1 = U_1$. In the proof of Lemma 6.13 we have seen that $\mathcal{D}$ sends $\widetilde{W}_x^+$ diffeomorphically onto a set of the form $(S^n \setminus a) \times b$. For any $y \in S^n \setminus a$ such that $\mathcal{D}(z) = y$ and $z \in \widetilde{W}^+_{x_1}$, $\mathcal{D}$ sends $\widetilde{W}^-_z$ diffeomorphically onto a set of the form $y \times (S^m \setminus \omega(y))$. So we get a well-defined map $\omega : S^n \setminus a \to S^m$.

Denote by $Gr(\omega)$ the graph of ω. Then the complement of $Gr(\omega)$ in $(S^n \setminus a) \times S^m$ is the open set $\mathcal{D}(U_1)$. So ω is continuous. By the definition of U_1 the inverse images of $\mathcal{D}\mid_{U_1}$ are all connected. Then by the existence of fine transverse sections $\mathcal{D}\mid_{U_1}$ is seen to be a fiber bundle of fiber $\mathbb{R}$. So the lemma is true for $k = 1$.

Suppose that the lemma is true for Ω_k. Then $\mathcal{D}\mid_{\Omega_k}$ is a fiber bundle with fiber $\mathbb{R}$ and $\mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of the graph of a C^0 map $u_k : S^n \to S^m$ or of the union of a vertical $a_k \times S^m$ and the graph of a C^0 map $u_k : S^n \setminus a_k \to S^m$.

In addition by the argument above, we know that $\mathcal{D}(U_{k+1})$ is the complement of the union of $b_{k+1} \times S^m$ and of the graph of a C^0 map $v_{k+1} : S^n \setminus b_{k+1} \to S^m$ and $\mathcal{D}\mid_{U_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Since $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is the complement in $S^n \times S^m$ of a finite union of topological submanifolds of codimension at least two, then $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ is connected and open.

Firstly we want to see that $\mathcal{D}\mid_{\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$. Take $x \in \Omega_k$ and $y \in U_{k+1}$ such that $\mathcal{D}(x) = \mathcal{D}(y)$. Since Ω_{k+1} is connected, then $\Omega_k \cap U_{k+1} \neq \emptyset$. So we can take $z \in \Omega_k \cap U_{k+1}$ and a C^0 curve γ in $\mathcal{D}(U_{k+1}) \cap \mathcal{D}(\Omega_k)$ connecting $\mathcal{D}(z)$ and $\mathcal{D}(x)$.

Since $\mathcal{D}\mid_{\Omega_k}$ and $\mathcal{D}\mid_{U_{k+1}}$ are fiber bundles with fiber $\mathbb{R}$, then we can lift γ to two C^0 curves $\gamma_1 \subseteq \Omega_k$ and $\gamma_2 \subseteq U_{k+1}$ such that $\gamma_1(0) = \gamma_2(0) = z$. Thus $\gamma_1(1)$ and x are contained in the same $\widetilde{\phi}_t$-orbit and so are $\gamma_2(1)$ and y.

Denote by Λ the subset of $t \in [0,1]$ such that $\gamma_1(t)$ and $\gamma_2(t)$ are in the same orbit of $\widetilde{\phi}_t$. By the section property, Λ is easily seen to be open in $[0,1]$. Suppose that $\{t_n\}_{n=1}^{\infty} \subseteq \Lambda$ and $t_n \to t$. If $\gamma_1(t)$ and $\gamma_2(t)$ are not in the same $\widetilde{\phi}_t$-orbit, then by Lemma 6.14 there exist disjoint $\widetilde{\phi}_t$-saturated open neighborhoods of $\gamma_1(t)$ and $\gamma_2(t)$. Thus for $n \gg 1$, $\gamma_1(t_n)$ and $\gamma_2(t_n)$ are not in the same orbit of $\widetilde{\phi}_t$, which is a contradiction. We deduce that Λ is closed. Thus $\Lambda = [0,1]$. So x and y are contained in the same $\widetilde{\phi}_t$-orbit. We deduce that $\mathcal{D}\mid_{\Omega_{k+1}}$ is a fiber bundle with fiber $\mathbb{R}$.

Now we want to see the form of $\mathcal{D}(\Omega_{k+1})$. Suppose at first that $\mathcal{D}(\Omega_k) = (Gr(u_k))^c$. Take $p \in S^n \setminus b_{k+1}$. If $u_k(p) \neq v_{k+1}(p)$, then $\mathcal{D}(\Omega_{k+1})$ contains the vertical $p \times S^m$. Since $p \neq b_{k+1}$, then there exists $x \in U_{k+1}$ such that $\mathcal{D}(\widetilde{W}^-_x) = p \times (S^m \setminus v_{k+1}(p))$. So there exists $y \in \Omega_k$ such that $p \times v_{k+1}(p) \in \mathcal{D}(\widetilde{W}^-_y)$. In particular, $\mathcal{D}(\widetilde{W}^-_x) \cap \mathcal{D}(\widetilde{W}^-_y) \neq \emptyset$. So there exists $t \in \mathbb{R}$ such that $\widetilde{\phi}_t(\widetilde{W}^-_x) = \widetilde{W}^-_y$. We deduce that $\mathcal{D}(\widetilde{W}^-_x) = p \times S^m$, which is absurd. So in this case, $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k))^c$.

Suppose that $\mathcal{D}(\Omega_k) = (Gr(u_k) \cup (a_k \times S^m))^c$. For each $p \in S^n \setminus \{a_k, b_{k+1}\}$ we get as above that $u_k(p) = v_{k+1}(p)$.

If $a_k \neq b_{k+1}$, then u_k and v_{k+1} can be extended to
the same continuous map $\bar{u}_k$ on S^n. In this case $\mathcal{D}(\Omega_{k+1}) = (Gr(\bar{u}_k))^c$.

If $a_k = b_{k+1}$, then we certainly have $\mathcal{D}(\Omega_{k+1}) = (Gr(u_k) \cup (a_k \times S^m))^c$. □

We deduce from the previous lemma that $\mathcal{D} : \widetilde{M} \to \mathcal{D}(\widetilde{M})$ is a C^∞ fiber bundle with fiber $\mathbb{R}$. So the transverse $(M_n \times M_m, S^n \times S^m)$-structure of Φ is complete. In addition by the proof of the previous lemma we see that if b_k is equal to b_1 for each $k \geq 1$ then $\mathcal{D}(\widetilde{M}) = (Gr(u_1) \cup (a_1 \times S^m))^c$. If there exists $k > 1$ such that $b_k \neq b_1$ then $\mathcal{D}(\widetilde{M}) = (Gr(\bar{u}_1))^c$.

By exchanging the roles of E^+ and E^- in the previous lemma, we get the following two cases:
(1) $\mathcal{D}(\widetilde{M}) = (S^n \diagdown a) \times (S^m \diagdown b)$.
(2) $\mathcal{D}(\widetilde{M}) = (Gr(f))^c$ where f is a homeomorphism of S^n onto S^m. In particular $n = m$ in this case.

Let us consider firstly Case (1). By changing the developing map we can suppose that $a = b = \infty$. Denote by CO_n the isometry group of the canonical conformal structure of $\mathbb{R}^n$. Then by Lemma 6.10 we get a compatible transverse $(CO_n \times CO_m, \mathbb{R}^n \times \mathbb{R}^m)$-structure of Φ. In particular, the weak stable and weak unstable foliations admit transverse affine structures. So by **[Pl2]** the flow ϕ_t admits a C^∞ global section Σ. Since the Poincaré map ϕ of Σ is also topologically transitive and quasiconformal, then by **[KS]**, ϕ is C^∞ conjugate to a finite factor of a hyperbolic automorphism of a torus. We deduce that up to finite covers, ϕ_t is C^∞ orbit equivalent to the suspension of a hyperbolic automorphism of a torus.

Now we consider Case (2). Denote by Γ the fundamental group of M. Then by Lemma 6.11 the group $\mathcal{H}(\Gamma)$ is discrete in $M_n \times M_n$. Define $\mathcal{H}_1 = pr_1 \circ \mathcal{H}$ and $\mathcal{H}_2 = pr_2 \circ \mathcal{H}$. Then we have

$$f \circ \mathcal{H}_1(\gamma) \circ f^{-1} = \mathcal{H}_2(\gamma),\ \forall\ \gamma \in \Gamma.$$

We deduce that $\mathcal{H}_1(\Gamma)$ and $\mathcal{H}_2(\Gamma)$ are both discrete in M_n. Denote $\mathcal{H}_1(\Gamma)$ by Γ_1 and $\mathcal{H}_2(\Gamma)$ by Γ_2. Since ϕ_t is topologically transitive, then Φ admits at least a simply connected leaf. We deduce that $\mathcal{H}$ is injective. So Γ and Γ_1 and Γ_2 are all isomorphic.

We can prove that Γ_1 is uniform in M_n as following. Suppose on the contrary that Γ_1 is not uniform. Then Γ_1 admits a finite index torsion free subgroup Γ_1' such that $cd(\Gamma_1') \leq n$, where $cd(\Gamma_1')$ denotes the cohomological dimension of Γ_1'. So by passing to a finite index subgroup if necessary, we can suppose that $cd(\Gamma) \leq n$.

Denote by $B\Gamma$ the classifying space of Γ and by $E\Gamma$ the universal covering space of $B\Gamma$. Then we have

$$B\Gamma \cong \Gamma_1 \diagdown \mathbb{H}^{n+1},\ E\Gamma \cong \mathbb{H}^{n+1},$$

where $\mathbb{H}^{n+1}$ denotes the simply connected hyperbolic space of dimension $n+1$. Denote by $E\Gamma \times_\Gamma \widetilde{M}$ the quotient manifold of $E\Gamma \times \widetilde{M}$ under the diagonal action of Γ. Then we have the following fibre bundle with fiber $\widetilde{M}$

$$\pi_1 : E\Gamma \times_\Gamma \widetilde{M} \to B\Gamma,$$

$$\Gamma((a, x)) \to \Gamma(a).$$

By using the cohomology Leray-Serre spectral sequence to this fibre bundle (see [**Mc**]), we get that

$$E_2^{p,q} = H^p(\Gamma, H^q(\widetilde{M}))$$

converges to $H^{p+q}(E\Gamma \times_\Gamma \widetilde{M})$. Since $\widetilde{M}$ is a fibre bundle with fiber $\mathbb{R}$ and base $(S^n \times S^n)\diagdown(Gr(f))$, then $\widetilde{M}$ is homotopically equivalent to the sphere S^n. Since we have in addition $cd(\Gamma) \leq n$, then we deduce from the spectral sequence above that $H^{2n+1}(E\Gamma \times_\Gamma \widetilde{M})$ is trivial.

However by projecting onto the second factor $E\Gamma \times_\Gamma \widetilde{M}$ is easily seen to be also a fibre bundle over M and with contractible fiber $E\Gamma$. So $E\Gamma \times_\Gamma \widetilde{M}$ is homotopically equivalent to M. We deduce that $H^{2n+1}(M)$ is trivial, which is absurd. So Γ_1 is uniform in M_n. Similarly Γ_2 is also uniform in M_n.

Since f conjugates Γ_1 to Γ_2, then by Mostow's rigidity theorem (see [**Mos**]) f is contained in M_n. So by replacing $\mathcal{D}$ by $(Id \times f^{-1}) \circ \mathcal{D}$, we can suppose that $f = Id$ and

$$\mathcal{D}(\widetilde{M}) = (S^n \times S^n)\diagdown\Delta,$$

where Δ denotes the diagonal of $S^n \times S^n$. In addition, we have $\mathcal{H}_1 = \mathcal{H}_2$. So by Lemma 6.10, Φ admits a compatible transverse $(M_n, (S^n \times S^n)\diagdown\Delta)$-structure with respect to the diagonal action of M_n on $(S^n \times S^n)\diagdown\Delta$.

Lift ϕ_t to a finite cover to eliminate the torsion of Γ and define $V = \mathcal{H}(\Gamma)\diagdown\mathbb{H}^{n+1}$. Then V is a closed hyperbolic manifold. In addition, the Γ-action on $Q_{\widetilde{\Phi}}$ is C^∞ conjugate to the $\mathcal{H}(\Gamma)$-action on the leaf space of the lifted geodesic flow of V under $\mathcal{D}$ and $\mathcal{H}$. Since the holonomy of each periodic orbit of ϕ_t is non-trivial, then the holonomy covering of each leaf of Φ is contractible. Denote by ψ_t the geodesic flow of V. So by Proposition 6.4 there exists a C^∞ homotopy equivalence h conjugating the leaf space of ϕ_t with that of ψ_t. However h is not in general a C^∞ diffeomorphism. In order to get a C^∞ orbit conjugacy between ϕ_t and ψ_t, we use a classical diffusion argument discovered by É. Ghys. Let us recall briefly this argument (see [**Gh3**] and [**Ba**] for details):

There exists a C^∞ function $u : \mathbb{R} \times M \to \mathbb{R}$ such that

$$h(\phi_t(x)) = \psi_{u(t,x)}(h(x)),\ \forall\ t \in \mathbb{R},\ \forall\ x \in M.$$

Define for $T \gg 1$, $u_T(x) = \frac{1}{T}\int_0^T u(s,x)ds$ and $h_T : M \to T^1V$ such that $h_T(x) = \psi_{u_T(x)}(h(x))$. If $T \gg 1$, then we can see that h_T satisfies the same conditions as h and is a C^∞ diffeomorphism.

So up to finite covers, ϕ_t is C^∞ orbit equivalent to the geodesic flow of a closed hyperbolic manifold, which finishes the proof of the first part of Theorem 6.2.

6.4.3 Smoothness blowing up

In this subsection we prove the second part of Theorem 6.2. Suppose that ϕ_t satisifes the conditions of Theorem 6.2 such that $E^+ \oplus E^-$ is in addition C^1. Because of the first part of Theorem 6.2, ϕ_t is seen to be volume-preserving. So in order to prove the second part of Theorem 6.2, we need only prove the C^∞ smoothness of $E^+ \oplus E^-$ and then use Theorem 6.1.

Lemma 6.16 – *Under the notations above, $E^+ \oplus E^-$ is C^∞.*

Proof – Suppose at first that ϕ_t is C^∞ orbit equivalent to the geodesic flow ψ_t of a hyperbolic manifold (up to finite covers). Denote by λ the canonical 1-form of ϕ_t and by X the generator of ψ_t. Up to C^∞ flow conjugacy we suppose that ϕ_t is generated by fX.

Since $E^+ \oplus E^-$ is supposed to be C^1, then λ is C^1 and $\lambda(X) = \frac{1}{f}$ is C^∞. It is easily seen that $d\lambda$ is ϕ_t-invariant. Then by the Anosov property of ϕ_t, we get $i_{fX}d\lambda = 0$. Thus $i_X d\lambda = 0$. We deduce that $d\lambda$ is ψ_t-invariant. Denote by λ' the canonical 1-form of ψ_t. Then by Proposition 4.2, there exists $a \in \mathbb{R}$ such that $d\lambda = a \cdot d\lambda'$.

Define $\beta = \lambda - a \cdot \lambda'$. Then β is a C^1 1-form such that $d\beta = 0$. In addition $\beta(X) = \lambda(X) - a$ is C^∞.

Sublemma – *Let ϕ_t be a C^∞ volume-preserving Anosov flow on M with generator X. If α is a C^1 1-form on M such that $d\alpha = 0$ and $\alpha(X)$ is C^∞, then α is C^∞.*

Proof – Since $d\alpha = 0$ and the Stokes formula is valid for C^1 forms (even for Lipchitz forms), then there exists a C^∞ 1-form β giving the same element of $(H_1(M,\mathbb{R}))^*$ as that given by α. So by integrating $(\alpha - \beta)$ along curves, we get a well-defined C^2 function f on M. Thus for any $x \in M$ and any $t \in \mathbb{R}$ we have

$$f(\phi_t(x)) - f(x) = \int_0^t (\alpha(X) - \beta(X)) \circ \phi_s(x) ds.$$

Since $\alpha(X)$ is supposed to be C^∞, then by Proposition 3.2, f is seen to be C^∞. However by the definition of f, we have $\alpha - \beta = df$. Thus α is C^∞. $\square$

We deduce from this sublemma that β is C^∞. Thus λ is C^∞. So $E^+ \oplus E^-$ is also C^∞.

If ϕ_t is C^∞ orbit equivalent to the suspension ψ_t of a hyperbolic automorphism of a torus (up to finite covers), then by similar arguments as above, we can see that $d\lambda$ is ψ_t-invariant.

Take a leaf Σ of the foliation of the sum of the strong stable and the strong unstable distributions of ψ_t and denote by ψ its Poincaré map. Then $\lambda \mid_\Sigma$ is C^1 and $d(\lambda \mid_\Sigma)$ is ψ-invariant. Thus by Lemma 4.3 we get $d(\lambda \mid_\Sigma) = 0$. We deduce that $d\lambda = 0$. Since in addition $\lambda(X) = \frac{1}{f}$ is C^∞, then by the previous sublemma λ is C^∞. Thus $E^+ \oplus E^-$ is also C^∞. $\square$

Proof of Proposition 6.1 – Suppose that ϕ_t satisfies the conditions of Proposition 6.1.1. Similar to the previous section, we can construct a C^∞ geometric structure

$(\mathcal{F}^{\pm}_{\Sigma}, \tau^{\pm}_{\Sigma})$ on each transverse section Σ of Φ. Similarly we can construct a family of transverse charts $\{(\Sigma_x, \phi_x)\}_{x \in M}$. Then because of the sphere-extension property, these charts are easily seen to be compatible with respect to the natural action of $M_2 \times M_m$ on $S^2 \times S^m$. So in this way we get a transverse $(M_2 \times M_m, S^2 \times S^m)$-structure on Φ. Then as in the previous subsection the proof splits into Case (1) and Case (2). Each of them is understood in the same manner as in the previous subsection.

6.5 Applications to the geodesic flows of closed hyperbolic manifolds

Now let us begin to prove Theorem 6.5.

Lemma 6.17 – *Let ϕ_t and ψ_t be two C^∞ Anosov flows which are C^1 orbit equivalent. If ψ_t is volume-preserving, then so is ϕ_t.*

Proof – By conjugating ϕ_t by the C^1 orbit conjugacy, we can suppose that ϕ_t is a C^1 flow and a time change of ψ_t. Denote by ν the ψ_t-invariant volume form and by X the generator of ψ_t. Then by taking $i_X \nu$ we get a family of Ψ-holonomy invariant volume forms on the transverse sections of Ψ, where Ψ denotes the orbit foliation of ψ_t. This family of transversal volume forms is also Φ-holonomy invariant. Denote by dt_ϕ the normalized foliated measure along the leaves of Φ such that $dt_\phi(Y) \equiv 1$, where Y denotes the generator of ϕ_t. In each flow box of ϕ_t we take the product measure $\nu_\Sigma \otimes dt_\phi$. Then it is easily seen that in the intersection of two flow boxes the two measures coincide. Then we can extend this family of local measures to a measure μ on M which is in the Lebesgue class and easily seen to be ϕ_t-invariant. □

Proof of Theorem 6.5 – Since ψ_t is conformal, then by Lemma 6.1, ϕ_t is quasiconformal. In addition by Lemmas 6.12 and 6.17, ϕ_t satisfies either the conditions of Theorem 6.2 or those of Proposition 6.1. So up to finite covers, ϕ_t is C^∞ orbit equivalent either to a suspension or to the geodesic flow of a hyperbolic manifold $\hat{\psi}_t$. Since ψ_t is contact, then it admits no C^1 global section. So up to finite covers, ϕ_t is C^∞ orbit equivalent to $\hat{\psi}_t$.

However in the proofs of Theorem 6.2 and Proposition 6.1, we passed to a finite cover only in order to eliminate the torsion in the fundamental group of M. But in the current case, the fundamental group has no torsion by the classical Cartan theorem. So ϕ_t is C^∞ orbit equivalent to $\hat{\psi}_t$. Then by Mostow's rigidity theorem (see [**Mos**] and [**Ma2**]) $\hat{\psi}_t$ is C^∞ flow equivalent to ψ_t. We deduce that ϕ_t is C^∞ orbit equivalent to ψ_t. □

Proof of Proposition 6.2 – Let us prove firstly (1). Suppose that the geodesic flow of M is Hölder-continuously orbit equivalent to that of a hyperbolic manifold

N. Since $n \geq 3$, then the fundamental group of M is isomorphic to that of N. Since $\pi_1(M)$ with its word metric is quasi-isometric to $\widetilde{M}$ and $\pi_1(N)$ is quasi-isometric to $\mathbb{H}^n$, then we deduce that $\widetilde{M}$ is quasi-isometric to $\mathbb{H}^n$.

Conversely, if $\widetilde{M}$ is quasi-isometric to $\mathbb{H}^n$, then $\pi_1(M)$ is also quasi-isometric to $\mathbb{H}^n$. Thus by **[Su]** and **[Tu]**, there exists a uniform lattice Γ in the isometric group of $\mathbb{H}^n$ and a surjective group homomorphism $\rho : \pi_1(M) \to \Gamma$ such that the kernal of ρ is finite. However by a classical result of É. Cartan, $\pi_1(M)$ is without torsion. We deduce that $\pi_1(M)$ is isomorphic to Γ. In particular, Γ is also without torsion. So $N = \Gamma \diagdown \mathbb{H}^n$ is a closed hyperbolic manifold.

Denote respectively by ϕ_t and ψ_t the geodesic flows of M and N. Since $\pi_1(M) \cong \pi_1(N)$, then by **[Gr]**, ϕ_t is C^0 orbit equivalent to ψ_t. Since each continuous orbit conjugacy between Anosov flows can be C^0 approximated by Hölder-continuous orbit conjugacies (see **[HK]**), then (1) is true.

Now let us prove (2). We need only prove the necessarity. Suppose that the geodesic flow ϕ_t of M is C^1 orbit equivalent to the geodesic flow ψ_t of a closed hyperbolic manifold N. Since ψ_t is conformal, then by Lemma 6.1, ϕ_t is quasiconformal. Thus by Corollary 6.4, it is C^∞ orbit equivalent to the geodesic flow of N. Since C^∞ orbit conjugacy preserves weak stable and weak unstable distributions, then ϕ_t is Anosov-smooth. So by **[BFL]**, it is C^∞ flow equivalent to the geodesic flow of a hyperbolic manifold. Then by **[BCG]**, M has constant negative curvature.
□

Chapter 7

On the Homogeneity of Affine Anosov-smooth Flows

Abstract – *In this chapter we prove a homogeneity result for affine Anosov-smooth flows, which should furnish the departing point for a future classification of such flows.*

7.1 Introduction

In the chapters above, a guiding idea was to construct C^∞ invariant linear connections from the given geometric structures. These (more or less) canonically constructed connections were in the center of our arguments. In this chapter we try to understand the general affine Anosov-smooth flows, i.e. the connection-preserving Anosov-smooth flows.

In Section 7.2 we formulate a conjecture about such flows. Then in Section 7.3 we prove the homogeneity of Parry time changes by using the results concerning rigid geometric strutures established in Chapter 2.

7.2 Invariant connections of Anosov flows and a conjecture

Let ϕ_t be an Anosov-smooth flow on M. If ∇^+ is a C^∞ ϕ_t-invariant linear connection along $\mathcal{F}^+$ and ∇^- is a C^∞ ϕ_t-invariant linear connection along $\mathcal{F}^-$, then (∇^+, ∇^-) is said to be a C^∞ *connection couple* of ϕ_t. We denote by $\Lambda(\phi_t)$ the set of C^∞ connection couples of ϕ_t. Let us prove firstly the following

Lemma 7.1 – *Let N be a C^∞ manifold and $\{E_i\}_{i=1}^k$ be some C^∞ distributions such that $TN = E_1 \oplus \cdots \oplus E_k$. For all $1 \leq i \leq k$, let ∇_i be a C^∞ linear connection* *of E_i along E_i. Then there exists a smooth linear connection ∇ on N such that $\nabla E_i \subseteq E_i$ for all $1 \leq i \leq k$ and in addition ∇ is preserved by each diffeomorphism preserving the decomposition and $\{\nabla_i\}_{i=1}^k$.*

Proof – For each $1 \leq i \leq k$, we denote by P^i the linear projection of TN onto E_i with respect to the decomposition. Then for arbitrary C^∞ vector fields Y and Z, we define

$$\nabla_Y Z = \sum_{1\leq i\neq j\leq k} P^j([P^iY, P^jZ]) + \sum_{1\leq l\leq k} (\nabla_l)_{P^lY} P^l Z.$$

Thus ∇ is easily seen to be a C^∞ linear connection fulfilling the conditions. □

Lemma 7.2 – *Suppose that ϕ_t is Anosov-smooth. Then ϕ_t preserves a C^∞ linear connection iff $\Lambda(\phi_t) \neq \emptyset$.*

Proof – Denote by P^+ and P^- the projections of TM onto E^+ and E^-. If ϕ_t preserves a C^∞ linear connection ∇, then it is easy to see that $(P^+ \circ \nabla, P^- \circ \nabla) \in \Lambda(\phi_t)$. So $\Lambda(\phi_t) \neq \emptyset$.

Conversely we suppose that $\Lambda(\phi_t) \neq \emptyset$. There exists a C^∞ ϕ_t-invariant connection of $\mathbb{R}X$ along $\mathbb{R}X$, ∇^0 such that $(\nabla^0)_X X = 0$. So by Lemma 7.1, we can construct a C^∞ ϕ_t-invariant linear connection from ∇^0 and each element of $\Lambda(\phi_t)$. Since $\Lambda(\phi_t) \neq \emptyset$, then ϕ_t preserves at least one C^∞ linear connection. □

The following lemma furnishes the basis of applying the *go-and-back* idea (see Section 4.5).

Lemma 7.3 – *Let ϕ_t and ψ_t be two Anosov-smooth flows such that they are time changes of each other. Then there exists a natural bijection between $\Lambda(\phi_t)$ and $\Lambda(\psi_t)$. In particular, ϕ_t is affine (i.e. connection-preserving) iff ψ_t is affine.*

Proof – Denote by $\bar{E}^+$ the strong unstable distribution of ψ_t and suppose that $(\nabla^+, \nabla^-) \in \Lambda(\phi_t)$.

Since ψ_t and ϕ_t are time changes of each other, i.e. they are C^∞ orbit equivalent, then for any $x \in M$, $W_x^{+,0} = \bar{W}_x^{+,0}$. Thus for any $y \in \bar{W}_x^+$, there exists $t \in \mathbb{R}$ such that $\phi_t(y) \in W_x^+$.

If $W_x^{+,0}$ contains no periodic orbit, this number t is unique for each y in $\bar{W}_x^+$. If $W_x^{+,0}$ contains a periodic orbit, then it contains exactly one periodic orbit (see Proposition 3.1). Denote by T its mininal positive period. So if $\phi_t(y) \in W_x^+$, then for all $k \in \mathbb{Z}$, $\phi_{t+k\cdot T}(y) \in W_x^+$.

Conversely if $\phi_{t_1}(y) \in W_x^+$ and $\phi_{t_2}(y) \in W_x^+$, then $\phi_{t_2-t_1} W_x^+ = W_x^+$. Thus $t_2 - t_1 \in T \cdot \mathbb{Z}$. So by associating $t + T\cdot\mathbb{Z}$ to y, we get a well-defined C^∞ map from $\bar{W}_x^+$ to $\mathbb{R}/T\mathbb{Z}$.

Thus by taking a lift if necessary, there exists a unique C^∞ map $\theta_x : \bar{W}_x^+ \to \mathbb{R}$ such that

$$\theta_x(x) = 0,\ \phi_{\theta_x(y)}(y) \in W_x^+,\ \forall\ y \in \bar{W}_x^+.$$

Define a C^∞ map $\Theta_x : \bar{W}_x^+ \to W_x^+$ such that

$$\Theta_x(y) = \phi_{\theta_x(y)}(y).$$

Then Θ_x is easily seen to be a local diffeomorphism. Thus we can find an open neighborhood U_x^+ of x in $\bar{W}_x^+$ such that $\Theta_x\mid_{U_x^+}$ is a diffeomorphism onto its image. So we get a family of maps $\{\Theta_x\mid_{U_x^+}\}_{x\in M}$.

For any $x, y \in M$, if $z \in U_x^+ \cap U_y^+$, then $\bar{W}_x^+ = \bar{W}_y^+$. Define

$$\alpha = \theta_y(z) - \theta_x(z),$$

then $\phi_\alpha(\Theta_x(z)) = \phi_{\theta_y(z)}(z) \in W_y^+$, i.e. $\phi_\alpha W_x^+ = W_y^+$. For all $w \in \bar{W}_y^+$,

$$\phi_{\alpha+\theta_x(w)}(w) \in \phi_\alpha W_x^+ = W_y^+.$$

So there exists $k \in \mathbb{Z}$ such that

$$\theta_y = \theta_x + \alpha + k \cdot T,$$

where T is taken to be zero if $W_y^{+,0}$ contains no periodic orbit. So we get

$$\Theta_y \circ \Theta_x^{-1} = \phi_{\alpha+k\cdot T}.$$

Since ∇^+ is ϕ_t-invariant, then we can pull it back by $\{\Theta_x\mid_{U_x^+}\}_{x\in M}$ to get a well-defined C^∞ linear connection along $\bar{\mathcal{F}}^+$ denoted by $\bar{\nabla}^+$.

Now let us prove that $\bar{\nabla}^+$ is ψ_t-invariant. At first there exists a C^∞ map $\beta : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\psi_t = \phi_{\beta(\cdot,t)}(\cdot).$$

For any $x \in M$ and any $t \in \mathbb{R}$, we define a C^∞ map $\eta : \bar{W}_x^+ \to \mathbb{R}$ such that

$$\eta = \theta_x + \beta(x,t) - \beta(\cdot,t).$$

Then for all $y \in \bar{W}_x^+$,

$$\phi_{\eta(y)}(\psi_t(y)) = \phi_{\theta_x(y)+\beta(x,t)}(y) \in \phi_{\beta(x,t)} W_x^+ = W_{\psi_t(x)}^+.$$

Since in addition $\eta(x) = 0$, then we have $\eta \circ \psi_{-t} = \theta_{\psi_t(x)}$, i.e. for all $y \in \bar{W}_x^+$,

$$\beta(x,t) + \theta_x(y) = \theta_{\psi_t(x)}(\psi_t(y)) + \beta(y,t).$$

Then it is easily verified that

$$\Theta_{\psi_t(x)} \circ \psi_t = \phi_{\beta(x,t)} \circ \Theta_x.$$

Thus $\bar{\nabla}^+$ is ψ_t-invariant. Similarly we get a C^∞ ψ_t-invariant connection along $\bar{\mathcal{F}}^-$ denoted by $\bar{\nabla}^-$.

In conclusion we have constructed an application from $\Lambda(\phi_t)$ to $\Lambda(\psi_t)$. Similarly we get an application from $\Lambda(\psi_t)$ to $\Lambda(\phi_t)$. By the construction, these two applications are the inverses of each other.

In particular, we get $\Lambda(\phi_t) \neq \emptyset$ iff $\Lambda(\psi_t) \neq \emptyset$. So by Lemma 7.2, ϕ_t is affine iff ψ_t is affine. $\square$

Corollary 7.1 – *Suppose that ϕ_t is a topologically transitive affine Anosov-smooth flow. If rank$(\phi_t) = 0$, then up to a constant change of time scale, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic infra-nilautomorphism.*

Proof – Since $\text{rank}(\phi_t) = 0$, then by Theorem 3.1 of **[Pl1]**, ϕ_t admits a C^∞ closed global section Σ. The induced diffeomorphism ϕ is easily seen to be Anosov-smooth (see Subsection 5.2.3). Since the suspension of ϕ is a C^∞ time change of ϕ_t and ϕ_t is supposed to be affine, then by Lemma 7.3, the suspension of ϕ is affine. So ϕ is also affine. We deduce by **[BL]** that ϕ is C^∞ conjugate to a hyperbolic infra-nilautomorphism.

Then again by **[Pl1]**, $E^+ \oplus E^-$ is integrable with compact leaves. Then the corollary follows by taking a leaf of the foliation of $E^+ \oplus E^-$ as Σ. $\square$

Lemma 7.4 – *Symmetric Anosov flows and their special time changes are all connection-preserving.*

Proof – By Lemma 7.3, we need only prove that each symmetric Anosov flow is connection-preserving. Denote by ϕ_t a symmetric Anosov flow on a m-dimensional manifold M. If $\text{rank}(\phi_t) = 0$ or $[\frac{m}{2}]$, it is certainly connection-preserving (see **[BFL2]** and **[BL]**). If $0 < \text{rank}(\phi_t) < [\frac{m}{2}]$, then by Proposition 3.6, ϕ_t is commensurable to an algebraic flow ψ_t defined on $\Gamma \diagdown V \rtimes_\rho H \diagup K$. Since $V \rtimes_\rho H \diagup K$ is canonically diffeomorphic to $V \times (H \diagup K)$, then we get two foliations of $V \rtimes_\rho H \diagup K$, $\mathcal{F}_1 = \{V \times *\}$ and $\mathcal{F}_2 = \{* \times (H \diagup K)\}$. The flow ψ_t induces naturally a flow $\bar{\psi}_t$ on $H \diagup K$ which is a lift of the geodesic flow of a hyperbolic space. So we can find a H-invariant and $\bar{\psi}_t$-invariant connection ∇_2 along $\mathcal{F}_2$. By combining the canonical flat connection along $\mathcal{F}_1$ and ∇_2, we can construct by Lemma 7.1 a left-invariant and ψ_t-invariant connection on $V \rtimes_\rho H \diagup K$. We deduce that ϕ_t is connection-preserving. $\square$

We have seen in Section 3.4 that each symmetric Anosov flow is volume-preserving and Anosov-smooth. So we arrive at the following

Conjecture 7.1 – *Let ϕ_t be a volume-preserving affine Anosov-smooth flow. Then it is commesuable to a canonical time change of a symmetric Anosov flow.*

At the moment, I can not prove this conjecture. But in the following section, we shall estalish a weaker result.

7.3 Homogeneity of Parry time changes

Suppose in this section that ϕ_t is a volume-preserving affine Anosov-smooth flow of which the Bowen-Margulis measure is in the Lebesgue class and $E^{\pm}$ are both orientable. Because of Corollary 7.1 and Lemma 3.7, we suppose in addition that ϕ_t is topologically mixing. The topological entropy of ϕ_t is denoted by h and the dimensions of E^+ and E^- are supposed to be n and m.

Define $a^+ = \frac{h}{n}$ and $a^- = -\frac{h}{m}$ and take an element $(\nabla^+, \nabla^-) \in \Lambda(\phi_t)$, which is not empty by Lemma 7.2. Then we can construct a C^∞ ϕ_t-invariant canonical linear connection ∇ (see Section 4.4) such that

$$\nabla X = 0,\ \nabla E^{\pm} \subseteq E^{\pm},\ \nabla_{Y^{\pm}} Z^{\mp} = P^{\mp}[Y^{\pm}, Z^{\mp}],$$

$$\nabla_{Y^{\pm}} Z^{\pm} = (\nabla^{\pm})_{Y^{\pm}} Z^{\pm},\ \nabla_X Y^{\pm} = [X, Y^{\pm}] + a^{\pm} Y^{\pm}.$$

So by Proposition 4.2, the line bundle $\wedge^n E^+$ admits a C^∞ nowhere-vanishing ∇-parallel section denoted by ω^+. Denote by Ω^+ the curvature form of $(\nabla, \wedge^n E^+)$. Since ω^+ is ∇-parallel, then we have $\Omega^+ \equiv 0$.

Denote by ν the C^∞ nowhere-vanishing invariant volume form of ϕ_t and define the underlying geometric structure of ϕ_t as

$$g = (X, E^+, E^-, \nabla, \nu).$$

Since linear connections are rigid, then g is a C^∞ rigid geometric structure of order two on M. In addition, g is ϕ_t-invariant.

Since ϕ_t preserves a volume-form, then it is topologically transitive and its periodic orbits are dense in M (see Subsection 3.2.2). So by Theorem 2.1, the pseudogroup of C^∞ local g-isometries I^{loc} admits a unique open-dense orbit denoted by Ω. Fix a periodic point x in Ω. Then by Section 2.4, we get two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. recall that $\mathfrak{g}$ denotes the space of germs at x of C^∞ local g-Killing fields.

For all $Y \in \mathfrak{g}$, we define

$$A_Y = \mathcal{L}_Y - \nabla_Y.$$

It is easy to verify that A_Y is a local C^∞ $(1,1)$-tensor. Then we get a linear map $j : \mathfrak{g} \to T_xM \times End(T_xM)$ such that $j(Y) = (Y_x, (A_Y)_x)$, which is easily seen to be injective. Since $\mathfrak{g} \mid_x = T_xM$ (see Subsection 2.4), then

$$pr_1(j(\mathfrak{g})) = T_xM.$$

If the Lie algebra structure of $\mathfrak{g}$ is pushed forward onto $j(\mathfrak{g})$ by j, then for all $(u, A),\ (u', A') \in j(\mathfrak{g})$, we have

$$[(u, A), (u', A')] = [Au' - A'u + T(u, u'), [A, A'] - R(u, u')],$$

where T and R denote the torsion tensor and the curvature tensor of ∇.

Define $i : I^{loc}_{x,x} \to GL(T_xM)$ such that $i(h) = D_xh$. Then i is easily seen to be injective and real algebraic (see Corollary 2.2). So $I^{loc}_{x,x}$ is identified under i to a closed subgroup of $GL(T_xM)$ (see **[OV]** and **[Bor]**). As in Subsection 2.5.4, the left-invariant Lie algebra of $I_{x,x}^{loc}$ can be identified to $\mathfrak{h}$ as following:

$$r : \mathfrak{g}_{I_{x,x}^{loc}} \to \mathfrak{h}$$

$$u \to (y \to \frac{\partial}{\partial t} \mid_{t=0} exp(-tu)(y)).$$

Then under this identification, it is easily verified that $Di = j \mid_{\mathfrak{h}}$.

Since x is chosen to be ϕ_t-periodic, then there exists $T > 0$ such that $\phi_T(x) = x$. Thus $D_x\phi_T \in i(I_{x,x}^{loc})$. Denote by L_0 the logarithm of the hyperbolic part in the complete Jordan decomposition of $D_x\phi_T$ (see **[He]** and **[Eb]**). Since $I_{x,x}^{loc}$ is real algebraic, then $L_0 \in j(\mathfrak{h})$.

In the following, we identify $\mathfrak{g}$ with $j(\mathfrak{g})$, $\mathfrak{h}$ with $j(\mathfrak{h})$ and $I_{x,x}^{loc}$ with $i(I_{x,x}^{loc})$. Replacing T by nT ($n \gg 1$) if nescessary, we can see by the Anosov property that the eigenvalues of L_0 on E_x^+ are strictly positive and its eigenvalues on E_x^- are strictly negative. In addition, we have $L_0(X_x) = 0$.

Define $d\chi : \mathfrak{g} \to \mathbb{R}$ such that $d\chi((u, A)) = Tr(A \mid_{E_x^+})$. Since for all $u, v \in TM$ we have $Tr(R(u, v) \mid_{E_x^+}) = \Omega^+(u, v)$ and $\Omega^+ \equiv 0$, then for all $(u, A), (v, B) \in \mathfrak{g}$, we get

$$d\chi([(u, A), (v, B)]) = Tr(([A, B] - R(u, v)) \mid_{E_x^+}) = 0.$$

So $d\chi$ is a character of $\mathfrak{g}$, i.e. a Lie algebra homomorphism from $\mathfrak{g}$ into $\mathbb{R}$. Define $\mathfrak{g}' = Ker(d\chi)$ and $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$. Since $d\chi(L_0) > 0$, then $\mathfrak{g}'$ and $\mathfrak{h}'$ are both codimension-one ideals in $\mathfrak{g}$ and $\mathfrak{h}$.

Lemma 7.5 – *Under the notations above, the center of* $\mathfrak{g}$ *is* $\mathbb{R}X$ *and the center of* $\mathfrak{g}'$ *is trivial.*

Proof – Certainly $\mathbb{R}X$ is contained in the center of $\mathfrak{g}$. Suppose that $\alpha = (u^+ + u^- + aX_x, A)$ is in the center of $\mathfrak{g}$. Then

$$[L_0, \alpha] = (L_0(u^+) + L_0(u^-) + aL_0(X_x), [L_0, A]) = 0.$$

Since the eigenvalues of L_0 on E_x^+ are strictly positive and its eigenvalues on E_x^- are strictly negative, then we get $u^+ = u^- = 0$. Thus $\alpha - aX$ ($\in \mathfrak{h}$) is in the center of $\mathfrak{g}$.

Since $pr_1(j(\mathfrak{g})) = T_xM$, then for all $u \in T_xM$, there exists $(u, B) \in \mathfrak{g}$. So

$$[(\alpha - aX), (u, B)] = ((\alpha - aX)(u), [\alpha - aX, B]) = 0.$$

Thus $(\alpha - aX)(u) = 0$ for all $u \in T_xM$. So $\alpha = aX$, i.e. the center of $\mathfrak{g}$ is $\mathbb{R}X$. By a simple calculation, we get $d\chi(X) = -h < 0$. So the center of $\mathfrak{g}'$ is trivial. □

Denote by $\bar{G}$ the connected and simply connected Lie group with left-invariant Lie algebra $\mathfrak{g}$ and by $\bar{H}$ the connected Lie subgroup of $\bar{G}$ integrating $\mathfrak{h}$. Then by Section 2.4, g is normal iff $\bar{H}$ is closed in $\bar{G}$.

Lemma 7.6 – *Under the notations above, g is normal, i.e. $\bar{H}$ is closed in $\bar{G}$.*

Proof – Define $I = \{h \in I^{loc}_{x,x} \mid h_*\omega^+ = \omega^+ \text{ as germs at } x\}$. Since h sends parallel sections of $\wedge^n E^+$ to parallel sections, then $h_*\omega^+ = \omega^+$ as germs at x iff $(h_*\omega^+)_x = \omega^+_x$. So I is a real algebraic subgroup of $I^{loc}_{x,x}$.

It is easy to see that for all $Y \in \mathfrak{g}$, $\mathcal{L}_Y\omega^+ = d\chi(Y)\cdot\omega^+$. So $\mathfrak{h}' = \{Y \in \mathfrak{h} \mid \mathcal{L}_Y\omega^+ = 0 \text{ as germ at } x\}$. Then the Lie algebra of I is identified to $\mathfrak{h}'$ under r.

Define $\rho : I^{loc}_{x,x} \to Aut(\mathfrak{g})$ such that $\rho(h)(Y) = Dh(Y)$. Since ρ is algebraic, then $\rho(I)$ is a closed in $Aut(\mathfrak{g})$.

On the other hand, we denote by $\bar{G}'$ and $\bar{H}'$ the connected Lie subgroups of $\bar{G}$ integrating $\mathfrak{g}'$ and $\mathfrak{h}'$. Then we have the restriction of the adjoint representation of $\bar{G}$ to $\bar{G}'$, $Ad : \bar{G}' \to Aut(\mathfrak{g})$. It is easily seen that $\rho(I)$ and $Ad(\bar{H}')$ have the same Lie algebra in $Aut(\mathfrak{g})$. Since in addition the center of $\mathfrak{g}'$ is trivial by Lemma 7.3.1, then

$$\bar{H}' = (Ad^{-1}((\rho(I))_0))_0.$$

So $\bar{H}'$ is closed in $\bar{G}'$.

Denote by $\mathbb{R}$ the one-parameter subgroup of $\bar{G}$ integrating $\mathbb{R}L_0$. Since $\mathfrak{g} \simeq \mathfrak{g}' \rtimes \mathbb{R}L_0$ and $\bar{G}$ is simply-connected, then $\bar{G}'$ and $\mathbb{R}$ are both closed in $\bar{G}$ and $\bar{G} \simeq \bar{G}' \rtimes \mathbb{R}$. We deduce that $\bar{H}' \rtimes \mathbb{R}$ is closed in $\bar{G}$.

Since $\bar{H}'$ is normal in $\bar{H}$ and $\mathbb{R} \subseteq \bar{H}$, then $\bar{H}' \rtimes \mathbb{R}$ is a Lie subgroup of the connected Lie group $\bar{H}$. Thus $\bar{H}' \rtimes \mathbb{R} = \bar{H}$ since they have the same dimension. So $\bar{H}$ is closed in $\bar{G}$. □

Denote by $\bar{M}$ the manifold $\bar{G}/\bar{H}$. Then by Proposition 2.1, we get on $\bar{M}$ a $\bar{G}$-invariant geometric structure $\bar{g}$ locally isomorphic to $g\mid_\Omega$. We have

$$\bar{g} = (\bar{X}, \bar{E}^+, \bar{E}^-, \bar{\nabla}, \bar{\nu}).$$

In addition by taking all the C^∞ local isometries from $g\mid_\Omega$ to $\bar{g}$, we get on Ω a $(I(\bar{g}), \bar{M})$-structure. For each connected and simply-connected open subset O of Ω, by classical arguments, there exists a developing map $\theta : O \to \bar{M}$ which is a C^∞ local diffeomorphism such that $\theta_*(g\mid_O) = \bar{g}$. The following arguments are largely inspired by those of **[BFL2]**.

Lemma 7.7 – *Under the notations above, the ∇-geodesics tangent to E^+ or E^- and the $\bar{\nabla}$-geodesics tangent to $\bar{E}^+$ or $\bar{E}^-$ are complete.*

Proof – By the same arguments as in Lemma 5.4, we can see that the ∇-geodesics tangent to E^+ or E^- are defined on $\mathbb{R}$. Now we want to see that the $\bar{\nabla}$-geodesics tangent to $\bar{E}^+$ or $\bar{E}^-$ are also complete. Define firstly

$$\Delta = \{y \in \Omega \mid W^+_y \subseteq \Omega,\ W^-_y \subseteq \Omega\}.$$

Since Ω is ϕ_t-invariant, then Δ is also ϕ_t-invariant. Since for all $y \in M$ we have

$$W^+_y = \{z \mid d(\phi_t(z), \phi_t(y)) \to 0,\ \text{if } t \to -\infty\},$$

then it is easy to see that $\mathrm{Per}(\phi_t) \cap \Omega \subseteq \Delta$. Thus Δ is dense in Ω.

Take $y \in \Delta$ and a $(I(\bar{g}), \bar{M})$-chart ϕ such that $\phi(y) = z$. Since each leaf of $\mathcal{F}^+$ is diffeomorphic to $\mathbb{R}^n$, then for all $u^+ \in E^+$ and all $T > 0$, we can find a connected and simply-connected open neighborhood O_T contained in Ω of the geodesic $\gamma_{u^+}^{\nabla} \mid_{[-T,T]}$. Then we can find a unique developing map $\theta_T : O_T \to \bar{M}$ such that the germ of θ_T at y is the same as that of ϕ at y. Since $(\theta_T)_*(g) = \bar{g}$, then the geodesic $\gamma_{D\phi(u^+)}^{\bar{\nabla}}$ can be defined on $[-T, T]$. We deduce that each $\bar{\nabla}$-geodesic containing z and tangent to $\bar{E}^+$ is defined on $\mathbb{R}$. Then by the homogeneity of $\bar{g}$, the $\bar{\nabla}$-geodesics tangent to $\bar{E}^+$ are complete. Similarly the $\bar{\nabla}$-geodesics tangent to $\bar{E}^-$ are also complete. $\square$

Using Lemma 7.7, we can construct *A-coordinates* and *A-starred open sets* as in **[BFL2]**. Let us recall some details. For each point $y \in M$, we define $\psi_y : T_yM \to M$ such that

$$\psi_y(Y^+ + tX_y + Y^-) = \phi_t(exp^{\nabla}(\tau_{Y^+}(Y^-))),$$

where exp^{∇} denotes the exponential map of ∇ and $\tau_{Y^+}(Y^-)$ denotes the image of parallel transport of Y^- along the ∇-geodesic $\gamma_{Y^+} \mid_{[0,1]}$. Because of Lemma 7.7, ψ_y is defined on T_yM. In an open neighborhood of zero, ψ_y gives a C^∞ coordinate, which is said to be the *A-coordinate with respect to* y. Similarly for all $\widetilde{y} \in \widetilde{M}$ and all $\bar{y} \in \bar{G}/\bar{H}$ we have the *A-coordinates* $\widetilde{\psi}_{\widetilde{y}}$ and $\bar{\psi}_{\bar{y}}$.

An open subset O of M is said to be *A-starred with respect to* y if there exists an open subset $U \subseteq T_yM$ such that
(1) ψ_y is a C^∞ diffeomorphism of U onto O.
(2) If $Y = Y^+ + tX_y + Y^- \in U$, then for all $s \in [0, 1]$, $Y^+ + tX_y + sY^-$ and $sY^+ + tX_y$ and stX_y are also contained in U.

If O is an A-starred open set with respect to y and θ is a developing map of O into $\bar{G}/\bar{H}$, then we have by construction $\theta \circ \psi_y = \bar{\psi}_{\theta(y)} \circ T_y\theta$.

Lemma 7.8 – *Let $y \in \Delta$ and O an A-starred open set with respect to y. Then there exists a open-dense subset O' of O such that O' is also A-starred with respect to y and $O' \subseteq \Omega$.*

Proof – Suppose that U is the open set of T_yM associated to O. Define

$$U' = \{Y = Y^+ + tX_y + Y^- \in U \mid \psi_y(Y^+ + tX_y + sY^-) \in \Omega, \ \forall \ s \in [0, 1]\}.$$

Define $O' = \psi_y(U')$. Then we have $O' \subseteq \Omega$ by definition. Since Ω is open, then U' is also open in U. Thus O' is open in O.

Suppose that $z = \psi_y(Y^+ + tX_y + Y^-) \in \Delta \cap O$. Then for all $s \in [0, 1]$,

$$\psi_y(Y^+ + tX_y + sY^-) = exp^{\nabla}(s \cdot \tau_{D\phi_t(Y^+)} D\phi_t(Y^-)) \in W_z^- \subseteq \Omega.$$

So we get $\Delta \cap O \subseteq O'$. Thus O' is dense in O.

If $Y^+ + tX_y + Y^- \in U'$, then certainly $Y^+ + tX_y + sY^- \in U'$ for all $s \in [0, 1]$. In addition, $sY^+ + tX_y$ and stX_y are contained in U' by the definition of U. So O' is

A-starred with respect to y. $\square$

Lemma 7.9 – *Under the notations above,* $\Omega = M$.

Proof – Take a cover of M by A-starred open sets $\{O_{y_i}\}$. Then for each i, we can find by Lemma 7.8 an A-starred open set O'_{y_i} in $O_{y_i} \cap \Omega$ which is dense in O_{y_i}.

Since O'_{y_i} is contractible and contained in Ω, then there exists a developing map θ of O'_{y_i} into $\bar{G}/\bar{H}$. Define $\bar{\theta} : O_{y_i} \to \bar{G}/\bar{H}$ such that

$$\bar{\theta} = \bar{\psi}_{\theta(y_i)} \circ (T_{y_i}\theta) \circ \psi_{y_i}^{-1}.$$

Then $\bar{\theta}$ is an extension of θ. Since O'_{y_i} is open-dense in O_{y_i}, then by the following sublemma $\bar{\theta}$ is seen to be a local isometry sending g to $\bar{g}$. Since $\bar{g}$ is homogeneous, then g is locally homogeneous, i.e. $\Omega = M$.

Sublemma – *Suppose that* θ *be a* C^∞ *map from* (M, ∇) *to* (N, ∇'), *where* ∇ *and* ∇' *denote two* C^∞ *linear connections. If there exists an open-dense subset* V *of* M *such that* $\theta\mid_V$ *is a local diffeomorphism sending* ∇ *to* ∇', *then* θ *is a local diffeomorphism on* M *sending* ∇ *to* ∇'.

Proof – For each $x \in M \setminus V$, we take a C^∞ curve l defined on $[-1, 1]$ in M such that $l(0) = x$ and $l(1) = y \in V$ and $\dot{l}(t) \neq 0$ for all $t \in [-1, 1]$. Define

$$A = \{s \in [0, 1] \mid (D\theta)_{l(\tau)} \text{ is bijective for all } \tau \in [s, 1]\},\ t = inf(A).$$

Since V is open and $\theta\mid_V$ is a local diffeomorphism, then A is not empty and t is well-defined. In order to see that θ is a local diffeomorphism on M, it is enough to prove that $t = 0$.

Suppose on the contrary that $t > 0$. Then by the definition of t, we get $(D\theta)_{l(\tau)}$ is bijective for all $\tau \in (t, 1]$. Define $\bar{l} = \theta \circ l$. Then $\bar{l}\mid_{(t,1]}$ gives a nonsingular curve in N.

For each $\tau \in (t, 1]$, we can find an open neighborhood O_τ of $l(\tau)$ such that $\theta\mid_{O_\tau}$ is a C^∞ diffeomorphism. It is well-known that $\theta\mid_{O_\tau}$ sends $\nabla\mid_{O_\tau}$ to ∇' iff its induced map on $\mathcal{F}O_\tau$ sends the horizontal distribution of ∇ to that of ∇', where $\mathcal{F}O_\tau$ denotes the frame bundle of O_τ. Since $V \cap O_\tau$ is dense in O_τ and $\theta\mid_V$ is affine, then we deduce that θ is affine on O_τ.

For each $u \in T_{l(t)}M$ we denote by $J(s)$ its parallel transport along l (on $[-1, 1]$). Then $\bar{J} = \theta_*(J)$ is parallel along $\bar{l}\mid_{(t,1]}$. Take a chart in a neighborhood of $\bar{l}(t)$ and consider the following linear system:

$$\dot{I}^i + \bar{\Gamma}^i_{kl}\dot{x}^k I^l = 0,\ \forall\ i,$$

where $\{\bar{\Gamma}^i_{kl}\}$ denote the Christoffel symbols of ∇' along $\bar{l}$ and $\{x^k\}$ denote the coordinates of $\bar{l}$. Fix ϵ such that $(t - \epsilon, t + \epsilon) \subseteq [-1, 1]$ and the system above is defined. Take $t_1 = t + \frac{\epsilon}{2}$. Then there exists a unique solution $\{I^i\}$ of system above defined on $(t-\epsilon, t+\epsilon)$ and satisfying $I^i(t_1) = \bar{J}^i(t_1)$ for all i. Thus by some classical results concerning linear differential systems, we get $I\mid_{(t,t+\epsilon)} = \bar{J}\mid_{(t,t+\epsilon)}$ and I vanishes nowhere on $(t-\epsilon, t+\epsilon)$. So we get

$$\theta_*(u) = \lim_{s \to t+} \bar{J}(s) = I(t) \neq 0.$$

Thus $(D\theta)_{l(t)}$ is bijective and $t > inf(A)$, which is a contradiction.

We deduce that $t = 0$. Thus θ is a local diffeomorphism on M. Now it is easy to see as above that θ sends ∇ to ∇'. $\square$

Since $\widetilde{g}$ is locally homogeneous by Lemma 7.9, then we get a C^∞ developing map $\theta : \widetilde{M} \to \bar{M}$. Since by Lemma 7.7, g is complete with respect to the geodesic structure in example (2) of Subsection 2.5.2, then θ is a surjective diffeomorphism by Proposition 2.4. So we get the following

Proposition 7.1 – *Suppose that ϕ_t is a C^∞ topologically mixing affine Anosov-smooth flow such that E^+ and E^- are both orientable. If its Bowen-Margulis measure is lebesgue, then ϕ_t preserves a C^∞ canonical linear connection ∇ such that $I(\widetilde{g})$ acts transitively on $\widetilde{M}$, where $\widetilde{g}$ denotes the lift on $\widetilde{M}$ of $g = (X, E^+, E^-, \nabla, \nu)$.*

See Section 4.4 for the definition of canonical connections. We deduce from Proposition 7.1 and Corollary 7.1 the following

Theorem 7.1 – *Let ϕ_t be a C^∞ volume-preserving affine Anosov-smooth flow on M. Denote by $\hat{\phi}_t$ its Parry time change. Then we have the following alternatives :*
(1) *Up to a constant change of time scale and finite covers, ϕ_t is C^∞ flow equivalent to the suspension of a hyperbolic nilautomorphism.*
(2) *$\hat{\phi}_t$ is topologically mixing and there exists a Lie group G containing the fundamental group Γ of M as a discrete subgroup, a closed subgroup H of G and a vector α in the Lie algebra of G, such that $\hat{\phi}_t$ is C^∞ flow equivalent to the flow $\psi_t : \Gamma\diagdown G\diagup H \to \Gamma\diagdown G\diagup H$ given by $\psi_t(\Gamma g H) = \Gamma(g \cdot exp(t\alpha))H$.*

We stop our Anosov-smooth journey here to take a little break.......

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