Etude quantitative des ensembles semi-pfaffiens

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N° d'Ordre : 2991

T H È S E

Présentée

DEVANT L'UNIVERSITÉ DE RENNES I

pour obtenir

le grade de DOCTEUR DE L'UNIVERSITÉ DE RENNES I

Mention Mathématiques et Applications

par

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TITRE DE LA THÈSE :

Étude quantitative des ensembles semi-pfaffiens

Soutenue le 12 décembre 2003 devant la Commission d'Examen

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Étude quantitative des ensembles semi-pfaffiens

Résumé

Dans la présente thèse, on établit des bornes supérieures sur les nombres de Betti des ensembles définis à l'aide de fonctions pfaffiennes, en fonction de la complexité pfaffienne (ou format) de ces ensembles.

Les fonctions pfaffiennes ont été définies par Khovanskii, comme solutions au comportement quasi-polynomial de certains systèmes polynomiaux d'équations différentielles. Les ensembles semi-pfaffiens satisfont une condition de signe booléene sur des fonctions pfaffiennes, et les ensembles sous-pfaffiens sont projections de semi-pfaffiens. Wilkie a démontré que les fonctions pfaffiennes engendrent une structure o-minimale, et Gabrielov a montré que cette structure pouvait être efficacement décrite par des ensembles pfaffiens limites.

À l'aide de la théorie de Morse, de déformations, de recurrences sur le niveau combinatoire et de suites spectrales, on donne dans cette thèse des bornes effectives pour toutes les catégories d'ensembles pré-citées.

Quantitative study of semi-Pfaffian sets

Abstract

In the present thesis, we establish upper-bounds on the Betti numbers of sets defined using Pfaffian functions, in terms of the natural Pfaffian complexity (or format) of those sets.

Pfaffian functions were introduced by Khovanskii, as solutions of certain polynomial differential systems that have polynomial-like behaviour over the real domain. Semi-Pfaffian sets are sets that satisfy a quantifier-free sign condition on such functions, and sub-Pfaffian sets are linear projection of semi-Pfaffian sets. Wilkie showed that Pfaffian functions generate an o-minimal structure, and Gabrielov showed that this structure could be effectively described by Pfaffian limit sets.

Using Morse theory, deformations, recursion on combinatorial levels and a spectral sequence associated to continuous surjections, we give in this thesis effective estimates for sets belonging to all of the above classes.

Contents

Index of notations

$\mathbb{N}$	the natural numbers. $\mathbb{N} = \{0, 1, 2, \cdots\}$
$\mathbb{R}$	the field of real numbers
$\boldsymbol{f}$	a Pfaffian chain $\boldsymbol{f} = (f_1, \dots, f_\ell)$
ℓ	the length of $\boldsymbol{f}$
α	the degree of $\boldsymbol{f}$
$\mathcal{U}$	a domain where $\boldsymbol{f}$ is defined
γ	the degree of $\mathcal{U}$, a measure of its topological complexity
q	a Pfaffian function in the chain $\boldsymbol{f}$; $q(x) = Q(x, f_1(x), \dots, f_\ell(x))$
$\deg_{\boldsymbol{f}} q$	the degree of q in the chain $\boldsymbol{f}$
$\mathcal{P}$	a finite collection of Pfaffian functions
β	a bound on $\deg_{\boldsymbol{f}} q$ for $q \in \mathcal{P}$
$\lvert x \rvert$	the Euclidean norm of x
$\overline{A}$	the closure of A for the Euclidean topology
∂A	the frontier of A, $\partial A = \overline{A} \setminus A$
$\Gamma(f)$	the graph of $f : \Gamma(f) = \{(x, y) \mid y = f(x)\}$
$\lambda \ll 1$	abbreviates $\exists \lambda_0, \forall \lambda \in (0, \lambda_0) \dots$
$H_i(X)$	the i-th singular homology group of X, with coefficients in $\mathbb{Z}$
$b_i(X)$	the i-th Betti number of X, $b_i(X) = \operatorname{rank} H_i(X)$
$b(X)$	the sum of the Betti numbers of X
X_λ	the fiber of X at $\lambda : X_\lambda = \{x \mid (x, \lambda) \in X\}$

Remerciements

De nombreuses personnes ont aidé à l'émergence de cette thèse.

Avant tout, je voudrais remercier Marie-Françoise Roy et Andrei Gabrielov pour avoir bien voulu dirriger mes travaux. Cela leur a sans doute demandé beaucoup de patience et de compréhension, et je leur suis très reconnaissant pour l'aide qu'ils m'ont apportée. Il va sans dire que cette thèse n'aurait jamais vu le jour sans eux.

Merci à Krysztof Kurdyka et à Jean-Philippe Rolin pour avoir accepté la tâche ingrate de rapporter cette thèse, et à Michel Coste et Jean-Marie Lion d'avoir accepté de figurer dans le jury. J'en profite aussi pour remercier Alex Eremenko, Leonard Lipshitz et Kenji Matsuki d'avoir bien voulu faire partie de mon commité de thèse à Purdue. L'aide spontanée de Mohab Safey El Din et d'Éric Schost a été cruciale pour faire la chasse aux fautes de frappes. Nul doute qu'il en subsiste malgré leur lecture attentive.

De nombreuses personnes ont eu l'amabilité de s'intéresser à ces travaux, de répondre à mes questions et plus généralement de m'encourager. Parmis celles-ci, je tiens à remercier, en plus des pré-citées, Ricky Pollack, Saugata Basu, Dan Richardson et Patrick Speissegger.

Je tiens aussi à remercier les personnels administratifs de Rennes, à l'IRMAR comme à la scolarité, qui ont toujours été prêts à m'aider à m'extraire des situations administrative les plus obscures.

Le *Mathematical Sciences Research Institute* de Berkeley, Californie, et le Ministère des Affaires Étrangères sont les deux institutions qui ont contribué à ce qu'une grande partie de ces travaux a été effectuée à l'étranger. Qu'ils en soient remerciés.

Enfin, je tiens à exprimer ma gratitude à mes parents et ceux qui, des deux côtés de l'Atlantique, m'ont donné leur amitié et leur soutien. Un merci particulier à Schmürtz, Spag, Mohab, Bob, Keumar, Délégator et les autres pour m'avoir supporté aussi longtemps. Nul doute que sans eux, cette thèse aurait été achevée beaucoup plus tôt.

Et merci à Inès pour la cravate…

Introduction

> *Je n'ai jamais été assez loin pour bien sentir l'application de l'algèbre la géométrie. Je n'aimais point cette manière d'opérer sans voir ce qu'on fait; et il me semblait que résoudre un problème de géométrie par les équations, c'était jouer un air en tournant une manivelle.*
>
> J.-J. Rousseau (*Les Confessions,* Livre VI)

Origine des fonctions pfaffiennes

Les fonctions pfaffiennes ont été définies par Khovanskii [Kh1, Kh2, Kh3] à la fin des années soixante-dix. Ce sont des fonctions analytiques réelles aux **propriétés de finitude** similaires aux polynômes. Si $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ sont des fonctions analytiques definies sur un domaine $\mathcal{U} \subseteq \mathbb{R}^n$, on dit qu'elles forment une **chaîne pfaffienne** sur $\mathcal{U}$ s'il existe des polynômes à coefficients réels $P_{i,j}$ tels que le système différentiel **triangulaire** suivant soit vérifié pour tout $x \in \mathcal{U}$.

$$df_i(x) = \sum_{j=1}^{n} P_{i,j}(x, f_1(x), \ldots, f_i(x))\, dx_j; \qquad \text{pour tout } 1 \leq i \leq \ell.$$

(Le système est bien triangulaire, puisque pour tout i, df_i ne dépend que des fonctions $f_1, \ldots, f_i$.)

Plus généralement, une **fonction pfaffienne** est une fonction analytique réelle q qui peut s'écrire sous la forme $q(x) = Q(x, f_1(x), \ldots, f_\ell(x))$, où Q est un polynôme et $(f_1, \ldots, f_\ell)$ une chaîne pfaffienne quelconque. Ces fonctions forment une classe importante qui comprend, en particulier, toutes les fonctions liouvilliennes et toutes les fonctions élémentaires ne comprenant que des sinus et cosinus définis sur des intervalles bornés. Une notion plus générale de fonction pfaffienne, en utilisant des chaînes de variétés intégrales de 1-formes à coefficients polynomiaux, est proposée dans [Kh3]. Ce cadre plus général donne localement les mêmes fonctions, aussi nous nous permettrons de conserver la première définition, qui est plus adaptée à notre travail, sans restreindre la portée de nos résultats.

Le résultat principal de la théorie des fonctions pfaffiennes est le suivant : tout système de n équations pfaffiennes en n variables $q_1(x) = \cdots = q_n(x) = 0$ n'a qu'un nombre fini de solutions réelles **non dégénérées**, c'est-à-dire des solutions où le déterminant jacobien $|\partial q_i/\partial x_j|$ ne s'annule pas. De plus, ce nombre de solutions peut être borné par une fonction explicite des paramètres entiers (degré, nombres de variables...) du système. Ce résultat est connu sous le nom de **theorème de Khovanskii**, et c'est lui qui donne aux fonctions pfaffiennes leur comportement quasi-polynomial. Dans le cas particulier où $q_1, \ldots, q_n$ sont des polynômes dont le degré est au plus d, l'**inégalité de Bézout** affirme que le nombre de solutions non-dégénérées du système est borné par d^n, et le principe du théorème est de se réduire à cette situation en remplaçant les fonctions de la chaîne pfaffienne par des variables et en utilisant le **théorème de Rolle** pour produire les équations manquantes.

Le fait que le système différentiel que satisfait la chaîne pfaffienne est *triangulaire* est crucial pour que le théorème de Khovanskii soit vérifié. Si on retire cette restriction (on obtient alors une chaîne de **fonctions noetheriennes**), on ne peut plus espérer avoir de finitude globale[1], comme on peut le constater avec l'exemple suivant : $f_1(x) = \cos x$, $f_2(x) = \sin x$. La chaîne (f_1, f_2) est noetherienne sur $\mathbb{R}$, puisqu'elle vérifie $f_1' = -f_2$ et $f_2' = f_1$, mais l'équation $f_1(x) = 0$ a une infinité de solutions non-dégénérées sur $\mathbb{R}$.

L'étude des fonctions pfaffiennes par Khovanskii était motivée par des questions liées à la deuxième partie du **seizième problème de Hilbert.** Ce problème considère un champ de vecteur polynomial dans le plan donné par

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}.$$

La question originelle de Hilbert était de savoir combien de *cycles limites* (solutions periodiques isolées) une telle équation peut avoir, et où ils se trouvent, en fonction des degrés de P et Q. Ce problème ainsi formulé reste ouvert, et son étude a donné naissance à de nombreuses questions annexes, dont certaines faisant intervenir les fonctions pfaffiennes. En particulier, elles ont joué un rôle clef dans la solution du **problème de Hilbert-Arnold local** pour les polycycles élémentaires. Ilyashenko et Yakovenko [IY] ont montré que le nombre de cycles limites générés par un polycycle élémentaire dans une famille de champs de vecteurs plans, lisse, générique et à k paramètres était fini, et Kaloshin [Kal] a établi une borne supérieure explicite en fonction de k. Pour plus de détails sur l'histoire du seizième problème et de ses variantes, voir [I].

Une autre application importante des fonctions pfaffiennes est la théorie des **fewnomials** (*oligonômes*), ou polynômes creux. On rappelle que pour un polynôme univarié $p(x) = \sum_{i=1}^{r} a_i x^{m_i}$ (où m_i est croissante et $a_i \neq 0$ pour tout i), la règle des signes de Descartes affirme que le nombre de racines réelles strictement positives de $p(x)$ (comptées avec multiplicité) est borné par le **nombre de changements de signes** de la suite a_i, c'est-à-dire le nombre d'indices i tels que $a_i a_{i+1} < 0$. En particulier, on peut en déduire que le nombre de zéro réels de $p(x)$ est borné par $2r - 1$, et donc est indépendant du degré m_r de p.

[1]Bien que la finitude soit toujours préservée localement, voir [GKh]

Les fonctions pfaffiennes permettent de généraliser ce type de bornes au cas de polynômes à plusieurs variables, et donnent une borne explicite qui est exponentielle en r^2 et polynomiale en le nombre de variables. Des résultats analogues existent pour des polynômes à faible **complexité additive**, la complexité additive d'un polynôme étant, informellement, le nombre minimum d'additions nécessaires pour évaluer ce polynôme. Ainsi, le polynôme $f(x) = (1 + x^p + x^q)^r$ a une complexité additive de 2 quelles que soient les valeurs des paramètres p, q, r.[2]

Topologie modérée des ensembles pfaffiens

Ce comportement quasi-polynomial des fonctions pfaffiennes entraîne que les propriétés des **ensembles pfaffiens**, les ensembles définis à partir de ces fonctions, sont géométriquement simples, évitant les exemples les plus pathologiques de la topologie. Par exemple, ces ensembles ont une notion de dimension bien définie, qui est un entier, et leurs caractéristiques géométriques et topologiques tendent à être finies. De la même façon que l'inégalité de Bézout permet d'établir des bornes explicites pour la complexité des ensembles semi-algébriques, on peut utiliser le théorème de Khovanskii pour transformer ces résultats de finitude en résultats quantitatifs.

> **Objet de la dissertation.** – *L'objet de cette thèse est d'appliquer la remarque précédente au cas des nombres de Betti des ensembles pfaffiens.*

Un tel projet nécéssite plus que simplement retranscrire les résultats déjà connus pour les semi-algébriques dans ce nouveau cadre. En effet, les ensembles pfaffiens ont des descriptions plus variées que les semi-algébriques[3], et chaque forme apparaît naturellement dans certains contextes et requiert un traitement different. Une description détaillée de ces différentes formes se trouve dans la section suivante.

Le cadre unifiant ces différents types d'ensembles pfaffiens est la théorie des **structures o-minimales**, qui permet de manipuler ces ensembles d'une manière plus uniforme et nous évitera d'avoir à résoudre certaines questions difficiles.

O-minimalité et fonctions pfaffiennes

Pour définir la structure o-minimale associée aux fonctions pfaffiennes et son intérêt pour nous, faisons un détour par les semi-algébriques.

Les sous-ensembles semi-algébriques de $\mathbb{R}^n$ sont, par définition, les ensembles de l'algèbre booléenne SA_n générée par les ensembles de la forme $\{q > 0\}$ pour tout polynôme réel q

[2] Ici, le fait que p soit un polynôme n'est pas crucial : ces résultats s'appliquent aussi aux séries de Laurent, ou au cas où les exposants sont des réels positifs.

[3] Les semi-algébriques peuvent toujours être définies sans quantificateurs, pas les ensembles pfaffiens.

à n variables. En particulier, SA_n est stable par **intersections finies, unions finies et complémentaire.** Quand on considère ces familles pour différentes dimensions n, deux propriétés supplémentaires viennent s'ajouter. Une stabilité par **produit cartésien** qui est assez évidente: on a clairement $SA_m \times SA_n \subseteq SA_{m+n}$; on a aussi un résultat moins trivial, le *théorème de Tarski-Seidenberg*, qui dit que ces familles sont **stables par projection linéaire** : si π est la projection canonique $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$, et si $A \in SA_{m+n}$, on a $\pi(A) \in SA_n$.

Les propriétés ci-dessus font que la famille des semi-algébriques est une **structure.** En pratique, cela signifie qu'en partant de données semi-algébriques et en faisant des opérations géométriques classiques, on s'attend à ne définir que des ensembles semi-algébriques.[4]

De la même façon, on définit les sous-ensembles **semi-pfaffiens** de $\mathbb{R}^n$ comme étant les éléments de l'algèbre booléenne engendrée par les ensembles de la forme $\{q > 0\}$, où q est une fonction pfaffienne en n variables. Les ensembles semi-pfaffiens ne sont pas stables par projection, contrairement aux semi-algébriques, comme le montre un contre-exemple classique de Osgood [Osg]. La projection d'un semi-pfaffien est appelée ensemble **sous-pfaffien.** Si $X \subseteq \mathbb{R}^n$ est sous-pfaffien, on ne sait pas non plus, en général, si $\mathbb{R}^n \setminus X$ est aussi sous-pfaffien.[5]

En général, une structure est dite **o-minimale** si tous les ensembles appartenant à cette structure ont un nombre fini de composantes connexes, et donc la structure des ensembles semi-algébriques est un exemple de structure o-minimale. La notion d'o-minimalité a été proposée en logique mathématique, plus précisément en théorie des modèles [D1, D4, KPS, PS]. Les ensembles appartenant à des structures o-minimales sont connus comme ayant une **topologie modérée** [6] : les hypothèses de stabilité sous les opérations de structure et la finitude du nombre de composante connexes de tous ces ensembles suffit pour que toute structure o-minimale admette une décomposition analogue à la **décomposition algébrique cylindrique** des ensembles semi-algébriques.[7] Il s'ensuit que les propriétés géométriques et topologiques de toutes ces structures sont très similaires. En particulier, notons que pour tout ensemble élément d'une structure o-minimale quelconque, la somme de ses nombres de Betti est toujours finie.

Puisque les opérations booléennes, les projections et les produits cartésiens apparaissent naturellement en géométrie, il est logique pour nous de s'intéresser à la **structure pfaffienne** : la plus petite collection d'ensembles contenant tous les semi-pfaffiens et stable par les opérations de structure. Et puisque les fonctions pfaffiennes ont un comportement très modéré, il était naturel d'espérer que cette structure soit o-minimale. Ce fait a été prouvé par Wilkie [W2], en se basant sur des idées de Charbonnel [Ch]. Ces résultats ont

[4] À comparer avec le fait que si on part de données *algébriques*, par exemple $V \subseteq \mathbb{R}^n$ l'ensemble des zéros communs d'un idéal de polynômes, les ensembles que l'on peut définir, comme par exemple l'ensemble des points non-singuliers $V^* \subseteq V$ est toujours semi-algébrique, mais pas en général algébrique.

[5] Si c'était le cas, les ensembles sous-pfaffiens formeraient une structure, voir la Remarque 1.66.

[6] L'expression a été proposée par Grothendieck dans [Gro], et reprise dans les travaux sur l'o-minimalité.

[7] Ce résultat est le fondement de la théorie des structures o-minimales, voir [C2, D4]

été généralisés dans [KM, LR2, Sp]. L'une des difficultés est de construire cette structure pfaffienne d'une manière plus explicite que la définition donnée plus haut. N'ayant pas de théorème du complément pour les ensembles sous-pfaffiens, Wilkie a utilisé la notion de **clôture à l'infini** pour obtenir une telle construction.

Ainsi, le problème posé au départ est raisonnable : la somme des nombres de Betti est finie pour tout ensemble de la structure pfaffienne, et on peut espérer en tirer des informations quantitatives. De plus, des arguments de topologie modérée vont beaucoup aider dans la manipulation de ces ensembles. Reste une difficulté : la construction de Wilkie n'est pas idéale pour y greffer des mesures de complexité. Les **ensembles limites** développés par Gabrielov [G6] offrent une construction alternative de la structure pfaffienne dans laquelle la notion de complexité est naturelle. La construction, expliquée en détails dans le Chapitre 1, est grosso-modo la suivante : si X est une **famille semi-pfaffienne à un paramètre** $\lambda > 0$, on pose $\check{X}$ la **limite de Hausdorff** des ensembles $\overline{X_\lambda}$ quand λ tend vers zéro. Un **couple semi-pfaffien** est la donnée de deux telles familles, satisfaisant certaines conditions supplémentaires. Quand ces conditions sont satisfaites, on définit la **clôture relative** du couple (X, Y) par $(X, Y)_0 = \check{X}\backslash\check{Y}$. Gabrielov prouve dans [G6] que tout ensemble de la structure pfaffienne peut s'exprimer comme un *ensemble limite* : une réunion finie de clôtures relatives. Les ensembles semi-pfaffiens ayant une notion naturelle de complexité, les ensembles limites en ont une aussi.

Différents types d'ensembles définissables

Un ensemble appartenant à une struture donnée est dit **définissable** dans cette structure, et dans la présente thèse, *définissable* sera utilisé presque exclusivement pour la structure pfaffienne. Du point de vue de la complexité, il est nécéssaire de distinguer dans la structure pfaffienne les ensembles suivants.

- Une **variété pfaffienne** V est un ensemble défini par une condition de la forme $q_1(x) = \cdots = q_r(x) = 0$. On écrira aussi $V = \mathcal{Z}(q_1, \ldots, q_r)$.
- Un **ensemble semi-pfaffien** est donné par une condition de signe booléenne portant sur des fonctions pfaffiennes.
- Un **ensemble sous-pfaffien** est la projection linéaire d'un ensemble semi-pfaffien.
- La **clôture relative** d'un couple semi-pfaffien (X, Y) est l'ensemble $(X, Y)_0 = \check{X}\backslash\check{Y}$.
- Un **ensemble pfaffien limite** est la réunion finie de clôtures relatives.

Pour chacun de ces types d'ensembles, on peut associer une notion de complexité, que nous appelerons **format**. C'est une suite d'entiers mesurant à la fois la complexité combinatoire (nombre de fonctions pfaffiennes utilisées) et algébrique (degrés, longueur de la chaîne, nombre de variables) de la description de l'ensemble, et toutes les bornes seront exprimées en fonction de ce format.

Un traitement détaillé de toutes les notions évoquées jusqu'ici se trouve dans le Chapitre 1. Ce traitement comprend les bases de la théorie de Khovanskii, bien sûr, mais aussi des résultats essentiels sur les structures o-minimales et sur le rôle qu'elles jouent dans le présent travail.

Présentation des résultats

Comme cela a déjà été mentionné, nous voulons établir des bornes supérieures pour la somme des nombres de Betti de chaque type d'ensemble définissable dans la structure pfaffienne. Pour ce faire, la remarque fondamentale est la suivante : si $V = \mathcal{Z}(q_1, \ldots, q_r)$ est une variété pfaffienne lisse et compacte, le théorème de Khovanskii nous permet de borner la somme de ses nombres de Betti en fonction des degrés des équations définissant V. En effet, cette somme est bornée par le nombre de points critiques de n'importe quelle fonction de Morse sur V, et ces points critiques sont solutions d'un système pfaffien, faisant intervenir les dérivées partielles des fonctions q_i. Le bon comportement des ensembles pfaffiens permet de généraliser ce résultat à n'importe quelle variété pfaffienne.

Cette borne pour les variétés sera notée $\mathcal{V}(\cdots)$ [8], et tous les résultats de cette thèse peuvent s'exprimer en fonction de cette quantité $\mathcal{V}$. Pour réduire les problèmes plus compliqués à des questions de variétés, nous utiliserons des techniques de topologie algébrique, des déformations et arguments de position générale, et l'o-minimalité de la structure pfaffienne.

Topologie des ensembles semi-pfaffiens (Chapitre 2)

Le Chapitre 2 est consacré à des résultats sur les ensembles semi-pfaffiens, et le contenu est de fait très proche de l'état de l'art sur les semi-algébriques.

On commence par établir la borne $\mathcal{V}$ pour les variétés pfaffiennes suivant les idées évoquées précédemment (Théorème 2.3). Ces méthodes ont été introduites dans le cadre algébrique par Oleinik, Petrovsky Thom et Milnor [O, OP, T, M2].

Du Théorème 2.3, on déduit (Théorème 2.17) une borne sur les nombres de Betti d'un ensemble semi-pfaffien compact défini par une **formule $\mathcal{P}$-fermée**[9]. Puis, le Théorème 2.25 établit une borne sur le nombre de **cellules de signe connexes** $\mathcal{C}(\mathcal{P})$ d'une famille de fonctions pfaffiennes $\mathcal{P}$. Ce nombre borne à la fois le nombre de composantes connexes d'un ensemble semi-pfaffien quelconque et le nombre de conditions de signe simultanées compatibles sur $\mathcal{P}$. On déduit de cela une borne sur le rang de **l'homologie de Borel-Moore** de tout ensemble semi-pfaffien localement fermé (Théorème 2.32). En particulier,

[8]où $\cdots$ est le format de la variété V, voir Definition 1.4.

[9]Si $\mathcal{P} = \{p_1, \ldots, p_s\}$ est un ensemble de fonctions pfaffiennes, une condition de signe $\mathcal{P}$-fermée est obtenue par conjonction et disjonction (mais **pas par négation**) d'atomes de la forme $\{p_i \geq 0\}$, $\{p_i \leq 0\}$ ou $\{p_i = 0\}$ pour $1 \leq i \leq s$.

on en déduit une borne sur les nombres de Betti de tout ensemble semi-pfaffien compact, même s'il n'est pas défini par une formule $\mathcal{P}$-fermée.

De tels résultats étaient déjà connus pour les semi-algébriques, voir par exemple [B2, BPR1, BPR3, Bürg, MMP, Yao].

Une constante du Chapitre 2 est l'utilisation de récurrences sur le **niveau combinatoire** d'une famille de fonctions $\mathcal{P}$. Cette notion apparaît dans les articles de Basu, Pollack et Roy; voir par exemple [BPR3]. Le niveau combinatoire de $\mathcal{P}$ est défini comme le plus grand entier m tel qu'il existe m fonctions distinctes dans $\mathcal{P}$ ayant un zéro commun. En se ramenant à des problèmes en position générale par déformation, on n'a pas besoin de considérer des niveau combinatoires supérieurs à la dimension de l'espace ambiant, et un niveau combinatoire de z'ero correspond à des variétés, et permet donc d'exprimer les résultats en fonction de $\mathcal{V}$. On obtient ainsi des bornes exactes [10], en plus des bornes asymptotiques.

Le Chapitre 2 s'achève par une récente application du Théorème 2.17 (donnée sans preuve, simplement pour référence) due à Gabrielov et Vorobjov : une borne sur la somme des nombres de Betti d'un ensemble semi-pfaffien arbitraire (sans hypothèse ni topologique ni sur la forme de la condition de signe définissant l'ensemble). Une borne simplement exponentielle dans ce cas n'était pas connue même dans le cas semi-algébrique.

Nombres de Betti des sensembles sous-pfaffiens (Chapitre 3)

Dans le Chapitre 3, nous présentons une **suite spectrale** $E^r_{p,q}$ qui existe pour toute surjection continue $f : X \to Y$. On prouve que cette suite converge vers l'homologie singulière de Y, $H_*(Y)$ pour toute f qui est un **recouvrement par les compacts**, c'est-à-dire si pour tout compact $L \subseteq Y$, on peut trouver $K \subseteq X$ compact aussi tel que $f(K) = L$. Une construction analogue est bien connue dans le cadre de la cohomologie des faisceaux, sous le nom de *descente cohomologique* (voir [Del]). Le terme $E^1_{p,q}$ est isomorphe à $H_q(W^p)$, où W^p est la $(p+1)$-ème puissance fibrée de X sur Y;

$$W^p = \underbrace{X \times_Y \cdots \times_Y X}_{(p+1)\ \text{termes}} = \{(\mathbf{x}_0, \dots, \mathbf{x}_p) \mid f(\mathbf{x}_0) = \cdots = f(\mathbf{x}_p)\}.$$

Ce résultat permet de donner une estimation des nombres de Betti de Y en fonction de ceux des ensembles W^p(Théorème 3.1).

Le Chapitre 3 applique ces résultats dans le cas où f est la projection d'un sous-ensemble semi-pfaffien X d'un cube, avec X ouvert ou fermé. Dans ce cas, la suite spectrale est convergente, et les produits fibrés correspondants $X \times_Y \cdots \times_Y X$ étant semi-pfaffiens, on peut utiliser les résultats du Chapitre 2 pour borner les nombres de Betti du **sous-pfaffien**

[10] *i.e.* qui ne dépendent pas de constantes inconnues.

$Y = f(X)$ (Théorème 3.20). On en déduit par dualité une borne pour un ensemble défini par un quantificateur **universel** (Corollaire 3.21).

Ces résultats peuvent être appliqués récursivement pour établir des bornes pour des sous-ensembles du cube définis par ν **blocs de quantificateurs** alternés. Les résultats ainsi obtenus (Corollaire 3.25 dans le cas pfaffien et Corollaire 3.26 dans le cas algébrique) améliorent les résultats connus quand ν est petit.

Topologie des ensembles limites (Chapitres 4 et 5)

Les chapitres 4 et 5 sont consacrés à l'étude des ensembles limites.

Le Chapitre 4 établit une borne simplement exponentielle sur le **nombre de composantes connexes** de la clôture relative $(X, Y)_0$ d'un couple pfaffien (X, Y), et par conséquent une borne pour tout ensemble limite.

Ce résultat est obtenu par l'étude des extrema locaux d'une fonction distance Φ restreinte à la fibre $X_\lambda \times Y_\lambda$ pour $\lambda \ll 1$, ce qui, dans le cas où X_λ et Y_λ sont lisses, se réduit à l'étude de la restriction de Φ à des ouverts de $X_\lambda \times (Y_\lambda)^p$, pour $1 \leq p \leq \dim(X) + 1$. La borne dans ce cas est donnée par le Théorème 4.4, et la borne dans le cas singulier est obtenue par déformation (Théorème 4.6).

Enfin, le Chapitre 5 s'attaque au problème des autres **nombres de Betti** des clôtures relatives. On commence par le cas $X_0 = (X, \varnothing)_0$, où la clôture est simplement la **limite de Hausdorff** des fibres compactes X_λ quand λ tend vers zéro. On établit un résultat général valable dans n'importe quelle structure o-minimale : les nombres de Betti de X_0 sont bornés par les nombres de Betti de diagonales épaissies de X_λ pour $\lambda \ll 1$. Ces diagonales étant semi-pfaffiennes dans le cas qui nous intéresse, on obtient via les résultats du Chapitre 2 des bornes explicites pour la clôture relative X_0 (Théorème 5.7).

Pour la clôture relative $(X, Y)_0$ avec Y **non vide**, on prouve que les nombres de Betti de $(X, Y)_0$ peuvent être explicitement bornés par une expression dépendant des formats de X_λ et Y_λ (Théorème 5.17). Ainsi, on confirme que c'est le format des fibres (plutôt que le format total des familles X et Y) qui mesure la complexité topologique des clôtures relatives.

La démonstration de ces résultats est basée sur la suite spectrale qui apparaît au Chapitre 3. On montre que si X est une famille définissable de compacts à un paramètre, on peut construire (de façon non-effective) une **famille de surjections continues** $f^\lambda : X_\lambda \to X_0$ pour $\lambda \ll 1$. Le cœur de la preuve est alors de montrer que les puissance fibrées venues de la suite spectrale peuvent être approximées par des diagonales épaissies.

Remarques sur les résultats obtenus

La borne donnée par le théorème de Khovanskii est généralement considérée comme très pessimiste quand le nombre ℓ de fonctions dans la chaîne pfaffienne est grand. En effet, la borne donnée par ce théorème provient de l'application de l'inégalité de Bézout dans un cas particulier où rien ne laisse penser que cette borne soit atteinte. Ainsi, toute amélioration de la borne de Khovanskii serait une amélioration de $\mathcal{V}$, et donc de tous les résultats présentés ici.

Les résultats de cette thèse apparaissent aussi dans : [Z1] pour les sections 2.1 and 2.2 du Chapitre 2, [GVZ] pour le Chapitre 3 et [GZ] pour le Chapitre 4. Les résultats du Chapitre 5 ont été acceptés pour publication sous une forme légèrement différente[11] dans [Z2]. Pour les clôtures relatives avec $Y \neq \varnothing$, un article offrant de meilleurs bornes est en préparation [Z3].

Enfin, il faut mentionner que le résultat récent de Gabrielov et Vorobjov [GV4] donnant des bornes presque optimales pour les ensembles semi-algébriques donnés par une condition de signe arbitraire est apparu alors que cette thèse était presque achevée, ne permettant pas de complètement l'incorporer dans le texte. Le résultat a cependant été utilisé quand il simplifiait substantiellement certaines difficultés techniques.

[11] L'article [Z2] généralise les résultats du Chapitre 5 à des limites de Hausdorff dans des familles à plus de un paramètre.

Chapter 1

Preliminaries

This chapter presents all the necessary background material about Pfaffian functions and o-minimal structures. The material is organized as follows. The first section introduces *Pfaffian functions* along with the bounds of Khovanskii about the number of solutions of a Pfaffian system. Section 2 deals with *semi and sub-Pfaffian sets,* and their formats; section 3 is about *o-minimal structures* on the real field and their basic geometric properties. At last, Pfaffian *limit sets* are introduced in section 4. This section finishes with some corollaries of the o-minimality of the structure of Pfaffian functions that will be widely used in the other chapters.

1.1 Pfaffian functions

In this section, we define Pfaffian functions following Khovanskii; we define the notion of complexity of Pfaffian functions and state the fundamental result in the theory: any system of Pfaffian equations has a finite number of *isolated*[1] solutions, that can be effectively estimated from above by an expression involving only the discrete parameters of the Pfaffian system (degrees, number of variables, and chain length). These parameters are often referred to as the *format* or *Pfaffian complexity* of the functions.

1.1.1 Definition and examples

Definition 1.1 (Pfaffian chain) *Let* $\boldsymbol{f} = (f_1, \dots, f_\ell)$ *be a sequence of real analytic functions defined on a domain* $\mathcal{U} \subseteq \mathbb{R}^n$. *We say that they constitute a* Pfaffian chain *if there exists polynomials* $P_{i,j}$, *each in* $n+i$ *indeterminates, such that the following equations*

$$\frac{\partial f_i}{\partial x_j}(x) = P_{i,j}(x, f_1(x), \dots, f_i(x)), \quad 1 \leq i \leq \ell,\ 1 \leq j \leq n, \tag{1.1}$$

hold for all $x \in \mathcal{U}$.

[1]Real soltions isolated over $\mathbb{C}$, that is.

This definition is sufficient when considering functions that are all simultaneously defined. However, in all generality, one should use the following definition.

Definition 1.2 (Pfaffian chain 2) *A sequence $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ of analytic functions in $\mathcal{U}$ is called a* Pfaffian chain *if it satisfies on $\mathcal{U}$ a differential system of the form:*

$$df_i = \sum_{j=1}^{n} P_{i,j}(x, f_1(x), \ldots, f_i(x))dx_j, \tag{1.2}$$

where each $P_{i,j}$ is a polynomial in $n + i$ indeterminates, and the following holds.

(P1) *The graph $\Gamma_i = \{t = f_i(x)\}$ of f_i is contained in a domain Ω_i defined by polynomial inequalities in $(x, f_1(x), \ldots, f_{i-1}(x), t)$, and such that $\partial\Gamma_i \subseteq \partial\Omega_i$.*

(P2) *Γ_i is a* separating submanifold *in Ω_i,* i.e. *$\Omega_i \setminus \Gamma_i$ is a disjoint union of the two sets $\Omega_i^+ = \{f_i > 0\}$ and $\Omega_i^- = \{f_i < 0\}$. (See [Kh3, p. 38]. This is also called the* Rolle leaf *condition in the terminology of [LR1, LR2].)*

Definition 1.3 (Pfaffian function) *Let $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ be a fixed Pfaffian chain, and $q(x)$ be an analytic function on the domain of that chain. We say that q is* a Pfaffian function in the chain $\boldsymbol{f}$ *if there exists a polynomial Q such that $q = Q(x, \boldsymbol{f})$,* i.e.

$$q(x) = Q(x, f_1(x), \ldots, f_\ell(x)) \quad \forall x \in \mathcal{U}. \tag{1.3}$$

Definition 1.4 (Format) *Let $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ be a Pfaffian chain. We call ℓ the* length *(also called* depth *or* order*) of $\boldsymbol{f}$. We say $\boldsymbol{f}$ is of* degree α *if all the polynomials $P_{i,j}$ appearing in* (1.1) *are of degree at most α. If Q is a polynomial of degree β in $n+\ell$ variables and $q = Q(x, \boldsymbol{f})$, we say that β is the* degree of q in $\boldsymbol{f}$, *and we will write $\beta = \deg_{\boldsymbol{f}}(q)$.*

Examples of Pfaffian functions

1. The polynomials are the Pfaffian functions such that $\ell = 0$.

2. The exponential function $f_1(x) = e^x$ is Pfaffian, with $\ell = 1$ and $\alpha = 1$, because of the equation $f_1' = f_1$. More generally, we can define the iterated exponential functions by the induction $f_r(x) = \exp(f_{r-1}(x))$ for all x. Then, $(f_1, \ldots, f_r)$ is a Pfaffian chain of length r and degree r for all r, since $f_r' = f_{r-1}' f_r = f_1 \cdots f_r$ (by induction).

3. Let $\mathcal{U} = \mathbb{R}\backslash\{0\}$, and let $f(x) = x^{-1}$ and $g(x) = \ln|x|$. Then, (f, g) is a Pfaffian chain of degree $\alpha = 2$ on $\mathcal{U}$, since we have $f' = -f^2$ and $g' = f$.

4. Let $f(x) = (x^2 + 1)^{-1}$ and $g(x) = \arctan x$. Then, (f, g) is a Pfaffian chain of degree $\alpha = 3$ on $\mathbb{R}$ since we have $f' = -2xf$ and $g' = f$.

5. Let $f(x) = \tan x$ and $g(x) = \cos^2 x$. We have $f' = 1 + f^2$ and $g' = -fg$, so (f, g) is a Pfaffian chain of degree $\alpha = 2$ on the domain $\{x \not\equiv \frac{\pi}{2}\,[\mathrm{mod}\,\pi]\}$. The function $h(x) = \cos(2x)$ is Pfaffian in this chain, since we have $h(x) = 2\,g(x) - 1$.

6. Let $m \geq 2$ be an integer, and f and g be as above. Then, the previous example shows that $(f(x/2m), g(x/2m))$ is a Pfaffian chain on the domain $\{x \not\equiv m\pi\,[\mathrm{mod}\,2m\pi]\}$. Then $\cos x$ is a Pfaffian function of degree m in that chain, since $\cos x$ is a polynomial of degree m in $\cos(x/m) = 2\,g(x/2m) - 1$.

7. $f(x) = \cos x$ is not Pfaffian on the whole real line, since $f(x) = 0$ has infinitely many isolated solutions (see Theorem 1.10).

More generally, if we consider the following functions (in any finite number of variables): polynomials, exponentials, trigonometric functions and their composition inverses wherever applicable. Then, the *real elementary functions* is the class obtained from these by taking the closure under arithmetical operations and composition. If f is in this class and the functions sin and cos appear in f only through their restriction to *bounded* intervals, the f is Pfaffian on its domain of definition (See [Kh3, §1]).

Still, one of the most important applications of Pfaffian functions is to polynomials themselves, and more specifically to the so-called *fewnomials.*

Definition 1.5 (Fewnomials) *Fix $\mathcal{K} = \{m_1, \dots, m_r\} \in \mathbb{N}^n$ a set of exponents. The polynomial q is a $\mathcal{K}$-*fewnomial *if it is of the form:*

$$q(x) = Q(x^{m_1}, \dots, x^{m_r}),$$

where Q is a polynomial in r variables. If $\beta = \deg(Q)$, we say that q has pseudo-degree β *in $\mathcal{K}$.*

Let $\ell = n + r$, and $\boldsymbol{f} = (f_1, \dots, f_\ell)$ be the chain defined by:

$$f_i(x) = \begin{cases} x_i^{-1} \text{ if } 1 \leq i \leq n, \\ x^{m_{i-n}} \text{ if } i > n. \end{cases} \tag{1.4}$$

It is easy to see that $\boldsymbol{f}$ is a Pfaffian chain of length ℓ and degree $\alpha = 2$ in the domain $\mathcal{U} = \{x_1 \cdots x_n \neq 0\}$, since we have:

$$\frac{\partial f_i}{\partial x_j} = \begin{cases} -f_i^2 \text{ if } i = j \leq n, \\ f_j f_i \text{ if } i > n. \end{cases}$$

Then, a $\mathcal{K}$-fewnomial q can be seen as a Pfaffian function in $\boldsymbol{f}$, with $\deg_{\boldsymbol{f}} q$ equal to the pseudo-degree of q, but its format is completely independent of the usual *degree* of q. This fact will enable us to generalize the well-known consequence of Descartes's rule: a univariate polynomial with m non-zero monomials has at most $m - 1$ positive roots.

Remark 1.6 *This is not necessarily the best way to see a $\mathcal{K}$-fewnomial as a Pfaffian function. On the first quadrant $(\mathbb{R}_+)^n$, one can make the change of variables $t_i = \log x_i$, and questions about $\mathcal{K}$-fewnomials can thus be reduced to questions about Pfaffian functions in the chain $(e^{m_1 \cdot t}, \dots, e^{m_r \cdot t})$, which is of length only r compared to the chain (1.4) which has length $n + r$.*

These considerations about fewnomials can be generalized in many ways: we do not need the exponents to be integers, and we can consider functions in these chains of degree larger than one. Though the number of monomials of such a function may depend on the values of $m_1, \dots, m_r$ they are still well-behaved. More generally, one can consider the additive complexity of polynomials.

Definition 1.7 (Additive complexity [BR, Kh3]) *Let $m \in \mathbb{N}^n$ and $c \in \mathbb{R}\backslash\{0\}$. Then, the polynomial $c + x^m$ is said to have* additive complexity *1. If q is a polynomial, we say its* additive complexity is bounded *by $k+1$ if $q(x) = c + x^{m_0} p_1(x)^{m_1} \cdots p_k(x)^{m_k}$, where $m_0 \in \mathbb{N}^n$, and for all $1 \leq i \leq k$, $m_i \in \mathbb{N}$ and p_i is a polynomial of additive complexity bounded by i.*

In particular, if p has an additive complexity bounded by k, it means that it can be evaluated using at most k additions. Since a function of the form $p(x)^m$ is Pfaffian with a complexity independent of m on the domain $\{p(x) \neq 0\}$, so this notion can be approached from the point of view of Pfaffian functions. Such an approach yields explicit bounds on the number of roots of such polynomials (see Theorem 1.13).

Proposition 1.8 *Let $\boldsymbol{f} = (f_1, \dots, f_\ell)$ be a Pfaffian chain on a domain $\mathcal{U} \subseteq \mathbb{R}^n$. Then, the algebra generated by $\boldsymbol{f}$ is stable under differentiation. Moreover, the degree in $\boldsymbol{f}$ of the sum, product, and partial derivatives of functions from this algebra can be estimated in terms of the format of the original functions.*

Proof: Let $g = G(x, \boldsymbol{f})$ and $h = H(x, \boldsymbol{f})$ be two functions from the algebra generated by $\boldsymbol{f}$, with $\deg(G) = \beta_1$ and $\deg(H) = \beta_2$. We have

$$(g+h)(x) = G(x, f_1(x), \dots, f_\ell(x)) + H(x, f_1(x), \dots, f_\ell(x)),$$

so $g + h$ is in the algebra generated by $\boldsymbol{f}$, and we have $\deg_{\boldsymbol{f}}(g+h) \leq \max(\beta_1, \beta_2)$.

Similarly, we have

$$(gh)(x) = G(x, f_1(x), \dots, f_\ell(x))\, H(x, f_1(x), \dots, f_\ell(x)),$$

so gh is in the algebra generated by $\boldsymbol{f}$ with $\deg_{\boldsymbol{f}}(gh) = \beta_1 + \beta_2$.

At last, we have by the chain rule,

$$\frac{\partial g}{\partial x_j}(x) = \frac{\partial G}{\partial X_j}(x, \boldsymbol{f}(x)) + \sum_{k=1}^{\ell} \frac{\partial G}{\partial Y_k}(x, \boldsymbol{f}(x))\, P_{k,j}(x, \boldsymbol{f}(x)).$$

The stability under derivation of the algebra generated by $\boldsymbol{f}$ follows. If the degree of the chain $\boldsymbol{f}$ is α, the degree of any first-order derivative of g is bounded by $\alpha + \beta_1 - 1$. □

Remark 1.9 *If $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are two Pfaffian chains defined on the same domain $\mathcal{U}$, of length respectively ℓ_1 and ℓ_2 and degree α_1 and α_2, the concatenation of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ gives a new Pfaffian chain $\boldsymbol{f}$ of length at most $\ell_1 + \ell_2$ and degree* $\max(\alpha_1, \alpha_2)$. *Thus, we can always work in the algebra generated by a fixed chain $\boldsymbol{f}$.*

1.1.2 Khovanskii's theorem

The fundamental result about Pfaffian functions is the following theorem.

Theorem 1.10 (Khovanskii) *Let $\boldsymbol{f}$ be a Pfaffian chain of length ℓ and degree α, with domain $\mathbb{R}^n$. Let $Q_1, \dots, Q_n$ be polynomials in $n+\ell$ variables of degrees respectively $\beta_1, \dots, \beta_n$, and let for all $1 \le i \le n$, $q_i(x) = Q_i(x, \boldsymbol{f})$. Then the number of solutions of the system*

$$q_1(x) = \cdots = q_n(x) = 0, \tag{1.5}$$

that are isolated in $\mathbb{C}^n$ is bounded from above by

$$2^{\ell(\ell-1)/2} \beta_1 \cdots \beta_n \, (\beta_1 + \cdots + \beta_n - n + \min(n, \ell)\alpha + 1)^{\ell}. \tag{1.6}$$

The above bound can be found in [Kh3, §3.12, Corollary 5]. It also holds when the domain of the functions is the quadrant $(\mathbb{R}_+)^n$. Over $\mathbb{C}^n$, the result is of course not true, since e^x is a Pfaffian function. The complex analogue of the above result is a bound on the multiplicity of the root of a system of complex Pfaffian functions [G2] (see also [GKh]).

Roughly, the method of proof is the following: one has to replace inductively the functions $f_i(x)$ by variables y_i, starting from $f_\ell(x)$. At each step, a Rolle-type argument allows to produce an extra polynomial Q_{n+i} so that the system

$$Q_j(x, f_1(x), \dots, f_{i-1}(x), y_i, \dots, y_\ell) = 0, \quad 1 \le j \le n+i;$$

has at least as many isolated solutions as the original system. Thus, after replacing $f_1(x)$, one obtains a system of $n + \ell$ polynomial equations in $n + \ell$ unknowns. The degrees of the polynomials $Q_{n+1}, \dots, Q_{n+\ell}$ can be effectively estimated, and by Bézout's theorem, the number of isolated solutions of the final system can be bounded.

Remark 1.11 *In [Kh3], Theorem 1.10 is formulated as a bound on the number of* non-degenerate *roots of the system* (1.5). *If q is the map $(q_1, \dots, q_n)$, the number of non-degenerate roots of the system is simply the number of points x in the preimage $q^{-1}(0)$ for which the rank of $dq(x)$ is maximal. The two formulations are clearly equivalent.*

Considering systems defined by sparse polynomials involving r non-zero monomials in the positive quadrant, one can use the change of variables $t_i = \log x_i$, – as explained in Remark 1.6, – to reduce the problem to a problem about systems involving r exponential functions. One can then bound the number of non-degenerate solutions independently of the degrees of the polynomials, to obtain the following estimate.

Corollary 1.12 (Fewnomial systems) *Let $q_1, \dots, q_n$ be polynomials in n variables such that r monomials appear with a non-zero coefficient in at least one of these polynomials. Then, the number of non-degenerate solutions of the system*

$$q_1(x) = \cdots = q_n(x) = 0,$$

in the quadrant $(\mathbb{R}_+)^n$ is bounded by

$$2^{r(r-1)/2}\,(n+1)^r. \tag{1.7}$$

For systems defined by polynomials of additive complexity bounded by k, (see Definition 1.7), there is a detailed proof in [BR, Chapter 4] that the number of non-degenerate roots admits a computable upper-bound in terms of k. In the case $n = 1$, the following explicit bound is given [BR, Theorem 4.2.4].

Theorem 1.13 (Bounded additive complexity) *Let $p(x)$ be a univariate polynomial of additive complexity bounded by k. The number of real roots of p is at most*

$$(k+2)^{2k+1}\,2^{2k^2+2k+1};$$

which is less than 5^{k^2} for k large enough.

1.1.3 Domains of bounded complexity

We will now define a class of domains $\mathcal{U}$ over which Khovanskii's result can be easily generalized. Note that in order to have the nice topological and geometrical properties we hope for, one cannot generalize these results to domains that would be too pathological.

Definition 1.14 (Domain of bounded complexity) *We say that $\mathcal{U}$ is a domain of bounded complexity γ for the Pfaffian chain $\boldsymbol{f} = (f_1, \dots, f_\ell)$ if there exists a function g of degree γ in the chain $\boldsymbol{f}$ such that the sets $\{g \geq \varepsilon\}$ form an exhausting family of compact subsets of $\mathcal{U}$ for $\varepsilon \ll 1$. We call g an* exhausting function *for $\mathcal{U}$.*

Example 1.15 *Let $\boldsymbol{f} = (f_1, \dots, f_\ell)$ be a Pfaffian chain defined on a* **bounded** *domain $\mathcal{U}$ of the form*

$$\mathcal{U} = \{x \in \mathbb{R}^n \mid g_1(x) > 0, \dots, g_r(x) > 0\}, \tag{1.8}$$

where $(g_1, \dots, g_r)$ are Pfaffian functions in the chain $\boldsymbol{f}$. Then, $\mathcal{U}$ is a domain of bounded complexity, since $g = g_1 \cdots g_r$ is clearly an exhausting function for $\mathcal{U}$.

Note that the assumption of boundedness of $\mathcal{U}$ can be dropped: let

$$\rho(x) = \frac{1}{1+|x|^2}. \tag{1.9}$$

The function ρ is a Pfaffian function defined on $\mathbb{R}^n$, with a degree $\alpha = 3$, since we have

$$d\rho(x) = -2\rho^2(x)\,(x_1 dx_1 + \cdots + x_n dx_n).$$

Moreover, $\rho(x) > 0$ on $\mathbb{R}^n$ and the sets $\{\rho \geq \varepsilon\}$ are compact for $0 < \varepsilon < 1$. So even an **unbounded** domain $\mathcal{U}$ of the form (1.8) is a domain of bounded complexity for any Pfaffian chain of the form $(\rho, f_1, \dots, f_\ell)$, with exhausting function $g = g_1 \cdots g_r + \rho$.

Over a domain of bounded complexity, Khovanskii's estimates becomes the following.

Theorem 1.16 (Khovanskii's theorem for a domain of bounded complexity) *Let $\boldsymbol{f}$ be a Pfaffian length of degree α and length ℓ defined on a domain $\mathcal{U}$ of bounded complexity γ for $\boldsymbol{f}$. Let $Q_1, \dots, Q_n$ be polynomials in $n + \ell$ variables of degree respectively $\beta_1, \dots, \beta_n$ and let $q_i = Q(x, \boldsymbol{f})$ for all i. Then, the number of solutions in $\mathcal{U}$ of the system*

$$q_1(x) = \cdots = q_n(x) = 0; \tag{1.10}$$

which are isolated in $\mathbb{C}^n$ is bounded by

$$2^{\ell(\ell-1)/2}\beta_1 \cdots \beta_n \frac{\gamma}{2}[\beta_1 + \cdots + \beta_n + \gamma - n + \min(n+1, \ell)\alpha]^\ell \tag{1.11}$$

Proof: Introduce a new variable t and consider the system given by

$$q_1(x) = 0, \dots, q_n(x) = 0,\ g(x) - t^2 = \varepsilon; \tag{1.12}$$

where ε is a fixed positive real number. Then, for any values of ε, every isolated solution of the system (1.10) that is contained in the domain $\Omega_\varepsilon = \mathcal{U} \cap \{g(x) > \varepsilon\}$ gives rise to exactly two isolated solutions for (1.12). So it is enough to bound the number of isolated solutions of (1.12) for a value of ε such that all the isolated solutions of (1.10) are contained in Ω_ε. The choice of the parameter ε does not affect the complexity of the new system, and the bound (1.11) can then be established following the results appearing in [Kh3]. □

1.2 Semi and sub-Pfaffian sets

Semi and sub-Pfaffian sets occur naturally in the study of Pfaffian functions: semi-Pfaffian sets are sets that can be defined by a quantifier-free sign condition on Pfaffian functions, and sub-Pfaffian sets are linear projections of semi-Pfaffian sets, or equivalently, defined by existential sign conditions on Pfaffian functions.

Example 1.17 *Let q be a Pfaffian function defined on a domain $\mathcal{U} \subseteq \mathbb{R}^n$. Then, the set of critical points of q is semi-Pfaffian and the set of its critical values is sub-Pfaffian.*

Proof: This is straightforward. The critical locus of q is defined by

$$X = \left\{ x \in \mathcal{U} \mid \frac{\partial q}{\partial x_1}(x) = \cdots = \frac{\partial q}{\partial x_n}(x) = 0 \right\};$$

and the set of critical values is

$$Y = \{y \in \mathbb{R} \mid \exists\, x \in X,\ q(x) = y\}.$$

Since the partial derivatives of q are again Pfaffian functions, it is clear that X is semi-Pfaffian and Y is sub-Pfaffian. □

Semi-Pfaffian and sub-Pfaffian sets have a lot of finiteness properties. The present section contains mainly definitions and relevant examples, and we refer the reader to the bibliography [GV1, G3, G5, G2, GV2, PV] for more details. A comprehensive survey [GV3] will be available soon.

From now on, $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ will be a fixed Pfaffian chain of degree α defined on a domain of bounded complexity $\mathcal{U} \subseteq \mathbb{R}^n$, and we will consider only functions fro, the algebra generated by $\boldsymbol{f}$.

1.2.1 Semi-Pfaffian sets

As mentioned in the beginning of this section, semi-Pfaffian sets are given by quantifier-free sign conditions on Pfaffian functions. We start by recalling the definition of quantifier-free formulas, and we define a notion of *format* for such formulas. This format will be all the data we need to establish bounds on the topological complexity of semi-Pfaffian sets.

Definition 1.18 (QF formula) *Let $\mathcal{P} = \{p_1, \ldots, p_s\}$ be a set of Pfaffian functions. A* quantifier-free *(QF) formula with atoms in $\mathcal{P}$ is constructed as follows:*

1. *An atom is of the form $p_i \star 0$, where $1 \le i \le s$ and $\star \in \{=, \le, \ge\}$. It is a QF formula;*
2. *If Φ is a QF formula, its* negation *$\neg\Phi$ is a formula;*
3. *If Φ and Ψ are QF formulas, then their* conjunction *$\Phi \wedge \Psi$ and their* disjunction *$\Phi \vee \Psi$ are QF formulas.*

Definition 1.19 (Format of a formula) *Let Φ be a QF formula as above. If the number of variables is n, the length of $\boldsymbol{f}$ is ℓ, the degrees of the polynomials $P_{i,j}$ in (1.1) is bounded by α, $s = |\mathcal{P}|$ and β is the maximum of the degrees in the chain of the functions in $\mathcal{P}$, we call $(n, \ell, \alpha, \beta, s)$ the* format *of Φ.*

Definition 1.20 ($\mathcal{P}$-closed formulas) *We will say that the formula Φ is $\mathcal{P}$-closed if it was derived without negations,* i.e. *using rules 1 and 3 only.*

Definition 1.21 (Semi-Pfaffian set) *A set $X \subseteq \mathcal{U}$ is called* semi-Pfaffian *if there exists a finite set $\mathcal{P}$ of Pfaffian functions and a QF formula Φ with atoms in $\mathcal{P}$ such that*

$$X = \{x \in \mathcal{U} \mid \Phi(x)\}.$$

We call format *of X the format of the defining formula Φ.*

The notion of format is important to establish the kind of quantitative bounds we want on the topology of semi-Pfaffian sets. But the above definition can be improved.

Indeed, taking example on the algebraic case, one expects equalities and inequalities to affect very differently bounds on the topology of the sets they define. If $V = \{x \in \mathbb{R}^n \mid p_1(x) = \cdots = p_s(x) = 0\}$ is a real algebraic variety defined by polynomials of degree at most d, we know by a result of Oleinik-Petrovsky Thom and Milnor [OP, O, M2, T] that the sum of its Betti numbers is bounded by $d(2d-1)^{n-1}$, so does not depend on s.

On the other hand, when dealing with inequalities, the number s of functions does matter, as the following example shows: take $p_i(x) = (x-i)^2$ and let $S = \{x \in \mathbb{R} \mid p_1(x) > 0, \dots, p_s(x) > 0\}$. Then $S = \mathbb{R} \backslash \{1, 2, \dots, s\}$, so it has $s+1$ connected components.

To make full use of that difference between equalities and inequalities in our formulas, we will introduce the following definitions.

Definition 1.22 (Variety) *The semi-Pfaffian set $V \subseteq \mathcal{U}$ is called a* variety *if it is defined using only equations. We will use the notation*

$$\mathcal{Z}(q_1, \dots, q_r) = \{x \in \mathcal{U} \mid q_1(x) = \cdots = q_r(x) = 0\}.$$

Definition 1.23 (Semi-Pfaffian subsets of a variety) *If $V = \mathcal{Z}(q_1, \dots, q_r)$ is a Pfaffian variety and Φ a QF formula, one can consider the semi-Pfaffian set $X = \{x \in V \mid \Phi(x)\}$. Then, the* format *of X is defined as $(n, \ell, \alpha, \max(\beta_1, \beta_2), s)$ where β_1 is a bound on the degrees of $q_1, \dots, q_r$ and $(n, \ell, \alpha, \beta_2, s)$ is the format of Φ.*

Remark 1.24 *Such a cumbersome definition and notion of format may seem strange, but we will see in Chapter 2 that this will allow us to establish more precise bounds on the topology of X, for which r is irrelevant and the parameter $d = \dim(V)$ plays a part.*

A more usual definition for semi-Pfaffian sets is to define them as finite unions of *basic* semi-Pfaffian sets, where a basic set is of the form

$$B = \{x \in \mathcal{U} \mid \varphi_1(x) = \cdots = \varphi_I(x) = 0,\, \psi_1(x) > 0, \dots,\, \psi_J(x) > 0\}; \tag{1.13}$$

for some Pfaffian functions $\varphi_1, \dots, \varphi_I$ and $\psi_1, \dots, \psi_J$. (Writing as semi-Pfaffian set as a union of basic ones is just putting the defining formula Φ in disjunctive normal form, so the two definitions are clearly equivalent.)

Remark 1.25 *Semi-Pfaffian sets presented as union of basic sets occur frequently in the literature. The reader should be aware that the definition of their* format *in that case is different. The format of a basic set of the form* (1.13) *is then defined as* $(I, J, n, \ell, \alpha, \beta)$, *and if* X *is the union of* N *basic sets* $B_1, \dots, B_N$, *of respective formats* $(I_i, J_i, n, \ell, \alpha, \beta)$, *the format of* X *is* $(N, I, J, n, \ell, \alpha, \beta)$, *where* $I = \max\{I_1, \dots, I_N\}$ *and* $J = \max\{J_1, \dots, J_N\}$. *The two notions of formats are comparable.*

Definition 1.26 (Effectively non-singular set) *If* X *is a basic semi-Pfaffian set, we'll say that* X *is* effectively *non-singular if the functions* $\varphi_1, \dots, \varphi_I$ *appearing in* (1.13) *verify*

$$\forall x \in X, \quad d\varphi_1(x) \wedge \cdots \wedge d\varphi_I(x) \neq 0.$$

If X *is effectively non singular, it is a smooth submanifold of* $\mathbb{R}^n$ *of dimension* $n - I$.

Basic sets appear rather naturally because they are easier to handle algorithmically. In particular, effectively non-singular basic sets is what is used in [GV1] to produce a weak stratification algorithm (using an oracle) for semi-Pfaffian sets.

Definition 1.27 (Restricted set) *We say that a semi-Pfaffian set* X *is* restricted *if it is relatively compact in* $\mathcal{U}$.

Let us introduce now the notations we will use for the topological invariants we want to bound.

Notation 1.28 *Throughout this thesis, if* X *is a topological space,* $H_i(X)$ *will denote its* i*-th homology group with integer coefficients,* $b_i(X)$ *will be the* i*-th Betti number of* X, *which is the rank of* $H_i(X)$, *and* $b(X)$ *will denote the sum* $\sum_i b_i(X)$.

If $\mathcal{P} = \{p_1, \dots, p_s\}$ a family of Pfaffian functions, we denote by $\mathfrak{S}$ be the set of strict sign conditions on $\mathcal{P}$. If $\sigma \in \mathfrak{S}$, we have

$$\sigma(x) = p_1(x)\sigma_1 0 \wedge \cdots \wedge p_s(x)\sigma_s 0; \quad \sigma_i \in \{<, >, =\} \text{ for } 1 \leq i \leq s. \tag{1.14}$$

Then, for any fixed Pfaffian variety V, we can consider the following.

Definition 1.29 (Connected sign cells) *A* cell *of the family* $\mathcal{P}$ *on the variety* V *is a connected component of the basic semi-Pfaffian set* $S(V; \sigma) = \{x \in V \mid \sigma(x)\}$ *for some* $\sigma \in \mathfrak{S}$.

Then, we define the *number of connected sign cells* of $\mathcal{P}$ over V simply as the sum

$$\mathcal{C}(V; \mathcal{P}) = \sum_{\sigma \in \mathfrak{S}} b_0(S(V; \sigma)). \tag{1.15}$$

Remark 1.30 *In particular, for any semi-Pfaffian set* $X = \{x \in V \mid \Phi(x)\}$, *if* Φ *as atoms in* $\mathcal{P}$, *the number of connected components of* X *is bounded by* $\mathcal{C}(V;\mathcal{P})$.

Proof: Indeed, we can assume without loss of generality that Φ is in disjunctive normal form, in which case it is of the form $\sigma_1 \vee \cdots \vee \sigma_N$ for some $\sigma_1, \ldots, \sigma_N \in \mathfrak{S}$. Then, $X = S(V;\sigma_1) \cup \cdots \cup S(V;\sigma_N)$, so we have

$$b_0(X) \leq b_0(S(V;\sigma_1)) + \cdots + b_0(S(V;\sigma_N)) \leq \mathcal{C}(V;\mathcal{P}).$$

□

Of course, higher Betti numbers are not sub-additive, so a similar procedure cannot be followed in general.

Definition 1.31 (Consistent sign assignment) *Let* V *be a Pfaffian variety,* $\mathcal{P}$ *a family of Pfaffian functions, and* $\mathfrak{S}$ *the set of strict sign conditions on* $\mathcal{P}$. *Then* $\sigma \in \mathfrak{S}$ *is a* consistent sign assignment *of* $\mathcal{P}$ *on* V *if the basic set* $S(V;\sigma)$ *is not empty.*

Then, $\mathcal{C}(V;\mathcal{P})$ bounds also the number of consistent sign assignments $\sigma \in \mathfrak{S}$. Theorem 2.25 shows that for a fixed variety V, $\mathcal{C}(V;\mathcal{P})$ is a polynomial in the number s of functions in $\mathcal{P}$, and thus, the number of consistent sign assignments is asymptotically much less than the trivial bound of 3^s.

1.2.2 Sub-Pfaffian sets

Definition 1.32 *The set* $Y \subseteq \mathbb{R}^n$ *is a* sub-Pfaffian set *if there exists a semi-Pfaffian set* $X \subseteq \mathcal{U} \subseteq \mathbb{R}^{n+p}$ *such that* Y *is the image of* X *by the canonical projection* $\pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$. *Equivalently, this can be formulated by using an existential formula;*

$$Y = \{y \in \mathbb{R}^n \mid \exists x \in \mathbb{R}^p,\ (x,y) \in X\}. \tag{1.16}$$

Unlike semi-algebraic sets, semi-Pfaffian sets are not stable under projections.

Example 1.33 (Osgood [Osg]) *The following sub-Pfaffian set is not semi-Pfaffian.*

$$X = \{(x,y,z) \in \mathbb{R}^3 \mid \exists u \in [0,1], y = xu, z = xe^u\}.$$

Proof: The set X is clearly a strict subset of $\mathbb{R}^3$ that contains 0. If X is semi-Pfaffian, there exists a non-zero analytic function F that vanishes on X in a neighbourhood of 0. We can write F as a convergent series of homogeneous polynomials F_d, were $\deg(F_d) = d$. Then, we must have for all $u \in [0,1]$,

$$F(x, xu, xe^u) = \sum_{d \geq 0} x^d F_d(1, u, e^u) = 0.$$

Thus, we must have $F_d(1,u,e^u) = 0$ for all $d \geq 0$ and all $u \in [0,1]$, which implies $F_d \equiv 0$ for all $d \geq 0$. Thus, $F \equiv 0$ is the only analytic function that vanishes on X in a neighbourhood of 0. Since X is a strict subset of $\mathbb{R}^3$, it cannot be semi-Pfaffian. □

Remark 1.34 (Sub-fewnomial sets) *Let $X \subseteq \mathbb{R}^{n+p}$ be a semi-algebraic set and $Y \subseteq \mathbb{R}^n$ be its projection. By the Tarski-Seidenberg theorem, Y is certainly semi-algebraic too. But even though X may be a fewnomial set, Y is only sub-fewnomial: describing Y with a fewnomial quantifier-free formula may not always be possible.*

Example 1.35 (Gabrielov [G4]) *Consider for all $m \in \mathbb{N}$ the set*

$$Y_m = \{(x, y) \in \mathbb{R}^2 \mid \exists t \in \mathbb{R},\, t^m - xt = 1,\, (y-t)^m - x(y-t) = 1\}. \tag{1.17}$$

Then, there is no quantifier-free fewnomial formula describing Y_m having a format independent of m.

Remark 1.36 (Open problem) *If Y is a sub-Pfaffian set and Y is not subanalytic, it is not known whether its complement is also sub-Pfaffian or not. This is one of the reasons that motivates the introduction of Pfaffian limit sets in Section 1.4.*

1.3 Basic properties of o-minimal structures

In this section, we describe the main definitions and results concerning o-minimal structures. O-minimal structures appear in model theory, and provide a framework for the ideas of *tame topology* [Gro]. Many surveys are available to the reader for more details, for instance [C2, D4, DM2]. For more details about model-theoretic aspects, see also [D3, D2].

1.3.1 O-minimal expansions of the real field

Definition 1.37 (o-minimal structure) *For all $n \in \mathbb{N}$ let $\mathcal{S}_n$ be a collection of subsets of $\mathbb{R}^n$, and let $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$. We say that $\mathcal{S}$ is an* o-minimal structure on the field $\mathbb{R}$ *if the following axioms hold.*

(O1) *For all n, $\mathcal{S}_n$ is a Boolean algebra.*

(O2) *If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$.*

(O3) *If $A \in \mathcal{S}_{n+1}$, and π is the canonical projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$, then $\pi(A) \in \mathcal{S}_n$.*

(O4) *$\mathcal{S}_n$ contains all the semi-algebraic subsets of $\mathbb{R}^n$.*

(O5) *All sets in $\mathcal{S}_1$ have a finite number of connected components.*

Recall that **(O1)** means that the collections $\mathcal{S}_n$ are stable by finite intersection, finite unions and taking complements. The axioms **(O1)** through **(O4)** mean that $\mathcal{S}$ is a *structure.* Axiom **(O5)** is called the *o-minimality axiom.*

Definition 1.38 (Definability) *Let $\mathcal{S}$ be a structure. If $A \in \mathcal{S}_n$, we say that A is $\mathcal{S}$-*definable. *A map $f : A \subseteq \mathbb{R}^m \to B \subseteq \mathbb{R}^n$ is called $\mathcal{S}$-*definable *if and only if its graph belongs to $\mathcal{S}_{m+n}$.*

Example 1.39 *Let $\mathcal{S}_n$ be the set of semi-algebraic subsets of $\mathbb{R}^n$. Then, $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ is an o-minimal structure.*

Proof: Recall that a subset of $\mathbb{R}^n$ is *semi-algebraic* if it can be defined by a quantifier-free sign condition on polynomials. Then, $\mathcal{S}$ is clearly a Boolean algebra and is stable by Cartesian products. Elements of $\mathcal{S}_1$ can only have finitely many connected components since polynomials in one variable have only finitely many zeros. Thus, the only non-trivial axiom is **(O3)**: stability by projection, which is the result of the classical Tarski-Seidenberg theorem [BCR]. □

Example 1.40 *Let $\mathcal{S}_n$ be the set of* globally subanalytic sets: *subsets of $\mathbb{R}^n$ that are subanalytic in $\mathbb{R}P^n$. Then, $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ is an o-minimal structure.*

Proof: Here, $\mathcal{S}$ is stable by projections by definition, and the fact that it is a Boolean algebra follows from Gabrielov's theorem of the complement [G1]. The axioms 2 and 4 are clear, and the finiteness of the number of connected components follows from the local properties of semi-analytic sets [Loj]. □

Definition 1.41 (Generated structure) *Let $\mathcal{S}$ be a structure and $\mathcal{A} = (\mathcal{A}_n)_{n\in\mathbb{N}}$ a collection of subsets of $\mathbb{R}^n$ for all $n \in \mathbb{N}$. If the closure of $\mathcal{A}$ under the Boolean operations, Cartesian product and linear projections is $\mathcal{S}$, we say that $\mathcal{S}$ is* generated *by $\mathcal{A}$.*

For example, the structure of semi-algebraic sets is generated by the sets $\{f = 0\}$ for all polynomials f, and the structure of globally subanalytic sets is generated by all the restrictions $f|_{[-1,1]^n}$ of all the graphs of functions f that are analytic in a neighbourhood of $[-1, 1]^n$.

After the general setting of o-minimal structures was introduced, a lot of effort was put into constructing new examples. Our main interest here is the fact that o-minimal functions do generate an o-minimal structure. This fact, proved first by Wilkie in [W2], is the object of the next section. For now, let us mention two other cases that seem of particular interest.

The structure $\mathbb{R}_{\exp}$ generated by the exponential function is o-minimal [W1]. This is of special interest since it relates to Tarski's problem about the decidability of real exponentiation [MW], which was one of the problems which first motivated the introduction of o-minimal structures.

More recently, Rolin, Speissegger and Wilkie [RSW] constructed new o-minimal structures using certain quasi-analytic Denjoy-Carleman classes. This construction allowed to settle two open problems: (1) if $\mathcal{A}_1$ and $\mathcal{A}_2$ generate o-minimal structures, the structure generated by $\mathcal{A}_1 \cup \mathcal{A}_2$ is not necessarily o-minimal, (and thus, there is no 'largest' o-minimal structure), and (2) there are some o-minimal structures that do not admit analytic cell decomposition (see Theorem 1.44 and Remark 1.45).

We will now describe the main properties of o-minimal structures. Essentially, such a structure has a geometrical and topological behaviour which is very similar to what is observed in semi-algebraic sets. For the remaining of the chapter, $\mathcal{S}$ will be a fixed o-minimal structure and we will write *definable* for $\mathcal{S}$-definable. In the next chapters, definable will always mean definable in the o-minimal structure generated by Pfaffian functions.

1.3.2 The cell decomposition theorem

The cell decomposition theorem is an o-minimal analogue of the cylindrical algebraic decomposition used in real algebraic geometry [BR, BCR]. Since most features of semi-algebraic sets follow from that decomposition, they will have an equivalent for definable sets in o-minimal structures. We fix $\mathcal{S}$ an o-minimal structure.

Definition 1.42 (Cylindrical cell) *Cylindrical cells are defined by induction on the dimension of the ambient space n. A subset C of $\mathbb{R}$ is a cell if and only if it is an open interval or a point. A set $C \subseteq \mathbb{R}^n$ is a cell if and only if there exists a cell $D \subseteq \mathbb{R}^{n-1}$ such that one of the two following conditions is true.*

1. *There exists a* continuous *definable function $f : D \to \mathbb{R}$ such that C is one of the following sets,*

$$C_0(f) = \{(x', x_n) \mid x_n = f(x')\},$$
$$C_+(f) = \{(x', x_n) \mid x_n > f(x')\},$$
$$C_-(f) = \{(x', x_n) \mid x_n < f(x')\}.$$

2. *There exists* continuous *definable functions f and g from D into $\mathbb{R}$ such that $f < g$ on D and*

$$C(f, g) = \{(x', x_n) \mid f(x') < x_n < g(x')\}.$$

Definition 1.43 (Cell decomposition) *A cell decomposition of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into open intervals and points. For $n > 1$, we say that a finite set $\mathcal{C}$ of cylindrical cells of $\mathbb{R}^n$ is a* cell decomposition *of $\mathbb{R}^n$ if $\mathcal{C}$ is a partition of $\mathbb{R}^n$ such that the collection $\{\pi(C) \mid C \in \mathcal{C}\}$ is a cell decomposition of $\mathbb{R}^{n-1}$. (Here again, π is the canonical projection $\mathbb{R}^n \to \mathbb{R}^{n-1}$.)*

If $A_1, \dots, A_k$ are definable subsets of $\mathbb{R}^n$ and $\mathcal{C}$ is a cell decomposition of $\mathbb{R}^n$, the partition $\mathcal{C}$ is said to be compatible *with $A_1, \dots, A_k$ if each A_i is a finite union of cells in $\mathcal{C}$.*

Theorem 1.44 (Cell decomposition theorem) *1. Let $A_1, \ldots, A_k$ be definable subsets of $\mathbb{R}^n$. Then there exists a cell decomposition of $\mathbb{R}^n$ compatible with $A_1, \ldots, A_k$.*

2. If $f : A \to \mathbb{R}$ is definable, $A \subseteq \mathbb{R}^n$, there exists a cell decomposition of $\mathbb{R}^n$ compatible with A such that for each cell $C \subseteq A$, the restriction $f|_C$ is continuous.

Remark 1.45 (C^k cell decompositions) *The notions of cells and cell decomposition can be generalized to the C^k setting for any $k \in \mathbb{N} \cup \{\infty, \omega\}$: we can define C^k-cells by requiring that the graphs that appear in the definition of the cells are graph of C^k functions. Then, a C^k cell decomposition of $\mathbb{R}^n$ would be of course a cell decomposition where all the cells considered are C^k. For any fixed $k \in \mathbb{N}$, a C^k analogue of Theorem 1.44 holds for any o-minimal structure. Although analytic cell decomposition holds in many known cases, including the Pfaffian case [LS1, DM1, Loi1], it does not hold in general [RSW].*

As mentioned previously, Theorem 1.44 can be interpreted as a generalization of the cylindrical algebraic decomposition. However, it is worth noting that the proof of Theorem 1.44 is much more technical (it takes about ten pages in [D4]).

In the course of proving Theorem 1.44, the following theorem is necessary.

Theorem 1.46 (Monotonicity theorem) *Let $-\infty \leq a < b \leq \infty$ and let $f : (a, b) \to \mathbb{R}$ be definable. Then, there exists $a_0 = a < a_1 < \cdots < a_k = b$ such that on each interval (a_i, a_{i+1}), the function f is either constant or strictly monotonous and continuous.*

The following results are immediate corollaries of the existence of cell decomposition.

Corollary 1.47 *Any definable set has a finite number of connected components.*

Proof: Let $A \subseteq \mathbb{R}^n$ be definable and $\mathcal{C}$ be a definable cell decomposition of $\mathbb{R}^n$ compatible with A. Each cell $C \in \mathcal{C}$ is connected, so the number of connected components of A is at most the number of cells $C \in \mathcal{C}$ such that $C \subseteq A$. □

By construction, all cells in a cell decomposition are definably homeomorphic to a cube $(0, 1)^d$ for some d. If C is a cell homeomorphic to $(0, 1)^d$, we let $d = \dim(C)$. Then, it is natural to define the dimension of a definable set $A \neq \varnothing$ as the maximum of $\dim(C)$ taken over all cells C contained in A, for a given cell decomposition $\mathcal{C}$ compatible with A. Then, the following holds.

Proposition 1.48 (Dimension is well behaved) *Let A be a definable set, $A \neq \varnothing$, and $f : A \to \mathbb{R}^m$ a definable map. The dimension of A is well-defined (independent of the choice of the cell decomposition $\mathcal{C}$) and dimensions verifies the following properties.*

1. $\dim(\partial A) < \dim(A)$;

2. $\dim(f(A)) \leq \dim(A)$.

Corollary 1.49 (Uniform bound on fibers) *Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be definable, and define for all $x \in \mathbb{R}^m$ the fiber*

$$A_x = \{y \in \mathbb{R}^n \mid (x, y) \in A\}.$$

Then, there exists N such that for all $x \in \mathbb{R}^m$ the fiber A_x has at most N connected components.

Proof: Let $\mathcal{C} = \{C_i\}$ be a cell decomposition of $\mathbb{R}^m \times \mathbb{R}^n$ compatible with A. If $A = C_1 \cup \cdots \cup C_N$, we have $A_x = (C_1)_x \cup \cdots \cup (C_N)_x$. Thus, it is enough to check that each $(C_i)_x$ is connected, which is an easy induction on n. □

Proposition 1.50 (Definable choice) *Let $A \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be definable. Denote by π the canonical projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, and let $B = \pi(A)$. Then, there exists a definable $s : B \to \mathbb{R}^n$ such that its graph $\Gamma(s)$ is contained in A.*

Proof: Suppose $n = 1$, and let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^m \times \mathbb{R}$ compatible with A. Fix $x \in B$; there exists a cell $C \in \mathcal{C}$ such that $C \subseteq A$ and $x \in \pi(C)$. According to the type of the cell C, we can define $s(x)$ so that $(x, s(x)) \in C$.

- If $C = C_0(f)$, we let $s(x) = f(x)$;
- if $C = C_+(f)$, we let $s(x) = f(x) + 1$;
- if $C = C_-(f)$, we let $s(x) = f(x) - 1$;
- and if $C = C(f, g)$, we let $s(x) = (f(x) + g(x))/2$.

This solves the case $n = 1$, since s is certainly a definable function $\pi(C) \to \mathbb{R}$.

We finish the proof by induction on n. Assume the result holds up to $n - 1$, and let π' be the canonical projection $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-1}$ and $A' = \pi'(A)$. By induction, there exists $s' : B \to \mathbb{R}^{n-1}$ definable such that $\Gamma(s') \subseteq A'$. We can use the case $n = 1$ for $\Gamma(s')$ now, so there exists $s'' : \Gamma(s') \to \mathbb{R}$ such that $\Gamma(s'') \subseteq A$. The projection π restricted to $\Gamma(s'')$ must be a bijection onto B, so $\Gamma(s'')$ is the graph of a definable function $s : B \to \mathbb{R}^n$. □

Corollary 1.51 (Curve lemma) *Let $A \subseteq \mathbb{R}^n$ be definable and $a \in \partial A$. Then, there exists a definable arc $\gamma : (0, 1) \to A$ such that $\lim_{t \to 0} \gamma(t) = a$.*

Proof: Let $a \in \partial A$, and let $B = \{|x - a|, x \in A\}$. The set B is definable and since $0 \in \overline{B}$, there must be an interval $(0, \varepsilon)$ contained in B. By the definable choice theorem above, there exists a function $\gamma : t \in B \mapsto \gamma(t) \in A$ such that $|a\gamma(t)| = t$. By Theorem 1.46, γ is continuous on an interval $(0, \delta)$ for some $\delta \leq \varepsilon$, and by rescaling the variable t, we can always assume that $\delta = 1$. □

1.3.3 Geometry of definable sets

We will now list some of the deeper consequences of the o-minimality axiom. First, asymptotic behaviour of definable functions is very controlled, and there is a dichotomy between *polynomially bounded* o-minimal structures and structures where e^x is definable [Mi]. Although the usual, polynomial Łojasiewicz inequality does not hold in o-minimal structures that are not polynomially bounded, the following version does hold.

Theorem 1.52 (Generalized Łojasiewicz inequality) *Let $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ be definable functions such that $\{f = 0\} \subseteq \{g = 0\}$ and A is compact. Then, there exists a definable C^p function φ such that $|\varphi \circ g(x)| \leq |f(x)|$ for all $x \in A$.*

Remark 1.53 *For Pfaffian functions, a more explicit inequality, the* exponential *Łojasiewicz inequality, holds. See Proposition 1.75.*

Definition 1.54 (Stratification) *Let $p \geq 1$ be an integer. A C^p stratification of a set A is a partition of A into strata such that each stratum is a C^p-smooth submanifold and if X and Y are two strata such that $X \cap \overline{Y} \neq \varnothing$, then we have $X \subseteq \overline{Y}$.*

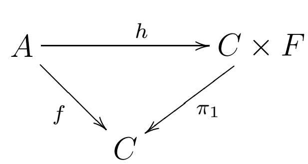
Theorem 1.55 (Existence of stratifications) *Let $p \geq 1$ be a fixed integer.*

1. *Let A be definable. There exists a definable C^p stratification of A.*
2. *Let A be a closed definable set and $f : A \to \mathbb{R}$ be a continuous definable function. Then, there exists a definable C^p stratification of A such that for each stratum X, the restriction $f|_X$ is C^p and of constant rank.*

Remark 1.56 *More precise results about stratification with specific regularity conditions exist: e.g. Whitney, Thom [Loi2], Verdier [Loi3], etc...*

The next result is about the local triviality of continuous definable maps. It originated with Hardt in the semi-algebraic case [H].

Definition 1.57 (Trivial map) *Let $f : A \to C$ be a definable map. The map f is called* (definably) trivial *if there exists a definable set F and a definable homeomorphism $h : A \to C \times F$ such that the following diagram commutes.*



where $\pi_1 : C \times F \to C$ is the canonical projection.

Theorem 1.58 (Generic triviality) *Let $f : A \to C$ be a continuous definable map. there exists a finite definable partition $C = C_1 \cup \cdots \cup C_r$ such that f is definably trivial over each C_i.*

We finish our discussion of the general properties of o-minimal structures with questions about triangulations of sets and maps, which will play a large role in the proofs of the results of Chapter 5. For simplicial complexes, we use the terminology of [C2] rather than [D4].

Definition 1.59 (Simplex) *Let $a_0, \dots, a_d$ be affine-independent points in $\mathbb{R}^n$, (not contained in any $(d-1)$-dimensional affine subspace). We define the* closed *simplex $\bar{\sigma} = [a_0, \dots, a_d]$ as the subset of $\mathbb{R}^n$ defined by*

$$\bar{\sigma} = \left\{ \sum_{i=0}^{d} w_i \, a_i \mid \sum_{i=0}^{d} w_i = 1, w_1 \geq 0, \dots, w_d \geq 0 \right\}. \tag{1.18}$$

The open *simplex $\sigma = (a_0, \dots, a_d)$ is defined as above, with the additional condition that all weights w_i are positive. The points $a_0, \dots, a_d$ are called* vertices *of the (open or closed) simplex. The* dimension *of the simplex is d.*

Note that the condition $w_0 + \cdots + w_d = 1$ in (1.18) implies that the weights w_i are uniquely determined.

Definition 1.60 (Faces) *If $\bar{\sigma} = [a_0, \dots, a_d]$ is a closed simplex, its faces are all the closed simplexes of the form $[a_i, i \in I]$ where I is any non-empty subset of $\{0, \dots, d\}$.*

Definition 1.61 (Simplicial complex) *A* (finite) simplicial complex *K of $\mathbb{R}^n$ is a finite collection $\{\bar{\sigma}_1, \dots, \bar{\sigma}_k\}$ of closed simplices of $\mathbb{R}^n$ such that the following two conditions hold.*

- *For any $i, j \in \{1, \dots, k\}$, the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ is a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$;*
- *K is closed under taking faces.*

We denote by $|K|$ the subset $\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_k$ of $\mathbb{R}^n$.

Theorem 1.62 (Triangulation of compact definable sets) *Let $A \subseteq \mathbb{R}^n$ be a compact definable set, and $B_1, \dots, B_k$ be definable subsets of A. There exists a finite simplicial complex K with vertices in $\mathbb{Q}^n$, sets $S_1, \dots, S_k$ of open simplices of K and a definable homeomorphism $\Phi : |K| \to A$ such that for each i, we have $B_i = \cup_{\sigma \in S_i} \Phi(\sigma)$.*

Note that the result still holds when A is not compact, provided we take the weaker notion of simplicial complex where we do not require K to be closed under taking faces. See [D4] for more details.

It is well-known that definable maps are not always triangulable: for example, the blow-up map $f(x, y) = (x, xy)$ is not. The result below says that definable continuous maps from a compact into $\mathbb{R}$ always are. Recall that the function $f : A \to \mathbb{R}$ is *triangulable* if there exists a finite simplicial complex K and a homeomorphism $\Phi : |K| \to A$ such that $f \circ \Phi$ is affine. By affine map, we mean the following.

Definition 1.63 (Affine map) *Let $g : \bar{\sigma} = [a_0, \ldots, a_d] \to \mathbb{R}$ be a function defined on a simplex. It is* affine *if it satisfies the equality*

$$g\left(\sum_{i=0}^{d} w_i\, a_i\right) = \sum_{i=0}^{d} w_i\, g(a_i); \tag{1.19}$$

for all non-negative $(w_0, \ldots, w_d)$ such that $w_0 + \cdots + w_d = 1$.

The proof of the following theorem in the o-minimal setting can be found in [C1, C2].

Theorem 1.64 (Triangulation of functions) *Let $A \subseteq \mathbb{R}^n$ be a compact definable subset in an o-minimal structure $\mathcal{S}$ and $f : A \to \mathbb{R}$ be a definable continuous function. Then, there exists a finite simplicial complex K in $\mathbb{R}^{n+1}$ and a definable homeomorphism $\Phi : |K| \to A$ such that $f \circ \Phi$ is affine on each simplex of K.*

Moreover, given finitely many definable subsets $B_1, \ldots, B_k$ of A, we can choose the triangulation $\Phi : |K| \to A$ so that each B_i is the union of images of open simplices of K.

1.4 Pfaffian functions and o-minimality

As mentioned earlier, works of Charbonnel [Ch] and Wilkie [W2] led to the following result, which we will use extensively in the present work. This theorem was then generalized extensively in [KM, Sp, LR2].

Theorem 1.65 (Wilkie) *The structure generated by Pfaffian functions is o-minimal.*

The main result in [W2] is a theorem of the complement: Wilkie shows that the Pfaffian structure can be obtained by starting from semi-Pfaffian sets and iterating the operations of closure under finite unions, projections and *closure at infinity,* where the last operation consists in considering all sets of the form $A_0 \cap \overline{A_1} \cap \cdots \cap \overline{A_p}$ for sets $A_0, \ldots, A_p$ already constructed. The end result is called the *Charbonnel closure*, and [W2, Theorem 1.8] says that the Charbonnel closure obtained from semi-Pfaffian sets is closed under complementation (and thus a *bona fide* structure) and o-minimal.

This construction of the Pfaffian structure, however, is not very convenient for quantitative purposes, especially since if $\mathcal{T}_m$ denotes the pre-structure obtained after m iteration and if $X \subseteq \mathbb{R}^n$ denotes a definable set that can be constructed within $\mathcal{T}_m$, there doesn't seem to be any way to derive from Wilkie's work an upper-bound a the number p such that $\mathbb{R}^n \backslash X$ can be constructed within $\mathcal{T}_{m+p}$. This is what made it desirable to find an alternative construction for the Pfaffian structure.

We will now describe in some details the construction of the Pfaffian structure via limit sets that was suggested by Gabrielov in [G6]. Limit sets will provide a notion of format for arbitrary definable sets, and we will show this format can effectively be used to derive upper-bounds (Chapter 4 and 5).

Remark 1.66 (About sub-Pfaffian sets) *We come back to the open problem evoked in Remark 1.36: if (C) is the statement:* the complement of any sub-Pfaffian set is again sub-Pfaffian, *we do not know whether (C) holds or not. If we knew that (C) was true, we could deduce easily that the Pfaffian structure is o-minimal, since then all definable sets would be sub-Pfaffian and semi-Pfaffian sets (and thus sub-Pfaffian sets) always have finitely many connected components.*

Proof: Let us show that (C) would imply that sub-Pfaffian sets form a structure. Since sub-Pfaffian sets are clearly stable under projection and Cartesian products. Also, sub-Pfaffian sets are always closed under finite unions since if X and Y are sub-Pfaffian, we can assume that $X = \pi(X_1)$ and $Y = \pi(Y_1)$ for some semi-Pfaffian subsets X_1 and Y_1 of $\mathbb{R}^{n+p}$, with π the canonical projection $\mathbb{R}^{n+p} \to \mathbb{R}^n$, and thus $X \cup Y = \pi(X_1 \cup Y_1)$, so it is clearly sub-Pfaffian.

Hence, all we have to show is that if X and Y are sub-Pfaffian and (C) holds, then $X \cap Y$ is sub-Pfaffian too. Since we're assuming (C), it is enough to show that the complement $\mathbb{R}^n \backslash (X \cap Y)$ is sub-Pfaffian. But this is obvious, since $\mathbb{R}^n \backslash (X \cap Y) = (\mathbb{R}^n \backslash X) \cup (\mathbb{R}^n \backslash Y)$: by (C), both $(\mathbb{R}^n \backslash X)$ and $(\mathbb{R}^n \backslash Y)$ are again sub-Pfaffian, and we have just showed that sub-Pfaffian sets were stable under finite unions. □

1.4.1 Relative closure and limit sets

From now on, we consider semi-Pfaffian subsets of $\mathbb{R}^n \times \mathbb{R}_+$ with a fixed Pfaffian chain $\boldsymbol{f} = (f_1, \dots, f_\ell)$ in a domain $\mathcal{U}$ of bounded complexity. We write $(x_1, \dots, x_n)$ for the coordinates in $\mathbb{R}^n$ and λ for the last coordinate (which we think of as a parameter.) If X is such a subset and $\lambda > 0$, X_λ is its fiber

$$X_\lambda = \{x \mid (x, \lambda) \in X\} \subseteq \mathbb{R}^n;$$

and we consider X as the family of its fibers X_λ. We let

$$X_+ = X \cap \{\lambda > 0\}, \text{ and } \check{X} = \{x \in \mathbb{R}^n \mid (x, 0) \in \overline{X_+}\}.$$

(Thus, $\check{X}$ is the Hausdorff limit of the family $\overline{X_\lambda}$ when λ goes to zero.) The following definitions appear in [G6].

Definition 1.67 (Semi-Pfaffian family) *Let X be a relatively compact semi-Pfaffian subset of $\mathbb{R}^n \times \mathbb{R}_+$. The family X_λ is said to be a* semi-Pfaffian family *if for any $\varepsilon > 0$, the set $X \cap \{\lambda > \varepsilon\}$ is restricted. (See Definition 1.27.) The* format $(n, \ell, \alpha, \beta, s)$ *of the family X is the format of the fiber X_λ for a small $\lambda > 0$.*

Remark 1.68 (Format) *Note that the format of X as a semi-Pfaffian set is different from its format as a semi-Pfaffian family. We will sometimes refer to it as the* fiber-wise format *to emphasize that fact. Note also that [G6] uses the format discussed in Remark 1.25 rather than the formula-based format, both being of course valid measures of the descriptive complexity of limit sets.*

Definition 1.69 (Semi-Pfaffian couple) *Let X and Y be semi-Pfaffian families in $\mathcal{U}$ with a common chain $(f_1, \ldots, f_\ell)$. They form a* semi-Pfaffian couple *if the following properties are verified:*

- $(\bar{Y})_+ = Y_+$;
- $(\partial X)_+ \subseteq Y$.

Then, the format *of the couple (X, Y) is the component-wise maximum of the format of the families X and Y.*

Definition 1.70 (Relative closure) *Let (X, Y) be a semi-Pfaffian couple in $\mathcal{U}$. We define the* relative closure *of (X, Y) at $\lambda = 0$ by*

$$(X, Y)_0 = \check{X} \setminus \check{Y} \subseteq \check{\mathcal{U}}. \tag{1.20}$$

We will use the notation $X_0 = (X, \partial X)_0$.

Definition 1.71 (Limit set) *Let $\Omega \subseteq \mathbb{R}^n$ be an open domain. A* limit set *in Ω is a set of the form $(X_1, Y_1)_0 \cup \cdots \cup (X_k, Y_k)_0$, where (X_i, Y_i) are semi-Pfaffian couples respectively defined in domains $\mathcal{U}_i \subseteq \mathbb{R}^n \times \mathbb{R}_+$, such that $\check{\mathcal{U}}_i = \Omega$ for $1 \leq i \leq k$. If the formats of the couples (X_i, Y_i) is bounded component-wise by $(n, \ell, \alpha, \beta, s)$ we say that the* format *of the limit set is $(n, \ell, \alpha, \beta, s, k)$*

Remark 1.72 *We assumed that the semi-Pfaffian families X are all relatively compact. This restriction allows us to avoid a separate treatment of infinity: we can see $\mathbb{R}^n$ as embedded in $\mathbb{R}P^n$, in which case any set we consider can be subdivided into pieces that are relatively compact in their own charts.*

Example 1.73 *Any (not necessarily restricted) semi-Pfaffian set X is a limit set.*

Proof: It is enough to prove the result for a basic set $X \subseteq \mathcal{U}$,

$$X = \{x \in \mathcal{U} \mid \varphi_1(x) = \cdots = \varphi_I(x) = 0,\ \psi_1(x) > 0, \ldots,\ \psi_J(x) > 0\};$$

Let $\psi = \psi_1 \cdots \psi_J$ and let g be an exhausting function for $\mathcal{U}$. Define the sets

$$\begin{aligned}
W &= \{(x, \lambda) \in X \times \Lambda \mid g(x) > \lambda\}\,; \\
Y_1 &= \{(x, \lambda) \in \mathcal{U} \times \Lambda \mid \varphi_1(x) = \cdots = \varphi_I(x) = 0,\ \psi(x) = 0,\ g(x) \geq \lambda\}\,; \\
Y_2 &= \{(x, \lambda) \in \mathcal{U} \times \Lambda \mid \varphi_1(x) = \cdots = \varphi_I(x) = 0,\ g(x) = \lambda\}\,.
\end{aligned}$$

where $\Lambda = (0, 1]$. If $Y = Y_1 \cup Y_2$, it is clear that (W, Y) satisfies the requirements of a semi-Pfaffian couple in Definition 1.69; its relative closure is clearly X. □

For all $n \in \mathbb{N}$ we let $\mathcal{S}_n$ be the collection of limit sets in $\mathbb{R}^n$, and $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$. The following theorem sums up the results in [G6, Theorems 2.9 and 5.1].

Theorem 1.74 *The collection $\mathcal{S}$ is a structure, and it is o-minimal. Moreover, if X is a definable set obtained by a combination of Boolean operations and projections of the limit sets $L_1, \ldots, L_N$, the set X can be presented as a limit set whose format is bounded by an effective function of the formats of $L_1, \ldots, L_N$.*

Moreover, it is clear that $\mathcal{S}$ coincides with the structure constructed by Wilkie. A key result to work with limit sets is the following inequality.

Proposition 1.75 (Exponential Łojasiewicz inequality) *Let $\boldsymbol{f}$ be a Pfaffian chain of length ℓ defined on a domain of bounded complexity $\mathcal{U}$ for $\boldsymbol{f}$. Let $q(x) = Q(x, \boldsymbol{f})$, and suppose that $0 \in \mathrm{cl}(X \cap \{q > 0\})$. Then, there exists $N \in \mathbb{N}$ such that*

$$0 \in \mathrm{cl}(\{x \in X \mid q(x) \geq 1/\exp_\ell(|x|^{-N})\});$$

where $\exp_\ell$ is the ℓ-th iterated exponential.

The proof relies on proving that the rank of the Hardy field generated by $\boldsymbol{f}$ at 0 is bounded by $\ell+1$ (see [Ros]). A detailed proof can be found in [G6], see also [Gri, L, LMP].

1.4.2 Special consequences of o-minimality

When giving bounds on the topology of sets defined using Pfaffian functions, one invokes constantly the o-minimality of the structure generated by those functions. In this section are gathered a few minor results that will be often used in the next chapters.

Lemma 1.76 (Existence of limits) *Let $f : (0, \varepsilon) \to \mathbb{R}$ be definable. Then, the function f has a well-defined limit in $\mathbb{R} \cup \{\pm\infty\}$.*

Proof: This is a simple consequence from the monotonicity theorem (Theorem 1.46). There exists $\delta > 0$ such that the restriction of f to $(0, \delta)$ is continuous, and one of strictly increasing, constant, or strictly decreasing. The case where f is constant on $(0, \delta)$ is trivial. If f is strictly increasing on that interval, then either it is bounded from above, and then f must have a finite limit at 0, or it is not bounded and the limit of f is $+\infty$. The case where f is decreasing is similar. □

Lemma 1.77 (Critical values) *Let $q : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a C^∞ definable function. Then, q has finitely many critical values.*

Proof: The set of critical values of q is a definable subset of $\mathbb{R}$. (If q is a Pfaffian function, this set is actually sub-Pfaffian, see Example 1.17.) By Sard's lemma, it must be of measure zero, and a definable subset of $\mathbb{R}$ of measure zero can only be finite. □

Recall that we denote by $b_i(X)$ the i-th Betti number of X (see Notation 1.28) and $b(X) = \sum_i b_i(X)$.

Lemma 1.78 (Deformation of basic sets) *Let $\mathcal{U}$ be a domain of bounded complexity, g an exhausting function for $\mathcal{U}$ and let $X = \{x \in \mathcal{U} \mid q_1(x) = \cdots = q_r(x) = 0, p_1(x) > 0, \ldots, p_s(x) > 0\}$ be a basic semi-Pfaffian set, and for $\varepsilon > 0$ and $t \in (\mathbb{R}_+)^s$, define $X_\varepsilon = \{x \in \mathcal{U} \mid q_1(x) = \cdots = q_r(x) = 0, p_1(x) \geq \varepsilon t_1, \ldots, p_s(x) \geq \varepsilon t_s, g(x) \geq \varepsilon\}$. Then for all $\varepsilon \ll 1$, $b(X_\varepsilon) = b(X)$.*

Proof: The groups $H_*(X)$ are the direct limit of the singular homology groups of the compact subsets of X ([Spa, Theorem 4.4.5].) Thus, $b(X) = \lim_{\varepsilon \to 0} b(X_\varepsilon)$, and by the generic triviality theorem (Theorem 1.58), this sequence is eventually stationary. □

Lemma 1.79 (topology of compact limits) *Let K_ε be a decreasing sequence of compact definable sets defined for $\varepsilon > 0$, and let K be their intersection. Then, for all $\varepsilon \ll 1$, and all $0 \leq i \leq n$, we have*

$$b_i(K_\varepsilon) = b_i(K)$$

Proof: Since all the sets considered are triangulable, their homological type is that of a polyhedron, and the Čech homology $\check{H}_*$ and the singular homology H_* are isomorphic. Since the sequence K_n is compact and decreasing and the limit is compact too, we have [ES]

$$H_*(K) = \varprojlim H_*(K_\varepsilon). \tag{1.21}$$

But by generic triviality, the sets K_ε are homeomorphic for $\varepsilon \ll 1$, hence the limit in (1.21) becomes eventually stationary. □

Chapter 2

Betti numbers of semi-Pfaffian sets

This chapter is devoted to the study of the possible complexity bounds that can be proved on the Betti numbers of semi-Pfaffian sets defined on a domain of bounded complexity. These results include bounds for the sum of Betti numbers of compact and non-compact Pfaffian varieties (Theorem 2.8 and Theorem 2.10), bounds for the sum of Betti numbers of basic semi-Pfaffian sets (Lemma 2.14) and semi-Pfaffian sets given by $\mathcal{P}$-closed formulas (the main result of this chapter, Theorem 2.17). Theorem 2.25 gives a bound on $\mathcal{C}(V;\mathcal{P})$, the number of connected sign cells of the family $\mathcal{P}$ on V that was introduced in Definition 1.29, and this allows to establish in Theorem 2.32 a bound on the Borel-Moore homology of arbitrary (locally closed) semi-Pfaffian sets. In particular, this last result provides an upper-bound for the sum of Betti numbers of any compact semi-Pfaffian set, without requiring the defining formula to be $\mathcal{P}$-closed.

Recently, Gabrielov and Vorobjov [GV4] generalized the results of the present chapter: they established a general, single-exponential bound for the sum of the Betti numbers of *any* semi-Pfaffian set, without any assumption on its topology or defining formula. Such a result was not known even in the algebraic case, and the precise statement was added at the end of this chapter (Theorem 2.34) for reference purposes.

The setting for the present chapter will be the following: we will consider a fixed Pfaffian chain $\boldsymbol{f}$ of length ℓ and degree α in a domain $\mathcal{U} \subseteq \mathbb{R}^n$ of bounded complexity for the chain $\boldsymbol{f}$. We will let g be an exhausting function for $\mathcal{U}$, and $\gamma = \deg_{\boldsymbol{f}} g$.

Throughout this chapter, $p_1, \ldots, p_s$, and $q_1, \ldots, q_r$, will be Pfaffian functions in the fixed chain $\boldsymbol{f}$, and we'll write $\mathcal{P}$ for $\{p_1, \ldots, p_s\}$ and $\deg_{\boldsymbol{f}} \mathcal{P}$ for $\max\{\deg_{\boldsymbol{f}} p_i \mid 1 \leq i \leq s\}$. The number β will be a common upper-bound for $\deg_{\boldsymbol{f}} p_i$ and $\deg_{\boldsymbol{f}} q_j$. We'll let $q = q_1^2 + \cdots + q_r^2$ and $V = \mathcal{Z}(q_1, \ldots, q_r) = \mathcal{Z}(q)$. The dimension of V will be denoted by d. We'll let

$$\mathcal{V}(n, \ell, \alpha, \beta, \gamma) = 2^{\ell(\ell-1)/2} \beta (\alpha + \beta - 1)^{n-1} \frac{\gamma}{2} [n(\alpha + \beta - 1) + \gamma + \min(n, \ell)\alpha]^{\ell}. \qquad (2.1)$$

The chapter is organized as follows.

- In the first section, we show that $b(V)$ can be bounded in terms of $\mathcal{V}(n, \ell, \alpha, \beta, \gamma)$.

- In section 2.2, we show that if X is a semi-Pfaffian subset of a compact variety V given by a $\mathcal{P}$-closed formula, $b(X) \leq (5s)^d\, \mathcal{V}(n,\ell,\alpha,2\beta,\gamma)$.
- In section 2.3, we show that $\mathcal{C}(V;\mathcal{P}) \leq \Sigma(s,d)\, \mathcal{V}(n,\ell,\alpha,\beta^\star,\gamma)$, where $\beta^\star = \max(\beta,\gamma)$ and
$$\Sigma(s,d) = \sum_{0\leq i\leq d} \binom{4s+1}{i}.$$
- The last section is devoted to proving that the rank of the Borel-Moore homology groups of a locally closed X with the above format is bounded by an expression of the form
$$b^{\mathrm{BM}}(X) \leq s^{2d}\, 2^{\ell(\ell-1)} O(n\beta + \min(n,\ell)\alpha)^{2(n+\ell)};$$
for some constant depending on $\mathcal{U}$.

The main results of this chapters are inspired by similar results in the semi-algebraic case by Basu, Pollack and Roy [B2, BPR1, BPR3]. Note that more analogues could also be formulated for more recent results in the same vein [B1, B4, B5]. Indeed, the o-minimality of the structure generated by Pfaffian functions ensures that most arguments can still be used. The use of infinitesimals in those papers can be avoided most of the time by placing oneself in a compact setting and replacing the infinitesimals in small real numbers. (The proof of Theorem 2.17 is an example of how one can compute a real number r such that the condition that ε is an infinitesimal can be replaced by $\varepsilon < r$.)

As in the work of Basu, Pollack and Roy, one of the ideas behind the bounds is the notion of *combinatorial level* of a family of functions $\mathcal{P}$.

Definition 2.1 (Combinatorial level) *Let $X \subseteq \mathcal{U}$ be a semi-Pfaffian set and $\mathcal{P}$ a family of functions on $\mathcal{U}$. The* combinatorial level *of the couple $(X,\mathcal{P})$ is the largest integer m such that there exists x in X and m functions in $\mathcal{P}$ vanishing at x.*

This leads to a combinatorial definition of the idea of general position.

Definition 2.2 (General position) *Let V be a Pfaffian variety. The set $\mathcal{P}$ is said to be in* general position *with V if the combinatorial level of $(V,\mathcal{P})$ is bounded by* $\dim(V)$.

2.1 Betti numbers of Pfaffian varieties

This section is devoted to proving the following analogue for Pfaffian varieties of the Oleinik-Petrovskii-Thom-Milnor upper bound [O, OP, T, M2] on the Betti numbers of real algebraic sets. As explained above, $\boldsymbol{f}$ is a fixed Pfaffian chain of degree α and length ℓ, and $\mathcal{U}$ is a domain of bounded complexity for $\boldsymbol{f}$ with an exhausting function g such that $\deg_{\boldsymbol{f}} g = \gamma$. We fix $V = \mathcal{Z}(q_1,\dots,q_r)$ a Pfaffian variety and let $q = q_1^2 + \cdots + q_r^2$. The result we will prove is the following.

Theorem 2.3 *Let $V = \mathcal{Z}(q)$ a Pfaffian variety with q as above, $\deg_{\boldsymbol{f}} q = 2\beta$. If V is compact, its Betti numbers verify*

$$b(V) \leq \mathcal{V}(n, \ell, \alpha, \beta, \gamma);$$

where $\mathcal{V}(n, \ell, \alpha, \beta, \gamma)$ is defined in (2.1).

If V is not compact, we have $b(V) \leq \mathcal{V}(n, \ell, \alpha, \beta^*, \gamma)$, where $\beta^* = \max(\beta, \frac{\gamma}{2})$.

In practice, it makes sense to assume that the chain $\boldsymbol{f}$ and the domain $\mathcal{U}$ are fixed, and to let the degree β go to infinity. We then obtain a more manageable estimate.

Corollary 2.4 *Let $\mathcal{U}$ be a fixed domain of bounded complexity for a Pfaffian chain $\boldsymbol{f}$. If $V = \mathcal{Z}(q_1, \ldots, q_r)$ is a Pfaffian variety and $\deg_{\boldsymbol{f}} q_i \leq \beta$ for all i, the following asymptotic estimate holds.*

$$b(V) \leq 2^{\ell(\ell-1)/2}\, O(n\beta + \min(n, \ell)\alpha)^{n+\ell}.$$

(Here, the constant in the O term depends on γ.)

2.1.1 Bound for compact Pfaffian varieties

In this section, we assume that the variety $V \subseteq \mathcal{U}$ is compact. Recall that if $V = \mathcal{Z}(q_1, \ldots, q_r)$, we let $q = q_1^2 + \cdots + q_r^2$. Let $K_r = \{x \in \mathcal{U} \mid q(x) \leq r\}$. According to Lemma 1.77, q has only a finite number of critical values, and so the K_r are smooth manifolds with boundaries for all but finitely many values of r. Let $K_r^* \subseteq K_r$ be the union of the connected components of K_r that intersect V. We want to show that $b(V)$ is equal to $b(K_r^*)$ for small values of r. We shall start by proving that K_r^* is compact if r is small enough.

Lemma 2.5 *Let $d_V(x)$ be the distance of x to V, and for all $\delta > 0$, let $T(\delta) = \{x \in \mathcal{U} \mid d_V(x) \leq \delta\}$. There exists $\delta_1 > 0$ such that $K_r^* = K_r \cap T(\delta_1)$ for $r \ll 1$.*

Proof: Define, for any set C, $\mathrm{dist}(C, V) = \min\{d_V(x) \mid x \in C\}$. Let $\delta_0 = \mathrm{dist}(\partial\mathcal{U}, V)$. Since V is compact, we have $\delta_0 > 0$. Fix $\delta_1 \in (0, \delta_0)$.

For all $r > 0$, let $C_r = K_r \backslash K_r^*$. This set is closed for all r. We will show that $\mathrm{dist}(C_r, V) > \delta_1$ when $r \ll 1$, by contradiction. If $C_r \cap T(\delta_1) \neq \varnothing$ for all $r > 0$, their intersection $\cap_{r>0}(C_r \cap T(\delta_1))$ must be non-empty too, since those sets are compact. But a point in this intersection cannot be in V. Since $V = \cap_{r>0} K_r$, we have a contradiction. □

Remark 2.6 *It is important to consider K_r^*, since K_r itself is not necessarily compact. The following example comes from [BR].*

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be the map:

$$P(x, y) = (x^2 + y^2)((y(x^2 + 1) - 1)^2 + y^2).$$

$P^{-1}(0) = \{0\}$ is compact, but as $P(x, (1 + x^2)^{-1})$ goes to 0 as x goes to infinity, the sets $\{P \le r\}$ are not bounded for $r > 0$.

Since the set $T(\delta_1)$ is compact, Lemma 2.5 implies that K_r^* is compact for $r \ll 1$. Since $V = \cap_{r>0} K_r^*$ is compact too, we can apply Lemma 1.79 to conclude that $b(V) = b(K_r^*)$ for $r \ll 1$. To obtain a bound on $b(V)$, we need to establish a relation between the topology of K_r^* and the topology of its boundary.

Lemma 2.7 *Let $K = K_r^*$. Then $b(\partial K) = 2b(K)$.*

Proof: Let $K^c = \mathbb{R}^n \backslash K$. The Mayer-Vietoris sequence in reduced homology of $(K, \overline{K^c})$ is:

$$\cdots \longrightarrow \tilde{H}_{i+1}(\mathbb{R}^n) \longrightarrow \tilde{H}_i(\partial K) \longrightarrow \tilde{H}_i(K) \oplus \tilde{H}_i(\overline{K^c}) \longrightarrow \tilde{H}_i(\mathbb{R}^n) \longrightarrow \cdots \tag{2.2}$$

As $\tilde{H}_*(\mathbb{R}^n) = 0$, this yields $\tilde{H}_i(\partial K) \cong \tilde{H}_i(K) \oplus \tilde{H}_i(\overline{K^c})$, and as ∂K has a collar in $\overline{K^c}$, we have $\tilde{H}_i(\overline{K^c}) \cong \tilde{H}_i(K^c)$.

Alexander duality gives $\tilde{H}_i(K^c) \cong \check{H}^{n-i-1}(K)$. This yields the relations

$$b_i(\partial K) = b_i(K) + b_{n-i-1}(K), \qquad 0 \le i \le n - 1. \tag{2.3}$$

We have $b_n(K) = b_n(\partial K) = 0$, so summing all the equalities in (2.3) gives the result $b(\partial K) = 2b(K)$. □

Theorem 2.8 (Compact varieties) *Let $\boldsymbol{f}$ be a Pfaffian chain of length ℓ and degree α defined in a domain $\mathcal{U}$ of bounded complexity γ. Let $q_1, \ldots, q_r$ be Pfaffian functions such that $\deg_{\boldsymbol{f}} q_i \le \beta$, and let $V = \mathcal{Z}(q_1, \ldots, q_r)$. If V is* compact, *we have*

$$b(V) \le \mathcal{V}(n, \ell, \alpha, \beta, \gamma); \tag{2.4}$$

where $\mathcal{V}(n, \ell, \alpha, \beta, \gamma)$ is defined in (2.1).

Proof: For $r \ll 1$, we know from Lemma 1.79 that $b(K_r^*) = b(V)$. According to Lemma 2.7, it is enough to estimate the Betti numbers of $W = \partial K_r^*$, which is a smooth compact manifold for $r \ll 1$.

Up to a rotation of the coordinate system, – which does not alter the complexity of V, – we can assume that the projection map π of W on the x_1 axis is a Morse function, *i.e.* has only non-degenerate critical points with distinct critical values. By standard Morse theory [M1], $b(W)$ is bounded by the number of critical points of π, which in turn is bounded by the number of (non-degenerate) solutions of the system;

$$q(x) - r = \frac{\partial q}{\partial x_2}(x) = \cdots = \frac{\partial q}{\partial x_n}(x) = 0.$$

The first equation has degree 2β in the chain $\boldsymbol{f}$, and the others have degree $\alpha + 2\beta - 1$. From Theorem 1.16, such a system has at most $2\,\mathcal{V}(n\ell, \alpha, \beta, \gamma)$ solutions, and the bound on $b(V)$ follows. □

2.1.2 The case of non-compact varieties

Assume now that $V \subseteq \mathcal{U}$ is not compact. Let g be an exhausting function for $\mathcal{U}$, and define for all $\varepsilon > 0$,

$$V_\varepsilon = \{x \in V \mid g(x) \geq \varepsilon\}. \tag{2.5}$$

The set V_ε is compact for all $\varepsilon > 0$.

Proposition 2.9 *For all $\varepsilon \ll 1$, we have $b(V_\varepsilon) = b(V)$.*

Proof: By generic triviality, there exists $\varepsilon_0 > 0$ such that g restricted to V is a trivial fibration over $(0, \varepsilon_0)$. In particular, this implies that for $0 < \varepsilon' < \varepsilon < \varepsilon_0$, the inclusion $V_\varepsilon \hookrightarrow V_{\varepsilon'}$ is a homotopy equivalence, and thus $b(V_\varepsilon)$ is constant for $\varepsilon \in (0, \varepsilon_0)$. By [Spa, Theorem 4.4.6], $H_*(V)$ is the inductive limit of the groups $H_*(V_\varepsilon)$, and the result follows. □

Theorem 2.10 (Non-compact varieties) *Let $\boldsymbol{f}$ be a Pfaffian chain of length ℓ and degree α defined in a domain $\mathcal{U}$ of bounded complexity γ. Let $q_1, \ldots, q_r$ be Pfaffian functions in $\boldsymbol{f}$ of degree at most β and $V = \mathcal{Z}(q_1, \ldots, q_r)$. If V is* not *compact, we have*

$$b(V) \leq \mathcal{V}(n, \ell, \alpha, \beta^*, \gamma); \tag{2.6}$$

where $\mathcal{V}$ is defined in (2.1) and $\beta^* = \max(\beta, \frac{\gamma}{2})$.

Proof: Choose $\varepsilon > 0$ such that $b(V_\varepsilon) = b(V)$, and define, for $\omega > 0$ and $\eta > 0$, the set $K = \{\omega q + g \geq \varepsilon - \eta\}$. Note that K is a compact subset of $\mathcal{U}$.

We can choose sequences ω_ν and η_ν such that the corresponding sets K_ν are smooth manifolds with boundary, but also such that the sequence is decreasing, and that $V_\varepsilon = \cap_\nu K_\nu$. In order to do that, it is enough to take a sequence η_ν that decreases to 0, and, if $M_\nu = \max_{K_\nu} q$, to choose $\omega_\nu \to_\nu \infty$ such that $(\omega_{\nu+1} - \omega_\nu) M_\nu \leq \eta_\nu - \eta_{\nu+1}$.

Since the decreasing sequence of compacts K_ν has the compact set V_ε as a limit, Lemma 1.79 gives that $b(V_\varepsilon) = \lim_{\nu\to\infty} b(K_\nu)$, and by the same arguments as in Lemma 2.7, we have $2b(K_\nu) = b(\partial K_\nu)$. As in the proof of Theorem 2.8, we reduced our the problem to the one of estimating the Betti numbers of a compact smooth hypersurface given by a single Pfaffian equation $\{\omega q + g = \varepsilon - \eta\}$. This estimate is established by the same method as in the compact case, by counting critical points of a projection on a coordinate axis. After a shift of coordinates, we must estimate the number of non-degenerate solutions of the system

$$h(x) - \varepsilon + \eta = \frac{\partial h}{\partial x_2}(x) = \cdots = \frac{\partial h}{\partial x_n}(x) = 0; \tag{2.7}$$

where $h(x) = \omega q(x) + g(x)$. Since $\deg_{\boldsymbol{f}} q = 2\beta$ and $\deg_{\boldsymbol{f}} g = \gamma$, we must have $\deg_{\boldsymbol{f}} h \leq \max(2\beta, \gamma)$, and Khovanskii's bound from Theorem 1.16 gives that the system (2.7) has at most $2\,\mathcal{V}(n, \ell, \alpha, \beta^*, \gamma)$ non-degenerate solutions. □

2.2 Betti numbers of semi-Pfaffian sets

In this section, $\boldsymbol{f}$ is still a Pfaffian chain defined on a domain of bounded complexity $\mathcal{U}$. Let V be a compact Pfaffian variety of dimension d and Φ be a $\mathcal{P}$-closed QF formula, with atoms in a finite set of Pfaffian functions $\mathcal{P} = \{p_1, \dots, p_s\}$.

2.2.1 Going to general position

Recall that $\mathcal{P}$ and V are said to be in general position if the combinatorial level $(V, \mathcal{P})$, introduced in Definition 2.1, is bounded by d. The following proposition shows that one can reduce to this case at a low complexity cost.

Proposition 2.11 (General position) *Let $X = \{x \in V \mid \Phi(x)\}$, where V and Φ are as above. Then there exists a set of $2s$ Pfaffian functions $\mathcal{P}^*$ and a $\mathcal{P}^*$-closed QF formula Φ^* such that the set $X^* = \{x \in V \mid \Phi^*(x)\}$ verifies $b(X) = b(X^*)$. Moreover, we have $\deg_{\boldsymbol{f}} \mathcal{P} = \deg_{\boldsymbol{f}} \mathcal{P}^*$.*

Proof: Of course, the result is non-trivial only if the combinatorial level of $(V, \mathcal{P})$ is at least $d+1$, which implies in particular that $s \geq d+1$.

Fix $t \in (\mathbb{R}_+)^s$ and let $\mathcal{P}_\varepsilon = \{p_1 \pm \varepsilon t_1, \dots, p_s \pm \varepsilon t_s\}$. For all $\varepsilon > 0$, we build from Φ a QF formula Φ_ε which is $\mathcal{P}_\varepsilon$-closed, replacing the atoms of Φ according to the following procedure.

- An atom of the form $\{p_i \geq 0\}$ is replaced by $\{p_i \geq -\varepsilon t_i\}$;
- an atom of the form $\{p_i \leq 0\}$ is replaced by $\{p_i \leq \varepsilon t_i\}$;
- an atom of the form $\{p_i = 0\}$ is replaced by the conjunction $\{p_i \leq \varepsilon t_i\} \wedge \{p_i \geq -\varepsilon t_i\}$.

Then, let $X_\varepsilon = \{x \in V \mid \Phi_\varepsilon(x)\}$. The sets X_ε are compact and $X = \cap_{\varepsilon > 0} X_\varepsilon$. By Lemma 1.79, there exists $\varepsilon \ll 1$ such that $b(X_\varepsilon) = b(X)$.

Assume that V is a C^1-smooth submanifold, and let $p = (p_1, \dots, p_s)$. By Sard's lemma, the set of critical values of $p|_V$ has measure zero. Hence, for a generic choice of $(t_1, \dots, t_s)$, we can find $\varepsilon > 0$ arbitrarily small such that any element of the form $(\pm\varepsilon t_1, \dots, \pm\varepsilon t_s)$ is a regular value of $p|_V$. For such a choice, $(V, \mathcal{P}_\varepsilon)$ is in general position and we can take $X^* = X_\varepsilon$. If V is not a submanifold, it can be stratified as a disjoint union of submanifolds, and we can choose a t that will work for every stratum. □

Proposition 2.12 (Mayer Vietoris inequalities) *Let X_1 and X_2 be two compact semi-Pfaffian sets. Then, for all i, the following inequalities hold.*

$$b_i(X_1) + b_i(X_2) \leq b_i(X_1 \cup X_2) + b_i(X_1 \cap X_2); \tag{2.8}$$
$$b_i(X_1 \cup X_2) \leq b_i(X_1) + b_i(X_2) + b_{i-1}(X_1 \cap X_2). \tag{2.9}$$

Proof: The Mayer-Vietoris sequence [Bred] for X_1 and X_2 is the following.

$$\cdots \to H_{i+1}(X_1 \cup X_2) \to H_i(X_1 \cap X_2) \to H_i(X_1) \oplus H_i(X_2) \to H_i(X_1 \cup X_2) \to \cdots$$

This sequence is exact when X_1 and X_2 are compact, and the above inequalities follow easily. $\square$

2.2.2 Betti numbers of a basic open set

If $\mathcal{P} = \{p_1, \ldots, p_s\}$, the basic open set defined by $\mathcal{P}$ on the variety V is the set

$$X(V;\mathcal{P}) = \{x \in V \mid p_1(x) > 0, \ldots, p_s(x) > 0\}. \tag{2.10}$$

Definition 2.13 *Let $\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m)$ be the maximum of $b(X)$ where $X = X(V;\mathcal{P})$ for some set of Pfaffian functions $\mathcal{P}$ on a Pfaffian variety $V = \mathcal{Z}(q_1, \ldots, q_r)$, with the following conditions.*

- *All functions are polynomial in some Pfaffian chain $\boldsymbol{f}$ of length ℓ and degree α, defined on a domain $\mathcal{U}$ of bounded complexity γ for $\boldsymbol{f}$;*
- *$|\mathcal{P}| = s$; and the combinatorial level of $(V, \mathcal{P})$ is m;*
- *$\deg_{\boldsymbol{f}} p_i$ and $\deg_{\boldsymbol{f}} q_j$ are bounded by β for $1 \leq i \leq s$ and $1 \leq j \leq r$.*

Then, $\mathcal{B}_0$ admits the following upper-bound.

Lemma 2.14 (Basic set bound) *Let $\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m)$ be as in Definition 2.13. Then,*

$$\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m) \leq 2^m \binom{s}{m} \mathcal{V}(n, \ell, \alpha, \beta, \gamma); \tag{2.11}$$

where $\mathcal{V}(n, \ell, \alpha, \beta, \gamma)$ is defined in (2.1). In particular, if $\mathcal{U}$ is fixed and $m \leq n$, we have

$$\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m) \leq \binom{s}{m} 2^{\ell(\ell-1)/2} O(n\beta + \min(n, \ell)\alpha)^{n+\ell}; \tag{2.12}$$

for a constant depending on γ.

Proof: Let X_ε be the set:

$$X_\varepsilon = \{x \in V \mid p_1(x) \geq \varepsilon, \ldots, p_s(x) \geq \varepsilon\}. \tag{2.13}$$

Applying Lemma 1.78, we have $b(X_\varepsilon) = b(X)$ for $\varepsilon \ll 1$. Consider now the sets:

$$\begin{aligned}
T &= \{x \in V \mid p_2 \geq \varepsilon, \dots, p_s \geq \varepsilon\} \supset X_\varepsilon,\\
X_\varepsilon^- &= T \cap \{p_1 \leq -\varepsilon\},\\
R &= X_\varepsilon \cup X_\varepsilon^-,\\
S &= T \cap \{-\varepsilon \leq p_1 \leq \varepsilon\}.\\
W^+ &= T \cap \{p_1 = \varepsilon\},\\
W^- &= T \cap \{p_1 = -\varepsilon\},\\
W &= W^+ \cup W^-.
\end{aligned}$$

As $T = R \cup S$ and $W = R \cap S$ and R and S are compact, the Mayer-Vietoris inequality (2.8) gives:

$$b(R) + b(S) \leq b(T) + b(W).$$

As the union $R = X_\varepsilon \cup X_\varepsilon^-$ is a disjoint union, the Mayer-Vietoris inequality (2.9) gives $b(R) = b(X_\varepsilon) + b(X_\varepsilon^-)$. This yields:

$$b(X) = b(X_\varepsilon) \leq b(R) \leq b(T) + b(W). \tag{2.14}$$

Let $\mathcal{P}_1 = \{p_2, \dots, p_s\}$. For $\varepsilon \ll 1$, the set T has the same Betti numbers as the basic set $X(V; \mathcal{P}_1)$, and $b(W) = b(X(V_1^+; \mathcal{P}_1)) + b(X(V_1^-; \mathcal{P}_1))$, where $V_1^+ = V \cap \mathcal{Z}(p_1 + \varepsilon)$ and $V_1^- = V \cap \mathcal{Z}(p_1 - \varepsilon)$. The set $\mathcal{P}_1$ has $s-1$ elements, and the corresponding combinatorial levels are bounded by m for $(V; \mathcal{P}_1)$ and by $m-1$ for $(V_1^+; \mathcal{P}_1)$ and $(V_1^-; \mathcal{P}_1)$. Thus, the relation (2.14) gives the following inequality.

$$\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m) \leq \mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s-1, m) + 2\, \mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s-1, m-1).$$

When $s = 0$, we have $X(V; \varnothing) = V$, and when $m = 0$, the functions p_i have constant sign on V, so that $X(V; \mathcal{P}) = V$ or $X(V; \mathcal{P}) = \varnothing$, depending on whether all functions of $\mathcal{P}$ are positive on V or not. Thus, we obtain the following initial conditions for the induction.

$$\begin{cases} \mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, 0, m) & \leq \mathcal{V}(n, \ell, \alpha, \beta, \gamma);\\ \mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, 0) & \leq \mathcal{V}(n, \ell, \alpha, \beta, \gamma). \end{cases} \tag{2.15}$$

We will prove (2.11) by induction on s, for all $m \leq s$. Assume that

$$\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s-1, m) \leq 2^m \binom{s-1}{m} \mathcal{V}(n, \ell, \alpha, \beta, \gamma);$$

holds for all integers $m \leq s-1$. We have:

$$\begin{aligned}
\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s, m) &\leq \mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s-1, m) + 2\mathcal{B}_0(n, \ell, \alpha, \beta, \gamma, s-1, m-1)\\
&\leq 2^m \binom{s-1}{m} \mathcal{V}(n, \ell, \alpha, \beta, \gamma) + 2 \cdot 2^{m-1} \binom{s-1}{m-1} \mathcal{V}(n, \ell, \alpha, \beta, \gamma)\\
&\leq 2^m \binom{s}{m} \mathcal{V}(n, \ell, \alpha, \beta, \gamma);
\end{aligned}$$

where the last line follows from the Newton identity. This proves the estimate (2.11), and the asymptotic estimate follows easily from this and Corollary 2.4. □

2.2.3 Bound for a $\mathcal{P}$-closed formula

Definition 2.15 *Let $\mathcal{B}(n, \ell, \alpha, \beta, \gamma, s, m)$ be the maximum of $b(X)$ where $X = \{x \in V \mid \Phi(x)\}$ for some $\mathcal{P}$-closed formula having atoms in a set of Pfaffian functions $\mathcal{P}$, where $V = \mathcal{Z}(q_1, \ldots, q_r)$ and the following holds.*

- *All functions are polynomial in some Pfaffian chain $\boldsymbol{f}$ of length ℓ and degree α, defined on a domain $\mathcal{U}$ of bounded complexity γ for $\boldsymbol{f}$;*
- *$|\mathcal{P}| = s$; and the combinatorial level of $(V, \mathcal{P})$ is m;*
- *$\deg_{\boldsymbol{f}} p_i$ and $\deg_{\boldsymbol{f}} q_j$ are bounded by β for $1 \leq i \leq s$ and $1 \leq j \leq r$.*

Recall that the notion of $\mathcal{P}$-closed formula was introduced in Definition 1.20. It is a quantifier-free formula with atoms of the form $\{p = 0\}$, $\{p \geq 0\}$ and $\{p \leq 0\}$ for $p \in \mathcal{P}$ that is derived without using negations.

Theorem 2.16 *Let $\mathcal{B}(n, \ell, \alpha, \beta, \gamma, s, m)$ be as in Definition 2.15. Then, the following inequality holds*

$$\mathcal{B}(n, \ell, \alpha, \beta, \gamma, s, m) \leq \mathcal{B}_0(n, \ell, \alpha, 2\beta, \gamma, s, m) + 3s\mathcal{B}(n, \ell, \alpha, \beta, \gamma, 3s, m-1); \tag{2.16}$$

where $\mathcal{B}_0$ is as in Definition 2.13.

Proof: Let $\mathcal{P} = \{p_1, \ldots, p_s\}$ be a family of Pfaffian functions and $X = \{x \in V \mid \Phi(x)\}$; where V is a Pfaffian variety and Φ is a $\mathcal{P}$-closed formula, and all the formats fit the requirements of Definition 2.15. We will decompose X into sets that do not involve the conditions $\{p_i = 0\}$

We will start by bounding $b(X)$ with Betti numbers of sets where the sign condition $\{p_1 = 0\}$ doesn't appear. Assume that $m < s$, and let

$$\mathcal{I} = \{I = (i_1, \ldots, i_m) \mid 2 \leq i_1 < \cdots < i_s \leq s\};$$

and for all $I \in \mathcal{I}$, define

$$Z_I = \{x \in V \mid p_{i_1}(x) = \cdots = p_{i_m}(x) = 0\}.$$

Let $Z = \cup_{I \in \mathcal{I}} Z_I$. The set Z is compact, and the restriction of p_1 to Z is never zero, or it would contradict the fact that the combinatorial level of $(V, \mathcal{P})$ is bounded by m. Thus, $\varepsilon_1 = \min_Z |p_1| > 0$. If $m = s$, $\mathcal{I}$ is empty and we can take any positive real for ε_1.

Let $0 < \eta_1 < \varepsilon_1/2$. and consider the sets:

$$\begin{aligned} R_1 &= \{x \in V \mid \Phi(x) \wedge (p_1(x) \leq -\eta_1 \vee p_1(x) \geq \eta_1)\}, \\ W_1^+ &= \{x \in V \mid \Phi(x) \wedge (p_1(x) = \eta_1)\}, \\ W_1^- &= \{x \in V \mid \Phi(x) \wedge (p_1(x) = -\eta_1)\}, \\ S_1 &= \{x \in V \mid \Phi(x) \wedge (-\eta_1 \leq p_1(x) \leq \eta_1)\}, \\ S_1' &= \{x \in V \mid \Phi(x) \wedge (p_1(x) = 0)\}. \end{aligned}$$

As $X = R_1 \cup S_1$ the Mayer-Vietoris inequality (2.9) yields $b(X) \leq b(R_1) + b(S_1) + b(R_1 \cap S_1)$, and since $R_1 \cap S_1 = W_1^+ \cup W_1^-$, we obtain $b(X) \leq b(R_1) + b(S_1) + b(W_1^+ \cup W_1^-)$.

By Lemma 1.79, we have for $\eta_1 \ll 1$, $B(S_1) = b(S_1')$, and since $W_1^+ \cap W_1^- = \varnothing$, we must have by the Mayer-Vietoris inequality (2.9) again that $b(W_1^+ \cup W_1^-) \leq b(W_1^+) + b(W_1^-)$. Thus, we have for $b(X)$ the following bound;

$$b(X) \leq b(R_1) + b(S_1') + b(W_1^+) + b(W_1^-). \tag{2.17}$$

Introduce the following varieties:

$$V_1 = V \cap \mathcal{Z}(p_1), \quad V_1^+ = V \cap \mathcal{Z}(p_1 - \eta_1), \quad V_1^- = V \cap \mathcal{Z}(p_1 + \eta_1).$$

S_1' is a semi-Pfaffian subset of the variety V_1 given by sign conditions on $\mathcal{P}_1 = \mathcal{P} \backslash \{p_1\}$. If $V_1 \neq \varnothing$, $(V_1, \mathcal{P}_1)$ has a combinatorial level that is at most $m-1$: if there is $x \in V_1$ and $p_{i_1}, \ldots, p_{i_m}$, in $\mathcal{P}_1 = \{p_2, \ldots, p_s\}$ such that $p_{i_1}(x) = \cdots = p_{i_m}(x) = 0$, then x is a point in V be such that $p_1(x) = p_{i_1}(x) = \cdots = p_{i_m}(x) = 0$. This contradicts the hypothesis that the combinatorial level of $(V, \mathcal{P})$ is bounded by m.

The sets W_1^+, and W_1^- are semi-Pfaffian subsets given by sign conditions over the family $\mathcal{P}$ on the varieties V_1^+ and V_1^- respectively. According to the choice made for η_1, those varieties do not meet the set $Z = \cup_{I \in \mathcal{I}} Z_I$, since each Z_I is a variety obtained by setting exactly m of the functions in $\{p_2, \ldots p_s\}$ to zero. Thus, the combinatorial level of $(V_1^+, \mathcal{P})$, – the system over which W_1^+ is defined, – is bounded by $m-1$. The same holds for W_1^-.

The above discussion for W_1^+ and W_1^- works for $s > m$ only, but when $m = s$, the combinatorial levels of $(V_1^+, \mathcal{P})$ and $(V_1^-, \mathcal{P})$ are still bounded by $m-1$: if m functions were to vanish at a point $x \in V_1^+$, p_1 would have to be one of them, but it's impossible since $p_1 \equiv \eta_1$ on V_1^+.

Thus, the relation (2.17) bounds $b(X)$ in terms of the Betti numbers of three sets that have a lower combinatorial level and one set, R_1 that can be defined by a sign condition that does not involve the atom $\{p_1 = 0\}$.

Now, the set R_1 is defined by sign conditions where the atom $p_1 = 0$ doesn't appear any more. We can use a similar treatment on this set to eliminate the atom $p_2 = 0$ by defining the following sets:

$$\begin{aligned} R_2 &= \{x \in R_1 \mid \Phi(x) \wedge (p_2(x) \leq -\eta_2 \vee p_2(x) \geq \eta_2)\}, \\ W_2^+ &= \{x \in R_1 \mid \Phi(x) \wedge (p_2(x) = \eta_2)\}, \\ W_2^- &= \{x \in R_1 \mid \Phi(x) \wedge (p_2(x) = -\eta_2)\}, \\ S_2 &= \{x \in R_1 \mid \Phi(x) \wedge (-\eta_2 \leq p_2(x) \leq \eta_2)\}, \\ S_2' &= \{x \in R_1 \mid \Phi(x) \wedge (p_2(x) = 0)\}. \end{aligned}$$

Here, η_2 is any positive real number smaller than $\varepsilon_2/2$, where ε_2 is the minimum of $|p_2|$ over all the varieties given by m equations on V chosen among $p_1 = \eta_1$, $p_1 = -\eta_1$, and $p_i = 0$ for $i \geq 3$. Repeating the previous arguments, we obtain

$$b(R_1) \leq b(R_2) + b(S_2') + b(W_2^+) + b(W_2^-) \tag{2.18}$$

when $\eta_2 \ll 1$. Note that again, the sets S_2', W_2^+ and W_2^- are defined with systems that have a combinatorial level at most $m-1$. Repeating this until all the functions p_i have been processed, we end up with a bound of the form:

$$b(X) \leq b(R_s) + \sum_{i=1}^{s} b(S_i') + b(W_i^+) + b(W_i^-). \tag{2.19}$$

In this relation, the sets S_i', W_i^+ and W_i^- are all defined by a system of combinatorial level at most $m-1$. All that remains is to estimate $b(R_s)$. We will show now that we can bound $b(R_s)$ by the sum of Betti numbers of a certain basic semi-Pfaffian set.

Let C be a connected component of R_s. Then, C is contained in one of the basic closed sets of the form

$$\{x \in V \mid p_1(x) \geq \pm\eta_1, \ldots, p_s(x) \geq \pm\eta_s\}.$$

Indeed, if it wasn't the case, there would be points y and z of C and an index i such that $p_i(y) \leq -\eta_i$ and $p_i(z) \geq \eta_i$. The points x and y would be joined by a curve contained in C that would have to intersect the variety $\mathcal{Z}(p_i)$. by construction, R_s does not meet any of the varieties $\mathcal{Z}(p_1), \ldots, \mathcal{Z}(p_s)$, so C is indeed contained in one of the sets of the form above.

Let's assume for simplicity that C is contained in the subset of V defined by $p_i(x) \geq \eta_i$ for all $1 \leq i \leq s$. By Lemma 1.78, the equality $b(C) = b(C')$ holds when $\eta_1, \ldots, \eta_s$ are small, where C' is the connected component of the basic set $\{x \in V \mid p_1(x) > 0, \ldots, p_s(x) > 0\}$ such that $C \subseteq C'$.

Define the basic set

$$\Sigma = \{x \in V \mid p_1^2(x) > 0, \ldots, \ p_s^2(x) > 0\};$$

and let Σ^* be the union of all connected components D of Σ such that $D \cap R_s \neq \varnothing$. Following the above arguments, we have $b(R_s) = b(\Sigma^*) \leq b(\Sigma)$ for $\eta_1, \ldots, \eta_s$ small enough. We can thus bound $b(R_s)$ using Lemma 2.14, and the inequality (2.16) follows. □

Theorem 2.17 *For $\mathcal{B}$ as in Definition 2.15, we have the following inequality.*

$$\mathcal{B}(n, \ell, \alpha, \beta, \gamma, s, m) \leq (5s)^m \, \mathcal{V}(n, \ell, \alpha, 2\beta, \gamma). \tag{2.20}$$

In particular, if $X \subseteq V$ is a semi-Pfaffian subset of a compact variety V given by a $\mathcal{P}$-closed formula of format $(n, \ell, \alpha, \beta, s)$, we have

$$b(X) \leq (10s)^d \, \mathcal{V}(n, \ell, \alpha, 2\beta, \gamma). \tag{2.21}$$

Proof: For $m = 0$, no function in $\mathcal{P}$ can change sign on X, so any connected component of V is either not in X or a connected component of X. For any space X, its singular homology is the direct sum of the homology of its connected components [Spa, Theorem 4.4.5]. Thus, for $m = 0$, we have $b(X) \leq b(V)$, and (2.20) holds by Theorem 2.8, since we certainly have $\mathcal{V}(n, \ell, \alpha, \beta, \gamma) \leq \mathcal{V}(n, \ell, \alpha, 2\beta, \gamma)$.

Assume (2.20) holds at rank $m-1$. Using the inductive relation proved in Theorem 2.16 and the bound on $\mathcal{B}_0$ from Lemma 2.14, we obtain;

$$\mathcal{B}(n, \ell, \alpha, \beta, \gamma, s, m) \leq \left[2^m \binom{s}{m} + 3 \cdot 5^{m-1} s^m \right] \mathcal{V}(n, \ell, \alpha, 2\beta, \gamma).$$

We can bound the binomial coefficient with:

$$\binom{s}{m} = \frac{s!}{m!(s-m)!} = \frac{s(s-1)\cdots(s-m+1)}{m!} \leq s^m;$$

which gives us:

$$2^m \binom{s}{m} + 3 \cdot 5^{m-1} s^m \leq \left(2^m + 3 \cdot 5^{m-1} \right) s^m \leq (5s)^m.$$

This concludes the induction, proving (2.20).

The inequality (2.21) follows from this and from the general position argument of Proposition 2.11. □

Corollary 2.18 *Let $V \subseteq \mathcal{U}$ compact Pfaffian variety, $d = \dim(V)$, and let $X = \{x \in V \mid \Phi(x)\}$ where Φ is a $\mathcal{P}$-closed Pfaffian formula. If the format of X is $(n, \ell, \alpha, \beta, \gamma, s)$, the following bound holds.*

$$b(X) \leq s^d \, 2^{\ell(\ell-1)/2} O(n\beta + \min(n, \ell)\alpha)^{n+\ell};$$

where the constant depends only on $\mathcal{U}$.

Proof: The result follows simply from (2.21) and the asymptotic estimates for $\mathcal{V}$ appearing in Corollary 2.4. □

2.2.4 Bounds for non-compact semi-Pfaffian sets

The results of this section can be extended to the case where V is not compact, including the case $V = \mathcal{U}$. Let Φ be a $\mathcal{P}$-closed formula and $V \subseteq \mathcal{U}$ be a non-compact Pfaffian variety. Let $X = \{x \in V \mid \Phi(x)\}$.

First, define $V_\varepsilon = V \cap \{g(x) \geq \varepsilon\}$, where g is an exhausting function for $\mathcal{U}$, and $X_\varepsilon = X \cap V_\varepsilon$. For $\varepsilon \ll 1$, we have $b(X) = b(X_\varepsilon)$, so we are reduced to estimating $b(X_\varepsilon)$.

Proposition 2.11 on general position can be repeated verbatim for X_ε instead of X, so we can construct X_ε^* compact defined by a $\mathcal{P}^*$ closed formula, where the combinatorial level of $(V, \mathcal{P}^*)$ is bounded by $\dim(V)$, and $|\mathcal{P}^*| \leq 2s + 2$.

If the combinatorial level of $(V, \mathcal{P}^*)$ is zero, we have $b(X^*) \leq b(V_\varepsilon)$. If $V \neq \mathcal{U}$, this can be estimated using our bounds on varieties. If $V = \mathcal{U}$, Lemma 2.7 indicates that $b(V_\varepsilon)$ can be estimated from $b(\mathcal{Z}(g - \varepsilon))$. Since $\mathcal{Z}(g - \varepsilon)$ is a compact variety, this last invariant can be estimated by Theorem 2.8 without problem.

Thus, the inductions can be initiated. The Mayer-Vietoris arguments also hold in this case: for instance, we can restrict all the set to $\{g(x) \geq \delta\}$ for $\delta \ll \varepsilon$, so that we keep compact sets at all times.

Thus, analogues of Lemma 2.14 and Theorem 2.17 hold for the case where V is not compact. The precise bounds are slightly different, but we obtain asymptotic bounds which are identical to Corollary 2.18. We will finish this discussion with the following result.

Corollary 2.19 (Complements of $\mathcal{P}$-closed sets) *Let $\boldsymbol{f}$ be a Pfaffian chain defined on a domain $\mathcal{U}$ of bounded complexity. Let Φ be a $\mathcal{P}$-closed Pfaffian formula of format $(n, \ell, \alpha, \beta, s)$ and let $X = \{x \in \mathcal{U} \mid \Phi(x)\}$. We have*

$$b(\mathbb{R}^n \backslash X) \leq s^d \, 2^{\ell(\ell-1)/2} O(n\beta + \min(n, \ell)\alpha)^{n+\ell};$$

where the constant depends on $\mathcal{U}$.

Proof: We can assume without loss of generality that X is compact. By Alexander duality [Bred], the equality $b(\mathbb{R}^n \backslash X) = b(X) + 1$ holds, so the result follows from Corollary 2.18. □

2.2.5 Applications to fewnomials

Now, we apply the results of this section to semi-algebraic sets defined in the positive quadrant $(\mathbb{R}_+)^n$. As explained in Remark 1.6, we can reduce the problem by a change of variables to a problem about Pfaffian functions in a chain of length r, where r is the number of non-zero monomials appearing in the polynomials defining the set. Thus, the following result follows from Corollary 2.18.

Corollary 2.20 *Let $X \subseteq V \subseteq (\mathbb{R}_+)^n$ be defined by a $\mathcal{P}$-closed formula, where $V = \mathcal{Z}(q_1, \dots, q_r)$ and p_i and q_j are polynomials. If $\dim(V) = d$ and the number of non-zero monomials appearing in the polynomials p_i and q_j is r, we have*

$$b(X) \le s^d \, 2^{O(n^2 r^4)}.$$

2.3 Counting the number of cells

Let $\boldsymbol{f}$ be a Pfaffian chain defined on a domain of bounded complexity $\mathcal{U}$, and let $q_1, \dots, q_r$, and $\mathcal{P} = \{p_1, \dots, p_s\}$ be Pfaffian functions in $\boldsymbol{f}$. We let $V = \mathcal{Z}(q_1, \dots, q_r)$. This section is devoted to giving an upper-bound on the number of cells $\mathcal{C}(V; \mathcal{P})$ introduced in Definition 1.29.

Recall that we denote by $\mathfrak{S}$ the set of conjunctions strict sign conditions σ on $\mathcal{P}$. For $\sigma \in \mathfrak{S}$, we let $S(V;\sigma)$ be the corresponding basic set $\{x \in V \mid \sigma(x)\}$.

To count the number of connected components of $S(V;\sigma)$, we construct a variety $V(\sigma) \subseteq S(V;\sigma)$ such that $b_0(V(\sigma)) \ge b_0(S(V;\sigma))$.

2.3.1 Components deformation

Fix positive numbers $a_1, \dots, a_s, b_1, \dots, b_s, \varepsilon$ and δ, and let $\mathcal{P}' = \{p_1 \pm \delta a_1, \dots, p_s \pm \delta a_s, p_1 \pm \eta b_1, \dots, p_s \pm \eta b_s, g - \varepsilon\}$. We let $V_e = V \cap \{g(x) \ge \varepsilon\}$, and we choose $\varepsilon \ll 1$ so that V_ε meets every connected component of every set $S(V;\sigma)$ for $\sigma \in \mathfrak{S}$.

Fix $\delta > 0$, and for any $\sigma \in \mathfrak{S}$, consider the set $C_1(\sigma) \subseteq V_\varepsilon$ defined on $\mathcal{P}'$ by replacing any atom $\{p_i > 0\}$ of σ by $\{p_i \ge \delta a_i\}$ and any atom $\{p_i < 0\}$ by $\{p_i \le -\delta a_i\}$.

Proposition 2.21 *There is $\delta_0 > 0$ such that for all $\delta \le \delta_0$ and for all strict sign condition $\sigma \in \mathfrak{S}$, we have $b_0(S(V;\sigma)) \le b_0(C_1(\sigma))$.*

Proof: It's enough to find a δ_0 for a fixed sign condition σ. Clearly, $C_1(\sigma) \subseteq S(V;\sigma)$, so all we need to do is prove that if D is a connected component of $S(V;\sigma)$, it meets $C_1(\sigma)$ when δ is small enough. Fix $x^* \in D \cap V_\varepsilon$. Then, $x^* \in C_1(\sigma)$ if and only if for all i such that $p_i(x^*) \ne 0$, we have $|p_i(x^*)| \ge \delta a_i$. Since $D \cap V_\varepsilon$ is compact, such a condition will hold for δ small enough. □

Fix $\eta > 0$ and for a sign condition $\sigma \in \mathfrak{S}$, define $C_2(\sigma)$ to be the set defined on V_ε by the following replacement rules: as in the definition of $C_1(\sigma)$, any atom $\{p_i > 0\}$ of σ by $\{p_i \ge \delta a_i\}$ and any atom $\{p_i < 0\}$ by $\{p_i \le -\delta a_i\}$. Moreover, the atoms of the type $\{p_i = 0\}$ are replaced by $\{-\eta b_i \le p_i \le \eta b_i\}$.

Proposition 2.22 *Let $\delta \le \delta_0$ be fixed. Then there exists a $\eta_0 > 0$ such that for all $\sigma \in \mathfrak{S}$, and for all $\eta \le \eta_0$, we have the equality $b_0(C_1(\sigma)) = b_0(C_2(\sigma))$.*

Proof: Again, it is enough to prove the result for a fixed $\sigma \in \mathfrak{S}$. The sets $C_2(\sigma)$ form a decreasing sequence of compacts converging to $C_1(\sigma)$ when η goes to zero, so the result follows readily. □

2.3.2 Varieties and cells

The following result allows to extract from sets defined by weak inequalities varieties that meet every connected components of those sets.

Proposition 2.23 *Let $p_1, \ldots, p_s$ be Pfaffian functions, V a Pfaffian variety and C be a connected component of the set $\{x \in V \mid p_1(x) \geq 0, \ldots, p_s(x) \geq 0\}$. Then, there exists $I \subseteq \{1, \ldots, s\}$, (possibly empty) such that C contains a connected component of the set $V_I = \{x \in V \mid p_i(x) = 0\ \forall i \in I\}$.*

Proof: Take I a set such that $C \cap V_I \neq \varnothing$ and I is maximal for inclusion. Let $x \in C \cap V_I$ and D be the connected component of V_I containing x. Assume $D \not\subseteq C$. Let y in $D \backslash C$: there exists an index $j \notin I$ such that $p_j(y) < 0$. Since $j \notin I$, and $x \in C$, it implies that $p_j(x) > 0$. Let $z(t)$ be a path connecting $x = z(0)$ to $y = z(1)$ in D; by the intermediate value theorem, there exists t_0 such that $p_j(z(t_0)) = 0$. If t_0 is the smallest with this property, we must have $z(t_0) \in C$. But we have $p_i(z(t_0)) = 0$ for all $i \in I \cup \{j\}$, and that contradicts the maximality of I since $j \notin I$. □

Proposition 2.24 *There exists $a_1, \ldots, a_s, \varepsilon$ positive real numbers such with $\varepsilon < \varepsilon_0$ such that, for all $0 < \delta < 1$ we can find positive real numbers $b_1, \ldots, b_s$ for which for all $0 < \eta < 1$, the family*

$$\mathcal{P}' = \{p_1 \pm \delta a_1, \ldots, p_s \pm \delta a_s, p_1 \pm \eta b_1, \ldots, p_s \pm \eta b_s, g - \varepsilon\};$$

is in general position over V.

Proof: This is essentially a repeat of the proof of Proposition 2.11. □

We can now state the main result.

Theorem 2.25 *Let $\boldsymbol{f}$ be a Pfaffian chain defined on a domain $\mathcal{U} \subseteq \mathbb{R}^n$, of bounded complexity γ. Let $\mathcal{P} = \{p_1, \ldots, p_s\}$ be Pfaffian functions defined in the chain $\boldsymbol{f}$ with degree β and V be a Pfaffian variety of dimension d given by equations of degree at most β in the same chain. Then,*

$$\mathcal{C}(V; \mathcal{P}) \leq \Sigma(s, d)\, \mathcal{V}(n, \ell, \alpha, \beta^\star, \gamma); \tag{2.22}$$

where $\beta^\star = \max(\beta, \gamma)$ and

$$\Sigma(s, d) = \sum_{0 \leq i \leq d} \binom{4s + 1}{i}.$$

Proof: According to the results proved in this section, it is enough to bound the number of connected components of all the sets of the form $C_2(\sigma)$ for a suitable choice of the real numbers $a_1, \ldots, a_s, \varepsilon, \delta, b_1, \ldots, b_s$ and η.

For a fixed $\sigma \in \mathfrak{S}$, we can bound the number of connected components of $C_2(\sigma)$, using Proposition 2.23, by counting the connected component of all the Pfaffian varieties defined on V by equations taken among the elements of $\mathcal{P}'$.

According to Proposition 2.24, we can assume that the sets of the form $C_2(\sigma)$ are given by functions which are in general position over V. Then, we have to count the connected components of sets of the form

$$\{x \in V \mid p_{i_1} = \star_{i_1}, \ldots, p_{i_k} = \star_{i_k}\} \text{ or } \{x \in V \mid g = \varepsilon, p_{i_2} = \star_{i_2}, \ldots, p_{i_k} = \star_{i_k}\},$$

where $\star_i \in \{-\delta a_i, -\eta b_i, \delta a_i, \eta b_i\}$, only for $0 \leq k \leq d$.

This gives $\Sigma(s,d)$ possible sets of equations over V. We can then apply Theorem 2.8 and the result follows. □

Remark 2.26 (Combinatorial lemma) *We have* $\Sigma(s,d) \leq (4s+1)^d$.

Proof: By definition, $\Sigma(s,d)$ is the number of subsets of cardinality at most d in a set with $4s+1$ elements. If f is a function from $A = \{1, \ldots, d\}$ to $B = \{1, \ldots, 4s+1\}$, we have $|f(A)| \leq d$, and thus $\Sigma(s,d)$ is bounded by the number of maps $f : A \to B$ which is $(4s+1)^d$. □

Remark 2.27 *As explained in Chapter 1, the bound* (2.22) *has two corollaries: it bounds* $b_0(X)$ *for any semi-Pfaffian set* X, *and bounds the cardinality of the set of consistent sign assignments:* $\{\sigma \in \mathfrak{S} \mid S(V;\sigma) \neq \varnothing\}$. *In particular, note that for a fixed* d, *the bound on* $\mathcal{C}(V;\mathcal{P})$ *is a polynomial in* s.

Corollary 2.28 (Fewnomial case) *Let* $\mathcal{K}$ *be a set of* r *exponents in* $\mathbb{N}^n$. *If* $V \subseteq \mathbb{R}^n$ *is a* d*-dimensional variety defined by* $\mathcal{K}$*-fewnomials and* $\mathcal{P}$ *is a set of* s $\mathcal{K}$*-fewnomials, the number of cells of* $\mathcal{P}$ *over* V *is bounded by*

$$\mathcal{C}(V;\mathcal{P}) \leq \binom{s}{d} 2^{O(n^2 r^4)}.$$

Proof: Divide $\mathbb{R}^n$ in 2^n quadrants and the n coordinate hyperplanes. By Theorem 2.25, a bound of this type holds for each quadrant, and we can iterate this on the coordinate hyperplanes. The number of cells is then bounded by the sum of the number of cells of the restriction to each set in the partition. Thus we get

$$\mathcal{C}(V;\mathcal{P}) \leq 2^n \binom{s}{d} 2^{O(n^2 r^4)} + n \binom{s}{d} 2^{O((n-1)^2 r^4)};$$

and the result follows. □

2.4 Borel-Moore homology of semi-Pfaffian sets

We conclude this chapter by estimates on the Borel-Moore Betti numbers of a locally closed semi-Pfaffian set. These estimates follow the techniques that appear in [Bürg, MMP, Yao] but yield a tighter bound even in the semi-algebraic case because of our use of the improved bound on the number of cells (Theorem 2.25 in the previous section) derived from [BPR1]. In particular, this estimate lets us bound $b(X)$ for X a compact semi-Pfaffian set which is *not* necessarily defined by a $\mathcal{P}$-closed formula. However, this estimate was recently outranked by recent work of Gabrielov and Vorobjov [GV4]. Their result is stated in Theorem 2.34 for reference purposes.

Throughout the rest of the section, we will assume without loss of generality that all sets under consideration are *bounded.*

Recall that a *locally closed* subset of $\mathbb{R}^n$ is any set that can be defined as the intersection of an open set and a closed set. In particular, any *basic* semi-Pfaffian set is locally closed, but a general semi-Pfaffian set is not necessarily so, since clearly the subset of $\mathbb{R}^2$ defined by $\{x < 0, y < 0\} \cup \{x \geq 0, y \geq 0\}$ is not locally closed.

Definition 2.29 (Borel-Moore homology) *Let X be a locally closed semi-Pfaffian set. We then define its Borel-Moore homology by*

$$H_*^{BM}(X) = H_*(\overline{X}, \partial X; \mathbb{Z}).$$

We will denote by $b^{BM}(X)$ the rank of $H_*^{BM}(X)$.

Note that when X is *compact,* we have of course $b(X) = b^{\mathrm{BM}}(X)$. The key property of Borel-Moore homology is the following result.

Lemma 2.30 *Let X be a locally closed semi-Pfaffian set and $Y \subseteq X$ be closed in X. Then, the following inequality holds.*

$$b^{BM}(X) \leq b^{BM}(X \backslash Y) + b^{BM}(Y). \tag{2.23}$$

Proof: (See also [BCR, §11.7].) Let $C \subseteq B \subseteq A$ be compact definable sets. We can triangulate A so that B and C are subcomplexes of A. This yields an exact sequence

$$\cdots \longrightarrow H_{i+1}(A, B) \longrightarrow H_i(B, C) \longrightarrow H_i(A, C) \longrightarrow H_i(A, B) \longrightarrow \cdots \tag{2.24}$$

Now, if X is bounded and locally closed, we have $X = U \cap F$ for U open and F closed. Thus, we have $\partial X = \partial U \cap F$, so ∂X is compact, and since Y is closed in X, the set $\partial X \cup Y$ is compact too. Thus, setting $A = \overline{X}$, $B = \partial X \cup Y$ and $C = \partial X$ in (2.24), we obtain the exact sequence

$$\cdots \longrightarrow H_{i+1}(\overline{X}, \partial X \cup Y) \longrightarrow H_i(\partial X \cup Y, \partial X) \longrightarrow H_i(\overline{X}, \partial X) \longrightarrow H_i(\overline{X}, \partial X \cup Y) \longrightarrow \cdots$$

Let Y' be the interior of Y in $\overline{X}$. It is easy to check that $\overline{X\backslash Y} = \overline{X}\backslash Y'$ and that $\partial(X\backslash Y) = \partial X \cup (Y\backslash Y') = (\partial X \cup Y)\backslash Y'$ (the last one since $\partial X \cap Y' = \varnothing$). Thus, by excision, we obtain for all i the following isomorphism

$$H_i(\overline{X}, \partial X \cup Y) \cong H_i(\overline{X}\backslash Y', (\partial X \cup Y)\backslash Y') = H^{\text{BM}}(X\backslash Y).$$

Similarly, since Y is closed in X, we have $\partial Y \subseteq \partial X$, so if Z is the interior of ∂X in $\partial X \cup Y$, we have $x \in \partial X \cap \partial Y$ if and only if $x \notin Z$. Hence, one obtains by excision that

$$H_i(\partial X \cup Y, \partial X) \cong H_i((\partial X \cup Y)\backslash Z, \partial X\backslash Z) = H^{\text{BM}}(Y).$$

Thus, we end up with the long exact sequence

$$\cdots \longrightarrow H^{\text{BM}}_{i+1}(X\backslash Y) \longrightarrow H^{\text{BM}}_i(Y) \longrightarrow H^{\text{BM}}_i(X) \longrightarrow H^{\text{BM}}_i(X\backslash Y) \longrightarrow \cdots$$

The inequality (2.23) then follows easily. □

This result allows to derive immediately an upper-bound for any basic set.

Proposition 2.31 *Let $\mathcal{P} = \{p_1, \dots, p_s\}$ be a family of Pfaffian functions in a given chain $\boldsymbol{f}$ of length ℓ and degree α, defined on a domain $\mathcal{U}$ of bounded complexity. Suppose that the maximum of $\deg_{\boldsymbol{f}} p_i$ is bounded by β, and let $\sigma \in \mathfrak{S}$ be a strict sign condition on $\mathcal{P}$. Then, if V is a Pfaffian variety of dimension d defined by equations of degree bounded by β, we have*

$$b^{BM}(S(V;\sigma)) \leq s^d\, 2^{\ell(\ell-1)/2} O(n\beta + \min(n,\ell)\alpha)^{n+\ell}; \tag{2.25}$$

where the constant depends only on the domain $\mathcal{U}$.

Proof: Without loss of generality, we can assume that we have

$$S(V;\sigma) = \{x \in V \mid p_1(x) = \cdots = p_r(x) = 0, p_{r+1} > 0, \dots, p_s(x) > 0\}.$$

Let $q = p_{r+1} \cdots p_s$, and define the sets

$$X = \{x \in V \mid p_1(x) = \cdots = p_r(x) = 0, p_{r+1} \geq 0, \dots, p_s(x) \geq 0\};$$
$$Y = \{x \in V \mid p_1(x) = \cdots = p_r(x) = 0,\ q(x) = 0,\ p_{r+1} \geq 0, \dots, p_s(x) \geq 0\}.$$

The sets X and Y are closed, with $Y \subseteq X$, and we have $S(V;\sigma) = X\backslash Y$. Thus, by Lemma 2.30, we have

$$b^{\text{BM}}(S(V;\sigma)) \leq b^{\text{BM}}(X) + b^{\text{BM}}(Y) = b(X) + b(Y);$$

(since X and Y are compact). The upper-bound follows from the estimates on the sum of Betti numbers for $\mathcal{P}$-closed formulas appearing in Corollary 2.18. □

Theorem 2.32 *Let V be a Pfaffian variety of dimension d and X be a locally closed semi-Pfaffian subset of V of format $(n,\ell,\alpha,\beta,\gamma,s)$. The rank of the Borel-Moore homology of X verify*

$$b^{BM}(X) \leq s^{2d}\, 2^{\ell(\ell-1)} O(n\beta + \min(n,\ell)\alpha)^{2(n+\ell)}; \tag{2.26}$$

where the constant depends only on the domain $\mathcal{U}$.

Proof: Let $\mathcal{P} = \{p_1,\dots,p_s\}$ be the set of possible functions appearing in the atoms of the formula defining X. Using Lemma 2.30 twice, we obtain

$$b^{\mathrm{BM}}(X) \leq b^{\mathrm{BM}}(X\cap\{p_1<0\}) + b^{\mathrm{BM}}(X\cap\{p_1=0\}) + b^{\mathrm{BM}}(X\cap\{p_1>0\}).$$

Repeating this inductively for $p_2,\dots,p_s$, we obtain

$$b^{\mathrm{BM}}(X) \leq \sum_{\sigma\in\mathfrak{S}} b^{\mathrm{BM}}(X\cap S(V;\sigma)). \tag{2.27}$$

Since X is defined on V by a sign condition on $\mathcal{P}$, the intersections $X\cap S(V;\sigma)$ are either empty or equal to $S(V;\sigma)$. The bound (2.25) is known for $b^{\mathrm{BM}}(S(V;\sigma))$, and since the number of terms appearing in the right-hand side of (2.27) is bounded by the number of cells $\mathcal{C}(V;\mathcal{P})$ of $\mathcal{P}$ on V, we can combine the bound from Theorem 2.25 to (2.25) to obtain the above upper-bound on $b^{\mathrm{BM}}(X)$. □

Remark 2.33 (Compact case) *As mentioned earlier, if X is a compact semi-Pfaffian set, it is certainly locally closed and verifies $H_*(X) = H_*^{BM}(X)$, and thus, in this case, Theorem 2.32 gives an upper-bound on $b(X)$.*

We conclude this chapter by giving, for reference purposes, a very recent result (Summer 2003) of Gabrielov and Vorobjov. It is the most general upper-bound known for the sum of the Betti numbers of semi-Pfaffian set, since it does not have require any hypothesis on the topology of the set or the shape of the defining formula. As mentioned earlier, it gives for compact sets a sharper bound than Theorem 2.32.

Theorem 2.34 (Gabrielov-Vorobjov [GV4]) *Let X be* any *semi-Pfaffian set defined by a quantifier free formula of format (n,ℓ,α,β,s). The sum of the Betti numbers of X admits a bound of the form*

$$b(X) \leq 2^{\ell(\ell-1)/2}\, s^{2n}\, O(n\beta + \min(n,\ell)\alpha)^{n+\ell}; \tag{2.28}$$

where the constant depends only on the definable domain $\mathcal{U}$.

The estimate (2.28) is obtained by constructing a set X_1 defined by a $\mathcal{P}^*$-closed formula, where

$$\mathcal{P}^* = \{h_i \mid 1\leq i\leq s\} \cup \{h_i^2 - \varepsilon_j \mid 1\leq i,j\leq s\}.$$

For a suitable choice $1 \gg \varepsilon_1 \gg \cdots \gg \varepsilon_s > 0$, we have $b(X_1) = b(X)$, and the result then follows from Theorem 2.17.

Chapter 3

Betti numbers of sub-Pfaffian sets

The first section of this chapter is devoted to proving the following theorem.

Theorem 3.1 *Let $f : X \to Y$ be a surjective continuous* compact covering[1] *map. Then, for all $k \in \mathbb{N}$, we have*

$$b_k(Y) \le \sum_{p+q=k} b_q(\mathcal{W}_f^p(X)); \tag{3.1}$$

where $\mathcal{W}_f^p(X)$ is the $(p+1)$-fold fibered product of X over f,

$$\mathcal{W}_f^p(X) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in X^{p+1} \mid f(\mathbf{x}_0) = \cdots = f(\mathbf{x}_p)\}. \tag{3.2}$$

The rest of the chapter contains applications of this result to establish upper-bounds on the Betti numbers of sub-Pfaffian subsets of the cube. Applications of this theorem to relative closures will be found in Chapter 5.

If $I = [0, 1]$ and X is a semi-Pfaffian set in the $N = n_0 + \cdots + n_\nu$-dimensional cube I^N, and $Q_1, \ldots, Q_\nu$ is a sequence of alternating quantifiers, the set

$$S = \{\mathbf{x}_0 \in I^{n_0} \mid Q_\nu \mathbf{x}_\nu \in I^{n_\nu} \ldots Q_1 \mathbf{x}_1 \in I^{n_1}, (\mathbf{x}_0, \ldots, \mathbf{x}_\nu) \in X\};$$

is a sub-Pfaffian set [G3]. If X is semi-algebraic, then S is semi-algebraic too. When ν is small, the bounds established here for $b(S)$ are better than the previously known bounds coming from cell decomposition [Col, GV2, PV] or quantifier elimination.

The chapter is organized as follows: the first section describes the construction of a spectral sequence $E^r_{p,q}$ that gives Theorem 3.1. Section 2 contains various topological lemmas, and section 3 applies the theorem to the case of a set defined by one quantifier block (Theorem 3.20 and Corollary 3.21), initiating the induction. In section 4, we establish an inductive relation for the bound for ν quantifiers (Theorem 3.24), and use it to deduce general upper-bounds (Corollary 3.25 for the Pfaffian case and Corollary 3.26 for the algebraic case). For semi-algebraic sets, a comparison is presented in section 5 between those bounds and the previously available ones using quantifier elimination.

[1] $f : X \to Y$ is compact covering if and only if for any compact $L \subseteq Y$, there exists a compact $K \subseteq X$ such that $f(K) = L$.

3.1 Spectral sequence of a surjective map

For a *closed* surjection f, Theorem 3.1 will be proved in the following way. We will construct a space $J^f(X)$ which is homotopically equivalent to Y, and has a natural filtration $J^f_p(X)$. This filtration gives rise to a spectral sequence $E^r_{p,q}$ converging to the homology of $J^f(X)$, as described in the Appendix. The first term of the sequence $E^1_{p,q}$ is isomorphic to $H_q(\mathcal{W}^p_f(X))$, which will prove the result. The convergence for a compact-covering map will be deduced from the closed case in Theorem 3.12.

In this section, X, Y and $P_0, \dots P_p$ are topological spaces, and $f_i : P_i \to Y$ are continuous surjective maps. We denote by Δ^p the standard p-simplex

$$\Delta^p = \{s = (s_0, \dots, s_p) \in \mathbb{R}^{p+1} \mid s_0 \geq 0, \dots, s_p \geq 0,\ s_0 + \cdots + s_p = 1\}.$$

Definition 3.2 (Join) *For a sequence $(P_0, \dots, P_p)$ of topological spaces, their* join $P_0 * \cdots * P_p$ *can be defined as the quotient*

$$P_0 \times \cdots \times P_p \times \Delta^p / \sim;$$

where $\sim$ is the join *relation*

$$(x_0, \dots, x_p, s) \sim (x'_0, \dots, x'_p, s') \text{ iff } s = s' \text{ and } (s_i \neq 0) \Rightarrow (x_i = x'_i). \tag{3.3}$$

Recall that if for all i, $f_i : P_i \to Y$ is a continuous surjective map, we can define the *fibered product*

$$P_0 \times_Y \cdots \times_Y P_p = \{(x_0, \dots, x_p) \in P_0 \times \cdots \times P_p \mid f_0(x_0) = \cdots = f_p(x_p)\}.$$

Note that there is a natural map

$$\begin{aligned} F : P_0 \times_Y \cdots \times_Y P_p &\to Y \\ (x_0, \dots, x_p) &\mapsto f_i(x_i); \quad \text{(taking any } 0 \leq i \leq p.) \end{aligned}$$

Definition 3.3 (Fibered join) *For $P_0, \dots, P_p$ as above, we define the* fibered join $P_0 *_Y \cdots *_Y P_p$ *as the quotient space of $P_0 \times_Y \cdots \times_Y P_p \times \Delta^p$ over the join relation* (3.3).

The map $F : P_0 \times_Y \cdots \times_Y P_p \to Y$ extends naturally to $P = P_0 *_Y \cdots *_Y P_p$. Indeed, for any $(x_0, \dots, x_p)$ and $(x'_0, \dots, x'_p)$ in $P_0 \times_Y \cdots \times_Y P_p \to Y$ such that $(x_0, \dots, x_p, s) \sim (x'_0, \dots, x'_p, s)$ for some $s \in \Delta^p$, we must have $x_i = x'_i$ for some $0 \leq i \leq p$, and thus $F(x_0, \dots, x_p) = F(x'_0, \dots, x'_p)$. We still denote this map by F. For any point $y \in Y$ the fiber $F^{-1}(y)$ coincides with the join $f_0^{-1}(y) * \cdots * f_p^{-1}(y)$ of the fibers of f_i.

The other natural map is the projection $\pi : P \to \Delta^p$. If s is in the interior of Δ^p, the equivalence relation $\sim$ is trivial over s, so we must have

$$\forall s \in \mathrm{int}(\Delta^p), \quad \pi^{-1}(s) = P_0 \times_Y \cdots \times_Y P_p.$$

For $0 \leq i \leq p$, define Q_i to be the fibered join

$$Q_i = P_0 *_Y \cdots *_Y P_{i-1} *_Y P_{i+1} *_Y \cdots *_Y P_p.$$

Then, one can define a map $\varphi_i : Q_i \to P$ by

$$\varphi_i(y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_p, t) = (y_0, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_p, s)$$

where $x_i \in P_i$ is any point and

$$s_j = \begin{cases} t_j & \text{if } j < i; \\ 0 & \text{if } j = i; \\ t_{j-1} & \text{if } j > i. \end{cases}$$

Lemma 3.4 *The map φ_i is an embedding $Q_i \to P$, and $\varphi_i(Q_i) = \pi^{-1}\{s_i = 0\}$. Moreover, the space*

$$P/\left(\bigcup_i \varphi_i(Q_i)\right);$$

is homotopy equivalent to the p-th suspensionof $P_0 \times_Y \cdots \times_Y P_p$.

Proof: The map φ_i sends Q_i to points $(x_0, \dots, x_p, s)$ of P such that $s_i = 0$ by construction. This means also that φ_i does not depend on the choice of the point $x_i \in P_i$. □

We now consider the case of a continuous surjection $f : X \to Y$. For all $p \in \mathbb{N}$, define $J_p^f(X)$ to be the fibered join of $p+1$ copies of X over f,

$$J_p^f(X) = \underbrace{X *_Y \cdots *_Y X}_{p+1 \text{ times}}. \tag{3.4}$$

Definition 3.5 (Join space) *If f is as above, the* join space *$J^f(X)$ is the quotient space*

$$\bigsqcup_p J_p^f(X)/\sim; \tag{3.5}$$

where we identify, for all $p \in \mathbb{N}$,

$$J_{p-1}^f(X) \sim \varphi_i(J_{p-1}^f(X)) \subseteq J_p^f(X), \quad \textit{for all } 0 \leq i \leq p.$$

When Y is a point, we write $J_p(X)$ instead of $J_p^f(X)$ and $J(X)$ instead of $J^f(X)$.

Lemma 3.6 *Let $\varphi : J_p(X) \to J(X)$ be the natural map induced by the maps φ_i. Then $\varphi(J_{p-1}(X))$ is contractible in $\varphi(J_p(X))$.*

Proof: Let x be a point in X. For $t \in [0,1]$, the maps $g_t(x, x_1, \ldots, x_p, s) \mapsto (x, x_1, \ldots, x_p, ts)$ define a contraction of $\varphi_0(J_{p-1}(X))$ to the point $x \in X$ where X is identified with its embedding in $J_p(X)$ as $\pi^{-1}(1, 0, \ldots, 0)$. It is easy to see that the maps g_t are compatible with the equivalence relations in Definition 3.5 and define a contraction of $\varphi(J_{p-1}(X))$ to a point in $\varphi(J_p(X))$. □

Proposition 3.7 *Let $f: X \to Y$ be a closed surjective continuous map, where X and Y are definable in an o-minimal structure. If $J^f(X)$ is the join space introduced in Definition 3.5, we have*

$$H_*(J^f(X)) \cong H_*(Y).$$

Proof: Let $F: J^f(X) \to Y$ be the natural map induced by f. Its fiber $F^{-1}y$ over a point $y \in Y$ coincides with the join space $J(f^{-1}y)$. According to Lemma 3.6, $\varphi(J_{p-1}(f^{-1}y))$ is contractible in $J(f^{-1}y)$, for each p, so $\bar{H}^*(J(f^{-1}y)) = 0$.

Since $J(f^{-1}y)$ is a definable quotient, it is locally contractible, so by Proposition A.5, we also have for the reduced Alexander cohomology $\widetilde{\bar{H}}^*(J(f^{-1}y)) = 0$.

Since $F: J^f(X) \to Y$ is a closed continuous surjection with fibers that are trivial for the Alexander cohomology, we can apply to F the Vietoris-Begle theorem (Theorem A.6) to obtain $\bar{H}^*(J^f(X)) \cong \bar{H}^*(Y)$, which implies that $H_*(J^f(X)) \cong H_*(Y)$. □

Notation 3.8 (Fibered products) *Throughout this chapter, for $f: X \to Y$ a continuous surjection, we will denote by $\mathcal{W}_f^p(X)$ the $(p+1)$-fold fibered product of X over f,*

$$\mathcal{W}_f^p(X) = \underbrace{X \times_Y \cdots \times_Y X}_{p+1 \text{ times}} = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in (X)^{p+1} \mid f(\mathbf{x}_0) = \cdots = f(\mathbf{x}_p)\}.$$

Moreover, if $A \subseteq X$, we will denote by $\mathcal{W}_f^p(A)$ the corresponding fibered product for the restriction $f|_A$.

Theorem 3.9 (Spectral sequence, closed case) *Let $f: X \to Y$ be a closed surjective continuous map, where X and Y are definable in an o-minimal structure. Then, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y)$ with*

$$E^1_{p,q} \cong H_q(\mathcal{W}_f^p(X)) \tag{3.6}$$

Proof: By Theorem A.1, the filtration of $J^f(X)$ by the spaces $J_p^f(X)$ gives rise to a spectral sequence $E^r_{p,q}$ converging to $H_*(J^f(X))$, which, by Proposition 3.7, is isomorphic to $H_*(Y)$.

The first term of the sequence is $E^1_{p,q} = H_{p+q}(A_p)$, where

$$A_p = J_p^f(X) / \left(\bigcup_{q<p} J_q^f(X) \right).$$

From Lemma 3.4, the space A_p is homotopy equivalent to the p-th suspension of $\mathcal{W}_f^p(X)$, and thus we have $E^1_{p,q} \cong H_q(\mathcal{W}_f^p(X))$, proving the theorem. □

One of the features of spectral sequences is that the rank of the limit of a spectral sequence is controlled by the rank of the initial terms. Such estimates are discussed in more details in Corollary A.2 in the appendix, and applying them to the present situation yields the estimates of Theorem 3.1 for a closed f.

Remark 3.10 *The condition of o-minimality is not really important here. If X is the difference between a finite CW-complex and one of its subcomplexes, and Y is of the same type, the spaces $J(f^{-1}y)$ are still locally contractible and the result still holds.*

For a locally split map [2] f, the convergence of the same spectral sequence can be derived from [DHI, Corollary 1.3]. However, this follows from a more general result: the spectral sequence converges when f is a *compact covering* map. Let us first recall the definition.

Definition 3.11 (Compact covering) *A map $f : X \to Y$ is called* compact covering *if for all compact $L \subseteq Y$, there exists a compact $K \subseteq X$ such that $f(K) = L$.*

Note that if f is *closed* or *locally split,* it is necessarily compact-covering.

Theorem 3.12 (Spectral sequence, compact covering case) *Let $f : X \to Y$ be a definable, compact covering surjection. Then, there exists a spectral sequence $E^r_{p,q}$ converging to $H_*(Y)$ with*

$$E^1_{p,q} \cong H_q(\mathcal{W}_f^p(X)). \tag{3.7}$$

Proof: Recall that the singular homology of a space is isomorphic to the direct limit of its compact subsets [Spa, Theorem 4.4.6]. Since f is compact covering, if K and L range over all compact subsets of X and Y respectively, the following inductive limits verify

$$\varinjlim H_*(f(K)) = \varinjlim H_*(L) \cong H_*(Y). \tag{3.8}$$

Let p be fixed and L_p be a compact subset of the fibered product $\mathcal{W}_f^p(X)$. If for all $0 \le i \le p$, π_i denotes the canonical projection $(\mathbf{x}_0, \dots, \mathbf{x}_p) \mapsto \mathbf{x}_i$, we let $K_p = \pi_0(L_p) \cup \dots \cup \pi_p(L_p)$. Observe then that the set $\mathcal{W}_f^p(K_p)$ is a compact subset of $\mathcal{W}_f^p(X)$ containing L_p. Thus, we also have the following equality

$$\varinjlim H_*(\mathcal{W}_f^p(K_p)) = \varinjlim H_*(L_p) \cong H_*(\mathcal{W}_f^p(X)). \tag{3.9}$$

For any compact subset K of X, the restriction $f|_K$ is closed, so by Theorem 3.9, there exists a spectral sequence $E^r_{p,q}(K)$ that converges to $H_*(f(K))$ and such that $E^1_{p,q}(K) \cong H_q(\mathcal{W}_f^p(K))$. By (3.8) and (3.9), the direct limit of $E^r_{p,q}(K)$ when K ranges over all compact subsets of X is a spectral sequence converging to $H_*(Y)$ and verifying (3.7). □

[2] A map $f : X \to Y$ is *locally split* if it admits continuous sections defined around any point $y \in Y$. In particular, the projection of an open set is always locally split.

Example 3.13 *Note that without an additional assumption on X and Y, the spectral sequence may not converge to $H_*(Y)$. For instance, consider for X any open segment in $\mathbb{R}^3$. Let $\{a, b\} = \partial X$, assume $a \neq b$ and let f be any projection such that f is 1-to-1 on X and there exists $c \in X$ such that $f(a) = f(b) = f(c)$. Then, if $Y = f(X)$, we have $b_1(Y) = 2$, but since X is contractible, $b_1(X) = 0$, and since f is 1-to-1 on X, we have $b_0(\mathcal{W}_f^1(X)) = 1$, so $b_1(Y) > b_1(X) + b_0(\mathcal{W}_f^1(X))$. The inequality of Theorem 3.1 does not hold in this case.*

Remark 3.14 *For a map f with 0-dimensional fibers, a similar spectral sequence, called "image computing spectral sequence", was applied to problems in theory of singularities and topology by Vassiliev [V], Goryunov-Mond [GoM], Goryunov [Go], Houston [Hou], and others. In sheaf cohomology, the corresponding spectral sequence is known as* cohomological descent *[Del].*

3.2 Topological lemmas

Throughout the rest chapter, I denotes the closed interval $[0, 1]$ and *open* and *closed* are meant in a cube I^m (for some m).

Lemma 3.15 *Let $X \subseteq I^{n+p}$ be closed (resp. open). Then, the sets*

$$Y = \{\mathbf{y} \mid \exists \mathbf{x} \in I^p, (\mathbf{x}, \mathbf{y}) \in X\}, \text{ and } \quad Z = \{\mathbf{y} \mid \forall \mathbf{x} \in I^p, (\mathbf{x}, \mathbf{y}) \in X\};$$

are both closed (resp. open).

Proof: Let π be the canonical projection $\mathbb{R}^{n+p} \to \mathbb{R}^n$. The sets Y and Z can be defined by $Y = \pi(X)$ and $I^n \backslash Z = \pi(I^{n+p} \backslash X)$. Since π is continuous, it sends closed sets to closed sets, since any closed subset of a cube is compact. Moreover, π also sends open sets to open sets. The result then follows easily. □

Lemma 3.16 (Alexander duality in the cube 1) *Let $X \subseteq I^n$ be a definable open set. For any $0 \leq q \leq n-1$, we have*

$$\tilde{H}_q(X \cup J^n) \cong H^{n-q-1}(I^n \backslash X); \tag{3.10}$$

where $J^n = (-\varepsilon, 1+\varepsilon)^n \backslash I^n$ for some $\varepsilon > 0$.

Proof: Let $S^n = \mathbb{R}^n \cup \{\infty\}$ be the one-point compactification of $\mathbb{R}^n$, and let K be the complement of $X \cup J^n$ in S^n. Since K is closed and not empty,we have by Alexander duality in S^n [Bred, Corollary VI.8.6];

$$\tilde{H}_q(X \cup J^n) = \tilde{H}_q(S^n \backslash K) \cong \tilde{\check{H}}^{n-q-1}(K); \tag{3.11}$$

and since K is triangulable, the right-hand side of this equation is isomorphic to $\tilde{H}^{n-q-1}(K)$. If $C_0, \ldots, C_N$ are the connected components of K, where $\infty \in C_0$, then $C_0 = S^n \backslash (-\varepsilon, 1+\varepsilon)^n$ is contractible and $C_1 \cup \cdots \cup C_N = I^n \backslash X$. This implies that

$$H^*(K) \cong H^*(I^n \backslash X) \oplus H^*(C_0);$$

and as C_0 is contractible, we obtain that $\tilde{H}^*(K) \cong H^*(I^n \backslash X)$. Substituting this result in (3.11) gives the lemma. □

To prove a similar result in the case where X is closed, we will need the following lemma.

Lemma 3.17 *Let I_0 be the open interval $(0,1)$, and let $X \subseteq I^n$ be closed. Then, we have $H_*(I_0^n \backslash X) \cong H_*(I^n \backslash X)$.*

Proof: Consider for $\delta > 0$ the set X_δ defined by

$$X_\delta = \{x \in I^n \mid \mathrm{dist}(x, X) < \delta\}.$$

Since $I^n \backslash X_\delta$ is a compact subset of $I^n \backslash X$, we have $H_*(I^n \backslash X) \cong H_*(I^n \backslash X_\delta)$ for $\delta \ll 1$. (The proof uses the same arguments as the proof of Lemma 1.78.) But clearly $I^n \backslash X_\delta$ is homotopy equivalent to $I_0^n \backslash X_\delta$, and we also have $H_*(I_0^n \backslash X_\delta) \cong H_*(I_0^n \backslash X)$ for $\delta \ll 1$. □

Lemma 3.18 (Alexander duality in the cube 2) *Let $X \subseteq I^n$ be a definable closed subset. For any $0 \leq q \leq n-1$, we have*

$$H_q(I^n \backslash X) \cong \tilde{H}^{n-q-1}(X \cup \partial I^n). \tag{3.12}$$

Proof: Let us consider the compact set $K = X \cup \partial I^n$. As in the proof of Lemma 3.16, Alexander duality in S^n gives

$$\tilde{H}_q(S^n \backslash K) \cong \tilde{\check{H}}^{n-q-1}(K). \tag{3.13}$$

Again, the right-hand side above is isomorphic to $\tilde{H}^{n-q-1}(K)$, since K is triangulable. Let $C_0, \ldots, C_N$ be the connected components of $S^n \backslash K$, with $\infty \in C_0$. The component C_0 is simply $S^n \backslash I^n$, and thus is contractible. As before, we can derive from this that $\tilde{H}_*(S^n \backslash K) \cong H_*(I^n \backslash K)$. If $I_0 = (0,1)$, we have $I^n \backslash K = I_0^n \backslash X$, and we can conclude using Lemma 3.17. □

Lemma 3.19 (Generalized Mayer-Vietoris inequalities) *Let $X_1, \ldots, X_m \subseteq I^n$ be all open or all closed in I^n. Then*

$$b_i\left(\bigcup_{1\leq j\leq m} X_j\right) \leq \sum_{J\subseteq\{1,\ldots,m\}} b_{i-|J|+1}\left(\bigcap_{j\in J} X_j\right); \tag{3.14}$$

and

$$b_i\left(\bigcap_{1\leq j\leq m} X_j\right) \leq \sum_{J\subseteq\{1,\ldots,m\}} b_{i+|J|-1}\left(\bigcup_{j\in J} X_j\right). \tag{3.15}$$

Proof: The proof is by induction by m. For $m = 2$, the result follows from the exactness of the Mayer-Vietoris sequence of (X_1, X_2). If the result is true up to $m-1$, then define $Y = X_2 \cup \cdots \cup X_m$ and $Z = X_2 \cap \cdots \cap X_m$. Then (3.14) and (3.15) follow from the Mayer-Vietoris sequences of (X_1, Y) and (X_1, Z) respectively. □

3.3 The one quantifier case

In this section, we apply the spectral sequence discussed earlier to sub-Pfaffian sets defined using a single quantifier.

Let $\boldsymbol{f}$ be a Pfaffian chain on a domain $\mathcal{U} \subseteq \mathbb{R}^{n_0+n_1}$ of bounded complexity γ for $\boldsymbol{f}$, such that $I^{n_0+n_1} \subseteq \mathcal{U}$. Let $\mathcal{P}$ be a set of Pfaffian functions in the chain $\boldsymbol{f}$ and $\Phi(\mathbf{x}_0, \mathbf{x}_1)$ be a $\mathcal{P}$-closed formula. We denote by π_0 the canonical projection $\mathbb{R}^{n_0+n_1} \to \mathbb{R}^{n_0}$.

Theorem 3.20 (Existential bound) *Let Φ be as above, and let $X = \{(\mathbf{x}_0, \mathbf{x}_1) \in I^{n_0+n_1} \mid \Phi(\mathbf{x}_0, \mathbf{x}_1)\}$ and $Y = \pi_0(X)$. Then, if the format of Φ is $(n_0 + n_1, \alpha, \beta, s)$, we have for all $k \in \mathbb{N}$,*

$$b_k(Y) \leq (ks + n_0 + kn_1)^N \, 2^{L(L-1)/2} \, O(\, N\beta + \min(N, L)\, \alpha)^{N+L}; \tag{3.16}$$

where $N = n_0 + (k+1)n_1$ and $L = (k+1)\ell$.

Moreover, if $X' = \{(\mathbf{x}_0, \mathbf{x}_1) \in I^{n_0+n_1} \mid \neg\Phi(\mathbf{x}_0, \mathbf{x}_1)\}$ and $Y' = \pi_0(X')$, the same bound holds for $b_k(Y')$.

Proof: Theorem 3.1 is applicable to the map π_0 restricted to X, giving

$$b_k(Y) \leq \sum_{p+q=k} b_q(\mathcal{W}^p(X));$$

where $\mathcal{W}^p(X)$ is the $(p+1)$-fold fibered product of X over Y. We can build from the Pfaffian chain $\boldsymbol{f}$ a chain $F = (\boldsymbol{f}(\mathbf{x}_0, \mathbf{y}_0), \ldots, \boldsymbol{f}(\mathbf{x}_0, \mathbf{y}_p))$ of length $(p+1)\ell$ and degree α by substituting successively each $\mathbf{y}_j$ for $\mathbf{x}_1$. The set $\mathcal{W}^p(X)$ is defined by the following quantifier-free formula in that chain.

$$(\mathbf{x}_0, \mathbf{y}_0, \ldots, \mathbf{y}_p) \in I^{n_0} \times I^{(p+1)n_1} \wedge \Phi(\mathbf{x}_0, \mathbf{y}_0) \wedge \cdots \wedge \Phi(\mathbf{x}_0, \mathbf{y}_p).$$

This formula is $\mathcal{P}'$-closed for some $\mathcal{P}'$, and its format is

$$(n_0 + (p+1)n_1, (p+1)\ell, \alpha, \beta, (p+1)s + 2[n_0 + (p+1)n_1]);$$

and by Corollary 2.18, we have

$$b(\mathcal{W}^p(X)) \leq (ps + n_0 + pn_1)^{n_0+(p+1)n_1} \, 2^{(p+1)\ell((p+1)\ell-1)/2}$$
$$O[(n_0 + pn_1)\beta + \min(n_0 + pn_1, p\ell)\alpha]^{n_0+(p+1)(n_1+\ell)}.$$

Summing the above for $0 \le p \le k$ gives (3.16).

When considering the case of Y', Theorem 3.1 is again applicable, so we still have $b_k(Y') \le \sum_{p+q=k} b_q(\mathcal{W}^p(X'))$, but here $\mathcal{W}^p(X')$ is defined by the formula

$$(\mathbf{x}_0, \mathbf{y}_0, \ldots, \mathbf{y}_p) \in I^{n_0} \times I^{(p+1)n_1} \wedge \neg\Phi(\mathbf{x}_0, \mathbf{y}_0) \wedge \cdots \wedge \neg\Phi(\mathbf{x}_0, \mathbf{y}_p);$$

which is neither a $\mathcal{P}$-closed formula nor the negation of one, but we can reduce to that case in the following way. Let I_ε be the interval $[\varepsilon, 1-\varepsilon]$, and let $\mathcal{W}^p_\varepsilon(X')$ be the set defined by

$$\mathcal{W}^p_\varepsilon(X') = \{(\mathbf{x}_0, \mathbf{y}_0, \ldots, \mathbf{y}_p) \in \mathrm{int}(I^{n_0+(p+1)n_1}) \mid \neg\Phi(\mathbf{x}_0, \mathbf{y}_0) \wedge \cdots \wedge \neg\Phi(\mathbf{x}_0, \mathbf{y}_p)\}.$$

For $\varepsilon \ll 1$, we have $b_q(\mathcal{W}^p(X')) = b_q(\mathcal{W}^p_\varepsilon(X'))$ for all q. Since $\mathcal{W}^p_\varepsilon(X')$ is given by the negation of a $\mathcal{P}'$-closed formula of format

$$(n_0 + (p+1)n_1, (p+1)\ell, \alpha, \beta, (p+1)s + 4[n_0 + (p+1)n_1]).$$

Corollary 2.19 is applicable, and yields the same asymptotic bound as the bound for $b(\mathcal{W}^p(X))$. □

Corollary 3.21 (Universal bound) *Let Φ be as above and let $Z = \{\mathbf{x}_0 \in I^{n_0} \mid \forall \mathbf{x}_1 \in I^{n_1}, \neg\Phi(\mathbf{x}_0, \mathbf{x}_1)\}$. Then, if the format of Φ is $(n_0 + n_1, \alpha, \beta, s)$, we have for all $k \in \mathbb{N}$,*

$$b_k(Z) \le (n_0 + (n_0 - k)(s + n_1))^{N^*}\, 2^{L^*(L^*-1)/2}\, O(\, N^*\beta + \min(N^*, L^*)\,\alpha)^{N^*+L^*}; \quad (3.17)$$

where $N^* = n_0 + (n_0 - k)n_1$, and $L^* = (n_0 - k)\ell$.

Moreover, if $Z' = \{\mathbf{x}_0 \in I^{n_0} \mid \forall \mathbf{x}_1 \in I^{n_1}, \Phi(\mathbf{x}_0, \mathbf{x}_1)\}$, the same bound holds for $b_k(Z')$.

Proof: Let F be the closed set $F = I^{n_0} \backslash Z$. We have $b_k(Z) = b_k(I^{n_0} \backslash Z)$, so by Lemma 3.18, it is enough to estimate $b_{n-k-1}(F \cup \partial I^{n_0})$ to estimate $b_k(Z)$. Let

$$X = \{(\mathbf{x}_0, \mathbf{x}_1) \in I^{n_0+n_1} \mid \Phi(\mathbf{x}_0, \mathbf{x}_1)\} \cup \partial I^{n_0} \times I^{n_1}.$$

Note that X can be given by a quantifier-free $\mathcal{P}$-closed formula. Moreover, we have $\pi_0(X) = F \cup \partial I^{n_0}$. Thus, we can apply Theorem 3.20 to estimate the Betti numbers of $F \cup \partial I^{n_0}$, and thus of Z. The case of Z' is identical. □

Remark 3.22 *The $\mathcal{P}$-closed formula hypothesis is not really necessary here. Note that similar estimates can be established for a compact set X defined by a formula Φ that is not $\mathcal{P}$-closed, replacing the estimates of Corollary 2.18 by the Borel-Moore estimates of Theorem 2.32.*

3.4 The case of two and more quantifiers

We will now generalize the results of the previous section to the case of an arbitrary number of quantifiers. Complementation (with the use of Alexander duality) allows to apply repeatedly the spectral sequence argument, by forcing the outer quantifier to be existential, and thus we can deduce estimates by induction.

Let us fix $n_0 \in \mathbb{N}$ and $\boldsymbol{n} = (n_1, n_2, \dots)$ a sequence of positive integers. For any $\nu \geq 0$, we let $N(\nu) = n_0 + \cdots + n_\nu$.

Definition 3.23 *For $\boldsymbol{n}$ as above, we let $\mathcal{E}(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s)$ be the maximum of $b(S)$, where $S \subseteq I^{n_0}$ is a sub-Pfaffian set defined as follows: Φ should be a quantifier-free formula with format $(N(\nu), \ell, \alpha, \beta, s)$ (see Definition 1.18 and Definition 1.19), for some Pfaffian chain $\boldsymbol{f}$ defined on a domain $\mathcal{U} \supseteq I^{N(\nu)}$. We assume furthermore that the semi-Pfaffian set $\{\mathbf{x} \in I^{N(\nu)} \mid \Phi(\mathbf{x})\}$ is either open or closed in $I^{N(\nu)}$. Then, if $Q_1, \dots, Q_\nu$ is a sequence of alternating quantifiers, the set S is defined by*

$$S = \{\mathbf{x}_0 \in I^{n_0} \mid Q_\nu \mathbf{x}_\nu \in I^{n_\nu} \dots Q_1 \mathbf{x}_1 \in I^{n_1}, \Phi(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\nu)\}. \tag{3.18}$$

If $\Phi_1, \dots, \Phi_M$ are quantifier-free formulas defined in the same Pfaffian chain, defining only open or only closed sets in the cube $I^{N(\nu)}$, and with the same format $(N(\nu), \ell, \alpha, \beta, s)$, and if $Q_1, \dots, Q_\nu$ is a fixed sequence of alternating quantifiers, we can define for all $1 \leq m \leq M$

$$S_m = \{\mathbf{x}_0 \in I^{n_0} \mid Q_\nu \mathbf{x}_\nu \in I^{n_\nu} \dots Q_1 \mathbf{x}_1 \in I^{n_1}, \Phi_m(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\nu)\}.$$

Then, we also define $\mathcal{E}^M(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s)$ to be the maximum of $b(S)$, where

- $S = S_1 \cup \cdots \cup S_M$, if $Q_\nu = \exists$; or
- $S = S_1 \cap \cdots \cap S_M$, if $Q_\nu = \forall$.

Theorem 3.24 *For any $\nu \geq 1$ and any values of the other parameters, the quantity $\mathcal{E}^M(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s)$ is bounded by*

$$(4Mn_0)^{n_0}\, \mathcal{E}^{n_0}(\boldsymbol{n}, (n_\nu + 1)n_0, \nu - 1, n_0\ell, \alpha, \beta, n_0 s). \tag{3.19}$$

Proof: Let $\Phi_1, \dots, \Phi_M$ be quantifier-free formulas as in Definition 3.23, and consider the sets

$$X_m = \{(\mathbf{x}_0, \mathbf{x}_1) \in I^{n_0+n_1} \mid \forall \mathbf{x}_{\nu-1} \in I^{n_{\nu-1}} \dots Q_1 \mathbf{x}_1 \in I^{n_1}, \Phi_m(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\nu)\}.$$

Let $X = X_1 \cup \cdots \cup X_M$, and let $S_m = \pi_1(X_m)$, where π_1 is the canonical projection $\mathbb{R}^{n_0+n_1} \to \mathbb{R}^{n_0}$. We will bound the Betti numbers of $S = S_1 \cup \cdots \cup S_M$. We can always reduce to this case by taking complement, in the same way as in the proof of Corollary 3.21.

Step one: *spectral sequence argument.* Note that union and projection commute, so we have $S = \pi_1(X)$. Since X is open or closed (by Lemma 3.15) we can apply Theorem 3.1 to π_1 to obtain

$$b_k(S) \leq \sum_{p+q=k} b_q(\mathcal{W}^p(X));$$

where $\mathcal{W}^p(X)$ is the corresponding fibered product. If for all p, $\mathbf{y}^p = (\mathbf{y}_0, \ldots, \mathbf{y}_p)$ denotes a bloc of $(p+1)$ times n_ν variables, and if $m_p = n_0 + (p+1)n_\nu$ denotes the total number of variables in $(\mathbf{x}_0, \mathbf{y}^p)$, we have

$$\mathcal{W}^p(X) = \{(x_0, \mathbf{y}^p) \in I^{m_p} \mid \bigwedge_{j=0}^{p} \bigvee_{m=1}^{M} (\mathbf{x}_0, \mathbf{y}_j) \in X_m\}. \tag{3.20}$$

Step two: *Mayer-Vietoris and duality.* If we define for $1 \leq m \leq M$ and $0 \leq j \leq p$,

$$Y_j^m = \{(\mathbf{x}_0, \mathbf{y}^p) \in I^{m_p} \mid (\mathbf{x}_0, \mathbf{y}_j) \in X_m\}; \tag{3.21}$$

we then have from (3.20)

$$\mathcal{W}^p(X) = \bigcap_{j=0}^{p} \bigcup_{m=1}^{M} Y_j^m. \tag{3.22}$$

We can use the generalized Mayer-Vietoris inequality (3.15) to transform the intersection above in a union; we obtain

$$b_q(\mathcal{W}^p(X)) \leq \sum_{J \subseteq \{0,\ldots,p\}} b_{q+|J|-1} \left(\bigcup_{j \in J} \bigcup_{m=1}^{M} Y_j^m \right) \tag{3.23}$$

Define $\widetilde{Q}_i$ as the opposite quantifier to Q_i, *i.e.* $\widetilde{Q}_i = \exists$ if $Q_i = \forall$ and vice-versa, and for all j and m, let Z_j^m be the subset defined by

$$Z_j^m = \{(\mathbf{x}_0, \mathbf{y}^p) \in I^{m_p} \mid (\mathbf{x}_0, \mathbf{y}_j) \notin X_m\} \tag{3.24}$$

$$= \{(\mathbf{x}_0, \mathbf{y}^p) \in I^{m_p} \mid \exists \mathbf{x}_{\nu-1} \in I^{n_{\nu-1}} \cdots \widetilde{Q}_1 \mathbf{x}_1 \in I^{n_1} \neg \Phi_m(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\nu)\}. \tag{3.25}$$

By comparing (3.21) and (3.24), we notice that $Z_j^m = I^{m_p} \backslash Y_j^m$. For all J, we have

$$\bigcap_{j \in J} \bigcap_{m=1}^{M} Z_j^m = \left(I^{m_p} \backslash \bigcup_{j \in J} \bigcup_{m=1}^{M} Y_j^m \right);$$

and so by Alexander duality (Lemma 3.16 or Lemma 3.18), we obtain (up to the boundary terms which we will neglect for the sake of simplifying the notations)

$$b_{q+|J|-1}\left(\bigcup_{j\in J}\bigcup_{m=1}^{M} Y_j^m\right) = b_{m_p-q-|J|+1}\left(\bigcap_{j\in J}\bigcap_{m=1}^{M} Z_j^m\right). \tag{3.26}$$

Now, using the other Mayer-Vietoris inequality (3.14), we have

$$b_{m_p-q-|J|+1}\left(\bigcap_{j\in J}\bigcap_{m=1}^{M} Z_j^m\right) \le \sum_{K\subseteq J\times\{1,\dots,M\}} b_{m_p-q-|J|+|K|}\left(\bigcup_{(j,m)\in K} Z_j^m\right). \tag{3.27}$$

Note that the sets Z_j^m are subsets of I^{m_p}, and therefore we have $b_i(Z_j^m)=0$ for $i\ge m_p$. Thus, we can restrict the sum above to subsets K such that $m_p-q-|J|+|K|\le m_p-1$, which gives

$$|K|\le q+|J|-1\le q+p. \tag{3.28}$$

Combining this fact with (3.26) and (3.27), we obtain

$$b_{q+|J|-1}\left(\bigcup_{j\in J}\bigcup_{m=1}^{M} Y_j^m\right) \le \sum_{K\subseteq J\times\{1,\dots,M\},|K|\le p+q} b_{m_p-q-|J|+|K|}\left(\bigcup_{(j,m)\in K} Z_j^m\right). \tag{3.29}$$

Step three: *combinatorial estimates for* $b_q(\mathcal{W}^p(X))$. Let p and q be fixed. We will estimate $b_q(\mathcal{W}^p(X))$ in terms of $\mathcal{E}$. For any $J\subseteq\{0,\dots,p\}$ and $K\subseteq\{0,\dots,p\}\times\{1,\dots,M\}$, we let

$$Y_J=\bigcup_{j\in J}\bigcup_{m=1}^{M} Y_j^m,\ \text{and}\ Z_K=\bigcup_{(j,m)\in K} Z_j^m.$$

We let $j_0=|J|$ and $k_0=|K|$, and we denote by $\mathcal{J}(j_0)$ and $\mathcal{K}(k_0)$ respectively the set of subsets of $\{0,\dots,p\}$ of cardinality j_0 and the set of subsets of $\{0,\dots,p\}\times\{1,\dots,M\}$ with cardinality k_0. With these notations, the inequality (3.29) can be written as

$$b_{q+j_0-1}(Y_J)\le\sum_{k_0=1}^{p+q}\sum_{K\in\mathcal{K}(k_0)} b_{m_p-q-j_0+k_0}(Z_K). \tag{3.30}$$

Let $K\subseteq\{0,\dots,p\}\times\{1,\dots,M\}$ be fixed and consider the set

$$\Sigma(j_0,K)=\{J\in\mathcal{J}(j_0)\mid K\subseteq J\times\{1,\dots,M\}\}.$$

Then, for every $J \in \Sigma(j_0, K)$, the term $b_{m_p-q-j_0+k_0}(Z_K)$ appears, – when bounding $b_{q+j_0-1}(Y_J)$, – on the right-hand side of (3.30). Thus, if $\sigma(j_0, K)$ denotes the cardinality of $\Sigma(j_0, K)$, we obtain, when summing (3.30) over all $J \in \mathcal{J}(j_0)$,

$$\sum_{J \in \mathcal{J}(j_0)} b_{q+j_0-1}(Y_J) \leq \sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} \sigma(j_0, K)\, b_{m_p-q-j_0+k_0}(Z_K).$$

Since $\Sigma(j_0, K) \subseteq \mathcal{J}(j_0)$, we have the trivial bound $\sigma(j_0, K) \leq 2^{j_0+1} \leq 2^{p+1}$. Using this in the above inequality, we get

$$\sum_{J \in \mathcal{J}(j_0)} b_{q+j_0-1}(Y_J) \leq 2^{p+1} \sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} b_{m_p-q-j_0+k_0}(Z_K). \tag{3.31}$$

Recall that from (3.23), we have

$$b_q(\mathcal{W}^p(X)) \leq \sum_{j_0=1}^{p+1} \sum_{J \in \mathcal{J}(j_0)} b_{q+j_0-1}(Y_J).$$

Thus, summing (3.31) for $1 \leq j_0 \leq p+1$, we obtain

$$b_q(\mathcal{W}^p(X)) \leq 2^{p+1} \sum_{j_0=1}^{p+1} \sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} b_{m_p-q-j_0+k_0}(Z_K).$$

We can change the ordering of the sums in the right hand side to obtain

$$b_q(\mathcal{W}^p(X)) \leq 2^{p+1} \sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} \sum_{j_0=1}^{p+1} b_{m_p-q-j_0+k_0}(Z_K)$$

and since obviously $\sum_{j_0=1}^{p+1} b_{m_p-q-j_0+k_0}(Z_K) \leq b(Z_K)$, we get

$$b_q(\mathcal{W}^p(X)) \leq 2^{p+1} \sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} b(Z_K). \tag{3.32}$$

Now, observe that every set Z_K is given by a union of $|K|$ sets Z_j^m, which by the formula (3.25) are sub-Pfaffian subsets of I^{m_p} given by an alternation of $\nu - 1$ quantifier, starting with $\exists$. The formula defining Z_j^m is $\Phi_m(\mathbf{x}_0, \ldots, \mathbf{x}_{\nu-1}, \mathbf{y}_j)$. Thus, defining Z_K may require up to $p+q$ such formulas (since $|K| \leq p+q$) for $1 \leq m \leq M$ and $0 \leq j \leq p$. Since each such formula involves s Pfaffian functions of degree at most β defined in a chain of length ℓ, the set Z_K can be defined with $(p+q)s$ functions which are defined in a Pfaffian chain of length $(p+1)\ell$. By definition of $\mathcal{E}^M$, it follows that

$$b(Z_K) \leq \mathcal{E}^{|K|}(\boldsymbol{n}, m_p, \nu-1, (p+1)\ell, \alpha, \beta, (p+q)s)$$

Since $|K| \leq p+q$, using this estimate in (3.32) gives that $b_q(\mathcal{W}^p(X))$ is bounded by

$$2^{p+1}[M(p+1)]^{p+q}\,\mathcal{E}^{p+q}(\boldsymbol{n}, m_p, \nu-1, (p+1)\ell, \alpha, \beta, (p+q)s); \tag{3.33}$$

since we have (as in Remark 2.26)

$$\sum_{k_0=1}^{p+q} \sum_{K \in \mathcal{K}(k_0)} 1 = \sum_{k_0=1}^{p+q} \binom{M(p+1)}{k_0} \leq [M(p+1)]^{p+q}.$$

Step four: *summing up.* To bound $b(S)$, all we need to do now is to sum up (3.33) for $0 \leq k \leq n_0 - 1$ and $p+q = k$. For all the terms in the sums, we have $p+q \leq n_0$, $p+1 \leq n_0$ and $m_p \leq (n_\nu+1)n_0$, so all the terms $\mathcal{E}^{p+q}$ from (3.33) will be bounded by

$$\mathcal{E}^{n_0}(\boldsymbol{n}, (n_\nu+1)n_0, \nu-1, n_0\ell, \alpha, \beta, n_0 s).$$

All that remains to be estimated is a term of the form

$$\sum_{k=0}^{n_0-1} \sum_{p+q=k} 2^{p+1}[M(p+1)]^{p+q}$$

which is clearly bounded by $(2M)^{n_0} \sum_{k=0}^{n_0-1} k n_0^k \leq (4Mn_0)^{n_0}$, and thus the bound (3.19) follows. □

Corollary 3.25 *Let $u_\nu = 2^\nu n_0 n_\nu \cdots n_1$ and $v_\nu = 2^{2\nu} n_0^2 n_\nu^2 \cdots n_3^2 n_2$. Then, we have*

$$\mathcal{E}(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s) \leq 2^{O(\nu u_\nu + \ell^2 v_\nu^2)} s^{O(u_\nu)} [u_\nu(\alpha+\beta)]^{O(u_\nu + \ell v_\nu)}.$$

Proof: Using the Gabrielov-Vorobjov estimate for arbitrary semi-Pfaffian sets (Theorem 2.34), we can generalize Theorem 3.20 and Corollary 3.21 to obtain, for $\nu = 1$, that

$$\mathcal{E}^M(\boldsymbol{n}, n_0, 1, \ell, \alpha, \beta, s) \leq 2^{n_0\ell(n_0\ell-1)/2}\,(sM)^{2n_0(n_1+1)}\,O(n_0 n_1(\alpha+\beta))^{n_0(n_1+1+\ell)}. \tag{3.34}$$

(Using the fact that union and existential quantifiers commute, as do intersections and universal quantifiers.)

Let us now apply Theorem 3.24 inductively. After i iterations, we will denote by N_i the number of free variables, s_i the number of Pfaffian functions, ℓ_i the length of the Pfaffian chain, M_i the number of sets, and a number F_i so that

$$\mathcal{E}^M(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s) \leq F_i\,\mathcal{E}^{M_i}(\boldsymbol{n}, N_i, \nu-i, \ell_i, \alpha, \beta, s_i).$$

We let $N_0 = n_0, s_0 = s, M_0 = M, F_0 = 1, \ell_0 = \ell$. From Theorem 3.24, we know that

$$N_1 = (n_\nu+1)n_0,\ s_1 = n_0 s,\ M_1 = n_0,\ F_1 = (4Mn_0)^{n_0} \text{ and } \ell_1 = n_0\ell.$$

Thus, we obtain for these parameters the following inductions

$$\begin{aligned} N_{i+1} &= (n_{\nu-i}+1)N_i \\ s_{i+1} &= N_i s_i = s N_0 \cdots N_i \\ M_{i+1} &= N_i \\ F_{i+1} &= F_i (4M_i N_i)^{N_i} \\ \ell_{i+1} &= N_i \ell + \ell_i = (N_0 + \cdots + N_i)\ell. \end{aligned}$$

From the induction on N_i, we obtain that $N_{i+1} \le 2N_i n_{\nu-i+1}$, and thus for all i,

$$N_i \le 2^i n_0 n_\nu \cdots n_{\nu-i+1};$$

After $\nu - 1$ iteration, and applying (3.34), one gets

$$\begin{aligned} \mathcal{E}^M(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s) &\le F_{\nu-1}\, \mathcal{E}^{M_{\nu-1}}(\boldsymbol{n}, N_{\nu-1}, 1, \ell_{\nu-1}, \alpha, \beta, s_{\nu-1}) \\ &\le F_{\nu-1}\, 2^{O(N_{\nu-1}^2 \ell_{\nu-1}^2)}\, (s_{\nu-1} M_{\nu-1})^{2N_{\nu-1}(n_1+1)}\, O(N_{\nu-1} n_1(\alpha+\beta))^{N_{\nu-1}(n_1+1+\ell_{\nu-1})}. \end{aligned}$$

Using the notations introduced in the statement of the corollary, we obtain

$$N_{\nu-1} n_1 = O(u_\nu), \text{ and } N_{\nu-1}\ell_{\nu-1} = N_{\nu-1}\ell \sum_{j=0}^{\nu-2} N_j \le N_{\nu-1}(\nu-1)N_{\nu-2} = O(v_\nu).$$

We also have $2N_{\nu-1}(n_1+1) = O(u_\nu)$, so we can bound the term $(s_{\nu-1}M_{\nu-1})^{2N_{\nu-1}(n_1+1)}$ by

$$(sN_0 \cdots N_{\nu-3} N_{\nu-2}^2)^{O(u_\nu)} \le (sN_{\nu-2}^\nu)^{O(u_\nu)} \le (su_\nu)^{O(u_\nu)};$$

and we also have

$$F_{\nu-1} = \prod_{i=0}^{\nu-1} (4M_i N_i)^{N_i} \le M^{N_0}\, (4N_{\nu-2}N_{\nu-1})^{N_0+\cdots+N_{\nu-1}} \le M^{n_0}\, 2^{O(\nu u_\nu)}.$$

Using the above estimates, one derives easily an upper-bound for $\mathcal{E}^M(\boldsymbol{n}, n_0, \nu, \ell, \alpha, \beta, s)$ in terms of the parameters $(M, n_0, \ldots, n_\nu, \ell, \alpha, \beta, s)$, and the stated result follows from the case where $M = M_0 = 1$. □

Corollary 3.26 (Semi-algebraic case) *Let $S \subseteq I^{n_0}$ be as in Definition 3.23, for a formula Φ having as atoms s polynomials of degree bounded by d. Then, we have*

$$b(S) \le [2^{\nu^2} d\, s\, n_0 n_1 \cdots n_\nu]^{O(2^\nu\, n_0 n_1 \cdots n_\nu)}.$$

Proof: The result follows from the proof of Corollary 3.25, replacing β by d and setting $\alpha = \ell = 0$. □

3.5 Comparison with quantifier elimination

We will now compare the results obtained in Corollary 3.26 with the bounds that can be established using quantifier elimination. Similar comparisons can be made in the Pfaffian case between Corollary 3.25 and effective cylindrical decomposition as appear in [GV2] and [PV]. However, we will restrict our attention to the algebraic case for simplicity.

Let X be a semi-algebraic subspace of $n_0 + \cdots + n_\nu$-space defined by s polynomials of degree bounded by d, and let

$$S = \{\mathbf{x}_0 \in \mathbb{R}^{n_0} \mid Q_\nu \mathbf{x}_\nu \in \mathbb{R}^{n_\nu} \dots Q_1 \mathbf{x}_1 \in \mathbb{R}^{n_1}, (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_\nu) \in X\}.$$

The set S can be effectively described by a quantifier-free formula $\Psi(\mathbf{x}_0)$. The best complexity results for Ψ appear in [BPR2] (see also [B3]). We have there

$$\Psi(\mathbf{x}_0) = \bigvee_{1 \le i \le I} \bigwedge_{1 \le j \le J_i} \mathrm{sign}(P_{i,j}(\mathbf{x}_0)) = \varepsilon_{i,j};$$

for some family of polynomials $P_{i,j}$ and some sign conditions $\varepsilon_{i,j} \in \{=0, >0, <0\}$, with

$$\begin{aligned} I &\le s^{\prod_{i \ge 0}(n_i+1)} d^{(n_0+1)\prod_{i \ge 1} O(n_i)}; \\ J_i &\le s^{\prod_{i \ge 1}(n_i+1)} d^{\prod_{i \ge 1} O(n_i)}; \\ \deg P_{i,j} &\le d^{\prod_{i \ge 1} O(n_i)}. \end{aligned}$$

From this, we can derive the following estimate.

Proposition 3.27 *Let S be as above. Then the sum of the Betti numbers of S verifies*

$$b(S) \le s^{4n_0(n_0+1)\prod_{i \ge 0}(n_i+1)} \, d^{O(n_0^2 n_1 \cdots n_\nu)}.$$

Proof: The set S is defined by a quantifier-free formula involving at most $\sigma = J_1 \cdots J_I$ polynomials of degree $\delta \le d^{\prod_{i \ge 1} O(n_i)}$. By [GV4], the sum of the Betti numbers is bounded by $O(\sigma^2 \delta)^{n_0}$. Bounding σ and δ gives the proposition. □

Thus, the estimate in Proposition 3.27 is better than Corollary 3.26 asymptotically in ν, but Corollary 3.26 is better for small values of ν. (For a fixed value of ν, Corollary 3.26 is better when n_0 goes to infinity.)

Remark 3.28 *Note that Proposition 3.27 is a much more general result than Corollary 3.26, since it is not necessary to make assumptions on the topology of X or S, nor is it necessary to restrict ourselves to cubes.*

Chapter 4

Connected components of limit sets

In this chapter, we will give effective estimates for the number of connected components of the relative closure $(X,Y)_0$ of a semi-Pfaffian couple (X,Y). This estimate is established first in the smooth case, by estimating the number of local extrema of the distance function $\operatorname{dist}(\cdot, Y_\lambda)$ on X_λ. In the singular case, deformation techniques are used to reduce to the case of smooth hypersurfaces.

Note that the case where $Y = \varnothing$ is trivial. Indeed, this implies that $(\partial X)_+ = \varnothing$, and since X is assumed to be relatively compact (see Remark 1.72), X_λ is compact for all λ and the number of connected components of X_0 is bounded by the number of connected components of a generic fiber X_λ for $\lambda \ll 1$. Since X_λ is semi-Pfaffian, Theorem 2.25 provides an estimate in that case.

Thus, we'll assume throughout the present chapter that $Y \neq \varnothing$. In the first section, we establish a property that proves the finiteness of $b_0((X,Y)_0)$. This is used in the second part to provide the quantitative estimates, first in the smooth case (Theorem 4.4) and then in the singular case (Theorem 4.6). These results are then used in the third section to give upper-bounds in the fewnomial case.

4.1 Finiteness of the number of connected components

We show here how to reduce the problem of counting the number of connected components of a limit set to a problem in the semi-Pfaffian setting.

Let Φ be the (squared) distance function on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{aligned} \Phi : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x,y) &\mapsto |x-y|^2 \end{aligned} \tag{4.1}$$

For any $\lambda > 0$, we can define the distance to Y_λ, Ψ_λ on X_λ by:

$$\Psi_\lambda(x) = \min_{y \in Y_\lambda} \Phi(x,y). \tag{4.2}$$

Define similarly for $x \in \check{X}$:

$$\Psi(x) = \min_{y \in \check{Y}} \Phi(x,y). \tag{4.3}$$

Theorem 4.1 *Let (X,Y) be a semi-Pfaffian couple. Then, there exists $\lambda \ll 1$ such that for every connected component C of $(X,Y)_0$, we can find a connected component D_λ of the set of local maxima of Ψ_λ such that D_λ is arbitrarily close to C.*

Proof: Let C be a connected component of $(X,Y)_0$. Note that by definition of the relative closure, if x is in C, it cannot be in $\check{Y}$. So we must have $\Phi(x,y) > 0$ for all $y \in \check{Y}$, and since $\check{Y}$ is compact, we must have $\Psi(x) > 0$. Also, any point in ∂C must be in $\check{X}$, but not in $(X,Y)_0$. So we must have $\partial C \subseteq \check{Y}$, hence $\Psi|_{\partial C} \equiv 0$. This means that the restriction of Ψ to C takes its maximum inside of C.

Choose $x_0 \in C$, and let $c = \Psi(x_0) > 0$. For a small λ, there is a point $x_\lambda \in X_\lambda$ close to x_0 such that $c_\lambda = \Psi_\lambda(x_\lambda)$ is close to c, and is greater than the maximum of the values of Ψ_λ over points of X_λ close to ∂C. Hence the set $\{x \in X_\lambda \mid \Psi_\lambda(x) \geq c_\lambda\}$ is nonempty, and the connected component A_λ of this set that contains x_λ is close to C. There exists a local maximum $x^*_\lambda \in A_\lambda$ of Ψ_λ. If D_λ is the connected component in the set of local maxima of Ψ_λ, it is contained in Z_λ and is close to C. □

From the above theorem, we can not only deduce that $(X,Y)_0$ has finitely many connected components, but also derive effective estimates.

4.2 Bounding the number of connected components

4.2.1 Finding local maxima of the distance function

We will now show how the number of connected components of the set of local maxima of Ψ_λ that appear in Theorem 4.1 can be estimated when the sets X_λ and Y_λ are smooth.

Define for all p,

$$Z^p_\lambda = \{(x, y_0, \dots, y_p) \in W^p_\lambda \mid \Phi(x,y_0) = \dots = \Phi(x,y_p)\}, \tag{4.4}$$

where

$$W^p_\lambda = \{(x, y_0, \dots, y_p) \in X_\lambda \times (Y_\lambda)^{p+1} \mid y_i \neq y_j, 0 \leq i < j \leq p\}. \tag{4.5}$$

Lemma 4.2 *Assume (X,Y) is a Pfaffian couple such that X_λ and Y_λ are smooth for all $\lambda > 0$. For a given $\lambda > 0$, let x^* be a local maximum of $\Psi_\lambda(x)$. Then, there exists $0 \leq p \leq \dim(X_\lambda)$ and a point $z^* = (x^*, y^*_0, \dots, y^*_p) \in Z^p_\lambda$ such that Z^p_λ is smooth at z^*, and z^* is a critical point of $\Phi(x,y_0)$ on Z^p_λ.*

Proof: Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y_0^* \in Y_\lambda$ such that $\Phi(x^*, y_0^*) = \min_{y \in Y_\lambda} \Phi(x^*, y) = \Psi_\lambda(x^*)$. In particular, $d_y\Phi(x, y) = 0$ at $(x, y) = (x^*, y_0^*)$. If (x^*, y_0^*) is a critical point of $\Phi(x, y)$ (this is always the case when $\dim(X_\lambda) = 0$) the statement holds for $p = 0$. Otherwise $d_x\Phi(x, y_0^*) \neq 0$ at $x = x^*$. Let ξ be a tangent vector to X at x^* such that $d_x\Phi(x^*, y_0^*)(\xi) > 0$.

Assume that for all $y \in Y_\lambda$ such that $\Phi(x^*, y) = \Psi_\lambda(x^*)$, we have $d_x\Phi(x, y)(\xi) > 0$ when $x = x^*$. Let $\gamma(t)$ be a curve on X_λ such that $\gamma(0) = x^*$ and $\dot\gamma(0) = \xi$. For all $y \in Y_\lambda$, there exists T_y such that for all $0 < t < T_y$, the inequality $\Phi(\gamma(t), y) > \Phi(x^*, y)$ holds. By compactness of Y_λ, this means we can find some t such that that inequality holds for all $y \in Y_\lambda$. Hence, $\Psi_\lambda(\gamma(t)) > \Psi_\lambda(x^*)$, which contradicts the hypothesis that Ψ_λ has a local maximum at x^*.

Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y_1^* \in Y_\lambda$ such that $d_x\Phi(x, y_1^*)(\xi) \leq 0$ at $x = x^*$ and $\Phi(x^*, y_1^*) = \Psi_\lambda(x^*)$. In particular, $y_1^* \neq y_0^*$, $\quad d_y\Phi(x^*, y) = 0$ at $y = y_1^*$, and $d_x\Phi(x, y_1^*) \neq d_x\Phi(x, y_0^*)$ at $x = x^*$. This implies that $(x^*, y_0^*, y_1^*) \in Z_\lambda^1$, and Z_λ^1 is smooth at (x^*, y_0^*, y_1^*). If (x^*, y_0^*, y_1^*) is a critical point of $\Phi(x, y_0)$ on Z_λ^1 (this is always the case when $\dim(X_\lambda) = 1$) the statement holds for $p = 1$. Otherwise $d_x\Phi(x, y_0^*)$ and $d_x\Phi(x, y_1^*)$ are linearly independent at $x = x^*$. Since $\dim(X_\lambda) \geq 2$, there exists a tangent vector ξ to X_λ at x^* such that $d_x\Phi(x^*, y_0^*)(\xi) > 0$ and $d_x\Phi(x^*, y_0^*)(\xi) > 0$. Since x^* is a local maximum of $\Psi_\lambda(x)$, there exists a point $y_2^* \in Y_\lambda$ such that $d_x\Phi(x, y_2^*)(\xi) \leq 0$ at $x = x^*$ and $\Phi(x^*, y_2^*) = \Psi_\lambda(x^*)$. This implies that $(x^*, y_0^*, y_1^*, y_2^*) \in Z_\lambda^2$, and Z_λ^2 is smooth at $(x^*, y_0^*, y_1^*, y_2^*)$. The above arguments can be repeated now for Z_λ^2, Z_λ^3, etc., to prove the statement for all $p \leq \dim(X_\lambda)$. □

Assume now that X_λ and Y_λ are *effectively* non-singular, *i.e.* they are of the following form:

$$\begin{aligned} X_\lambda &= \{x \in \mathbb{R}^n \mid p_1(x, \lambda) = \cdots = p_{n-d}(x, \lambda) = 0\}; \\ Y_\lambda &= \{y \in \mathbb{R}^n \mid q_1(y, \lambda) = \cdots = q_{n-k}(y, \lambda) = 0\}; \end{aligned} \tag{4.6}$$

where, for all $\lambda > 0$, we assume that $d_xp_1 \wedge \cdots \wedge d_xp_{n-d} \neq 0$ on X_λ and that $d_yq_1 \wedge \cdots \wedge d_yq_{n-k} \neq 0$ on Y_λ. In particular, we have $\dim(X_\lambda) = d$ and $\dim(Y_\lambda) = k$.

Remark 4.3 *Note that we assume that no inequalities appear in* (4.6). *We can clearly make that assumption for Y_λ, since that set has to be closed for all $\lambda > 0$. For X_λ, we observe the following: if C is a connected component of C_λ^p, the critical set of $\Phi|_{Z_\lambda^p}$, the function Φ is constant on C. If C contains a local maximum for Ψ_λ, it cannot meet ∂X_λ because $\partial X_\lambda \subseteq Y_\lambda$. Hence, we do not need to take into account the inequalities appearing in the definition of X_λ.*

Let us now define for all p,

$$\theta_p : (y_0, \ldots, y_p) \in (Y_\lambda)^{p+1} \mapsto \sum_{0 \leq i < j \leq p} |y_i - y_j|^2. \tag{4.7}$$

Then, for X_λ and Y_λ as in (4.6), the sets Z^p_λ are defined for all p by the following conditions.

$$\begin{cases} p_1(x,\lambda) = \cdots = p_{n-d}(x,\lambda) = 0; \\ q_1(y_i,\lambda) = \cdots = q_{n-k}(y_i,\lambda) = 0, \quad 0 \le i \le p; \\ \Phi(x,y_i) - \Phi(x,y_j) = 0, \quad 0 \le i < j \le p; \end{cases} \tag{4.8}$$

and the inequality

$$\theta_p(y_0,\dots,y_p) > 0. \tag{4.9}$$

Under these hypotheses, we obtain the following bound.

Theorem 4.4 (Basic smooth case) *Let (X,Y) be a semi-Pfaffian couple defined in a domain of bounded complexity γ, such that for all small $\lambda > 0$, X_λ and Y_λ are effectively non-singular basic sets of dimension respectively d and k. If the fiber-wise format of (X,Y) is (n,ℓ,α,β,s), the number of connected components of $(X,Y)_0$ is bounded by*

$$2\sum_{p=0}^{d} \mathcal{V}((p+2)n,(p+2)\ell,\alpha,\beta_p,\gamma); \tag{4.10}$$

where $\beta_p = \max\{1+(n-k)(\alpha+\beta-1), 1+(n-d+p)(\alpha+\beta-1)\}$, and $\mathcal{V}$ is defined in (2.1).

Proof: According to Theorem 4.1, we can chose $\lambda > 0$ such that for any connected component C of $(X,Y)_0$, we can find a connected component D_λ of the set of local maxima of Ψ_λ such that D_λ is close to C. We see that for λ small enough, two connected components C and C' of $(X,Y)_0$ cannot share the same connected component D_λ, since D_λ cannot meet $\check{Y}$ for λ small enough. Indeed, the distance from D_λ to $\check{Y}$ is bounded from below by the distance from D_λ to Y_λ, – which is at least c_λ, – minus the distance between Y_λ and $\check{Y}$. But the latter distance goes to zero, whereas the former goes to a positive constant c when λ goes to zero.

Once that λ is fixed, all we need to do is estimate the number of connected components of the set of local maxima of Ψ_λ. According to Lemma 4.2, we can reduce to estimating the number of connected components of the critical sets C^p_λ of the restriction $\Phi|_{Z^p_\lambda}$ for $0 \le \lambda \le d$.

For the sake of concision, we will drop λ from the notations in this proof, writing Z^p for Z^p_λ, $p_i(x)$ for $p_i(x,\lambda)$, *etc...*

A point $z = (x,y_0,\dots,y_p) \in Z^p$ is in C^p if and only if the following conditions are satisfied:

$$\begin{cases} d_y\Phi(x,y_j) = 0, \quad 0 \le j \le p; \\ \operatorname{rank}\left(d_x\Phi(x,y_0),\dots,d_x\Phi(x,y_p)\right) \le p. \end{cases} \tag{4.11}$$

For X_λ and Y_λ as in (4.6), those conditions become:

$$\begin{cases} \operatorname{rank}\{\nabla_y q_1(y_i),\ldots,\nabla_y q_{n-k}(y_i),\nabla_y \Phi(x,y_i)\} \le n-k, \quad 0\le i\le p; \\ \operatorname{rank}\{\nabla_x p_1(x),\ldots,\nabla_x p_{n-d}(x),\nabla_x \Phi(x,y_0),\ldots,\nabla_x \Phi(x,y_0)\} \le n-d+p. \end{cases} \tag{4.12}$$

Those conditions translate into all the maximal minors of the corresponding matrices vanishing. These minors are Pfaffian functions in the chain used to define X and Y. Their degrees are respectively $1+(n-k)(\alpha+\beta-1)$ and $1+(n-d+p)(\alpha+\beta-1)$.

The number of connected components of C^p is bounded by the number of connected components of the set D^p defined by the conditions in (4.8) and (4.9), and the vanishing of the maximal minors corresponding to the conditions in (4.12).

Let E^p be the set defined by the equations (4.8) and (4.12), so that $D^p = E^p \cap \{\theta_p > 0\}$. Then, the number of connected components of D^p is bounded by the number of connected components of E^p plus the number of connected components of $E^p \cap \{\theta_p = \varepsilon\}$ for a choice of $\varepsilon > 0$ small enough.

Hence, we're reduced to the problem of estimating the number of connected components of two varieties in $\mathbb{R}^{(p+2)n}$ defined in a Pfaffian chain of degree α and length $(p+2)\ell$. Using Theorem 2.8, we obtain the bound (4.10). □

4.2.2 Bounds for the singular case

Let's consider now the case where X_λ and Y_λ may be singular. We can use deformation techniques to reduce to the smooth case. First, the following lemma shows we can reduce to the case where X_λ is a basic set.

Lemma 4.5 *Let X_1, X_2 and Y be semi-Pfaffian sets such that (X_1, Y) and (X_2, Y) are Pfaffian families. Then, $(X_1 \cup X_2, Y)_0 = (X_1, Y)_0 \cup (X_2, Y)_0$.*

The proof follows from the definition of the relative closure.

Theorem 4.6 (Singular case) *Let (X, Y) be a semi-Pfaffian couple defined in a domain of bounded complexity γ. Assume X_λ and Y_λ are unions of basic sets of format $(n, \ell, \alpha, \beta, s)$. If the number of basic sets in X_λ is M and the number of basic sets in Y_λ is N, then the number of connected components of $(X, Y)_0$ is bounded by*

$$2\, MN \sum_{p=0}^{n-1} \mathcal{V}((p+2)n, (p+2)\ell, \alpha, \beta_p^*, \gamma); \tag{4.13}$$

where $\beta_p^* = 1 + (p+1)(\alpha + 2\beta - 1)$ and $\mathcal{V}$ is defined in (2.1).

Proof: Again, we want to estimate the number of local maxima of the function Ψ_λ defined in (4.2).

By Lemma 4.5, we can restrict ourselves to the case where X is basic. Let $Y = Y_1 \cup \cdots \cup Y_N$, where all the sets Y_i are basic. For each basic set, we take the sum of squares of the equations defining it: the corresponding positive functions, which we denote by p and $q_1, \ldots, q_N$, have degree 2β in the chain. Fix $\varepsilon_i > 0$, for $0 \leq i \leq N$, and $\lambda > 0$, and let $\mathcal{X} = \{p(x,\lambda) = \varepsilon_0\}$ and for all $1 \leq i \leq N$, let $\mathcal{Y}_i = \{q_i(x,\lambda) = \varepsilon_i\}$.

Since Y_λ is compact, if x^* is a point in X_λ such that Ψ_λ has a local maximum at $x = x^*$, there is a point y^* in some $(Y_i)_\lambda$ such that $\Phi(x,y) = \Psi_\lambda(x)$. Then, we can find a couple $(x', y') \in \mathcal{X}_\lambda \times (\mathcal{Y}_i)_\lambda$ close to (x^*, y^*) such that $\Phi(x', y')$ is a local maximum of the distance (measured by Φ) from $\mathcal{X}_\lambda$ to $(\mathcal{Y}_i)_\lambda$.

Since for small enough $\varepsilon_0, \ldots, \varepsilon_N$, the sets $\mathcal{X}_\lambda$ and $(\mathcal{Y}_i)_\lambda$ are effectively non-singular hypersurfaces, the number of local maxima of the distance of $\mathcal{X}_\lambda$ to $(\mathcal{Y}_i)_\lambda$ can be bounded by (4.10), for appropriate values of the parameters. The estimate (4.13) follows. □

Corollary 4.7 *Let $\mathcal{U}$ be a fixed domain of bounded complexity, and let (X,Y) be a semi-Pfaffian couple defined in $\mathcal{U}$. If X_λ and Y_λ are unions of basic sets of format (n,ℓ,α,β,s), where X_λ is the union of M basic sets and Y_λ is the union of N basic sets, the number of connected components of $(X,Y)_0$ is bounded by*

$$MN\, 2^{(n\ell)^2}\, O(n^2(\alpha+\beta))^{(n+1)\ell};$$

for a constant that depends on $\mathcal{U}$.

4.3 Application to fewnomials

In this section, we will apply our previous results to the case where the Pfaffian functions we consider are fewnomials.

Recall from Definition 1.5 that we can consider the restriction of any polynomial q to $\mathcal{U} = \{x_1 \cdots x_n \neq 0\}$ as a Pfaffian function whose complexity depends only on the number of non zero monomials in q. Fix $\mathcal{K} = \{m_1, \ldots, m_r\} \in \mathbb{N}^n$ a set of exponents, and let $\ell = n + r$, and $\boldsymbol{f} = (f_1, \ldots, f_\ell)$ be the functions defined by:

$$f_i(x) = \begin{cases} x_i^{-1} \text{ if } 1 \leq i \leq n, \\ x^{m_{i-n}} \text{ if } i > n. \end{cases} \tag{4.14}$$

Then, if q is a polynomial whose non-zero coefficients are in $\mathcal{K}$, it can be written as a Pfaffian function in $\boldsymbol{f}$ with degree $\beta = 1$.

Let now $S \subseteq \mathcal{U}$ be a bounded semi-algebraic set. We can define from S a semi-Pfaffian family $X \subseteq \mathbb{R}^n \times \mathbb{R}$ by:

$$X = \{(x,\lambda) \in \mathcal{U} \times \mathbb{R}_+ \mid x \in S,\ x_1 > \lambda, \ldots, x_n > \lambda\}. \tag{4.15}$$

If S is defined by $\mathcal{K}$-fewnomials, we can apply the results from Theorems 4.4 and 4.6 to X, to obtain a bound on the number of connected components of $\overline{S} \cap \partial\mathcal{U}$. Note that from Example 1.35, one can build a $\mathcal{K}$-fewnomial set S such that $\overline{S}$ is not a $\mathcal{K}$-fewnomial set (see [G4]).

Theorem 4.8 *Let* (X, Y) *be a semi-Pfaffian couple defined by degree 1 functions in the chain* (4.14)*. If* X *and* Y *are the union of respectively* M *and* N *basic sets, and letting* $q = p + 2$*, the number of connected components of* $(X, Y)_0$ *is bounded by*

$$MN \sum_{p=0}^{n-1} 2^{q^2(n+r)^2/2}(6n+6)^{q(3n+2r)} q^{q(n+r)}. \tag{4.16}$$

Proof: This bound is obtained using Theorem 4.6 for $\alpha = 2$, $\beta = 1$ and $\ell = n + r$. □

Let X be a semi-Pfaffian family such that for all $\lambda > 0$, the set X_λ is defined by $\mathcal{K}$-fewnomials. By definition of a family, ∂X_λ is restricted for all $\lambda > 0$. By the results contained in [G5], this set is semi-Pfaffian set in the same chain, and the format of ∂X_λ can be estimated from the format of X_λ. Applying those results together with those of Theorem 4.6, we can give estimates for the number of connected components of $(X, \partial X)_0$.

Theorem 4.9 *Let* $\mathcal{K}$ *be fixed and* X *be a semi-Pfaffian family in the chain* (4.14)*. If* X *is the union of* N *basic sets of format* $(n, \ell = n + r, \alpha = 2, \beta = 1, s)$*, the number of connected components of* $X_0 = (X, \partial X)_0$ *is bounded by*

$$N^2 s^{N+rO(n^2)} n^{(n+r)^{n^{O(n^2+nr)}}}. \tag{4.17}$$

Proof: Following [G5], the set ∂X_λ can be defined using the same Pfaffian chain as X_λ, using N' basic sets and functions of degree at most β', where, under the hypotheses above, the following bounds hold.

$$\beta' \leq n^{(n+r)^{O(n)}}, \quad \text{and} \quad N' \leq N s^{N+rO(n^2)} N^{(n+r)^{rO(n)}}.$$

The bound on the number of connected components follows readily. □

Chapter 5

Topology of Hausdorff limits

In this chapter, we consider the following: $\mathcal{U}$ is a domain of bounded complexity in $\mathbb{R}^n \times \mathbb{R}_+$, for a Pfaffian chain $\boldsymbol{f}$ of length ℓ and degree α, and V is a Pfaffian variety in $\mathcal{U}$ of dimension $d+1$, with V_λ compact for $\lambda > 0$.

Let $X \subseteq V$ be a semi-Pfaffian family that is defined by a $\mathcal{P}$-closed formula Φ on V. If $\partial X_\lambda = \varnothing$ for $\lambda > 0$, we can consider relative closure of X, $X_0 = (X, \varnothing)_0$, which is the Hausdorff limit of the family of compact sets X_λ. We will give in Theorem 5.7 an explicit upper-bound on $b_k(X_0)$ for any $k > 0$. This allows in turn to establish an upper-bound for the Betti numbers of any relative closure $(X, Y)_0$, even when $Y_\lambda \neq \varnothing$ (Theorem 5.17). In both cases, the bounds depend only on the format of generic fibers, and are not affected by the dependence in the parameter λ.

The proof relies on the spectral sequence for closed surjections developed in Chapter 3. Using triangulation, we construct a surjection $f^\lambda : X_\lambda \to X_0$ for $\lambda \ll 1$. Then, we approximate the corresponding fibered products by semi-Pfaffian sets.

Remark 5.1 *In the more general setting of o-minimal structures, the result of the present chapter allow to estimate the Betti numbers of any Hausdorff limit in a definable family in terms of* simple *definable sets (deformations of diagonals in Cartesian products). This is the point of view adopted in [Z2]. One can reduce to the one parameter case using the main result of [LS2].*

5.1 Constructions with simplicial complexes

We describe here some constructions that involve simplicial complexes and PL-maps on them. Using the fact that continuous definable functions in an o-minimal structure can be triangulated, we will be able to use these construction in the next section.

It is important to note that the constructions done in this section *are not explicit.* They can be achieved using general arguments from o-minimality, but we do not claim to be able to give an effective procedure to construct triangulations in the Pfaffian case. Consequently, these constructions in themselves will not suffice to establish any Betti number bounds.

First, let us recall some of the notations used in the discussion of definable triangulations in section 1.3.3. See Definition 1.59 and Definition 1.61 for more details. If $a_0, \ldots, a_d$ are affine-independent points in $\mathbb{R}^n$, we denote by $\sigma = (a_0, \ldots, a_d)$ the *open* simplex and $\bar{\sigma} = [a_0, \ldots, a_d]$ the *closed* simplex defined by those points. We say that $K = \{\bar{\sigma}_1, \ldots, \bar{\sigma}_k\}$ is a *simplicial complex* if it is closed under taking faces and for all i, j $\bar{\sigma}_i \cap \bar{\sigma}_j$ is a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$. We denote by $|K|$ the *geometric realization* of K.

5.1.1 Retraction on a subcomplex

Let K be a simplicial complex in $\mathbb{R}^n$, and let $L \subseteq K$ be a subcomplex, *i.e.* L is also a simplicial complex.

Let $S = \mathrm{st}_K(L)$ be the *star* of L in K, *i.e.* the union of all open σ such that $\bar{\sigma} \in K$ and has at least one vertex in L. We will define a continuous retraction F from S to L.

If $a_0, \ldots, a_d$ are vertices of K ordered such that $a_0, \ldots, a_k$ are in L and $a_{k+1}, \ldots, a_d$ are not in L, (for some k such that $0 < k < d$), the open simplex $\sigma = (a_0, \ldots, a_d)$ is contained in S and we will define F on σ by

$$F\left(\sum_{i=0}^{d} w_i\, a_i\right) = \frac{1}{\sum_{i=0}^{k} w_i} \quad \sum_{i=0}^{k} w_i\, a_i. \tag{5.1}$$

Proposition 5.2 *The formula* (5.1) *defines a continuous retraction F from S to $|L|$ that maps all points on the segment $[x, F(x)]$ to $F(x)$.*

Proof: Let σ be an open simplex appearing in S. Then σ is of the form $(a_0, \ldots, a_d)$, where the vertices a_i are ordered as above so that the vertices in L are exactly $a_0, \ldots, a_k$ for some $0 < k < d$.

Fix $x \in \sigma$, and let $s = \sum_{i=0}^{k} w_i$. Since all the weights w_i are positive, the inequality $0 < k < d$ implies that $0 < s < 1$. Thus, the formula (5.1) clearly defines a continuous function from σ to $|L|$.

Let $y = \theta x + (1-\theta)F(x)$ for $\theta \in (0,1)$ be a point on the open segment $(x, F(x))$. We have $y = \sum_{i=0}^{k} w'_i\, a_i$, where

$$w'_i = \begin{cases} \theta w_i + (1-\theta)\frac{w_i}{s} & \text{if } 0 \le i \le k; \\ \theta w_i & \text{if } k+1 \le i \le d. \end{cases}$$

To prove that $F(x) = F(y)$, we must prove that for all $0 \leq i \leq k$,

$$\frac{w_i}{\sum_{j=0}^{k} w_j} = \frac{w'_i}{\sum_{j=0}^{k} w'_j}.$$

Cross-multiplying, we get the following quantities.

$$w_i \sum_{j=0}^{k} w'_j = w_i \sum_{j=0}^{k} \left(\theta w_j + (1-\theta)\frac{w_j}{s}\right) = w_i \left(1 - \theta + \theta s\right);$$

and

$$w'_i \sum_{j=0}^{k} w_j = \left(\theta w_i + (1-\theta)\frac{w_i}{s}\right) s = (\theta s + (1-\theta))w_i.$$

The two cross-multiplied quantities are indeed equal, so $F(x) = F(y)$. □

For any $x \in S\backslash|L|$, we denote by τ_x the open segment $(x, F(x))$.

Proposition 5.3 *Let x and y be points of $S\backslash|L|$ such that τ_x and τ_y intersect. Then, we have $\tau_x \subseteq \tau_y$ or $\tau_y \subseteq \tau_x$.*

Proof: Let $z \in \tau_x \cap \tau_y$. By Proposition 5.2, we have $F(x) = F(y) = F(z)$. Thus, τ_x and τ_y have one endpoint in common, and at least one point in common. One must be contained in the other. □

5.1.2 Level sets of a PL-function

Assume now that there exists a continuous function $\pi : |K| \to \mathbb{R}$ with the following properties.

- π is affine on each simplex $\bar{\sigma}$ of K;
- π is positive on K;
- $|L| = \pi^{-1}(0)$.

For all $\lambda > 0$, we will denote by $|K|_\lambda$ the level set $\pi^{-1}(\lambda)$. We define also

$$\lambda_0 = \min\{\pi(a) \mid a \text{ is a vertex of } K, a \notin L\}. \tag{5.2}$$

Remark 5.4 *Note that for all $0 < \lambda < \lambda_0$, the level set $|K|_\lambda$ is contained in the star S. Indeed, if $\sigma = (a_0, \ldots, a_d)$ is a simplex that is not in S, we must have $\pi(a_i) \geq \lambda_0$ for all i since none of the a_i are in L, and π being affine on $\bar{\sigma}$, it follows that $\pi(x) \geq \lambda_0$ for all $x \in \sigma$.*

We want to describe for $0 < \lambda < \lambda_0$, the restriction of the retraction F to the level set $|K|_\lambda$. We will denote by F^λ this restriction. A similar construction is outlined in [C2, Exercise 4.11].

Proposition 5.5 *For all $0 < \lambda' < \lambda < \lambda_0$, there exists a homeomorphism $H : |K|_\lambda \to |K|_{\lambda'}$ such that $F^\lambda \circ H = F^{\lambda'}$.*

Proof: Let $x \in |K|_\lambda$. For any $z \in \tau_x$, if $z = \theta x + (1-\theta)F(x)$, we have

$$\pi(z) = \theta\pi(x) + (1-\theta)\pi(F(x)) = \theta\lambda;$$

since z is in any simplex $\bar{\sigma}$ of K that contains x and $F(x)$ and since π is affine on the simplices of K. Thus, $z \in |K|_{\lambda'}$ if and only if $\theta = \lambda'/\lambda$, and so the map h defined by

$$H(x) = \frac{\lambda'}{\lambda}\, x + \left(1 - \frac{\lambda'}{\lambda}\right) F(x); \tag{5.3}$$

maps $|K|_\lambda$ to $|K|_{\lambda'}$.

The map H is certainly injective, since by Proposition 5.3, two segments τ_x and τ_y cannot intersect if x and y are two distinct points of $|K|_\lambda$. It is also surjective, since for $z \in |K|_{\lambda'}$, it is easy to verify that the point x defined by

$$x = \frac{\lambda}{\lambda'} z - \left(\frac{\lambda}{\lambda'} - 1\right) F(z);$$

is a point in $|K|_\lambda$ such that $H(x) = z$.

The continuity of H follows from the continuity of F. Since $H(x) \in \tau_x$ by construction, Proposition 5.2 implies that $F(H(x)) = F(x)$. □

Proposition 5.6 *For F^λ defined as above, we have*

$$\lim_{\lambda \to 0} \max_{x \in |K|_\lambda} |x - F^\lambda(x)| = 0. \tag{5.4}$$

Proof: Let $\sigma = (a_0, \ldots, a_d)$ be an open simplex contained in S, such that $\sigma \not\subseteq |L|$. As before, we can assume that the vertices of σ that are in L are $a_0, \ldots, a_k$, where $0 \le k < d$ Fix $x = \sum_{i=0}^d w_i\, a_i$ is σ, and let $s = \sum_{i=0}^k w_i$. We have

$$\sum_{i=k+1}^{d} w_i = \sum_{i=0}^{d} w_i - \sum_{i=0}^{k} w_i = 1 - s;$$

and

$$x - F(x) = \sum_{i=0}^{d} w_i\, a_i - \frac{1}{s}\sum_{i=0}^{k} w_i\, a_i = \left(1 - \frac{1}{s}\right)\left(\sum_{i=0}^{k} w_i\, a_i\right) + \sum_{i=k+1}^{d} w_i\, a_i.$$

By the triangle inequality, we obtain

$$|x - F(x)| \leq \max_{o \leq i \leq d} |a_i| \left(\left| 1 - \frac{1}{s} \right| \left(\sum_{i=0}^{k} w_i \right) + \sum_{i=k+1}^{d} w_i \right) = 2(1-s) \max_{0 \leq i \leq d} |a_i|. \tag{5.5}$$

If $x \in |K|_\lambda$, we have $\pi(x) = \sum_{i=k+1}^{d} w_i\, \pi(a_i) = \lambda$. Since $\pi(a_i) \geq \lambda_0$ for all $i \geq k+1$, it follows that

$$\lambda = \sum_{i=k+1}^{d} w_i\, \pi(a_i) \geq \lambda_0 \left(\sum_{i=k+1}^{d} w_i \right) = \lambda_0(1-s). \tag{5.6}$$

It follows that $1 - s \leq \dfrac{\lambda}{\lambda_0}$. Combining this with (5.5), we obtain

$$|x - F(x)| \leq 2\, \frac{\lambda}{\lambda_0} \max_{0 \leq i \leq d} |a_i| \leq 2\, \frac{\lambda}{\lambda_0} \max\{|a|,\, a \text{ vertex of } K\}.$$

Thus, $|x - F(x)|$ is bounded by a quantity independent of x that goes to zero when λ goes to zero, and the result follows. □

5.2 Bounds on the Betti numbers of Hausdorff limits

Fix $\mathcal{U}$ a domain of bounded complexity γ for a Pfaffian chain $\boldsymbol{f}$. Let X be a semi-Pfaffian family with compact fibers defined on a variety V such that $\dim(V) = d+1$. Assume that for all $\lambda \in (0,1)$, X_λ is compact, so that $X_0 = (X, \varnothing)_0$. The main result of this chapter is the following.

Theorem 5.7 *Let X be a semi-Pfaffian family with compact fibers as above. If the format of X is $(n, \ell, \alpha, \beta, s)$, we have for all $0 \leq k \leq d$,*

$$b_k(X_0) \leq \sum_{p=0}^{k} (10s)^{(p+1)d}\, \mathcal{V}((p+1)n, (p+1)\ell, \alpha, 2\beta, \gamma); \tag{5.7}$$

where $\mathcal{V}$ is defined in (2.1)*. In particular, we have*

$$b_k(X_0) \leq s^{d(k+1)}\, 2^{(k\ell)^2} O(kn\beta + k\min(n,\ell)\alpha)^{(k+1)(n+\ell)};$$

where the constant depends on $\mathcal{U}$.

Remark 5.8 *If X is not defined by a $\mathcal{P}$-closed formula, the method of proof is still valid, and one can still establish bounds on $b_k(X_0)$, using the Borel-Moore estimates from Chapter 2. In that case, the bound obtained is*

$$b_k(X_0) \leq s^{2d(k+1)}\, 2^{2(k\ell)^2} O(kn\beta + k\min(n,\ell)\alpha)^{2(k+1)(n+\ell)};$$

In the process of proving Theorem 5.7, we will actually prove a much more general result. Before stating it, we need the following notation: for any integer p, we let ρ_p be the function on $(p+1)$-tuples $(\mathbf{x}_0, \ldots, \mathbf{x}_p)$ of points in $\mathbb{R}^n$ defined by

$$\rho_p(\mathbf{x}_0, \ldots, \mathbf{x}_p) = \sum_{0 \leq i < j \leq p} |\mathbf{x}_i - \mathbf{x}_j|^2;$$

Then we will prove the following theorem.

Theorem 5.9 *Let $X \subseteq \mathbb{R}^n \times \mathbb{R}_+$ be a bounded set definable in* any *o-minimal structure, such that the fibers X_λ are compact for all $\lambda > 0$. Let X_0 be the Hausdorff limit of those fibers when λ goes to zero. Then, there exists $\lambda > 0$ such that for any integer k, we have*

$$b_k(X_0) \leq \sum_{p+q=k} b_q(D^p_\lambda(\delta));$$

for some $\delta > 0$, where the set $D^p_\lambda(\delta)$ is the expanded diagonal

$$D^p_\lambda(\delta) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in (X_\lambda)^{p+1} \mid \rho_p(\mathbf{x}_0, \ldots, \mathbf{x}_p) \leq \delta\}.$$

Remark 5.10 *Using results on the definability of Hausdorff limits in o-minimal structures (see for instance [LS2]), one can even generalize the above further: we can estimate in this way the Betti numbers of any Hausdorff limit of a sequence of compact fibers in a p-parameter definable family. See [Z2] for more details.*

5.2.1 Triangulation of the projection on λ

Let A be the closure of $X \cap \{0 < \lambda < 1\}$. By Theorem 1.64, there exists a simplicial complex K such that $|K| \subseteq \mathbb{R}^{n+1}$, a subcomplex $L \subseteq K$ and a homeomorphism $\Phi : |K| \to A$ such that $\Phi(L) = X_0$ and such that $\pi_\lambda \circ \Phi$ is affine on each simplex of K.

Denote by F the retraction constructed in the previous section. For all $\lambda < \lambda_0$, let $f^\lambda = \Phi^{-1} \circ F^\lambda$.

Proposition 5.11 *For all $\lambda < \lambda_0$, the map f^λ is a continuous surjection from X_λ to X_0. Moreover, we have*

$$\lim_{\lambda \to 0} \max_{x \in X_\lambda} |x - f^\lambda(x)| = 0. \tag{5.8}$$

Proof: Since Φ is uniformly continuous, Proposition 5.6 implies (5.8). □

Define for $p \in \mathbb{N}$ and $\lambda \in (0, \lambda_0)$,

$$W^p_\lambda = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in (X_\lambda)^{p+1} \mid f^\lambda(\mathbf{x}_0) = \cdots = f^\lambda(\mathbf{x}_p)\}. \tag{5.9}$$

From Theorem 3.1 we have for any $\lambda \in (0, \lambda_0)$,

$$b_k(X_0) \le \sum_{p+q=k} b_q(W^p_\lambda). \tag{5.10}$$

Thus, the problem is reduced to estimating the Betti numbers of the sets W^p_λ. The first step in that direction is the following.

Proposition 5.12 *For all $0 < \lambda' < \lambda < \lambda_0$, the sets W^p_λ and $W^p_{\lambda'}$ are homeomorphic.*

Proof: Let $H : |K|_\lambda \to |K|_{\lambda'}$ be the homeomorphism described in Proposition 5.5. Then, the map $h = \Phi \circ H \circ \Phi^{-1}$ is a homeomorphism between X_λ and $X_{\lambda'}$, and since $F^\lambda \circ H = F^{\lambda'}$, we also have $f^\lambda \circ h = f^{\lambda'}$. It is then easy to check that the map $h^p : (X_\lambda)^{p+1} \to (X_{\lambda'})^{p+1}$ defined by

$$h^p(\mathbf{x}_0, \ldots, \mathbf{x}_p) = (h(\mathbf{x}_0), \ldots, h(\mathbf{x}_p)); \tag{5.11}$$

maps W^p_λ homeomorphically onto $W^p_{\lambda'}$. □

5.2.2 Approximating W^p

For $p \in N$ and $\mathbf{x}_0, \ldots, \mathbf{x}_p \in \mathbb{R}^n$, let ρ_p be the polynomial

$$\rho_p(\mathbf{x}_0, \ldots, \mathbf{x}_p) = \sum_{0 \le i < j \le p} |\mathbf{x}_i - \mathbf{x}_j|^2. \tag{5.12}$$

For $\lambda \in (0, \lambda_0)$, $\varepsilon > 0$ and $\delta > 0$, we define the following sets.

$$W^p_\lambda(\varepsilon) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in (X_\lambda)^{p+1} \mid \rho_p(f^\lambda(\mathbf{x}_0), \ldots, f^\lambda(\mathbf{x}_p)) \le \varepsilon\};$$
$$D^p_\lambda(\delta) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p) \in (X_\lambda)^{p+1} \mid \rho_p(\mathbf{x}_0, \ldots, \mathbf{x}_p) \le \delta\}.$$

We will use these sets to approximate the sets W^p. Namely, we will show that for any $p \in \mathbb{N}$, we can find appropriate values of λ and δ such that the Betti numbers of W^p_λ and $D^p_\lambda(\delta)$ coincide.

Proposition 5.13 *Let $p \in \mathbb{N}$ be fixed. There exists $\varepsilon_0 > 0$, such that for all $\lambda \in (0, \lambda_0)$ and all $0 < \varepsilon' < \varepsilon < \varepsilon_0$, the inclusion $W^p_\lambda(\varepsilon') \hookrightarrow W^p_\lambda(\varepsilon)$ is a homotopy equivalence. In particular, this implies that*

$$b_q(W^p_\lambda(\varepsilon)) = b_q(W^p_\lambda);$$

for all $\lambda \in (0, \lambda_0)$ and all $\varepsilon \in (0, \varepsilon_0)$.

Proof: First, notice that it is enough to prove the result for a fixed $\lambda \in (0, \lambda_0)$, since if $0 < \lambda' < \lambda < \lambda_0$ are fixed, the map h^p introduced in (5.11) induces a homeomorphism between $W^p_\lambda(\varepsilon)$ and $W^p_{\lambda'}(\varepsilon)$ for any $\varepsilon > 0$.

Let us fix $\lambda \in (0, \lambda_0)$. By the generic triviality theorem (Theorem 1.58), there exists $\varepsilon_0 > 0$ such that the projection

$$\{(\mathbf{x}_0, \ldots, \mathbf{x}_p, \varepsilon) \mid \varepsilon \in (0, \varepsilon_0) \text{ and } (\mathbf{x}_0, \ldots, \mathbf{x}_p) \in W_\lambda(\varepsilon)\} \mapsto \varepsilon;$$

is a trivial fibration. It follows that for all $0 < \varepsilon' < \varepsilon < \varepsilon_0$, the inclusion $W^p_\lambda(\varepsilon') \hookrightarrow W^p_\lambda(\varepsilon)$ is a homotopy equivalence, and thus the homology groups $H_*(W^p_\lambda(\varepsilon))$ are isomorphic for all $\varepsilon \in (0, \varepsilon_0)$.

The sets W^p_λ and $W^p_\lambda(\varepsilon)$ being compact definable sets, they are homeomorphic to finite simplicial complexes. This means that their singular and Čech homologies coincide, and since $W^p_\lambda = \cap_{\varepsilon>0} W^p_\lambda(\varepsilon)$, the continuity property of the Čech homology implies that $H_*(W^p_\lambda)$ is the projective limit of $H_*(W^p_\lambda(\varepsilon))$. Since the latter groups are constant when $\varepsilon \in (0, \varepsilon_0)$, the result follows. □

Proposition 5.14 *Let $p \in \mathbb{N}$ be fixed. For $\lambda \ll 1$, there exist definable functions $\delta_0(\lambda)$ and $\delta_1(\lambda)$ such that $\lim_{\lambda\to 0} \delta_0(\lambda) = 0$, $\lim_{\lambda\to 0} \delta_1(\lambda) \neq 0$, and such that for all $\delta_0(\lambda) < \delta' < \delta < \delta_1(\lambda)$, the inclusion $D^p_\lambda(\delta') \hookrightarrow D^p_\lambda(\delta)$ is a homotopy equivalence.*

Proof: Let $\lambda \in (0, \lambda_0)$ be fixed. By the same local triviality argument as above, there exists $d_0 = 0 < d_1 < \cdots < d_m < d_{m+1} = \infty$ such that for all $0 \le i \le m$ and all $d_i < \delta' < \delta < d_{i+1}$, the inclusion $D^p_\lambda(\delta') \hookrightarrow D^p_\lambda(\delta)$ is a homotopy equivalence. When λ varies, the values $d_i(\lambda)$ can be taken as definable functions of the variable λ, so by Lemma 1.76 each has a well-defined if possibly infinite limit when λ goes to zero. We take $\delta_0(\lambda) = d_j(\lambda)$, where j is the largest index such that $\lim_{\lambda\to 0} d_j(\lambda) = 0$, and take $\delta_1(\lambda) = d_{j+1}(\lambda)$. □

We define for $p \in \mathbb{N}$,

$$\eta_p(\lambda) = p(p+1)\left(4R \max_{x\in X_\lambda} |x - f^\lambda(x)| + 2\,(\max_{x\in X_\lambda} |x - f^\lambda(x)|)^2\right); \tag{5.13}$$

where, as before, R is a constant such that $X_\lambda \subseteq B(0, R)$ for all $\lambda > 0$. By Proposition 5.11, we have

$$\lim_{\lambda\to 0} \eta_p(\lambda) = 0.$$

Lemma 5.15 *For all $\lambda \in (0, \lambda_0)$, $\delta > 0$ and $\varepsilon > 0$, the following inclusions hold.*

$$D^p_\lambda(\delta) \subseteq W^p_\lambda(\delta + \eta_p(\lambda)), \text{ and } W^p_\lambda(\varepsilon) \subseteq D^p_\lambda(\varepsilon + \eta_p(\lambda)).$$

Proof: Let $m(\lambda) = \max_{x \in X_\lambda} |x - f^\lambda(x)|$. For any $\mathbf{x}_i, \mathbf{x}_j$ in X_λ, the triangle inequality gives

$$\begin{aligned} |f^\lambda(\mathbf{x}_i) - f^\lambda(\mathbf{x}_j)|^2 &\leq [|f^\lambda(\mathbf{x}_i) - \mathbf{x}_i| + |\mathbf{x}_i - \mathbf{x}_j| + |\mathbf{x}_j - f^\lambda(\mathbf{x}_j)|]^2 \\ &\leq [|\mathbf{x}_i - \mathbf{x}_j| + 2m(\lambda)]^2 \\ &\leq |\mathbf{x}_i - \mathbf{x}_j|^2 + 8R\, m(\lambda) + 4m(\lambda)^2. \end{aligned}$$

Summing this inequality for all $0 \leq i < j \leq p$, we obtain that for any $\mathbf{x}_0, \ldots, \mathbf{x}_p$ in X_λ,

$$\rho_p(f^\lambda(\mathbf{x}_0), \ldots, f^\lambda(\mathbf{x}_p)) \leq \rho_p(\mathbf{x}_0, \ldots, \mathbf{x}_p) + \eta_p(\lambda).$$

The first inclusion follows easily from this inequality. The second inclusion follows from a similar reasoning. □

Proposition 5.16 *For any $p \in \mathbb{N}$, there exists $\lambda \in (0, \lambda_0)$, $\varepsilon \in (0, \varepsilon_0)$ and $\delta > 0$ such that*

$$H_*(W_\lambda^p(\varepsilon)) \cong H_*(D_\lambda^p(\delta)). \tag{5.14}$$

Proof: Let $\delta_0(\lambda)$ and $\delta_1(\lambda)$ be the functions defined in Proposition 5.14. Since the limit when λ goes to zero of $\delta_1(\lambda) - \delta_0(\lambda)$ is not zero, whereas the limit of $\eta_p(\lambda)$ is zero, we can choose $\lambda > 0$ such that $\delta_1(\lambda) - \delta_0(\lambda) > 2\eta_p(\lambda)$. Then, we can choose $\delta' > 0$ such that $\delta_0(\lambda) < \delta' < \delta' + 2\eta_p(\lambda) < \delta_1(\lambda)$. Taking a smaller λ if necessary, we can also assume that $\delta' + 3\eta_p(\lambda) < \varepsilon_0$.

Let $\varepsilon = \delta' + \eta_p(\lambda)$, $\delta = \delta' + 2\eta_p(\lambda)$ and $\varepsilon' = \delta' + 3\eta_p(\lambda)$. From Lemma 5.15, we have the following sequence of inclusions;

$$D_\lambda^p(\delta') \overset{i}{\hookrightarrow} W_\lambda^p(\varepsilon) \overset{j}{\hookrightarrow} D_\lambda^p(\delta) \overset{k}{\hookrightarrow} W_\lambda^p(\varepsilon').$$

By the choice of $\varepsilon, \varepsilon'$ and λ, λ', the inclusions $k \circ j$ and $j \circ i$ are homotopy equivalences. The resulting diagram in homology is the following;

$$\begin{array}{ccccccc} H_*(D_\lambda(\delta')) & & \xrightarrow[\cong]{(j\circ i)_*} & & H_*(D_\lambda(\delta)) & & \\ & \searrow^{i_*} & & \nearrow^{j_*} & & \searrow^{k_*} & \\ & & H_*(W_\lambda(\varepsilon)) & & \xrightarrow[\cong]{(k\circ j)_*} & & H_*(W_\lambda(\varepsilon')) \end{array}$$

Since $(j \circ i)_* = j_* \circ i_*$, is an isomorphism, j_* must be surjective, and similarly, the fact that $(k \circ j)_* = k_* \circ j_*$ is an isomorphism implies that j_* is injective. Hence, j_* is an isomorphism between $H_*(W_\lambda(\varepsilon))$ and $H_*(D_\lambda(\delta))$, as required. □

5.2.3 Proof of Theorem 5.7

Recall that from the spectral sequence inequality (5.10), all we need to do to bound $b_k(X_0)$ is to give explicit estimates on the Betti numbers of the sets W^p_λ for all $0 \leq p \leq k$.

Let $p \in \mathbb{N}$ be fixed, and choose ε, λ and δ as in Proposition 5.16. Since $\varepsilon < \varepsilon_0$, the Betti numbers of $W^p_\lambda(\varepsilon)$ and W^p_λ are the same. Thus, we are reduced to estimating $b_q(D^p_\lambda(\delta))$. This set is a semi-Pfaffian subset of V^{p+1}_λ defined by a $\mathcal{Q}$-closed formula, where $\mathcal{Q}$ is a set of $s(p+1)+1$ Pfaffian functions in $n(p+1)$ variables, of degree bounded by β in a chain of length $(p+1)\ell$ and degree α. More explicitly, the chain under consideration is

$$\boldsymbol{f}^p = (f_1(\mathbf{x}_0, \lambda), \dots, f_\ell(\mathbf{x}_0, \lambda), \dots, f_1(\mathbf{x}_p, \lambda), \dots, f_\ell(\mathbf{x}_p, \lambda)),$$

where λ is kept constant. It follows from Theorem 2.17 that

$$b_q(D^p_\lambda(\delta)) \leq b(D^p_\lambda(\delta)) \leq (10s)^{d(p+1)}\, \mathcal{V}((p+1)n, (p+1)\ell, \alpha, 2\beta, \gamma).$$

The bound (5.7) follows. □

5.3 Betti numbers of a relative closure

In addition to the bound of Theorem 5.7, the techniques developed in the present chapter allow us to give a rough estimate for the Betti numbers of a relative closure of a semi-Pfaffian couple. In that case, – as for the Hausdorff limits, – the Betti numbers of the limit depends on the format of the fibers, but not on the families' dependence on the parameter λ.

Theorem 5.17 *Let (X, Y) be a semi-Pfaffian couple. Then, the Betti numbers of its relative closure $(X, Y)_0$ can be bounded in terms of the format of the fibers X_λ and Y_λ for $\lambda \ll 1$.*

Proof: Let $\delta_0 > 0$ and define

$$K_0 = \{x \in \check{X} \mid \mathrm{dist}(x, \check{Y}) \geq \delta_0\}.$$

Recall that $(X, Y)_0 = \{x \in \check{X} \mid \mathrm{dist}(x, \check{Y}) > 0\}$, so K_0 is a compact subset of $(X, Y)_0$. For $\delta_0 \ll 1$, the Betti numbers of K_0 and $(X, Y)_0$ coincide. Let $\delta(\lambda)$ be any definable function such that $\lim_{\lambda \to 0} \delta(\lambda) = \delta_0$, and let

$$K = \{(x, \lambda) \in X \mid \mathrm{dist}(x, Y_\lambda) \geq \delta(\lambda)\}.$$

The set K is a definable family with compact fibers for $\lambda > 0$. The Hausdorff limit of this family when λ goes to zero is K_0. By Theorem 5.9, we have for all $k \in \mathbb{N}$ and all $\lambda \ll 1$,

$$b_k(K_0) \leq \sum_{p+q=k} b_q(D^p_\lambda(\eta));$$

where $\eta > 0$ is fixed and

$$D^p_\lambda(\eta) = \{(\mathbf{x}_0, \dots, \mathbf{x}_p) \in (K_\lambda)^{p+1} \mid \rho_p(\mathbf{x}_0, \dots, \mathbf{x}_p) \le \eta\}.$$

Consider now the Cartesian product

$$T^p_\lambda = \{(\mathbf{x}_0, \dots, \mathbf{x}_p, \mathbf{y}_0, \dots, \mathbf{y}_p) \mid \mathbf{x}_i \in X_\lambda, \mathbf{y}_j \in Y_\lambda\}.$$

We can consider a cylindrical cell decomposition of T^p_λ that would be compatible with the subsets $\{\rho_p(\mathbf{x}_0, \dots, \mathbf{x}_p) = \eta\}$ and $\{|\mathbf{x}_i - \mathbf{y}_i| = \delta(\lambda)\}$. The number of cells in such a decomposition depends only on p and on the formats of X_λ and Y_λ, and the projection of this decomposition on the variables $(\mathbf{x}_0, \dots, \mathbf{x}_p)$ is compatible with D^p_λ. Thus, the total number of cells in the decomposition of T^p_λ bounds $b(D^p_\lambda)$, and an upper-bound on $b_k(K_0) = b_k((X, Y)_0)$ follows. Explicit bounds, which would be doubly exponential in kn, can be derived from [GV2, PV]. □

Remark 5.18 *The explicit bound that would be obtained by the above method is very bad (doubly exponential). In particular, the bounds obtained are worse than those obtained in Chapter 4 in the case where $k = 0$. Better estimates, that coincide with Chapter 4 when $k = 0$, are work in progress [Z3].*

Appendix A

Spectral sequence associated to a filtered chain complex

A.1 Homology spectral sequence

We consider here only first quadrant homology spectral sequences. Assume $\{E^r_{p,q}\}$ are modules over some ring R, which are non zero only for $p, q, r \geq 0$.

This is a spectral sequence if for all r, p, q there is a differential $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ such that $d^r \circ d^r = 0$, and such that

$$E^{r+1}_{p,q} = \frac{\ker(d^r_{p,q})}{d^r_{p+r,q-r+1}\left(E^r_{p+r,q-r+1}\right)}. \tag{A.1}$$

Note that for $r > p$ we have $\ker(d^r_{p,q}) = E^r_{p,q}$ since the image is a term that lies in $p < 0$. Similarly, for $r > q+1$, the module $E^r_{p+r,q-r+1}$ is zero, hence $d^r_{p+r,q-r+1}\left(E^r_{p+r,q-r+1}\right)$ is zero too. It follows from (A.1) that for all $r > \max(p, q+1)$, we must have $E^{r+1}_{p,q} = E^r_{p,q}$. We denote by $E^\infty_{p,q}$ the term at which $E^r_{p,q}$ stabilizes.

Let H be a chain complex with a an increasing filtration

$$0 \subseteq F_0H \subseteq F_1H \subseteq \cdots \subseteq F_pH \subseteq \cdots \subseteq H$$

such that $\cup_p F_pH = H$. We say that $E^r_{p,q}$ converges to H if for all p and q we have

$$E^\infty_{p,q} \cong \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}} \tag{A.2}$$

It is usually denoted by $E^r_{p,q} \Rightarrow H$.

A.2 Sequence associated with a filtered complex

Let C be a chain complex such that $C_n = 0$ for $n < 0$. Denote by d_n the differential from C_n to C_{n-1}. Assume that there is a filtration of C by subcomplexes $\{F_p\}_{p\in\mathbb{N}}$ such that F_p is increasing and $\cup_p F_p = C$.

For $n \in \mathbb{N}$, denote by $Z_n = \ker(d_n)$ the cycles in C_n and by $B_n = d_{n+1}C_{n+1}$ the boundaries. Let $H_n = Z_n/B_n$. We can use the filtration F_p to approximate Z_n and B_n, in the following fashion: define, for all $r \geq 0$,

$$A_p^r = \{c \in F_p \mid dc \in F_{p-r}\}.$$

The elements of A_p^r are cycles 'up to F_{p-r}.' For $r > p$, they are really cycles.

One then defines approximate cycles $Z_p^r = A_p^r/F_{p-1}$ and $B_p^r = dA_{p+r-1}^{r-1}/F_{p-1}$. Note that the indices are chosen so that both are submodules of F_p/F_{p-1}, and we have the following inclusions:

$$0 \subseteq B_p^0 \subseteq B_p^1 \subseteq \cdots \subseteq B_p^\infty \subseteq Z_p^\infty \subseteq \cdots \subseteq Z_p^1 \subseteq Z_p^0 = F_p/F_{p-1}.$$

Here, B_p^∞ denotes the increasing union of the modules B_p^r and Z_p^∞ is the decreasing intersection of the modules Z_p^r. We then define

$$E_{p,q}^r = (Z_p^r)_{p+q}/(B_p^r)_{p+q}.$$

Theorem A.1 *With the above definitions, $E_{p,q}^r$ is a spectral sequence that converges to $H(C)$.*

See [McCl] for a proof.

Corollary A.2 *Let C be a chain complex such that $C_n = 0$ for $n < 0$, such that there is an increasing exhaustive filtration of C and let $E_{p,q}^r$ be the associated homology spectral sequence. Then, we have for all n,*

$$\text{rank } H_n(C) \leq \sum_{p+q=n} \text{rank}\,(E_{p,q}^1).$$

Proof: From (A.1), it is clear that $\text{rank}\, E_{p,q}^r$ is a decreasing sequence when p and q are fixed and r increases. Thus, we have $\text{rank}\, E_{p,q}^\infty \leq \text{rank}\, E_{p,q}^1$. Since $E_{p,q}^r \Rightarrow H(C)$, it follows from (A.2) that

$$H_n(C) \cong \bigoplus_{p+q=n} E_{p,q}^\infty.$$

The result follows easily. □

A.3 Alexander cohomolgy and applications

This section contains a short list of auxiliary results that play a role in the construction of the filtration that gives rise to the spectral sequence of Theorem 3.9. We refer the reader to Chapter 6 of [Spa] for more general statements, proofs and additional details.

In this section, $\bar{H}^*$ denotes the Alexander cohomology.

Definition A.3 *A topological space X is said to be* homologically connected *if for all $x \in X$ and all neighbourhood U of x, there exists a neighbourhood $V \subseteq U$ such that the map $\widetilde{H}_q(V) \to \widetilde{H}_q(U)$ given by inclusion is trivial for all q.*

Definition A.4 *A topological space X is said to be* locally contractible *if for all $x \in X$ and all neighbourhood U of x, there exists a neighbourhood $V \subseteq U$ such that V can be deformed to x in U.*

If X is locally contractible, it is homologically connected. In particular, all sets that are definable in some o-minimal structures are locally contractible.

Proposition A.5 *Let X be homologically connected. We have $\bar{H}^*(X) \cong H^*(X)$, where $H^*(X)$ is the singular cohomology of X.*

Theorem A.6 (Vietoris-Begle) *Let $F : A \to B$ be a closed, continuous surjection between paracompact Hausdorff spaces. If for all q, we have $\widetilde{\bar{H}}^q(F^{-1}y) = 0$ for all $y \in B$, the map $F^* : \bar{H}^*(B) \to \bar{H}^*(A)$ is an isomorphism.*

See [Spa, p. 344] for a proof. Example 16 on the same page shows that the theorem does not hold if F is not closed.

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