Algebraic descriptions of weighted directed graphs and their applications

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COMPOSITION DU JURY :

M.	J-L.	Loday	Rapporteur
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Contents

Chapter 1

Introduction

This introduction is voluntarily written in a rather loose style so that the main ideas, stemming essentially from Physics can be easily read by theoretical physicists as well. In the sequel, I would like to outline the motivations which led me to introduce coalgebras equipped with at least two coproducts and how the different chapters are entwined.

1.1 Preliminaries

The recent development of quantum information theory demonstrated the necessity to study the behaviour of random walks on directed graphs. The first results recently obtained, see for instance Campanino and Petritis [6, 7] yield very surprising behaviour for those random walks. There exists an impressive quantity of results concerning random walk on groups, i.e., via the Cayley graph on non-directed graphs. However, as soon as the invertibility of each element is no longer guaranteed, i.e., the graph become directed, the existing literature is mainly reduced to the general theory of Markov chains. In this general framework, the detailed exploration is slowed down by a lack of algebraic structures.

We propose, in the sequel, an algebraic formalism for weighted directed graphs and show unexpected relationships between new kinds of algebras firstly introduce by Loday and later by Loday and Ronco. All began by simple remarks on the use of directed graphs in algebra.

Before giving more details, let us just recall the definitions of a coassociative coalgebra and the convolution products. Other definitions can be found in Chapter 2.

Denote by k, the real or the complex field. A *coassociative coalgebra* (C, Δ, ϵ) (with counit) over k [57, 49], is a k-vector space such that the counit map $\epsilon : C \to k$ and the coproduct map $\Delta : C \to C \otimes C$ verify:

1. The coassociativity equation: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

2. The counit equation: $(id \otimes \epsilon)\Delta = id = (\epsilon \otimes id)\Delta$.

A Hopf algebra is both an algebra and a coassociative coalgebra, its coproduct and counit being homomorphisms and is equipped with a special map called antipode.

Let (A, m) be an associative algebra. Consider the k-vector space of linear functions which map (C, Δ), a coassociative coalgebra, to A. Let $f, g : C \to A$ be such maps. Such a space can be embedded into an associative algebra by defining the so-called convolution product $f * g := m(f \otimes g)\Delta$. Therefore, to construct algebras with some particular properties, it can be interesting to construct their co-versions. In the sequel, we will construct a (weighted) directed graph on each coassociative coalgebra. To make this introduction easier to read, by a slight abuse, we will identify graph with coalgebra and speak in the same time of coalgebras and their corresponding algebras (their products being defined by the convolutions products).

1.2 A static description

There already exist at least two algebraic structures on (not weighted) directed graphs. The first one is embedded into the field of Hopf algebras. The property that a given path $(v_1, \ldots, v_n)$ of a directed graph can be decomposed into two parts, say $(v_1, \ldots, v_p)$ and $(v_{p+1}, \ldots, v_n)$, in several ways is a starting point to define a coassociative coproduct. More information can be found, for instance, in the works of Cibils and Rosso [9, 10] and the references therein. Another approach relates directed graphs to C^*-algebras. Initially introduced in a work of Cuntz and Kreiger [14], a class of C^*-algebras related to topological Markov chains is constructed. In later works, directed graphs are associated with these so-called Cuntz-Kreiger algebras. Conversely, directed graphs yield Cuntz-Kreiger families and C^*-algebras are constructed, the topology being induced by groupoid argument coming from the fact that a path can be already decomposed into several parts. The reader can find the definition of a Cuntz-Kreiger family in Chapter 2.

At this present stage, it is worth making two remarks. Firstly, these two approaches do not authorise weights necessary to deal with random walks for instance. Secondly, the decomposition of a path of a directed graph, used to define for instance a coassociative coproduct, is reminiscent of a scan. I will refer to such a decomposition as *the static point of view*.

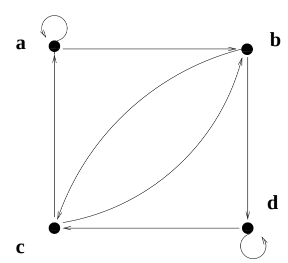
How to develop a *dynamical point of view*, implicitly used in weighted directed graphs? To answer this question, let us make a detour to Hopf algebras. In [49], Majid shows that Hopf algebras can be related to classical (and quantum) random walks. For instance, via the Cayley graph, studying random walks on a non-directed graph is equivalent to study random walks on a group. Since a Hopf algebra can be constructed on groups, the use of Hopf algebras turn out to be powerful to evaluate probabilistic quantities such as mean of a function in a given state.

It is known since the works of Rota and Joni that coassociative coalgebras are closely related to combinatorics, the coproduct coding the combinatorial transformations in algebraic substitutions. How can we code the combinatorics of a weighted directed graph? Motivated by the intrusion of coassociative structures in combinatorics and discrete Markov processes, I decided to reverse the point of view and construct weighted directed graphs from algebraic structures such as coassociative coalgebras.

1.3 Locality and directed graphs

The simplest way to construct a directed graph from a coassociative coalgebra (C, Δ), spanned as a k-vector space by the basis C_0 is to consider C_0 as the vertex set and as arrow set, the mutual relations between elements of C_0, defined by the coproduct Δ. Therefore, we identify $c \in C_0$ with the vertex: $c \bullet$ and tensor products appearing in the definition of Δ, say $\lambda c_1 \otimes c_2$, $c_1, c_2 \in C_0$ and $\lambda \in k \setminus \{0\}$, by a weighted arrow: $c_1 \bullet \overset{\lambda}{\to} \bullet\, c_2$. We obtain easily a weighted directed graph, $Gr(C)$, which will be called the geometric support of (C, Δ). The (linear) maps s, t, source and terminus are naturally defined such as $s(c_1 \otimes c_2) = c_1$ and $t(c_1 \otimes c_2) = c_2$, with $c_1, c_2 \in C_0$. In the sequel, to make the reading of this introduction easier, we will underpin our concepts through simple examples.

Example 1.3.1 Denote here by $\mathcal{F}$, the k-vector space spanned by the basis a, b, c and d with coassociative coproduct $\Delta : \mathcal{F} \to \mathcal{F}^{\otimes 2}$ given by: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$. The directed graph, $Gr(\mathcal{F})$, associated with $\mathcal{F}$ is:



Directed graph associated with $\mathcal{F}$: $Gr(\mathcal{F})$.

From a physical point of view, there is something odd in this approach. When a directed graph, for example $Gr(\mathcal{F})$, is given, it is usual to name the near neighbourhood of a vertex v by looking around it and selecting vertices belonging to the set $t(s^{-1}(\{v\}))$. For instance, the nearest neighbours of a are a and b. Another remark comes from the following *Gedanken* experiment. Suppose $Gr(\mathcal{F})$ is a model of space-time and an experimentalist stands on a and sends light to communicate with other experimentalists standing on b and c. Taking the locality, i.e., the usual definition of near neighbourhood, for granted, we expect the light he sends, follows the trajectory $a \to a$ and $a \to b$. Now, if we observe the way this graph has been constructed, we remark that a is related by the coproduct Δ to $a \to a$, as expected but also to $b \to c$. Therefore, if the experimentalist sends light from a, the trajectory followed by the light is the following. It starts in the same time, from a and b and arrive to a and c. This *Gedanken* experiment shows that our perception of the usual locality on directed graph has to be improved since there is, apparently, a non-local transmission of light, if we see the coproduct as a propagator. Two relevant questions arise.

1. Is our definition of locality on directed graph correct or does it need to be generalised?

2. How can we recover the usual locality?

1.4 A dynamical description

To answer these two questions, we need to regard algebraic descriptions of directed graphs in a non-static way. To recover locality, we have to break the coassociativity, apparent responsible for this *a priori* non-physical model of space-time, and to introduce a dynamical description restoring locality. How can we do that?

The simplest way is to code in an algebraic way, the physical notions of Past, Present, Future, i.e., to investigate algebraic spaces equipped with two propagators, one relating the Present to the Future, the other relating the Present from the Past.

Mathematically, this is equivalent to introduce, what is called a Markov L-coalgebra $\mathcal{G}$, i.e., a k-vector space, generated by an independent spanning set $\mathcal{G}_0$, equipped with two coproducts $\Delta_M : \mathcal{G} \to \mathcal{G}^{\otimes 2}$ and $\tilde{\Delta}_M : \mathcal{G} \to \mathcal{G}^{\otimes 2}$ defined formally by:

1. The future structure: $\Delta_M(\text{Present}) = \text{Present} \otimes \text{Future}$.
2. The past structure: $\tilde{\Delta}_M(\text{Present}) \quad = \text{Past} \otimes \text{Present}$.

The Past, the Present and the Future being related by the following equation:

$$(\text{Past} \otimes \text{Present}) \otimes \text{Future} = \text{Past} \otimes (\text{Present} \otimes \text{Future}).$$

More precisely, this leads us to define $\Delta_M v := v \otimes \sum_{v_i \in t(s^{-1}(\{v\}))} v_i$ and $\tilde{\Delta}_M v := \sum_{v_j \in s(t^{-1}(\{v\}))} v_j \otimes v$, where the v are vertices of a given directed graph. The coproducts are related by the so-called coassociativity breaking equation:

$$(\tilde{\Delta}_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M. \tag{1.1}$$

Observe that starting with this Markov L-coalgebra, we can easily recover the (weighted) directed graph we started with. Moreover, the usual definition of a near neighbourhood of a vertex is recovered and given by the Markovian coproducts.

1.5 Outline of the results

What is the most important in this framework is that the "symmetry" (or degeneracy) associated with a coassociative coproduct (in some sense, equality between Past and Future) is broken (or split) into two (*a priori* non-coassociative) coproducts coding any weighted directed graph through its Past, Present and Future. We now generalise this framework and define a L-coalgebra, that is a space $(L, \Delta, \tilde{\Delta})$ equipped with two coproducts verifying (1.1). A weighted directed graph, called its geometric support can be associated with it. By imposing the degeneracy $\Delta = \tilde{\Delta}$, we recover the usual coassociative framework and the construction of graphs previously described.

Observe that this setting, via convolution products, leads to study algebras equipped with two products $\vdash$, $\dashv$, obeying $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. Another quite remarkable thing, also exposed in Chapter 2, is the following. To code a bi-directed graph, that can be considered as a non-directed graph in some sense, we simply embed it into its Markov L-coalgebra and remark that $\Delta_M = \tau\tilde{\Delta}_M$, τ being the switch map. This equality defining the cocommutator space in the degenerate case, we will call, $\ker(\Delta - \tau\tilde{\Delta})$, the L-cocommutator of a L-coalgebra. Algebraically speaking, requiring that a Markov L-coalgebra, i.e., the usual algebraic model of a directed graph, is included into its L-cocommutator means that the graph is bi-directed. Therefore, this leads us to study a particular commutator $[x, y] = x \dashv y - y \vdash x$. Requiring, as usual, that $[x, \cdot]$ and $[\cdot, y]$ behaves like Leibniz derivatives imply special conditions on the L-coalgebra.

These conditions has been found by Loday. Motivated by periodicity phenomena in algebraic K-theory, he introduced [45] the notion of "non-commutative Lie algebra", called Leibniz algebra. Such algebras D are described by a bracket $[\cdot, z]$ verifying the so-called Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and the Leibniz algebra turns out to be a Lie algebra. A way to construct such a Leibniz algebra is to start with an associative dialgebra, that is a k-vector space D equipped with two associative products, $\vdash$ and $\dashv$, such that for all $x, y, z \in D$,

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
3. $(x \vdash y) \vdash z = (x \dashv y) \vdash z$.

The associative dialgebra is then a Leibniz algebra by defining $[x, y] := x \dashv y - y \vdash x$, for all $x, y \in D$. The operad of associative dialgebras is Koszul dual to the operad [1] of dendriform algebras, a *dendriform algebra* E being a k-vector space equipped with two binary operations, $\prec$, $\succ$: $E \otimes E \to E$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
2. $(a \succ b) \prec c = a \succ (b \prec c)$,
3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

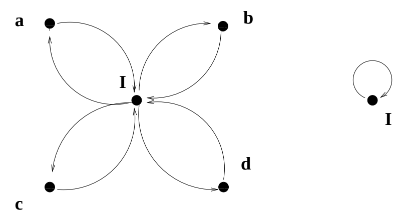
This notion dichotomizes the notion of associativity since the product $a * b = a \prec b + a \succ b$, for all $a, b \in E$, is associative.

Observe that these two types of algebras verify, in their coversions, the equation (1.1). Motivated by these unexpected similarities, we have found among weighted directed graphs,

[1] Roughly speaking, an operad is an algebraic notion allowing a presentation of free algebras equipped with several laws and constraints in terms of generators and relations. The "Koszulness" of an operad guaranteed "good properties". As an indication, the operad *As* (resp. *Poiss*) of associative algebras (resp. Poisson algebras) are Koszul self-dual. The operad *Com* of commutative and associative algebras is Koszul dual to *Lie*, the operad associated with Lie algebras. When a type of algebra is discovered, one can look for its Koszulness and its dual. The dual of the corresponding operad will give another type of algebra. This process was used by Loday and Ronco to discover the algebras used here. It is quite remarkable that by a graphical point of view, most of them can be also rediscovered.

viewed from the L-coalgebra framework, (co)-dialgebras and dendriform (co)-algebras. Since then, several types of algebras have been defined by Loday and after by Loday and Ronco. These algebras will be described (and recovered) through out this thesis. Another motivation to study them can be found in combinatorics, since for instance the free dendriform algebra or the free dual version of coassociative L-coalgebras are related to planar binary trees [2].

The first (Markovian) coassociative codialgebra I met, via graph theory, is a non-expected one. Let A be a unital associative algebra, with unit I and $a \in A$. From the equality $(I \cdot a) \cdot I = I \cdot (a \cdot I)$, A carries a non-trivial finite Markov L-bialgebra, with coproducts $\delta_f(a) := a \otimes I$ and $\tilde{\delta}_f(a) := I \otimes a$, for all $a \in A$. In the case when A is generated by a set A_0, its geometric support is called the flower graph, because it is the concatenation of petals:



Example of geometric support associated with an algebra $k\langle a, b, c, d \rangle \oplus kI$.

Equipped with these coproducts, the algebra A is a coassociative codialgebra. Observe also that for all $a \in A$ different from I, $a \mapsto \delta_f(a) + \tilde{\delta}_f(a)$ and $I \mapsto I \otimes I$ is a coassociative cocommutative coproduct and that $a \mapsto \delta_f(a) - \tilde{\delta}_f(a)$ is also a well-know differential map, known as the Karoubi differential.

Chapter 3 starts with two remarks. When a bialgebra is given, it is by definition both a coassociative coalgebra and an algebra with unit. In terms of graphs, this means that two graphs are given. That constructed from the coalgebra structure and that defined by the underlying algebra. Let us represent these structures by their respective coproducts Δ, δ_f and $\tilde{\delta}_f$. To compare the two graphs, the simplest thing to do is a substraction, say $\Delta - \delta_f$ and $\Delta - \tilde{\delta}_f$. A particularity of these new coproducts was put forwards by Hudson [22], in terms of Leibniz-Ito derivatives. Let A be an algebra with unit I, M a A-bimodule and $\rho : A \to M$, a linear map. Such a map is a Leibniz-Ito derivative if $\rho(I) = 0$ and $\rho(xy) = x\rho(y) + \rho(x)y + \rho(x)\rho(y)$, for all $x, y \in A$. It is Leibniz-Ito derivatives and not Leibniz derivatives which naturally occur from graph theory. Motivated by a work of Quillen [61], Chapter 3 translates the Leibniz-Ito property in terms of Quillen curvature $\omega(x, y) := \rho(xy) - \rho(x)\rho(y) = x\rho(y) + \rho(x)y$, which looks like Leibniz derivative. (Roughly speaking, it is worth noticing that $\omega(x, y)$ stands for $\omega(x \otimes y)$ and thus, with the graphical viewpoint in mind, the Quillen curvature is taken on the arrow $x \longrightarrow y$, whereas a Leibniz derivative is $D(xy) := D(m(x \otimes y))$ is taken on the vertex $m(x \otimes y)$, reminiscent of a tool from graph theory named line-extension (of a graph).)

What is interesting in [61], is that Quillen searches to define common tools used in algebra in terms of the Hochschild complex. In that way, an elegant definition of Leibniz derivatives is given. How can we characterise, with the same elegance, a Leibniz-Ito derivative, an object

[2] The fact that the free associative L-algebras is related to planar binary trees is a very recent result, see Pirashvili [51].

commonly arising in classical and quantum stochastic calculi? What will occur, if in algebraic fields such as non-commutative geometry, instead of the Leibniz property to define a differential map, the Leibniz-Ito property was chosen?

Chapter 3 is an attempt to answer these questions from the coproducts defining the flower graph. Briefly, we show the usefulness of the Bianchi identity, how to construct super-algebras and differential associative dialgebras from the Quillen curvature of a Leibniz-Ito map. As Connes [12] did for the Leibniz map, $[F, \cdot]$, with F a well-chosen operator, we show also that cyclic cocycles can be constructed from the Quillen curvature of a Leibniz-Ito map. Their relations with periodic orbits of the flower graph are also given. In Appendix B, let us mention that the Hochschild complex can be interpreted from reading the periodic orbits of the flower graph.

Before continuing on the interplay between extented graph theory, via L-coalgebras, and the works of Loday and Ronco in Chapter 5, let us mention the rôle of Chapter 4. This chapter is the fundamental one in the understanding of L-coalgebras from a graphical point of view. It is the first time, the extended graph theory plays an important rôle in applications and where the notion of indexed coproduct appears. Motivated by links between combinatorics and coassociative coalgebras introduced by Joni and Rota [24], we describe a combinatorics generated by a quantum random walk, the so-called Hadamard walk over $\mathbb{Z}$ which is a quantisation of the Bernoulli walk, in terms of a coassociative coalgebra having the same geometric support than the Markov L-coalgebra involved in the definition of this walk. Relationships between the classical chaotic map $f: \ x \mapsto 2x \mod 1, \ x \in [0, 1]$, a quantisation of a Bernoulli walk over $\mathbb{Z}$ and De-Bruijn graphs are given. An important notion developed from the coproduct point of view is the notion of coassociative grammar. We show that two different coproducts, the one which comes from the usual Markovian point of view and the one associating with a coassociative coalgebra supply the same description of this quantum walk and give the same dictionnary. We use this fact to embed the set of periodic orbits generated by the chaotic map f into a language whose production rules are substitutions given by a coassociative coproduct. The set of classical periodic orbits of f can then be viewed as a "coassociative" context free grammar, i.e., a grammar whose production rules are given from a coassociative coproduct. We point out relations between periodic orbits of the classical chaotic map f and non-commutative polynomials describing the Hadamard walk.

Chapter 5 is inspired from Chapters 2 and 4. Roughly speaking, at the end of Chapter 2, the Markov L-coalgebras are compared to coassociative coalgebras. For instance, the rôle of primitive elements in coassociative coalgebras are played by isolated periodic orbits in Markov L-coalgebras. To establish such a comparison we are forced to study coproducts from the vertex set to the arrow set in the case of usual coassociative coalgebras and to define coproducts from the arrow set to the set of the paths of length two in the case of Markov L-coalgebras. In this setting, the arrow set (resp. the set of the paths of length two) of a Markov L-coalgebra "plays the rôle" of the vertex set (resp. the arrow set) of a coassociative coalgebra. In usual graph theory, such a process is named the line-extension [3] of a graph. An indication related to such a result is the following one. There have to exist geometric supports of Markov L-coalgebras whose line-extensions recover the geometric supports of coassociative coalgebras. Indeed, it is the case for usual geometric supports of coassociative coalgebras, whose coproducts are defined by a "n by n matrix product". They can be recovered from line-extensions of a Markov L-

[3] This graphic tool is often used and implictly related to the definition of indexed coproducts.

coalgebras called the $(n,1)$-De Bruijn graphs [4]. In fact, we obtain more information from the line-extension. By actions of a shift, we can produce a tiling of the $(n^2,1)$-De Bruijn graph by defining n-coproducts Δ_i verifying:

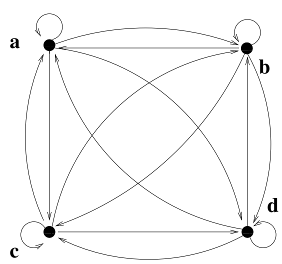
$$(\Delta_i \otimes id)\Delta_j = (id \otimes \Delta_j)\Delta_i, \quad i,j = 0,\ldots,n-1. \tag{1.2}$$

In the literature, the free algebra, whose co-version is defined by (1.2) has been studied in the case $n=2$ by Richter [53] and by Loday and Ronco [48] in the case $n=3$ under the name cubical trialgebra. Here we have, what I called hypercube n-(co)algebras. We observe also that other examples of hypercube n-(co)algebras can be constructed by letting $M_n(k)$ act on the vectors ${}^t(0,\ldots,0,\Delta_i,0,\ldots,0)$, i.e., on the axioms (1.2). The transformation so obtained has another consequence. The new coproducts can be viewed as coassociative clusters whose splitting gives n coassociative coproducts. Splitting (co)associative clusters [5] is an important notion which will be developed in [35], see also [46]. It is reminiscent of particule physics when particules split in collider, with obvious notation, this splitting $\Delta'_i \longrightarrow \sum_j \lambda_j \Delta_j$ (or decay) can be compared to the dendriform product $* \longrightarrow \prec + \succ$. Readers prefering a mathematical point of view will observe that coproducts verifying (1.2) define a k-vector space with dimension n.

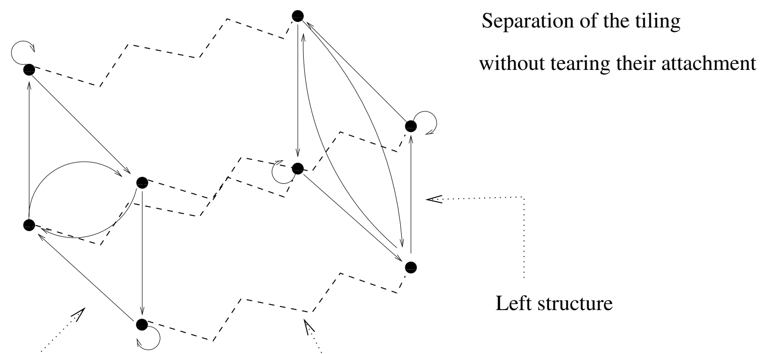
Another important notion in this chapter is the notion of *(a)chirality*. Axioms defining L-coalgebras $(L,\Delta,\tilde{\Delta})$, are usually not invariant by exchanging the right structure (L,Δ), (also called the Future structure, if we have a physical point of view in mind) with the left one $(L,\tilde{\Delta})$ (also called the Past), i.e., via the transformation: $\Delta \leftrightarrow \tilde{\Delta}$. If axioms defining $(L,\Delta,\tilde{\Delta})$ can distinguish the left structure from the right one, then the L-coalgebra is said *chiral*. On the contrary, the L-coalgebra is said *achiral*. What we have obtained in Chapter 5 is a tiling of the n^2-De Bruijn graph by n-coassociative structures $(\Delta_i)_{0\leq i \leq n-1}$, any two of them being related by (1.2), i.e., by an achiral relation.

At the begining, the equation (1.2) was called the coassociativity breaking equation. Starting with a non-locality problem on directed graphs, we had to break the symmetry of the coassociative coproduct to restore the usual Past and Future on a graph. Now with the tiling obtained in Chapter 5, we have to re-interpret this fundamental equation. How can we do that? It is the starting point of coassociative manifolds and L-molecules, developed in Chapter 7. Before continuing with Chapter 7, let us explain a "drawback" which appeared in the consequences of such tilings. Applying this tiling, for instance to the Hopf algebra $Sl_q(2)$ gives a left part to $Sl_q(2)$. However, the algebra underlying $Sl_q(2)$ does not embed the left coproduct into a homomorphism. In terms of graph, we can interpret this consequence as a kind of "repulsion". The two geometric supports are glued together on a same graph called the $(2,1)$-De Bruijn graph and are "too closed" to share the same algebra underlying the right coalgebra, i.e., the usual $Sl_q(2)$. Therefore, we have to take the left part away, in keeping the fact that these two structures are glued. It is a kind of topological problem: Deform the two parts of this tiling without tearing them.

[4] Viewed as Markov L-coalgebras, such graphs are also the geometric supports for coassociative codialgebras.
[5] En français: fission d'amas associatif.



The (4,1)–De Bruijn graph



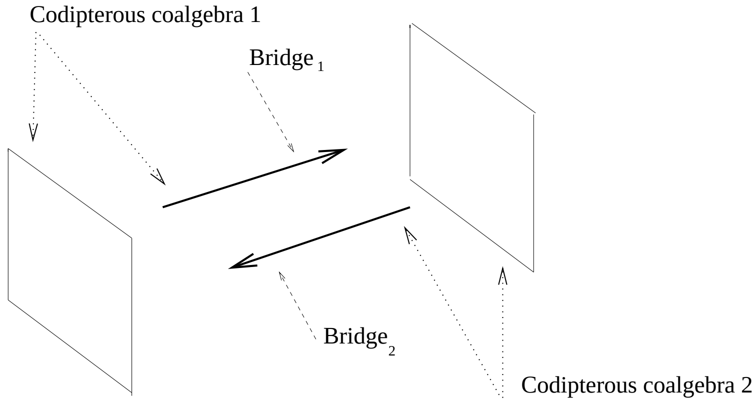
Tiling the $(4,1)$-De Bruijn graph with an achiral structure.

Graphically, we have represented the geometric supports of the right and left structures of $Sl_q(2)$. They produce a tiling of the $(4,1)$-De Bruijn graph. To produce such a separation, we have to create bridges, i.e., others algebraic structures between the two left and right stuctures. The new interpretation of (1.1) we are looking for becomes apparent now. A bridge has to be a directed graph, therefore defined by a coproduct. It has for support, or is hold up, by a coassociative coalgebra. This is the usual definition of what will be called a codipterous coalgebra. Therefore, we have two supports, each one holding up a bridge. We say that the two algebraic structures are entangled, if the coproducts defining bridges verify the previously named coassociativity breaking equation. This equation will be from now on called the entanglement equation. Two coproducts will be said (chiral) entangled if they verify the entanglement equation and (achiral) entangled if this equation is invariant by inverting them. More precisely, this leads us to define a k-vector space $\mathbb{D}$ equipped with two coproducts $\Delta, \delta : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ verifying:

1. Coas: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.
2. Codip: $(\Delta \otimes id)\delta = (id \otimes \delta)\delta$.

It is called a *codipterous coalgebra* [6]. Similarly, we call an *anti-codipterous coalgebra* a k-vector space $\hat{\mathbb{D}}$ equipped with two coproducts $\Delta, \hat{\delta} : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ such that Δ is coassociative and $(id \otimes \Delta)\hat{\delta} = (\hat{\delta} \otimes id)\hat{\delta}$.

[6] A codipterous coalgebra (resp. anti-codipterous coalgebra) can also be viewed as a coassociative coalgebra with an extra left (resp. right) comodule on itself.



Entanglement of two codipterous coalgebras.

As a consequence, other algebraic structures, discovered independently of Loday and Ronco called pre-dendriform (co)algebras are found. Pre-dendriform coalgebra is the entanglement of two coproducts which gravitate around a coassociative coproduct, i.e., a k-vector space $\mathbb{D}$ equipped with three coproducts $\Delta, \delta, \hat{\delta} : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ verifying:

1. Coas: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

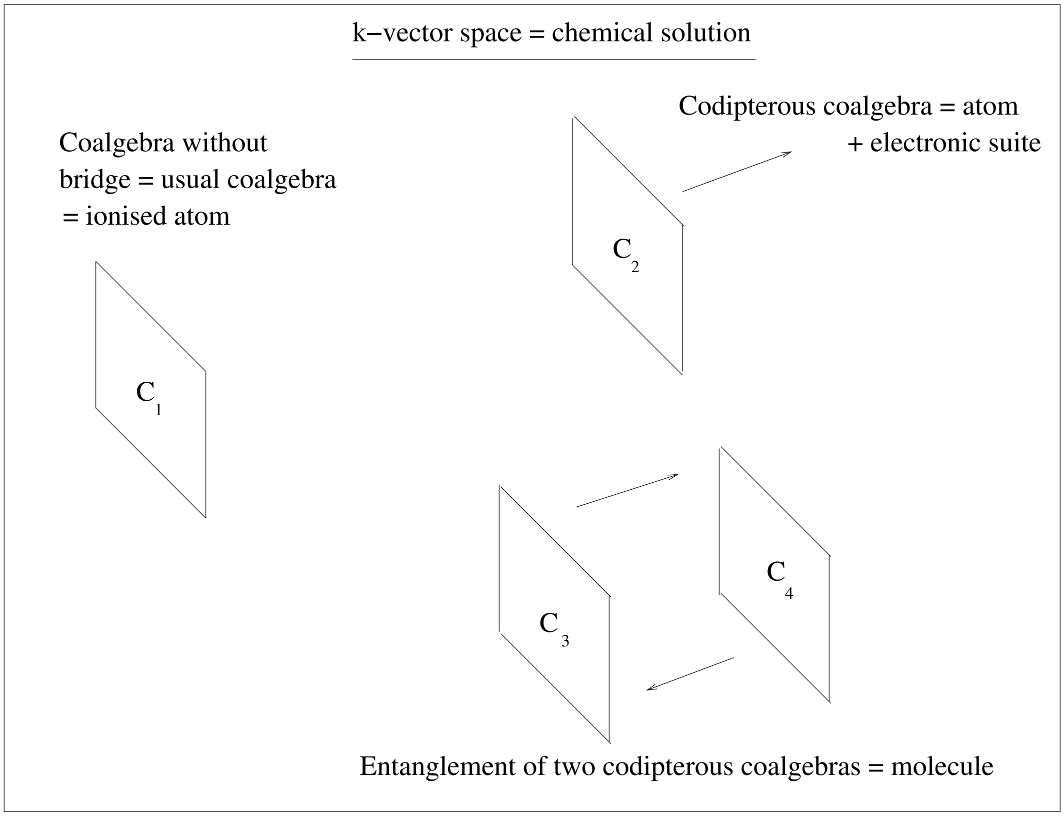
2. Codip: $(\Delta \otimes id)\delta = (id \otimes \delta)\delta$.

3. Anti-codip: $(id \otimes \Delta)\hat{\delta} = (\hat{\delta} \otimes id)\hat{\delta}$.

4. Entanglement equation: $(id \otimes \hat{\delta})\delta = (\delta \otimes id)\hat{\delta}$.

Therefore, a pre-dendriform coalgebra can be viewed as an entanglement of a codipterous coalgebra and an anticodipterous coalgebra supporting the same coassociative law. The coproducts gravitating around this coassociative coproduct are also called bridges.

What is fundamental is the notion of entanglement of (anti)codipterous coalgebras. They are the basis of more and more general algebraic structures and here comes the main idea. A k-vector space can be equipped with several coproducts defining coassociative coalgebras, bialgebras or Hopf algebras (if underlying algebras are also defined). Graphically, all these structures can be represented by weighted directed graphs reminiscent of atoms in a chemical solution. To construct molecules, atoms "share" their electronic suites, here modelled by bridges. Therefore, as an analogy, a coassociative coalgebra can be identified with an ionised atom, i.e., an atom which has no electronic suite, a coassociative (anti)codipterous with an atom and entanglement of two such structures with a molecule.



Shematic description of geometric supports associated with different algebraic structures

Obviously, entangling several (anti)codipterous coalgebras leads to complicated symmetries in mathematics (or polymers with a chemistry point of view). A consequence of such an interpretation is the construction of coassociative codialgebras, coassociative cotrialgebras, Leibniz algebras, Poisson dialgebras and also of dendriform codialgebras by a graphical notion of self-covering explained in Chapter 7.

Let us end Chapter 7 on the notion of coassociative manifolds. A graphical particularity of all the algebraic structures constructed with the notion of entanglement is that on the intersection of any two geometric supports representing coassociative coalgebras, the two respective coproducts are equal. The notion of open sets covering a topological manifold is played by the notion of coassociative coalgebra and multiplying two functions on the intersection of two such "open sets" is possible because of the degeneracy of convolution products. This leads us naturally to enlarge our graphical constructions to the notion of coassociative manifolds.

Chapter 6 has apparently nothing to do with other chapters. The main result of this chapter is to prove that there are solutions of the Yang-Baxter equation which code weighted directed graphs. In fact all weighted directed graphs produce at least a solution of the Yang-Baxter equation and thus, provide new representations of the braid groups. As a consequence, weighted directed graphs are also related to special bialgebras, see [36].

In Appendix A, we try to bring another point of view on commutativity of positive operators. It is well known that commutativity of positive (self-adjoint) operators is essential in quantum mechanics since it guarantees coherent results in measure processes. With the growing field of quantum information, a distance, called the Bures distance has been used. This distance uses a law, called the quantum fidelity, which is self-distributive on positive operators which commute. Is it possible to establish the converse? Otherwise stated, are three positive operators, such that the law coming from the Bures distance is self-distributive, commuting? If this is true, the commutation of two positive operators (three with the identity element) could be interpreted as a move called the third Reidmeister move in knot theory. We address this question in several cases. Other results are obtained concerning how the third Reidmeister move, i.e., the self-distributivity of this law, can be viewed from a quantum system. It is known that information gained on such systems are done via the trace map which authorise commutativity at short distance. What are the algebraic laws verifying the self-distributivity condition up to commutativity at short distance? We show that the Wigner-Yanase distance can be related in some sense to this new way to see self-distributivity. We find also other examples of self-distributive laws.

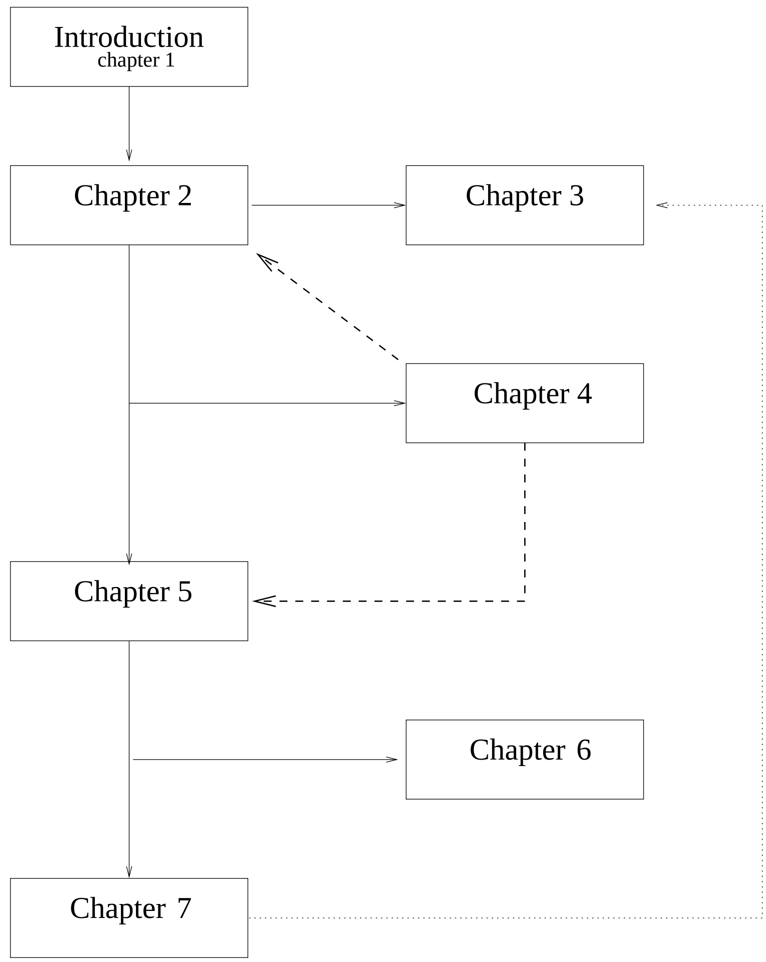
Appendix B shows how to recover Hochschild complex by reading periodic orbits of the flower graph associated with an unital algebra. The idea of reading periodic orbits, associated with another directed graph, is also used in Chapter 4.

Appendix C deals with special combinatorics defined from completely positive maps, a tool mainly used in quantum information theory to model quantum channels or in mathematics in the study of (non)-commutative contractions and their relations with Markov L-coalgebras. We give new ways to produce combinatorics defining completely positive maps and non-commutative contractions.

Appendix D is an exciting, although very speculative, consequence of my work on weighted directed graphs. A graph can be viewed as a model of space-time. Is it possible that mathematics we used to model physics is dictated by the structure of space-time at a given energy level? Otherwise stated and rather shortly, does the graph, via its L-coalgebra setting impose the algebraic structures we have to discover in theoretical physics? In some sense, via convolution products, the graph dictates how two functions have to be multiplied, but can we do better?

Following this idea, a graph via its L-coalgebra structure is used to create algebraic products. A consequence of this fact is the possibility to create probabilistic algebraic products and dynamical L-coalgebras. The idea is simple. The graph, modelling space-time, is a dynamical object. If this graph contracts becoming one of its own subgraph, the algebra on which we worked before its contraction has to change. We say that there is a mutation of the algebraic product. For instance, we can model a matrix projection from a special contraction or dealing with dynamical complex and homology. This dynamical idea has another counterpart. Suppose the graph, modelling space-time is equipped with a probability measure. According the realisation of the stochastic process on this graph, the product will change and will behave as a probabilistic product. These two ideas can be modelled on polynomial algebras whose product is inherited from a coproduct of a L-coalgebra.

1.6 Mutual relations between chapters



Chapter 2

An algebraic framework of weighted directed graphs

Abstract [1]**:**

We show that an algebraic formulation of weighted directed graphs leads to introduce a k-vector space equipped with two coproducts Δ and $\tilde{\Delta}$ verifying the so-called coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. Such a space is called a L-coalgebra. Explicit examples of such spaces are constructed and links between graph theory and coassociative coalgebras are given.

2.1 Introduction

On the one hand, motivated by periodicity phenomena in algebraic K-theory, J-L. Loday [45] introduced the notion of "non-commutative Lie algebra", called Leibniz algebra. Such algebras D are described by a bracket $[-, z]$ verifying the Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and the Leibniz algebra turns out to be a Lie algebra. A way to construct such a Leibniz algebra is to start with an associative dialgebra, that is a k-vector space D equipped with two associative products, $\vdash$ and $\dashv$, such that for all $x, y, z \in D$,

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
3. $(x \vdash y) \vdash z = (x \dashv y) \vdash z$.

[1] *1991 Mathematics Subject Classification:* 16A24, 60J15, 05C20

Key words and phrases: Coalgebras, weighted directed graphs, Leibniz-Ito derivatives, L-coalgebras.

The associative dialgebra is then a Leibniz algebra by defining $[x,y] := x \dashv y - y \vdash x$, for all $x, y \in D$. The operad of associative dialgebras is Koszul dual to the operad of dendriform algebras, a *dendriform algebra* E being a k-vector space equipped with two binary operations, $\prec$, $\succ$: $E \otimes E \rightarrow E$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
2. $(a \succ b) \prec c = a \succ (b \prec c)$,
3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

This notion dichotomizes the notion of associativity since the product $a * b = a \prec b + a \succ b$, for all $a, b \in E$, is associative. By dualizing, we can easily define the notions of coassociative codialgebras and dendriform coalgebras. Coassociative codialgebras are then k-vector spaces equipped with two coassociative coproducts Δ_{dias} and $\tilde{\Delta}_{dias}$ verifying $(\tilde{\Delta}_{dias} \otimes id)\Delta_{dias} = (id \otimes \Delta_{dias})\tilde{\Delta}_{dias}$, in addition with two others axioms easily obtained from the definition of a dialgebra. Similarly, a dendriform coalgebra is a k-vector space equipped with two coproducts, not necessarily coassociative, Δ_{dend} and $\tilde{\Delta}_{dend}$, still verifying the same equation $(\tilde{\Delta}_{dend} \otimes id)\Delta_{dend} = (id \otimes \Delta_{dend})\tilde{\Delta}_{dend}$.

On the other hand, motivated by classical random walks on directed graphs and more generally by weighted directed graphs, we are lead to introduce an algebraic framework based on a particular k-vector space equipped with two coproducts Δ and $\tilde{\Delta}$ verifying $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, called in the sequel the coassociativity breaking equation. Such a space is called a L-coalgebra. In this setting, bi-directed graphs are characterised by the equation $\Delta = \tau\tilde{\Delta}$, where τ is the switch map. This leads us to introduce a special cocommutator $\ker(\Delta - \tau\tilde{\Delta})$. By dualizing, we also obtain an algebraic framework of algebras equipped with two products $\vdash$ and $\dashv$ and the commutator becomes $[x, y] := x \dashv y - y \vdash x$. Requiring that $[-, z]$ verifies the “Leibniz identity” implies to study particular coalgebras called coassociative co-dialgebras. Therefore, requiring an algebraic framework for weighted directed graph leads also to consider special algebras equipped with two products.

These particular L-coalgebras, the coassociative codialgebras and dendriform codialgebras, in addition with those coming from weighted graphs theory are exciting motivations to investigate further algebras, (resp. coalgebras) equipped with two or more products (resp. coproducts). Indeed, to construct associative dialgebras and dendriform algebras, a way is to start with constructing their co-versions and to consider the k-vector space of linear maps defined on such spaces and taking values into an associative algebra. The convolution products defined from these two coproducts will yield associative dialgebras and dendriform algebras.

This paper is the first of a series of 7 papers [40, 41, 39, 38, 34, 37] on the constructions of L-coalgebras, via graph theory. Let us very briefly summarise the results obtained. In [40], we focus on unital algebras viewed as L-bialgebras and show the existence of differential associative dialgebra associated with each curvature, in the sense of D. Quillen [61], of a Leibniz-Ito derivative, i.e., a linear map $\rho : A \rightarrow M$ such that for all $x, y \in A$, $\rho(xy) = x\rho(y) + \rho(x)y + \rho(x)\rho(y)$, with A an algebra with unit 1, M a A-bimodule and $\rho(1) = 0$. Motivated by a work of S.A. Joni and G.-C. Rota [24] on combinatorics, the L-coalgebra formalism is also applied in [39]. We prove that the combinatorics generated by a quantum random walk over $\mathbb{Z}$, called the Hadamard walk, can be recovered from periodic orbits of a classical chaotic system. There exists a bijection between these periodic orbits and periodic orbits of a particular directed graph whose associated L-coalgebra is such that $\tilde{\Delta} = \Delta$. In [38], we show that any weighted directed graph, through its associated (Markov) L-coalgebra yields solutions of the Yang-Baxter equation and thus provide representations of the braid groups. In [41], we construct (Markov) coassociative co-dialgebras and show a relationship between these codialgebras and a class of well-known coassociative coalgebras by considering a tool from graph theory called the line-extension. We also exhibit a tiling of directed graphs, called the $(n^2, 1)$-De Bruijn graphs made with n coassociative coalgebras. These constructions were our first examples of coassociative manifolds [34]. We also obtain examples of cubical trialgebra, a notion defined in [48] and more generally examples of hypercube n-algebra, i.e., a k-vector space V equipped with n products verifying: $(x \bullet_i y) \bullet_j z = x \bullet_i (y \bullet_j z), \;\; x, y, z \in V, \; i, j = 0, \ldots, n-1$, as well as associative products which split into several ones, i.e., $x \star y = \sum_i x \star_i y$, for all $x, y \in A$, with $\star_i$ associative, for all $i = 0, \ldots, n-1$. In [34], we construct L-Hopf algebras, coassociative codialgebras, coassociative cotrialgebras see [48] for the definition, dendriform coalgebras and Poisson algebras. All theses constructions led us to the notion of coassociative manifold developed in [34].

Let us now introduce the first part of the work on L-coalgebras. We start this paper in Section 2.2 with recalling the definition of a weighted directed graph and with introducing axioms of L-coalgebras. We construct from any weighted directed graphs a L-coalgebra, called a Markov L-coalgebra. Then, we enlarge the definition of a directed graph and construct to any L-coalgebra its weighted directed graph. As a coassociative coalgebra is a trivial L-coalgebra, we study some consequences of this association, one of them being the non-locality of the coassociative coproduct over such a graph. We prove also that any coassociative coalgebra (C, Δ) equipped with a group-like element can be viewed as a non-trivial L-coalgebra $(C, \overrightarrow{d}, \overleftarrow{d})$. The case of a bialgebra is then investigated and the new coproducts, $\overrightarrow{d}, \overleftarrow{d}$, turn out to be Leibniz-Ito derivatives. Motivated by this construction, we construct in Section 2.4, L-coalgebras, from any Markov L-coalgebras $(L, \Delta, \tilde{\Delta})$ whose new coproducts, $\overrightarrow{d}$ and $\overleftarrow{d}$ become Leibniz-Ito derivatives if coproducts Δ and $\tilde{\Delta}$ are unital homomorphism. We yield also a comparison between usual graph theory and coassociative coalgebras. This work ends with the example of the quaternions algebra embedded into a Markov L-Hopf algebra (of degree 2) whose associated directed graph is the directed triangle.

2.2 Definitions and notation

We denote by k, the field $\mathbb{R}$ or $\mathbb{C}$ and consider only unital associative algebras. The Sweedler notation $\Delta a = \sum_a a_{(1)} \otimes a_{(2)}$ will be also used. The *transposition map* will be denoted by $\tau : \; V_1 \otimes V_2 \otimes \ldots \otimes V_n \to V_n \otimes V_1 \otimes \ldots \otimes V_{n-1}$, such that $\tau(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = x_n \otimes x_1 \otimes \ldots \otimes x_{n-1}$, where $V_1, V_2, \ldots, V_n$ are n k-vector spaces. We recall that a *unital associative algebra* is a k-vector space (A, m, η) equipped with a product $m : A \otimes A \to A$ verifying $m(m \otimes id) = m(id \otimes m)$ (associativity) and a unit map $\eta : k \to A, \; \lambda \mapsto \lambda 1_A$. Dualising the previous definition, we obtain a *coassociative coalgebra* over k [57, 49], i.e., a k-vector space (C, Δ, ϵ) such that the counit map $\epsilon : C \to k$ and the coproduct map $\Delta : C \to C \otimes C$ verify:

1. The coassociativity equation: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.
2. The counit equation: $(id \otimes \epsilon)\Delta = id = (\epsilon \otimes id)\Delta$.

A coalgebra is said *cocommutative* if $\Delta = \tau\Delta$. Similarly, a *bialgebra* $(C, m, \eta, \Delta, \epsilon, k)$ over k is a k-vector space such that (C, Δ, ϵ) is a coalgebra and (C, m, η) is an algebra such that the coproduct and counit are algebra homomorphisms. A *Hopf algebra* $(H, m, \eta, \Delta, \epsilon, S, k)$ is a bialgebra with a k-linear map $S : H \to H$ called antipode which verifies: $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\epsilon$. An antipode is unique and is a unital antialgebra map and an anticoalgebra map, i.e., for all $x \in H$, $(S \otimes S)\Delta x = \tau\Delta S(x)$.

Definition 2.2.1 [Directed Graph] A *directed graph* G is a quadruple [54], (G_0, G_1, s, t) where G_0 and G_1 are two denumerable sets respectively called the *vertex set* and the *arrow set*. The two maps, $s,\ t : G_1 \to G_0$ are respectively called *source* and *terminus*. A vertex $v \in G_0$ is a *source* (resp. a *sink*) if $t^{-1}(\{v\})$ (resp. $s^{-1}(\{v\})$) is empty. A graph G is said *locally-finite*, (resp. *row-finite*) if $t^{-1}(\{v\})$ is finite (resp. $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1,\ s(a) = v\}$. A *weight* associated with the vertex v is a map $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a *weighted directed graph*.

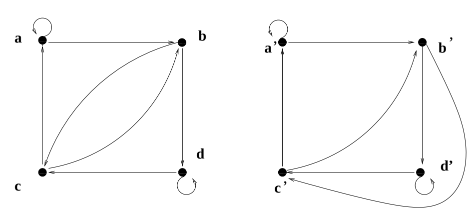
Remark: The case of non-directed graphs can be dealt in this framework by imposing that for each arrow $a \in G_1$, such that $s(a) = u$ and $t(a) = v$, there exists a unique $\bar{a} \in G_1$ with $s(\bar{a}) = v$ and $t(\bar{a}) = u$. We then identify a with $\bar{a}$. Should this identification be omitted the graph is directed, the condition of existence of $\bar{a}$ meaning that every arrow has an inverse.

This subsection ends by recalling the definition of the line-extension of a directed graph and the graph-isomorphism.

Definition 2.2.2 [Line-extension] The *line-extension* of a directed graph $G := (G_0, G_1, s, t)$, with a denumerable vertex set G_0 and a denumerable arrow set $G_1 \subseteq G_0 \times G_0$ is the directed graph with vertex set G_1 and arrow set $G_2 \subseteq G_1 \times G_1$ defined by $(v, w) \in G_1 \times G_1$ belongs to G_2 if and only if $t(v) = s(w)$. This directed graph, called the line-directed graph of G, is denoted by $E(G)$.

Definition 2.2.3 [Graph isomorphism] A *graph isomorphism* $f : G \to H$ between two graphs $G = (G_0, G_1, s_G, t_G)$ and $H = (H_0, H_1, s_H, t_H)$ is a pair of bijection $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ such that $f_0(s_G(a)) = s_H(f_1(a))$ and $f_0(t_G(a)) = t_H(f_1(a))$ for all $a \in G_1$. All the directed graphs in this formalism will be considered up to a graph isomorphism.

Example 2.2.4 The two following directed graphs are isomorphic.



2.3 Coassociativity breaking

Let us introduce L-coalgebras and show why this notion is interesting.

2.3.1 Axioms

Definition 2.3.1 [**L-coalgebra**] A *L-coalgebra* $(L, \Delta, \tilde{\Delta})$ is a k-vector space equipped with a right coproduct $\Delta : L \to L^{\otimes 2}$ and a left coproduct $\tilde{\Delta} : L \to L^{\otimes 2}$, verifying the *coassociativity breaking equation*: $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. A L-coalgebra may have two counits. The right counit $\epsilon : L \to k$ verifying $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : L \to k$ verifying $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$. A L-coalgebra is called *coassociative* if its two coproducts are coassociative. In this case, the equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is called the *entanglement equation*, see [41, 34].

Proposition 2.3.2 *Any coassociative coalgebra is a L-coalgebra.*

Proof: Let C be a coassociative coalgebra and Δ its coproduct. Set $\tilde{\Delta} := \Delta$. The two coproducts verify $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. □

The case $\tilde{\Delta} := \Delta$ will be called the *degenerate case*. To discriminate between the different types of L-coalgebras, we define:

Definition 2.3.3 [Finite Markov L-coalgebra] A *Markov L-coalgebra* is a L-coalgebra $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$, which is of dimension $\dim \mathcal{G}$ as a k-vector space, with a basis $\mathcal{G}_0 := (v_i)_{1 \leq i \leq \dim \mathcal{G}}$ equipped with:

1. A set $\mathcal{G}_1 := \{v_i \otimes v_j;\ (v_i, v_j) \in \mathcal{G}_0 \times \mathcal{G}_0\ \}$,
2. Two subsets I_{v_i} and J_{v_i} of $\mathcal{G}_1$, and maps $w_{v_i} : I_{v_i} \to k$ and $\tilde{w}_{v_i} : J_{v_i} \to k$ called *weights*, for any $1 \leq i \leq \dim \mathcal{G}$, verifying that:

$$\Delta_M(v_i) = \sum_{k:\ v_i \otimes v_k \in I_{v_i}} w_{v_i}(v_i \otimes v_k)\ v_i \otimes v_k \quad \text{and} \quad \tilde{\Delta}_M(v_i) = \sum_{j:\ v_j \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_j \otimes v_i)\ v_j \otimes v_i.$$

A Markov L-coalgebra is said to be *finite* when I_{v_i} and J_{v_i} are finite sets, for all i. The linear maps $s, t:\ k\mathcal{G}_1 \to k\mathcal{G}_0$ given by $s(v_i \otimes v_j) = v_i$ and $t(v_i \otimes v_j) = v_j$, for all $v_i,\ v_j \in \mathcal{G}_0$, are still called source and terminus, respectively. Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra and $v_i \in \mathcal{G}_0$. The future of v_i is defined as $t(\Delta_M(v_i))$ and the past of v_i as $s(\tilde{\Delta}_M(v_i))$.

Remark: The definition of a Markov L-coalgebra is basis dependent. With the viewpoint of discrete Markov processes in mind, the definitions of future and past are constructed on linear superpositions of usual classical future(s) (or past(s)) and are reminiscent of quantum definitions of future and past, see also [39].

Theorem 2.3.4 *Any weighted directed graph $G = (G_0, G_1, s, t)$, supposed to be locally-finite, row-finite, without sink and source, equipped with a family of weights $(w_v)_{v\in G_0}$, gives a finite Markov L-coalgebra.*

Proof: Let $G = (G_0, G_1, s, t)$ be a directed graph, supposed to be locally-finite, row-finite, without sink and source, equipped with a family of weights $(w_v)_{v\in G_0}$. Let us consider the free k-vector space kG_0. Identify any directed arrow $v \longrightarrow w$ of G_1 with $v\otimes w$. The set G_1 is then viewed as a subset of $kG_0^{\otimes 2}$. The family of weights $(w_v)_{v\in G_0}$ is then viewed as a family of maps $w_v : F_v \to k$, where $F_v := \{a \in G_1, s(a) = v\}$. Define the coproducts $\Delta_M, \tilde{\Delta}_M : kG_0 \to kG_0^{\otimes 2}$ as follows:

$$\Delta_M(v) := \sum_{i:a_i\in F_v} w_v(a_i)\ v \otimes t(a_i) \ \text{ and } \ \tilde{\Delta}_M(v) := \sum_{i:a_i\in P_v} w_{s(a_i)}(a_i)\ s(a_i) \otimes v,$$

where $P_v := \{a \in G_1, t(a) = v\}$, for all $v \in G_0$. For all $v \in G_0$, the maps $\tilde{w}_v : P_v \to k$ is such that $\tilde{w}_v(a_i) = w_{s(a_i)}(a_i)$, for all $a_i \in P_v$. With these definitions the free k-vector space kG_0, equipped with coproducts Δ_M and $\tilde{\Delta}_M$ is a finite Markov L-coalgebra. □

Motivated by this theorem, we construct a weighted directed graph from each L-coalgebra.

Definition 2.3.5 [Geometric support] Let $(L, \Delta, \tilde{\Delta})$ be a L-coalgebra generated, as a k-vector space, by an independent spanning set L_0. To any $v, w \in L_0$ such that the coefficient λ of the element $v \otimes w$ is different from zero either in $\Delta(z)$ or in $\tilde{\Delta}(z)$, for some $z \in L_0$, we associate a weighted directed arrow $v \xrightarrow{\lambda} w$. The weighted directed graph so obtained, denoted by $Gr(L)$, is called the *geometric support* of L. Its vertex set is L_0 and its arrow set, the set of those tensor products $v \otimes w$, $v, w \in L_0$, appearing in the definition of the coproducts. This construction is basis dependent.

Definition 2.3.6 [L-cocommutator] If (C, Δ_C) is a coassociative coalgebra, $\ker(\Delta_C - \tau\Delta_C)$ represents the cocommutator subspace of C. Similarly, a L-coalgebra $(L, \Delta, \tilde{\Delta})$ will be said L-*cocommutative* if $L^\natural = L$, where $L^\natural := \ker(\Delta - \tau\tilde{\Delta})$ is called the L-*cocommutator* subspace of L.

Theorem 2.3.7 *Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra, generated, as a k-vector space, by an independent spanning set $\mathcal{G}_0$, with families of weights $(w_v)_{v\in\mathcal{G}_0}$ and $(w'_v)_{v\in\mathcal{G}_0}$. If $v \in \mathcal{G}_0$ such that for each arrow $a \in Gr(\mathcal{G})_1$ emerging from v, with a given weight $w_v(a)$, there exists a unique arrow $b \in Gr(\mathcal{G})_1$, such that $s(b) = t(a)$, $t(b) = v$ and $w_v(a) = w'_v(b)$, then $v \in \mathcal{G}^\natural$.*

Proof: Straightforward. □

Remark: The L-cocommutativity describes algebraically the fact that a directed graph can be bi-directed. The characterisation of bi-directed graphs leads naturally to construct Leibniz bracket, see the introduction and also [45, 34].

Theorem 2.3.8 *Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra, generated as a k-vector space by an independent spanning set $\mathcal{G}_0$ equipped with two families of weights $(w_v)_{v\in\mathcal{G}_0}$ and $(\tilde{w}_v)_{v\in\mathcal{G}_0}$.*

Suppose that for all $v \in \mathcal{G}_0$, w_v and $\tilde{w}_v$ takes values in $\mathbb{R}_+$. The family of weights $(w_v)_{v\in\mathcal{G}_0}$ describes a family of probability vectors on $Gr(\mathcal{G})$ if and only if the right counit $v \mapsto \epsilon(v) := 1$, for all $v \in \mathcal{G}_0$, exists.

Proof: Straightforward. □

Theorem 2.3.9 *Let $G = (G_0, G_1, s, t)$ be a directed graph supposed to be locally-finite, row-finite, without sink and source, equipped with a family of probability vectors $(\Pi_v)_{v\in G_0}$. Then G can be seen as the geometric support from a finite Markov L-coalgebra equipped with right and left counits.*

Proof: Let $G = (G_0, G_1, s, t)$ be a directed graph supposed to be locally-finite, row-finite, without sink and source, equipped with a family of probability vectors $(\Pi_v)_{v\in G_0}$. Let us consider the free k-vector space kG_0. For all $v \in G_0$, define the right coproduct $\Delta_M : kG_0 \longrightarrow kG_0^{\otimes 2}$ such that for all $v \in G_0$, $\Delta_M(v) = \sum_{a\in F_v} \Pi_v(a)\ v \otimes t(a)$, the left coproduct $\tilde{\Delta}_M : kG_0 \longrightarrow kG_0^{\otimes 2}$ verifying for all $v \in G_0$, $\tilde{\Delta}_M(v) = \frac{1}{\mathsf{card}(P_v)} \sum_{a\in P_v} s(a) \otimes v$, the right (resp. the left) counit ϵ (resp. $\tilde{\epsilon}$) such that $\tilde{\epsilon}(v) = 1 = \epsilon(v)$ for all $v \in G_0$. This finite Markov L-coalgebra has two counits, and its geometric support is G. □

Proposition 2.3.10 *Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra generated, as a k-vector space, by an independent spanning set $\mathcal{G}_0$. The sequence $(\Delta_M)_1 \equiv \Delta_M, (\Delta_M)_2 = id\otimes\Delta_M, (\Delta_M)_3 = id\otimes id\otimes\Delta_M, \ldots$ generates all possible weighted paths in $Gr(\mathcal{G})$, starting at any vertex. Similarly, the sequence $(\tilde{\Delta}_M)_{n\geq 0}$, generates all the possible weighted paths in $Gr(\mathcal{G})$ arriving at a given vertex.*

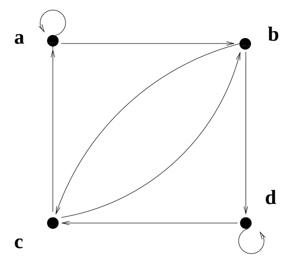
Proof: Straightforward. □

The algebraic framework of finite Markov L-coalgebras extends the classical setting of weighted directed graphs. Usually, a directed graph $G = (G_0, G_1, s, t)$ is also coded through an adjacency matrix, i.e., a square matrix A_G, with $A_G[v_i,\ v_j] = 1$ if the directed arrow $v_i \longrightarrow v_j \in G_1$. With the viewpoint of random walk on directed graphs, the adjacency matrix codes the (Markovian) neighbourhood of a given vertex, i.e., vertices present in the sum $t\Delta_M$ for the future and in the sum $s\tilde{\Delta}_M$ for the past, when we view the graph through its Markov L-coalgebra. The coproducts Δ_M and $\tilde{\Delta}_M$ can be called the *propagators* on the geometric support of the Markov L-coalgebra. The locality is respected. If $v_1, \ldots, v_n$ denote the vertices of the graph G, then observe that $t(\Delta_M(v)) = A_G \cdot X$, where the adjacency matrix A_G is written in the basis $(v_1, \ldots, v_n)$, and X is the vector ${}^t v :=\ {}^t(\lambda_1, \ldots, \lambda_n)$ in the same basis.

Motivated by this framework, to any L-coalgebra $(L, \Delta, \tilde{\Delta})$ generated, as a k-vector space by an independent spanning set L_0, a unique directed graph $Gr(L)$, called the geometric support of L has been constructed. Regarding the coproducts of L as a propagator of a walk generated by the sequences $(\Delta_n)_{n>0}$ and $(\tilde{\Delta}_n)_{n>0}$, we define the future of $v \in L_0$ by $t(\Delta(v))$ and the past by $s(\tilde{\Delta}(v))$. Let see what occurs in the case of a coassociative coalgebra.

First of all, we enlarge our graphical construction to bialgebra. Let $(A := k\langle\, A_0 \,\rangle \diagup \mathcal{R}, \Delta, m, \eta)$ be a bialgebra such that $(A := k\{A_0\} \diagup \mathcal{R}, m, \eta)$ is an associative algebra generated by a denumerable set A_0 verifying a set of relations $\mathcal{R}$. Consider the subvector-space of A spanned by the set

A_0 and denoted by kA_0. If A_0 is an independent spanning set and $\Delta : kA_0 \to kA_0^{\otimes 2}$, then we construct its geometric support as before. In other terms, the vertex set is composed by the generators of the algebra A and the arrows still represent the tensor products appearing in the definition of Δ. Fix an invertible element $q \in k$. The Hopf algebra $Sl_q(2)$ is generated by a, b, c and d such that $ba = qab,\ ca = qac,\ dc = qcd,\ db = qbd,\ bc = cb,\ ad - da = (q^{-1} - q)bc,\ ad - q^{-1}bc = 1$. The antipode map is described by the linear map $S : Sl_q(2) \to Sl_q(2)$ such that $S(a) = d$, $S(d) = a$, $S(b) = -qb$ and $S(c) = -q^{-1}c$. The well-known coalgebra structure is described by $\Delta_{Sl}(a) = a \otimes a + b \otimes c,\ \ \Delta_{Sl}(b) = a \otimes b + b \otimes d,\ \ \Delta_{Sl}(c) = d \otimes c + c \otimes a,\ \ \Delta_{Sl}(d) = d \otimes d + c \otimes b$. The directed graph associated with $Sl_q(2)$ is:



Directed graph associated with $Sl_q(2)$.

The geometric support of $Sl_q(2)$, whose future and past of a given vertex are coded by its coproduct Δ_{Sl} behave in a non-local way. For the sake of an example, notice that on the directed graph $Gr(Sl_q(2))$, the future of a is not a and b as expected in usual graph theory. To the contrary, the future of a is a and c, its past being a and b.

Observe also that the antipode map, as an anticoalgebra map, has also an interesting interpretation since it realises a time reversal. (The future becomes the past and conversely.)

Proposition 2.3.11 *If (C, Δ_C) is a cocommutative coassociative coalgebra generated, as a k-vector space, by an independent spanning set C_0, then its geometric support $Gr(C)$ can be viewed as a non-directed graph.*

Proof: Let (C, Δ_C) be a cocommutative coassociative coalgebra generated, as a k-vector space, by an independent spanning set C_0. Let $a, b, x \in C_0$ and suppose that the term $a \otimes b$ appears in the description of Δx. The same must be true for $b \otimes a$, since $\Delta_C = \tau \Delta_C$. On the geometric support $Gr(C)$, an arrow emerges from a to b and from b to a. We have just proved that the graph is bi-directed. By identifying the arrow emerging from a to b with that from b to a, we obtain a non-directed graph. $\square$

To embed any directed graph into an algebraic framework, we use the formalism of L-coalgebra. This point of view has the advantage to manipulate weighted directed graphs and deals with future and past in an algebraic way and to generalise these notions to any L-coalgebra. Let us mention that any directed graph can also be embedded into a coassociative coalgebra, by considering its path space instead of its vertex space [10]. This result can be also recovered by the following theorem.

Theorem 2.3.12 *Let G be a non-empty set consisting of a distinguished subset $G^{(0)} \subset G$, two* *maps $t, s : G \to G^{(0)}$ and a law of composition,*

$$\circ : G^{(2)} = \{(\gamma_1, \gamma_2) \in G \times G;\ s(\gamma_1) = t(\gamma_2)\} \to G,$$

such that:

1. $s(\gamma_1 \circ \gamma_2) = s(\gamma_2),\ t(\gamma_1 \circ \gamma_2) = t(\gamma_1),\ \forall (\gamma_1, \gamma_2) \in G^{(2)}$,
2. $\forall x \in G^{(0)},\ s(x) = t(x) = x;\ \forall \gamma \in G,\ \gamma \circ s(\gamma) = \gamma,\ t(\gamma) \circ \gamma = \gamma$,
3. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$,
4. *The family* $(G_\gamma := \{(\gamma_1, \gamma_2) \in G^{(2)};\ \gamma = \gamma_1 \circ \gamma_2\})_{\gamma \in G}$ *is a family of finite sets.*

Let C be a k-vector space equipped with a right action $\alpha : C \times G \to C$. If $\tilde{C} := \{c_\gamma = \alpha(c, \gamma), \gamma \in G\}$ is the k-vector space of orbits associated with α, then $\tilde{C}$ has a coassociative coalgebra structure.

Proof: Fix $c_\gamma \in \tilde{C}$ and define $\Delta c_\gamma := \sum_{\gamma_1 \circ \gamma_2 = \gamma} c_{\gamma_1} \otimes c_{\gamma_2}$. From Cond. 2, one gets that there exists at least an element $(\gamma_1, \gamma_2, \gamma_3) \in G^{\times 3}$ such that $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$. By definition of the coproduct and the associativity of the product $\circ$ we get,

$$\begin{aligned}
\sum_{\gamma_1 \circ \gamma_2 = \gamma} c_{\gamma_1} \otimes \Delta(c_{\gamma_2}) &= \sum_{\gamma_1 \circ \gamma_2 = \gamma} \sum_{\gamma'_1 \circ \gamma'_2 = \gamma_2} c_{\gamma_1} \otimes (c_{\gamma'_1} \otimes c_{\gamma'_2}) = \sum_{\gamma_1 \circ (\gamma'_1 \circ \gamma'_2) = \gamma} c_{\gamma_1} \otimes (c_{\gamma'_1} \otimes c_{\gamma'_2}), \\
\sum_{\gamma_1 \circ \gamma_2 = \gamma} \Delta(c_{\gamma_1}) \otimes c_{\gamma_2} &= \sum_{\gamma_1 \circ \gamma_2 = \gamma} \sum_{\gamma''_1 \circ \gamma''_2 = \gamma_1} (c_{\gamma''_1} \otimes c_{\gamma''_2}) \otimes c_{\gamma_2} = \sum_{(\gamma''_1 \circ \gamma''_2) \circ \gamma_2 = \gamma} (c_{\gamma''_1} \otimes c_{\gamma''_2}) \otimes c_{\gamma_2},
\end{aligned}$$

proving that $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$, since the sums involved are over all possible decompositions of γ in three parts. As $\gamma = \gamma \circ s(\gamma) = t(\gamma) \circ \gamma$, we define $\epsilon(c_\gamma) = 0$, if $\gamma \in G \setminus G^{(0)}$ and $\epsilon(c_\gamma) = 1$ otherwise. We have,

$$\Delta(c_\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma;\ \gamma \in G \setminus G^{(0)}} c_{\gamma_1} \otimes c_{\gamma_2} + c_{t(\gamma)} \otimes c_\gamma + c_\gamma \otimes c_{s(\gamma)},$$

thus $(id \otimes \epsilon)\Delta = (\epsilon \otimes id)\Delta = id$. □

Remark: Observe that a directed graph, as a geometric object, can be a geometric support for several L-coalgebras. For instance the geometric support of the coassociative coalgebra $(\mathcal{F}, \Delta)$, spanned as a k-vector space by a basis a, b, c and d and whose coproduct Δ is given by: $\Delta(a) = a \otimes a + b \otimes c,\quad \Delta(b) = a \otimes b + b \otimes d,\quad \Delta(c) = d \otimes c + c \otimes a,\quad \Delta(d) = d \otimes d + c \otimes b$, is the same that, or isomorphic to, the geometric support of the finite Markov L-coalgebra spanned, as a k-vector space, by a basis a, b, c, d and described by the right coproduct: $\Delta_M(a) = a \otimes (a + b),\quad \Delta_M(b) = b \otimes (c + d),\quad \Delta_M(c) = c \otimes (a + b),\quad \Delta_M(d) = d \otimes (c + d)$ and the left coproduct: $\tilde{\Delta}_M(a) = (a+c) \otimes a,\ \ \tilde{\Delta}_M(b) = (a+c) \otimes b,\ \ \tilde{\Delta}_M(c) = (b+d) \otimes c,\ \ \tilde{\Delta}_M(d) = (b+d) \otimes d$.

To turn a coassociative coalgebra into a non-degenerate L-coalgebra, we will use a generalisation of an idea applied by R.L. Hudson [22].

Proposition 2.3.13 *Let (C, Δ_C) be a coassociative coalgebra, with a group-like element e. Define the coproducts $\tilde{\delta}_f,\ \delta_f : C \to C^{\otimes 2}$ such that for all $c \in C$, $\tilde{\delta}_f(c) := e \otimes c$ and $\delta_f(c) := c \otimes e$. Then $(C, \tilde{\delta}_f, \delta_f)$ is a finite Markov L-coalgebra which is in addition a coassociative codialgebra.*

Proof: Straightforward. □

Proposition 2.3.14 *Any coassociative coalgebra (C, Δ_C), with a group-like element gives rise to a non-degenerate L-coalgebra structure on C.*

Proof: Let (C, Δ_C) be a coassociative coalgebra. Suppose e is a group-like element, i.e., $\Delta_C(e) = e \otimes e$. Define as in Proposition 2.3.13, the coassociative coproducts $\delta_f(c) := c \otimes e$ and $\tilde{\delta}_f(c) := e \otimes c$, for all $c \in C$. Define also two linear maps $\overrightarrow{d},\ \overleftarrow{d} : C \to C \otimes C$ by $\overrightarrow{d}(c) = \Delta_C(c) - \delta_f(c)$ and $\overleftarrow{d}(c) = \Delta_C(c) - \tilde{\delta}_f(c)$, for all $c \in C$. These linear maps, $\overleftarrow{d}$ and $\overrightarrow{d}$, turn the coassociative coalgebra C into a non-degenerate L-coalgebra. Indeed, if $c \in C$ such that $\Delta_C(c) = \sum c_{(1)} \otimes c_{(2)}$ then,

$$c \xrightarrow{\overleftarrow{d}} \sum c_{(1)} \otimes c_{(2)} - e \otimes c \xrightarrow{id \otimes \overrightarrow{d}} \sum c_{(1)} \otimes \Delta_C(c_{(2)}) - \Delta_C(c) \otimes e - e \otimes \Delta_C(c) + e \otimes c \otimes e,$$

$$c \xrightarrow{\overrightarrow{d}} \sum c_{(1)} \otimes c_{(2)} - c \otimes e \xrightarrow{\overleftarrow{d} \otimes id} \sum \Delta_C(c_{(1)}) \otimes c_{(2)} - \Delta_C(c) \otimes e - e \otimes \Delta_C(c) + e \otimes c \otimes e.$$

Moreover $\overleftarrow{d}$ is obviously not equal to $\overrightarrow{d}$ on the whole coalgebra. Therefore $(C, \overrightarrow{d}, \overleftarrow{d})$ is a non-degenerate L-coalgebra. □

Remark: Let $(C, \Delta_C, \epsilon_C)$ be a coassociative coalgebra with counit ϵ_C. Observe that $c \in C \cap \ker \epsilon_C$ and $(\overleftarrow{d} \otimes id)\overrightarrow{d}(c) = 0$ if and only if c is a primitive element of C.

Corollary 2.3.15 *Any bialgebra can be viewed as a non-trivial L-coalgebra.*

Proof: Straightforward since the identity element is group-like. □

We can construct another interesting class of L-coalgebra.

Definition 2.3.16 [C**-bimodule**] Let (C, Δ_C) be a bialgebra. From Proposition 2.3.14, $(C, \overleftarrow{d}, \overrightarrow{d})$ is a L-coalgebra. Define on $C^{\otimes 2}$ two structures of C-*bimodule* given by the following products: for $x, y \in C$, $c \in C^{\otimes 2}$, $x \tilde{\cdot} c = \tilde{\delta}_f(x)c,\ c \tilde{\cdot} y = c\tilde{\delta}_f(y)$, and $x \cdot c = \delta_f(x)c,\ c \cdot y = c\delta_f(y)$.

Let A be an algebra with unit 1 and M a A-bimodule. A *Leibniz-Ito derivative* is a linear map $\rho : A \to M$ such that for all $x, y \in A$, $\rho(xy) = x\rho(y) + \rho(x)y + \rho(x)\rho(y)$ and $\rho(1) = 0$.

Theorem 2.3.17 *Let (C, Δ_C) be a bialgebra. As Δ_C is a unital homomorphism, the coproducts $\overleftarrow{d}, \overrightarrow{d}$ turns out to be Leibniz-Ito derivatives.*

Proof: Let (C, Δ_C) be a bialgebra and $x, y \in C$. The relation $\overleftarrow{d}(1) = 0 = \overrightarrow{d}(1)$ holds. Moreover, $\overrightarrow{d}(x)\overrightarrow{d}(y) = \Delta_C(xy) + xy \otimes 1 - \Delta_C(x)(y \otimes 1) - (x \otimes 1)\Delta(y)$ implies $\overrightarrow{d}(xy) = \overrightarrow{d}(x)\overrightarrow{d}(y) + \overrightarrow{d}(x) \cdot y + x \cdot \overrightarrow{d}(y)$. The same equation holds for the other coproduct. □

If (C, Δ_C) is bialgebra, we call $(C, \overleftarrow{d}, \overrightarrow{d})$, a *Leibniz-Ito L-coalgebra*. This kind of L-coalgebra plays an important rôle in quantum stochastic processes, see [22, 21].

Let us yield here two ways to construct L-coalgebras from known ones.

Bicomodule over C, for a coassociative coalgebra C

Let (C, Δ_C) be a coassociative coalgebra. Let B be a C-bicomodule i.e., there exist linear maps δ and $\tilde{\delta}$ such that the following diagram commute:

$$\begin{array}{ccc} B & \xrightarrow{\delta} & B \otimes C \\ \tilde{\delta}\downarrow & & \downarrow \tilde{\delta}\otimes id_C \\ C \otimes B & \xrightarrow{id_C \otimes \delta} & C \otimes B \otimes C \end{array}$$

and such that $(id_B \otimes \Delta_C)\delta = (\delta \otimes id_C)\delta$ and $(\Delta_C \otimes id_B)\tilde{\delta} = (id_C \otimes \tilde{\delta})\tilde{\delta}$.

Proposition 2.3.18 *Let (C, Δ_C) be a coassociative coalgebra and B be a bicomodule over C. Let $A := C \otimes B \otimes C$. Keeping the previous notation, the linear maps δ and $\tilde{\delta}$ induce coproducts Δ and $\tilde{\Delta}$ on A given by:*

$$\Delta(u \otimes b \otimes v) := (u \otimes \delta(b)) \otimes (v \otimes e \otimes f),$$

$$\tilde{\Delta}(u \otimes b \otimes v) := (g \otimes h \otimes u) \otimes (\tilde{\delta}(b) \otimes v),$$

for $u \otimes b \otimes v \in A$, and fixed elements $g, f \in C$ and $h, e \in B$. That means that there exists a natural structure of L-coalgebra on A, for any couple of pairs $(g, f) \in C^{\times 2}$ and $(h, e) \in B^{\times 2}$.

Proof: Let (C, Δ_C) be a coassociative coalgebra and B be a bicomodule over C. Fix $g, f \in C$ and $h, e \in B$. Set $A := C \otimes B \otimes C$. The coproducts $\delta, \tilde{\delta}$ induce coproducts $\Delta, \tilde{\Delta}$ defined above such that the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \tilde{\Delta}\downarrow & & \downarrow \tilde{\Delta}\otimes id_A \\ A \otimes A & \xrightarrow{id_A \otimes \Delta} & A \otimes A \otimes A \end{array}$$

Indeed,

$$\begin{array}{ccc} u \otimes b \otimes v & \xrightarrow{\Delta} & (u \otimes \delta(b)) \otimes (v \otimes e \otimes f) \\ \tilde{\Delta}\downarrow & & \downarrow \tilde{\Delta}\otimes id_A \\ (g \otimes h \otimes u) \otimes (\tilde{\delta}(b) \otimes v) & \xrightarrow{id_A \otimes \Delta} & (g \otimes h \otimes u) \otimes (\tilde{\delta} \otimes id_C)\delta(b)) \otimes (v \otimes e \otimes f), \end{array}$$

□

Tensor product

Let $(L, \Delta_L, \tilde{\Delta}_L)$, $(M, \Delta_M, \tilde{\Delta}_M)$ be two L-coalgebras. Define the right coproduct $\Delta_{L\otimes M}$ to be the composite:

$$L \otimes M \xrightarrow{\Delta_L \otimes \Delta_M} (L \otimes L) \otimes (M \otimes M) \xrightarrow{id_L \otimes \tau \otimes id_M} (L \otimes M) \otimes (L \otimes M)$$

and the left coproduct $\tilde{\Delta}_{L\otimes M}$ by:

$$L \otimes M \xrightarrow{\tilde{\Delta}_L \otimes \tilde{\Delta}_M} (L \otimes L) \otimes (M \otimes M) \xrightarrow{id_L \otimes \tau \otimes id_M} (L \otimes M) \otimes (L \otimes M).$$

With this setting $(L \otimes M, \Delta_{L\otimes M}, \tilde{\Delta}_{L\otimes M})$ becomes a L-coalgebra over k. If both L-coalgebras have counits, we define the right counit $\epsilon_{L\otimes M}$ as $L \otimes M \xrightarrow{\epsilon_L \otimes \epsilon_M} k \otimes k \simeq k$ and the left counit $\tilde{\epsilon}_{L\otimes M}$ as $L \otimes M \xrightarrow{\tilde{\epsilon}_L \otimes \tilde{\epsilon}_M} k \otimes k \simeq k$.

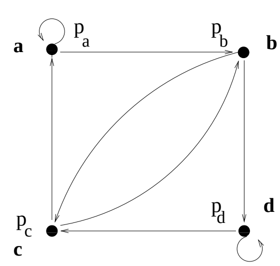
Indexed L-coalgebra

Let $(L, \Delta, \tilde{\Delta})$ be a L-coalgebra, spanned by a set L_0 and V be a k-vector space such that $\dim V := \dim L < \infty$. With each $v_j \in L_0$, $1 \leq j \leq \dim L$, is associated a generator of V denoted, for convenience, by $|v_j\,\rangle$. Consider the space $\hat{L}$ spanned by $(|v_j\,\rangle \otimes v_j)_{1 \leq j \leq \dim L}$. Define two new coproducts $\Delta^i, \tilde{\Delta}^i : \hat{L} \to \hat{L}^{\otimes 2}$ such as, if $\Delta(v_j) := \sum \lambda_{(j,(1),(2))}(v_j)_{(1)} \otimes (v_j)_{(2)}$, with $(v_j)_{(1)}, (v_j)_{(2)} \in L_0$, $\Delta^i(v_j) := \sum \lambda_{(j,(1),(2))}(|(v_j)_{(1)}\,\rangle \otimes (v_j)_{(1)}) \otimes (|(v_j)_{(2)}\,\rangle \otimes (v_j)_{(2)})$, for $1 \leq j \leq \dim L$. Similarly, $\tilde{\Delta}^i(v_j) := \sum \lambda_{(j,(\bar{1}),(\bar{2}))}(|(v_j)_{(\bar{1})}\,\rangle \otimes (v_j)_{(\bar{1})}) \otimes (|(v_j)_{(\bar{2})}\,\rangle \otimes (v_j)_{(\bar{2})})$, if $\tilde{\Delta}(v_j) := \sum \lambda_{(j,(\bar{1}),(\bar{2}))}(v_j)_{(\bar{1})} \otimes (v_j)_{(\bar{2})}$, with the $(v_j)_{(\bar{1})}, (v_j)_{(\bar{2})} \in L_0$. The use of indexed L-coalgebras can be found in [39].

Proposition 2.3.19 *The space $(\hat{L}, \Delta^i, \tilde{\Delta}^i)$ is a L-coalgebra called the indexed L-coalgebra associated with $(L, \Delta, \tilde{\Delta})$ and is defined up to a k-vector space isomorphism.*

Proof: Straightforward. □

Remark: Indexed L-coalgebras are useful for dealing with coassociative coalgebras, whose vertices of their geometric supports are weighted, without breaking the coassociativity of the underlying coproducts.



Example of indexed coassociative coalgebra. The vertices $(\mathcal{F}, \Delta)$ are supposed to be weighted. The following definition $\Delta(a) := (p_a a) \otimes (p_a a) + (p_b b) \otimes (p_c c) \ldots$ breaks the coassociativity of the coproduct Δ. It is restored by defining the indexed coproduct $\Delta^i(|p_a\,\rangle \otimes a) := (|p_a\,\rangle \otimes a) \otimes (|p_a\,\rangle \otimes a) + (|p_b\,\rangle \otimes b) \otimes (|p_c\,\rangle \otimes c) \ldots$.

The weights, on the vertex set, are played by the k-vector space V and a linear map $w : V \to k$. For a higher visibility, the basis of $\hat{L}$ can be renamed as follows $|v_j\rangle \otimes v_j \equiv |j,\ w(v_j)\rangle \otimes v_j$, for all j. Weights could be also defined on the arrow set via a map $\hat{w} : \hat{\mathcal{G}}_1 \subset \hat{\mathcal{G}}^{\otimes 2} \to k$. For instance, a possible choice could be $\hat{w}((|j,\ w(v_j)\rangle \otimes v_j) \otimes (|l,\ w(v_l)\rangle \otimes v_l)) := w(v_j)w(v_l)$.

Let the $\langle v_j|\cdot\rangle : V \to k$, to be the dual basis of V, i.e., $\langle v_j|v_l\rangle := 1$ if $l = j$ and 0 otherwise. Another asset of indexed L-coalgebras is the possiblity to select a trajectory or an arrow on the geometric support associated with a L-coalgebra by applying, for instance, projectors $(|v_{j_1}\rangle\langle v_{j_1}| \otimes id) \otimes (|v_{j_2}\rangle\langle v_{j_2}| \otimes id) \ldots$ to the set of trajectories defined by the operators $\ldots(id\otimes id\otimes\Delta)(id\otimes\Delta)\Delta(\cdot)$. This remark could be also useful for recovering the classical notions of future or past from the definition we gave in terms of the Markovian coproducts.

2.3.2 L-Bialgebras, L-Hopf algebras

The construction of vector spaces equipped with two coproducts entails the generalisation of definitions such as bialgebras and Hopf algebras.

Definition 2.3.20 [**L-bialgebra**] A *L-bialgebra* with counits ϵ and $\tilde{\epsilon}$ is a L-coalgebra with counits $(L, \Delta, \tilde{\Delta}, \epsilon, \tilde{\epsilon})$ equipped with an extra-structure of unital algebra over k, such that the coproducts and counits are algebra homomorphisms.

Definition 2.3.21 [**L-Hopf algebra**] A *L-Hopf algebra* is a L-bialgebra with counits $(H, \Delta, \tilde{\Delta}, \epsilon, \tilde{\epsilon})$ equipped with two linear maps $S,\ \tilde{S} : H \to H$ called right and left antipodes which verify the equalities: $m(id \otimes S)\Delta = \eta\epsilon$ and $m(\tilde{S} \otimes id)\tilde{\Delta} = \eta\tilde{\epsilon}$.

Remark: In the sequel, all our L-Hopf algebras will verify the previous equalities. However, it is worth noticing [34] we can also construct another type of L-Hopf algebras verifying the equalities: $m(S \otimes id)\Delta = \eta\epsilon$ and $m(id \otimes \tilde{S})\tilde{\Delta} = \eta\tilde{\epsilon}$.

Let us give two examples of L-bialgebras.

Example 2.3.22 [The Cuntz-Krieger algebra] In [14, 32], a C^*-algebra, called a *Cuntz-Krieger algebra*, is associated with each directed graph $G = (G_0, G_1, s, t)$. If G is a row-finite (i.e., $\forall v \in G_0,\ s^{-1}(\{v\})$ is finite) directed graph, a Cuntz-Krieger G-family consists of a set $\{P_v : v \in G_0\}$ of mutually orthogonal projections and a set $\{S_e : e \in G_1\}$ of partial isometries satisfying,

$$\forall (e, v) \in G_1 \times G_0,\ S_e^* S_e = P_{t(e)},\ P_v = \sum_{e:s(e)=v} S_e S_e^*,$$

where, for all $e \in G_1$, S_e^* denotes the adjoint of S_e.

Proposition 2.3.23 *A Cuntz-Krieger algebra CK associated with a graph without sinks and loops and whose vertex set is finite is a finite Markov L-bialgebra.*

Proof: As usual, by defining:

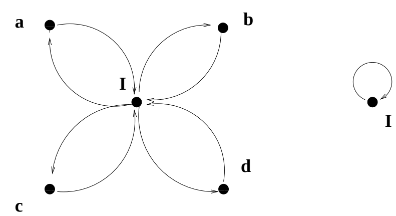
$$\Delta_M(P_v) = \sum_{v_1 \in t(s^{-1}(\{v\}))} P_v \otimes P_{v_1} \text{ and } \tilde{\Delta}_M(P_v) = \sum_{v_0 \in s(t^{-1}(\{v\}))} P_{v_0} \otimes P_v,$$

we turn CK into a Markov L-coalgebra. Thanks to the mutual orthogonality property of the projectors, we get for instance,

$$\Delta_M(P_v)\Delta_M(P_{v'}) = \sum_{v_1 \in t(s^{-1}(v))} \sum_{v'_1 \in t(s^{-1}(v'))} P_v P_{v'} \otimes P_{v_1} P_{v'_1} = \delta(v, v')\Delta_M(P_v) = \Delta_M(P_v P_{v'}),$$

where δ is the Kronecker symbol. Since the vertex set is finite, CK has an identity element $\sum_{v \in G_0} P_v := I$. In general, we do not have $\Delta_M(I) = I \otimes I = \tilde{\Delta}_M(I)$. □

Example 2.3.24 [Unital algebra] Let A be a unital algebra with unit I. From the equality $(I \cdot a) \cdot I = I \cdot (a \cdot I)$, A carries a non-trivial finite Markov L-bialgebra, with coproducts $\delta_f(a) = a \otimes I$ and $\tilde{\delta}_f(a) = I \otimes a$, for all $a \in A$. If $A := k\{A_0\} / \mathcal{R}$ is an associative algebra generated by a denumerable set A_0 verifying a set of relations $\mathcal{R}$, then its geometric support can be constructed. We call it the flower graph.



Example of geometric support associated with an algebra $k\{a, b, c, d\} \oplus kI$**.**

Observe that for all $a \in A$ different from I, $a \mapsto \delta_f(a) + \tilde{\delta}_f(a)$ and $I \mapsto I \otimes I$ is a coassociative cocommutative coproduct.

2.4 Finite Markov L-coalgebra and periodic orbits

The Propositions 2.3.14 and 2.3.17 assert that a coassociative coalgebra (C, Δ_C) with a group-like element e yields two coproducts $\overleftarrow{d}$ and $\overrightarrow{d}$ constructed from the coproduct Δ_C which turn out to be Leibniz-Ito derivatives if Δ_C is a unital homomorphism and if $e = I$. In this framework, if we replace a coassociative coalgebra by a finite Markov L-coalgebra, how can we produce two new coproducts such that, if the old ones are unital homomorphisms, the new ones become Leibniz-Ito derivatives?

The answer to this question is to find, in the proof of the Proposition 2.3.14, into the term: $e \otimes x \otimes e$. Graphically, this term describes the path $e \longrightarrow x \longrightarrow e$, i.e., we have made one complete turn around the orbit (e, x, e) of the flower graph. In a general finite Markov L-coalgebra $\mathcal{G}$ generated by an independent spanning set $\mathcal{G}_0$, there does not exist such a possibility. Therefore we have to create it. To do so, we need to consider the arrow set $\mathcal{G}_1$ of the geometric support of a finite Markov L-coalgebra $\mathcal{G}$ and fix an arrow, say $a \to b$ which is associated with $a \otimes b$ in $\mathcal{G}^{\otimes 2}$. By this way, we can construct two virtual periodic orbits of period 2, either $a \to b \to a$ or $b \to a \to b$. Before going on, let us introduce a notion, inspired from line-extension of graph: the L-coalgebras of degree n.

Definition 2.4.1 Let $n \in \mathbb{N} \setminus \{0\}$. The k-vector space $(Z, \Delta_n, \tilde{\Delta}_n)$, is a L-coalgebra of degree n over k if the following diagram,

$$\begin{array}{ccc} Z^{\otimes n} & \xrightarrow{\Delta_n} & Z^{\otimes n+1} \\ \tilde{\Delta}_n \downarrow & & \downarrow \tilde{\Delta}_n \otimes id \\ Z^{\otimes n+1} & \xrightarrow{id \otimes \Delta_n} & Z^{\otimes n+2} \end{array}$$

commutes, i.e., $(\tilde{\Delta}_n \otimes id)\Delta_n = (id \otimes \Delta_n)\tilde{\Delta}_n$. Such a space may have a right counit $\epsilon_n : Z^{\otimes n} \to Z^{\otimes n-1}$ such that: $(id \otimes \epsilon_n)\Delta_n = id$ and a left counit $\tilde{\epsilon}_n : Z^{\otimes n} \to Z^{\otimes n-1}$ such that: $(\tilde{\epsilon}_n \otimes id)\tilde{\Delta}_n = id$. By convention $Z^{\otimes 0} := k$.

Proposition 2.4.2 *A finite Markov L-coalgebra $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ is a finite Markov L-coalgebra of degree n, for any $n > 0$.*

Proof: Let $\Delta_M, \tilde{\Delta}_M, \epsilon, \tilde{\epsilon}$ be the coproducts and the possible counits of a finite Markov L-coalgebra $\mathcal{G}$ and define the following operators:

$$(\Delta_M)_n = \underbrace{id \otimes ... \otimes id}_{n-1} \otimes \Delta_M, \ \ (\tilde{\Delta}_M)_n = \tilde{\Delta}_M \otimes \underbrace{id \otimes ... \otimes id}_{n-1}, \ \epsilon_n = \underbrace{id \otimes ... \otimes id}_{n-1} \otimes \epsilon, \ \tilde{\epsilon}_n = \tilde{\epsilon} \otimes \underbrace{id \otimes ... \otimes id}_{n-1}.$$

Equipped with these maps, $\mathcal{G}$ is a finite Markov L-coalgebra of degree n. □

From now on, we consider the special case $n = 2$. The Markov L-coalgebra $\mathcal{G}$ can be embedded into a Markov L-coalgebra of degree 2. Let us see now the link with the line-extension in graph theory. Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a Markov L-coalgebra generated by an independent spanning set $\mathcal{G}_0$, whose geometric support is denoted by $Gr(\mathcal{G})$. Fix $u \longrightarrow v \in Gr(\mathcal{G})_1$. Then $(\Delta_M)_2(u \otimes v) := u \otimes \Delta_M(v)$. Therefore, the line-extension of $Gr(\mathcal{G})$, denoted by $E(Gr(\mathcal{G}))$, is a Markov L-coalgebra, with coproduct $\Delta_E(u \otimes v) := (u \otimes v) \otimes \Delta_M(v)$. Recall that s is the source map. Therefore, $(id \otimes s \otimes id)\Delta_E := (\Delta_M)_2$, since $(id \otimes s \otimes id)\Delta_E(u \otimes v) := u \otimes s(v \otimes v) \otimes t(\Delta_M(v)) := u \otimes \Delta_M(v) := (\Delta_M)_2(u \otimes v)$.

Definition 2.4.3 Let $\mathcal{G}$ be a k-vector space generated by an independent spanning set $\mathcal{G}_0$. Define the coproducts: $\delta_R, \ \delta_L : \ \mathcal{G}^{\otimes 2} \to \mathcal{G}^{\otimes 3}$, such that $\delta_R(a \otimes b) = a \otimes b \otimes a, \ \delta_L(a \otimes b) = b \otimes a \otimes b$, for all $a, b \in \mathcal{G}_0$.

Proposition 2.4.4 *Let $\mathcal{G}$ be a k-vector space generated by an independent spanning set $\mathcal{G}_0$. The coproducts δ_L and δ_R verify the coassociativity breaking equation $(\delta_L \otimes id)\delta_R = (id \otimes \delta_R)\delta_L$. Moreover δ_L, δ_R are both homomorphisms if $\mathcal{G}$ has also an extra-structure of algebra over k.*

Proof: Let $\mathcal{G}$ be a k-vector space generated by an independent spanning set $\mathcal{G}_0$. Fix $a, b \in \mathcal{G}_0$. We get, $a \otimes b \xrightarrow{\delta_L} b \otimes a \otimes b \xrightarrow{id \otimes \delta_R} b \otimes (a \otimes b \otimes a)$ and $a \otimes b \xrightarrow{\delta_R} a \otimes b \otimes a \xrightarrow{\delta_L \otimes id} (b \otimes a \otimes b) \otimes a$. If $\mathcal{G}$ is also an algebra generated by $\mathcal{G}_0$, fix $(a,b,c,d) \in \mathcal{G}_0$, then $\delta_L(a\otimes b)\delta_L(c\otimes d) = (b\otimes a\otimes b)(d\otimes c\otimes d) = (bd\otimes ac\otimes bd) = \delta_L(ac\otimes bd) = \delta_L((a\otimes b)(c\otimes d))$. The same computation is used for proving that δ_R is a homomorphism. □

Theorem 2.4.5 *Let $\mathcal{G}$ be a finite Markov L-coalgebra, generated, as a k-vector space, by an independent spanning set $\mathcal{G}_0$, equipped with coproducts Δ_M and $\tilde{\Delta}_M$. Set $(\Delta_M)_2 := id \otimes \Delta_M$ and $(\tilde{\Delta}_M)_2 := \tilde{\Delta}_M \otimes id$. Define the two coproducts $\overleftarrow{d}_{\mathcal{G}}$, $\overrightarrow{d}_{\mathcal{G}}$ as: $\overrightarrow{d}_{\mathcal{G}} = (\Delta_M)_2 - \delta_R$ and $\overleftarrow{d}_{\mathcal{G}} = (\tilde{\Delta}_M)_2 - \delta_L$. These two coproducts verify the coassociativity breaking equation $(\overleftarrow{d}_{\mathcal{G}} \otimes id)\overrightarrow{d}_{\mathcal{G}} = (id \otimes \overrightarrow{d}_{\mathcal{G}})\overleftarrow{d}_{\mathcal{G}}$. Moreover $\overrightarrow{d}_{\mathcal{G}} = 0 = \overleftarrow{d}_{\mathcal{G}}$ on (trivial weighted) isolated periodic orbits of period two.*

Proof: Straightforward by noticing that: $(\delta_L \otimes id)(\Delta_M)_2 = (id \otimes (\Delta_M)_2)\delta_L$ and that $((\tilde{\Delta}_M)_2 \otimes id)\delta_R = (id \otimes \delta_R)(\tilde{\Delta}_M)_2$. Let $x, y \in \mathcal{G}_0$ representing a (trivial weighted) isolated periodic orbit of period two on $Gr(\mathcal{G})$. Such an orbit verifies that $(\Delta_M)_2(x\otimes y) = x\otimes y\otimes x$, and $(\tilde{\Delta}_M)_2(x\otimes y) = y\otimes x\otimes y$, which implies that $\overleftarrow{d}_{\mathcal{G}}$ and $\overrightarrow{d}_{\mathcal{G}}$ vanishes on such an element. □

Remark: With the Markov processes in mind, directed graphs equipped with probability vectors have always their isolated periodic orbits trivial weighted, i.e., all the weights are equal to 1 on each arrow of the orbit. From now on, such orbits will always be supposed trivial weighted.

Theorem 2.4.6 *Let $\mathcal{G}$ be a Markov L-bialgebra generated by a set $\mathcal{G}_0$, equipped with unital coproducts Δ_M and $\tilde{\Delta}_M$. Then $\overrightarrow{d}_{\mathcal{G}}$, $\overleftarrow{d}_{\mathcal{G}}$ behave as Leibniz-Ito derivatives, i.e., verify $\overrightarrow{d}_{\mathcal{G}}(x)\overrightarrow{d}_{\mathcal{G}}(y) = \overrightarrow{d}_{\mathcal{G}}(xy) - \overrightarrow{d}_{\mathcal{G}}(x)\delta_R(y) - \delta_R(x)\overrightarrow{d}_{\mathcal{G}}(y)$ and $\overrightarrow{d}_{\mathcal{G}}(I\otimes I) = 0$. Similarly, these two equalities hold for $\overleftarrow{d}_{\mathcal{G}}$.*

Proof: Let $\mathcal{G}$ be a Markov L-bialgebra generated by a set $\mathcal{G}_0$. Fix $a, b, c, d \in \mathcal{G}_0$. Define $x := a\otimes b$ and $y := c\otimes d$. We get: $\overleftarrow{d}_{\mathcal{G}}(x)\overleftarrow{d}_{\mathcal{G}}(y) = ((\tilde{\Delta}_M)_2(x) - \delta_L(x))((\tilde{\Delta}_M)_2(y) - \delta_L(y)) = (\tilde{\Delta}_M)_2(xy) - (\tilde{\Delta}_M)_2(x)\delta_L(y) - \delta_L(x)(\tilde{\Delta}_M)_2(y) + \delta_L(x)\delta_L(y) + (\delta_L(x)\delta_L(y) - \delta_L(x)\delta_L(y)) = \overleftarrow{d}_{\mathcal{G}}(xy) - \overleftarrow{d}_{\mathcal{G}}(x)\delta_L(y) - \delta_L(x)\overleftarrow{d}_{\mathcal{G}}(y)$. Similarly, for the coproduct $\overrightarrow{d}$, we can show that $\overrightarrow{d}_{\mathcal{G}}(x)\overrightarrow{d}_{\mathcal{G}}(y) = \overrightarrow{d}_{\mathcal{G}}(xy) - \overrightarrow{d}_{\mathcal{G}}(x)\delta_R(y) - \delta_R(x)\overrightarrow{d}_{\mathcal{G}}(y)$. Moreover $\overrightarrow{d}_{\mathcal{G}}(I\otimes I) = 0 = \overleftarrow{d}_{\mathcal{G}}(I\otimes I)$. These equations are reminiscent of those of Theorem 2.3.17, when δ_R is played by δ_f and δ_L by $\tilde{\delta}_f$. □

Let $\mathcal{G}$ be a finite Markov L-coalgebra, generated as a k-vector space by an independent spanning set $\mathcal{G}_0$, equipped with coproducts Δ_M and $\tilde{\Delta}_M$. The present setting can be easily generalised. Fix $n > 1$ and generalise the definition of δ_R and δ_L as follow: $\delta_{R,n}, \ \delta_{L,n} : \mathcal{G}^{\otimes n} \to \mathcal{G}^{\otimes(n+1)}$, defined by $\delta_{R,n}(a_1, \dots, a_n) = (a_1, \dots, a_n) \otimes a_1$ and $\delta_{L,n}(a_1, \dots, a_n) = a_n \otimes (a_1, \dots, a_n)$, where $a_1, \dots, a_n \in \mathcal{G}_0$.

Theorem 2.4.7 *Let $\mathcal{G}$ be a finite Markov L-coalgebra generated as a k-vector space by an independent spanning set $\mathcal{G}_0$ and equipped with coproducts Δ_M and $\tilde{\Delta}_M$. We obtain,*

1. *$(\delta_{L,n} \otimes id)\delta_{R,n} = (id \otimes \delta_{R,n})\delta_{L,n}$ and $\delta_{R,n}, \delta_{L,n}$ are homomorphisms.*
2. *If we define $\overrightarrow{d}_{\mathcal{G},n} = (\Delta_M)_n - \delta_{R,n}$ and $\overleftarrow{d}_{\mathcal{G},n} = (\tilde{\Delta}_M)_n - \delta_{L,n}$, then $(\overleftarrow{d}_{\mathcal{G},n} \otimes id)\overrightarrow{d}_{\mathcal{G},n} = (id \otimes \overrightarrow{d}_{\mathcal{G},n})\overleftarrow{d}_{\mathcal{G},n}$.*

3. *The equality* $\overleftarrow{d}_{\mathcal{G},n}[w] = 0 = \overrightarrow{d}_{\mathcal{G},n}[w]$ *holds if the tensor product* $[w] = (a_1 \otimes \ldots \otimes a_n)$, $a_1, \ldots, a_n \in \mathcal{G}_0$, *represents an isolated periodic orbit of period* n *on the geometric support of the finite Markov L-coalgebra.*

4. *If* $\mathcal{G}$ *is also a Markov L-bialgebra. The coproducts* $(\delta_{R,n}, \delta_{L,n})$ *are homomoprhism. If the coproducts of* $\mathcal{G}$ *are unital then* $\overleftarrow{d}_{\mathcal{G},n}$, $\overrightarrow{d}_{\mathcal{G},n}$ *behave as Leibniz-Ito derivatives, i.e., verify* $\overrightarrow{d}_{\mathcal{G},n}(x)\overrightarrow{d}_{\mathcal{G},n}(y) = \overrightarrow{d}_{\mathcal{G},n}(xy) - \overrightarrow{d}_{\mathcal{G},n}(x)\delta_R(y) - \delta_R(x)\overrightarrow{d}_{\mathcal{G},n}(y)$, *the same equality holding for* $\overleftarrow{d}_{\mathcal{G},n}$.

Proof: Straightforward. □

If C is a coassociative coalgebra with a group-like element e, then two important coproducts $\overrightarrow{d}, \overleftarrow{d} : C \to C^{\otimes 2}$ can be constructed. If $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ is a finite Markov L-coalgebra, generated as a k-vector space by an independent spanning set $\mathcal{G}_0$ then two other important coproducts $\overrightarrow{d}_{\mathcal{G},2}, \overleftarrow{d}_{\mathcal{G},2} : \mathcal{G}^{\otimes 2} \to \mathcal{G}^{\otimes 3}$ can be also constructed. This remark suggests the fact that some geometric supports of coassociative coalgebras can be obtained by line-extension of some geometric supports of finite Markov L-coalgebras. Indeed, in [41], we have constructed from line-extension of geometric supports associated with a class of coassociative coalgebras, Markov coassociative co-dialgebras.

It is interesting to notice that the rôle played by the coproducts δ_f and $\tilde{\delta}_f$ of the flower graph in the case of a coassociative coalgebra is played by the coproducts $\delta_{R,n}$ and $\delta_{L,n}$, creating virtual periodic orbits of period n in the case of a finite Markov L-coalgebra of degree n. Observe also that $(\overleftarrow{d}_{\mathcal{G},n} \otimes id)\overrightarrow{d}_{\mathcal{G},n}[w] = 0$ implies that the path of length n, represented algebraically by the tensor $[w]$, has to be a trivial weighted isolated periodic orbit of period n. We sum up briefly some results in the following array:

	Coassociative coalgebra C	Markov L-coalgebra $\mathcal{G}$
Coproducts	Δ_C	$\Delta_{\mathcal{G},n}$, $\tilde{\Delta}_{\mathcal{G},n}$
Markovian coproducts	e group-like, $\delta_f(x) := x \otimes e$, $\tilde{\delta}_f(x) := e \otimes x$	$\delta_{R,n}$, $\delta_{L,n}$
$(\overleftarrow{d} \otimes id)\overrightarrow{d} = 0$	Over primitive elements	over isolated orbits of period n
$\overleftarrow{d}$, $\overrightarrow{d}$	Leibniz-Ito derivative	behave as Leibniz-Ito derivative

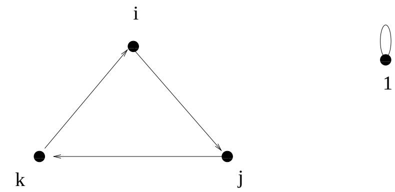
2.4.1 Examples

In the following examples, from a known algebra, we construct a finite Markov L-coalgebra so as to the algebra turns it into a finite Markov L-Hopf-algebra of degree 2. Set $id_n := \underbrace{id \otimes id \otimes \ldots \otimes id}_{n}$, $n > 0$. A L-Hopf algebra of degree n, $(H, \Delta_H, \tilde{\Delta}_H)$, is by definition a L-bialgebra of degree n, equipped with right and left counits, $\tilde{\epsilon}_H$, ϵ_H of degree n, such that its antipodes $S, \tilde{S} :\ H \to H$ verify $(id_{n-1} \otimes m)(id_n \otimes S)\Delta_H = \eta_n \epsilon_H$ and $(m \otimes id)(\tilde{S} \otimes id_n)\tilde{\Delta}_H = \tilde{\eta}_n \tilde{\epsilon}_H$, with $\eta_n, \tilde{\eta}_n : H^{\otimes(n-1)} \to H^{\otimes n}$ such that $\eta_n(h) := h \otimes 1_H$ and $\tilde{\eta}_n(h) := 1_H \otimes h$, $h \in H^{\otimes(n-1)}$.

Example 2.4.8 [The triangle graph and quaternions] Here $k = \mathbb{R}$. Recall that quaternions are defined by the associative algebra $\mathbb{H} := \mathbb{R}\{1, i, j, k\}/\mathcal{R}$ where the set of relations $\mathcal{R}$ is defined by:

$$ij = k, \quad jk = i, \quad ki = j, \quad ii = jj = kk = -1.$$

The quaternions fit the present formalism by considering the directed triangle graph,



Defining $x_0 \equiv i$, $x_1 \equiv j$, $x_2 \equiv k$ and adding subscripts $\alpha, \beta \in \{0, 1, 2\} \mod 3$ i.e., $x_{\alpha+\beta} \equiv x_{\alpha+\beta \bmod 3}$, we define the Markovian coproducts associated with this directed triangle as: $\Delta_M(x_\alpha) = x_\alpha \otimes x_{\alpha+1}$, $\Delta_M(1) = \tilde{\Delta}_M(1) = 1 \otimes 1$, $\tilde{\Delta}_M(x_\alpha) = x_{\alpha-1} \otimes x_\alpha$. Therefore,

$$\begin{aligned} (\Delta_M)_2(x_\alpha \otimes x_\beta) &= x_\alpha \otimes x_\beta \otimes x_{\beta+1}, \quad \epsilon_2(x_\alpha \otimes x_\beta) = x_\alpha, \\ (\tilde{\Delta}_M)_2(x_\alpha \otimes x_\beta) &= x_{\alpha-1} \otimes x_\alpha \otimes x_\beta, \quad \tilde{\epsilon}_2(x_\alpha \otimes x_\beta) = x_\beta, \end{aligned}$$

embed the directed triangle graph into a finite Markov L-coalgebra of degree 2.

Theorem 2.4.9 *The algebra of quaternions,*

1. *embeds the triangle graph into a L-bialgebra of degree 2.*
2. *Defining linear maps $S, \tilde{S} : \mathbb{H} \to \mathbb{H}$ by $S(x_i) = -x_{i-1}$ and $\tilde{S}(x_{i-1}) = -x_i$ for every $i \in \{0, 1, 2\}$, the L-bialgebra $\mathbb{H}$ becomes a L-Hopf algebra of degree 2, with S and $\tilde{S}$ playing the rôle of the right and left antipodes, respectively.*
3. *The linear maps $S, \tilde{S}$ are unital antialgebra maps and satisfy $S\tilde{S} = id = \tilde{S}S$. They are the unique right and left antipodes of $\mathbb{H}$, viewed as a L-Hopf algebra of degree 2.*

Proof: Let $i, i', j, j' \in \{0, 1, 2\}$. In whet follows we make computations with the right coproduct. Let us show that $(\Delta_M)_2$ is a unital algebra map. We get,

$$\begin{aligned} (\Delta_M)_2(x_i \otimes x_j)(\Delta_M)_2(x_{i'} \otimes x_{j'}) &= (x_i \otimes x_j \otimes x_{j+1})(x_{i'} \otimes x_{j'} \otimes x_{j'+1}) = x_i x_{i'} \otimes x_j x_{j'} \otimes x_{j+1} x_{j'+1}, \\ \text{and} \quad (\Delta_M)_2(x_i \otimes x_j)(x_{i'} \otimes x_{j'}) &= (\Delta_M)_2(x_i x_{i'} \otimes x_j x_{j'}) = x_i x_{i'} \otimes x_j x_{j'} \otimes t(\Delta_M(x_j x_{j'})). \end{aligned}$$

Therefore, we have to prove that $t(\Delta_M(x_j x_{j'})) = x_{j+1} x_{j'+1}$, which is straightforward by the following geometric proof. Suppose $j \neq j'$ and $(x_j, x_{j'})$ defines an edge of the triangle. This entails that $(x_{j+1}, x_{j'+1})$ defines the sole edge following it when we turn in a trigonometrical way. Now we observe that up to a sign the concatenation of an edge, that is the product of its source and its terminus gives the third vertex of the triangle. Hence by rotation the concatenation of $(x_{j+1}, x_{j'+1})$ will give the vertex just after. Therefore, up to a sign $t(\Delta_M(x_j x_{j'})) = x_{j+1} x_{j'+1}$. The sign is easily obtained by noticing that if $(x_j, x_{j'})$ is an arrow of the triangle so is $(x_{j+1}, x_{j'+1})$ and the sign is plus in both case when the concatenation is realised. If the direction of $(x_j, x_{j'})$ is in the opposite sens of an existing arrow, so is $(x_{j+1}, x_{j'+1})$ and the concatenation will give a minus sign in both cases. In the case when $x_{j'}$ or x_j is the identity element the proof is obvious since there is a loop on the identity. The case $x_{j'} = x_j$ is also trivial.

The coproducts $(\Delta_M)_2, (\tilde{\Delta}_M)_2$ are thus unital homomorphisms. The counits ϵ_2 and $\tilde{\epsilon}_2$ are also unital algebra maps. To prove the L-Hopf algebra part, we must check

$$(id \otimes m)(id \otimes id \otimes S)(\Delta_M)_2 = \eta_2\epsilon_2,\ (m \otimes id)(\tilde{S} \otimes id \otimes id)(\tilde{\Delta}_M)_2 = \tilde{\eta}_2\tilde{\epsilon}_2,$$

which is straightforward with the choice we made for the right and left antipodes. The map S is an anti-unital map since by definition, $-x_i = S(x_{i+1}) = S(x_{i-1}x_i)$ and $S(x_i)S(x_{i-1}) = (-x_{i-1})(-x_{i-2}) = (x_{i-1})(x_{i-2}) = -(x_{i-2})(x_{i-1}) = -(x_i)$, so $S(x_ix_j) = S(x_j)S(x_i)$. Moreover $S(x_ix_j) = S(x_j)S(x_i) = x_{j-1}x_{i-1}$ and $S(x_ix_j) = -S(x_jx_i) = -S(x_i)S(x_j) = -x_{i-1}x_{j-1} = x_{j-1}x_{i-1}$ proving that S is well defined. The map S is unital since $S(1) = S(x_i(-x_i)) = S(-x_i)S(x_i) = -(-x_{i+1})(-x_{i+1}) = 1$. The map S is unique since if S_1, S_2 are two such right antipodes we must get $x_iS_1(x_{i+1}) = x_iS_2(x_{i+1}) = 1$ but $x_ix_i = -1$ so $S_1(x_i) = S_2(x_i)$. As S_1, S_2 are equal on the generators of the quaternions, $S_1 = S_2$. Moreover, $S\tilde{S}(x_i) = S(-x_{i+1}) = -(-x_i) = x_i$ and $\tilde{S}S(x_i) = \tilde{S}(-x_{i-1}) = -(-x_i) = x_i$. □

Remark: The right and left antipodes S and $\tilde{S}$ are not unital anticoalgebra maps.

Remark: As a coproduct, Δ_M is well defined on the directed triangle graph, but is not an homomorphism for the quaternion product. If it were the case, we would get, for example $-\Delta_M(k) = -\Delta_M(ij) = \Delta_M(i)\Delta_M(j) = -ij \otimes jk = -k \otimes i$ which is true and $\Delta_M(-k) = \Delta_M(ji) = \Delta_M(j)\Delta_M(i) = ji \otimes kj = (-k) \otimes (-i)$ which is still true. Yet, the k-linearity is lost.

Example 2.4.10 [The Pauli matrices] Here $k = \mathbb{C}$. The Pauli matrices:

$$1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

verify the relations $\gamma_k\gamma_{k+1} = i\gamma_{k+2}$, $\gamma_k\gamma_k = 1_2$ and $\gamma_k\gamma_{k+1} = -\gamma_{k+1}\gamma_k$. Recall that $M_2(k)$ is the algebra generated by the Pauli matrices. This algebra fits the present formalism by considering the directed triangle graph with a loop on 1_2 not represented here,

The first graph is to recall that $\gamma_k\gamma_{k+1} = i\gamma_{k+2}$, but it is the second one which we are interested in because $(i\gamma_{k+1})(i\gamma_k) = (i\gamma_{k+2})$. Defining $x_0 \equiv i\gamma_0$, $x_1 \equiv i\gamma_1$, $x_2 \equiv i\gamma_2$ and adding subscripts $\alpha, \beta \in \{0, 1, 2\} \mod 3$ i.e., $x_{\alpha+\beta} \equiv x_{\alpha+\beta \mathrm{mod} 3}$, we define, $\Delta_M(x_\alpha) = x_\alpha \otimes x_{\alpha+1}$, $\Delta_M(1) = \tilde{\Delta}_M(1) = 1 \otimes 1$, $\tilde{\Delta}_M(x_\alpha) = x_{\alpha-1} \otimes x_\alpha$. The following coproducts,

$$\begin{aligned} (\Delta_M)_2(x_\alpha \otimes x_\beta) &= x_\alpha \otimes x_\beta \otimes x_{\beta+1}, \quad \epsilon_2(x_\alpha \otimes x_\beta) = x_\alpha, \\ (\tilde{\Delta}_M)_2(x_\alpha \otimes x_\beta) &= x_{\alpha-1} \otimes x_\alpha \otimes x_\beta, \quad \tilde{\epsilon}_2(x_\alpha \otimes x_\beta) = x_\beta, \end{aligned}$$

embed the triangle graph into a Markov L-coalgebra of degree 2.

Theorem 2.4.11 *The algebra generated by the Pauli matrices, i.e., $M_2(k)$,*

1. *embeds the triangle graph into a L-bialgebra of degree 2.*

2. *Defining linear maps* $S, \tilde{S} : M_2(k) \to M_2(k)$ *by* $S(x_i) = -x_{i-1}$ *and* $\tilde{S}(x_{i-1}) = -x_i$ *for every* $i \in \{0, 1, 2\}$, *the L-bialgebra* $M_2(k)$ *becomes a L-Hopf algebra of degree 2, with* S *and* $\tilde{S}$ *playing the rôle of right and left antipodes, respectively.*

3. *The maps* $S, \tilde{S}$ *are unital antialgebra maps and satisfy* $S\tilde{S} = id = \tilde{S}S$. *They are the unique right and left antipodes of* $\mathbb{H}$ *viewed as a L-Hopf algebra of degree 2.*

Proof: The proof is a corollary from the quaternion example. We only stress for instance that, $S(x_k) = -x_{k-1}$ implies the equality $x_k S(x_{k+1}) = -x_k x_k = -(i\gamma_k)(i\gamma_k) = (\gamma_k)(\gamma_k) = 1_2$, usefull for computing the antipodes equalities. □

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Chapter 3

Leibniz derivatives versus Leibniz-Ito derivatives

Abstract [1]**:**

We investigate, in an algebraic way, the notion of Leibniz-Ito derivative, a mathematical object arising in stochastic calculus comparable in some sense with the usual Leibniz derivatives. To compare these two notions, we view a unital associative algebra (A, m) as a unital bi-dialgebra $(A, \delta_f, \tilde{\delta}_f, m)$ and construct from its coassociative coproducts a differential equation admitting Leibniz and Leibniz-Ito derivatives as solutions. Motivated by this result, we construct a sequence of differential equations whose solutions can be constructed from the Quillen curvature of a Leibniz-Ito derivative and its associated Bianchi identity. From the Quillen curvature of a Leibniz-Ito derivative, we construct a differential associative dialgebra whose integral calculus yields cyclic cocycles, reminiscent of what was constructed by Connes in the context of linear maps characterising by the Leibniz property. Finally, we relate the Leibniz-Ito property to a distributivity defect of a certain law with respect to the associative product m of the unital associative algebra (A, m).

3.1 Introduction

In this article, k is either the real field or the complex field.

Motivated by periodicity phenomena in algebraic K-theory, J-L. Loday [45] introduced the notion of "non-commutative Lie algebra", called Leibniz algebra. Such algebras D are described by a Leibniz bracket $[-,-]_L$ verifying the Leibniz identity:

$$[[x, y]_L, z]_L = [[x, z]_L, y]_L + [x, [y, z]_L]_L, \quad \forall x, y, z \in D.$$

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When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and the Leibniz algebra turns out to be a Lie algebra. A way to construct such a Leibniz algebra is to start with an associative dialgebra, that is a k-vector space D equipped with two associative products $\vdash$ and $\dashv$ such that for all $x, y, z \in D$,

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
3. $(x \vdash y) \vdash z = (x \dashv y) \vdash z$,

The associative dialgebra is then a Leibniz algebra by defining a Leibniz commutator: $[x, y]_L := x \dashv y - y \vdash x$, for all $x, y \in D$.

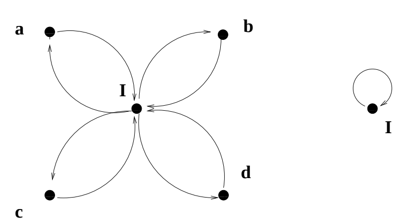
The operad associated with associative dialgebras is Koszul dual to the operad associated with dendriform algebras, a *dendriform algebra* E being a k-vector space equipped with two binary operations, $\prec$, $\succ$: $E \otimes E \to E$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
2. $(a \succ b) \prec c = a \succ (b \prec c)$,
3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

This notion dichotomises the notion of associativity since the product $a * b = a \prec b + a \succ b$, for all $a, b \in E$ is associative. As explained in the appendix, with any associative dialgebra can be associated a dendriform algebra.

Dualising the properties defining dialgebra and dendriform algebras, we obtain easily axioms characterising codialgebra and dendriform coalgebras, which are particular cases of L-coalgebras.

To describe weighted directed graphs, we introduced [40] the notion of a L-coalgebra over a field k, i.e., a k-vector space $(L, \Delta, \tilde{\Delta})$ equipped with two coproducts Δ, $\tilde{\Delta} : L \to L^{\otimes 2}$, which obey the coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. Conversely, for any L-coalgebra generated as a k-vector space by an independent spanning set L_0, a weighted directed graph, called its geometric support can be constructed [40]. Let A be a unital algebra with unit I. It has been shown [40] that A carries a non-trivial L-bialgebra, which is also a bi-dialgebra, i.e., can be viewed as a k-vector space equipped with two coassociative coproducts, being both unital homomorphisms, $\delta_f(a) := a \otimes I$ and $\tilde{\delta}_f(a) := I \otimes a$, for all $a \in A$. It has for geometric support, the flower graph when it is generated, as a k-algebra, by a set A_0, i.e., $A := kA_0 \oplus kI$.



Example of geometric support associated with an algebra $k\{a, b, c, d\} \oplus kI$.

When a unital bialgebra (B, Δ, m) is given, it is by definition both a coassociative coalgebra and an algebra with unit I. This unital bialgebra (B, Δ, m) can then be viewed as an algebra equipped with three coassociative coproducts $\Delta, \delta_f, \tilde{\delta}_f : B \to B^{\otimes 2}$, which are m-homomorphisms. It is then natural to introduce two co-operations $\overrightarrow{d} := \Delta - \delta_f$ and $\overleftarrow{d} := \Delta - \tilde{\delta}_f$, which embed (B, Δ, m) into a L-bialgebra $(B, \overrightarrow{d}, \overleftarrow{d}, m)$ [40]. A particularity of such co-operations was put forward by Hudson [22]. They both obey the Leibniz-Ito property.

Derivatives or differentials used in algebra are often characterised by the Leibniz property. However, in classical and quantum stochastic dynamics, the Leibniz-Ito property is often required to formalise a stochastic calculus, see for instance [22] in the quantum case. Recall that Leibniz-Ito derivative is a notion firstly introduced in classical stochastic calculus. Stochastic integral arises when we are interested in integrating functions along trajectories of a stochastic process. Assuming for the moment that such trajectories are smooth (of class C^1) such an integral reduces into the Riemann-Stieltjes integral along smooth curves. The problem arising in stochastic integration is that in the most interesting stochastic processes like Brownian motion, the trajectories of the process are almost everywhere continuous but nowhere differentiable. It turns out that if $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is of class C^1 with respect with the first variable and of class C^2 with respect with the second variable, its differential reads $df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt$, where (B_t) is the Brownian motion and where formally $dt := dB_t dB_t$. Let A be an algebra unit I and M be a A-bimodule. Placed in an algebraic setting, the linear map $d_{Ito} : A \to M$ is said to be a Leibniz-Ito derivative if $\rho(I) = 0$ and obeys the *Leibniz-Ito property* $d_{Ito}(xy) = x d_{Ito}(y) + d_{Ito}(x) y + d_{Ito}(x) d_{Ito}(y)$, for all $x, y \in A$.

Motivated by, but independently from the work of Hudson, we pursue our investigations on the Leibniz-Ito property. In Section 3.2, we introduce notation. In Section 3.3, inspired by a work of Quillen [61], we translate the Leibniz-Ito property of a Leibniz-Ito derivative in terms of Quillen curvature $\omega(x, y) := d_{Ito}(xy) - d_{Ito}(x) d_{Ito}(y) = x d_{Ito}(y) + d_{Ito}(x) y$, for all $x, y \in A$. From the coassociative coproducts of a unital associative algebra (A, m) viewed as a L-bialgebra $(A, \delta_f, \tilde{\delta}_f, m)$, we construct a differential operator which characterises Leibniz and Leibniz-Ito derivatives. In Subsection 3.3.1, motivated by this new differential operator, we produce another complex, closely related to Hochschild's one and show the usefulness of the Bianchi identity applied to the Quillen curvature of a Leibniz-Ito derivative. We also yield explicit solutions of differential equations constructed from these differential operators. Motivated by works of Connes [12] on the construction of a Leibniz graded differential algebra from a well chosen operator F, we construct from the Quillen curvature of a Leibniz-Ito derivative a graded differential associative dialgebra in Section 3.3.3 and produce a closed trace which vanishes on Leibniz commutator. Applied to n-forms, the integral calculus so obtained, yields cyclic cocycles. We show also how to relate this Subsection to Subsection 3.3.1. We end this paper with Section 3.4 by giving an interpretation of the Leibniz-Ito property in terms of a distributivity defect of a certain law with respect to the associative product m.

3.2 Quillen curvature

In [61], Quillen defines the notion of curvature of a linear map relative to the Hochschild homology of an associative algebra A with product m. Here we study only the case of a unital

algebra. The unit element will be denoted by I. For the convenience of the reader, we remind briefly the definition of the boundary operators b and b', see for instance [43] for more details.
Notation: For every $n \in \mathbb{N}$, we denote $(a_1, a_2, ..., a_n)$ the tensor product: $a_1 \otimes a_2 \otimes ... \otimes a_n$ and for $n > 1$, we define $b' : A^{\otimes n} \to A^{\otimes (n-1)}$, by:

$$b'(a_1, ..., a_n) := \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, ..., a_i a_{i+1}, ..., a_n).$$

Since A is unital, the b'-complex,

$$... \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \to 0,$$

is exact. The *differential*, $b : A^{\otimes n} \to A^{\otimes (n-1)}$ is usually defined by:

$$b(a_1, ..., a_n) = b'(a_1, ..., a_n) + (-1)^{n-1} (a_n a_1, ..., a_{n-1}).$$

When A is unital, the Hochschild homology $H(A, A)$, is usually computed from the b-complex:

$$... \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A \to 0.$$

Let A be a unital associative algebra, we denote by $F(A) = k \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \dots$ the *tensor space* associated with A. The coassociative coproduct usually used on $F(A)$ is $\Delta : F(A) \to F(A) \otimes F(A)$, such that

$$\Delta(a_1, ..., a_n) := \sum_{i=0}^{n} (a_1, ..., a_i) \otimes (a_{i+1}, ..., a_n).$$

By a n-cochain on A, we mean a multilinear function $f(a_1, ..., a_n)$ with values in some vector space V or equivalently a linear map from $A^{\otimes n}$ to V. These cochains form a complex $\mathsf{Hom}(F(A), V)$, where the differential is $\hat{\delta}(f) = -(-1)^n f b', \quad f \in \mathsf{Hom}(A^{\otimes n}, V)$. Suppose L is a unital associative algebra, the complex of cochains $\mathsf{Hom}(F(A), L)$ has a convolution product defined by, $fg = m(f \otimes g)\Delta$, where $m : L \otimes L \to L$ is the product of L. If f and g have respectively degrees p and q, we define the associative product,

$$(fg)(a_1, ..., a_n) = (-1)^{pq} f(a_1, ..., a_p) g(a_{p+1}, ..., a_{p+q}).$$

As an important example, we have:

Definition 3.2.1 [Quillen curvature of a 1-cochain [61]] Let ρ be a 1-cochain, that is a linear map from A to L. We can view ρ as a connection form and construct its *Quillen curvature* : $\omega := \hat{\delta}\rho + \rho^2$, which will be a 2-cochain. Then, the *Quillen curvature* of ρ,

$$\omega(a_1, a_2) := (\hat{\delta}\rho + \rho^2)(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2), \quad \forall a_1, a_2 \in A,$$

quantifies how close ρ is to a homomorphism.

3.2.1 About cyclic cocycles

We still follow Quillen [61]. Let $\sigma : L \to V$ be a trace on the algebra L with values in the vector space V, i.e., a linear map vanishing on the commutator subspace $[L, L]$. The image of the map $m - m\tau : L \otimes L \to L$ giving $[L, L]$, we can say that a trace is realy a linear map defined on the commutator quotient space:

$$L_\natural := L/[L, L] = \text{coker}\{m - m\tau : L \otimes L \to L\}.$$

Therefore, naturally associated with the bar construction of A is its cocommutator subspace,

$$F(A)^\natural := \ker\{\Delta - \tau\Delta : F(A) \to F(A) \otimes F(A)\}.$$

If $\natural : F(A)^\natural \to F(A)$ denotes the inclusion map, it is the universal cotrace in the sense that a cotrace $L \to F(A)$ is the same as a linear map with values in $F(A)^\natural$. By combining the trace σ with the universal cotrace $\natural$, Quillen defines a morphism of complexes:

$$\sigma^\natural : \text{Hom}(F(A), L) \to \text{Hom}(F(A)^\natural, V), \quad \sigma^\natural(f) = \sigma f \natural,$$

which is a trace on the DG algebra of cochains i.e., vanishes on $[f, g] = fg - (-1)^{deg(f)deg(g)} gf$. Furthermore, let us recall that:

$$\sigma^\natural(\omega(a_1, a_2), \ldots, \omega(a_{2n-1}, a_{2n}) = n\sigma(\omega(a_1, a_2), \ldots, \omega(a_{2n-1}, a_{2n}) - \omega(a_{2n}, a_1), \ldots, \omega(a_{2n-2}, a_{2n-1})),$$

is a cyclic cocycle of degree $2n - 1$.

3.3 Leibniz-Ito derivatives

Let A be an associative algebra with unit I and with associative product $m : A^{\otimes 2} \to A$. Denote by $\eta : k \to A$, the unit map such that $\lambda \mapsto \lambda I$. Let M be a A-bimodule, that is a vector space with left and right product maps $m_l : A \otimes M \to M, \quad m_r : M \otimes A \to M$, defining left and right module structures which commute that is $m_r(m_l \otimes id) = m_l(id \otimes m_r)$. Consider $A \otimes V \otimes A$, where V is a vector space, as a A-bimodule with $m_r = id \otimes id \otimes m$ and $m_l = m \otimes id \otimes id$. We have the following Proposition, which can be also found in [61]:

Proposition 3.3.1 *There is a one-to-one correspondence between linear maps $h : V \to M$ and bimodule morphisms $\tilde{h} : A \otimes V \otimes A \to M$ given by:*

$$\tilde{h} = m_r(m_l \otimes id)(id \otimes h \otimes id), \quad h = \tilde{h}(\eta \otimes id \otimes \eta).$$

Consider the exact b'-complex:

$$\ldots \to A^{\otimes 4} \xrightarrow{\square} A^{\otimes 3} \xrightarrow{m\otimes id - id\otimes m} A^{\otimes 2} \xrightarrow{m} A \to 0,$$

where $\square := m \otimes id \otimes id - id \otimes m \otimes id + id \otimes id \otimes m$. The bimodule Ω_A of the (non-commutative) differentials over A is defined to be the kernel of m. We say that the linear mapping $D : A \to$

M, where M is a bimodule, is a *Leibniz derivative*, if the corresponding bimodule morphism $\tilde{D} : A^{\otimes 3} \to M$ satisfies:

$$\tilde{D}\Box = 0,$$

that is: $0 = \tilde{D}\Box(I, x, y, I) = \tilde{D}(x, y, I) - \tilde{D}(I, xy, I) + \tilde{D}(I, x, y)$, or $0 = xD(y) - D(xy) + (Dx)y$, for all $x, y \in A$. This Proposition yields an elegant way to characterise Leibniz derivatives. From now on, let us construct an analogue of the operator $\Box$ to characterise, with the same elegance the Leibniz-Ito property. Recall, see for instance [22], that a linear map $d_{\text{Ito}} : A \to M$, where M is a A-bimodule, is called a *Leibniz-Ito derivative* if $d_{\text{Ito}}(I) = 0$ and verifies the Leibniz-Ito property $d_{\text{Ito}}(xy) = d_{\text{Ito}}(x)y + xd_{\text{Ito}}(y) + d_{\text{Ito}}(x)d_{\text{Ito}}(y)$, for all $x, y \in A$. The Quillen curvature of such a derivative will be denoted by ω_{Ito}. For all $x \in A$, if $d_{\text{Ito}} : A \to A$ is a Leibniz-Ito derivative with Quillen curvature ω_{Ito}, then $\omega_{\text{Ito}}(I, x) = \omega_{\text{Ito}}(x, I) = d_{\text{Ito}}(x)$. We now adapt Proposition 3.3.1 to $V := A^{\otimes 2}$ and denote by Ξ the *pairing isomorphism*, $\Xi : A^{\otimes 4} \to A \otimes (A^{\otimes 2}) \otimes A$, such that $(a_1, a_2, a_3, a_4) \mapsto a_1 \otimes (a_2, a_3) \otimes a_4$.

Definition 3.3.2 Let A be a unital associative algebra with unit I, viewed as a L-bialgebra $(A, \delta_f, \tilde{\delta}_f, m)$. We define the map $\Box_* := \Xi\Box'$,

$$A^{\otimes 2} \xrightarrow{\Box'} A^{\otimes 4} \xrightarrow{\Xi} A \otimes (A^{\otimes 2}) \otimes A,$$

with $\Box' = \delta_f \otimes \delta_f - \delta_f \otimes \tilde{\delta}_f - \tilde{\delta}_f \otimes \delta_f + \tilde{\delta}_f \otimes \tilde{\delta}_f$.

Remark: Observe that $\Box' = d \otimes d$ and $\Box' d = 0$, where $d = \delta_f - \tilde{\delta}_f : \; x \mapsto x \otimes I - I \otimes x$, for all $x \in A$, is the usual differential.

Recall that thanks to the Proposition 3.3.1, $\tilde{h} : A \otimes (A^{\otimes 2}) \otimes A \to M$ is in one-to-one correspondence with $h : A^{\otimes 2} \to M$. Define $\tilde{m}, \tilde{\omega}_\gamma, \tilde{\omega}_{\text{Ito}}, \widetilde{Dm}$, the correspondence between the associative product m of A, the Quillen curvature of an homomorphism γ, the Quillen curvature of a Leibniz-Ito derivative d_{Ito} and the product m composed by a Leibniz derivative D.

Theorem 3.3.3 *Let (A, m) be a unital associative algebra with unit I and M be a A-bimodule. If $\tilde{F} : A \otimes (A^{\otimes 2}) \otimes A \to M$ stands for the maps $\tilde{m}$, $\tilde{\omega}_\gamma$, $\tilde{\omega}_{Ito}$ or $\widetilde{Dm}$, then:*

$$\tilde{F}\Box_* = 0.$$

Proof: Let (A, m) be a unital associative algebra with unit I and M be a A-bimodule. Suppose that $\tilde{F} : A \otimes (A^{\otimes 2}) \otimes A \to M$ verifies $\tilde{F}\Box_* = 0$, with $\Box_* = \Xi(\delta_f \otimes \delta_f - \delta_f \otimes \tilde{\delta}_f - \tilde{\delta}_f \otimes \delta_f + \tilde{\delta}_f \otimes \tilde{\delta}_f)$ that is:

$$0 = \tilde{F}(x, (I, y), I) - \tilde{F}(x, (I, I), y) - \tilde{F}(I, (x, y), I) + \tilde{F}(I, (x, I), y).$$

Therefore, $0 = x \cdot F(I, y) \cdot I - x \cdot F(I, I) \cdot y - I \cdot F(x, y) \cdot I + I \cdot F(x, I) \cdot y$, where $x, y \in A$.

- $F = m$ yields: $0 = xy - xy - xy + xy$.
- $F = \omega_\gamma$ yields: $0 = 0 - 0 - \omega_\gamma(x, y) + 0$.
- $F = \omega_{\text{Ito}}$ yields: $0 = xd_{\text{Ito}}(y) - 0 - (d_{\text{Ito}}(xy) - d_{\text{Ito}}(x)d_{\text{Ito}}(y)) + d_{\text{Ito}}(x)y$.
- $F = Dm$ yields: $0 = xD(y) - 0 - D(xy) + D(x)y$. $\Box$

Theorem 3.3.4 *Let (A, m) be a unital associative algebra with unit I and M be a A-bimodule. Suppose the Quillen curvature ω of a linear map $\rho : A \to M$ verifies $\tilde{\omega}\Box_* = 0$.*

- *If ρ is unital, then ρ is a homomorphism.*
- *If $\rho(I) = 0$, then ρ is a Leibniz-Ito derivative.*

Proof: Straightforward. □

Thanks to the differential operator $\Box_*$, we characterise Leibniz and Leibniz-Ito properties. We now relate homomorphisms to Leibniz-Ito derivatives.

Theorem 3.3.5 *Let A be a unital associative algebra with unit I. The set of Leibniz-Ito derivatives from A to A is in bijection with the set of unital homomorphisms from A to A.*

Proof: Let A be a unital associative algebra with unit I. Let ρ be a unital homomorphism from A to A. The linear map $d = \rho - id$ is a Leibniz-Ito derivative since for $x, y \in A$, we have:

$$dxdy = (\rho(x) - x)(\rho(y) - y) = \rho(xy) - (xy - xy) - x\rho(y) - \rho(x)y + xy,$$

that is: $dxdy = d(xy) - xdy - d(x)y$ and $d(I) = 0$. Let d be a Leibniz-Ito derivative from A to A. The linear map $\rho = d + id$ is a unital homomorphism since $d(x)d(y) = d(xy) - xd(y) - d(x)y$ and $\rho(x)\rho(y) = d(x)d(y) + xd(y) + d(x)y + xy = d(xy) + xy = \rho(xy)$, for $x, y \in A$ and $\rho(I) = I$. □

Theorem 3.3.6 *Let A be a unital associative algebra with unit I. Let ρ be a unital linear map from A to A with Quillen curvature ω_ρ. Decompose $\rho = \zeta + id$, where ζ is a linear map mapping I to 0 with Quillen curvature ω_ζ. We have: $\tilde{\omega}_\rho\Box_* = \tilde{\omega}_\zeta\Box_*$.*

Proof: Let $x, y \in A$.

$$\begin{aligned}
\tilde{\omega}_\rho\Box_*(x, y) = -\omega_\rho(x, y) &= -\rho(xy) + \rho(x)\rho(y) \\
&= -((\zeta(xy) + xy) - (\zeta(x) + x)(\zeta(y) + y)) \\
&= -(\zeta(xy) - \zeta(x)y - x\zeta(y) - \zeta(x)\zeta(y)) \\
&= x \cdot \omega_\zeta(I, y) \cdot I - x \cdot \omega_\zeta(I, I) \cdot y - I \cdot \omega_\zeta(x, y) \cdot I + I \cdot \omega_\zeta(x, I) \cdot y \\
&= \tilde{\omega}_\zeta\Box_*(x, y),
\end{aligned}$$

which completes the proof. □

Remark: Let A be a unital associative algebra with unit I_A. Let $\mathsf{Hom}_{I_A}(A)$ be the set of unital linear maps from A to A and $\mathsf{Hom}_{0_A}(A)$ be the set of linear maps which map I_A to 0_A. The one-to-one mapping Ψ:

$$\mathsf{Hom}_{0_A}(A) \xrightarrow{\Psi} \mathsf{Hom}_{I_A}(A)$$

$$\zeta \mapsto \Psi(\zeta) := \zeta + id := \rho,$$

leaves the Quillen curvature of the maps involved invariant by $\Box_*$, i.e., $\tilde{\omega}_\zeta\Box_*(x, y) = \tilde{\omega}_{\Psi(\zeta)}\Box_*(x, y)$, for all $x, y \in A$.

3.3.1 Bianchi identity and Hochschild complex

Let (A, m) be a unital associative algebra with unit I. By establishing a link between $A^{\otimes 2}$ and its free bimodule $A \otimes (A^{\otimes 2}) \otimes A$, we showed a common point between the Quillen curvature of Leibniz-Ito derivatives and Leibniz derivatives. We go further by showing the usefulness of the Bianchi identity applied to the Quillen curvature of a Leibniz-Ito derivative. Recall that if $\rho : A \to A$ is a 1-cochain, its Quillen curvature is defined by the equation $\omega = \hat{\delta}\rho + \rho^2$ where $\hat{\delta}$ is related to the Hochschild boundary b'. The Bianchi identity reads $\hat{\delta}\omega = -[\rho, \omega]$, with:

$$[\rho, \omega](a_1, a_2, a_3) = \rho(a_1)\omega(a_2, a_3) - \omega(a_1, a_2)\rho(a_3) = \omega(a_1a_2, a_3) - \omega(a_1, a_2a_3), \quad \forall a_i \in A.$$

If ρ is a Leibniz-Ito derivative, we have $\omega(a_0, a_1) = a_0\rho(a_1) + \rho(a_0)a_1$, for all $a_0,\ a_1 \in A$. Set $A \otimes (A^{\otimes n}) \otimes A := \hat{A}^{\otimes n}$. To explain the following Theorem, recall we showed $\tilde{\omega}\Box_* = 0$. Is it possible to construct multilinear maps $f_n : A^{\otimes n} \to A$ and differential operators $\Box_{*n} : A^{\otimes n} \to \hat{A}^{\otimes n}$ such that $\tilde{f}_n\Box_{*n} = 0$? In the following, we set $\tilde{A}^{\otimes n} := \mathsf{Im}(\Box_{*n})$.

Theorem 3.3.7 *Let ρ be a Leibniz-Ito derivative with Quillen curvature ω. We have the following complex between the Hochschild complex with boundary b' and its associated free bimodule:*

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \leftarrow & A & \xleftarrow{b'_2} & A^{\otimes 2} & \xleftarrow{b'_3} & A^{\otimes 3} & \xleftarrow{b'_4} & A^{\otimes 4} & & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \Box_{*2}} & & \downarrow{\scriptstyle \Box_{*3}} & & \downarrow{\scriptstyle \Box_{*4}} & & \\
0 & \leftarrow & \tilde{A} & \xleftarrow[b'_{*2}]{} & \tilde{A}^{\otimes 2} & \xleftarrow[b'_{*3}]{} & \tilde{A}^{\otimes 3} & \xleftarrow[b'_{*4}]{} & \tilde{A}^{\otimes 4} & & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where for all n, $b'_n := b'$. Define for all $n > 1$, the linear maps $\Xi_n : A^{\otimes(n+2)} \to \hat{A}^{\otimes n}$ $(a_0, \dots, a_{n+1}) \mapsto a_0 \otimes (a_1 \dots a_n) \otimes a_{n+1}$ and $\Box_{*n} := \Xi_n \circ (\underbrace{d \otimes id \dots id \otimes d}_{n \quad terms})$, where $d = \delta_f - \tilde{\delta}_f$ is the usual differential. Let us define $f_2 = \omega$, $f_3 = \hat{\delta}\omega$, and for all $n > 3$:

$$\begin{array}{rcl}
f_{2n} & = & \hat{\delta}\omega\rho^{2n-3} + \rho^2\hat{\delta}\omega\rho^{2n-5} + \rho^4\hat{\delta}\omega\rho^{2n-7} + \rho^6\hat{\delta}\omega\rho^{2n-9} + \dots + \rho^{2(n-2)}\hat{\delta}\omega\rho + \rho^{2(n-1)}\omega, \\
f_{2n+1} & = & \hat{\delta}\omega\rho^{(2n+1)-3} + \rho^2\hat{\delta}\omega\rho^{(2n+1)-5} + \rho^4\hat{\delta}\omega\rho^{(2n+1)-7} + \rho^6\hat{\delta}\omega\rho^{(2n+1)-9} + \dots + \rho^{(2n+1)-3}\hat{\delta}\omega.
\end{array}$$

Then, for all $n > 1$, $\tilde{f}_n\Box_{*n} = 0$, $\hat{\delta}f_{2n} = f_{2n+1}$ and $\hat{\delta}f_{2n+1} = 0$.

Remark: For $n = 2$, it is already proved. The whole Proposition derives from the following remark. Fix $a_0, a_1 \in A$, we get:

$$\begin{array}{rcl}
a_0\rho(a_1)\rho(a_2) & = & \omega(a_0, a_1)\rho(a_2) \\
 & & -\rho(a_0)\omega(a_1, a_2) \\
 & & +\rho(a_0)\rho(a_1)a_2.
\end{array}$$

This equality shows the usefulness of the Bianchi identity when it is computed on a Leibniz-Ito derivative. The idea is then to find a differential operator, here $\Box_{*3}$, and f_3, here $\widetilde{\hat{\delta}\omega}$, such that $\widetilde{\hat{\delta}\omega}\Box_{*3}(a_0, a_1, a_2) = 0$, for all $a_0, a_1, a_2 \in A$.

Proof: We now fix $n > 2$ and remark that for all $a_0, \ldots, a_{2n-1} \in A$:

$$\begin{aligned} a_0\rho(a_1)\ldots\rho(a_{2n-1}) &= \hat{\delta}\omega(a_0, a_1, a_2)\rho(a_3)\ldots\rho(a_{2n-1}) \\ &+\rho(a_0)\rho(a_1)\hat{\delta}\omega(a_2, a_3, a_4)\rho(a_5)\ldots\rho(a_{2n-1}) + \ldots \\ &+\rho(a_0)\ldots\rho(a_{2n-3})\omega(a_{2n-2}, a_{2n-1}) \qquad (4) \\ &+\rho(a_0)\ldots\rho(a_{2n-2})a_{2n-1}. \end{aligned}$$

which explains the definition of f_{2n}. Moreover,

$$\begin{aligned} \Box_{*n}(a_0, \ldots, a_{2n-1}) &= \Xi_n \circ (d \otimes id \ldots id \otimes d)(a_0, \ldots, a_{2n-1}) \\ &= +a_0 \otimes (I, a_1, \ldots, a_{2n-2}, a_{2n-1}) \otimes I \qquad (5) \\ &\quad -a_0 \otimes (I, a_1, \ldots, a_{2n-2}, I) \otimes a_{2n-1} \qquad (6) \\ &\quad -I \otimes (a_0, a_1, \ldots, a_{2n-2}, a_{2n-1}) \otimes I \qquad (7) \\ &\quad +I \otimes (a_0, a_1, \ldots, a_{2n-2}, I) \otimes a_{2n-1} \qquad (8) \end{aligned}$$

Since $\rho(I) = 0$, we observe that $\widetilde{\hat{\delta}\rho^{2n-3}}$ yields zero when applied to equations (6) and (8). When applied to (5), we obtain $+a_0\omega(I, a_1)\rho(a_2)\ldots\rho(a_{2n-1})$.
Applied to (7), we get $-\hat{\delta}\omega(a_0, a_1, a_2)\rho(a_3)\ldots\rho(a_{2n-1})$. The other terms of the definition of $\tilde{f}_{2n}$ only apply to equation (7) and give the other terms of the sum (4), except the last term $\widetilde{\rho^{2n-2}\omega}$ which, when applied on equations (7) and (8), yields the two last terms of the sum (4). We obtain $\tilde{f}_{2n}\Box_{*2n}(a_0, \ldots, a_{2n-1}) = 0$ i.e., the sum (4). The same remark is used to prove the odd case. To prove $\hat{\delta}f_{2n} = f_{2n+1}$, we remark that for all $n > 2$, $f_{2n+1} = \hat{\delta}(\rho^{2n})$, hence $\hat{\delta}f_{2n+1} = 0$. Moreover, $f_{2n} = \hat{\delta}f_{2n-1}\rho + \rho^{2n-2}\omega$.

$$\begin{aligned} \hat{\delta}f_{2n} &= \hat{\delta}(\hat{\delta}f_{2n-1}\rho + \rho^{2n-2}\omega) \\ &= ((-1)^{2n-1}\hat{\delta}f_{2n-1}\hat{\delta}\rho) + (\hat{\delta}\rho^{2n-2}\omega + \rho^{2n-2}\hat{\delta}\omega) \\ &= (-\hat{\delta}\rho^{2n-2}(\omega - \rho^2) + (\hat{\delta}\rho^{2n-2}\omega + \rho^{2n-2}\hat{\delta}\rho^2) \\ &= \hat{\delta}\rho^{2n-2}\rho^2 + \rho^{2n-2}\hat{\delta}\rho^2 \\ &= \hat{\delta}\rho^{2n} = f_{2n+1}. \end{aligned}$$

We must be careful about sign. Recall that,

$$(f.g)(a_1, \ldots, a_{p+q}) = (-1)^{pq}f(a_1, \ldots, a_p)g(a_{p+1}, \ldots a_{p+q}).$$

In our case f_n are defined up to a sign, without importance for the result. We restore the right sign by noticing that for all $n > 0$, $f'_{4n} \equiv -f_{4n}$ and $f'_{4n+1} \equiv -f_{4n+1}$. All the other, $f'_{4n+l} \equiv f_{4n+l}$, with $l = 2$ or 3, are correctely defined.

Nothing has been said about the operators b'_{*n}. We define $b'_{*n} := \Box_{*(n-1)}(m \otimes id \ldots id \otimes m)(id \otimes b_n \otimes id)J$, where m denotes the associative product of A. The aim of the "projection" J is to select the equation (6) among the four possibilities (5), (6), (7), (8) that is for all $a_0, a_1, \ldots, a_{n-1} \in A$,

$$\begin{aligned}\Box_{*n}(a_0, \ldots, a_n) &= \Xi_n \circ (d \otimes id \ldots id \otimes d)(a_0, \ldots, a_n) \\ &\xrightarrow{J} \Xi_n \circ (\delta_f \otimes id \ldots id \otimes \tilde{\delta}_f)(a_0, \ldots, a_n) \\ &= (a_0, (I, a_1, \ldots, a_{n-2}, I), a_{n-1}).\end{aligned}$$

Recall that, δ_f and $\tilde{\delta}_f$ are the coproducts of the unital algebra A, viewed as a L-bialgebra $(A, \delta_f, \tilde{\delta}_f, m)$. Now if we prove that for all $a_0, a_1, \ldots, a_{n-1} \in A$:

$$(m \otimes id \ldots id \otimes m)(id \otimes b_n \otimes id)(a_0, (I, a_1, \ldots, a_{n-2}, I), a_{n-1}) = b_n(a_0, a_1, \ldots, a_{n-2}, a_{n-1}),$$

the commutativity of the sequence above will be proved and since

$$b'_{*n}(b'_{*(n+1)}\Box_{*(n+1)}) = b'_{*n}(\Box_{*n}b'_{(n+1)}) = (\Box_{*(n-1)}b'_n)b'_{(n+1)} = 0,$$

we will get $b'_{*n}b'_{*(n+1)} = 0$. The sequence will be a complex as claimed in the Theorem. Fix $a_0, a_1, \ldots, a_n \in A$. By definition,

$$b'_n = m \otimes id \ldots \otimes id - id \otimes m \ldots \otimes id + \ldots + (-1)^{n+1}\ id \otimes id \ldots \otimes m.$$

Hence,

$$\begin{aligned}(m \otimes id \ldots id \otimes m)(id \otimes b'_n \otimes id) &= \\ &m(id \otimes m) \otimes id \ldots \otimes id \otimes m(id \otimes id) \\ &-m(id \otimes id) \otimes m \ldots \otimes id \otimes m(id \otimes id) \\ &+ \ldots + (-1)^{n+1}\ m(id \otimes id) \otimes id \otimes id \ldots id \otimes m(m \otimes id).\end{aligned}$$

Yet,

- $m(id \otimes m)\delta_f(a_0) \otimes a_1 = m(a_0 \otimes a_1)$,
- $(id \otimes m)a_{(n-2)} \otimes \tilde{\delta}_f a_{(n-1)} = a_{(n-2)} \otimes a_{(n-1)}$,

proving that:

$$(m(id \otimes m) \otimes id \ldots \otimes id \otimes m(id \otimes id))(\delta_f \otimes id \ldots id \otimes \tilde{\delta}_f)(a_0, \ldots, a_n) = (m \otimes id \ldots \otimes id)(a_0, \ldots, a_n).$$

Moreover,

- $m(id \otimes id)\delta_f(a_0) = a_0$,
- $(id \otimes m)a_{(n-2)} \otimes \tilde{\delta}_f a_{(n-1)} = a_{(n-2)} \otimes a_{(n-1)}$,

proving that:

$$-(m(id \otimes id) \otimes m \ldots \otimes id \otimes m(id \otimes id))(\delta_f \otimes id \ldots id \otimes \tilde{\delta}_f)(a_0, \ldots, a_n) = -(id \otimes m \ldots \otimes id)(a_0, \ldots, a_n),$$

and the equality between all the other terms of the sum except the last one. However,

- $m(m\otimes id)a_{(n-2)}\otimes\tilde{\delta}_f a_{(n-1)} = m(a_{(n-2)}\otimes a_{(n-1)})$,

proving that $((-1)^{n+1}\ m(id\otimes id)\otimes id\otimes id\dots id\otimes m(m\otimes id))(\delta_f\otimes id\dots id\otimes\tilde{\delta}_f)(a_0,\dots,a_n) = ((-1)^{n+1}\ id\otimes id\dots\otimes m)(a_0,\dots,a_n)$. This concludes the proof. $\square$

Remark: Related to the Hochschild complex with boundary b' is the following complex,

$$0\to \mathsf{Hom}(\tilde{A},A)\xrightarrow{\hat{\delta}}\mathsf{Hom}(\tilde{A}^{\otimes 2},A)\xrightarrow{\hat{\delta}}\mathsf{Hom}(\tilde{A}^{\otimes 3},A)\xrightarrow{\hat{\delta}}\mathsf{Hom}(\tilde{A}^{\otimes 4},A)\xrightarrow{\hat{\delta}}\dots$$

It is worth noticing that the equations above admit Leibniz-Ito derivatives ρ but also Leibniz derivatives D, if we associate formally with the Quillen curvature ω, the bilinear map Dm.

Motivated by the construction of the operator $\square_*$, we have constructed another complex related to the Hochschild complex by the operators $\square_{*n}$, $n>1$. Moreover, for all $n>1$, we have produced solutions of equations $F\square_{*n}=0$ and showed how these solutions were related by the Hochschild coboundary $\hat{\delta}$. We will recover these solutions in Subsection 3.3.3.

3.3.2 A Leibniz-Ito graded differential algebra

We pursue our analogy Leibniz derivative versus Leibniz-Ito derivatives. Let A be a unital associative algebra with unit I. This short Subsection is an attempt to adapt what was done in the case of cyclic cocycles to the Leibniz-Ito case. In [12], Connes defines an operator F such that $F^2=I$ and constructs a Leibniz graded differential algebra Ω^* with n-forms $a_0[F,a_1]\dots[F,a_n]$, $n\geq 0$. Fix $n\geq 0$. The differential d_C maps forms $a_0[F,a_1]\dots[F,a_n]$ to $[F,a_0][F,a_1]\dots[F,a_n]$. Set $D(a)=[F,a]$, $a\in A$.

$$\begin{aligned} a_0D(a_1)\dots D(a_k)a_{k+1} &= \\ &(-1)^k\sum_{j=1}^{k}(-1)^j a_0D(a_1)\dots D(a_{i-1})D(a_ia_{i+1})D(a_{i+2})\dots D(a_{k+1}) \\ &+\ (-1)^k a_0a_1D(a_2)\dots D(a_{k+1}),\quad \forall a_j\in A. \end{aligned}$$

The product of two forms is then associative thanks to the Leibniz property of the maps involved.

Recall that if $\rho: A\to A$ is a Leibniz-Ito derivative, i.e., $\rho(I)=0,\quad \rho(ab)=\rho(a)b+a\rho(b)+\rho(a)\rho(b)$, with Quillen curvature ω, we get $\omega(a_1,a_2)=a_1\rho(a_2)+\rho(a_1)a_2$. This property is nearly the same as the Leibniz one. An idea would be to replace $D(a)$ by $\rho(a)$ and construct the space of forms from,

$$\eta = a_0\rho(a_1)\dots\rho(a_k),\quad \forall a_j\in A.$$

There are two drawbacks in this naive framework. The first one is that

$$\rho(ab)=\rho(a)b+a\rho(b)+\rho(a)\rho(b),$$

mixing 1-forms and 2-forms. However, since $\omega(a,I)=\omega(I,a)=\rho(a)$, we get rid of this obstacle by denoting $\omega(I,a)\equiv\omega(a)$. Hence, we define for $k>0$, Ω^k the linear span of the operators

$$\eta = a_0\omega(a_1)\dots\omega(a_k),\quad \forall a_j\in A.$$

For $k = 0$, $\Omega^0 = A$. Then the k-vector space Ω^* is defined as $\Omega^* := \bigoplus \Omega^k$. As in the Leibniz case, we remark that

$$\begin{aligned} a_0\omega(a_1)\dots\omega(a_k)a_{k+1} &= \\ & (-1)^k \sum_{j=1}^{k} (-1)^j a_0\omega(a_1)\dots\omega(a_{i-1})\omega(a_i,a_{i+1})\omega(a_{i+2})\dots\omega(a_{k+1}) \\ &+ \ (-1)^k a_0 a_1\omega(a_2)\dots\omega(a_{k+1}), \quad \forall a_j \in A. \end{aligned}$$

Therefore, Ω^* can be embedded into an algebra structure. Here is the second drawback. The product of two forms is no longer associative. Nevertheless, we give the following result.

If $\eta_1 \in \Omega^{k_1}$ and $\eta_2 \in \Omega^{k_2}$ then $\eta_1\eta_2 \in \Omega^{k_1+k_2}$ and, for all k, Ω^k is a A-bimodule. The differential $d_I : \Omega^* \to \Omega^*$ is defined as follows:

$$d_I(a_0\omega(a_1)\dots\omega(a_k)) = \omega(a_0)\omega(a_1)\dots\omega(a_k).$$

Proposition 3.3.8 *By construction* $d_I^2 = 0$ *and*

$$d_I(\eta_1\eta_2) = d_I(\eta_1)\eta_2 + (-1)^{k_1}\eta_1 d_I(\eta_2) + (-1)^{k_1} d_I(\eta_1)d_I(\eta_2), \quad \forall \eta_j \in \Omega^{k_j}.$$

Proof: The equality $d_I^2 = 0$ is straightforward since $\omega(I) \equiv \omega(I,I) = 0$. Let $\eta_1 = a_0\omega(a_1)\dots\omega(a_{k_1})$ and $\eta_2 = b_0\omega(b_1)\dots\omega(b_{k_2})$. For convenience we rename for all j, b_j as a_{k_1+j+1} so that $\eta_2 = a_{k_1+1}\omega(a_{k_1+2})\dots\omega(a_{k_1+k_2+1})$. Then,

$$\begin{aligned} d_I(\eta_1\eta_2) &= (-1)^{k_1}\sum_{j=1}^{k_1}(-1)^j\omega(a_0)\rho(a_1)\dots\omega(a_{i-1})\omega(a_i,a_{i+1})\omega(a_{i+2})\dots\omega(a_{k_1})\omega(a_{k_1+1}) \\ & \dots\omega(a_{k_1+k_2+1}) + (-1)^{k_1}\omega(a_0a_1)\omega(a_2)\dots\omega(a_{k_1})\omega(a_{k_1+1})\dots\omega(a_{k_1+k_2+1}) \end{aligned}$$

Yet by definition, $\omega(a_0a_1) = \omega(a_0,a_1) - \omega(a_0)\omega(a_1)$ and

$$\begin{aligned} d_I(\eta_1\eta_2) &= (-1)^{k_1}\sum_{j=0}^{k_1}(-1)^j\omega(a_0)\omega(a_1)\dots\omega(a_{i-1})\omega(a_i,a_{i+1})\omega(a_{i+2})\dots\omega(a_{k_1})\omega(a_{k_1+1}) \\ & \dots\omega(a_{k_1+k_2+1}) + (-1)^{k_1}\omega(a_0)\omega(a_1)\omega(a_2)\dots\omega(a_{k_1})\omega(a_{k_1+1})\dots\omega(a_{k_1+k_2+1}). \end{aligned}$$

However,

$$\begin{aligned} d_I(\eta_1)\eta_2 &= \omega(a_0)\omega(a_1)\omega(a_2)\dots\omega(a_{k_1})a_{k_1+1}\omega(a_{k_1+2})\dots\omega(a_{k_1+k_2+1}) \\ &= (-1)^{k_1}\sum_{j=0}^{k_1}(-1)^j\omega(a_0)\omega(a_1)\dots\omega(a_{i-1})\omega(a_i,a_{i+1})\omega(a_{i+2})\dots\omega(a_{k_1+k_2+1}) \\ & -(-1)^{k_1}a_0\omega(a_1)\omega(a_2)\dots\omega(a_{k_1})a_{k_1+1}\omega(a_{k_1+2})\dots\omega(a_{k_1+k_2+1}). \end{aligned}$$

$$\begin{aligned} (-1)^{k_1}\eta_1 d_I(\eta_2) &= (-1)^{k_1}a_0\omega(a_1)\omega(a_2)\dots\omega(a_{k_1})\omega(a_{k_1+1})\omega(a_{k_1+2})\dots\omega(a_{k_1+k_2+1}). \\ (-1)^{k_1}d_I(\eta_1)d_I(\eta_2) &= (-1)^{k_1}\omega(a_0)\omega(a_1)\omega(a_2)\dots\omega(a_{k_1})\omega(a_{k_1+1})\omega(a_{k_1+2})\dots\omega(a_{k_1+k_2+1}). \end{aligned}$$

Remark: We could define then a graded Quillen curvature $\omega(\eta_1, \eta_2) := d_I(\eta_1\eta_2) - (-1)^{k_1} d(\eta_1)d(\eta_2)$. □

However, to recover an associative product, we have to modify the present setting.

3.3.3 Quillen curvature of a Leibniz-Ito derivative and differential associative (super)-dialgebra

We construct from the Quillen curvature of a Leibniz-Ito derivative, an anti-$\mathbb{Z}_2$ graded differential algebra of non-commutative forms. Then, we show this algebra is an associative dialgebra [45] and that the products of this dialgebra embed it into a di-superalgebra. From this dialgebra, we construct a special dendriform algebra which allow the construction of closed forms. We produce also a closed trace which vanishes on the Leibinz commutator, naturally associated with this graded differential associative dialgebra.

In the sequel, if x is a form whose degree is $\deg(x)$, then $(-1)^x$ reads $(-1)^{\deg(x)}$. Let us start with some definitions.

Definition 3.3.9 A *super-algebra*, see for instance [60], is a k-vector space $S = S_+ \oplus S_-$ of even and odd elements, belonging respectively to S_+ and S_-, equipped with an associative product which respects this $\mathbb{Z}_2$ grading, i.e., $aa' \in S_+$ if and only if a and a' are both even or both odd and $aa' \in S_-$ otherwise.

An *anti-superalgebra* $As = As_+ \oplus As_-$ is a k-vector space of even and odd elements belonging respectively to As_+ and As_-, equipped with an associative product such that $aa' \in As_-$ if and only if a and a' are both even or both odd and $aa' \in As_+$ otherwise.

Let us construct from an associative unital algebra A with unit I and the Quillen curvature ω of a Leibniz-Ito derivative $\rho : A \to A$ a differential associative dialgebra which respects the $\mathbb{Z}_2$-grading. Recall that $\omega(I, I) = 0$ and $\omega(I, a) = \omega(a, I) = \rho(a)$, for all $a \in A$. Let $\Omega^* = \bigoplus \Omega^k$, with for all $k > 0$, Ω^k be the A-bimodule constructed over the linear span of the operators:

$$a_0 \underbrace{\omega(a_1, a_2) \dots \omega(a_{2k-1}, a_{2k})}_{\text{k}} a_{2k+1}.$$

For $k = 0$, set $\Omega^0 = A$. The product $\star$ between a k-form and a l-form is defined from the Quillen curvature ω by:

$$(a_0\omega(a_1, a_2) \dots \omega(a_{2k-1}, a_{2k})a_{2k+1}) \star (b_0\omega(b_1, b_2) \dots \omega(b_{2l-1}, b_{2l})b_{2l+1}) = \\ a_0\omega(a_1, a_2) \dots \omega(a_{2k+1}, b_0) \dots \omega(b_{2k-1}, b_{2k})b_{2l+1},$$

where $a_0, \dots, a_{2k+1}, b_0, \dots, b_{2l+1} \in A$.

Remark: To preserve the associativity of the product $\star$, the product $\star$ between 0-forms and other forms is not defined, except for the identity element I. Observe that the product embeds two forms of degree k and l into a form of degree $k + l + 1$.

Definition 3.3.10 The differential $\bar{d} : \Omega^k \to \Omega^{k+1}$ is defined for 0-forms as $\bar{d}(a) := \omega(I, a) = I \star a = a \star I$ and for forms of higher order by $\bar{d}(a_0\omega(a_1, a_2)\dots\omega(a_{2k-1}, a_{2k})a_{2k+1}) = \omega(I, a_0)\omega(a_1, a_2)\dots\omega(a_{2k-1}, a_{2k})a_{2k+1} + (-1)^k a_0\omega(a_1, a_2)\dots\omega(a_{2k-1}, a_{2k})\omega(a_{2k+1}, I)$.

Proposition 3.3.11 *The operator $\bar{d}$ verifies for all $x, y \in \Omega^{\deg(x)} \times \Omega^{\deg(y)}$,*

$$\bar{d}^2 = 0 \quad \textit{and} \quad \bar{d}(x \star y) = \bar{d}(x) \star y + (-1)^{x+1} x \star \bar{d}(y) = \bar{d}(x) \star y - (-1)^x x \star \bar{d}(y).$$

Proof: Recall that $\omega(I, I) = 0$. Fix a k-form $x := a_0\omega(a_1, a_2)\dots\omega(a_{2k-1}, a_{2k})a_{2k+1}$, with $k = \deg(x)$. We have, $\bar{d}^2(x) = \bar{d}(I\omega(I, a_0)\omega(a_1, a_2)\dots\omega(a_{2k-1}, a_{2k})a_{2k+1} + (-1)^k a_0\omega(a_1, a_2)\dots \omega(a_{2k-1}, a_{2k})\omega(a_{2k+1}, I)I) = +(-1)^{k+1} I\omega(I, a_0)\dots\omega(a_{2k+1}, I) + (-1)^k \omega(I, a_0)\dots\omega(a_{2k+1}, I)I = 0$. The remaining property follows by straightforward computations. □

Recall that the notions of dialgebra and dendriform algebra are defined in introduction.

Theorem 3.3.12 *Define the operations $\dashv$ and $\vdash$ such that $x \dashv y := -x \star \bar{d}(y)$ and $x \vdash y := (-1)^{x+1}\bar{d}(x) \star y$, for all $x, y \in \Omega^*$. Then, $(\Omega^*, \dashv, \vdash)$ is embedded into an associative dialgebra.*

Proof: Fix $x, y, z \in \Omega^*$. By xy, we mean $x \star y$. The associativity is straightforward. On the one hand, $x \dashv (y \dashv z) = x \dashv y\bar{d}z = x\bar{d}(y\bar{d}z) = x\bar{d}y\bar{d}z$ and $(x \dashv y) \dashv z = x\bar{d}y \dashv z = x\bar{d}y\bar{d}z$. On the other hand, $x \vdash (y \vdash z) = x \vdash (-1)^{y+1}\bar{d}(y)z = (-1)^{x+1}(-1)^{y+1}\bar{d}x\bar{d}(y)z = (-1)^{x+y}\bar{d}x\bar{d}(y)z$ and $(x \vdash y) \vdash z = (-1)^{x+1}(\bar{d}x)y \vdash z = (-1)^{x+1}(-1)^{(x+(y+1)+1)+1}\bar{d}((\bar{d}x)y)z = (-1)^{x+1}(-1)^{((x+1)+y+1)+1}(-1)^{x+2}\bar{d}x\bar{d}yz = (-1)^{x+y}\bar{d}x\bar{d}(y)z$. Here we must be careful with minus sign. Indeed, $\bar{d}(x)y$ is a $x+y+1+1$ form because $\bar{d}$ maps forms of degree k into forms of degree $k+1$ as does the product itself too. The end of the proof is straightforward. □

Remark: Observe that the differential can be re-written: $(-1)^{x+1}\bar{d}(x \star y) = x \vdash y - x \dashv y$, for all $x, y \in \Omega^*$.

Remark: Fix $x, y \in \Omega^*$. Recall that in the case of an associative dialgebra, the bracket $[x, y]_L := x \dashv y - y \vdash x$ defines a Leibniz bracket which can be re-written as $[x, y]_L = (-1)^y\bar{d}(y) \star x - x \star \bar{d}(y)$. The following Proposition allows us to consider for a differential calculus just one product.

Proposition 3.3.13 *Let $x, y \in \Omega^*$, we have: $\bar{d}(x \dashv y) = -\bar{d}x \star \bar{d}y = \bar{d}(x \vdash y)$, $\bar{d}[x, y]_L = \bar{d}(y) \star \bar{d}(x) - \bar{d}(x) \star \bar{d}(y) = \bar{d}(x \dashv y - y \dashv x)$.*

Proof: Straigtforward. □

Remark: In our example, the associative products $(\dashv\,, \vdash)$ respect the $\mathbb{Z}_2$ grading of Ω^*. In addition to being an associative dialgebra, Ω^* is a (di)-superalgebra. If we embed $(\Omega^*, \dashv\,, \vdash)$ into a dendriform algebra (see appendix), with for example $\succ\, \equiv\, \vdash$ associative, we will get $\bar{d}(a \prec b) := \bar{d}(a \dashv b - a \vdash b) = 0$, i.e., $(\bar{d} \prec) : \Omega^* \otimes \Omega^* \to \Omega^*$ will give closed forms. Observe that in the case of the usual graded Leibniz algebra, the (associative) Fedosov product [2] turns it into a superalgebra. It is also an associative di-algebra with $x \dashv y := x\bar{d}(y)$ and $x \vdash y := (-1)^x\bar{d}(x)y$, see [45].

[2]In the case of a graded Leibniz algebra, the product of two forms x, y defined by $xy \pm (-1)^{\deg x}\bar{d}x\bar{d}y$ is associative. This product is the Fedosov product when the minus sign is chosen.

Theorem 3.3.14 *The k-vector space $(\Omega^*, \dashv)$ is an associative superalgebra. For all $x \in \Omega^*$, we define the linear map $x \mapsto \mathsf{Tr}(x) = \sigma^\natural(\bar{d}(x) \star I)$, where σ is a trace on A and $\natural$ is the universal cotrace defined in [61]. In this case, Tr is a closed trace on $(\Omega^*, \dashv)$ and vanishes on the Leibniz commutator, i.e., $\mathsf{Tr}[x, y]_L = 0$.*

Proof: Let $x, y \in (\Omega^*, \dashv)$, we have $\mathsf{Tr}(dx) = 0$ since $\bar{d}^2 = 0$. Fix the forms $x := a_0\omega(a_1, a_2)\dots \omega(a_{2n-1}, a_{2n})a_{2n+1}$ and $y := b_0\omega(b_1, b_2)\dots\omega(b_{2m-1}, b_{2m})b_{2m+1}$, we obtain:

$$\bar{d}(x) \star \bar{d}(y) \star I = \omega(a_0, I)\dots\omega(a_{2n+1}, I)\omega(b_0, I)\dots\omega(b_{2m+1}, I).$$

Therefore, $\mathsf{Tr}(x \dashv y) = -\tau^\natural(\bar{d}(x) \star \bar{d}(y) \star I) = -(-1)^{2(x+2)2(y+2)}\tau^\natural(\bar{d}(y) \star \bar{d}(x) \star I) = \mathsf{Tr}(y \dashv x)$, since in the graded algebra defined at the begining of this Section the Quillen curvature ω is a two-cochain. $\square$

Corollary 3.3.15 *Let $x, y \in (\Omega^*, \dashv)$, then $\mathsf{Tr}(x)$ is a cyclic cocycle of degree $2(\deg(x) + 2) - 1$ and $\mathsf{Tr}\ (x \dashv y)$ is a cyclic cocycle of degree $2(\deg(x) + \deg(y) + 4) - 1$.*

Proof: This is a consequence from [61] or Subsection 3.2.1. $\square$

Remark: Let $w = (a_0, (I, a_1, I, \dots, I, a_{n-2}), a_{n-1}) \in A \otimes A^{\otimes 2(n-2)} \otimes A$.
Define $x([w]) = a_0\omega(I, a_1)\omega(I, a_2)\dots\omega(I, a_{n-2})a_{n-1} \in \Omega^*$, then the functions f_n defined in Section 3.3.1 can be expressed in term of $\bar{d}x([w])$. Since $\omega(I, a_1) = \omega(a_1, I) = \rho(a_1)$ we get,

$$\begin{aligned} \bar{d}x([w]) &= \omega(I, a_0)\omega(I, a_1)\omega(I, a_2)\dots\omega(I, a_{n-2})a_{n-1} \\ &\quad +(-1)^{(n-2)+1}\omega(I, a_1)\omega(I, a_2)\dots\omega(I, a_{n-2})\omega(I, a_{n-1}) \\ &= \rho(a_0)\rho(a_1)\rho(a_2)\dots\rho(a_{n-2})a_{n-1} - (-1)^n a_0\rho(a_1)\rho(a_2)\dots\rho(a_{n-2})\rho(a_{n-1}), \end{aligned}$$

to be compared with $\tilde{f}_n\square_{*n}(a_0, a_1, \dots, a_{n-2}, a_{n-1}) = 0$.

3.4 The distributivity defect of a Leibniz-Ito derivative

Let (A, m) be a unital associative algebra with unit I. For all $a, b \in A$, we set $m(a \otimes b) := a \cdot b$.

We end the analogy between Leibniz and Leibniz-Ito derivatives by observing that for any element $a \in A$, $[a, \cdot] : b \mapsto a \cdot b - b \cdot a$ is a Leibniz derivative. Therefore, we obtain an embedding $L : A \to \mathsf{Hom}(A, A), \quad a \mapsto [a, \cdot]$. Defining the law $\circ$ such that for all $a, b \in A$, $a \circ b := [a, b]$, we get: $(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y$, meaning that the lack of associativity of the product $\circ$ can be controlled.

In the case of Leibniz-Ito derivatives, can we embed A into $\mathsf{Hom}(A, A)$ such that the maps involved describe Leibniz-Ito derivatives, instead of Leibniz derivatives ? Let $x, y \in A$. Let us re-write the Leibniz-Ito property by defining the law $*$ such that $x * y := \rho_x(y)$, where ρ_x is a Leibniz-Ito derivative. Re-writting the Leibniz-Ito property, we get:

Leibniz-Ito property for the law $\cdot$: $\quad x * (y \cdot z) - (x * y) \cdot (x * z) = (x * y) \cdot z + y \cdot (x * z), \ \forall y, z \in A,$

which means that the lack of distributivity of the product $*$ with respect to the associative product $\cdot$ of the algebra A can be controlled. In addition, as the neutral element for the law $\cdot$ is I we obtain $x * I := \rho_x(I) = 0$. As an example, let A^* be the group of invertible elements of A. Fix $x \in A^*$. Define the map $y \mapsto \rho_x(y) := xyx^{-1} - y := x * y$ from A to A. Then, ρ_x is a Leibniz-Ito derivative, i.e., the law $*$ verifies the Leibniz-Ito property for the law $\cdot$.

3.5 Conclusion

Motivated by a previous work [40], we have viewed a unital associative algebra (A, m) as a bi-dialgebra, $(A, \delta_f, \tilde{\delta}_f, m)$, where coassociative coproduct δ_f and $\tilde{\delta}_f$ were defined naturally from the definition of the unit element and the use of associativity of the product m.

Motivated by the use of Leibniz-Ito derivative in classical and quantum stochastic calculi, instead of the usual Leibniz derivatives, we have proposed an algebraic framework to compare these two types of derivatives.

From the two coproducts δ_f and $\tilde{\delta}_f$ of a unital associative algebra $(A, \delta_f, \tilde{\delta}_f, m)$, we have constructed a differential operator $\Box_*$, whose equation $\tilde{F}\Box_* = 0$ admits, roughly speaking, Leibniz derivatives but also Leibniz-Ito derivatives and homomorphisms from A to A, via their Quillen curvatures. Motivated by this differential operator and via Bianchi identity, we construct another complex and differential equations admitting as solutions functions constructed from Leibniz-Ito derivatives and the Bianchi identity naturally associated with their Quillen curvatures.

Reminiscent of what was done in the construction of cyclic cocycles from a Leibniz derivative, $x \mapsto [F, x]$, with F a well-chosen operator [12], we have constructed a differential associative di-algebra from the Quillen curvature of a Leibniz-Ito derivative. We have proved that the integral calculus defined from a closed trace yields cyclic cocycles and vanishes on particular commutator, not skew symmetric, natural generalisation of Lie commutator which appears in the theory of associative dialgebra [45]. We also relate the results of these cyclic cocycles on particular elements $(a_0, (I, a_1, I, \dots, I, a_{n-2}), a_{n-1}) \in A \otimes A^{2(n-2)} \otimes A$ to solutions of differential equations obtained in Subsection 3.3.1. Observe that if $A := kA_0 \oplus kI$ where A_0 is a set, then such elements $(a_0, (I, a_1, I, \dots, I, a_{n-2}), a_{n-1})$, with $a_i \in A_0$ can be interpreted as a periodic orbit $\dots, a_{n-2}, (I, a_1, I, \dots, I, a_{n-2}), I, a_1, \dots$ of the flower graph naturally associated with such algebras, see the introduction.

The Leibniz derivative versus Leibniz-Ito derivative can be pushed further by expressing the Leibniz-Ito property in terms of laws showing a distributivity defect of special laws with respect to the associative product m of the algebra A.

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3.6 Appendix

The aim of this appendix is to show a relation between associative dialgebras and dendriform algebras. Recall that an associative dialgebra is a k-vector space D equipped with two associative products $\vdash$ and $\dashv$ such that for all $x, y, z \in D$,

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
3. $(x \vdash y) \vdash z = (x \dashv y) \vdash z$,

Based on this idea we define, an associative *pre-dialgebra* of type I, (respectively of type III), i.e., a k-vector space equipped with two associative products verifying all the conditions of an associative dialgebra but maybe the last one (respectively the first one). Recall also that a dendriform algebra E is a k-vector space equipped with two binary operations, $\prec$, $\succ$: $E \otimes E \rightarrow E$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
2. $(a \succ b) \prec c = a \succ (b \prec c)$,
3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

Theorem 3.6.1 *Let $(D, \dashv, \vdash)$ be an associative pre-dialgebra of type I and let $a, b \in D$. The relations,*

$$a \prec b = a \dashv b, \qquad \textit{and} \qquad a \succ b = a \vdash b - a \dashv b,$$

embed D into a dendriform algebra. Similarly, if $(D, \dashv, \vdash)$ is an associative pre-dialgebra of type III, the relations,

$$a \succ b = a \vdash b, \qquad \textit{and} \qquad a \prec b = a \dashv b - a \vdash b,$$

embed D into a dendriform algebra. Conversely, any dendriform algebra with $\prec$ associative is an associative pre-dialgebra of type I and any dendriform algebra with $\succ$ associative is an associative pre-dialgebra of type III.

Before giving the proof, we need two auxiliary results.

Lemma 3.6.2 *Let $(D, \dashv, \vdash)$ be an associative pre-dialgebra of type I. Fix $a, b \in D$ With the relations,*

$$a \prec b = a \dashv b, \qquad \textit{and} \qquad a \succ b = a \vdash b - a \dashv b,$$

the first, respectively the second, axiom of an associative pre-dialgebra of type I is equivalent to the first, respectively the second, axiom of a dendriform algebra with $\prec$ associative.

Proof: Fix $a, b, c \in D$. If the law $\prec$ is associative we get $a \prec (b \succ c) = 0$. However, $a \prec (b \succ c) = 0$ is equivalent to $a \dashv (b \vdash c - b \dashv c) = 0$. This proves that the first axiom of an associative pre-dialgebra of type I is equivalent to the first axiom of a dendriform algebra. Similarly, $(a \succ b) \prec c - a \succ (b \prec c) = 0$ is equivalent to $(a \vdash b) \dashv c - a \vdash (b \dashv c) = 0$, since the product $\prec$ is supposed to be associative. $\square$

Lemma 3.6.3 *Let $(D, \dashv, \vdash)$ be an associative pre-dialgebra of type III. Fix $a, b \in D$. With the relations,*

$$a \succ b = a \vdash b, \qquad \textit{and} \qquad a \prec b = a \dashv b - a \vdash b,$$

the third, respectively the second, axiom of an associative pre-dialgebra of type III is equivalent to the third, respectively the second, axiom of a dendriform algebra with $\succ$ associative.

Proof: The proof is the same. We verify that $(a \prec b) \succ c = 0 \Leftrightarrow (a \dashv b) \vdash c - (a \vdash b) \vdash c = 0$ and so on. $\square$

Proof: (of Theorem 3.6.1) In the case of an associative pre-dialgebra of type I, the proof is completed by noticing that the third axiom of a dendriform algebra with $\prec$ associative is not enough to prove the third axiom of an associative dialgebra. Therefore from the axioms of a dendriform algebra with $\prec$ associative, we prove only the axioms of an associative pre-dialgebra of type I, i.e., the axioms 1 and 2 of an associative dialgebra. (Similarly for a dendriform algebra with $\succ$ associative.) $\square$

Chapter 4

Coassociative grammar, periodic orbits and quantum random walk over $\mathbb{Z}$

Abstract [1]**:**

Inspired by a work of Joni and Rota, we show that the combinatorics generated by a quantisation of the Bernoulli random walk over $\mathbb{Z}$ can be described from a coassociative coalgebra. Relationships between this coalgebra and the set of periodic orbits of the classical chaotic system $x \mapsto 2x \mod 1$, $x \in [0, 1]$ is also given.

4.1 Introduction and Notation

Motivated by the success of classical random walks and chaotic dynamical systems, we study the quantisation of the random walk over $\mathbb{Z}$ and its relationships with a classical chaotic system $x \mapsto 2x \mod 1$, $x \in [0, 1]$. In the physics literature, quantum random walk has been studied for instance, by Ambainis et al. [17], Konno et al., in [27, 28]. In [27], Konno shows that a quantum random walk over $\mathbb{Z}$, called also the Hadamard random walk, generates a particular combinatorics.

In [24], Joni and Rota showed that some combinatorics can be recovered from coproducts of coassociative coalgebras. Therefore, is it possible to create a coassociative coalgebra which recovers the combinatorics generated by the Hadamard random walk? We start in Section 4.2 by briefly recalling a new formalism, inspired by weighted directed graph theory. In Section 4.3, we present a mathematical framework for studying the Hadamard random walk over $\mathbb{Z}$. In Section 4.4, we construct a coassociative coalgebra based on results on graphs developed in Section 4.2. We show that the combinatorics generated by the Hadamard random walk over $\mathbb{Z}$

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can be recovered from this coalgebra. In Section 4.5, we present the notion of quantum graph developed in [50] and point out a relation between a quantum graph, the classical Bernoulli random walk, the Hadamard random walk and the periodic orbits of the classical chaotic system $x \mapsto 2x \mod 1$ with $x \in [0,1]$. Briefly speaking, we show how to relate periodic orbits of this classical chaotic system to polynomials describing a quantisation of the Bernoulli walk.

4.2 L-Coalgebras

Definition 4.2.1 [Directed graph] A *directed graph* G is a quadruple [54], (G_0, G_1, s, t) where G_0 and G_1 are two countable sets respectively called the *vertex set* and the *arrow set*. The two mappings, $s,\ t : G_1 \to G_0$ are respectively called *source* and *terminus*. A vertex $v \in G_0$ is a *source* (resp. a *sink*) if $t^{-1}(\{v\})$ (resp. $s^{-1}(\{v\})$) is empty. A graph G is said *locally-finite*, (resp. *row-finite*) if $t^{-1}(\{v\})$ is finite (resp. $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1,\ s(a) = v\}$. A *weight* associated with the vertex v is a mapping $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a *weighted directed graph*.

In the sequel, directed graphs will be supposed locally-finite and row-finite. Let us introduce particular coalgebras named L-coalgebras [2] and explain why this notion is interesting.

Definition 4.2.2 [L-coalgebra] A *L-coalgebra* $(L, \Delta, \tilde{\Delta})$ over a field k is a k-vector space composed of a right part (L, Δ), where $\Delta : L \to L^{\otimes 2}$, is called the right coproduct and a left part $(L, \tilde{\Delta})$, where $\tilde{\Delta} : L \to L^{\otimes 2}$, is called the left coproduct such that the coassociativity breaking equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is verified. If $\Delta = \tilde{\Delta}$, the L-coalgebra is said *degenerate*. A L-coalgebra may have two counits, the right counit $\epsilon : L \to k$, verifying $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : L \to k$, verifying $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$. A L-coalgebra is said *coassociative* if its two coproducts are coassociative. In this case the equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is called the **entanglement equation** and we will say that its right part (L, Δ) is entangled to its left part $(L, \tilde{\Delta})$. Denote by τ the *transposition* mapping, i.e., $L^{\otimes 2} \xrightarrow{\tau} L^{\otimes 2}$ such that $\tau(x \otimes y) = y \otimes x$, for all $x, y \in L$. The L-coalgebra L is said to be *L-cocommutative* if for all $v \in L$, $(\Delta - \tau\tilde{\Delta})v = 0$. A *$L$-bialgebra* (with counits $\epsilon,\ \tilde{\epsilon}$), is a L-coalgebra (with counits) and an unital algebra such that its coproducts and counits are homomorphisms. A *L-Hopf algebra*, H, is a L-bialgebra with counits equipped with right and left antipodes $S,\ \tilde{S} : H \to H$, such that: $m(id \otimes S)\Delta = \eta\epsilon$ and $m(\tilde{S} \otimes id)\tilde{\Delta} = \eta\tilde{\epsilon}$ or $m(S \otimes id)\Delta = \eta\epsilon$ and $m(id \otimes \tilde{S})\tilde{\Delta} = \eta\tilde{\epsilon}$.

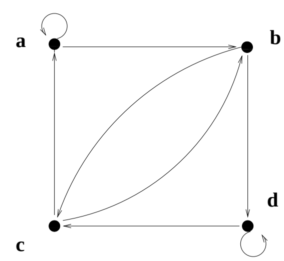
Let $G = (G_0, G_1, s, t)$ be a directed graph equipped with a family of weights $(w_v)_{v \in G_0}$. Let us consider the free k-vector space kG_0. The set G_1 is then viewed as a subset of $(kG_0)^{\otimes 2}$ by identifying $a \in G_1$ with $s(a) \otimes t(a)$. The mappings source and terminus are then linear mappings still called source and terminus $s, t :\ (kG_0)^{\otimes 2} \to kG_0$, such that $s(u \otimes v) = u$ and $t(u \otimes v) = v$, for all $u, v \in G_0$. The family of weights is given by $(w_v : F_v \to k)_{v \in G_0}$. Let $v \in G_0$. Define the right coproduct $\Delta_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\Delta_M(v) := \sum_{i : a_i \in F_v} \ w_v(a_i)\ v \otimes t(a_i)$ and the left coproduct $\tilde{\Delta}_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\tilde{\Delta}_M(v) := \sum_{i : a_i \in P_v} \ w_{s(a_i)}(a_i)\ s(a_i) \otimes v$, where P_v is the set $\{a \in G_1,\ t(a) = v\}$. With these definitions the k-vector space kG_0 is a L-coalgebra

[2]This notion has been introduced in [40] and developed in [40, 34, 39, 38].

called a *(finite) Markov L-coalgebra* since its coproducts Δ_M and $\tilde{\Delta}_M$ verify the coassociativity breaking equation $(\tilde{\Delta}_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M$. In addition, this particular coalgebra is called *finite* since for all $v \in G_0$, the sets F_v and P_v are finite and the coproducts have the form $\Delta_M(v) := v \otimes \cdots$ and $\tilde{\Delta}_M(v) := \cdots \otimes v$.

Assume we consider the Markov L-coalgebra just described and associate with each tensor product $\lambda u \otimes v$, where $\lambda \in k$ and $u, v \in G_0$, appearing in the definition of the coproducts, a directed arrow $u \xrightarrow{\lambda} v$. The weighted directed graph so obtained, called the *geometric support* of this L-coalgebra, is up to a graph isomorphism [3], the directed graph we start with. Therefore, general L-coalgebras generalise the notion of weighted directed graph. If $(L, \Delta, \tilde{\Delta})$ is a L-coalgebra generated as a k-vector space by an independent spanning set L_0, then its geometric support $Gr(L)$, is a directed graph with vertex set $Gr(L)_0 = L_0$ and with arrow set $Gr(L)_1$, the set of tensor products $u \otimes v$, with $u, v \in L_0$, appearing in the definition of the coproducts of L. As a coassociative coalgebra is a particular L-coalgebra, we naturally construct its directed graph. We draw attention to the fact that a directed graph can be the geometric support of different L-coalgebras.

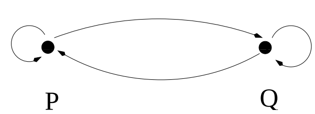
Example 4.2.3 The directed graph:



The graph $Gr(\mathcal{E})$.

is the geometric support associated with the degenerate L-coalgebra or coassociative coalgebra $\mathcal{E}$, spanned by the basis a, b, c and d, as a k-vector space, and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$ and the geometric support of the finite Markov L-coalgebra, spanned by the basis a, b, c and d, as a k-vector space, and described by the right coproduct: $\Delta_M a = a \otimes (a + b)$, $\Delta_M b = b \otimes (c + d)$, $\Delta_M c = c \otimes (a + b)$, $\Delta_M d = d \otimes (c + d)$ and the left coproduct: $\tilde{\Delta}_M a = (a + c) \otimes a$, $\tilde{\Delta}_M b = (a + c) \otimes b$, $\tilde{\Delta}_M c = (b + d) \otimes c$, $\tilde{\Delta}_M d = (b + d) \otimes d$.

Example 4.2.4 [The $(2,1)$-De Bruijn graph]



The $(2,1)$-De Bruijn graph

[3]A *graph isomorphism* $f : G \to H$ between two graphs G and H is a pair of bijections $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ such that $f_0(s_G(a)) = s_H(f_1(a))$ and $f_0(t_G(a)) = t_H(f_1(a))$ for all $a \in G_1$. All the directed graphs in this formalism will be considered up to a graph isomorphism.

This graph, also called the $(2,1)$-De Bruijn graph, is the geometric support of a Markov L-coalgebra, spanned by the basis P and Q. The (coassociative) coproducts are: $\Delta P = P\otimes P + P\otimes Q$ and $\Delta Q = Q\otimes P + Q\otimes Q$, $\tilde{\Delta} P = Q\otimes P + P\otimes P$ and $\tilde{\Delta} Q = P\otimes Q + Q\otimes Q$.

Remark: Let G be a finite Markov L-coalgebra. If the family of weights used for describing right and left coproducts take values in $\mathbb{R}_+$ and if the right counit $\epsilon : v \mapsto 1$ exists, then the geometric support associated with G is a directed graph equipped with a family of probability vectors. In addition, to enlarge the coassociative coalgebra setting, this algebraic formalism takes also into account the description of weighted paths on a given directed graph. We recall that the sequence, $\Delta_1 \equiv \Delta, \Delta_2 = id\otimes\Delta, \Delta_3 = id\otimes id\otimes \Delta, \ldots$, generates all possible weighted paths starting at any vertex. Similarly, the sequence of powers of $\tilde{\Delta}$, generates all the possible weighted paths arriving at a given vertex. We end this part on graph theory by recalling the definition of the line-extension.

Definition 4.2.5 [Line-extension] The *line-extension* of a directed graph $G := (G_0, G_1, s, t)$, with a denumerable vertex set G_0 and a denumerable arrow set $G_1 \subseteq G_0 \times G_0$ is the directed graph with vertex set G_1 and arrow set $G_2 \subseteq G_1 \times G_1$ defined by $(v, w) \in G_1 \times G_1$ belongs to G_2 if and only if $t(v) = s(w)$. This directed graph, called the line-directed graph of G, is denoted by $E(G)$.

Remark: The line-extension of the $(2,1)$-De Bruijn graph is $Gr(\mathcal{E})$, see also [41].

4.3 Quantum random walk over $\mathbb{Z}$

In the physics literature, quantum random walks were studied for instance, by Ambainis et al. [17], Konno et al., in [27, 28]. Here, we propose a mathematical framework for the quantum random walk over $\mathbb{Z}$ and show that the combinatorics [4] of this walk can be recovered by using the coproduct of $\mathcal{E}$.

Let $\mathcal{H}$ be a separable Hilbert space of infinite dimension with $(\,|n\rangle\,)_{n\in\mathbb{Z}}$ as an orthonormal basis. Consider the trivial tensor bundle $\mathcal{H}\otimes M_2(\mathbb{C})$. Fix a unitary matrix U and consider the operators P and Q, such that $U = P + Q$, with:

$$P = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ \gamma & \delta \end{pmatrix}.$$

They verify the following algebraic relations [27] $P^2 = \alpha P,\ Q^2 = \delta Q,\ PQP = \beta\gamma P,\ QPQ = \beta\gamma Q$.

Proposition 4.3.1 *Suppose $\alpha\delta \neq 0$. Consider $e_1 := \frac{1}{\alpha}P$ and $e_2 := \frac{1}{\delta}Q$. We get $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2e_1 = \lambda e_1$ and $e_2e_1e_2 = \lambda e_2$, where $\lambda := \frac{\gamma\beta}{\delta\alpha}$, i.e., the algebra generated by e_1, e_2 is a Jones algebra [23].*

[4] We keep the notation of [27].

Proof: Straightforward. □

Consider the algebra $\mathbb{C}\langle P,\ Q\rangle$, i.e., the non-commutative polynomials in P and Q and denote by $\mathcal{D}_-,\ \mathcal{D}_+$ the diffusion operators, i.e., the linear maps

$$\mathcal{D}_-,\ \mathcal{D}_+ :\ \mathcal{H}\otimes\mathbb{C}\langle P,\ Q\rangle \rightarrow \mathcal{H}\otimes\mathbb{C}\langle P,\ Q\rangle$$

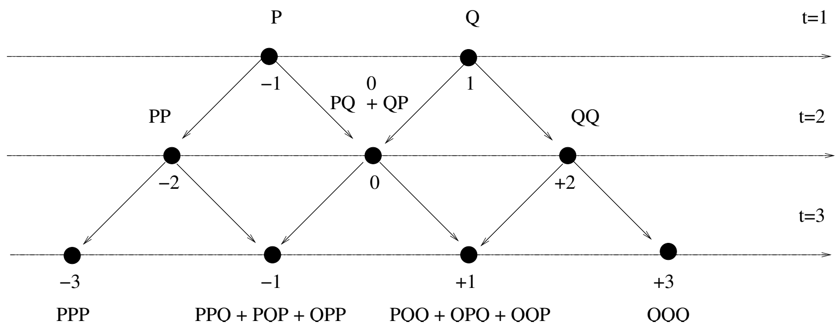
which are defined for all $k\in\mathbb{Z}$ and for all discrete time $n\in\mathbb{Z}$,

$$\mathcal{D}_-(\ |k+1\rangle\otimes\Xi_{[k+1;n]}\ ) = |k\rangle\otimes\Xi_{[k+1;n]}P,\ \mathcal{D}_+(\ |k-1\rangle\otimes\Xi_{[k-1;n]}\ ) = |k\rangle\otimes\Xi_{[k-1;n]}Q,$$

where $\Xi_{[0;0]} = id$, $\Xi_{[-1;1]} = P$, $\Xi_{[+1;1]} = Q$ and so on. The dynamics is defined by:

$$|k\rangle\otimes\Xi_{[k;n+1]} := \mathcal{D}_-(\ |k+1\rangle\otimes\Xi_{[k+1;n]}\ ) + \mathcal{D}_+(\ |k-1\rangle\otimes\Xi_{[k-1;n]}\ ).$$

That is [5]:



Combinatorics from the quantum random walk over $\mathbb{Z}$, up to $t = 3$.

Example 4.3.2 We yield here the nonzero polynomials $\Xi_{[k;n+1]}$ at time $t = 0,\ldots,4$. At time $t = 0$, we get, by convention, $\Xi_{[0;0]} = id$. At time $t = 1$, $\Xi_{[-1;1]} = P$, $\Xi_{[+1;1]} = Q$. At time $t = 2$, $\Xi_{[-2;2]} = P^2$, $\Xi_{[0;2]} = PQ + QP$ and $\Xi_{[+2;2]} = Q^2$. At time $t = 3$, $\Xi_{[-3;3]} = P^3$, $\Xi_{[-1;3]} = QP^2 + PQP + P^2Q$, $\Xi_{[+1;3]} = PQ^2 + QPQ + Q^2P$ and $\Xi_{[3;3]} = Q^3$. At time $t = 4$, $\Xi_{[-4;4]} = P^4$, $\Xi_{[-2;4]} = QP^3 + PQP^2 + P^2QP + P^3Q$, $\Xi_{[0;4]} = P^2Q^2 + PQPQ + PQ^2P + Q^2P^2 + QPQP + QP^2Q$, $\Xi_{[+2;4]} = PQ^3 + QPQ^2 + Q^2PQ + Q^3P$ and $\Xi_{[+4;4]} = Q^4$.

Denote by $S(\mathbb{C}^2)$, the set of vectors ψ of $S(\mathbb{C}^2)$ such that $\psi^\dagger\psi = 1$. The quantum random walk over $\mathbb{Z}$ from a state $\psi\in S(\mathbb{C}^2)$ is defined by the initial condition $\Psi_{space=0,time=0} := |0\rangle\otimes\psi$. At time n, this state will spread and the probability amplitude at position k described by $|k\rangle$ will be $\Psi_{k,n} := |k\rangle\otimes\Xi_{[k;n+1]}\psi$, (since $P^\dagger P + Q^\dagger Q = I$, the norm of the initial state is preserved.). We have an action from the bundle $\mathcal{H}_\mathbb{Z}\otimes\mathbb{C}\langle P,\ Q\rangle$ on $\mathbb{C}^2$ described by:

$$\mathcal{RW} :\ \mathcal{H}_\mathbb{Z}\otimes\mathbb{C}\langle P,\ Q\rangle\times\mathbb{C}^2 \rightarrow\ \mathcal{H}_\mathbb{Z}\otimes\mathbb{C}^2;\qquad (|k\rangle\otimes\Xi_{[k;n+1]},\psi)\mapsto |k\rangle\otimes\Xi_{[k;n+1]}\psi.$$

The total state is $\Psi^n_{total} := \sum_k \Psi_{k,n}$.

Proposition 4.3.3 *For all $x\in\mathbb{C}\langle P,\ Q\rangle$, we define the right polynomial multiplication R_x : $\mathbb{C}\langle P,\ Q\rangle\rightarrow\mathbb{C}\langle P,\ Q\rangle$, $y\mapsto yx$, we have, $[\mathcal{D}_+,\ \mathcal{D}_-] = id\otimes R_{[Q,P]}$.*

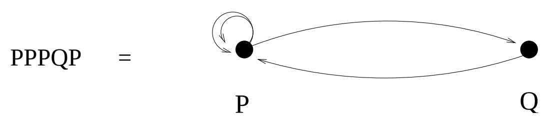
Proof: Straightforward. □

[5]Graphically, the Hilbert space $\mathcal{H}$ will be represented by the usual representation of $\mathbb{Z}$, a vertex $n\in\mathbb{Z}$ meaning $|n\rangle\in\mathcal{H}$.

4.4 Non-commutative polynomials and the reading of periodic orbits of $Gr(\mathcal{E})$

Remark: From now on, we forget the algebraic relation between P, Q. These monomials will be treaded simply as non-commutative symbols with no relation between them. Observe also that any monomial in $\mathbb{C}\langle P,\ Q\rangle$ is in one-to-one correspondence with a path of the $(2,1)$-De Bruijn graph.

Example 4.4.1 The monomial $PPPQP$ corresponds to the path:



Path associated with $PPPQP$**.**

of the $(2,1)$-De Bruijn graph.

The aim of this Section is to recover the polynomials $\Xi_{[k;n]}$, involved in the quantum random walk, from the periodic orbits of $Gr(\mathcal{E})$. We will show that the periodic orbits of this directed graph allow us to recover the combinatorics generated by the quantum random walk over $\mathbb{Z}$ and that this combinatorics is generated by the coassociative coproduct of $\mathcal{E}$. Recall that this coproduct is defined by:

$$\Delta a = a \otimes a + b \otimes c, \quad \Delta b = a \otimes b + b \otimes d, \quad \Delta c = d \otimes c + c \otimes a, \quad \Delta d = d \otimes d + c \otimes b.$$

However this directed graph can be also embedded into its natural Markov L-coalgebra. Recall that by definition, the right coproduct Δ_M verifies:

$$\Delta_M a = a \otimes (a+b), \quad \Delta_M b = b \otimes (c+d), \quad \Delta_M c = c \otimes (a+b), \quad \Delta_M d = d \otimes (c+d).$$

Definition 4.4.2 [Path space] Let us denote by $\mathcal{F}_n$ the free k-vector space spanned by all the monomials $x_1 \otimes \ldots \otimes x_n$, where for all $1 \leq i \leq n$, x_i stands for a, b, c, d and such that $x_1 \ldots x_n$ represents a path of length n of the graph $Gr(\mathcal{E})$. For instance $\mathcal{F}_0 := k\langle a, b, c, d\rangle$. We call the *path space* of the graph $Gr(\mathcal{E})$ the k-vector space $\mathcal{F} := \bigoplus_{n \geq 0} \mathcal{F}_n$.

Remark: As described in Section 4.2, we associate with each tensor product $P \otimes Q$, appearing in the definition of the coproducts, a directed arrow $P \to Q$, the relationship between the $(2,1)$-De Bruijn graph, whose vertex set is $\{P, Q\}$ and its line-extension is thus given by setting $a := P \otimes P$, $b := P \otimes Q$, $c := Q \otimes P$ and $d := Q \otimes Q$.

Often, for simplifying notation and only in the path space $\mathcal{F}$, we will write for instance xy instead of $x \otimes y$, where x, y stand for a, b, c, d. No confusion is possible since, in the sequel we forget the algebraic relations between letters. Define now the contraction map.

Definition 4.4.3 [Contraction map] For all $n > 1$, the *contraction map* is the linear map

$$\mathcal{C}: \ \mathcal{F} \to \mathbb{C}\langle P, \ Q \rangle, (y_1 \otimes y_2) \otimes (y_2 \otimes y_3) \otimes (y_3 \otimes y_4) \dots (y_{n-1} \otimes y_n) \mapsto y_1 y_2 \dots y_n,$$

where for all $1 \leq i \leq n$, y_i stands for P and Q.

Example 4.4.4 For instance, the contraction of $a \otimes b \otimes c := (P \otimes P) \otimes (P \otimes Q) \otimes (Q \otimes P)$ is equal to $PPQP$.

Proposition 4.4.5 *Fix a time $t > 1$. To any monomial Ξ constructed from P, Q in the algebra $\mathbb{C}\langle P, \ Q \rangle$, excepted of course P and Q, corresponds a unique monomial ω in $\mathcal{F}$ such that $\mathcal{C}(\omega) = \Xi$.*

Proof: Any monomial Ξ constructed from P, Q in the algebra $\mathbb{C}\langle P, \ Q \rangle$ corresponds to a unique path of the $(2,1)$-De Bruijn graph, i.e., a unique path of its line-extension. □

Lemma 4.4.6 *If x stands for a, b, c or d, we have the following equalities:*

$$\mathcal{C}(x \otimes a)P = \mathcal{C}(x \otimes a \otimes a); \qquad \mathcal{C}(x \otimes a)Q = \mathcal{C}(x \otimes a \otimes b);$$

$$\mathcal{C}(x \otimes b)P = \mathcal{C}(x \otimes b \otimes c); \qquad \mathcal{C}(x \otimes b)Q = \mathcal{C}(x \otimes b \otimes d);$$

$$\mathcal{C}(x \otimes c)P = \mathcal{C}(x \otimes c \otimes a); \qquad \mathcal{C}(x \otimes c)Q = \mathcal{C}(x \otimes c \otimes b);$$

$$\mathcal{C}(x \otimes d)P = \mathcal{C}(x \otimes d \otimes c); \qquad \mathcal{C}(x \otimes d)Q = \mathcal{C}(x \otimes d \otimes d).$$

Let x stands for a, b, c, d. We have $\mathcal{C}(x)(P + Q) = \mathcal{C}(\Delta_M x)$.

Proof: Straightforward. □

Define $(\Delta_M)_0 = id$, $(\Delta_M)_1 := \Delta_M$, $(\Delta_M)_2 := (id \otimes \Delta_M)\Delta_M$ and more generally, for all $n > 0$, $(\Delta_M)_n := (\underbrace{id \otimes \dots \otimes id}_{n-1} \otimes \Delta_M)(\Delta_M)_{n-1}$, similarly for the coassociative coproduct Δ.

Proposition 4.4.7 *We get $\Delta_M(a + b) = \Delta(a + b)$ and $\Delta_M(c + d) = \Delta(c + d)$. Moreover, $(id \otimes \Delta)\Delta_M = (id \otimes \Delta_M)\Delta_M$. The equalities $(id \otimes \Delta)\Delta(a + b) = (id \otimes \Delta_M)\Delta_M(a + b)$ and $(id \otimes \Delta)\Delta(c + d) = (id \otimes \Delta_M)\Delta_M(c + d)$ imply that $(\Delta_M)_n(a + b + c + d) = (\Delta)_n(a + b + c + d)$ for all $n \geq 0$.*

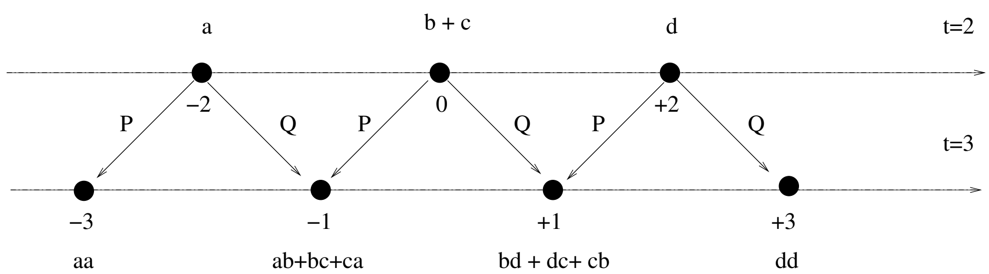
Proof: Straightforward. □

Corollary 4.4.8 *Fix $n > 1$. Denote by $\Sigma_n := \sum_{-n \leq k \leq n} \Xi_{[k,n]}$. Then $\Sigma_n = \mathcal{C}((\Delta_M)_{n-2}(a + b + c + d)) = \mathcal{C}((\Delta)_{n-2}(a + b + c + d))$.*

Proof: Straightforward. □

Remark: Set $n > 1$. The polynomial Σ_n is the sum of all the monomials $\Xi_{[k,n]}$, appearing exactly one time because of the Lemma 4.4.6, generated by the combinatorics of the quantum random walk over $\mathbb{Z}$. This sum can be computed by contraction of all the monomials from $\mathcal{F}$ present at time n and obtained either by applying the operator $(\Delta_M)_{n-2} \dots (\Delta_M)_2(\Delta_M)_1$ to $a+b+c+d$ or by applying the operator $\Delta_{n-2} \dots \Delta_2\Delta_1$ to $a+b+c+d$.

Here is the beginning of the combinatorics generated by the quantum random walk over $\mathbb{Z}$, viewed from the the path space $\mathcal{F}$.



The quantum random walk, coded in terms of the path space $\mathcal{F}$.

For the moment, we get all the sums Σ_n of monomials created by the walk. If a monomial is picked up from $\mathcal{F}$ or from a sum Σ_n, how can we say that it has to belong to such or such vertex? We have to enlarge the definition of $\mathcal{F}$ by defining an index map and an index path space. From now on, we denote by convention $x_{-1,-1} := a$, $x_{-1,+1} := b$, $x_{+1,-1} := c$, $x_{+1,+1} := d$ and observe that a monomial from $\mathcal{F}$ can be always written like $\omega := x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}$. The *index path space* $\hat{\mathcal{F}}$ is by definition the space $\mathcal{H} \otimes \mathcal{F}$.

Definition 4.4.9 [Index map] Let $\omega \in \mathcal{F}$, say, $\omega := x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}$. we define the (linear) *index map* as: $\widehat{\text{ind}} : \mathcal{F} \to \hat{\mathcal{F}}$, $\omega \mapsto (|\text{ind}(\omega)\,\rangle \otimes \omega)$, with $\text{ind}(\omega) = \text{ind}(x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}) := \sum_{k=1}^{n} i_k$.

Proposition 4.4.10 *Let $\omega := x_{i_1 i_2} \dots x_{i_{n-1} i_n} \in \mathcal{F}$. The index ind$(\omega)$ is equal to the number of Q minus the number of P obtained in the contraction of the monomial ω. Therefore, the index map fixes the vertex attributed by the quantum random walk over $\mathbb{Z}$.*

Proof: We will proceed by induction. It is true for $n = 2$, i.e., for a, b, c, d. Let ω be a monomial present at vertex k and at time $t = n > 2$. We suppose $\omega = x_{i_1 i_2} \dots x_{i_{n-1} i_n}$ and the index $\text{ind}(\omega) = k$ does indicate the number of Q minus the number of P obtained in the contraction of this monomial. At time $t = n + 1$, $\omega \mapsto \omega \otimes x_{i_n, i_{n+1}}$. By definition of the quantum random walk this monomial will be at vertex $k + 1$ if $x_{i_n, i_{n+1}}$ is equal to Q or $k - 1$ if $x_{i_n, i_{n+1}}$ is equal to P. Now $\text{ind}(\omega \otimes x_{i_n, i_{n+1}}) = \text{ind}(\omega) + i_{n+1}$. By definition, $i_{n+1} = +1$ for b and d which are monomials finishing by Q and $i_{n+1} = -1$ for a and c which are monomials finishing by P. □

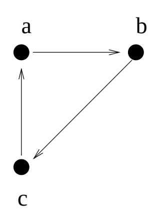
Example 4.4.11 For instance, $ind(a) = -2$ and $\mathcal{C}(a) = P^2$. Therefore, the monomial a has to be present at time $t = 2$. Moreover its contraction yielding the monomial P^2, a is at vertex -2, as expected.

By using the projector $|k\rangle\langle k| \otimes id$, we will recover all the monomials ω of the sum Σ_n with index $\mathrm{ind}(\omega) = k$. The next question is how can we produce all these monomials of $\mathcal{F}$ from the notion of periodic orbits of the $(2,1)$-De Bruijn graph.

Definition 4.4.12 [Periodic orbits, pattern] We define the equivalence relation $\sim$ in $\mathcal{F}$ by saying that $\omega_1 \sim \omega_2$ if and only if $\omega_1 = x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_1}$, for some n and $\exists m,\ \tau^m(\omega_1) = \omega_2$, where $\tau^m : \mathcal{F} \to \mathcal{F}$, is such that $y_1 \dots f_m \mapsto f_{p-m+1} \dots f_p f_1 \dots f_{p-m}$. The set $\mathbb{PO} = \mathcal{F}/\sim$ is the set of *periodic orbits* of the directed graph $Gr(\mathcal{E})$. We denote by $<\omega>$ the *pattern* of an equivalence classe associated with ω and its permutations, i.e., $<\omega>:=<x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_1}>$. A *periodic orbit* is just the graphical representation of the pattern. Often, we will confound the two words.

Remark: Fix a time $t > 1$. It is straightforward that the length of the pattern of a periodic orbit $<\omega>$ present at t denoted by $l(<\omega>)$, is equal to t.

Example 4.4.13 We have $a \otimes b \otimes c\ \sim\ c \otimes a \otimes b\ \sim\ b \otimes c \otimes a$. The equivalent classe is designed by the pattern $<a \otimes b \otimes c>$ and the associated periodic orbit is $\dots abcabcabcabc \dots$.



Periodic orbit, $<a \otimes b \otimes c>$, with pattern of length 3 .

This periodic orbit can be also represented by the pattern $<a \otimes b \otimes c \otimes a \otimes b \otimes c>$, i.e., we cover two times the triangle. Similarly, we have to enlarge the vector space of the periodic orbits $\mathbb{PO}$ to keep the notion of vertex attributed by the quantum walk to each periodic orbit. Denote by $\widehat{\mathbb{PO}} := \mathcal{H} \otimes \mathbb{PO}$ such a set.

Definition 4.4.14 [Index map in $\mathbb{PO}$] Define the (linear) *index map* $\widehat{\mathrm{Ind}}:\ \mathbb{PO} \to \widehat{\mathbb{PO}}$, $<\omega> \mapsto |\mathrm{Ind}(<\omega>)\rangle \otimes <\omega>$ with,

$$\mathrm{Ind}(<\omega>) := \mathrm{Ind}(<x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_1}>) := \frac{1}{2}((i_1 + i_2) + (i_2 + i_3) + \dots (i_{n-1} + i_1)) = \sum_{k=1}^{n-1} i_k.$$

This definition does not depend on the choice of the representative of the equivalent classe. Once we have the definition of periodic orbits, we have to read them to obtain information.

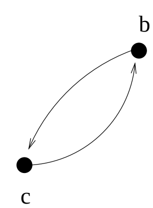
Definition 4.4.15 [Reading map] Let $<\omega>:=<x_{i_1i_2}x_{i_2i_3}\dots x_{i_ni_1}>$ be a periodic orbit. The *reading map* is denoted by $R:\mathbb{PO}\to F$ with $x_{i_1i_2}x_{i_2i_3}\dots x_{i_ni_1}\mapsto\sum_{k=1}^{N}x_{i_ki_{k+1}}x_{i_{k+1}i_{k+2}}\dots x_{i_{n+k-2}i_{k+n-1}}$, the labels being understood modulo n. Define $X_k:=x_{i_ki_{k+1}}x_{i_{k+1}i_{k+2}}\dots x_{i_{n+k-2}i_{k+n-1}}$ for all $1\le k\le n$, the integer N is equal to $\min\{k;X_i\neq X_j,1\le i,j\le k\}$. The reading map does not depend on the choice of the representative of the equivalent classe.

Proposition 4.4.16 *With the notation in the definition of the reading map we get* $X_{N+k \mod n}=X_k$.

Proof: Let us show that $X_{N+1}=X_1$. Indeed, if $k\le N$ is such that $X_{N+1}=X_k$ then it is straightforward to show that $X_{N+2-k}=X_1$ and $N+1-k=\min\{l;X_i\neq X_j,1\le i,j\le l\}$. Therefore, we get $N+2-k=N+1$, only possible for $k=1$. □

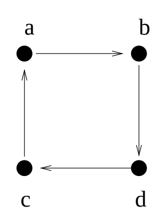
Example 4.4.17 Consider the periodic orbit $<abc>$. Its index is -1 and its reading yields $ab+bc+ca$. By contraction we obtain $PPQ+PQP+QPP$, which is exactly the polynomial expected at time $t=3$ and at vertex -1.

Example 4.4.18 The reading of the periodic orbit $\langle\, b\otimes c\otimes b\otimes c\,\rangle$, with a pattern of length 4, yields the monomials $X_1=b\otimes c\otimes b$, $X_2=c\otimes b\otimes c$, $X_3=b\otimes c\otimes b$ and $X_4=c\otimes b\otimes c$. Therefore, $N=2$ and $R\langle\, b\otimes c\otimes b\otimes c\,\rangle:=b\otimes c\otimes b+c\otimes b\otimes c$. By contraction, we obtain $PQPQ+QPQP$. Moreover its index is 0.



Periodic orbits of period 4, $<bcbc>$, with pattern of length 4.

The reading of the periodic orbit $\langle\, a\otimes b\otimes d\otimes c\,\rangle$, with pattern of length 4, yields $a\otimes b\otimes d+b\otimes d\otimes c+d\otimes c\otimes a+c\otimes a\otimes b$. By contraction, we obtain $PPQQ+PQQP+QQPP+QPPQ$. Its index is 0.



Periodic orbit, $<abdc>$, with pattern of length period 4.

That is the sum of these two periodic orbits give the polynomials expected at time $t=4$ and at vertex 0.

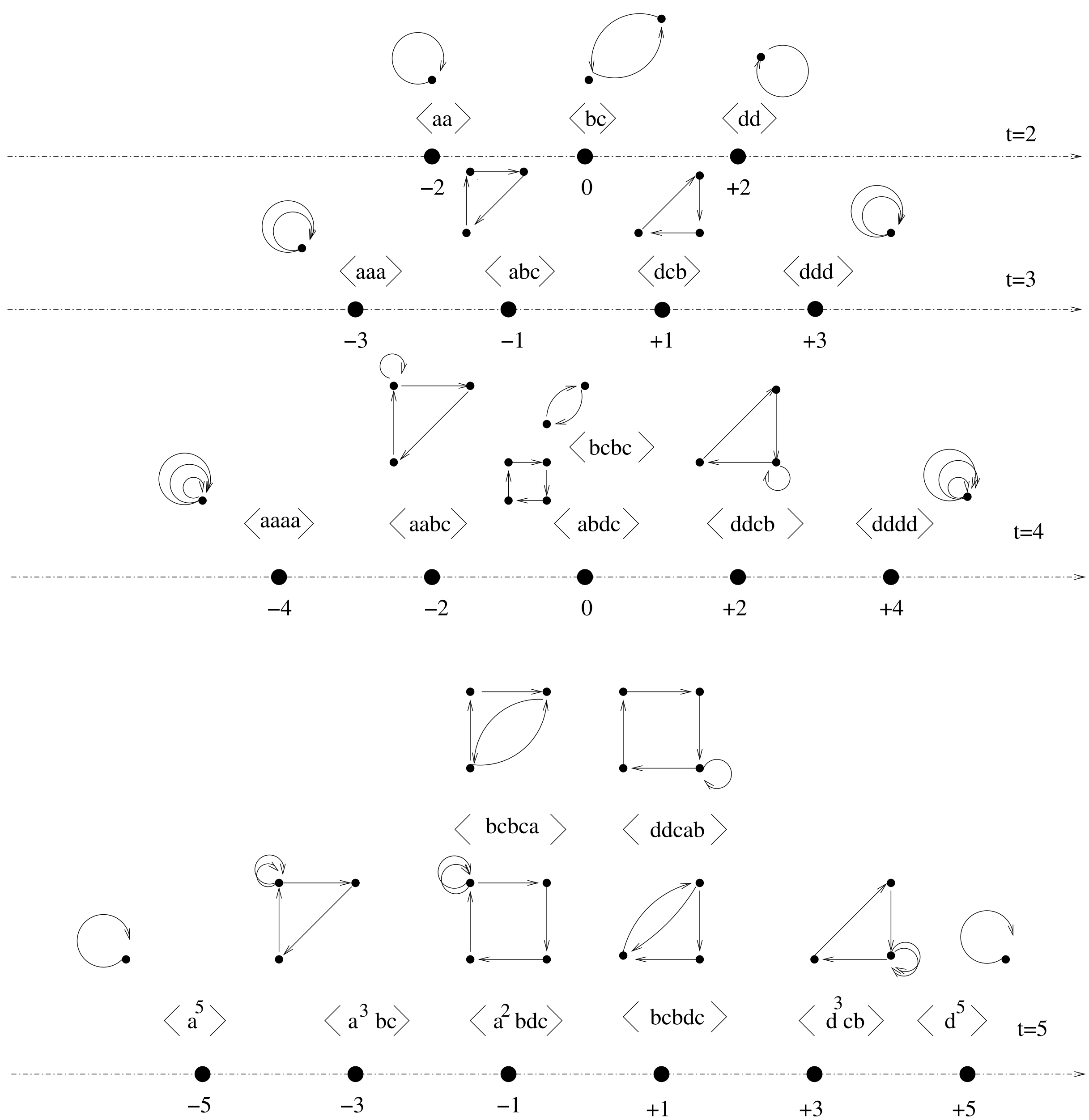
So as to give a minorant of the number of periodic orbits at a given time n at vertex k, we have to link the number k to the number x_P of P and the number x_Q of Q. Suppose k positive [6]. This means that $x_Q = x_P + k$. As $x_P + x_Q = n$, we get $x_P = \frac{n-k}{2}$. As this solution has to be an integer, this will fix the possibilites of k, i.e., the possible vertices reached by the quantum random walk at time n. Set $\kappa := \frac{n-k}{2}$.

Proposition 4.4.19 *Set* $\varpi := \frac{1}{n}\frac{n!}{\kappa!(n-\kappa)!}$. *The number of periodic orbits at the vertex* k *and at time* n *is greater than or equal to* ϖ *if this number is an integer and greater than or equal to the integer part of* $\varpi + 1$ *if* ϖ *is not an integer.*

Proof: For a given vertex k and a given time n, we have to have $\frac{n!}{\kappa!(n-\kappa)!}$ polynomials in P and Q and by definition we know that the reading of these periodic orbits yields at most n different monomials. □

In the following picture, we indicate the periodic orbits (of the line-extension) of the $(2, 1)$-De Bruijn graph involved in the quantum walk over $\mathbb{Z}$, up to t=5.

[6] As the walk is symetric, we have as many polynomials at vertex k as at vertex $-k$.



Periodic orbits and their associated pictures.

Proposition 4.4.20 *Let $< \omega > := < x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_n i_1} >$ be the pattern of a periodic orbit. Its reading yields $R < \omega > := \sum_{k=1}^{N} X_k$, where X_k is the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \dots$ $x_{i_{k-2+n \mod n} i_{k-1+n \mod n}}$. We have Ind$(< \omega >) :=$ ind(X_k), for all $k = 1, \dots, n$, i.e., the reading map decomposes the periodic orbit into monomials with same index.*

Proof: Let $< \omega > := < x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_n i_1} >$ be a periodic orbit and $X_1, \dots, X_N$ its decomposition under the reading map. We have $s_1 := \text{ind}(X_1) := \text{ind}(x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}) = \sum_{k=1}^{n} \ i_k$, $s_2 := \text{ind}(X_2) := \text{ind}(x_{i_2 i_3} x_{i_3 i_4} \dots x_{i_{n-1} i_n} x_{i_n i_1}) = \sum_{k=2}^{n} \ i_k + i_1$. Thus $s_1 = s_2$. Similarly, $s_k := \text{ind}(X_k) := i_k + i_{k+1} + i_{k+2} + \dots i_n + i_1 + \dots i_{k-1} = s_1$. Besides, by definition $\text{Ind}(< \omega >) := \frac{1}{2}(2\text{ind}(\omega))$. $\square$

Remark: A periodic orbit with index k will be assigned to the vertex k, by applying for example the projector $|k\rangle\langle k| \otimes id : \widehat{\mathbb{PO}} \to \widehat{\mathbb{PO}}$. Its reading will yield monomials of index k. For instance, the reading of the periodic orbit $< abc >$ is: $R(< abc >) := ab + bc + ca$. We have $\text{Ind}(< abc >) = -1$ and $\text{ind}(ab) = \text{ind}(ca) = \text{ind}(bc) = -1$.

Definition 4.4.21 [Completion] At time $t = n$ and a given vertex k, all the polynomials can be recovered by the contraction of monomials from $\mathcal{F}$ and having the same index k. Suppose we pick up one of the monomials present at this vertex k, say $x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}$. The *completion* Comp maps $x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n}$ to $< x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n} x_{i_n i_1} >$, thus it is a map from $\mathcal{F}$ to $\mathbb{PO}$. Similarly, we can define $\widehat{\text{Comp}} : \hat{\mathcal{F}} \to \widehat{\mathbb{PO}}$ by $\widehat{\text{Comp}} := id \otimes \text{Comp}$ thanks to the following Proposition.

Proposition 4.4.22 *$Ind(Comp(x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n})) = ind(x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{n-1} i_n})$.*

Proof: Straightforward. □

Remark: The reading of the completion of a monomial of $\mathcal{F}$ present at vertex k will yield many monomials of $\mathcal{F}$ present at this vertex. By contraction we will recover polynomials in P, Q present at vertex k. As all these polynomials in P, Q are coded bijectively through monomials of $\mathcal{F}$ present at the vertex k, the completion of all these monomials will form all the necessary periodic orbits whose reading will yield all the monomials of $\mathcal{F}$ present at k. For the time being, we started with the polynomial algebra $\mathbb{C}\langle P,\ Q\rangle$ and arrived at the periodic orbits set of the $(2,1)$-De Bruijn graph. We showed that the reading of all the patterns of periodic orbits present at a vertex k yielded all the monomials of $\mathcal{F}$ present at k and whose contraction gave the polynomials in P, Q.

Let us now equip the periodic orbits with the coproduct of $\mathcal{E}$. Thanks to this coproduct, we will be able to speak of the growth of periodic orbits and to recover the quantum random walk over $\mathbb{Z}$ from their reading.

Definition 4.4.23 [Growth of periodic orbit] We denote by $\mathcal{G} :\ \mathbb{PO} \to \mathbb{PO}$ the *growth operator*:

$$\mathcal{G}(< \omega >) = \sum_{k=1}^{l(<\omega>)} < \ {}^k\delta(\omega)\ >,$$

where ${}^k\delta = id \otimes \dots \otimes \underbrace{\Delta}_{k} \otimes \dots \otimes id$ and $l(< \omega >)$ is the length of the pattern of the periodic orbit $< \omega >$.

Remark: As $\Delta x_{ij} := \sum_{m=+1,-1} x_{im} \otimes x_{mj}$, it is easy to see that $\mathcal{G} :\ \mathbb{PO} \to \mathbb{PO}$. We take into account all the possible substitutions coming from the coassociative coproduct. Observe that $\sum_{k=1}^{l(<\omega>)} < \ {}^k\delta(\omega)\ >$ can be decomposed into $\sum_{k=1}^{l(<\omega>)} Y_{+1}^k$ and $\sum_{k=1}^{l(<\omega>)} Y_{-1}^k$ where $Y_{\pm 1}^k \in \mathbb{PO}$ and where ± 1 corresponds to the decomposition of the coproduct Δ.

Remark: Observe also that the coproduct transforms a letter of index k into two letters of index $k+1$ and $k-1$, i.e., the coassociative coproduct leaves the index invariant. For instance $\Delta b = ab + bd$ and $\text{Ind}(b) = 0 \mapsto (\text{Ind}(ab) = -1) + (\text{Ind}(bd) = +1)$.

Theorem 4.4.24 *The growth operator applied to all the periodic orbits at time $t = n$ will yield all the periodic orbits, perhaps with repetitions, at time $t = n + 1$.*

Proof: All the labels i in x_i will be understood modulo n, i.e., $x_{i \mod n}$. Let $\omega = x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n}$ be a monomial of $\mathcal{F}$ present at time $t = n$, its completion yields the periodic orbit $< x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} x_{i_n i_1} >$ whose reading will give us the monomials $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \ldots x_{i_{k+n-2} i_{k+n-1}}$, for $k = 1, \ldots, N$. By definition of the quantum random walk, we have to multiply them by P and Q to have the new polynomials present at time $t = n + 1$. Let see how it works on ω itself. We get $\mathcal{C}(x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n})P$ and $\mathcal{C}(x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n})Q$. These two polynomials come from the contraction of two monomials present at time $t = n+1$, $x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} x_{i_n i_m}$ and $x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} x_{i_n i_{m'}}$ with $m \neq m'$. By completion of these two monomials we get $< x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} x_{i_n i_m} x_{i_m i_1} >$ and $< x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} x_{i_n i_{m'}} x_{i_{m'} i_1} >$. The sum of these two periodic orbits is obviously equal to $< x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n} \Delta x_{i_n i_1} >$. Now, fix k. By computing the labels modulo n, the contraction of the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \ldots x_{i_{k+n-2} i_{k+n-1}}$, multiplied by P and Q come from the contraction of the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \ldots x_{i_{k+n-2} i_{k+n-1}} x_{i_{k+n-1} i_m}$ and the monomial $x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \ldots x_{i_{k+n-2} i_{k+n-1}} x_{i_{k+n-1} i_{m'}}$. The completion of these two monomials comes from the periodic orbit $< x_{i_k i_{k+1}} x_{i_{k+1} i_{k+2}} \ldots x_{i_{k+n-2} i_{k+n-1}} \Delta x_{i_{k+n-1} i_k} >$. The last term $\Delta x_{i_{k+n-1} i_k}$ is equal to $\Delta x_{i_{k-1} i_k}$. That is why the growth operator works on periodic orbits to recover monomials at time $t = n + 1$ from the periodic orbits at time $t = n$. □

Remark: It is worth noticing that the Markovian coproduct is closely related to the coassociative coproduct. Indeed, a monomial $x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{n-1} i_n}$, multiplied by P and Q will give the monomials $x_{i_1 i_2} x_{i_2 i_3} \ldots \Delta_M(x_{i_{n-1} i_n})$, see Lemma 4.4.6, whereas its completion will use the coassociative coproduct. For instance, consider the monomial ab. The contraction of ab multiplied by P and Q come from the contraction of abc and abd. Its completion will yield $abca$ and $abdc$, which is $< ab\Delta c >$. Thus $ab \to abc + abd = a\Delta_M(b) \xrightarrow{\text{Comp}} < ab\Delta c >$.

4.4.1 Reconstruction of the quantum walk from periodic orbits of $Gr(\mathcal{E})$

With the results we have obtained, we will now start with the set of periodic orbits of $Gr(\mathcal{E})$ to recover the combinatorics of the quantum random walk. We now consider the space $\hat{\mathbb{PO}}$ subvector space of $\mathcal{H} \otimes \mathbb{PO} := \widehat{\mathbb{PO}}$ spanned by the vectors $|\text{Ind}(< \omega >)\,\rangle \otimes < \omega >$, where $< \omega > \in \mathbb{PO}$. The growth operator is now extended to $\hat{\mathbb{PO}}$. $\hat{\mathcal{G}} : \hat{\mathbb{PO}} \to \hat{\mathbb{PO}}$, such that $\hat{\mathcal{G}}|\text{Ind}(< \omega >)\,\rangle \otimes < \omega > := \sum_{k=1}^{l(<\omega>)} (|\text{Ind}(Y_{+1}^k)\,\rangle \otimes Y_{+1}^k + |\text{Ind}(Y_{-1}^k)\,\rangle \otimes Y_{-1}^k)$. Define for all $n > 1$, $\mathbb{PO}_n$ the set of all patterns of index n and by convention $\mathbb{PO}_2 := \{< aa >, < bc >, < dd >\}$. Denote by $\Sigma'_n := \sum_{<\omega> \in \mathbb{PO}_{n-1}} \hat{\mathcal{G}}(\text{Ind}(< \omega >)\,\rangle \otimes < \omega >)$, $n > 2$. This sum is the sum of all patterns of periodic orbits, obtained perharps with repetition, present at time n. To avoid redundancy of information, define the (non-linear) operator $J : \Sigma'_n := \sum \lambda |\text{Ind}(< \omega >)\,\rangle \otimes < \omega > \mapsto \sum |\text{Ind}(< \omega >)\,\rangle \otimes < \omega >$, where λ are integers, in such a way that a pattern $< \omega >$, present in Σ'_n is present only one time in $J\Sigma'_n$. Therefore, for all $n > 1$, $J\Sigma'_n$ is the sum of all patterns of periodic orbits, present at time n. Apply now the projector $|k\,\rangle\langle\, k| \otimes id$ on $J\Sigma'_n$. Such a projector will yield all the orbits present at vertex k and at time n. By reading them, all the monomials in $\mathcal{F}$ present at time n at vertex k will be obtained. By contraction, we will obtain all the polynomials at time n at vertex k generated by the quantum random walk.

At time $t = 2$, we have two loops $< aa >$ and $< dd >$, and $< bc >$. Their readings yield a, d

and $b+c$. Their indexes are -2, 2 and 0. Their contractions yield PP, QQ and $PQ+QP$. At time $t=3$, the growth operator yields $\hat{\mathcal{G}}(|(-2)\rangle\otimes <aa>) \mapsto 2|(-3)\rangle\otimes <aaa> +2|(-1)\rangle\otimes <bca>$ and $\hat{\mathcal{G}}(|(2)\rangle\otimes <dd>) \mapsto 2|(3)\rangle\otimes <ddd> +2|(1)\rangle\otimes <cbd>$. Similarly, we obtain $\hat{\mathcal{G}}(|(0)\rangle\otimes <bc>)$. By applying J to Σ_3' we find $J\Sigma_3' = |(-3)\rangle\otimes <aaa> +|(-1)\rangle\otimes <bca> +|(3)\rangle\otimes <ddd> +|(1)\rangle\otimes <cbd>$.

For instance, apply the projector $|-1\rangle\langle -1|\otimes id$ to $J\Sigma_3'$. We find $|(-1)\rangle\otimes <bca>$. This pattern has to be present at time $t=3$ and at vertex -1. Its reading yields $ab+bc+ca$ all of index -1. Its contraction yields at time $t=3$ and at vertex -1, the polynomial $PPQ+PQP+QPP$ as expected. By applying the growth operator at time $t=3$, we will still obtain all the orbits present at time $t=4$, and so forth.

4.5 Relationships with classical systems

We would like to establish a link between the quantum random walk we studied and their classical counterparts. For convenience, we recall in the following two Subsections some results.

4.5.1 Classical random walk over $\mathbb{Z}$ and the Bernoulli shift

We consider the random walk over $\mathbb{Z}$, i.e., we consider $\Omega = \{-1,1\}^{\mathbb{N}}$ equipped with the product measure $\mu^{\otimes\mathbb{N}}$, where $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. We consider the sequence of iid random variables $(X_n)_{n\in\mathbb{N}}$, with:

$$X_n : \Omega \longrightarrow \{-1,+1\}, \quad \text{such that } X_n(\omega) = \omega_n.$$

It is well known that this process and the symbolic dynamics generated by $x \mapsto 2x \mod 1$ are isomorphic. Indeed consider the iid process defined by $Y_n : \Omega \longrightarrow \{0,+1\}, \quad$ such that $Y_n(\omega) = \omega_n' := \frac{\omega_n+1}{2}$. $\Omega = \{-1,1\}^{\mathbb{N}}$ becomes $\Omega' = \{0,1\}^{\mathbb{N}}$ and μ becomes $\mu' = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Define the measurable function:

$$\Phi : \Omega' \longrightarrow [0,1[, \quad \omega' \mapsto \sum_{n=0}^{\infty} \frac{\omega_n'}{2^{n+1}}.$$

The cylinder $C = [Y_0 = \omega_0'; \ldots; Y_l = \omega_l']$, being the set of sequences starting by $(\omega_0'; \ldots; \omega_l')$, will cover the interval $[\sum_{n=0}^{l} \frac{\omega_n'}{2^{n+1}}; \sum_{n=0}^{l} \frac{\omega_n'}{2^{n+1}} + \sum_{n=l+1}^{\infty} \frac{1}{2^{n+1}}]$. We notice that $\mathrm{Leb}(C) = \frac{1}{2^{l+1}}$, where Leb is the Lebesgue measure. Let us consider the shift θ defined by $(\theta\omega')_n = \omega_{n+1}'$. This shift leaves the Lebesgue measure of the cylinder C invariant if we write $\theta([Y_0 = \omega_0'; \ldots; Y_l = \omega_l']) = ([Y_1 = \omega_0'; \ldots; Y_{l+1} = \omega_l'])$. Moreover we have $\Phi(\theta(\omega')) = 2\Phi(\omega') \mod 1$.

The random walk over $\mathbb{Z}$ described by $(\Omega = \{-1,1\}^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}, \theta)$ is isomorphic to $(\Omega' = \{0,1\}^{\mathbb{N}}, \mu'^{\otimes\mathbb{N}}, \theta')$ which is isomorphic to the chaotic system $([0,1[, \beta[0,1[, \mathrm{Leb}, f : x \mapsto 2x \mod 1)$. As $\Phi\circ\theta' = f\circ\Phi$, the following diagram,

$$\begin{array}{ccc}
(\{0,1\}^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}) & \xrightarrow{\theta} & (\{0,1\}^{\mathbb{N}}, \mu^{\otimes\mathbb{N}}) \\
\Phi\big\downarrow & & \big\downarrow\Phi \\
([0,1[, \mathrm{Leb}) & \xrightarrow{f} & ([0,1[, \mathrm{Leb})
\end{array}$$

is commutative.

In [4], Biane proposes a non-commutative version of the Bernoulli process. Set $\Omega := \{+1, -1\}$, the probability space and define the probability $\mathbb{P}(\{+1\}) = p$ and $\mathbb{P}(\{-1\}) = q$. The process $X : \Omega \to \mathbb{R}$ is defined as $X(+1) = +1$ and $X(-1) = -1$. By identifying $(1,0)$ with $(4p)^{-\frac{1}{2}}(1+X)$ and $(0,1)$ with $(4q)^{-\frac{1}{2}}(1-X)$, the space $L^2(\Omega, \mathbb{P})$ is isomorphic to $\mathbb{C}^2$. We notice that the algebra $L^\infty(\Omega, \mathbb{P})$, acting on L^2 can be identified with the algebra of diagonal matrices of $M_2(\mathbb{C})$. A natural non-commutative generalisation consists in lifting this commutative algebra into a non-commutative one, i.e., $M_2(\mathbb{C})$. Notice that a subalgebra of $M_2(\mathbb{C})$ is also used in this framework.

4.5.2 Quantum graphs

Let B be a bistochastic matrix representing a directed graph, i.e., two vertices x_i and x_j are linked if and only if $B_{ij} \neq 0$. B is said unistochastic if there exists a unitary matrix U such that $B_{ij} = |U_{ij}|^2$. In this case, we say that the graph can be quantized. The notion of quantum graph was introduced as a toy model for studying quantum chaos by Kottos and Smilansky [29, 30]. This notion was also studied by Tanner [58] and by Barra and Gaspard [2] [3]. In this article we will follow another appraoch leading to quantum graphs put forward by Pakonski, Zyczkowski and Kus [50] concerning one dimensional dynamical systems. They consider a one-dimensional mapping f acting on $I = [0,1]$ such that $f : I \to I$, is piecewise linear. Moreover f verifies the following three conditions:

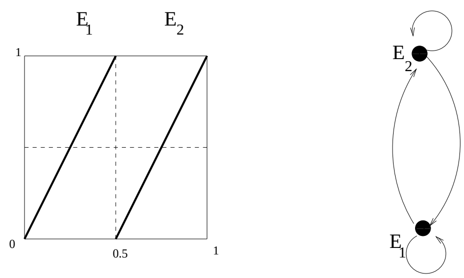
1. There exists a Markov partition of the interval I into M equal cells $E_i := [\frac{i-1}{M}, \frac{i}{M})$, $i = 1 \dots M$, with M a positive integer and f is linear on each cell E_i.
2. For all $y \in I$,
$$\sum_{x \in f^{-1}(y)} \frac{1}{f'(x)} = 1,$$
where f' is the right derivative, (defined almost everywhere on I).
3. The finite transfer matrix B, describing the action of f on the cells E_i is unistochastic.

Remark: On each cell, f concides with the function $f_i : [\frac{i-1}{M}, \frac{i}{M}) \to I$, $x \mapsto c_i x + b_i$ where the c_i have to be nonzero integers and the b_i are rational. The unistochastic matrix B is a M by M matrix and $B_{ij} = \frac{1}{|c_i|}$. Therefore, the probability of visiting the cell E_j from E_i is equal to $\frac{1}{f'(x)}$, with $x \in E_i$ and $f(x) \in E_j$.

Remark: The Kolmogorov-Sinai-entropy of the Markov chain generated by the bistochatsic matrix B [26] is
$$H_{\mathrm{KS}} = -\sum_{i=1}^{M} \tilde{p}_i \sum_{j=1}^{M} B_{ij} \log B_{ij},$$
where $\tilde{p}$ is the normalized left eigenvector of B such that $\tilde{p}B = \tilde{p}$, with $\sum_{i=1}^{M} \tilde{p}_i = 1$. This equation gives the dynamical entropy of the system since the Markov partition on M equal cells is a generating partition of the system. As the transition matrix is bistochastic, all the

components of $\tilde{p}$ are equal to $\frac{1}{M}$. Thus $H_{\mathrm{KS}} = 0$ if and only if all the $B_{ij} \in \{0, 1\}$. This entails that $|f'(x)| = 1$, i.e., the system is regular. With the conditions stated above, the converse is true.

Example 4.5.1 [Chaotic system, the Bernoulli shift]
The Bernoulli shift is described by $f : x \mapsto 2x \mod 1$, where $x \in [0, 1]$. This map is associated with the unistochastic matrix $B_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

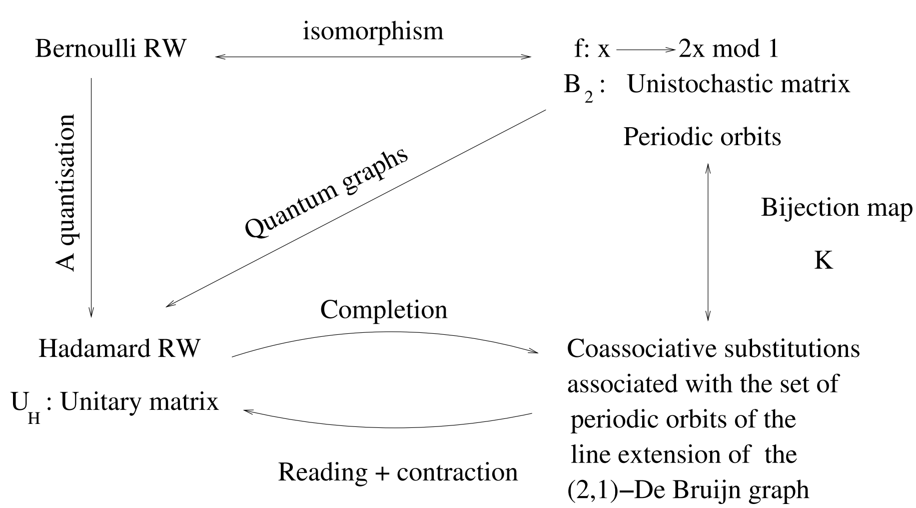


We recall an important result of [50].

Theorem 4.5.2 *With the assumptions described in 4.5.2 on the piecewise linear map f, to every periodic orbit of period n of the dynamical system described by f, corresponds a unique periodic orbit of period n of the directed graph described by the associated unistochastic matrix, i.e., by the $(2, 1)$-De Bruijn graph.*

We denote by K such a bijection. That is $K : \mathbb{PO} \to \mathbb{PO}[x \mapsto 2x \mod 1]$, where $\mathbb{PO}[x \mapsto 2x \mod 1]$ denote the k-vector space of patterns associated with the periodic orbits of the classical system $x \mapsto 2x \mod 1$.

Remark: Let B_2 be the unistochastic matrix associated with the chaotic map $x \mapsto 2x \mod 1$. One of the possible quantisation of this chaotic map is the so-called Hadamard matrix, $U_{\mathrm{H}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We now summarise our results in the following diagram:



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4.6 Link with language theory

In this appendix, we recall briefly the notion of grammar and define "linear extension of a grammar". We briefly show the points between grammar theory and the previous work. See also [42].

Recall that a *grammar* with non-terminal symbols is a 4-tuple $G = (W, V, U, n_0)$, where W is a finite set (its elements are called non-terminal symbols), V is also a finite set such that $V \cap W = \emptyset$. The production set U is a subset of pairs $u = (\alpha \to \beta)$ where $\alpha \in S = W \cup V$ containing at least one symbol of W and $\beta \in S^*$. The finite set S plays the rôle of the alphabet and S^* is the set of all finite strings over the alphabet S including the empty one. The symbol n_0 belongs to W.

A grammar is called *right linear* if all the productions are of the form $n \to \alpha m$, where $n, m \in W$ and $\alpha \in V^*$. It is called *context free* if all the productions are of the form $n \to \alpha$, where $n \in W$ and $\alpha \in V^*$. The *language* $L(G)$ *generated by* G is the set of all sentences generated by G. Two sentences being concatenation of symbols of S.

In this chapter, we use implicitely grammar theory and its "linear extension" defined as follows. Let us embed $S = W \cup V := \{A, B, C, D\} \cup \{a, b, c, d\}$ into its free k-vector space. Consider the Fock space $F(G) := \bigoplus_n kS^{\otimes n}$. Define the concatenation product of two symbols of S, β and α as $\beta \otimes \alpha$. The space S^* is then the space of all the finite sentences $\beta_1 \otimes \ldots \otimes \beta_k$, $\beta_i \in S$. The *language* $L(G)$ *generated by* G is the k-vector space generated by all the sentences generated by G. The element n_0 is then viewed as an element of kW.

Now consider the following production rules:

$$\begin{aligned} U_M(\mathcal{F}) \ &:= \ \{A \mapsto a \otimes A,\ A \mapsto a \otimes B,\ B \mapsto b \otimes C,\ B \mapsto b \otimes D,\ C \mapsto c \otimes A, \\ &\quad\ C \mapsto c \otimes B,\ D \mapsto d \otimes D,\ D \mapsto d \otimes C\}, \end{aligned}$$

and the so-called *coassociative production rules*:

$$\begin{aligned} U_c(\mathcal{F}) \ &:= \ \{A \mapsto a \otimes A,\ A \mapsto b \otimes C,\ B \mapsto b \otimes D,\ B \mapsto a \otimes B,\ C \mapsto c \otimes A, \\ &\quad\ C \mapsto d \otimes C,\ D \mapsto d \otimes D,\ D \mapsto c \otimes B\}. \end{aligned}$$

Consider the non-terminal symbol START $n_0 := A+B+C+D$. Then the language $L(G, U_M(\mathcal{F}), n_0)$ generated by n_0, i.e., all the sentences of $L(G)$ obtained by applying the production rules $U_M(\mathcal{F})$ to the start symbol $n_0 := A+B+C+D$ is the same that the language $L(G, U_c(\mathcal{F}), n_0)$ generated by n_0, see Proposition 4.4.8. In addition, these grammars are right linear.

Observe also that the grammar constructed on the set of periodic orbits, or patterns of the graph $\mathcal{F}$ is context free. The production rules are still given by the coassociative coproduct of $\mathcal{F}$,

$$\begin{aligned} U_c(\text{Per orbs}) \ &:= \ \{A \mapsto A \otimes A,\ A \mapsto B \otimes C,\ B \mapsto B \otimes D,\ B \mapsto A \otimes B,\ C \mapsto C \otimes A, \\ &\quad\ C \mapsto D \otimes C,\ D \mapsto D \otimes D,\ D \mapsto C \otimes B\}. \end{aligned}$$

The language of periodic orbits is the language generated by the sentence $A \otimes A + B \otimes C + D \otimes D$.

Chapter 5

Tiling the $(n^2, 1)$-De Bruijn graph with n coassociative coalgebras

Abstract [1] **:**

We construct, via usual graph theory a class of associative dialgebras as well as a class of coassociative L-coalgebras, the two classes being related by the graph theoretical tools called the line-extension. As a Corollary, a tiling of the n^2-De Bruijn graph with n (geometric supports of) coassociative coalgebras is obtained. We get, via the tilling of the $(3, 1)$-De Bruijn graph, an example of cubical trialgebra defined by Loday and Ronco. Other examples are obtained by letting $M_n(k)$ act on the axioms defining such tilings. Examples of associative products which split into several associative ones are also given.

5.1 Introduction

The field k stands for the real field or the complex field. Moreover, all the vector spaces considered in this paper will have a finite or a denumerable basis. Let us recall the general setting of this article by summarising the main steps of our previous work [40, 34, 39, 38].

Definition 5.1.1 [Directed graph] A *directed graph* G is a quadruple, see for instance [54], (G_0, G_1, s, t) where G_0 and G_1 are two denumerable sets respectively called the *vertex set* and the *arrow set*. The two mappings, $s,\ t : G_1 \to G_0$ are respectively called *source* and *terminus*. A vertex $v \in G_0$ is a *source* (resp. a *sink*) if $t^{-1}(\{v\})$ (resp. $s^{-1}(\{v\})$) is empty. A graph G is said *locally-finite*, (resp. *row-finite*) if $t^{-1}(\{v\})$ is finite (resp. $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1,\ s(a) = v\}$. A *weight* associated with the vertex v is a mapping $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a *weighted directed graph*.

[1] *2000 Mathematics Subject Classification: 16W30; 05C20; 05C90. Key words and phrases:* directed graphs, Hopf algebra, (Markov) L-coalgebra, coassociative co-dialgebra, cubical trialgebra, achirality.

In the sequel, directed graphs will be supposed locally-finite and row-finite. Let us introduce particular coalgebras named L-coalgebras [2] and explain why this notion is interesting.

Definition 5.1.2 [**L-coalgebra**] A *L-coalgebra* $(L, \Delta, \tilde{\Delta})$ over a field k is a k-vector space composed of a right part (L, Δ), where $\Delta : L \to L^{\otimes 2}$, is called the right coproduct and a left part $(L, \tilde{\Delta})$, where $\tilde{\Delta} : L \to L^{\otimes 2}$, is called the left coproduct such that the coassociativity breaking equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is verified. If $\Delta = \tilde{\Delta}$, the L-coalgebra is said *degenerate*. A L-coalgebra may have two counits, the right counit $\epsilon : L \to k$, verifying $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : L \to k$, verifying $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$. A L-coalgebra is said *coassociative* if its two coproducts are coassociative. In this case the equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is called the **entanglement equation** and we will say that its right part (L, Δ) is entangled to its left part $(L, \tilde{\Delta})$. Denote by τ, the *transposition* mapping, i.e., $L^{\otimes 2} \xrightarrow{\tau} L^{\otimes 2}$ such that $\tau(x \otimes y) = y \otimes x$, for all $x, y \in L$. The L-coalgebra L is said to be *L-cocommutative* if for all $v \in L$, $(\Delta - \tau\tilde{\Delta})v = 0$. A *$L$-bialgebra* (with counits ϵ, $\tilde{\epsilon}$), is a L-coalgebra (with counits) and an unital algebra such that its coproducts and counits are homomorphisms. A *L-Hopf algebra*, H, is a L-bialgebra with counits equipped with right and left antipodes S, $\tilde{S} : H \to H$, such that: $m(id \otimes S)\Delta = \eta\epsilon$ and $m(\tilde{S} \otimes id)\tilde{\Delta} = \eta\tilde{\epsilon}$ or $m(S \otimes id)\Delta = \eta\epsilon$ and $m(id \otimes \tilde{S})\tilde{\Delta} = \eta\tilde{\epsilon}$.

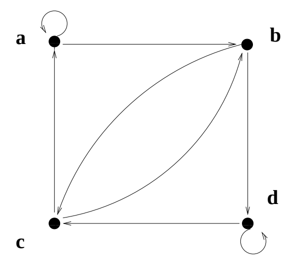
Let $G = (G_0, G_1, s, t)$ be a directed graph equipped with a family of weights $(w_v)_{v \in G_0}$. Let us consider the free k-vector space kG_0. The set G_1 is then viewed as a subset of $(kG_0)^{\otimes 2}$ by identifying $a \in G_1$ with $s(a) \otimes t(a)$. The mappings source and terminus are then linear mappings still called source and terminus $s, t :\ (kG_0)^{\otimes 2} \to kG_0$, such that $s(u \otimes v) = u$ and $t(u \otimes v) = v$, for all $u, v \in G_0$. The family of weights is given by $(w_v : F_v \to k)_{v \in G_0}$. Let $v \in G_0$. Define the right coproduct $\Delta_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\Delta_M(v) := \sum_{i: a_i \in F_v}\ w_v(a_i)\ v \otimes t(a_i)$ and the left coproduct $\tilde{\Delta}_M : kG_0 \to (kG_0)^{\otimes 2}$, such that $\tilde{\Delta}_M(v) := \sum_{i: a_i \in P_v}\ w_{s(a_i)}(a_i)\ s(a_i) \otimes v$, where P_v is the set $\{a \in G_1,\ t(a) = v\}$. With these definitions the k-vector space kG_0 is a L-coalgebra called a finite Markov L-coalgebra since its coproducts Δ_M and $\tilde{\Delta}_M$ verify the coassociativity breaking equation $(\tilde{\Delta}_M \otimes id)\Delta_M = (id \otimes \Delta_M)\tilde{\Delta}_M$. This particular coalgebra is called in addition finite Markov (L-coalgebra) because for all $v \in G_0$, the sets F_v and P_v are finite and the coproducts have the form $\Delta_M(v) := v \otimes \cdots$ and $\tilde{\Delta}_M(v) := \cdots \otimes v$.

Assume we consider the Markov L-coalgebra just described and associate with each tensor product $\lambda u \otimes v$, where $\lambda \in k$ and $u, v \in G_0$, appearing in the definition of the coproducts, a directed arrow $u \xrightarrow{\lambda} v$. The weighted directed graph so obtained, called the *geometric support* of this L-coalgebra, is up to a graph isomorphism [3], the directed graph we start with. Therefore, general L-coalgebras generalise the notion of weighted directed graph. If $(L, \Delta, \tilde{\Delta})$ is a L-coalgebra generated as a k-vector space by an independent spanning set L_0, then its geometric support $Gr(L)$ is a directed graph with vertex set $Gr(L)_0 = L_0$ and with arrow set $Gr(L)_1$, the set of tensor products $u \otimes v$, with $u, v \in L_0$, appearing in the definition of the coproducts of L. As a coassociative coalgebra is a particular L-coalgebra, we naturally construct its directed graph. We draw attention to the fact that a directed graph can be the geometric support of different L-coalgebras.

[2]This notion has been introduced in [40] and developed in [40, 34, 39, 38].

[3]A *graph isomorphism* $f : G \to H$ between two graphs G and H is a pair of bijection $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ such that $f_0(s_G(a)) = s_H(f_1(a))$ and $f_0(t_G(a)) = t_H(f_1(a))$ for all $a \in G_1$. All the directed graphs in this formalism will be considered up to a graph isomorphism.

Example 5.1.3 The directed graph:



is the geometric support associated with the degenerate L-coalgebra or coassociative coalgebra $\mathcal{F}$, spanned by the basis a, b, c and d, as a k-vector space, and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$ and the geometric support of the finite Markov L-coalgebra, spanned by the basis a, b, c and d, as a k-vector space, and described by the right coproduct: $\Delta_M a = a \otimes (a + b)$, $\Delta_M b = b \otimes (c + d)$, $\Delta_M c = c \otimes (a + b)$, $\Delta_M d = d \otimes (c + d)$ and the left coproduct: $\tilde{\Delta}_M a = (a + c) \otimes a$, $\tilde{\Delta}_M b = (a + c) \otimes b$, $\tilde{\Delta}_M c = (b + d) \otimes c$, $\tilde{\Delta}_M d = (b + d) \otimes d$.

Remark: Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra generated as a k-vector space by an independent spanning set $\mathcal{G}_0$. If the family of weights $(w_v)_{v \in \mathcal{G}_0}$, $(\tilde{w}_v)_{v \in \mathcal{G}_0}$, used for describing right and left coproducts, take values into $\mathbb{R}_+$ and if the right counit $\epsilon : v \mapsto 1$ exists, then the geometric support associated with $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ is a directed graph equipped with a family of probability vectors described by $(w_v)_{v \in \mathcal{G}_0}$.

The L-cocommutativity can be interpreted, in the case of a finite Markov L-coalgebra, in the following way. A directed graph is said bi-directed if for any arrow from a vertex v_1 to a vertex v_2, there exists an arrow from v_2 to v_1. To take into account the bi-orientation of a directed graph in an algebraic way, we first embed this directed graph into the finite Markov L-coalgebra described above. We then notice that a directed graph is bi-directed if and only if $\Delta_M = \tau \tilde{\Delta}_M$. Therefore, in this algebraic framework, we are naturally led to consider the L-cocommutator $\ker(\Delta - \tau\tilde{\Delta})$. Dualizing this formula leads to consider an (L-)algebra D equipped with two products $\vdash$ and $\dashv$, verifying $(x \vdash y) \dashv z = x \vdash (y \dashv z)$, $x, y, z \in D$, and to consider the particular commutator $[x, y] := x \dashv y - y \vdash x$. The bracket, $[-, z]$, verifies the analogue of the "Jacobi identity", called the Leibniz identity, i.e., $[[x, y], z] = [[x, z], y] + [x, [y, z]]$, if D is an algebra called an associative dialgebra [45].

Another motivation concerning associative dialgebras is the following. In a long-standing project whose ultimate aim is to study periodicity phenomena in algebraic K-theory, Loday in [45], and Loday and Ronco in [48] introduce several kind of algebras, one of which is the "non-commutative Lie algebras", called *Leibniz algebras*. Such algebras D are described by a bracket $[-, z]$ verifying the so called Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and the Leibniz algebra turns out to be a Lie algebra. A way to construct such Leibniz algebra is to start with an *associative dialgebra*, that is a k-vector space D equipped with two associative products, $\vdash$ and $\dashv$, such that for all $x, y, z \in D$

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
3. $(x \dashv y) \vdash z = (x \vdash y) \vdash z$.

The associative dialgebra is then a Leibniz algebra by defining the bracket $[x, y] := x \dashv y - y \vdash x$, for all $x, y \in D$. The operad associated with associative dialgebras is then Koszul dual to the operad associated with dendriform algebras, a *dendriform algebra* Z being a k-vector space equipped with two binary operations, $\prec$, $\succ$: $Z \otimes Z \to Z$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
2. $(a \succ b) \prec c = a \succ (b \prec c)$,
3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

This notion dichotomises the notion of associativity since the product $a * b = a \prec b + a \succ b$, for all $a, b \in Z$ is associative. Otherwise stated, the associative product $*$ splits into two operations $\prec$ and $\succ$.

The first result of this paper is the construction, via Markov L-coalgebras, i.e., via usual graph theory of a class of L-cocommutative and coassociative codialgebras as well as a class of coassociative L-coalgebras, the two classes being related by a tool from graph theory called the line-extension. The second one is the construction of a tiling of the $(n^2, 1)$-De Bruijn graph with n (geometric supports of) coassociative coalgebras. As a Corollary, we obtain examples of cubical trialgebras, a notion developed by Loday and Ronco in [48] and splittings of associative products into several associative ones.

Let us briefly introduce the organisation of the paper. In Section 5.2, we display the notion of coassociative co-dialgebras and recall the definition of the De-Bruijn graphs. We prove that the $(2, 1)$-De Bruijn graph, viewed as a Markov co-dialgebra, yields, by line-extension, the geometric support of the coassociative coalgebra $\mathcal{F}$, spanned by the basis a, b, c and d, as a k-vector space, and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$. Inspired by a previous work [39], relationships between the Markovian coproduct of the $(2, 1)$-De Bruijn graph and the coassociative coproduct of $\mathcal{F}$ are also given. In Section 5.3, we prove the existence of another structure associated with $\mathcal{F}$. This structure is also a coassociative coalgebra, called by convention the left structure of $\mathcal{F}$. As the two structures of $\mathcal{F}$ are entangled by the entanglement equation , we conclude that $\mathcal{F}$ is a coassociative L-coalgebra. Moreover, gluing their two associated geometric supports yields the $(4, 1)$-De Bruijn graph. Since the intersection of the two arrow sets of their geometric supports is empty, we assert that the $(4, 1)$-De Bruijn graph can be tiled by the two geometric supports of the coassociative L-coalgebra $\mathcal{F}$.

The consequences of such a left structure on $Sl_q(2)$ are also explored. If $\mathcal{F}$ stands for $Sl_q(2)$, we notice that the usual algebraic relations of the Hopf algebra $Sl_q(2)$ do not embed this new structure into a bialgebra. However, it allows us to construct a map which verifies the same axioms as an antipode map. We yield also a left structure for $SU_q(2)$.

This work ends, in Section 5.4, by showing that the $(n^2, 1)$-De Bruijn graph can also be tiled with n (geometric supports of) coassociative coalgebras. As a consequence, we give examples of cubical trialgebras and put forward an important notion which is the *achirality* of a L-coalgebra. We give also Leibniz co-derivative on these directed graphs.

5.2 On the De Bruijn graph families

The description of weighted directed graphs and coassociative coalgebras can be embedded into the L-coalgebra framework. In [40], we establish an important Theorem.

Proposition 5.2.1 *Any coassociative coalgebra (C, Δ_C), with a group-like element can be embedded into a non-degenerate L-coalgebra.*

Proof: Let (C, Δ_C) be a coassociative coalgebra. Suppose e is a group-like element, i.e., $\Delta_C e = e \otimes e$. Define the coassociative coproducts $\delta(c) := c \otimes e$ and $\tilde{\delta}(c) := e \otimes c$ for all $c \in C$. Define also the linear map $\overrightarrow{d} : C \to C \otimes C$ such that $\overrightarrow{d}(c) = \Delta_C c - \delta_f(c)$ and the linear map $\overleftarrow{d} : C \to C \otimes C$ such that $\overleftarrow{d}(c) = \Delta_C c - \tilde{\delta}_f(c)$. These linear maps, $\overleftarrow{d}$ and $\overrightarrow{d}$, turn the coassociative coalgebra (C, Δ_C) into a non-degenerate L-coalgebra $(C, \overrightarrow{d}, \overleftarrow{d})$. $\square$

Let C be a bialgebra with unit e. The two new coproducts, $\overleftarrow{d}, \overrightarrow{d} : C \to C^{\otimes 2}$, turn out to be a Leibniz-Ito derivative [4]. Starting with a Markov L-coalgebra $\mathcal{G}$, we can recover a similar Theorem. However, whereas in the coassociative coalgebra case, these new two coproducts map "the vertex set" C into the "arrow sets" $C^{\otimes 2}$, the new coproducts, $\overleftarrow{d}_{\mathcal{G}}$ and $\overrightarrow{d}_{\mathcal{G}}$, of a Markov L-coalgebra $\mathcal{G}$ map "the arrow set" $C^{\otimes 2}$ into the " paths of lenght 2" modelised by $C^{\otimes 3}$. This observation suggests the necessity of studying the line-extension of directed graphs, a notion defined in the sequel.

5.2.1 Line-extension of directed graphs

The aim of this Subsection is to show that some De Bruijn directed graphs, seen as Markov L-coalgebras are also coassociative co-dialgebras and that the line-extension of these directed graphs can be viewed as geometric supports associated with some well-known coassociative coalgebras. We start with two definitions.

Definition 5.2.2 [De Bruijn graph] A (p, n) De-Bruijn sequence on the alphabet $\Sigma = \{a_1, \dots, a_p\}$ is a sequence $(s_1, \dots, s_m)$ of $m = p^n$ elements $s_i \in \Sigma$ such that subsequences of length n of the form $(s_i, \dots, s_{i+n-1})$ are distinct, the addition of subscripts being done modulo m. A (p, n)-*De Bruijn graph* is a directed graph whose vertices correspond to all possible strings $s_1 s_2 \dots s_n$ of n symbols from Σ. There are p arcs leaving the vertex $s_1 s_2 \dots s_n$ and leading to the adjacent node $s_2 s_3 \dots s_n \alpha$, $\alpha \in \Sigma$. Therefore the $(p, 1)$-*De Bruijn graph* is the directed graph with p vertices, complete, with a loop at each vertex.

[4] Let A be an associative algebra with unit e, M be a A-bimodule and $f : A \to M$ be a linear map. The map f is said to be a Leibniz-Ito derivative if $f(e) = 0$ and $f(xy) = f(x)f(y) + xf(y) + f(x)y$, for all $x, y \in A$.

Fix $n \geq 1$. Let $D_{(n,1)} = ((D_{(n,1)})_0, (D_{(n,1)})_1, s, t)$ be the $(n,1)$-De Bruijn graph. Its natural Markov L-coalgebra is defined as follows. Denote by v_i, $1 \leq i \leq n$, the vertices of $D_{(n,1)}$. Embed $(D_{(n,1)})_0$ into its free k-vector space. Define the coproducts Δ_M, $\tilde{\Delta}_M : k(D_{(n,1)})_0 \to k(D_{(n,1)})_0^{\otimes 2}$, such that for all i, $\Delta_M v_i := v_i \otimes \sum_j v_j$ and $\tilde{\Delta}_M v_i := \sum_j v_j \otimes v_i$. There are obvious left and right counits, $\tilde{\epsilon}(v_i) = \epsilon(v_i) = \frac{1}{n}$, for all $v_i \in (D_{(n,1)})_0$.

Definition 5.2.3 [Line-extension] The *line-extension* of a directed graph $G := (G_0, G_1, s, t)$, with a denumerable vertex set G_0 and a denumerable arrow set $G_1 \subseteq G_0 \times G_0$ is the directed graph with vertex set G_1 and arrow set $G_2 \subseteq G_1 \times G_1$ defined by $(v, w) \in G_1 \times G_1$ belongs to G_2 if and only if $t(v) = s(w)$. This directed graph, called the line-directed graph of G, is denoted by $E(G)$.

The notion of associative dialgebra, introduced in [45], is a notion which generalises the notion of algebra. Associative dialgebras, via the notion of dendriform algebra [5], are closely related to (planar binary) trees, which became an important tool in quantum field theory [31, 13]. Here, we are interested in the notion of coassociative co-dialgebra.

Definition 5.2.4 [Coassociative co-dialgebra of degree n] Motivated by line-extension of (geometric support of) Markov L-coalgebra, we define in [40], (Markov) L-coalgebra of degree n, $n > 0$. Similarly, let Δ_n and $\tilde{\Delta}_n$ be two n-linear mappings $D^{\otimes n} \to D^{\otimes n+1}$, where D is a k-vector space. The k-vector space $(D, \Delta_n, \tilde{\Delta}_n)$ is said to be a *coassociative co-dialgebra* of degree n if the following axioms are verified:

1. Δ_n and $\tilde{\Delta}_n$ are coassociative,
2. $(id \otimes \Delta_n)\Delta_n = (id \otimes \tilde{\Delta}_n)\Delta_n$,
3. $(\tilde{\Delta}_n \otimes id)\tilde{\Delta}_n = (\Delta_n \otimes id)\tilde{\Delta}_n$,
4. $(\tilde{\Delta}_n \otimes id)\Delta_n = (id \otimes \Delta_n)\tilde{\Delta}_n$.

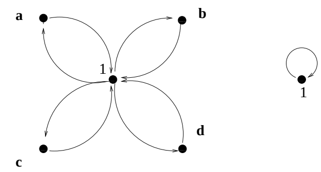
As in the L-coalgebra case, such a space may have a right counit $\epsilon_n : D^{\otimes n} \to D^{\otimes n-1}$ such that: $(id \otimes \epsilon_n)\Delta_n = id$ and a left counit $\tilde{\epsilon}_n : D^{\otimes n} \to D^{\otimes n-1}$ such that: $(\tilde{\epsilon}_n \otimes id)\tilde{\Delta}_n = id$. By convention $D^{\otimes 0} := k$. A coassociative co-dialgebra of degree 1 will be also called a coassociative co-dialgebra.

Proposition 5.2.5 *A k-vector space C equipped with two coproducts δ_f and $\tilde{\delta}_f$ such that $\delta_f(c) = c \otimes e$ and $\tilde{\delta}_f(c) = e \otimes c$, for all $c \in C$ is a Markov coassociative codialgebra.*

Proof: Straightforward since $\delta_f(e) = \tilde{\delta}_f(e)$. □

Example 5.2.6 [The flower graph] A unital algebra, A with unit 1, carries a non-trivial Markov L-bialgebra, obtained from the equality, $(1 \cdot a) \cdot 1 = 1 \cdot (a \cdot 1)$. The coproducts are $\delta_f(a) = a \otimes 1$ and $\tilde{\delta}_f(a) = 1 \otimes a$, $a \in A$. Its geometric support is called the flower graph.

[5] The operads Dias (associative dialgebra) and Dend (dendriform algebra) are dual in the operad sense, see [45].



Example of geometric support associated with an algebra $k\langle a,b,c,d\rangle \oplus k1$.

Theorem 5.2.7 *Any coassociative coalgebra (C,Δ), (respectively bialgebra, Hopf algebra), can be embedded into a L-coalgebra of degree n, (respectively, L-bialgebra of degree n, L-Hopf algebra* [6] *of degree n), with $n>1$.*

Proof: Fix $n>1$. Consider Δ, the coproduct of such a coassociative coalgebra C. Set: $id_n := \underbrace{id \otimes id \otimes \ldots \otimes id}_{n}$ and $\Delta_n := id_{n-1} \otimes \Delta$ and $\tilde{\Delta}_n := \Delta \otimes id_{n-1}$. The right and left coassociative coproducts Δ_n and $\tilde{\Delta}_n$ map $C^{\otimes n}$ into $C^{\otimes n+1}$ and the entanglement equation is realised. The right and left counits are $\epsilon_n := id_{n-1} \otimes \epsilon$ and $\tilde{\epsilon}_n := \epsilon \otimes id_{n-1}$, since for instance $(id \otimes \epsilon_n)\Delta_n = id_n$. If the counit and the coproduct of C are unital homomorphisms, so are the new coproducts and counits. If C is a Hopf algebra with antipode s, we embed C into a L-Hopf algebra of degree n since $(id_{n-1} \otimes m)(id_n \otimes s)\Delta_n = \eta_n \epsilon_n$ and $(m \otimes id_{n-1})(s \otimes id_n)\Delta_n = \tilde{\eta}_n \tilde{\epsilon}_n$. $\square$

Theorem 5.2.8 *If $n=2$, any coassociative coalgebra (C,Δ), can be viewed as a coassociative co-dialgebra of degree 2.*

Proof: Straightforward by using the definition of Δ_2 and the fact that Δ is coassociative. $\square$

Proposition 5.2.9 *Let $D_{(n,1)} = ((D_{(n,1)})_0,\ (D_{(n,1)})_1)$, be the $(n,1)$-De Bruijn graph and consider its free k-vector space $k(D_{(n,1)})_0$. The Markovian coproducts of the Markov L-coalgebra $k(D_{(n,1)})_0$ associated with the $(n,1)$-De Bruijn graph define a coassociative co-dialgebra.*

Proof: Straightforward. $\square$

Proposition 5.2.10 *There exists a coassociative coalgebra whose geometric support is the line-extension of the $(n,1)$-De Bruijn graph.*

Proof: Let us denote the arrow emerging from a given vertex v_i to a vertex v_j, with $i,j=1,\ldots,n$ of the $(n,1)$-De Bruijn graph $D_{(n,1)}$ by a_{ij}. The new vertices of $E(D_{(n,1)})$ are denoted by a_{ij} and the arrows are denoted by $((il),(lj))$. Consider the free k-vector space spanned by the set $E(D_{(n,1)})_0 := \{a_{ij};\ i,j=1\ldots n\}$ and define $\Delta a_{ij} = \sum_l a_{il} \otimes a_{lj}$, this coproduct is coassociative and the geometric support associating with the coassociative coalgebra $(kE(D_{(n,1)})_0, \Delta)$ is easily seen to be $E(D_{(n,1)})$. It has an obvious counit, $a_{ij} \mapsto 0$ if $i \neq j$ and $a_{ij} \mapsto 1$ otherwise. $\square$

[6] Set $id_n := \underbrace{id \otimes id \otimes \ldots \otimes id}_{n}$, $n>0$. A L-Hopf algebra of degree n, $(H, \Delta_H, \tilde{\Delta}_H)$, is by definition a L-bialgebra of degree n, equipped with right and left counits, $\tilde{\epsilon}_H$, ϵ_H of degree n, such that its antipodes $S, \tilde{S}:\ H \to H$ verify $(id_{n-1} \otimes m)(id_n \otimes S)\Delta_H = \eta_n \epsilon_H$ and $(m \otimes id)(\tilde{S} \otimes id_n)\tilde{\Delta}_H = \tilde{\eta}_n \tilde{\epsilon}_H$, with $\eta_n, \tilde{\eta}_n : H^{\otimes(n-1)} \to H^{\otimes n}$ such that $\eta_n(h) := h \otimes 1_H$ and $\tilde{\eta}_n(h) := 1_H \otimes h$, $h \in H^{\otimes(n-1)}$.

Corollary 5.2.11 *Recall that $\mathcal{F}$ is spanned by the basis a, b, c and d, as a k-vector space, and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$. The line-extension of the $(2,1)$-De Bruijn graph can be equipped with the coassociative coproduct associated with the coalgebra $\mathcal{F}$.*

Proposition 5.2.12 *Let B be a (Markov) coassociative co-dialgebra with coproducts $\tilde{\Delta}$ and Δ and C be a coassociative coalgebra with coproduct Δ_C. Then $B \otimes C$ is a coassociative co-dialgebra with coproduct $\delta_{B\otimes C} := (id \otimes \tau \otimes id)\Delta \otimes \Delta_C$ and $\tilde{\delta}_{B\otimes C} := (id \otimes \tau \otimes id)\tilde{\Delta} \otimes \Delta_C$. Similarly, $C \otimes B$ is a coassociative co-dialgebra with $\delta_{C\otimes B} := (id \otimes \tau \otimes id)\Delta_C \otimes \Delta$ and $\tilde{\delta}_{C\otimes B} := (id \otimes \tau \otimes id)\Delta_C \otimes \tilde{\Delta}$.*

Proof: Straightforward. □

We end this Section on De-Bruijn graphs and coassociative codialgebras, by constructing another family of coassociative co-dialgebra. The idea is to give the $(n,1)$-De Bruijn graph an attractor rôle.

Proposition 5.2.13 *Let n and m be two integers different from zero. Let D be the k-vector space $k\langle x_1, \ldots, x_m \rangle \oplus k\langle \alpha_1, \ldots, \alpha_n \rangle$. Define for all $i = 1, \ldots, m$, $\Delta x_i = \sum_{j=1}^{n} x_i \otimes \alpha_j$, $\tilde{\Delta} x_i = \sum_{j=1}^{n} \alpha_j \otimes x_i$ and for all $j = 1, \ldots, n$, $\Delta \alpha_j = \sum_{p=1}^{n} \alpha_j \otimes \alpha_p$, $\tilde{\Delta} \alpha_j = \sum_{p=1}^{n} \alpha_p \otimes \alpha_j$. These Markovian coproducts embed D into a coassociative co-dialgebra.*

Proof: Straightforward. □

Remark: Consider the k-vector space of linear maps $L(D, A)$ which map $(D, \Delta, \tilde{\Delta})$, a coassociative coalgebra into an associative algebra A equipped with a product m. The space $L(D, A)$ is then embedded into an associative dialgebra by defining for all $f, g \in L(D, A)$, the two convolution products: $f \dashv g := m(f \otimes g)\Delta$ and $f \vdash g := m(f \otimes g)\tilde{\Delta}$.

5.2.2 Relationships between the $(2,1)$-De Bruijn L-coalgebra and $\mathcal{F}$

Motivated by a previous work [39], the aim of this Subsection is to study the relationship between the $(2,1)$-De Bruijn graph, seen as a Markov L-coalgebra and its line-extension seen as the (geometric support of the) coassociative coalgebra $\mathcal{F}$. Let X, Y be two non-commuting operators, we consider the non-commutative algebra $A = k\langle X,\ Y \rangle \oplus k1$, with $XY = \eta\, YX$ and $\eta \in k \setminus \{0\}$. We equip A with the following Markovian coproducts $\Delta_M X = X \otimes X + X \otimes Y$ and $\Delta_M Y = Y \otimes X + Y \otimes Y$, $\tilde{\Delta}_M X = Y \otimes X + X \otimes X$, $\tilde{\Delta}_M Y = X \otimes Y + Y \otimes Y$ and set $\tilde{\Delta}_M 1 = \Delta_M 1 = 1 \otimes 1$. Set $a = X \otimes X$, $b = X \otimes Y$, $c = Y \otimes X, d = Y \otimes Y$. We would like to find operators which give the coassociative coproduct of $\mathcal{F}$ from the coassociative co-dialgebra A. For that, we define: $\overrightarrow{\Delta}_M : A \to A^{\otimes 2} \times A^{\otimes 2}$, $X \mapsto \overrightarrow{\Delta}_M(X) = \begin{pmatrix} X \otimes X \\ X \otimes Y \end{pmatrix}, Y \mapsto \overrightarrow{\Delta}_M(Y) = \begin{pmatrix} Y \otimes Y \\ Y \otimes X \end{pmatrix}, 1 \mapsto \overrightarrow{\Delta}_M(1) = \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes 1 \end{pmatrix}$ and $\overrightarrow{\tilde{\Delta}}_M : A \to A^{\otimes 2} \times A^{\otimes 2}$, $X \mapsto \overrightarrow{\tilde{\Delta}}_M(X) = \begin{pmatrix} X \otimes X \\ Y \otimes X \end{pmatrix}, Y \mapsto \overrightarrow{\tilde{\Delta}}_M(Y) = \begin{pmatrix} Y \otimes Y \\ X \otimes Y \end{pmatrix}, 1 \mapsto \overrightarrow{\tilde{\Delta}}_M(1) = \begin{pmatrix} 1 \otimes 1 \\ 1 \otimes 1 \end{pmatrix}$. If we define the bilinear map $\langle \cdot\, ;\, \cdot \rangle_* :\ (A^{\otimes 2} \times A^{\otimes 2})^{\times 2} \to$

$A^{\otimes 2}, ((z_1, z_2), (z_3, z_4)) \mapsto z_1 z_3 + z_2 z_4$, we recover $\Delta_M(X) = \langle \overrightarrow{\Delta}_M(X), \overrightarrow{\Delta}_M(1) \rangle_*$ and $\Delta_M(Y) = \langle \overrightarrow{\Delta}_M(Y), \overrightarrow{\Delta}_M(1) \rangle_*$. Define the bilinear map $\Box : A^{\otimes 2} \times A^{\otimes 2} \to A^{\otimes 2} \otimes A^{\otimes 2}$ such that:

$$(y_1 \otimes y_2)\Box(y_3 \otimes y_4) := (id \otimes \tau \otimes id)((y_1 \otimes y_2) \otimes (y_3 \otimes y_4) := (y_1 \otimes y_3) \otimes (y_2 \otimes y_4).$$

Define also $\langle \cdot \ ; \ \cdot \rangle$ and per : $(A^{\otimes 2} \times A^{\otimes 2})^{\times 2} \to A^{\otimes 2} \otimes A^{\otimes 2}$ such that $((z_1, z_2), (z_3, z_4)) \mapsto z_1 \Box z_3 + z_2 \Box z_4$ and the "permanent" $\mathrm{per}((z_1, z_2), (z_3, z_4)) := z_1 \Box z_4 + z_2 \Box z_3$. Let us express the relations between Markovian coproducts of the $(2,1)$-De Bruijn graph and the coassociative coproduct of $\mathcal{F}$.

Proposition 5.2.14 *The relations are:* $\langle \overrightarrow{\Delta}_M(X), \overrightarrow{\tilde{\Delta}}_M(X) \rangle = \Delta(a)$, $\langle \overrightarrow{\Delta}_M(Y), \ \overrightarrow{\tilde{\Delta}}_M(Y) \rangle = \Delta(d)$,
$per(\overrightarrow{\Delta}_M(X), \overrightarrow{\tilde{\Delta}}_M(Y)) = \Delta(b)$, $per(\overrightarrow{\Delta}_M(Y), \overrightarrow{\tilde{\Delta}}_M(X)) = \Delta(c)$.

Proof: Straightforward. For instance, $\langle \overrightarrow{\Delta}_M(X), \overrightarrow{\tilde{\Delta}}_M(X) \rangle = (X \otimes X)\Box(X \otimes X) + (X \otimes Y)\Box(Y \otimes X) = a \otimes a + b \otimes c = \Delta(a)$. $\Box$

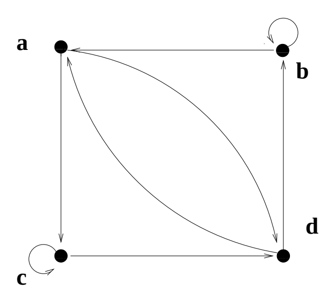
We can also recover algebraic relations of $Sl_q(2)$, except the q-determinant which is equal to zero, i.e., $ad - q^{-1}bc = 0$ instead of one. For checking the algebraic relations, we contract the arrows a, b, c, d into the vertices $\bar{a} = X^2, \bar{b} = XY, \bar{c} = YX, \bar{d} = Y^2$, thanks to the usual product of A, (recall that such a product maps the arrow set $A^{\otimes 2}$ into the vertex set A).

Proposition 5.2.15 *With* $XY = \eta YX$, *we obtain* $\bar{a}\bar{b} = \eta^2 \bar{b}\bar{a}, \quad \bar{c}\bar{b} = \bar{b}\bar{c}, \quad \bar{a}\bar{c} = \eta^2 \bar{c}\bar{a}, \quad \bar{a}\bar{d} = \eta^2 \bar{b}\bar{c}, \quad \bar{b}\bar{d} = \eta^2 \bar{d}\bar{b}, \quad \bar{c}\bar{d} = \eta^2 \bar{d}\bar{c}, \quad \bar{a}\bar{d} - \bar{d}\bar{a} = (\eta^2 - \eta^{-2})\bar{b}\bar{c}, \quad \bar{a}\bar{d} - \eta^2 \bar{b}\bar{c} = 0.$

Proof: For instance, $\bar{a}\bar{b} = XXXY = \eta^2 XYXX = \eta^2 \bar{b}\bar{a}$, $\bar{c}\bar{d} = YXYY = \eta^2 YYYX = \eta^2 \bar{d}\bar{c}$, $\bar{a}\bar{d} = XXYY = \eta^2 XYYX = \eta^2 \bar{b}\bar{c}$, and so on. By setting $\eta^2 = q^{-1}$, we recover the usual algebraic relations for $Sl_q(2)$, except the q-determinant which is equal to 0. $\Box$

5.3 The left part of $\mathcal{F}$

The aim of this Section is to prove that there exists a coassociative coalgebra $(\mathcal{F}, \tilde{\Delta})$ entangled to the usual coassociative coalgebra $(\mathcal{F}, \Delta)$, i.e., to prove that $(\mathcal{F}, \Delta, \tilde{\Delta})$ is a coassociative L-coalgebra. By convention, we call $(\mathcal{F}, \tilde{\Delta})$, the left part of $(\mathcal{F}, \Delta, \tilde{\Delta})$. Its associated geometric support is:



Geometric support associated with the left part $(\mathcal{F}, \tilde{\Delta})$.

This structure can be obtained, for instance, by inverting the map $\langle \cdot, \cdot \rangle$ and the map per in the equations defining the usual coproduct Δ of $\mathcal{F}$ obtained from the $(2,1)$-De Bruijn graph in Proposition 5.2.14. The relations for the new coproduct $\tilde{\Delta}$ are,

$$\tilde{\Delta}b = b \otimes b + a \otimes d, \ \tilde{\Delta}c = c \otimes c + d \otimes a, \ \tilde{\Delta}a = b \otimes a + a \otimes c, \ \tilde{\Delta}d = c \otimes d + d \otimes b.$$

Remark: In [40], we defined a matrix product $U \bar{\otimes} W$, where U, W are two matrices and $(U \bar{\otimes} W)_{ij} := \sum_k U_{ik} \otimes W_{kj}$. With:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} b & a \\ d & c \end{pmatrix},$$

the operation, $\tilde{}$, meaning here the permutation of the two columns, the definition of the two coproducts can be recovered. Indeed, $(U \bar{\otimes} U)_{ij} := \sum_k U_{ik} \otimes \underbrace{U_{kj} :=}\Delta U_{ij}$ yields the right coproduct and the left coproduct $\tilde{\Delta}$ can be recovered by computing $\tilde{U} \bar{\otimes} \tilde{U}$.

Remark: The linear map $\tilde{\epsilon} : \mathcal{F} \to k$, such that:

$$\tilde{\epsilon}(a) = \tilde{\epsilon}(d) = 0, \qquad \tilde{\epsilon}(b) = \tilde{\epsilon}(c) = 1,$$

is a counit map for $\tilde{\Delta}$, i.e., $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = (id \otimes \tilde{\epsilon})\tilde{\Delta} = id$.

Theorem 5.3.1 *The new coproduct $\tilde{\Delta}$ is coassociative and verifies the entanglement equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. Moreover the two coproducts verify also $(\Delta \otimes id)\tilde{\Delta} = (id \otimes \tilde{\Delta})\Delta$.*

Proof: The coassociativity is straightforward. The proof of the two equations is also straightforward by using the matrix product defined above. □

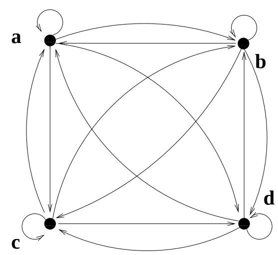
Remark: The axioms of coassociative co-dialgebras are not satisfied.

Definition 5.3.2 [Chiral, Achiral] Let $(L, \Delta, \tilde{\Delta})$ be a coassociative L-coalgebra with right coproduct Δ and left coproduct $\tilde{\Delta}$. The k-vector space $(L, \Delta, \tilde{\Delta})$ is *chiral* if $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$ and $(\Delta \otimes id)\tilde{\Delta} \neq (id \otimes \tilde{\Delta})\Delta$. This means that the entanglement equation "differentiates" the left part $(L, \tilde{\Delta})$ from the right part (L, Δ). On the contrary, $(L, \Delta, \tilde{\Delta})$ is said to be *achiral* if $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$ and $(\Delta \otimes id)\tilde{\Delta} = (id \otimes \tilde{\Delta})\Delta$, i.e., the axioms of an achiral coassociative L-coalgebra are globally invariant under the permutation $\tilde{\Delta} \leftrightarrow \Delta$. More generally, an algebra $(A, \bullet_1, \ldots, \bullet_n)$ equipped with n operations $\bullet_1, \ldots, \bullet_n : A^{\otimes 2} \to A$, verifying axioms $AX_1, \ldots AX_p$ is said to be achiral if $(A, \bullet_{\sigma(1)}, \ldots, \bullet_{\sigma(n)})$, where σ is a permutation, verifies also the same axioms, i.e., the axioms $AX_1, \ldots AX_p$ are globally invariant under the action of any permutation σ. The dualisation of this definition is straightforward.

Remark: In the Theorem 5.3.1, the entanglement equation is verified, even by inverting the rôle of the left and right coproducts. We say that $(\mathcal{F}, \Delta, \tilde{\Delta})$ is an achiral L-coalgebra, since the left and right parts are entangled by the entanglement equation which do not differentiate them. On the contrary, for instance, observe that the two coproducts δ_f and $\tilde{\delta}_f$ associated with an unital associative algebra, embed the algebra into a chiral L-bialgebra, see also [34].

Theorem 5.3.3 *The two geometric supports associated with the left part* $(\mathcal{F}, \tilde{\Delta})$ *and the right part* $(\mathcal{F}, \Delta)$ *of the coassociative L-coalgebra* $(\mathcal{F}, \Delta, \tilde{\Delta})$*, obtained from the line-extension of the* $(2,1)$*-De Bruijn graph, glued together yield the* $(4,1)$*-De Bruijn graph. Moreover the intersection of their arrow sets is empty.*

Proof: [**The gluing of the left and right parts of** $(\mathcal{F}, \Delta, \tilde{\Delta})$] When we glue the geometric support associated with $(\mathcal{F}, \Delta)$ with its left part $Gr(\mathcal{F}, \tilde{\Delta})$, we obtain the $(4,1)$-De Bruijn directed graph.



The $(4,1)$**-De Bruijn graph.**

The intersection of the arrow sets of these two graphs is empty. □

The $(4,1)$-De Bruijn graph is tiled with two (geometric supports of) entangled coassociative coalgebras, the entanglement being achiral.

5.3.1 Consequences

An example of achiral L-Hopf algebra

Proposition 5.3.4 *If* $(\mathcal{F}, \Delta, \tilde{\Delta})$ *is also a unital algebra and* a, b, c, d *commute pairewise. Then* $(\mathcal{F}, \Delta, \tilde{\Delta})$ *will be an achiral L-bialgebra. Moreover, if* $ad - bc = 1$*, define the linear map* $S_{\mathcal{F}}$ *which maps* $a \mapsto d$*,* $d \mapsto a$*,* $b \mapsto -b$ *and* $c \mapsto -c$ *and the linear map* $\tilde{S}_{\mathcal{F}}$ *which maps* $b \mapsto -c$*,* $c \mapsto -b$*,* $a \mapsto a$*,* $d \mapsto d$*, then the right (resp. left) part of* $(\mathcal{F}, \Delta, \tilde{\Delta})$ *is a Hopf algebra, i.e.,* $(\mathcal{F}, \Delta, \tilde{\Delta})$ *is an achiral L-Hopf algebra.*

Proof: Straightforward. □

An example of chiral L-bialgebra

Consider the algebra generated by x, p and g such that $pg = gp$, $xg = \eta gx$, $xp = \mu px$, with η, μ two scalars different from zero. Define the (left) part by: $\tilde{\Delta}x = x \otimes p + g \otimes x$, $\tilde{\Delta}p = p \otimes p$, $\tilde{\Delta}g = g \otimes g$ with counit $\tilde{\epsilon}$: $x \mapsto 0$ and $p, g \mapsto 1$. Define the (right) part by: $\Delta x = x \otimes x$, $\Delta p = p \otimes x$, $\Delta g = g \otimes x$.

Proposition 5.3.5 *The two coproducts are coassociative and verify the entanglement equation* $(id\otimes\Delta)\tilde{\Delta} = (\tilde{\Delta}\otimes id)\Delta$. *Moreover the two coproducts* Δ, $\tilde{\Delta}$ *and the counit* $\tilde{\epsilon}$ *are homomorphisms.*

Proof: Tedious but straightforward by noticing that it is a consequence of the previous subsection on the left part of $\mathcal{F}$, by setting formally $b = 0$. For instance, we check the entanglement equation on x. We get, $x \xrightarrow{\Delta} x\otimes x \xrightarrow{\tilde{\Delta}\otimes id} x\otimes p\otimes x+g\otimes x\otimes x$ and $x \xrightarrow{\tilde{\Delta}} x\otimes p+g\otimes x \xrightarrow{id\otimes\Delta} x\otimes p\otimes x+g\otimes x\otimes x$. For the homomorphism property, we get for instance $\Delta xg = \Delta x\cdot\Delta g = xg\otimes xx$ and $\Delta gx = \Delta g\cdot\Delta x = gx\otimes xx$, thus $\Delta xg = \eta\Delta gx$ and so forth. □

Consequences for $Sl_q(2)$

Proposition 5.3.6 *With the usual algebraic relations of the Hopf algebra* $Sl_q(2)$, *whose usual product will be denoted by* m, *the linear map* $\tilde{S} : Sl_q(2) \to Sl_q(2)$ *defined by* $b \mapsto -q^{-1}c; c \mapsto -qb; a \mapsto a; d \mapsto d$, *verifies* $m(id\otimes\tilde{S})\tilde{\Delta} = m(\tilde{S}\otimes id)\tilde{\Delta} = 1\cdot\tilde{\epsilon}$. *Moreover, the map* $\tilde{S}$ *can be extended to an unital algebra map. As* $1, a, d$ *and* bc *are fixed points of* $\tilde{S}$, *the q-determinant of the matrix* U *is invariant by applying* $\tilde{S}$.

Proof: Let us check the antipode property. For that, multiply $\tilde{U}$ by the following matrix

$$\tilde{S}(\tilde{U}) = \begin{pmatrix} -q^{-1}c & a \\ d & -qb \end{pmatrix},$$

which is the matrix obtained from $\tilde{U}$ by the left antipode $\tilde{S}$. We compute

$$\widetilde{\tilde{U}\tilde{S}(\tilde{U})} = \begin{pmatrix} ba - qab & ad - q^{-1}bc \\ da - qcb & cd - q^{-1}dc \end{pmatrix}.$$

This matrix must be equal to:

$$\tilde{\epsilon}(U) = \begin{pmatrix} \tilde{\epsilon}(a) = 0 & \tilde{\epsilon}(b) = 1 \\ \tilde{\epsilon}(c) = 1 & \tilde{\epsilon}(d) = 0 \end{pmatrix},$$

which is the case since we know that $da = 1 + qbc$. We obtain the same result if we consider $\widetilde{\tilde{S}(\tilde{U})\tilde{U}}$. The extension of the definition of $\tilde{S}$ to a homomorphism is straightforward. □

Remark: The algebraic relations of $Sl_q(2)$ do not embed its left part into a bialgebra. This fact will be a motivation for introducing coassociative manifolds in [34].

The left part of $SU_q(2)$

A consequence of what was done for the coassociative coalgebra $\mathcal{F}$ is the existence of a left part for the Hopf algebra $SU_q(2)$. We recall that $SU_q(2)$ is an $*$-algebra with two generators, a and c such that $ac = qca$, $ac^* = qc^*a$, $cc^* = c^*c$, $a^*a + c^*c = 1$, $aa^* + q^2cc^* = 1$, with q a real different from 0. The coproduct is given by $\Delta_1 a = a\otimes a - qc^*\otimes c$ and $\Delta_1 c = c\otimes a + a^*\otimes c$ and the counit is $\epsilon_1(a) = 1$ and $\epsilon_1(c) = 0$.

Proposition 5.3.7 *There exists a coassociative coalgebra which is achiral entangled to the usual coassociative coalgebra part of $SU_q(2)$.*

Proof: Define the (left) coproduct $\tilde{\Delta}_1$ by the $*$-linear map $a \mapsto \tilde{\Delta}_1 a := c^* \otimes a + a \otimes c$ and $c \mapsto \tilde{\Delta}_1 c := c \otimes c - q^{-1} a^* \otimes a$. This coproduct is coassociative. Moreover, we have $(\tilde{\Delta}_1 \otimes id)\Delta_1 = (id \otimes \Delta_1)\tilde{\Delta}_1$ and $(\Delta_1 \otimes id)\tilde{\Delta}_1 = (id \otimes \tilde{\Delta}_1)\Delta_1$. For instance, $(\Delta_1 \otimes id)\tilde{\Delta}_1 a = c^* \otimes a^* \otimes a + a \otimes c^* \otimes a + a \otimes a \otimes c - qc^* \otimes c \otimes c$ and $(id \otimes \tilde{\Delta}_1)\Delta_1 a = a \otimes c^* \otimes a + a \otimes a \otimes c - q(c^* \otimes c \otimes c - q^{-1} c^* \otimes a^* \otimes a)$, and so forth. The $*$-linear map $\tilde{\epsilon}_1$ is defined by $c \mapsto 1$ and $a \mapsto 0$ is the (left) counit. □

Remark: The usual algebraic relations of $SU_q(2)$ do not embed the new coproduct and the new counit into homomorphisms.

5.3.2 Splitting the coassociative coproducts and achiral L-coalgebras

The example of dendriform algebras [7] shows that there exists associative algebras whose associative product can split into two (a priori non-associative) products. In the case of achiral L-coalgebras, we have also a splitting of a coassociative coproduct into two coassociative ones. Let us see how it works.

Recall that an achiral L-coalgebra $(L, \Delta, \tilde{\Delta})$ has two coproducts Δ and $\tilde{\Delta}$ which verify the axioms:

1. Δ and $\tilde{\Delta}$ are coassociative.
2. $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$.
3. $(\Delta \otimes id)\tilde{\Delta} = (id \otimes \tilde{\Delta})\Delta$,

i.e., the axioms are globally invariant by the permutation $\Delta \leftrightarrow \tilde{\Delta}$.

Proposition 5.3.8 *Suppose L is an achiral L-coalgebra. Let $u, v, w, z \in k$. These axioms are invariant under the transformation* $\begin{pmatrix} \Delta \\ \tilde{\Delta} \end{pmatrix} \mapsto \begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} \Delta \\ \tilde{\Delta} \end{pmatrix}$. *For instance, the coproduct $\Delta' := u\Delta + v\tilde{\Delta}$ is still coassociative.*

Proof: Straightforward. □

5.4 Tiling the $(n^2, 1)$-De Bruijn co-dialgebra with n coassociative coalgebras

We can generalize the previous procedure to any coassociative coalgebras whose geometric supports are obtained by line-extension of the $(n, 1)$-De Bruijn graphs viewed as geometric supports

[7] See the introduction or [45].

of Markov co-dialgebras. Fix $n > 1$, the number of vertices of the $(n, 1)$-De Bruijn graph, denoted by U_i, $i = 1, \dots n$. We have seen in Proposition 5.2.10 that the line-extension of such a graph yields a geometric support of a coassociative coalgebra whose coproduct is denoted by Δ. The coproduct was recovered by computing $(U \bar{\otimes} U)$, where U_{ij} is the vertex associated with the arrow going from the vertex U_i to U_j in the $(n, 1)$-De Bruijn graph. Denote by p, the shift which maps j into $j + 1 \mod n$, for all $j = 1, \dots, n$. Let α be an integer equal to $0, \dots, n - 1$. We denote by $\mathcal{P}^{\alpha}(U)$, (resp. $\mathcal{P}^{-\alpha}(U)$), the matrix obtained from U by letting the shift p^{α}, (resp. $p^{-\alpha}$) acts on the columns of U, i.e., $\mathcal{P}^{\alpha}(U)_{ij} := U_{ip^{\alpha}(j)}$ and $\mathcal{P}^{-\alpha}(U)_{ij} := U_{ip^{-\alpha}(j)}$.

Lemma 5.4.1 *Let A, B be two n by n matrices and α be an integer equal to $0, \dots, n - 1$. We get $\mathcal{P}^{\alpha}(AB) = A\mathcal{P}^{\alpha}(B)$ and $\mathcal{P}^{-\alpha}(AB) = A\mathcal{P}^{-\alpha}(B)$. Therefore, we obtain, $\mathcal{P}^{\alpha}(U)U = \mathcal{P}^{\alpha}(\mathcal{P}^{\alpha}(U)\mathcal{P}^{-\alpha}(U))$.*

Proof: Notice that $\mathcal{P}^{\alpha}(AB)_{ij} = (AB)_{ip^{\alpha}(j)} = \sum_k A_{ik}B_{kp^{\alpha}(j)} = (A\mathcal{P}^{\alpha}(B))_{ij}$, which proved the first equality. The sequel is now straightforward. $\square$

Remark: This Lemma is also valid by replacing the usual product by $\bar{\otimes}$.

Definition 5.4.2 [Coproducts] As $\mathcal{P}^0 = id$, we rename the usual coproduct Δ by $\Delta_{[0]}$. Its explicit definition, as we have seen, is closely related to the matrix U. We define also the coproducts $\Delta_{[\alpha]}$, for all α, by $\Delta_{[\alpha]}U_{ij} = (\mathcal{P}^{\alpha}(\mathcal{P}^{\alpha}(U)\bar{\otimes}\mathcal{P}^{-\alpha}(U))_{ij} = (\mathcal{P}^{\alpha}(U)U)_{ij}$. Denote by $(\mathcal{F}_n, \Delta_{[\alpha]})$, $0 \leq \alpha \leq n - 1$, the coalgebra so obtained. (Observe that $\mathcal{F}_2 := \mathcal{F}$.)

Theorem 5.4.3 *The new coproducts $\Delta_{[\alpha]}$ are coassociative. Moreover, for all $\alpha, \beta = 0, \dots, n-1$, $\Delta_{[\alpha]}$ and $\Delta_{[\beta]}$ obey the entanglement equation.*

Proof: We define for two matrices A, B the following product $A *_{\alpha} B = (\mathcal{P}^{\alpha}(\mathcal{P}^{\alpha}(A)\bar{\otimes}\mathcal{P}^{-\alpha}(B))$. From the straightforward equality $A *_{\alpha} (B *_{\alpha} C) = (A *_{\alpha} B) *_{\alpha} C$, we obtain in the case where $A = B = C = U$, the coassociativity equation $(\Delta_{[\alpha]} \otimes id)\Delta_{[\alpha]}U_{ij} = (id \otimes \Delta_{[\alpha]})\Delta_{[\alpha]}U_{ij}$. To prove that two coproducts obey the entanglement equation, we have to show that $(\Delta_{[\alpha]} \otimes id)\Delta_{[\beta]}U_{ij} = (id \otimes \Delta_{[\beta]})\Delta_{[\alpha]}U_{ij}$, that is $U *_{\alpha} (U *_{\beta} U) = (U *_{\alpha} U) *_{\beta} U$, which is also straightforward. $\square$

Remark: We have showed that in the case of a coassociative coalgebra obtained by line-extension of the $(n, 1)$-De Bruijn graphs, we can construct others coassociative coalgebras whose coproducts verify the entanglement equation. Precisely, the coassociative coalgebras $(\mathcal{F}_n, \Delta_{[\alpha]})$ and $(\mathcal{F}_n, \Delta_{[\beta]})$, $0 \leq \alpha, \beta \leq n - 1$, are entangled, the entanglement being achiral. Therefore, the k-vector space $(\mathcal{F}_n, \Delta_{[0]}, \dots, \Delta_{[n-1]})$ is achiral[8].

Proposition 5.4.4 *Let $(C, \Delta_{[0]}, \dots, \Delta_{[n-1]})$ be a k-vector space equipped with n linear maps $\Delta_{[i]} : C \to C^{\otimes 2}$, $i = 0, \dots n - 1$, such that:*

$$(\Delta_i \otimes id)\Delta_j = (id \otimes \Delta_j)\Delta_i, \quad i, j = 0, \dots, n - 1.$$

For $i = 0, \dots n - 1$, set x_i the vector equal to ${}^t(0, \dots, 0, \Delta_i, 0, \dots, 0)$ and fix $Z \in M_n(k)$, a n by n matrix. Then $(C, \Delta'_{[0]} := Zx_0, \dots, \Delta'_{[n-1]} := Zx_{n-1})$ verify also:

$$(\Delta'_i \otimes id)\Delta'_j = (id \otimes \Delta'_j)\Delta'_i, \quad i, j = 0, \dots, n - 1.$$

[8] In [34], such algebraic objects will be called coassociative manifolds.

Proof: Straightforward. □

Remark: Observe that this Theorem allows us to produce splittings of associative laws into associative ones.

Remark: Applying this Proposition to the tiling of the $(3,1)$-De Bruijn graph gives examples of cubical cotrialgebras, i.e., a coalgebra equipped with 3 coproducts Δ_i, $i = 0,1,2$, such that:

$$(\Delta_i \otimes id)\Delta_j = (id \otimes \Delta_j)\Delta_i, \quad i,j = 0,1,2.$$

The notion of cubical trialgebra is defined in [48]. We recover easily cubical trialgebras from cubical cotrialgebras by considering the convolution products. The operad $Tricub$ on one generator associated with cubical trialgebra is Koszul and self-dual. The operad associated with the so-called hypercube n-algebra, i.e., a k-vector space equipped with n products verifying:

$$(x \bullet_i y) \bullet_j z = x \bullet_i (y \bullet_j z), \quad x,y,z \in A, \ i,j = 0,\dots,n-1,$$

is conjectured to be Kozul and self-dual (observe that there are n^2 operations and two possible choices of parentheses, thus $2n^2 - n^2 = n^2$).

Remark: The counit $\epsilon_{[\alpha]}$, associated with the coassociative coproduct $\Delta_{[\alpha]}$, is obviously defined by $U_{ip^\alpha(i)} \mapsto 1$ and $U_{ip^\alpha(j)} \mapsto 0$ if $i \neq j$.

Theorem 5.4.5 *The intersection of the arrow sets of the geometric supports of* $(\mathcal{F}_n, \Delta_{[\alpha]})$, $0 \leq \alpha \leq n-1$, *are empty, i.e.,* $Gr((\mathcal{F}_n, \Delta_{[\alpha]}))_1 \cap Gr((\mathcal{F}_n, \Delta_{[\beta]}))_1 = \emptyset$, $0 \leq \alpha,\ \beta \leq n-1$ *and* $\alpha \neq \beta$. *Gluing them yields the* $(n^2,1)$*-De Bruijn directed graph, i.e.,* $\bigcup_{0\leq\alpha\leq n-1} Gr((\mathcal{F}_n, \Delta_{[\alpha]})) = D_{(n^2,1)}$.

Proof: We call $(U_{ij})_{i,j=1,\dots,n}$, the vertices of the $(n^2,1)$-De Bruijn graph. Let us prove that the gluing of all the geometric supports of $(\mathcal{F}_n, \Delta_{[\alpha]})$, $0 \leq \alpha \leq n-1$, yields the $(n^2,1)$-De Bruijn directed graph. Every arrow of the $(n^2,1)$-De Bruijn directed graph can be described by $U_{ik} \otimes U_{lj}$, with $k,l,i,j = 1,\dots,n$. As the shift is one-to-one, there exists a unique integer α such that $p^\alpha(l) = k$ i.e., $U_{ik} \otimes U_{lj} = U_{ip^\alpha(l)} \otimes U_{lj}$, i.e., this arrow belongs to the definition of the coproduct $\Delta_[\alpha]$, see Lemma 5.4.1. Therefore, the $(n^2,1)$-De Bruijn graph is a part of the gluing of the geometric supports associated with the n coassociative coalgebras. The reversal is obvious.

Fix α and β two different integers. As the shift $p^{\beta-\alpha}$ has no fixed point, no arrow defined in the coproduct $\Delta_{[\alpha]}$ is present in the definition of the coproduct of $\Delta_{[\beta]}$. Therefore, the intersection of arrow sets of $Gr(\mathcal{F}_n, \Delta_{[\alpha]})$ and $Gr(\mathcal{F}_n, \Delta_{[\beta]})$ is empty. □

Remark: Via their geometric supports, we get:
$(n,1)$-De Bruijn co-dialgebra $\xrightarrow{\text{Line-extension}}$ n coassociative coalgebras [9] $\xrightarrow{\text{Gluing}}$ $(n^2,1)$-De Bruijn co-dialgebra.

We have yielded a tiling of the $(n^2,1)$-De Bruijn graph into n coassociative coalgebras, each $(\mathcal{F}_n, \Delta_{[\alpha]}, \Delta_{[\beta]})$, $0 \leq \alpha,\beta \leq n-1$, being an achiral coassociative L-coalgebra. The case $n = 1$ is trivial. Indeed, the $(1,1)$-De Bruijn graph and its line-extension are loops and a loop is coassociative, since it is of the form $x \mapsto x \otimes x$.

[9] entangled by the achiral entanglement equation.

Remark: Fix $n \geq 1$. The link between the Markovian coproducts of the coassociative codialgebra $(k(D_{(n,1)})_0, \Delta_M, \tilde{\Delta}_M)$, associated with the $(n,1)$-De Bruijn graph and the coassociative coproduct of $(\mathcal{F}_n, \Delta_{[0]})$ can be seen as follows. Recall that $D_{(n,1)}$ is the $(n,1)$-De Bruijn graph, with vertex set $(D_{(n,1)})_0 := \{U_i;\ 1 \leq i \leq n\}$. Consider its free k-vector space $k(D_{(n,1)})_0$. Its Markovian coproducts verify $\Delta_M U_i := U_i \otimes \sum_k U_k$ and $\tilde{\Delta}_M U_i := \sum_k U_k \otimes U_i$. Set for all $i, j = 0, \ldots, n-1$, $U_{ij} := U_i \otimes U_j$. Define the map $\boxminus : k(D_{(n,1)})_0^{\otimes 2} \times k(D_{(n,1)})_0^{\otimes 2} \to k(D_{(n,1)})_0^{\otimes 2} \otimes k(D_{(n,1)})_0^{\otimes 2}$ such that $(z_1 \otimes z_2) \boxminus (z_3 \otimes z_4) := (id \otimes \Psi \otimes id)(z_1 \otimes z_2 \otimes z_3 \otimes z_4)$, where $\Psi(z_i \otimes z_j) = z_i \otimes z_i$ if $i = j$, and 0 otherwise. Observe that:

$$\Delta_M(U_i) \boxminus \tilde{\Delta}_M(U_j) := \Delta_{[0]} U_{ij}, \quad i, j = 0, \ldots n-1.$$

As in L-coalgebra theory, the notion of algebraic product is not assumed, we end this Section by a Propositon on the Leibniz coderivation and an example applied on $\mathcal{F}$.

Definition 5.4.6 [Coderivation] Let C be a coassociative coalgebra with coproduct Δ. A linear map $D:\ C \to C$ is called a (Leibniz) *coderivation* with respect to the coproduct Δ if it verifies:

$$\Delta D = (id \otimes D)\Delta + (D \otimes id)\Delta.$$

Proposition 5.4.7 *Let C be a coassociative coalgebra with coproduct Δ, such that for all the elements $U_{ij} \in C$, $\Delta U_{ij} = \sum_{k=1}^{n} U_{ik} \otimes U_{kj}$, where n is a fixed integer. Define the linear map D by $U_{ij} \mapsto \sum_k^n U_{kj} - U_{ik}$. Then D is a Leibniz coderivation.*

Proof: Fix U_{ij}. We have $(id \otimes D)\Delta + (D \otimes id)\Delta(U_{ij}) = \sum_{k,l=1}^{n} U_{lk} \otimes U_{kj} - U_{ik} \otimes U_{kl}$ which is equal to $\Delta(\sum_{l=1}^{n} U_{lj} - U_{il})$. □

Example 5.4.8 [A coderivation for $(\mathcal{F}, \Delta, \tilde{\Delta})$]
For $(\mathcal{F}, \Delta)$, we get $D(a) = c - b = -D(d)$, $D(b) = d - a = -D(c)$. As $(\mathcal{F}, \Delta, \tilde{\Delta})$ is a L-coalgebra, we yield also a coderivation $\tilde{D}$ for $(\mathcal{F}, \tilde{\Delta})$. Define $\tilde{D} = D$. A straightforward computation shows that D is also a coderivation with respect to the coproduct $\tilde{\Delta}$.

5.5 Conclusion

The first result obtained in this paper is the possibility to construct from directed graphs, families of L-cocommutative coassociative co-dialgebras and therefore, via convolution products families of associative dialgebras. The second result is the possibility to recover from the line-extension of the geometric supports of these coassociative co-dialgebras, known coassociative coalgebras and in the case of the $(n^2, 1)$-De Bruijn graphs, $n > 0$, to obtain a tiling of these Markovian objects by n (geometric supports of) coassociative coalgebras. An important notion, called the achirality has been put forward. We have shown that actions of $M_n(k)$ on the coproducts defining the tiling of the $(n^2, 1)$-De Bruijn graph let globally invariant the relations between them. We gave consequences of such tilings and found examples of cubical trialgebras and more generally, examples of hypercube n-algebras. In addition, this allowed us to construct associative laws which split into several associative ones.

This paper has been pursued in [34]. In [34], the notion of codipterous coalgebras and pre-dendriform coalgebras [10] are established. These spaces constructed from coassociative coalgebra theory extend the notions developed so far and are the elementary boxes of coassociative manifolds. Via these notions, we construct Poisson algebras, dendriform algebras, associative dialgebras (which are not Markovian), associative trialgebras [48]. Notably, the tilings constructed so far will yield examples of coassociative manifolds [34].

These tilings give nice examples of hypercube n-algebras. Let us remark that for all $n > 0$, $M_n(A)$, where A is an associative algebra, is also a hypercube n-algebra.

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[10]These notions have been discovered by J-L Loday and M. Ronco [47] and rediscovered independently, via graph theory by the author [11].

Chapter 6

On representations of braid groups determined by weighted directed graphs

Abstract [1]**:** We prove that any non-L-cocommutative finite Markov L-coalgebra yields at least two solutions of the Yang-Baxter equation and therefore at least two representations of the braid groups.

6.1 Introduction

In this article, k is either the real field or the complex field. Moreover, all the involved vector spaces will have a finite or a denumerable basis.

The first part recalls the main notions on L-coalgebras introduced in [40] and developed in [40, 41, 39, 34]. The second part uses these objects to construct representations of braid groups.

6.2 L-Coalgebras

Definition 6.2.1 [**L-coalgebra**] A *L-coalgebra* $(L, \Delta, \tilde{\Delta})$ over a field k is a k-vector space equipped with a right coproduct, $\Delta : L \to L^{\otimes 2}$ and a left coproduct, $\tilde{\Delta} : L \to L^{\otimes 2}$, verifying the coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. If $\Delta = \tilde{\Delta}$, the L-coalgebra is said *degenerate*. A L-coalgebra is called coassociative if its two coproducts are coassociative. In this case, the equation, $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$, is called the entanglement equation, see [41, 34]. A L-coalgebra may have two counits, the right counit $\epsilon : L \to k$, verifying $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : L \to k$, verifying $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$. Denote by τ, the *transposition* mapping,

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i.e., $L^{\otimes 2} \stackrel{\tau}{\to} L^{\otimes 2}$ such that $\tau(x \otimes y) = y \otimes x$ for all $x, y \in L$. A L-coalgebra $(L, \Delta, \tilde{\Delta})$ is said *L-cocommutative* if and only if for all $v \in L$, $(\Delta - \tau\tilde{\Delta})v = 0$.

Definition 6.2.2 [Finite Markov L-coalgebra] A *Markov L-coalgebra* $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ with dimension $\dim \mathcal{G}$, is a k-vector space generated as a k-vector space by an independent spanning set $\mathcal{G}_0 := (v_i)_{1 \le i \le \dim \mathcal{G}}$ and a L-coalgebra such that for all $v_i \in \mathcal{G}_0$,

$$\Delta_M v_i = \sum_{k:\ v_i \otimes v_k \in I_{v_i}} w_{v_i}(v_i \otimes v_k)\ v_i \otimes v_k \ \text{ and } \ \tilde{\Delta}_M v_i = \sum_{j:\ v_j \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_j \otimes v_i)\ v_j \otimes v_i,$$

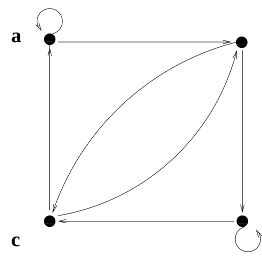
where I_{v_i} are subsets from $\{v_i \otimes v_k, \ \ v_k \in \mathcal{G}_0\}$, J_{v_i} are subsets from $\{v_j \otimes v_i, \ \ v_j \in \mathcal{G}_0\}$ and $w_{v_i} : I_{v_i} \to k$ and $\tilde{w}_{v_i} : J_{v_i} \to k$ are maps called *weights*. Such a L-coalgebra is said *finite* if I_{v_i}, J_{v_i} are finite.

Definition 6.2.3 [Directed graph] A *directed graph* G is a quadruple [54], (G_0, G_1, s, t) where G_0 and G_1 are two denumerable sets respectively called the *vertex set* and the *arrow set*. The two mappings, $s,\ t : G_1 \to G_0$ are respectively called *source* and *terminus*. A vertex $v \in G_0$ is a *source* (resp. a *sink*) if $t^{-1}(\{v\})$ (resp. $s^{-1}(\{v\})$) is empty. A graph G is said *locally finite*, (resp. *row-finite*) if $t^{-1}(\{v\})$ is finite (resp. $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1,\ s(a) = v\}$. A *weight* associated with the vertex v is a mapping $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a weighted graph.

In the sequel, directed graphs will be supposed locally finite and row-finite without sink and source. We recall a Theorem from [40]. Let $G := (G_0, G_1, s, t)$ be a directed graph equipped with a family of weights $(w_v)_{v \in G_0}$. Let us consider the free k-vector space kG_0. The set G_1 is then viewed as a sub-vector space of $kG_0^{\otimes 2}$, still denoted by G_1, by identifying $u \stackrel{\lambda}{\to} v$ with $\lambda u \otimes v$, where $\lambda \in k$ and $u, v \in G_0$. The mappings source and terminus are then linear mappings still called source and terminus $s, t :\ kG_0^{\otimes 2} \to kG_0$, such that $s(u \otimes v) = u$ and $t(u \otimes v) = v$ for all $u, v \in G_0$. The family of weights $(w_v)_{v \in G_0}$ is then viewed as a family of linear mappings from F_v to k. Let $v \in G_0$ and define the right coproduct Δ_M such that $\Delta_M(v) := \sum_{i: a_i \in F_v}\ w_v(a_i)\ v \otimes t(a_i)$ and the left coproduct $\tilde{\Delta}_M$ such that $\tilde{\Delta}_M(v) := \sum_{i: a_i \in P_v}\ w_{s(a_i)}(a_i)\ s(a_i) \otimes v$, where P_v is the set $\{a \in G_1,\ t(a) = v\}$. Define, for all $v \in G_0$, the linear mappings $\tilde{w}_v : P_v \to k$ such that $\tilde{w}_v(a_i) = w_{s(a_i)}(a_i)$ for all $a_i \in P_v$. With these definitions the vector space $(kG_0, \Delta_M, \tilde{\Delta}_M)$ is a finite Markov L-coalgebra. In the sequel, kG_0 will be identified with G.

Remark: [Geometric representation] Let $(L, \Delta, \tilde{\Delta})$ be a L-coalgebra generated as a k-vector space by an independent spanning set L_0. Associate with each tensor product $\lambda v \otimes w$, where $v, w \in L_0$ and $\lambda \in k$, appearing in the definition of the coproducts, a directed arrow $v \stackrel{\lambda}{\to} w$. The weighted directed graph so obtained, denoted by $Gr(L)$, is called the *geometric support* of L. Its vertex set is L_0 and its arrow set, the set of those tensor products $v \otimes w,\ v, w \in L_0$ appearing in the definition of the coproducts. The advantage of this formalism is to generalise the notion of directed graph. We draw attention to the fact that a directed graph can be the geometric support of different L-coalgebras.

Example 6.2.4 The directed graph:



is the geometric support of the degenerate L-coalgebra or coassociative coalgebra, spanned by the basis a, b, c and d, as a k-vector space and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$ and the geometric support of the finite Markov L-coalgebra, spanned by the basis a, b, c and d, as a k-vector space and described by the right coproduct: $\Delta_M a = a \otimes (a+b)$, $\Delta_M b = b \otimes (c+d)$, $\Delta_M c = c \otimes (a+b)$, $\Delta_M d = d \otimes (c+d)$ and the left coproduct: $\tilde{\Delta}_M a = (a+c) \otimes a$, $\tilde{\Delta}_M b = (a+c) \otimes b$, $\tilde{\Delta}_M c = (b+d) \otimes c$, $\tilde{\Delta}_M d = (b+d) \otimes d$.

Remark: Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra. If the family of weights used for describing right and left coproducts take values into $\mathbb{R}_+$ and if the right counit $\epsilon : v \mapsto 1$ exists, then the geometric support associated with $\mathcal{G}$ is a directed graph equipped with a family of probability vectors.

6.3 Representation of braid groups determined by weighted directed graphs

Definition 6.3.1 [Yang-Baxter equation] Let us consider a k-vector space V. Let $\hat{\Psi}$ be an automorphism on $V^{\otimes 2}$, $\hat{\Psi}$ verifies the *Yang-Baxter equation* (YBE) if [25]:

$$(\hat{\Psi} \otimes id)(id \otimes \hat{\Psi})(\hat{\Psi} \otimes id) = (id \otimes \hat{\Psi})(\hat{\Psi} \otimes id)(id \otimes \hat{\Psi}).$$

Such a solution is also called a R-matrix. Let us denote by S the set of solutions of YBE and by $\mathrm{Aut}(V)$ the linear automorphisms group of V.

Remark: Let us recall that any solution of YBE supplies a representation of braid groups, see for instance [25][49]. The aim of this article is to show that Markovian coproducts, used to code the paths of directed graphs, yield solutions of YBE and thus representation of braid groups.

Theorem 6.3.2 *Let V be a k vector space. The mapping $\widehat{\ } : Aut(V) \to S \times S$ defined by $\Psi \mapsto (\hat{\Psi}_1,\ \hat{\Psi}_2)$ where $\hat{\Psi}_1 := \tau(id \otimes \Psi)$ et $\hat{\Psi}_2 := \tau(\Psi \otimes id)$ is injective.*

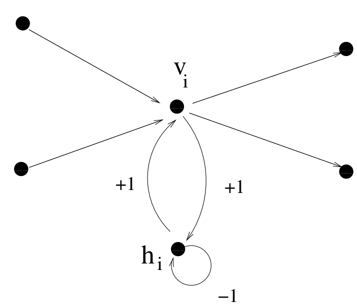
Proof: Straightforward. □

Theorem 6.3.3 *Let V be a k-vector space. If A and B are automorphisms on V, we define $c(A, B) := \tau(A \otimes B)$. Let Ψ_1, Ψ_2, Ψ_3 and Ψ_4 be four automorphisms on V. Then, $c(\Psi_1, \Psi_2)$ is a solution of YBE if and only if $\Psi_1\Psi_2 = \Psi_2\Psi_1$. If $c(\Psi_1, \Psi_2)$ and $c(\Psi_3, \Psi_4)$ are solutions of YBE and $[\Psi_1\Psi_4, \Psi_2\Psi_3] = 0$, then $\tau c(\Psi_1, \Psi_2)c(\Psi_3, \Psi_4)$ is still a solution of YBE.*

Proof: Let Ψ_1, Ψ_2, Ψ_3 and Ψ_4 be four automorphisms on V. The first claim is straightforward and so is the second one by noticing that $\tau c(\Psi_1, \Psi_2) c(\Psi_3, \Psi_4) = c(\Psi_2\Psi_3, \Psi_1\Psi_4)$. $\square$

Definition 6.3.4 [The companion graph] Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra, generated as a k-vector space by an independent spanning set $\mathcal{G}_0 := (v_i)_{i=1,\dots,\dim\mathcal{G}}$. These coproducts define a directed graph $Gr(\mathcal{G})$ without sink and source, such that $Gr(\mathcal{G})_0 := \mathcal{G}_0$, equipped with two family of weights $(w_{v_i})_{v_i\in\mathcal{G}_0}$ and $(\tilde{w}_{v_i})_{v_i\in\mathcal{G}_0}$. By definition, for all $v_i \in \mathcal{G}_0$, there exists finite sets I_{v_i} and J_{v_i} such that $\tilde{\Delta}_M v_i = \sum_{k:\ v_k\otimes v_i\in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k \otimes v_i$ and $\Delta_M v_i = \sum_{k:\ v_i\otimes v_k\in I_{v_i}} w_{v_i}(v_i \otimes v_k)\ v_i \otimes v_k$, for all i, with $1 \le i \le \dim\mathcal{G}$.

Let H be a k-vector space such that dim H = dim $\mathcal{G}$, generated as a k-vector space by an independent spanning set $H_0 := (h_i)_{i=1,\dots,\dim\mathcal{G}}$. With each vector $v_i \in \mathcal{G}_0$, let us associate a unique $h_i \in H_0$. Denote by $\mathcal{G}_* := \mathcal{G} \oplus H$, the associated finite Markov L-coalgebra, such that for $i = 1, \dots, \dim\mathcal{G}$, the left coproduct is defined by $\tilde{\Delta}_{M_*}(v_i) := \tilde{\Delta}_M v_i + h_i \otimes v_i$, $\tilde{\Delta}_{M_*} h_i = v_i \otimes h_i - h_i \otimes h_i$ and the right coproduct is defined by $\Delta_{M_*} v_i = \Delta_M v_i + v_i \otimes h_i$, $\Delta_{M_*} h_i = h_i \otimes v_i - h_i \otimes h_i$. The finite Markov L-coalgebra $(\mathcal{G}_*, \Delta_{M_*}, \tilde{\Delta}_{M_*})$ is called *the companion* of $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$. It is defined up to an isomorphism of k-vector space. The directed graph so obtained is called the *companion graph.*



Geometric representation of the companion graph at v_i.

Lemma 6.3.5 *Let $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ be a finite Markov L-coalgebra, generated as a k-vector space by an independent spanning set $\mathcal{G}_0 := (v_i)_{1\le i\le\dim\mathcal{G}}$ and such that the left coproduct is labelled on finite sets J_{v_i}. Denote by $\tilde{w}$ the family of weights $(\tilde{w}_{v_i})_{v_i\in\mathcal{G}}$ necessary to the definition of the left coproduct of $\mathcal{G}$. The linear mapping $\Psi_{\tilde{w}} : \mathcal{G}_* \to \mathcal{G}_*$ defined by $v_i \mapsto \sum_{k:\ v_k\otimes v_i\in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i$, $h_i \mapsto \sum_{k:\ v_k\otimes v_i\in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i + v_i$ and the linear mapping $\Phi_{\tilde{w}} : \mathcal{G}_* \to \mathcal{G}_*$ defined by $v_i \mapsto h_i - v_i$ and $h_i \mapsto \sum_{k:\ v_k\otimes v_i\in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (v_k - h_k) + v_i$, for all $1 \le i \le \dim\mathcal{G}$. Then $\Phi_{\tilde{w}} = (\Psi_{\tilde{w}})^{-1}$.*

Proof: Fix i, $1 \leq i \leq \dim \mathcal{G}$ and let us prove that $\Psi_{\tilde{w}}$ is invertible.

$$
\begin{aligned}
v_i \quad & \xrightarrow{\Psi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i \\
& \xrightarrow{\Phi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (h_k - v_k) + \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (v_k - h_k) + v_i = v_i. \\
v_i \quad & \xrightarrow{\Phi_{\tilde{w}}} \quad h_i - v_i \\
& \xrightarrow{\Psi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i + v_i - \left(\sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i \right) = v_i.
\end{aligned}
$$

$$
\begin{aligned}
h_i \quad & \xrightarrow{\Psi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i + v_i \\
& \xrightarrow{\Phi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (h_k - v_k) + \left(\sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (v_k - h_k) + v_i \right) \\
& \qquad + (h_i - v_i) = h_i. \\
h_i \quad & \xrightarrow{\Phi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ (v_k - h_k) + v_i \\
& \xrightarrow{\Psi_{\tilde{w}}} \quad \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i) \left(\sum_{l:\ v_l \otimes v_k \in J_{v_k}} \tilde{w}_{v_k}(v_l \otimes v_k)\ v_l + h_k \right) \\
& \qquad - \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i) \left(\sum_{l:\ v_l \otimes v_k \in J_{v_k}} \tilde{w}_{v_k}(v_l \otimes v_k)\ v_l + h_k + v_k \right) \\
& \qquad + \sum_{k:\ v_k \otimes v_i \in J_{v_i}} \tilde{w}_{v_i}(v_k \otimes v_i)\ v_k + h_i = h_i.
\end{aligned}
$$

□

Remark: There exists a unique couple of automorphisms $(\Psi_{\tilde{w}}, \Phi_{\tilde{w}})$ from $\mathrm{Aut}(\mathcal{G}_*)$ such that for all $v_i \in \mathcal{G}_0$, $\tilde{\Delta}_{M_*} v_i = \Psi_{\tilde{w}}(v_i) \otimes v_i$ and for all $h_i \in H_0$, $\tilde{\Delta}_{M_*} h_i = -\Phi_{\tilde{w}}(v_i) \otimes h_i$ and $\Phi_{\tilde{w}} = (\Psi_{\tilde{w}})^{-1}$. What was done with the left coproduct $\tilde{\Delta}_{M_*}$ remains exact with the right coproduct Δ_{M_*}. The equations remain the same except the labels of the sums $\{k :\ v_k \otimes v_i \in J_{v_i}\}$ which obviously become $\{k :\ v_i \otimes v_k \in I_{v_i}\}$ and where the family of weights $\tilde{w}$ is removed by the family of weights w. There always exists a unique couple of automorphisms (Ψ_w, Φ_w) of $\mathrm{Aut}(\mathcal{G}_*)$ such that for all $v_i \in \mathcal{G}_0$, $\Delta_{M_*} v_i = v_i \otimes \Psi_w(v_i)$ and for all $h_i \in H_0$, $\Delta_{M_*} h_i = h_i \otimes -\Phi_w(v_i)$ and $\Phi_w = \Psi_w^{-1}$.

Theorem 6.3.6 *With each non-L-cocommutative finite Markov L-coalgebra $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ are associated at least two couples of representations of braid groups determined by its coproducts. If $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$ is a L-cocommutative finite Markov L-coalgebra, then the two couples of solutions built on the coproducts are equal.*

Proof: With any finite Markov L-coalgebra $(\mathcal{G}, \Delta_M, \tilde{\Delta}_M)$, is associated, up to an isomorphism, a unique finite Markov L-coalgebra $(\mathcal{G}_*, \Delta_{M_*}, \tilde{\Delta}_{M_*})$. Therefore, there exists, up to an isomorphism, two different automorphisms Ψ_w and $\Psi_{\tilde{w}}$ of $\mathrm{Aut}(\mathcal{G}_*)$, since the L-coalgebra is not

L-cocommutative, coding the information contained into the coproducts of $\mathcal{G}_*$. Therefore, $\hat{\Psi}_w$ and $\hat{\Psi}_{\tilde{w}}$ are two couples of solutions of YBE thanks to the Theorem 6.3.2. $\square$

Remark: Therefore, there exist, among all the representations of the braid groups, representations which code weighted directed graphs.

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Chapter 7

From entangled codipterous coalgebras to coassociative manifolds

Abstract [1]**:**

In our previous works, we associated with each Hopf algebra, bialgebra and coassociative coalgebra, a directed graph. Describing how two coassociative coalgebras, via their directed graphs, can be entangled, leads to consider special coalgebras, called codipterous coalgebras. We yield a graphical interpretation of the notion of codipterous coalgebra and explain the necessity to study them. By gluing these objects, we easily obtain, particular objets named coassociative codialgebras and coassociative cotrialgebras and yield, thanks to their directed graph, a simple interpretation of these structures. Similarly, we construct Poisson algebras and dendriform coalgebras. From these constructions, we yield a new look on coassociative coalgebras and construct an analogue of topological manifolds called coassociative manifolds. Links with non-directed graphs are also given.

7.1 Introduction

By k, we mean either the real field or the complex field. Moreover, all the vector spaces will have a finite or a denumerable basis.

In this article, we are led to manipulate several kind of coalgebras. These coalgebras will be visualised through a directed graph. Let us recall for the convenience of the reader, the usual definition of a directed graph.

Definition 7.1.1 [Directed graph] A *directed graph* G is a quadruple [54], (G_0, G_1, s, t) where G_0 and G_1 are two denumerable sets respectively called the *vertex set* and the *arrow set*. The two mappings, $s,\ t : G_1 \to G_0$ are respectively called *source* and *terminus*. A vertex $v \in G_0$

[1] *2000 Mathematics Subject Classification: 16W30; 05C20; 05C90. Key words and phrases:* directed graphs, Poisson algebra, Hopf algebra, dendriform algebra, associative dialgebra, associative trialgebra, codipterous coalgebra, pre-dendriform coalgebra, coassociative L-coalgebra.

is a *source* (resp. a *sink*) if $t^{-1}(\{v\})$ (resp. $s^{-1}(\{v\})$) is empty. A graph G is said *locally-finite*, (resp. *row-finite*) if $t^{-1}(\{v\})$ is finite (resp. $s^{-1}(\{v\})$ is finite). Let us fix a vertex $v \in G_0$. Define the set $F_v := \{a \in G_1,\ s(a) = v\}$. A *weight* associated with the vertex v is a mapping $w_v : F_v \to k$. A directed graph equipped with a family of weights $w := (w_v)_{v \in G_0}$ is called a *weighted graph*.

In the sequel, directed graphs will be supposed locally-finite and row-finite. Let us introduce particular coalgebras named L-coalgebras [2] and explain why this notion is interesting.

Definition 7.1.2 [**L-coalgebra**] A *L-coalgebra* G over a field k is a k-vector space equipped with a right coproduct, $\Delta : G \to G^{\otimes 2}$ and a left coproduct, $\tilde{\Delta} : G \to G^{\otimes 2}$, verifying the coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. If $\Delta = \tilde{\Delta}$, the coalgebra is said *degenerate*. A L-coalgebra may have two counits, the right counit $\epsilon : G \to k$, verifying $(id \otimes \epsilon)\Delta = id$ and the left counit $\tilde{\epsilon} : G \to k$, verifying $(\tilde{\epsilon} \otimes id)\tilde{\Delta} = id$. A L-coalgebra is said *coassociative* if its two coproducts are coassociative, in this case the equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$ is called the **entanglement equation**. Denote by τ, the *transposition* mapping, i.e., $G^{\otimes 2} \xrightarrow{\tau} G^{\otimes 2}$ such that $\tau(x \otimes y) = y \otimes x$, for all $x, y \in G$. The L-coalgebra G is said to be *L-cocommutative* if for all $v \in G$, $(\Delta - \tau\tilde{\Delta})v = 0$.

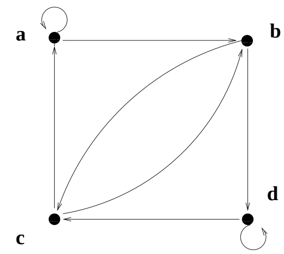
Let G be a directed graph equipped with a family of weights $(w_v)_{v \in G_0}$. Let us consider the free vector space kG_0. The set G_1 is then viewed as a subset of $(kG_0)^{\otimes 2}$ by identifying $a \in G_1$ with $s(a) \otimes t(a)$. The mappings source and terminus are then linear mappings still called source and terminus $s, t :\ (kG_0)^{\otimes 2} \to kG_0$, such that $s(u \otimes v) = u$ and $t(u \otimes v) = v$, for all $u, v \in G_0$. The family of weights $(w_v : F_v \to k)_{v \in G_0}$ is then viewed as a family of linear mappings. Let $v \in G_0$. Define the right coproduct $\Delta : kG_0 \to (kG_0)^{\otimes 2}$, such that $\Delta(v) := \sum_{i: a_i \in F_v}\ w_v(a_i)\ v \otimes t(a_i)$ and the left coproduct $\tilde{\Delta} : kG_0 \to (kG_0)^{\otimes 2}$, such that $\tilde{\Delta}(v) := \sum_{i: a_i \in P_v}\ w_{s(a_i)}(a_i)\ s(a_i) \otimes v$, where P_v is the set $\{a \in G_1,\ t(a) = v\}$. With these definitions the vector space kG_0 is a L-coalgebra called a finite Markov L-coalgebra since its coproducts Δ and $\tilde{\Delta}$ verify the coassociativity breaking equation $(\tilde{\Delta} \otimes id)\Delta = (id \otimes \Delta)\tilde{\Delta}$. This particular coalgebra is called in addition finite Markov (L-coalgebra) because for all $v \in G_0$, the sets F_v and P_v are finite and the coproducts are of the form $\Delta(v) := v \otimes \cdots$ and $\tilde{\Delta}(v) := \cdots \otimes v$.

Assume that we start with the Markov L-coalgebra just described and associate with each tensor product $\lambda x \otimes y$, where $\lambda \in k$ and $x, y \in G_0$, appearing in the definition of the coproducts, a directed arrow $x \xrightarrow{\lambda} y$. The directed graph so obtained, called the *geometric support* of this L-coalgebra, is up to a graph isomorphism [3], the directed graph we start with. Therefore, general L-coalgebras generalise the notion of directed graph. If G is a L-coalgebra generated as a k-vector space by an independent spanning set G_0, then its geometric support $Gr(G)$ is a directed graph with vertex set $Gr(G)_0 = G_0$ and with arrow set $Gr(G)_1$, the set of tensor products $a \otimes b$, with $a, b \in G_0$, appearing in the definition of the coproducts of G. As a coassociative coalgebra is a particular L-coalgebra, we naturally construct its directed graph. We draw attention to the fact that a directed graph can be the geometric support of different L-coalgebras.

[2] This notion has been introduced in [40] and developed in [40][41][39][38].

[3] A *graph isomorphism* $f : G \to H$ between two graphs G and H is a pair of bijection $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ such that $f_0(s_G(a)) = s_H(f_1(a))$ and $f_0(t_G(a)) = t_H(f_1(a))$ for all $a \in G_1$. All the directed graphs in this formalism will be considered up to a graph isomorphism.

Example 7.1.3 The directed graph:



is the geometric support associated with the degenerate L-coalgebra or coassociative coalgebra, spanned as a k-vector space by the basis a, b, c and d and described by the following coproduct: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$ and the geometric support of the finite Markov L-coalgebra, spanned as a k-vector space by the basis a, b, c and d, and described by the right coproduct: $\Delta_M a = a \otimes (a + b)$, $\Delta_M b = b \otimes (c + d)$, $\Delta_M c = c \otimes (a + b)$, $\Delta_M d = d \otimes (c + d)$ and the left coproduct: $\tilde{\Delta}_M a = (a + c) \otimes a$, $\tilde{\Delta}_M b = (a + c) \otimes b$, $\tilde{\Delta}_M c = (b + d) \otimes c$, $\tilde{\Delta}_M d = (b + d) \otimes d$.

Remark: Let G be a finite Markov L-coalgebra. If the family of weights used for describing right and left coproducts take values into $\mathbb{R}_+$ and if the right counit $\epsilon : v \mapsto 1$ exists, then the geometric support associated with G is a directed graph equipped with a family of probability vectors.

Before going on, let us interpret what represents the L-cocommutativity in the case of a finite Markov L-coalgebra. A directed graph is said bi-directed if for any arrow from a vertex v_1 to a vertex v_2, there exists an arrow from v_2 to v_1. To take into account the bi-orientation of a directed graph in an algebraic way, we first embed this directed graph into the finite Markov L-coalgebra described above. We then notice that a directed graph is bi-directed if and only if $\Delta = \tau\tilde{\Delta}$. Therefore, in this algebraic framework, we are naturally led to consider the L-cocommutator space $\ker(\Delta - \tau\tilde{\Delta})$. Dualizing this formula leads to consider an algebra D equipped with two products $\vdash$ and $\dashv$ and to consider the particular commutator space $[x, y] := x \dashv y - y \vdash x$. The bracket, $[-, z]$, verifies the "Jacobi identity", i.e., $[[x, y], z] = [[x, z], y] + [x, [y, z]]$, if D is an algebra called an associative dialgebra [45].

Another motivation concerning associative dialgebras is the following. In a long-standing project whose ultimate aim is to study periodicity phenomena in algebraic K-theory, J-L. Loday in [45], and J-L. Loday and M. Ronco in [48] introduce several kind of algebras, one of which is the "non-commutative Lie algebras", called *Leibniz algebras*. Such algebras D are described by a bracket $[-, z]$ verifying the Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and the Leibniz algebra turns out to be a Lie algebra. A way to construct such Leibniz algebra is to start from an *associative dialgebra*, that is a k-vector space D equipped with two associative products, $\vdash$ and $\dashv$, such that for all $x, y, z \in D$

1. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$,

2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,

3. $(x \dashv y) \vdash z = (x \vdash y) \vdash z$.

The associative dialgebra is then a Leibniz algebra by defining the bracket $[x, y] := x \dashv y - y \vdash x$, for all $x, y \in D$. The operad associated with associative dialgebras is then Koszul dual to the operad associated with dendriform algebras, a *dendriform algebra* Z being a k-vector space equipped with two binary operations, $\prec$, $\succ$: $Z \otimes Z \rightarrow Z$, satisfying the following axioms:

1. $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,

2. $(a \succ b) \prec c = a \succ (b \prec c)$,

3. $(a \prec b) \succ c + (a \succ b) \succ c = a \succ (b \succ c)$.

This notion dichotomizes the notion of associativity since the product $a * b = a \prec b + a \succ b$, for all $a, b \in Z$ is associative. Before continuing, let us recall the following Proposition from [40],

Proposition 7.1.4 *Any associative dialgebra can be viewed as a dendriform algebra.*

Proof: Let $(D, \vdash, \dashv)$ be an associative dialgebra. Let $a, b \in D$. The relations, $a \prec b = a \dashv b$ and $a \succ b = a \vdash b - a \dashv b$, embed D into a dendriform algebra. We notice that $a * b := a \prec b + a \succ b = a \vdash b$ is associative. For instance, $(a \prec b) \prec c = a \prec (b * c)$ means $(a \dashv b) \dashv c = a \dashv (b \vdash c)$ and so forth. □

By dualizing these notions, we can easily define *coassociative codialgebras* and *dendriform coalgebras*.

Definition 7.1.5 [Coassociative codialgebra] A *coassociative codialgebra* D is a k-vector space equipped with two coproducts $\delta, \hat{\delta}: D \rightarrow D^{\otimes 2}$, verifying:

1. δ and $\hat{\delta}$ are coassociative,

2. $(id \otimes \hat{\delta})\hat{\delta} = (id \otimes \delta)\hat{\delta}$,

3. $(\delta \otimes id)\delta = (\hat{\delta} \otimes id)\delta$,

4. $(\delta \otimes id)\hat{\delta} = (id \otimes \hat{\delta})\delta$.

Definition 7.1.6 [Dendriform coalgebra] A *dendriform coalgebra* Z is a k-vector space equipped with two coproducts $\delta, \hat{\delta}: Z \rightarrow Z^{\otimes 2}$, verifying:

1. $(id \otimes (\delta + \hat{\delta}))\hat{\delta} = (\hat{\delta} \otimes id)\hat{\delta}$,

2. $(id \otimes \hat{\delta})\delta = (\delta \otimes id)\hat{\delta}$,

3. $((\hat{\delta} + \delta) \otimes id)\delta = (id \otimes \delta)\delta$.

This notion dichotomizes the notion of coassociativity since $(\delta + \hat{\delta})$ is a coassociative coproduct.

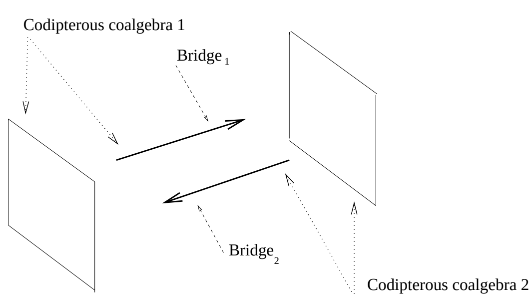
Similarly, so as to define a *"non-commutative version"* of Poisson algebra, J-L. Loday and M. Ronco introduce in [48], the *associative trialgebras*. Let us just mention that the operad associated with trialgebras is Koszul dual to the operad associated with dendriform trialgebras, dendriform trialgebras being k-vector spaces equipped with three laws $\prec$, $\succ$, $\cdot$, verifying special axioms, see also [33]. Similarly, the law $*$ such that $x * y := x \prec y + x \succ y + x \cdot y$, will be associative. Here, we are interested in contructions of coassociative cotrialgebras.

Definition 7.1.7 [Coassociative cotrialgebra] A *coassociative cotrialgebra* T is a k-vector space equipped with three coassociative coproducts $\Delta, \delta, \hat{\delta} : T \to T^{\otimes 2}$, verifying:

1. $(T, \delta, \hat{\delta})$ is a coassociative codialgebra,
2. $(\hat{\delta} \otimes id)\hat{\delta} = (id \otimes \Delta)\hat{\delta}$,
3. $(\Delta \otimes id)\hat{\delta} = (id \otimes \hat{\delta})\Delta$,
4. $(\hat{\delta} \otimes id)\Delta = (id \otimes \delta)\Delta$,
5. $(\delta \otimes id)\Delta = (id \otimes \Delta)\delta$,
6. $(\Delta \otimes id)\delta = (id \otimes \delta)\delta$.

Let us notice that all these vector spaces are equipped with two coproducts verifying the entanglement equation. Therefore, there exists a graphical representation of these objets determined by directed graphs. These directed graphs will be the geometric supports associated with these particular L-coalgebras.

In addition to a graphical representation of the axioms defined above, the main results of this article lie in the constructions of L-Hopf algebras, i.e., L-bialgebras equipped with two linear maps σ and $\tilde{\sigma}$ such that $m(id \otimes \tilde{\sigma})\tilde{\Delta} := 1\tilde{\epsilon}$ and $m(\sigma \otimes id)\Delta := 1\epsilon$. We construct also coassociative codialgebras as well as coassociative cotrialgebras. For that, we start with two coassociative coalgebras and construct bridges, i.e., coproducts, between them. Graphically speaking, the rôle of these bridges is to establish a connection between the two coalgebras. Therefore, we are led to consider a vector space equipped with two coproducts, a coassociative one and a bridge, i.e., an extra left (or right) comodule on itself. Such a vector space will be called a codipterous (or anti-codipterous) coalgebra.



Entanglement of two codipterous coalgebras.

Schematically speaking, a codipterous coalgebra is a face represented by a square symbolizing a coassociative coalgebra and by a bridge, symbolizing the left comodule. We realize an entanglement by "gluing" two codipterous coalgebras, such that their bridges verify the entanglement equation. There are two important interpretations of such a construction. Firstly, suppose that we consider a k-vector space E, with dimension large enough to include several codipteous coalgebras. Suppose we view a codipterous coalgebra like an atom in chemistry and the k-vector space E as a chemical solution. The entanglement of two codipterous coalgebras will yield a so-called L-molecule. By entangling several codipterous coalgebras, we can construct more and more complicated L-molecules. Secondly, what we have obtained in the construction of such a L-molecule, can be viewed as an interlocking of several coassociative coalgebras. Graphically, as we associate a directed graph with each coalgebra, a L-molecule is viewed as a special directed graph which admits a "coassociative" covering. By analogy with the topological manifolds, which are covered with open sets, we define coassociative manifolds. Indeed, the directed graph associated with a coassociative manifold will admit a covering with directed graphs, geometric support of coassociative coalgebras. Hence, the rôle of an open set will be played by a coassociative coalgebra.

Let us briefly introduce the organisation of the paper. In Section 7.2, we define the notion of self-entanglement and construct dialgebras, trialgebras, Poisson and Leibniz algebra and dendriform algebras. In Section 7.3 and 7.4, we construct other types of entanglement which lead to L-Hopf algebras and end this paper to the notion of coassociative manifold and L-molecule. We show links between non-directed graphs and Markovian coassociative manifolds.

7.2 Entanglement of codipterous coalgebras

The aim of this Section is to construct associative dialgebras, associative trialgebras, Poisson and Leibniz algebras and L-Hopf algebras. Let us start with the definition of a codipterous coalgebra and the notion of pre-dendriform coalgebra.

Definition 7.2.1 [codipterous coalgebra, anti-codipterous coalgebra] A k-vector space $\mathbb{D}$ equipped with two coproducts $\Delta, \delta : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ verifying:

1. Coas: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.
2. Codip: $(\Delta \otimes id)\delta = (id \otimes \delta)\delta$.

is called a *codipterous coalgebra* [4]. We call *bridge*, the coproduct δ. Similarly, we call an *anti-codipterous coalgebra* a k-vector space $\hat{\mathbb{D}}$ equipped with two coproducts $\Delta, \hat{\delta} : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ such that Δ is coassociative and $(id \otimes \Delta)\hat{\delta} = (\hat{\delta} \otimes id)\hat{\delta}$.

Definition 7.2.2 [Entanglement of codipterous coalgebras] Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two subspaces of a vector space E. Suppose $(\mathbb{D}_1, \Delta_1, \delta_1)$ and $(\mathbb{D}_2, \Delta_2, \delta_2)$ are two codipterous coalgebras.

[4] A codipterous coalgebra (resp. anti-codipterous coalgebra) can also be viewed as a coassociative coalgebra with an extra left (resp. right) comodule on itself.

The spaces $\mathbb{D}_1$ and $\mathbb{D}_2$ are said *entangled* if the bridges $\delta_1,\ \delta_2$ verify the entanglement equation. The entanglement is said *achiral* if the entanglement equation is verified whatever the positions occupied by δ_1 and δ_2 and *chiral* otherwise. The achiral (resp. chiral) entanglement is denoted by $[\mathbb{D}_1 \overset{\delta_1,\ \delta_2}{\longleftrightarrow} \mathbb{D}_2]$ (resp. $[\mathbb{D}_1 \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \mathbb{D}_2]$, if $(id \otimes \delta_2)\delta_1 = (\delta_1 \otimes id)\delta_2$ and $[\mathbb{D}_1 \underset{\delta_1}{\overset{\delta_2}{\rightleftharpoons}} \mathbb{D}_2]$, if $(id \otimes \delta_1)\delta_2 = (\delta_2 \otimes id)\delta_1)$.

Similarly, a codipterous coalgebra $(\mathbb{D}, \Delta, \delta_1)$ is entangled with an anti-codipterous coalgebra $(\mathbb{D}, \Delta, \hat{\delta}_1)$ if the bridges δ_1 and $\hat{\delta}_1$ verify the entanglement equation. More generally, the notion of entanglement between two coproducts means that they verify the entanglement equation.

Theorem 7.2.3 *Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two sub-spaces of a k-vector space E. Suppose $(\mathbb{D}_1, \Delta_1, \delta_1)$ (resp. $(\mathbb{D}_1, \Delta_1, \hat{\delta}_1)$) and $(\mathbb{D}_2, \Delta_2, \delta_2)$ (resp. $(\mathbb{D}_2, \Delta_2, \hat{\delta}_2)$) are two codipterous coalgebras (resp. anti-codipterous coalgebras). The entanglement so obtained is still a codipterous coalgebra (resp. anti-codipterous coalgebra).*

Proof: Denote by Δ_*, the coassociative coproduct such that $\Delta_* := \Delta_1$ over $\mathbb{D}_1$, (resp. $:= \Delta_2$ over $\mathbb{D}_2$) and $\delta_* := \delta_1$, (resp. $:= \delta_2$) over $\mathbb{D}_1$ (resp. over $\mathbb{D}_2$). The space $(\mathbb{D} := \mathbb{D}_1 + \mathbb{D}_2, \Delta_*, \delta_*)$ is a codipterous coalgebra. Similarly for the entanglement of two anti-codipterous coalgebras. □

Definition 7.2.4 [Pre-dendriform coalgebra] A k-vector space $\mathbb{D}$ equipped with three coproducts $\Delta, \delta, \hat{\delta} : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ verifying:

1. Coas: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.
2. Codip: $(\Delta \otimes id)\delta = (id \otimes \delta)\delta$.
3. Anti-codip: $(id \otimes \Delta)\hat{\delta} = (\hat{\delta} \otimes id)\hat{\delta}$.
4. Entanglement equation: $(id \otimes \hat{\delta})\delta = (\delta \otimes id)\hat{\delta}$.

is called a *pre-dendriform coalgebra*. Similarly, the coproducts δ and $\hat{\delta}$ are called *bridges*. Notice also that if $\Delta = \delta + \hat{\delta}$, then the pre-dendriform coalgebra becomes a dendriform coalgebra.

Remark: With the notion of entanglement, the reader may view the pre-dendriform coalgebra as the entanglement of the codipterous coalgebra $(\mathbb{D}, \Delta, \delta)$ with the anti-codipterous coalgebra $(\mathbb{D}, \Delta, \hat{\delta})$.

Remark: The axioms and terminology of dipterous algebras were discovered by J-L. Loday and M. Ronco [11, 47]. The axioms of pre-dendriform coalgebras were independently discovered by the author to describe the entanglement of two directed graphs. The denomination, pre-dendriform coalgebra, was also suggested by J-L. Loday [11]. If the coassociative law Δ is equal to $\delta + \hat{\delta}$, then a pre-dendriform coalgebra turns out to be a dendriform coalgebra. In terms of directed graphs, this means that the directed graphs determined by the bridges δ and $\hat{\delta}$ realize a covering of the directed graph determined by Δ. The covering becomes a tiling if the intersection of the arrow sets covering the graph determined by Δ is empty. The reader will notice then the interest to break coassociativity in several coproducts. Fix an integer n, Let $(\delta_i)_{1 \leq i \leq n}$ be n coproducts breaking the coassociativity of a coassociative coproduct Δ, i.e., n coproducts such that $\sum_{1\leq i\leq n} \delta_i = \Delta$. This decomposition can be viewed as a covering of the directed graph associated with Δ by n directed graphs associated with the coproducts δ_i. The difficulty is to find convenient covering, i.e., convenient relations between coproducts such as by dualizing, the operads so obtained are Koszul. It is the case with the non-Σ operad of dendriform algebras and with the non-Σ operad of dendriform trialgebras [5].

Definition 7.2.5 [Face, side] In this article, we will manipulate k-vector spaces [6] $(M, (\delta_i)_{i\in I})$ equipped with coassociative coproducts δ_i, $i \in I$. With the graphical viewpoint in mind, a *face* of such a space is $(M, (\delta_i)_{i\in J})$, $J \subset I$, a *side* is just a coassociative coalgebra (M, δ_{i_0}), i.e., $J := \{i_0\}$.

A way to produce bridges is to consider a channel map.

Definition 7.2.6 [Channel map] Let E be a k-vector space and C_1, C_2 two subspaces of E, such that $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$, are two coassociative coalgebras, resp. two bialgebras, resp. two Hopf algebras. A linear map Φ is said to be a *channel map* if $\Phi: C_1 \to C_2$ is invertible and a coalgebra morphism, resp. a bialgebra morphism, resp. a Hopf algebra morphism, i.e., $\Delta_2\Phi = (\Phi \otimes \Phi)\Delta_1$ and $\epsilon_2\Phi = \epsilon_1$ and so on. If C_1 and C_2 are also algebras with units, then the channel must be unital.

7.2.1 Self-entanglement

We now study the self-entanglement, i.e., the entanglement of two copies of a same codipterous coalgebra. A copy is produced thanks to a channel map. This kind of entanglement will yield associative dialgebras and associative trialgebras.

Theorem 7.2.7 *Let $(C_1, \Delta_1, \epsilon_1)$ and $(C_2, \Delta_2, \epsilon_2)$ be two coassociative coalgebras of a k-vector space E, with $\Phi: C_1 \to C_2$, a channel map without fixed point. Suppose $\Delta_1 = \Delta_2$ over $C_1 \cap C_2$. Consider the subspace $\mathbb{D} := C_1 \oplus C_2$. Denote by $\Delta^*: \mathbb{D} \to \mathbb{D}^{\otimes 2}$, the coproduct such that over C_1, $\Delta_* := \Delta_1$ and over C_2, $\Delta_* := \Delta_2$. Denote by $\delta_1: \mathbb{D} \to \mathbb{D}^{\otimes 2}$, the coproduct such that over C_1, $\delta_1 := \Delta_1$ and $\delta_1\Phi := (id \otimes \Phi)\Delta_1$. Then $(\mathbb{D}, \Delta_*, \delta_1)$ is a codipterous coalgebra. Moreover $(\epsilon_1 \otimes id)\delta_1 = id$. Denote by $\hat{\delta}_1: \mathbb{D} \to \mathbb{D}^{\otimes 2}$ the coproduct such that over C_1, $\hat{\delta}_1 := \Delta_1$ and $\hat{\delta}_1\Phi := (\Phi \otimes id)\Delta_1$. Then $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ is a (chiral) pre-dendriform coalgebra. Moreover $(id \otimes \epsilon_1)\hat{\delta}_1 = id$.*

Proof: Notice that Δ_* is coassociative. The equalities $(\epsilon_1 \otimes id)\delta_1 = id$ and $(id \otimes \epsilon_1)\hat{\delta}_1 = id$ are straightforward. Let us check $(\Delta_* \otimes id)\delta_1 = (id \otimes \delta_1)\delta_1$, which holds over C_1. Over C_2, we observe that the right hand side is equal to $(id \otimes id \otimes \Phi)(\Delta_1 \otimes id)\Delta_1$ and the left hand side is equal to $(id \otimes id \otimes \Phi)(id \otimes \Delta_1)\Delta_1$. We also easily obtain $(id \otimes \Delta_*)\hat{\delta}_1 = (\hat{\delta}_1 \otimes id)\hat{\delta}_1$. We can check that $(id \otimes \hat{\delta}_1)\delta_1 = (\delta_1 \otimes id)\hat{\delta}_1$, holds over C_1. Over C_2, the equalities $(id \otimes \hat{\delta}_1)\delta_1 = (id \otimes \Phi \otimes id)(id \otimes \Delta_1)\Delta_1$ and $(\delta_1 \otimes id)\hat{\delta}_1 = (id \otimes \Phi \otimes id)(\Delta_1 \otimes id)\Delta_1$ prove that $\mathbb{D}$ is a (chiral) pre-dendriform coalgebra. □

[5] See also J-L. Loday [46].

[6] Such k-vector space will be called coassociative manifolds at the end of this paper.

Remark: To construct the bridges $\hat{\delta}_1$ and δ_1 we decided to produce a copy C_2 of C_1 via the channel map Φ. Similarly, we construct bridges from C_2 by using Φ^{-1} instead of Φ. By reversing the point of view, we get $\hat{\delta}_2 := \delta_2 := \Delta_2$ over C_2. Over C_1, $\hat{\delta}_2\Phi^{-1} := (\Phi^{-1} \otimes id)\Delta_2$ and $\delta_2\Phi^{-1} := (id \otimes \Phi^{-1})\Delta_2$. Notice that results holding for the pre-dendriform coalgebra $(\mathbb{D},\ \Delta_*,\ \delta_1,\ \hat{\delta}_1)$ are still valid with the pre-dendriform coalgebra $(\mathbb{D},\ \Delta_*,\ \delta_2,\ \hat{\delta}_2)$.

Theorem 7.2.8 *With the hypotheses, the notation of the Theorem 7.2.7, and the previous remark, the codipterous coalgebra $(\mathbb{D},\ \Delta_*,\ \delta_1)$ is (chiral) entangled with the codipterous coalgebra $(\mathbb{D},\ \Delta_*,\ \delta_2)$. Such an entanglement is called a self-entanglement, (since over C_1, $\delta_2 = \hat{\delta}_1\Phi$) and is denoted by $[C_1 \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} C_1]$.*

Proof: The Theorem 7.2.7 and the previous remark imply that $(\mathbb{D},\ \Delta_*,\ \delta_2,\ (\hat{\delta}_2))$ and $(\mathbb{D},\ \Delta_*,\ \delta_1,\ (\hat{\delta}_1))$ are codipterous coalgebras. The straightforward equality $(id \otimes \delta_2)\delta_1 = (\delta_1 \otimes id)\delta_2$ entails that $[C_1 \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} C_1]$ is a self entangled codipterous coalgebra. □

Remark: The two coalgebras C_1 and C_2 are called the boundaries of the entangled codipterous coalgebras $[C_1 \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} C_1]$. The boundary of a codipterous coalgebra will be denoted by ∂. Therefore, $\partial[C_1 \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} C_1] = C_1 \cup C_2$.

Proposition 7.2.9 *With the hypotheses and the notation of the Theorem 7.2.7, we get $(\hat{\delta}_1 + \delta_1)\Phi = (\Phi \otimes id)\Delta_1 + (id \otimes \Phi)\Delta_1$.*

Proof: Straightforward. □

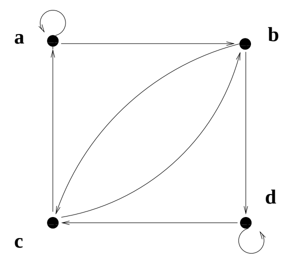
Remark: [Interpretation] Let us interpret the previous equality. Let (C, Δ) be a coassociative coalgebra as well as a unital algebra with unit 1. We recall that a Leibniz coderivative is a linear map $D : C \to C$ such that $D(1) = 0$ and $\Delta D := (D \otimes id)\Delta + (id \otimes D)\Delta$. Suppose now that $(C, \delta, \hat{\delta})$ is a dendriform coalgebra. As $\Delta := \delta + \hat{\delta}$ is coassociative, we get $(\delta + \hat{\delta})D := (D \otimes id)\Delta + (id \otimes D)\Delta$.

Definition 7.2.10 [Leibniz coderivative on a pre-dendriform coalgebra] With the hypotheses and the notation of the Theorem 7.2.7, suppose $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ is a pre-dendriform coalgebra as well as an algebra with unit 1. A Leibniz coderivative on the pre-dendriform coalgebra $\mathbb{D}$ is a linear map $D_I : C_1 \to C_2$ such that $(\delta_1 + \hat{\delta}_1)D_I := (D_I \otimes id)\Delta_1 + (id \otimes D_I)\Delta_1$ and $D_I(1) = 0$.

Proposition 7.2.11 *With the hypotheses and the notation of the Theorem 7.2.7, the map $D_I := \Phi - id$ is a Leibniz coderivative on the pre-dendriform coalgebra $\mathbb{D}$. If $\mathbb{D}$ is a pre-dendriform bialgebra then D_I is a Leibniz-Ito derivative.*

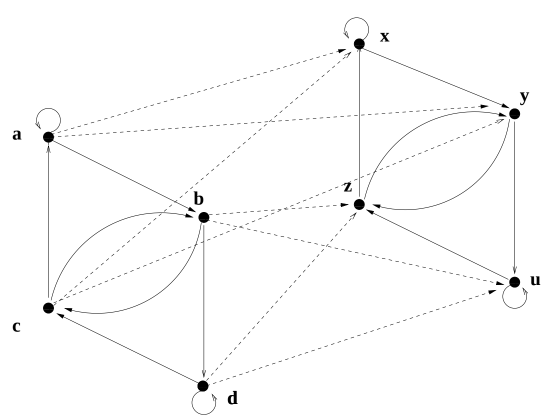
Proof: Let $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ be the pre-dendriform coalgebra described in Theorem 7.2.7. By definition of the channel map, $D_I(1) = 0$ is straightforward. The Leibniz coderivative equality comes from the definitions of the bridges. If $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ is a pre-dendriform bialgebra, as Φ is a homomorphism, we easily check the Leibniz-Ito property: $D_I(xy) - D_I(x)D_I(y) = xD_I(y) + D_I(x)y$, for all $x, y \in C_1$. $\square$

Example 7.2.12 [Axioms of codipterous coalgebras: a graphical interpretation] Let E be the k-vector space spanned by the basis a, b, c, d, x, y, z and u. Let $\mathcal{F}$ be the subspace spanned by a, b, c and d. Define $\Delta : \mathcal{F} \to \mathcal{F}^{\otimes 2}$ such that: $\Delta a = a \otimes a + b \otimes c$, $\Delta b = a \otimes b + b \otimes d$, $\Delta c = d \otimes c + c \otimes a$, $\Delta d = d \otimes d + c \otimes b$.



Geometric support of $\mathcal{F}$.

Consider the coassociative coalgebra $\mathcal{F}'$ spanned as a k-vector space by the basis x, y, z and u, with coproduct $\Delta' : \mathcal{F}' \to \mathcal{F}'^{\otimes 2}$ defined via the channel map $\Phi : \mathcal{F} \to \mathcal{F}'$, i.e., $\Delta'\Phi := (\Phi \otimes \Phi)\Delta$, with $\Phi(a) = x$, $\Phi(b) = y$, $\Phi(c) = z$, $\Phi(d) = u$. The geometric support of the entangled codipterous coalgebras constructed in the Theorem 7.2.7 has two directed graphs, $\mathcal{F}$ and $\mathcal{F}'$, with four bridges between them δ_1, $\hat{\delta}_1$, δ_2 and $\hat{\delta}_2$. The bridge δ_1 is defined over $\mathcal{F}'$ such that: $\delta_1 x = a \otimes x + b \otimes z$, $\delta_1 y = a \otimes y + b \otimes u$, $\delta_1 z = d \otimes z + c \otimes x$, $\delta_1 u = d \otimes u + c \otimes y$. The bridge $\hat{\delta}_1$ is defined over $\mathcal{F}'$ such that: $\hat{\delta}_1 x = x \otimes a + y \otimes c$, $\hat{\delta}_1 y = x \otimes b + y \otimes d$, $\hat{\delta}_1 z = u \otimes c + z \otimes a$, $\hat{\delta}_1 u = u \otimes d + z \otimes b$. We get $\partial[\mathcal{F} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \mathcal{F}] = \mathcal{F} \cup \mathcal{F}'$. The geometric support of $[\mathcal{F} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \mathcal{F}]$ is:



Geometric support of $[\mathcal{F} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \mathcal{F}]$. Among the bridges, only δ_1 is represented.

Since the chanel map has no fixed point, as a k-vector space, $[\mathcal{F} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \mathcal{F}] := F \oplus \Phi(F) := E$.

Theorem 7.2.13 *With the hypotheses and the notation of the Theorem 7.2.7, the pre-dendriform coalgebra* $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ *is a coassociative codialgebra and* $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ *is a coassociative cotrialgebra.*

Proof: Let us prove that the pre-dendriform coalgebra $(\mathbb{D}_1, \delta_1, \hat{\delta}_1)$ is a coassociative co-dialgebra. The bridges δ_1 and $\hat{\delta}_1$ are coassociative. This holds over C_1. Over C_2, since $\delta_1 = \Delta_1$ over C_1, we get $(\delta_1 \otimes id)\delta_1 = (\Delta_1 \otimes id)\delta_1 = (id \otimes \delta_1)\delta_1$, which proves the coassociativity of δ_1. Similarly, by using the anti-codipterous coalgebra axiom, we get the coassociativity of $\hat{\delta}_1$. The entanglement equation $(id \otimes \hat{\delta}_1)\delta_1 = (\delta_1 \otimes id)\hat{\delta}_1$ is assumed in the definition of a pre-dendriform coalgebra. Over C_2, the equalities $(\delta_1 \otimes id)\delta_1 = (\hat{\delta}_1 \otimes id)\delta_1$ and $(id \otimes \delta_1)\hat{\delta}_1 = (id \otimes \hat{\delta}_1)\hat{\delta}_1$ hold, since $\delta_1 = \hat{\delta}_1 = \Delta_1$ over C_1. This proves that $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ is a coassociative co-dialgebra. Similarly, the proof that $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ is a coassociative co-trialgebra is complete by checking the axioms of the definition 7.1.7. □

Remark: The previous Theorem is still valid for the pre-dendriform coalgebra $(\mathbb{D}, \Delta_*, \delta_2, \hat{\delta}_2)$.

Corollary 7.2.14 *With the hypotheses and the notation of Theorem 7.2.7, let* (A, m) *be an associative algebra over* k*. Denote by* $L(\mathbb{D}, A)$*, the* k*-vector space of the linear maps which map* $\mathbb{D}$ *into* A*. Let* $f, g \in L(\mathbb{D}, A)$*. Denote by* $f \perp g := m(f \otimes g)\Delta^*$*,* $f \dashv g := m(f \otimes g)\hat{\delta}_1$ *and* $f \vdash g := m(f \otimes g)\delta_1$*, the convolution products. Then* $(L(\mathbb{D}, A), \dashv, \vdash)$ *is an associative dialgebra. Define for all* $f, g \in L(\mathbb{D}, A)$*, the bracket* $[f, g] := f \dashv g - f \vdash g$*. This bracket embeds* $L(\mathbb{D}, A)$ *into a Leibniz algebra. Moreover* $(L(\mathbb{D}, A), \dashv, \vdash, \perp)$ *is an associative trialgebra.*

Proof: Straightforward. □

Remark: With the hypotheses and the notation of Theorem 7.2.7, notice there exist directed graphs, which are geometric supports of particular pre-dendriform coalgebras $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$, constructed from coassociative coalgebras via a channel map Φ, whose algebra of linear maps $L(\mathbb{D}, A)$, where A is an algebra with unit 1_A, has a structure of associative di or trialgebra. The Leibniz bracket allows us to deal with differential structures on these directed graphs. Notice that on the boundary C_1 of such a pre-dendriform coalgebra, the coproducts $\delta_1 = \hat{\delta}_1 = \Delta_1$, therefore the Leibniz algebra turns out to be a Lie algebra on the boundary C_1. With associative trialgebras, we can also construct *"non-commutative version" of Poisson algebras*. Defined in [48], a non-commutative version of Poisson algebra P is a k-vector space equipped with a Leibniz bracket and an associative operation $x \bullet y$ (not necessarily commutative) such that its relationship with the Leibniz bracket is given by:

$$\forall x, y, z \in P, \ [x \bullet y, z] = x \bullet [y, z] + [x, z] \bullet y, \quad [x, y \bullet z - z \bullet y] = [x, [y, z]].$$

By defining for all $f, g \in L(\mathbb{D}, A)$, $f \bullet g := f \perp g$, the associative trialgebra $L(\mathbb{D}, A)$ becomes a Poisson algebra.

Remark: In [45], J-L. Loday defines the notion of a bar-unit in an associative dialgebra $(X, \vdash, \dashv)$. An element $e \in X$ is said to be a *bar-unit* of X, if for all $x \in X$, $x \dashv e = x = e \vdash x$. The set of bar-units is called the *halo*. Let us still work with the hypotheses and the notation of the Theorem 7.2.7. Denote by $\eta_A : k \to A$ such that $\lambda \mapsto \lambda 1_A$. If the involved coassociative coalgebra C_1 has a counit $\epsilon : \ C_1 \to k$, then the map $e := \eta_A \epsilon_*$, where $\epsilon_* = \epsilon$ over C_1 and $\epsilon_* \Phi = \epsilon$ over C_2, is a bar-unit in $L(D, A)$, since if $f \in L(D, A)$, then $e \vdash f := m(e \otimes f)\delta_1 = m(id \otimes f)(\eta_A \epsilon_* \otimes id)\delta_1 = f$.

Similarly, $f \dashv v := m(f \otimes e)\hat{\delta}_1 = m(f \otimes id)(id \otimes \eta_A \epsilon_*)\hat{\delta}_1 = f$. As the counit ϵ is unique, there will exist a unique bar-unit on $L((\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1), A)$, i.e., the halo is a singleton [7].

Theorem 7.2.15 *Consider the hypotheses and the notation of the Theorem 7.2.7. If C_1, (resp. C_2) is spanned as a k-vector space by the basis $(v_i)_{1 \leq i \leq \dim C_1}$, (resp. $(w_i)_{1 \leq i \leq \dim C_1}$), with $\dim C_1 < \infty$, we denote by $(v_i^*)_{1 \leq i \leq \dim C_1}$, (resp. $(w_i^*)_{1 \leq i \leq \dim C_1}$), the dual basis, i.e., $v_i^*(v_j) = 1$ (resp. $w_i^*(w_j) = 1$) if $i = j$ and 0 otherwise. Denote by B_{ij}^k and C_{ij}^k the structure constants, i.e., scalars such that $[w_i^*, v_j^*] = \sum_k B_{ij}^k w_k^*$ and $[v_i^*, v_j^*] = \sum_k C_{ij}^k v_k^*$. The structure constants are determined by the geometric support of the pre-dendriform coalgebra $\mathbb{D}$, i.e., $v_i^* \otimes v_j^* \neq 0$ if and only if $v_i \to v_j$ is an arrow of the directed graph associated with $\mathbb{D}$, similarly for $w_i^* \otimes v_j^*$ and $v_i^* \otimes w_j^*$, for all $1 \leq i, j \leq \dim C_1$. Moreover, $[v_j^*, w_i^*] = 0$, for all $1 \leq i, j \leq \dim C_1$.*

Proof: Fix $i, j \in 1 \leq i \leq \dim C_1$. Denote by m, the product of the field k. Let v_j, (resp. w_i) be an element of the basis of C_1 (resp. of C_2). By definition, $[v_i^*, w_j^*] := v_i^* \dashv w_j^* - w_j^* \vdash v_i^* = m(v_i^* \otimes w_j^*)\hat{\delta}_1 - m(w_j^* \otimes v_i^*)\delta_1$, hence $[v_i^*, w_j^*] = 0$. □

Example 7.2.16 In the case of the pre-dendriform coalgebra $\mathbb{D} := [\mathcal{F} \underset{\delta_2}{\overset{\delta_1}{\rightleftarrows}} \mathcal{F}]$, on the boundary C_1, the Leibniz bracket turns out to be the Lie bracket determined by the convolution product associated with $\Delta_* := \Delta_1$. For instance we get $[a^*, b^*] = b^*$, $[b^*, c^*] = a^* - d^*$, which behaves as a Lie bracket and $[a^*, x^*] = 0$, $[x^*, a^*] = x^*$, $[y^*, c^*] = x^* - u^*$, $[c^*, y^*] = 0$, out of the boundary C_1, which behaves as a Leibniz bracket.

Theorem 7.2.17 *Consider the hypotheses and the notation of the Theorem 7.2.7. In addition, suppose that the coalgebras involved are bialgebras. As the channel Φ is a homomorphism, then so are the bridges δ_1 and $\hat{\delta}_1$.*

Proof: Let us prove the Theorem for the bridge δ_1. Over C_1, $\delta_1 := \Delta_1$, is obviously a unital homomorphism. Let $x, y \in C_2$. There exist unique $a, b \in C_1$, such that $\Phi(a) = x$ and $\Phi(b) = y$. Then $\delta_1(xy) = \delta_1\Phi(ab) := (id \otimes \Phi)\Delta_1(ab)$. With the Sweedler's notation, we write $\Delta_1(a) := \sum a_{(1)} \otimes a_{(2)}$ and $\Delta_1(b) := \sum b_{(1)} \otimes b_{(2)}$. Therefore, $(id \otimes \Phi)\Delta_1(ab) := \sum a_{(1)}b_{(1)} \otimes \Phi(a_{(2)}b_{(2)}) = (\delta_1\Phi(a))(\delta_1\Phi(b))$, which proves the homomorphism property of the bridge δ_1. Since the channel Φ is by definition unital, so is the bridge δ_1. □

Theorem 7.2.18 *Consider the hypotheses and the notation of the Theorem 7.2.7. In addition, suppose the coalgebras involved are Hopf algebras with antipodes S_1 and S_2 and recall that the channel Φ is a morphism of Hopf algebra. Denote by S_*, the linear map that equals to S_1 over C_1 and to S_2 over C_2. Denote by σ_1, the linear map that equals to S_1 over C_1 and to $\Phi^{-1}S_2$ over C_2. Then the bridges δ_1 and $\hat{\delta}_1$ verifies $m(id \otimes \sigma_1)\delta_1 = 1\epsilon_*$ and $m(\sigma_1 \otimes id)\hat{\delta}_1 = 1\epsilon_*$. Similarly, denote by σ_2, the linear map that equals to S_2 over C_2 and to ΦS_1 over C_1. Then the bridges δ_2 and $\hat{\delta}_2$ verify $m(id \otimes \sigma_2)\delta_2 = 1\epsilon_*$ and $m(\sigma_2 \otimes id)\hat{\delta}_2 = 1\epsilon_*$. Therefore, the pre-dendriform bialgebras $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$, $(\mathbb{D}, \Delta_*, \delta_2, \hat{\delta}_2)$ and $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_2)$ are L-Hopf algebras.*

[7] By entangling several codialgebras, constructed with this method, we could obtain a larger halo.

Proof: Each side of the pre-dendriform bialgebra $(\mathbb{D}, \delta_1, \hat{\delta}_1)$, i.e., $(\mathbb{D}, \delta_1)$ and $(\mathbb{D}, \hat{\delta}_1)$, has a map σ_1, defined in this Theorem, such that $id \vdash \sigma_1 = e$ and $\sigma_1 \dashv id = e$, where e is the bar unit of $L(\mathbb{D}, \mathbb{D})$. As their coproducts verify the entanglement equation, $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ can be viewed as a L-Hopf algebra [40]. Similarly for the space $(\mathbb{D}, \delta_2, \hat{\delta}_2)$. For the space $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_2)$, as the entanglement equation $(id \otimes \hat{\delta}_2)\delta_1 = (\delta_1 \otimes id)\hat{\delta}_2$ is verifyied, the space $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_2)$ has the structure of L-Hopf algebra [8]. □

Let $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ be the pre-dendriform coalgebra defined in the Theorem 7.2.7. Inspired with results in [22], we define $\overrightarrow{d} : \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$, such that $\overrightarrow{d} = \Delta_* - \delta_1$ and $\overleftarrow{d} : \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$, such that $\overleftarrow{d} = \Delta_* - \hat{\delta}_1$.

Theorem 7.2.19 *Let $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ be the pre-dendriform coalgebra defined in Theorem 7.2.7. We get: $(id \otimes \overleftarrow{d})\overrightarrow{d} = (\overrightarrow{d} \otimes id)\overleftarrow{d}$. Moreover, $(\overrightarrow{d} \otimes id)\Delta_* = (id \otimes \Delta_*)\overrightarrow{d}$ and $(id \otimes \overleftarrow{d})\Delta_* = (\Delta_* \otimes id)\overleftarrow{d}$.*

Proof: Straightforward by using the following equality $(id \otimes \hat{\delta})\delta = (\delta \otimes id)\hat{\delta}$. □

Definition 7.2.20 [bimodule] Let $(\mathbb{D}, \Delta_*, \delta_1, \hat{\delta}_1)$ be the pre-dendriform coalgebra defined in the Theorem 7.2.7. Recall that the coproducts δ_1 and $\hat{\delta}_1$ are coassociative. Suppose $\mathbb{D}$ is a pre-dendriform bialgebra. Embed $\mathbb{D}^{\otimes 2}$ into a $\mathbb{D}$-*bimodule* by defining for all $c, x, y \in \mathbb{D}$,

$$x \,\hat{\circ}\, \overleftarrow{d}(c) \;=\; \hat{\delta}_1(x)\overleftarrow{d}(c); \qquad \overleftarrow{d}(c) \,\hat{\circ}\, y = \overleftarrow{d}(c)\hat{\delta}_1(y).$$

Similarly, $\mathbb{D}^{\otimes 2}$ can be embedded into another $\mathbb{D}$-*bimodule* by defining for all $c, x, y \in \mathbb{D}$,

$$x \circ \overrightarrow{d}(c) \;=\; \delta_1(x)\overrightarrow{d}(c); \qquad \overrightarrow{d}(c) \circ y = \overrightarrow{d}(c)\delta_1(y).$$

Theorem 7.2.21 *Let $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ be the pre-dendriform bialgebra defined in Theorem 7.2.7. Then $\overleftarrow{d}, \overrightarrow{d}$ are Leibniz-Ito derivatives.*

Proof: Let $x, y \in \mathbb{D}$. We have $\overleftarrow{d}(1) = 0 = \overrightarrow{d}(1)$. Moreover, $\overrightarrow{d}(x)\overrightarrow{d}(y) = \Delta_*(xy) + \delta_1(xy) - \Delta_*(x)\delta_1(y) - \delta_1(x)\Delta_*(y)$, i.e., $\overrightarrow{d}(xy) = \overrightarrow{d}(x)\overrightarrow{d}(y) + \overrightarrow{d}(x) \circ y + x \circ \overrightarrow{d}(y)$. Similarly, $\overleftarrow{d}(xy) = \overleftarrow{d}(x)\overleftarrow{d}(y) + \overleftarrow{d}(x)\hat{\delta}_1(y) + \hat{\delta}_1(x)\overleftarrow{d}(y)$, that is $\overleftarrow{d}(xy) = \overleftarrow{d}(x)\overleftarrow{d}(y) + \overleftarrow{d}(x) \,\hat{\circ}\, y + x \,\hat{\circ}\, \overleftarrow{d}(y)$. □

Remark: Let $(\mathbb{D}, \delta_1, \hat{\delta}_1)$ be the pre-dendriform coalgebra defined in Theorem 7.2.7. If $x \in \mathbb{D}$ verifies $\Delta_*(x) = x \otimes 1 + 1 \otimes x$, then $(\overleftarrow{d} \otimes id)\overrightarrow{d}(x) = 0$.

7.2.2 Self-tilings and dendriform coalgebras

We now obtain dendriform coalgebras from the self-entanglement and apply these results on particular coalgebras.

Theorem 7.2.22 *With the hypotheses, the notation of the Theorem 7.2.7, as usual, define the bridge $\delta_d := \Delta$ over C_1 such that $\delta_d \Phi := (id \otimes \Phi)\Delta$ over C_2. Define also $\hat{\delta}_d \Phi := (\Phi \otimes id)\Delta$ over*

[8] It is not a coassociative codialgebra.

C_2 and $\hat{\delta}_d := 0$ otherwise. Define also $\Delta_*\Phi := (\Phi \otimes \Phi)\Delta$ over C_2 and $\Delta_* := \Delta$ over C_1. Then the coproduct $\bar{\Delta} := \delta_d + \hat{\delta}_d$ is coassociative and equipped with the coproducts δ_d, $\hat{\delta}_d$, E becomes a dendriform coalgebra.

Proof: Straightforward by checking axioms. □

Remark: The dendriform coalgebra is trivial on the boundary C_1. The directed graph constructed from $\bar{\Delta}$ is tiled by the directed graphs constructed from its decomposition, that is by the coproducts δ_d and $\hat{\delta}_d$.

Let us yield an application of entanglement given by two channels, inspired from a work of C. Cibils [9]. Fix an integer $n > 0$ and an invertible scalar $q \in k$. Let E be a k-vector space spanned by the basis $(a_i)_{0\leq i\leq n-1}$ and by $(x_i)_{0\leq j\leq n-1}$. Define the following coassociative coproduct Δ on $k\langle\, (a_i)_{0\leq i\leq n-1} \,\rangle$ such that $\Delta(a_i) = \sum_{j+k=i} a_j \otimes a_k$. Define the channel map $\Phi : k\langle\, (a_i)_{0\leq i\leq n-1} \,\rangle \to k\langle\, (x_i)_{0\leq i\leq n-1} \,\rangle$ such that $\Phi(a_i) = q^{-i}x_i$. Then the bridge $\hat{\delta}$ verifies $\hat{\delta}(x_i) = \sum_{j+k=i} q^k x_j \otimes a_k$ and $\hat{\delta}(a_i) = \sum_{j+k=i} a_j \otimes a_k$ and is coassociative. The map $\epsilon(a_i) = 0$, if $i \neq 0$, $\epsilon(a_0) = 1$ and $\epsilon(x_i) = 0$ is a right counit. Moreover $(E, \Delta_*, \hat{\delta})$, where $\Delta_*\Phi := (\Phi \otimes \Phi)\Delta$ over $k\langle\, (x_i)_{0\leq i\leq n-1} \,\rangle$ is an anti-codipterous coalgebra. Similarly, define the channel map $\Psi : k\langle\, (a_i)_{0\leq i\leq n-1} \,\rangle \to k\langle\, (x_i)_{0\leq i\leq n-1} \,\rangle$ such that $\Psi(a_i) = x_i$. Then the bridge δ verifies $\delta(x_i) = \sum_{j+k=i} a_j \otimes x_k$ and $\delta(a_i) = \sum_{j+k=i} a_j \otimes a_k$ and is coassociative. The map $\epsilon(a_i) = 0$, if $i \neq 0$, $\epsilon(a_0) = 1$ and $\epsilon(x_i) = 0$ is a left counit. Moreover (E, Δ^*, δ), where $\Delta^*\Psi := (\Psi \otimes \Psi)\Delta$ over $k\langle\, (x_i)_{0\leq i\leq n-1} \,\rangle$ is a codipterous coalgebra.

Proposition 7.2.23 *The k-vector space E equipped with the two coproducts δ, $\hat{\delta}$ is a coassociative co-dialgebra and $(E, \Delta_*, \hat{\delta})$ and (E, Δ^*, δ) are chiral entangled.*

Proof: Notice [9] that $\Delta^* := \Delta_*$ and that $(id \otimes \hat{\delta})\delta = (\delta \otimes id)\hat{\delta}$. The proof is complete by checking the axioms of a co-dialgebra. □

Theorem 7.2.24 *Define now the coproduct δ such that for all $0 \leq i \leq n-1$, $\delta(a_i) := \Delta(a_i)$ and $\delta(x_i) := \sum_{j+k=i} a_j \otimes x_k$. Similarly, define the coproduct $\hat{\delta}$ such that for all $0 \leq i \leq n-1$, $\hat{\delta}(a_i) := 0$ and $\hat{\delta}(x_i) := \sum_{j+k=i} q^k x_j \otimes a_k$. Then $(E, \delta, \hat{\delta})$ is a dendriform coalgebra.*

Proof: Notice that $\Delta := \hat{\delta} + \delta$ is coassociative [9]. It is trivial on $(a_k)_{0\leq k\leq n-1}$. Let us check the axioms on $(x_k)_{0\leq k\leq n-1}$. For the codipterous coalgebra axiom: $(id \otimes \delta)\delta(x_i) := \sum_{j+(l+m)=i} a_j \otimes a_l \otimes x_m$ which is equal to $(\Delta \otimes id)\delta(x_i) := \sum_{l'+j'+k=i} a_{j'} \otimes a_{l'} \otimes x_k$ since the sum is over all possible decompositions of i into three integers. Similarly for the anti-codipterous coalgebra axiom: $(\hat{\delta} \otimes id)\hat{\delta}(x_i) := \sum_{j'+l'+k=i} q^k(q^{l'} x_{j'} \otimes a_{l'}) \otimes a_k$ which is equal to $(id \otimes \Delta)\hat{\delta}(x_i) := \sum_{j+(h'+m')=i} q^{(h'+m')} x_j \otimes a_{h'} \otimes a_{m'}$, since the sum is over all possible decompositions of i into three integers. The entanglement equation is also verified. □

[9] This can be considered as a generalisation of the self-entanglement and works thanks to the particular form of the involved coproduct.

7.3 Entanglement of two different codipterous coalgebras

Another way to construct entangled codipterous coalgebra is to start, for instance, with an achiral coassociative L-coalgebra, i.e., a L-coalgebra whose two coproducts Δ and $\tilde{\Delta}$ verify the entanglement equation whatever the position of the two coproducts are, i.e., $(\Delta \otimes id)\tilde{\Delta} = (id \otimes \tilde{\Delta})\Delta$ and $(id \otimes \Delta)\tilde{\Delta} = (\tilde{\Delta} \otimes id)\Delta$. Otherwise stated and more generally, an algebra $(A, \bullet_1, \ldots, \bullet_n)$ equipped with n operations $\bullet_1, \ldots, \bullet_n : A^{\otimes 2} \to A$, verifying axioms $AX_1, \ldots AX_p$ is said to be achiral if $(A, \bullet_{\sigma(1)}, \ldots, \bullet_{\sigma(n)})$, where σ is a permutation, verifies also the same axioms, i.e., the axioms $AX_1, \ldots AX_p$ are globally invariant under the action of any permutation σ. The dualisation of this definition is straightforward. Such algebras allow the construction of n-hypercube algebras [10] and associative products which split into several one, see [41].

Theorem 7.3.1 *Let G be a sub-vector space of a k-vector space E, with $\dim E \geq 2 \dim G$, if $\dim G$ is finite. Suppose $(G, \Delta, \tilde{\Delta}, \epsilon, \tilde{\epsilon})$ is an achiral coassociative L-coalgebra. Define a channel map $\Phi : G \to E$, with no fixed point, verifying $\tilde{\Delta}\Phi := (\Phi \otimes \Phi)\tilde{\Delta}$. Define $C_1 := G$, $C_2 := \Phi(G)$, $\Delta_1 := \Delta$ and $\tilde{\Delta}_2 := \tilde{\Delta}\Phi$. Define $\mathbb{D} := C_1 \oplus C_2$ and $\delta_1 : \mathbb{D} \to \mathbb{D}^{\otimes 2}$ such that $\delta_1 := \Delta_1$ over C_1 and $\delta_1 \Phi = (id \otimes \Phi)\Delta_1$ over C_2. Similarly, define $\tilde{\delta}_2 : \mathbb{D} \to \mathbb{D}^{\otimes 2}$, verifying $\tilde{\delta}_2 = \tilde{\Delta}_2$ over C_2 and $\tilde{\delta}_2 \Phi^{-1} = (\Phi^{-1} \otimes id)\tilde{\Delta}_2$ over C_1. Denote by $\Delta_* := \Delta_1$ over C_1 and $\Delta_* = \tilde{\Delta}_2$ over C_2. The space $(\mathbb{D}, \Delta_*, \delta_1, \tilde{\delta}_2)$ is the entanglement of two codipterous coalgebras $(\mathbb{D}, \Delta_*, \delta_1)$ and $(\mathbb{D}, \Delta_*, \tilde{\delta}_2)$. This (chiral) entanglement is denoted by $[G \underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}} \tilde{G}]$, where $\tilde{G} = \Phi(G)$.*

Proof: We check that $(\Delta_1 \otimes id)\delta_1 = (\Delta_* \otimes id)\delta_1 = (id \otimes \delta_1)\delta_1$ and $(\tilde{\Delta}_2 \otimes id)\tilde{\delta}_2 = (\tilde{\Delta}_* \otimes id)\tilde{\delta}_2 = (id \otimes \tilde{\delta}_2)\tilde{\delta}_2$. Moreover $(\tilde{\delta}_2 \otimes id)\delta_1 = (id \otimes \delta_1)\tilde{\delta}_2$ is straightforward. □

Theorem 7.3.2 *Consider the hypotheses and the notation of Theorem 7.3.1. In addition, suppose the coalgebras involved are Hopf algebras with antipodes S_1 and $\tilde{S}_2$ and recall that the channel Φ is a morphism of Hopf algebras. Denote by S_*, the linear map equals to S_1 over C_1 and to $\tilde{S}_2$ over C_2. Denote by σ_1, the linear map equals to S_1 over C_1 and to $\Phi^{-1} \circ \tilde{S}_2$ over C_2. Define the map $\epsilon_* := \epsilon$ over C_1 and $\epsilon_* \Phi := \tilde{\epsilon}$ over C_2, the bridge δ_1 verifies $m(id \otimes \sigma_1)\delta_1 = 1\epsilon_*$. Similarly, denote by $\tilde{\sigma}_2$, the linear map equals to $\tilde{S}_2$ over C_2 and to $\Phi \circ S_1$ over C_1. Then the bridge $\tilde{\delta}_2$ verifies $m(id \otimes \tilde{\sigma}_2)\tilde{\delta}_2 = 1\epsilon_*$. Therefore, the space $[G \underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}} \tilde{G}]$ is a L-Hopf algebra.*

Proof: The proof is similar to the proof of the Theorem 7.2.18. Even if such an entanglement does not yield a coassociative trialgebra, the element $e' := \eta_A \epsilon_*$, with ϵ_* is defined in this Theorem, plays the rôle of a bar-unit. □

Theorem 7.3.3 *Let $(\mathbb{D}, \Delta_*, \delta_1, \tilde{\delta}_2)$ be the entangled codipterous coalgebra defined in Theorem 7.3.1. Let us define the coproduct Δ^* (resp. $\tilde{\Delta}^*$) such that $\Delta^* := \Delta_1$ (resp. $:= \tilde{\Delta}_1$) over C_1 and equals to Δ_2 such that $\Delta_2 \Phi := (\Phi \otimes \Phi)\Delta_1$ (resp. equals to $\tilde{\Delta}_2$ such that $:= \tilde{\Delta}_2 \Phi := (\Phi \otimes \Phi)\tilde{\Delta}_1$) over C_2. The coalgebra $(C_1 \oplus C_2, \Delta^*, \tilde{\Delta}^*)$ is then an achiral L-coalgebra. Recall that the bridges δ_1 and $\hat{\delta}_2$ are defined by $\delta_1 := \Delta_1$ over C_1, such that $\delta_1 \Phi := (id \otimes \Phi)\Delta_1$ and $\hat{\tilde{\delta}}_2 := \tilde{\Delta}_2$ over C_2, such that $\hat{\tilde{\delta}}_2 \Phi^{-1} := (\Phi^{-1} \otimes id)\tilde{\Delta}_2$. Therefore, the codipterous coalgebra $(\mathbb{D}, \Delta^*, \delta_1)$ and the*

[10] The free 2-hypercube (resp. 3-hypercube) algebra have been studied in [53], (resp. in [48]).

anti-codipterous coalgebra $(\mathbb{D}, \tilde{\Delta}^*, \hat{\hat{\delta}}_2)$ *are entangled since* $(id \otimes \hat{\hat{\delta}}_2)\delta_1 = (\delta_1 \otimes id)\hat{\hat{\delta}}_2$ *is verified. Let us define the maps* $\overrightarrow{d} : \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$, *such that* $\overrightarrow{d} = \Delta^* - \delta_1$ *and* $\overleftarrow{d} : \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$, *such that* $\overleftarrow{d} = \tilde{\Delta}^* - \hat{\hat{\delta}}_2$. *We get:* $(id \otimes \overleftarrow{d})\overrightarrow{d} = (\overrightarrow{d} \otimes id)\overleftarrow{d}$. *Moreover,* $(\overrightarrow{d} \otimes id)\Delta^* = (id \otimes \Delta^*)\overrightarrow{d}$ *and* $(id \otimes \overleftarrow{d})\tilde{\Delta}^* = (\tilde{\Delta}^* \otimes id)\overleftarrow{d}$.

Proof: Straightforward. □

Definition 7.3.4 [bimodule] Let the codipterous coalgebra $(\mathbb{D}, \Delta^*, \delta_1)$ and the anti-codipterous coalgebra $(\mathbb{D}, \tilde{\Delta}^*, \hat{\hat{\delta}}_2)$ be entangled as in the previous Theorem 7.3.3. Suppose all the involved coproducts are unital homomorphisms. As bridges are coassociative, we embed $\mathbb{D}^{\otimes 2}$ into a $\mathbb{D}$-*bimodule* by defining the following products: Let $c, x, y \in \mathbb{D}$,

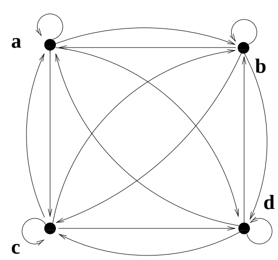
$$x \,\hat{\hat{\circ}}\, \overleftarrow{d}(c) = \hat{\hat{\delta}}_2(x)\overleftarrow{d}(c); \qquad \overleftarrow{d}(c) \,\hat{\hat{\circ}}\, y = \overleftarrow{d}(c)\hat{\hat{\delta}}_2(y).$$

$$x \circ \overrightarrow{d}(c) = \delta_1(x)\overrightarrow{d}(c); \qquad \overrightarrow{d}(c) \circ y = \overrightarrow{d}(c)\delta_1(y).$$

Theorem 7.3.5 *Let the codipterous coalgebra* $(\mathbb{D}, \Delta^*, \delta_1)$ *and the anti-codipterous coalgebra* $(\mathbb{D}, \tilde{\Delta}^*, \hat{\hat{\delta}}_2)$ *be entangled as in the previous Theorem 7.3.3. Suppose all the involved coproducts are unital homomorphisms. Then* $\overleftarrow{d}, \overrightarrow{d}$ *are Leibniz-Ito derivatives.*

Proof: Let $x, y \in \mathbb{D}$. We have $\overleftarrow{d}(1) = 0 = \overrightarrow{d}(1)$. Moreover, $\overrightarrow{d}(x)\overrightarrow{d}(y) = \Delta_*(xy) + \delta_1(xy) - \Delta_*(x)\delta_1(y) - \delta_1(x)\Delta_*(y)$, i.e., $\overrightarrow{d}(xy) = \overrightarrow{d}(x)\overrightarrow{d}(y) + \overrightarrow{d}(x) \circ y + x \circ \overrightarrow{d}(y)$. Similarly, $\overleftarrow{d}(xy) = \overleftarrow{d}(x)\overleftarrow{d}(y) + \overleftarrow{d}(x)\hat{\hat{\delta}}_2(y) + \hat{\hat{\delta}}_2(x)\overleftarrow{d}(y)$, that is $\overleftarrow{d}(xy) = \overleftarrow{d}(x)\overleftarrow{d}(y) + \overleftarrow{d}(x) \,\hat{\hat{\circ}}\, y + x \,\hat{\hat{\circ}}\, \overleftarrow{d}(y)$. □

Example 7.3.6 $[Sl_q(2)\underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftarrows}}\widetilde{Sl_q(2)}]$ In [41], the $(4,1)$-De Bruijn graph,

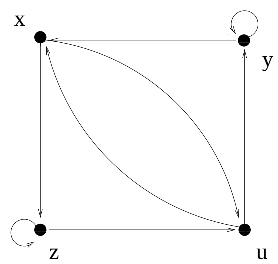


The $(4,1)$-De Bruijn graph.

was tiled with the geometric supports of two coassociative coalgebras, defining an achiral L-coalgebra, the coassociative coalgebra represented by $\mathcal{F}$ and the coassociative coalgebra represented by the directed graph defined by $\tilde{\Delta}$ verifying $\tilde{\Delta}b = b \otimes b + a \otimes d$, $\tilde{\Delta}c = c \otimes c + d \otimes a$, $\tilde{\Delta}a = b \otimes a + a \otimes c$, $\tilde{\Delta}d = c \otimes d + d \otimes b$.

Let q be an invertible scalar of k. Recall that the Hopf algebra $(Sl_q(2), \ \Delta_1)$ is generated as an algebra by a, b, c, d obeying the algebraic relations: $ba = qab$, $ca = qac$, $bc = cb$, $dc =$

$qcd,\ db = qbd, ad - da = (q^{-1} - q)bc, ad - q^{-1}bc = 1$. The coproduct Δ_1 verifies $\Delta_1 a = a \otimes a + b \otimes c,\ \Delta_1 b = a \otimes b + b \otimes d,\ \Delta_1 c = d \otimes c + c \otimes a,\ \Delta_1 d = d \otimes d + c \otimes b$. The antipode map S_1 is such that $S_1(a) = d,\ S_1(d) = a,\ S_1(b) = -qb,\ S_1(c) = -q^{-1}c$. Similarly, define $\widetilde{Sl_q(2)}$, the Hopf algebra generated by x, y, z, u with relations $xy = qyx,\ uy = qyu,\ xu = ux,\ zu = quz,\ zx = qxz,\ yz - zy = (q^{-1} - q)xu,\ yz - q^{-1}xu = 1$, equipped with the following coproduct: $\tilde{\Delta}_2 y = y \otimes y + x \otimes u,\ \tilde{\Delta}_2 z = z \otimes z + u \otimes x,\ \tilde{\Delta}_2 x = y \otimes x + x \otimes z,\ \tilde{\Delta}_2 u = z \otimes u + u \otimes y$, with directed graph:



Geometric support of $\widetilde{Sl_q(2)}$.

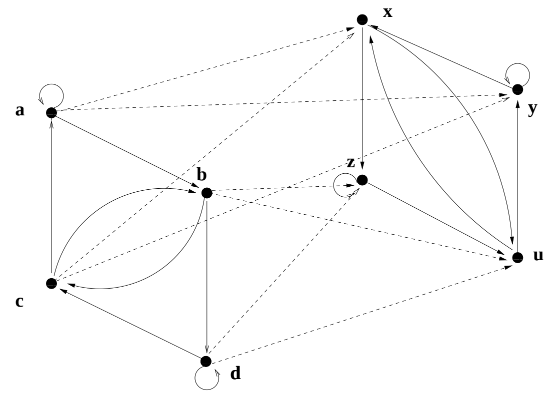
The antipode map is defined by $\tilde{S}_2(y) = z,\ \tilde{S}_2(z) = y,\ \tilde{S}_2(x) = -qx,\ \tilde{S}_2(u) = -q^{-1}u$.

Let E be the associative algebra generated by a, b, c, d, x, y, z and u with relations described above. Define, as a channel map, the homomorphism $M : E \to E$ such that $M(a) = y,\ M(b) = x,\ M(c) = u,\ M(d) = z$. Notice that $\tilde{\Delta}_2 := (M \otimes M)\Delta M^{-1}$. The bridges are defined by $\delta_1(x) := a \otimes x + b \otimes z,\ \delta_1(y) := a \otimes y + b \otimes u,\ \delta_1(z) := d \otimes z + c \otimes x,\ \delta_1(u) := d \otimes u + c \otimes y$, on $\widetilde{Sl_q(2)}$ and by Δ_1 over $Sl_q(2)$. Similarly, $\tilde{\delta}_2(a) := y \otimes a + x \otimes c,\ \tilde{\delta}_2(b) := y \otimes b + x \otimes d,\ \tilde{\delta}_2(c) := z \otimes c + u \otimes a,\ \tilde{\delta}_2(d) := z \otimes d + u \otimes b$, over $Sl_q(2)$ and by $\tilde{\Delta}_2$ over $\widetilde{Sl_q(2)}$.

Proposition 7.3.7 *Define the coproduct $\Delta_* = \Delta_1$ over $Sl_q(2)$ and $\Delta_* = \tilde{\Delta}_2$ over $\widetilde{Sl_q(2)}$. Equipped with the bridges δ_1 for $Sl_q(2)$ and $\tilde{\delta}_2$, the codipterous coalgebra $(Sl_q(2), \Delta_*, \delta_1)$ is entangled with the codipterous coalgebra $(\widetilde{Sl_q(2)}, \Delta_*, \tilde{\delta}_2)$. Moreover the bridges are unital homomorphisms.*

Proof: The bridges are unital homomorphisms comes from the homomorphism property of M.
□

The space $[Sl_q(2) \underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}} \widetilde{Sl_q(2)}]$ has the following geometric support:



Geometric support of $[Sl_q(2)\underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}}\widetilde{Sl_q(2)}]$. **(Among the bridges, only** δ_1 **is represented.)**

Example 7.3.8 $[SU_2(q)\underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}}\widetilde{SU_2(q)}]$ The same results hold for $SU_2(q)$. Indeed, let E be the algebra generated by $a,\ a^*,\ c,\ c^*,\ x,\ z^*,\ z,\ x^*$. Recall that the Hopf algebra $SU_2(q)$, generated by $a,\ a^*,\ c,\ c^*$, whose coassociative coproduct Δ_1 is described by $\Delta_1 a = a \otimes a - qc^* \otimes c$ and $\Delta_1 c = c \otimes a + a^* \otimes c$ is in fact a part of an achiral L-coalgebra whose the other coproduct [41] is described by $\tilde{\Delta}_1 a := c^* \otimes a + a \otimes c$ and $\tilde{\Delta}_1 c := c \otimes c - q^{-1} a^* \otimes a$. Similarly, define, as a channel, the $*$-homomorphism M such that $M(a) = z^*,\ M(c^*) = -q^{-1}x$. Define also $\widetilde{SU_2(q)}$, generated by $x,\ z^*,\ z,\ x^*$, such that $\tilde{\Delta}_2 := (M \otimes M)\Delta_1 M^{-1}$ we can also construct the bridges δ_1 and $\tilde{\delta_2}$, which are unital homomorphisms and define the entanglement $[SU_2(q)\underset{\delta_1}{\overset{\tilde{\delta}_2}{\rightleftharpoons}}\widetilde{SU_2(q)}]$.

Remark: In [41], we have shown that the $(n^2, 1)$-De Bruijn graph was tiled by (the geometric supports of) n coassociative coalgebras. We can create n transformations, via channel maps of $Sl_n(q)$ and glue them together. The space so obtained will have n boundaries.

7.4 Entanglement with a coassociative Markov L-coalgebra

7.4.1 Entanglement with a $(n, 1)$-De Bruijn codialgebra

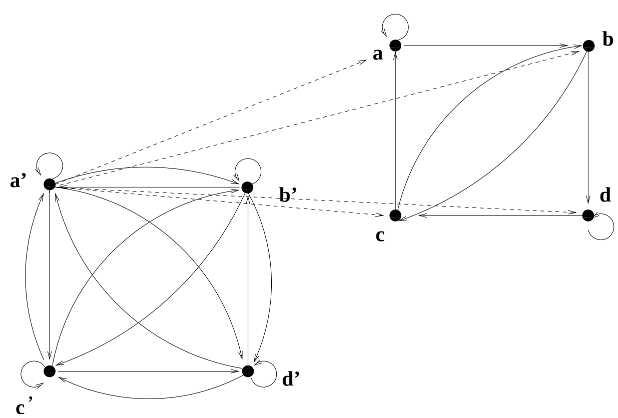
In [41], we have constructed L-cocommutative coassociative codialgebras via coassociative Markov L-coalgebras which are for instance the $(n, 1)$-De Bruijn graphs and unital associative algebras. By $(D_{(n,1)}, (x_i)_{1\leq i\leq n})$, we mean the coassociative codialgebra spanned as a k-vector space by the basis $(x_i)_{1\leq i\leq n}$ and whose coproducts are $\Delta_M x_i := x_i \otimes \Sigma$, where $\Sigma := \sum_{1\leq j\leq n} x_j$ and $\tilde{\Delta}_M x_i := \Sigma \otimes x_i$, for all $1 \leq i \leq n$. We call such a codialgebra the $(n, 1)$-De Bruijn codialgebra.

Proposition 7.4.1 *Fix $n > 0$. Let $E := G \oplus C$ be a k-vector space, with $\dim E = 2n$, with basis $((x_i)_{1\leq i\leq n}), (a_i)_{1\leq i\leq n})$, where $(G := D_{(n,1)},\ (x_i)_{1\leq i\leq n}))$ is the $(n, 1)$-De Bruijn codialgebra and (C, Δ) is a coassociative coalgebra, such that $\Phi : G \to C$ is a channel. Define the bridges δ_M (resp. $\tilde{\delta}_M$), such that $\delta_M := \Delta_M$, (resp. $\tilde{\delta}_M := \tilde{\Delta}_M$) over G and $\delta_M \Phi := (id \otimes \Phi)\Delta_M$, (resp. $\tilde{\delta}_M \Phi := (id \otimes \Phi)\tilde{\Delta}_M$). Define the bridge δ verifying $\delta := \Delta$ over C and $\delta\Phi^{-1} := (id \otimes \Phi^{-1})\Delta$.*

Define the coproduct Δ_*, by $\Delta_* = \Delta_M$ over G and $\Delta_* = \Delta$ over C. We get $(id \otimes \delta_M)\delta = (\delta \otimes id)\delta_M$, i.e., the two codipterous coalgebras $(C \oplus G, \delta, \Delta_*)$ and $(C \oplus G, \delta_M, \Delta_*)$ are entangled. Such an entanglement is denoted by $[G \underset{\delta_M}{\overset{\delta}{\rightleftharpoons}} C]$. Similarly, since $(\tilde{\delta}_M \otimes id)\delta = (id \otimes \delta)\tilde{\delta}_M$, by defining the coproduct Δ^*, such that $\Delta^* = \tilde{\Delta}_M$ over G and $\Delta^* = \Delta$ over C, the two codipterous coalgebras $(C \oplus G, \delta, \Delta^*)$ and $(C \oplus G, \tilde{\delta}_M, \Delta^*)$ are entangled. Such an entanglement is denoted by $[\tilde{G} \underset{\delta}{\overset{\tilde{\delta}_M}{\rightleftharpoons}} C]$.

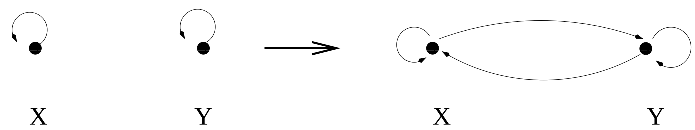
Proof: Fix $n > 0$ and denote by $G := D_{(n,1)}$ the $(n,1)$-De Bruijn codialgebra spanned as a k-vector space by the basis $(x_i)_{1 \leq i \leq n}$. Denote also by $(a_j)_{1 \leq j \leq n}$, the basis of C. With the hypotheses and notation of the Proposition 7.4.1, we check that $(\Delta_* \otimes id)\delta_M = (\Delta_M \otimes id)\delta_M = (id \otimes \delta_M)\delta_M$ and that $(\Delta_* \otimes id)\delta = (\Delta \otimes id)\delta = (id \otimes \delta)\delta$. Let us prove the equality $(id \otimes \delta_M)\delta = (\delta \otimes id)\delta_M$. Recall that for all i, with $1 \leq i \leq n$, the Markovian coproduct Δ_M is such that $\Delta_M x_i := x_i \otimes \Sigma$, where $\Sigma := \sum_{1 \leq j \leq n} x_j$. Let $x_i \in G$, with $1 \leq i \leq n$, there exists a unique $c \in C$, such that $\Phi^{-1}(c) = x_i$. Using the Sweedler's notation, $\Delta c := \sum c_{(1)} \otimes c_{(2)}$, we get $x_i \xrightarrow{\delta} \sum c_{(1)} \otimes \Phi^{-1}(c_{(2)}) \xrightarrow{(id \otimes \delta_M)} \sum c_{(1)} \otimes (\Phi^{-1}(c_{(2)}) \otimes \Sigma)$ and $x_i \xrightarrow{\delta_M} \sum x_i \otimes \Sigma \xrightarrow{(\delta \otimes id)} (\sum c_{(1)} \otimes \Phi^{-1}(c_{(2)})) \otimes \Sigma$. Similarly, we show that the same equality over C. The last assertion of the Theorem is straightforward (recall that $\tilde{\Delta}_M := \tau \Delta_M$). □

Example 7.4.2 $[D_{(4,1)} \underset{\delta_M}{\overset{\delta}{\rightleftharpoons}} \mathcal{F}]$ We have represented a part of the geometric support of $[D_{(4,1)} \underset{\delta_M}{\overset{\delta}{\rightleftharpoons}} \mathcal{F}]$, where $D_{(4,1)}$ is the $(4,1)$-De Bruijn codialgebra spanned as a k-vector space by the basis a', b', c', d'.



A part of the geometric support of $[D_{(4,1)} \underset{\delta_M}{\overset{\delta}{\rightleftharpoons}} \mathcal{F}]$**.**

Example 7.4.3 $[D_{(1,1)} \underset{\delta_M}{\overset{\delta}{\rightleftharpoons}} D_{(1,1)}]$ Similarly, we can draw the self-entanglement of the $(1,1)$-De Bruijn codialgebra. Suppose the k-vector space E is spanned by two independent elements X, Y and the channel map $\Phi' : E \to E$ such that $\Phi'(X) = Y$, maps $(D_{(1,1)}, (X))$ into $(D_{(1,1)}, (Y))$.



Right hand side: $\partial[D_{(1,1)} \underset{\delta_M}{\overset{\delta}{\leftrightarrows}} D_{(1,1)}]$. **Left hand side:** $[D_{(1,1)} \underset{\delta_M}{\overset{\delta}{\leftrightarrows}} D_{(1,1)}]$.

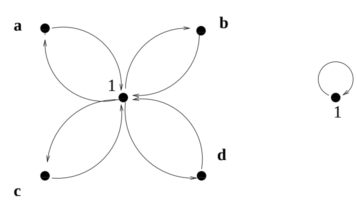
Recall that $(D_{(1,1)}, (X))$ spanned as a k-vector space by X, has a coassociative coproduct defined by $\Delta_M(X) := X \otimes X$. Therefore $(D_{(1,1)}, (Y))$, spanned as a k-vector space by $\Phi'(X) := Y$, is defined by the coassociative coproduct [11] $\Delta'_M(Y) := Y \otimes Y$, hence the geometric support of the boundaries of $[D_{(1,1)} \underset{\delta_M}{\overset{\delta}{\leftrightarrows}} D_{(1,1)}]$. The bridge on $(D_{(1,1)}, (X))$ is defined as $\delta_M := \Delta_M$ over $(D_{(1,1)}, (X)$ and $\delta_M(Y) := \delta_M \Phi(X) := X \otimes \Phi'(X) := X \otimes Y$. Similarly, the bridge on $(D_{(1,1)}, (Y))$ is defined as $\delta'_M := \Delta'_M$ over $(D_{(1,1)}, (Y))$ and $\delta'_M(X) := \delta'_M \Phi'^{-1}(Y) := Y \otimes \Phi'^{-1}(Y) := Y \otimes X$, hence the $(2,1)$-De Bruijn graph as geometric support of $[(D_{(1,1)}, \Delta_M, \delta_M) \underset{\delta_M}{\overset{\delta}{\leftrightarrows}} (D_{(1,1)}, \Delta'_M, \delta'_M)]$. This result is general.

Proposition 7.4.4 *Fix $n > 0$. Let $E := k\langle x_1, \dots x_n \rangle \oplus k\langle y_1, \dots y_n \rangle$ be a k-vector space, consider the $(n,1)$-De Bruijn codialgebra $(D_{(n,1)}, (x_j)_{1 \leq j \leq n})$. Consider a channel $\Phi' : k\langle x_1, \dots x_n \rangle \to k\langle y_1, \dots y_n \rangle$ such that $\Phi'(x_i) = y_i$ for all i. Then, the geometric support of $[D_{(n,1)} \underset{\delta_M}{\overset{\delta}{\leftrightarrows}} D_{(n,1)}]$ is the $(2n,1)$-De Bruijn graph.*

Proof: Straightforward. □

7.4.2 Entanglement with a unital associative algebra

Let A be a unital algebra with unit 1. We recall from [40] that A carries a non-trivial Markov bi-dialgebra, its coproducts δ_f and $\tilde{\delta}_f$ being defined by $\delta_f(a) = a \otimes 1$ and $\tilde{\delta}_f(a) = 1 \otimes a$, for all $a \in A$. If $A := k\langle a_1, \dots a_n \rangle \oplus k1$ divided by some relations $\mathcal{R}$, then such a space has a geometric support called *flower graph* because it is the concatenation of petals:



Geometric support of an algebra $A := k\langle a, b, c, d \rangle \oplus k1$.

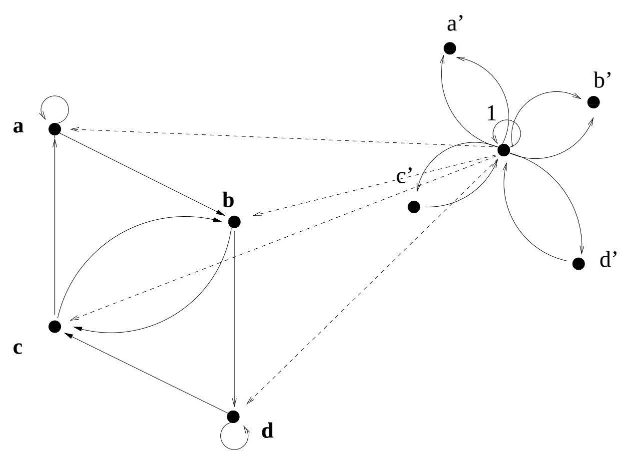
Recall also that $\Delta_f(1) := 1 \otimes 1$ and $\Delta_f(x) := x \otimes 1 + 1 \otimes x$, for all $x \in A$ different from 1 is a coassociative coproduct.

[11] By definition, a channel is a morphism of coalgebra.

Proposition 7.4.5 *Let E be a unital associative algebra, composed by the bialgebra (C,Δ) and by a unital associative algebra $(A,\tilde{\delta}_f,\delta_f)$, viewed as a L-bialgebra such that A and C are related by the isomorphism $\Phi : A \to C$, as a chanel with no fixed point except 1. Extend the coproduct $\tilde{\delta}_f$ over the coassociative bialgebra C by requiring that $\tilde{\delta}_f\Phi = (id\otimes\Phi)\tilde{\delta}_f$. In addition, define the bridge δ such that $\delta := \Delta$ over C, and $\delta\Phi^{-1} = (id\otimes\Phi^{-1})\Delta$. If $\Delta_* := \Delta_f$ over A and $\Delta_* := \Delta$ over C, then the codipterous bialgebra $(E,\Delta_*,\tilde{\delta}_f)$ is entangled to the codipterous bialgebra (E,Δ_*,δ), since $(\tilde{\delta}_f\otimes id)\delta = (id\otimes\delta)\tilde{\delta}_f$. Such a chiral entanglement is denoted by $[\tilde{A}\underset{\delta}{\overset{\tilde{\delta}_f}{\rightleftharpoons}}C]$. Similarly, since $(id\otimes\delta_f)\delta = (\delta\otimes id)\delta_f$, $[A\underset{\delta_f}{\overset{\delta}{\leftrightharpoons}}C]$ is also a chiral entangled codipterous bialgebra.*

Proof: Straightforward. □

Example 7.4.6 $[\tilde{A}\underset{\delta}{\overset{\tilde{\delta}_f}{\rightleftharpoons}}Sl_q(2)]$ Let the associative algebra E be composed by the Hopf algebra $Sl_q(2) := k\langle a,b,c,d\rangle$ and the algebra $A := k\langle a',b',c',d'\rangle$ related by the channel $\Phi : A \to Sl_q(2)$ defined by $\Phi(a) := a', \Phi(b) := b', \Phi(c) := c', \Phi(d) := d'$. We represent here a face of the geometric support of $[\tilde{A}\underset{\delta}{\overset{\tilde{\delta}_f}{\rightleftharpoons}}Sl_q(2)]$.



Face of the geometric support of $[\tilde{A}\underset{\delta}{\overset{\tilde{\delta}_f}{\rightleftharpoons}}Sl_q(2)]$. Among bridges, only $\tilde{\delta}_f$ is represented.

We recall a result from [40] inspired from a Theorem from [22].

Theorem 7.4.7 *Let (C,Δ) be a bialgebra, then the coproducts $\overrightarrow{d} := \Delta - \delta_f$ and $\overleftarrow{d} := \Delta - \tilde{\delta}_f$ are Leibniz-Ito derivatives and verify $(id\otimes\overrightarrow{d})\overleftarrow{d} = (\overleftarrow{d}\otimes id)\overrightarrow{d}$.*

7.4.3 Complex based on Leibniz-Ito coproducts of a bialgebra

Notation: In this Subsection, $\sum_i id\otimes\ldots\otimes \text{EXP}\otimes\ldots id$ means that the expression EXP is placed at position i. Recall that $\Delta_f := \delta_f + \tilde{\delta}_f$.

Let (C, Δ) be a bialgebra with unit 1. Let us establish a complex whose boundaries are based on the difference [12] $\Delta - \Delta_f$ and the Leibniz-Ito coproducts $\overrightarrow{d} := \Delta - \delta_f$ and $\overleftarrow{d} := \Delta - \tilde{\delta}_f$, where Δ_f is the coassociative coproduct defined from the Markovian coproducts of the flower graph, naturally attached to C as an algebra. In this study, the primitive elements, i.e., $x \in C$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x = \Delta_f(x)$ will vanish.

Lemma 7.4.8 *Let us denote $\partial'_0 = \Delta$ and for all $n > 0$,*

$$\partial'_n = \sum_{i=1}^{n+1} (-1)^{i+1} \underbrace{id \otimes \ldots \otimes id \otimes \Delta \otimes id \ldots \otimes id}_{n+1 \quad terms}.$$

Fix $n > 0$ and $a, a_1, \ldots, a_n \in C$. Then,

1. $\partial'_{n+1}\tilde{\delta}_f(a_1) \otimes a_2 \otimes \ldots \otimes a_n = \Delta(1) \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n - 1 \otimes \partial'_n(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$.
2. $(-1)^n \partial'_{n+1} a_1 \otimes a_2 \otimes \ldots \otimes \delta_f(a_n) = (-1)^n \partial'_n(a_1 \otimes a_2 \otimes \ldots \otimes a_n) \otimes 1 + a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes \Delta(1)$.
3. $(\tilde{\delta}_f \otimes id)\tilde{\delta}_f = \Delta(1) \otimes id = (\Delta \otimes id)\tilde{\delta}_f$.
4. $(id \otimes \delta_f)\delta_f = id \otimes \Delta(1) = (id \otimes \Delta)\delta_f$.
5. $(\Delta \otimes id)\delta_f(a) = \Delta(a) \otimes 1, \quad (id \otimes \Delta)\tilde{\delta}_f(a) = 1 \otimes \Delta(a)$.

Proof: The proof is complete by noticing that $\Delta(1) = 1 \otimes 1$ and by using the definitions of the coproducts $\delta_f(a) = a \otimes 1, \quad \tilde{\delta}_f(a) = 1 \otimes a$, for all $a \in C$. □

Theorem 7.4.9 *Recall that $\overleftarrow{d} := \Delta - \tilde{\delta}_f$ and $\overrightarrow{d} := \Delta - \delta_f$. The sequence,*

$$0 \to C \xrightarrow{\Delta - \Delta_f} C^{\otimes 2} \xrightarrow{\partial_1 = \overleftarrow{d} \otimes id - id \otimes \overrightarrow{d}} C^{\otimes 3} \xrightarrow{\partial_2} C^{\otimes 4} \xrightarrow{\partial_3} \ldots$$

with $\forall n > 0$,

$$\partial_n := \underbrace{\overleftarrow{d} \otimes id \otimes \ldots \otimes id}_{n+1 \quad terms} + \sum_{j=2}^{n-1} (-1)^{j+1}\, id \otimes \ldots \otimes id \otimes \Delta \otimes id \otimes \ldots \otimes id + (-1)^{n+1} id \otimes id \otimes \ldots \otimes id \otimes \overrightarrow{d},$$

defines a complex. The boundary operators verify:

1. $\forall n > 0 \quad \partial_{n+1}\partial_n = 0$ *and* $\partial_1(\Delta - \Delta_f) = 0$.
2. $\forall n > 0 \quad \partial_n(x_1, \ldots, x_n)$ *is a multilinear map which is a Leibniz-Ito derivative in the first and last variables and a homomorphism in others variables.*

[12] Such a difference is used in the definition of connexe bialgebras.

Proof: We only have to prove the first item, since the second one comes from the very definition of the boundary operators. From coassociativity coalgebra theory, we know that $\partial'_{k+1}\partial'_k = 0, \quad \forall k \in \mathbb{N}$. Fix $n > 0$.

$$\begin{aligned}\partial_{n+1}\partial_n &= \partial'_{n+1} - \tilde{\delta}_f \otimes id \ldots \otimes id + (-1)^{n+1} id \otimes \ldots \otimes id \otimes \delta_f)(\partial'_n \\ &\quad -\tilde{\delta}_f \otimes id \ldots \otimes id + (-1)^n id \otimes \ldots \otimes id \otimes \delta_f) \\ &= \partial'_{n+1}\partial'_n - \partial'_{n+1}(\tilde{\delta}_f \otimes id \ldots \otimes id) + (-1)^n \partial'_{n+1}(id \otimes \ldots \otimes id \otimes \delta_f) \\ &\quad -1 \otimes \partial'_n + (\tilde{\delta}_f \otimes 1)\tilde{\delta}_f \otimes id \ldots \otimes id \\ &\quad +(-1)^{n+1}\partial'_n \otimes 1 - id \otimes \ldots \otimes id \otimes (id \otimes \delta_f)\delta_f.\end{aligned}$$

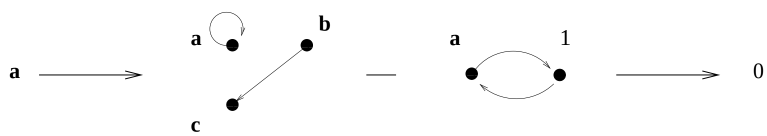
The equality $\partial_1(\Delta - \Delta_f) = 0$ is straightforward. □

We can also express the boundary operators only in terms of $\Delta - \Delta_f$.

Theorem 7.4.10 *For all $n > 0$, $\partial_n = \sum_{i=1}^{n+1} (-1)^{n+1}\ id \otimes \ldots \otimes (\Delta - \Delta_f) \otimes \ldots \otimes id$.*

Proof: The proof is straightforward by noticing that $id \otimes \tilde{\delta}_f = \delta_f \otimes id$. □

Example 7.4.11 Consider the geometric support of $Sl_2(q)$. The difference $\Delta - \Delta_f$, at a yields:



and vanishes when the operator $\partial_1 = \overleftarrow{d} \otimes id - id \otimes \overrightarrow{d}$ is applied.

Remark: Notice that all the results of this Subsection hold for a coassociative coalgebra equipped with a group-like element e, by removing 1 by e. Of course, the coproducts $\overleftarrow{d}, \overrightarrow{d}$ are no longer Leibniz-Ito derivatives.

7.5 Towards coassociative manifolds and L-molecules

Thanks to the notion of entanglement of codipterous coalgebras and anti-codipterous coalgebras, we construct more and more complicated directed graphs as well as a generalisation of coassociative coalgebras, bialgebras and Hopf algebras by considering them as simple structures, reminiscent of atoms in physics. The entanglement of atoms leads to molecules. We define:

Definition 7.5.1 [**L-molecule**] Let E be a k-vector space. A *L-molecule* in E is an entanglement of several codipterous coalgebras and / or anti-codipterous coalgebras constructed from coassociative coalgebras via channel maps.

Example 7.5.2 For instance, $[Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \widetilde{Sl_q(2)} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2)]$ or $[Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \cdots \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2)]$.

By dualizing all these definitions, we can construct algebras with several laws and can envisage to study their associated operads.

Conjecture: Any operad, associated with a dualisation of a L-molecule is Koszul.

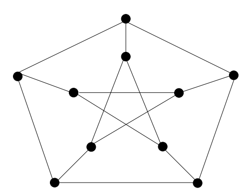
Let us end with a geometric interpretation of the notion of coproduct of a L-molecule M, spanned as a k-vector space by a basis $(v_i)_{1\leq i\leq \dim M}$. Recall that the family $(v_i)_{1\leq i\leq \dim M}$ plays the rôle of the vertex set and $v_i \otimes v_j$ is symbolized by a directed arrow $v_i \to v_j$. In addition, by the intersection of two directed graphs, we mean the intersection of their arrow sets.

Definition 7.5.3 [Coassociative covering] Let W be a k-vector space, with basis $(v_i)_{1\leq i\leq \dim W}$. Consider a subspace $\mathcal{G}$ of $W^{\otimes 2}$. Denote by G_δ, the directed graph defined by the coassociative coproduct δ. Let $\delta_1, \dots, \delta_n : W \to \mathcal{G}$ be n coassociative coproducts. The directed graph G, associated with $\mathcal{G}$ has a *coassociative covering* if $\cup G_{\delta_i} = G$ and $\delta_i = \delta_j$ over $s(G_{\delta_i} \cap G_{\delta_j}) \cup t(G_{\delta_i} \cap G_{\delta_j})$. Such a space is called a *coassociative manifold.*

Theorem 7.5.4 *Any geometric support of a L-molecule is a coassociative manifold.*

Proof: Straightforward by the very definition of a L-molecule. □

Let us show now how to embed any non-directed graph into a coassociative manifold. Let $G = (G_0, G_1)$ be a non-directed graph, such that given two vertices u, v of G_0, they are linked either with no edge or with a unique edge of G_1. Let $G^\natural$ be the directed graph such that the vertex set $G^\natural = G_0$. The arrow set $G_1^\natural$ is defined as follow: each edge, $\bullet - - - \bullet$, of G_1 which is not a loop, is removed by a bi-directed arrow $\bullet \leftrightarrows \bullet$. In addition, put a loop on each vertex without loop in G.



Example of non-directed graph: The Petersen graph.

Theorem 7.5.5 *Let $G = (G_0, G_1)$ be a non-directed graph such that two vertices u, v of G_0, are linked either with no edge or with a unique edge of G_1. Then $G^\natural$ is the geometric support of a L-molecule and thus of a coassociative manifold.*

Proof: Embed $G_0^\natural$ into its free k-vector space $kG_0^\natural$. Define the coassociative coproduct $\Delta_l v := v \otimes v$ for all $v \in G_0^\natural$. Let $v, w \in G_0^\natural$, with $v \neq w$. If $v \longrightarrow w$ is an arrow of $G_1^\natural$, define the chanel map $\phi_{vw} : k\langle v \rangle \to k\langle w \rangle$ such that $\phi_{vw}(v) = w$. Define the bridges $\delta_{vw} w := v \otimes w$ and $\delta_{vw} v := v \otimes v = \Delta_l v$. Then the family of bridges $(\delta_{s(a)t(a)})_{a\in G_1^\natural}$ is a family of coassociative coproducts. Moreover, $(kG_0^\natural, \Delta_l, \delta_{s(a)t(a)})_{a\in G_1^\natural}$ is a family of codipterous coalgebras and for all

$v, w \in G_1$, with $v \neq w$, the space $(k\langle v \rangle \oplus k\langle w \rangle, \Delta_l, \delta_{vw}, \delta_{wv})$ is (chiral) entangled, according to Section 7.2. Therefore, the directed graph $G^\natural$ is covered by geometric supports of (Markovian) coassociative coalgebras. As the common intersection between two geometric supports is either a loop or the empty set, $G^\natural$ is then a coassociative manifold since the coproducts are equal to Δ_l on a loop. □

Remark: Notice also that, "locally" two vertices related by a bi-directed arrow looks like a $(2,1)$-De Bruijn graph. Moreover, the map $\epsilon : kG_0^\natural \to k$ such that $\epsilon(v) = 1$ for all $v \in G_0^\natural$ is a left counit.

7.6 Conclusion

In this "coassociative geometry", the rôle of all the coproducts of a coassociative manifold can be better understood. As a comparison, a topological manifold can be covered by open sets, each one being homeomorphic to an open set of $\mathbb{R}^n$. Here, the rôle of an open set is played by a coassociative coalgebra, being a side of the L-molecule M. Denote by $L(M, A)$, the space of linear mappings defined over M, with values in an algebra A, on such side or "open set" of M, there will be a unique convolution product. By "travelling" over such a manifold, we will change progressively the convolution product since $\delta_i = \delta_j$ over $G_{\delta_i} \cap G_{\delta_j}$. As an example, recall that the self-entanglement of $\mathcal{F}$, denoted by M, yields two pre-dendriform coalgebras, each one having three coproducts. Consider $L(M, A)$, where A is an associative algebra. Suppose we start with the side δ_1 of M. This means that we live in an associative algebra $(L(M, A), \vdash)$ whose convolution product of two linear maps f, g, $f \vdash g$ is defined by δ_1. Continuing our travel, suppose we arrive on the side $(\mathcal{F}, \Delta_1)$. As $\delta_1 := \Delta_1$ on the side $(\mathcal{F}, \Delta_1)$, we will obtain $f \vdash g := f \perp g$ $(= f \dashv g)$. Leaving the side $(\mathcal{F}, \Delta_1)$ to go to the side $\hat{\delta}_1$, we will arrive in an associative algebra $(L(M, A), \dashv)$ whose convolution product is now determined by $\hat{\delta}_1$. Therefore, turning around the sides of the associative trialgebra $(L(M, A), \vdash, \dashv, \perp)$ leads us to pass "progressively" from $(L(M, A), \vdash)$ to $(L(M, A), \perp)$ to $(L(M, A), \dashv)$. Besides this dynamical point of view, the formalism developed so far could have several applications in graph theory but also in non-commutative stochastic processes. Quantum Levy processes on bialgebras are now well understood, see [62, 19] for instance. Would it be possible to develop such a theory on coassociative manifolds, or quantum Levy processes on bialgebras with several parameters, and thus with several coproducts? This work will pursued in [37].

Acknowledgments: The author wishes to thank D. Petritis for fruitful advice for the redaction of this paper as well as J-L. Loday and U. Franz and M. Schurmann for useful discussions.

Chapter 8

Conclusion and openings

8.1 Summary

We summarise briefly the ideas contained in Chapters 2, 5 and 7.

Motivated by the intrusion of the coalgebraic formalism in combinatorics and discrete Markov processes, we decided to reverse the point. We constructed a weighted directed graph over each coassociative coalgebra, generated as a k-vector space by an independent spanning set. This led us to observe, *a priori*, non-locality problems since the neighbourhood of a given vertex was not as expected. To recover the usual neighbourhood, we decided to break the coassociativity and to introduce a coalgebraic formalism with two coproducts. These types of coalgebra are called L-coalgebras.

Coding bi-directed graphs with this formalism led us to consider the notion of L-cocommutator, making contact with Leibniz algebras, introduced by Loday [44] ten years ago (1993) and motivated by K-theory. To understand Leibniz algebras, a breakthrough is made in 2001 via the notion of associative dialgebras and dendriform algebras [45]. These types of algebra are all L-coalgebras.

Meanwhile, other types of algebras arose. To understand relationships between the family of standard simplices and the family of Stasheff polytopes, Loday and Ronco [48] introduced other types of algebra called associative trialgebras and dendriform trialgebras (2002). In the same paper, they introduced also cubical trialgebras as a generalisation of cubical dialgebras [53]. Since then, other types of algebra have been studied. The question was then following one.

Could we construct, by our graphical formalism, these new types of algebras and thus give other motivations for their studies?

A first attempt was concretized via a tool from graph theory called line-extension. Briefly, matrix-coalgebras are related to the $(n, 1)$-De Bruijn graphs. A better understanding of their interplays led us to the notion of **coassociative tilings** of the $(n^2, 1)$-De Bruijn graphs. Writting how the different faces, each one modelled by a coassociative coproduct, were glued or entwined, led us to construct families of cubical di or tri-coalgebras and more generally to supply nice examples of hypercube n-algebras, see Chapter 4.

Via these tilings, we made two observations. As we showed, the L-coalgebra setting covers both the usual coalgebraic formalism and more generally the weighted directed graphs. The first observation was that some usual graphs, like De-Bruijn graphs, could be covered by other graphs, geometric supports of coassociative coalgebras. The second one lied in the translation of geometric properties of these tilings into algebraic equations. We could construct other types of algebra, consequences of K-theory motivations.

After the notion of coassociative tiling, could we deal with coassociative covering? What did this notion imply? Chapter 7 showed that **coassociative coverings** of directed graphs (by geometric supports of coassociative coalgebras) is a deep notion. Indeed, it allowed graphically and naturally the construction of coassociative codialgebras, dendriform coalgebras, Leibniz algebras, Poisson dialgebras, coassociative L-coalgebras, via a simple notion which is the entanglement of codipterous coalgebras. In addition, coassociative coverings of directed graphs allowed also the construction of new types of algebras and were related to the notion of L-molecules, see Chapter 7 and below.

8.2 Brief summary of Chapters 2, 5 and 7

1. Via the L-coalgebra formalism, we propose a new way to view weighted directed graphs.
2. It covers coassociative coalgebras as well.
3. Via simple notions such as,
    - **Coassociative tilings and,**
    - **Coassociative coverings,**

of directed graphs, we can naturally recover, construct and motivate other (co)-algebraic structures which are: cubical, associative and dendriform (co)-dialgebras; associative, cubical (co)-trialgebras; Leibniz algebras; Poisson (di)-algebras; associative L-(co)-algebras; dipterous and pre-dendriform (co)-algebras, and enlarge these notions to other structures called (co)-associative manifolds.

8.3 Towards a chemistry of associative algebras: The clusters!

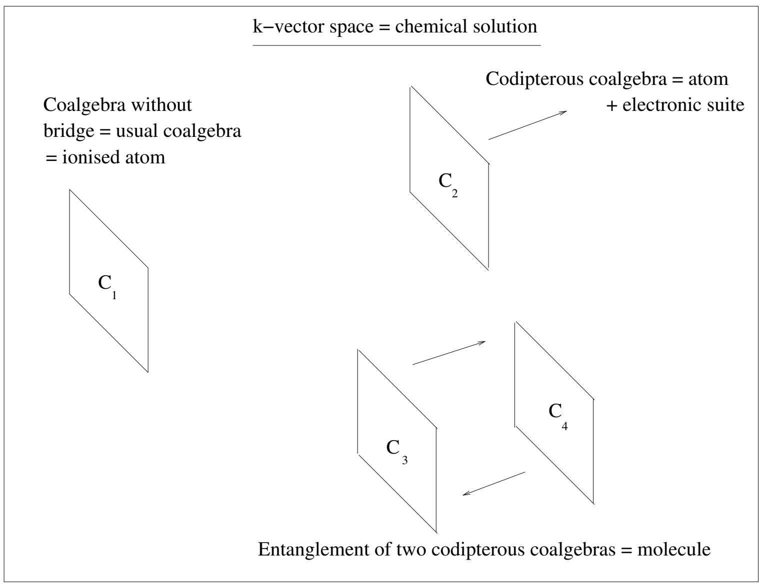
To construct molecule, atoms share their electronic suites, modelled in our formalism by the notion of bridges, (see Chapter 7). Reminiscent of what was identified in physics, we have proposed the following dictionary, (see Chapter 7 and introduction).

1. Coassociative coalgebra $\equiv$ ionised atom or nucleus.

2. Codipterous coalgebra $\equiv$ atom := nucleus + electrons.

3. Entanglement of two (anti)codipterous coalgebras $\equiv$ molecule.

4. Entanglement of n (anti)codipterous coalgebras $\equiv$ polymers, periodic net, etc. For instance,
$$[Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \widetilde{Sl_q(2)} \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2)], \quad [Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2) \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} \cdots \underset{\delta_2}{\overset{\delta_1}{\rightleftharpoons}} Sl_q(2)].$$
.



In Physics, the nucleus (played here by a coassociative coalgebra) splits. It is called nuclear fission. There exists a similar phenomenon in associative algebra suggested either by Koszul duality, or by self-covering of a pre-dendriform structure, called [33] associative clusters [1].

This phenomenon can be met in dendriform (tri)-algebras, which are associative algebras whose product splits in several operations:

$$\star \longrightarrow \prec + \succ + (\circ).$$

Similarly for hypercube n-algebras,
$$\star \longrightarrow \sum_i \star_i.$$

Very recently, other types of algebra have arisen, all presenting the same splitting phenomenon. Quadri-algebras [1] are also associative algebras whose product is a cluster of four operations which splits with precise rules:

$$\star \longrightarrow \nwarrow + \swarrow + \searrow + \nearrow .$$

[1] En français, amas associatifs.

They have intriguing properties and are closely related to pre-Lie algebras. There exists another structure imitating quadri-algebras, also closely related to pre-Lie algebras. They are called t-ennea-algebras [33] (for $t = 0$, we recover quadri-algebras) and allow the construction of nested dendriform trialgebras. They still are associative algebras whose product is a cluster of nine operations:

$$\bar{\star} \longrightarrow \nwarrow + \swarrow + \searrow + \nearrow + \uparrow + \downarrow + \prec + \succ + \circ\, .$$

Let us mention that here again, for all t, weighted directed graphs give very nice examples of such algebras.

The question is then the following one:

By analogy with,

1. Chemistry (Mendeleiev classification),
2. Nuclear Physics,
3. Particules Physics,
4. Astrophysics (stardusts [2]),

can we propose a classification of associative algebras, viewed as "molecular and nuclear" clusters, suggested by our graphical interpretation of Loday and Ronco's works?

[2] Amas stellaires.

Appendix A

Commutativity in quantum mechanics and the third Reidemeister move

Abstract: In quantum mechanics, a simultaneous measure modelled by observables makes sense if the observables involved commute pairewise. The aim of this work is to relate commutativity of positive operators to a topological move in knot theory, known as the third Reidemeister move *via* a new way to view quantum fidelity in quantum information theory.

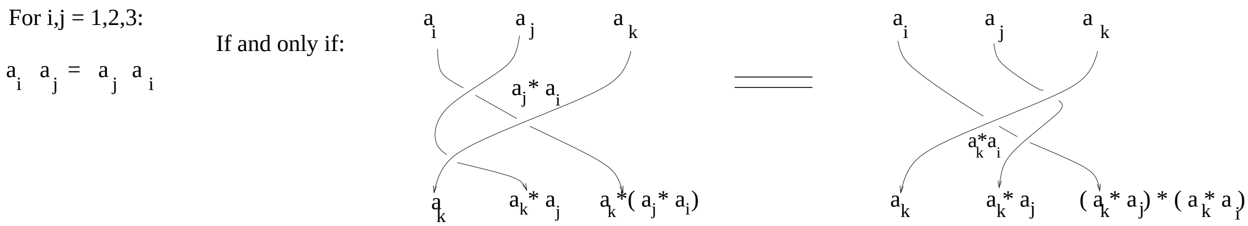
A.1 Introduction

On the one hand, let us recall that quandle sets and left-distributive laws arise naturally in knot theory. It has been shown by Reidemeister that all the possible moves we can do on a given knot can be decomposed into three topological moves called the Reidemeister moves. Expressing them in an algebraically way yield the axioms for quandle sets, the third Reidemeister move being algebraically related to left-distributive laws. A law $\star$ is said left distributive if for all a, b, c, belonging to a given set, the following relation holds:

$$a \star (b \star c) = (a \star b) \star (a \star c).$$

The aim of this work is to relate commutativity of two positive operators to the third Reidemeister move in knot theory.

Let $B(\mathcal{H})$ be the algebra of bounded operators acting on a separable Hilbert space. In quantum information theory, one can define for positive operators, $a, b \in B(\mathcal{H})$, the following product: $a * b := (b^{\frac{1}{2}} a b^{\frac{1}{2}})^{\frac{1}{2}}$, known as the *quantum fidelity* when both operators have a trace equal to one, i.e., model quantum systems. Via the quantum fidelity, a distance called the *Bures distance* between two density operators $a, b \in B(\mathcal{H})$ is then defined [5, 59, 16] as $d_B(a, b) := (2 - 2\,\mathrm{tr}(a * b))^{\frac{1}{2}}$. It can be checked that if three positive operators commute pairwise the quantum fidelity, viewed through the law $*$ turns out to be a left-distributive law. On the other hand, in quantum mechanics, the notion of a n-tuple of self-adjoint operators which commute pairwise is capital to model simultaneous measurements. We propose here to relate commutativity of positive operators to a left-distributivity law. In a family of pairwise commuting positive operators, we observe that such a set, equipped with the quantum fidelity law $*$ generates a lelf-distributive set, i.e., the third Reidemeister move is possible among the given positive operators. We conjecture that the reverse also holds, that is the quantum fidelity law $*$ is left-distributive between three given positive operators entails that these three operators commute pairwise.



Conjecture expressed in terms of the third Reidmeister move.

Should the left-distributivity of the law $*$ be not possible, we interpret it as an obstruction to commutativity. We prove this conjecture in Section A.4 for important cases.

In quantum mechanics, information are obtained *via* the trace map which authorises commutativity at "short distance" since $tr(ab) = tr(ba)$. In Subsection A.4, we define quasi left-distributive laws, that is laws $\star$ which are left-distributive when this commutativity at "short distance" is authorised, that is:

$$\mathrm{tr}(a \star (b \star c) - (a \star b) \star (a \star c)) = 0.$$

We relate the Wigner-Yanase information to such laws and interpret it as a defect of $\star$-morphism. In Section A.5, we relate the Bures distance to another concept, introduced by Rylov, which replaces Riemannian geometry. The idea is to construct an analogue of Euclidean or Riemannian geometries only in terms of a world function obeying few axioms. Embedding space of density matrices, which model quantum systems, with such a geometry we construct a world function from the Bures distance and relate the quantum fidelity law, which is left-distributive when density matrices commute pairewise and volumes of this new geometry to 2-cocycles coming from quandle set theory. In Section A.2, we recall some definitions and in Section A.3, we construct new left-distributive laws.

A.2 Quandle algebras and LDRI systems

Notation: All over this paper, we denote by $\mathbb{R}_+^*$, the set of strictly positive reals and by $\mathbb{C}^*$, the set of complexes different from zero. The field k will denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition A.2.1 [Quandle algebras and LDRI systems] Let S be a set equipped with a product $* : S \times S \to S$. The set S is said a *quandle algebra* [18] if it verifies the axioms R_1, R_2, R_3 or $R_1, R_2, R_{3'}$, with:

R_1 : Idempotent law or the first Reidemeister move, i.e., for all $a \in S$, $a * a = a$.

R_2 : The second Reidemeister move, i.e., for all $a, b \in S$, there exists a unique $c \in S$ such that $b = a * c$.

R_3 : Left distributivity or the third Reidemeister move, i.e., for all $a, b, c \in S$, $a*(b*c) = (a*b)*(a*c)$.

$R_{3'}$: Right distributivity or the third Reidemeister move, i.e., for all $a, b, c \in S$, $(b*c)*a = (b*a)*(c*a)$.

S is said a *left distributive (LD) system* [15] if it verifies at least the third axiom of a quandle algebra and a *(LDI) system* if it verifies R_1 and R_3. A *right distributive (RD) system* S verifies $R_{3'}$. The definition of a *LDRI system* is now straightforward.

Remark: Define for all $a \in S$, $\Psi_a : S \to S$, $x \mapsto \Psi_a(x) = a * x$. Let $a \in S$, R_3 means that Ψ_a is an $*$-homomorphism since for all $b, c \in S$, we get $\Psi_a(b * c) = \Psi_a(b) * \Psi_a(c)$. The axiom R_1 means that any Ψ_a has at least a fixed point a and R_2 means that any map Ψ_a is injective. Stated otherwise we can say that $*$ is left cancellative i.e., $a * c = a * c'$ implies $c = c'$.

Definition A.2.2 [Entropic law] A set S is called an *entropic* [15] or a *medial system* if and only if S is equipped with a product such that all its elements verify $(xy)(uv) = (xu)(yv)$.

A.3 New examples

Convention: Fix $z \in \mathbb{C}^*$ with argument $\theta \in [0,\ 2\pi[$ and $a \in]0, 1[$. Among all the possible roots of z^a, we choose $|z|^a \exp(ia\theta)$.

Few examples of LD systems are known in the literature. We present here some new examples.

Example A.3.1 [**$\mathbb{C}$ and $\mathbb{R}_+$**]

Theorem A.3.2 *Let $a, b \in \mathbb{R}$. The map $\mathbb{C} \times \mathbb{C} \xrightarrow{\star} \mathbb{C}$, $(y, z) \mapsto y^a z^b$ embeds $\mathbb{C}$ into a LDRI system and $\mathbb{C}^*$ into a quandle algebra if and only if $a + b = 1$. In this case $R_1 \Leftrightarrow R_3$. If $a, b \in \mathbb{R}_+$, the same result holds, i.e., $(\mathbb{R}_+, \star)$ is a LDRI set and $(\mathbb{R}_+^*, \star)$ a quandle algebra. Moreover this product is entropic.*

Proof: Straightforward. □

Remark: Equipped with the law $a \lhd b = a^{-1}ba$, any group can be embedded into a quandle algebra. However, in the case of commutative group, this quandle is trival.

Before going on, let us recall the Schur product [20] of two matrices. Let $M_n(k)$ be the algebra of n by n matrices over the field k. Let $x, y \in M_n(k)$, the Schur product of x and y, denoted by $x \circ y$ is defined by: $(x \circ y)_{i,j} = x_{i,j} y_{i,j}$, for all $1 \le i, j \le n$. If x and y are positive, so is $x \circ y$. Let $a \ge 0$. In the sequel, $(x)^{\circ a}$ will mean x to the power a in the sense of the Schur product.

Example A.3.3 [Non-negative matrices] Let $x \in M_n(\mathbb{R})$. The matrix x is said non-negative if all its components are non-negative. Denote by NN the set of non-negative matrices. The set NN can be embedded into a LDRI set by defining for all $x, y \in NN$, the law $*$ such that $x * y := x^{\circ a} \circ y^{\circ b}$, with $a + b = 1$ and a, b positive.

Example A.3.4 [Self-adjoint matrices] By observing that the Schur product of two self-adjoint matrices is still self-adjoint, the set of self-adjoint matrices can be embedded into a LDRI set by defining for all x, y self-adjoints, the law $*$ such that $x * y = x^{\circ a} \circ y^{\circ b}$, with $a + b = 1$ and a, b positive.

A.4 Obstruction to commutativity

This Subsection is an attempt to generalise what was said in Theorem A.3.2 into a non-commutative algebra. We will see that a correct generalisation of the $\star$ product is what is called the quantum fidelity in quantum information theory. We will show also that an obstruction of the third Reidemeister move, for this new law, can be viewed as an obstruction to the commutativity of positive operators.

Let $\mathcal{H}$ be a separable Hilbert space. We denote by $B(\mathcal{H})$ the set of bounded operators acting on $\mathcal{H}$, by $P(\mathcal{H})$ the set of bounded positive operators and by $P(\mathcal{H})_+$ the set of strictly bounded positive operators. Two brackets will be used, $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$ for any $a, b \in B(\mathcal{H})$. We denote by $\{a\}'$, the commutant of $a \in B(\mathcal{H})$, i.e., the set of bounded operators which commute with a and by I the unit of $B(\mathcal{H})$. By $\mathbb{D}$, we mean the set of density operators, i.e., the set of trace one positive operators. Let $a \in B(\mathcal{H})$. By $\mathsf{Spec}(a)$, we mean the spectrum of an operator a and by $a^\dagger$, the adjoint of an operator a.

Definition A.4.1 [The $*$ product] Let $a, b \in P(\mathcal{H})$, we define the $*$ product by: $a * b = (b^{\frac{1}{2}} a b^{\frac{1}{2}})^{\frac{1}{2}}$.

Remark: If $\lambda \geq 0$ and $a, b \in P(\mathcal{H})$, then $a * (\lambda b) = (\lambda a) * b = \sqrt{\lambda}(a * b)$. In quantum information theory, this product is called the quantum fidelity if both a and b verify $\mathrm{tr}\ a = 1 = \mathrm{tr}\ b$. Quantum fidelity is closely related to the Bures distance $d_B(a, b) := (2 - 2\,\mathrm{tr}(a * b))^{\frac{1}{2}}$, also used in quantum information theory. We will now view quantum fidelity as an algebraic law.

Theorem A.4.2 *Let $I \subseteq \mathbb{N}$ and $(a_i)_{i \in I}$ a family of positive operators. The set $((a_i)_{i \in I}, *)$ is a LDRI set. It is a quandle algebra if all the operators a_i, $i \in I$, are strictly positive.*

Proof: Straightforward. □

Remark: Let $a, b, c \in P(\mathcal{H})$. If the operators a, b, c commute pairwise then R_3 holds on the set $(\{a, b, c\}, *)$, i.e., the third Reidemeister move is possible on it.

We study the converse of this Theorem to view non-commutativity of positive operators as an obstruction to the third Reidemeister move. Though the converse is for the moment a conjecture, we prove in the sequel some particular, but important, cases.

Conjecture Let $(a,b,c) \in P(\mathcal{H})$. The quantum fidelity law is still denoted by $*$. If R_3 holds on the set $(\{a,b,c\},*)$, then a,b,c have to commute pairwise. A graphical picture of this conjecture can be found in the introduction of this work.

For the study of some particular cases, we need the following Theorem.

Theorem A.4.3 *Let X be a normal bounded operator. Then $X = BC$, where B, C are self-adjoint bounded operators, entails that B and C commute.*

Proof: Let X be a normal bounded operator. Since X is normal, it commutes with its adjoint. We have, say $Xx = \lambda_x x$, where $x \in \mathcal{H}$ and λ_x is a nonzero scalar. Since $Xx \neq 0$, $Cx \neq 0$. Moreover $CX = CBC$ self-adjoint implies $x^\dagger CXx = \lambda_x x^\dagger Cx$. Since C and CX are self-adjoint operators, all the eigenvalues of X must be real. That is $X = X^\dagger$. In this case $BC = CB$. $\square$

For $a,b \in P(\mathcal{H})$, we define $[a,b]_* := a*b - b*a$.

Theorem A.4.4 *Let $a,b \in P(\mathcal{H})$.*

$$[a,b]_* = 0 \Leftrightarrow [a,b] = 0.$$

Proof: Let $a,b \in P(\mathcal{H})$. By unicity of the square root of a positive operator, the equation, $a*b = b*a$, is equivalent to $b^{\frac{1}{2}}ab^{\frac{1}{2}} = a^{\frac{1}{2}}ba^{\frac{1}{2}}$, i.e., $X := a^{\frac{1}{2}}b^{\frac{1}{2}}$ is normal, i.e., $[a,b] = 0$. $\square$

Theorem A.4.5 *Let $a,b,c \in P(\mathcal{H})$, with a,b strictly positive. If $[a,b] = 0$ and $[a,c] = 0$, then the third Reidemeister move between (a,b,c) is possible if and only if $[b,c] = 0$.*

Proof: Among all the writtings of R_3, we choose to study what constraints imply the following choice:

$$b*(a*c) = (b*a)*(b*c).$$

Since $[a,c]_* = 0$, $[(b*a),(b*c)]_* = 0$, i.e., $[(b*a),(b*c)] = 0$, i.e., $[b,c] = 0$. $\square$

Remark: This Theorem claims that two strictly positive operators in the commutant of a turns R_3 possible if and only if b and c belong to $\{b\}^{'} \cap \{c\}^{'}$. Stated otherwise the non-validity of R_3 on the set $(\{a,b,c\},*)$ can be viewed, in the commutant of a, as an obstruction to the commutativity of b with respect to c.

Corollary A.4.6 *Let $b,c \in P(\mathcal{H})$, with b invertible. Then, $[b,c] = 0$ if and only if R_3 holds on the set $(\{I,b,c\},*)$.*

Corollary A.4.7 *Let $b,c \in P(\mathcal{H})$, with b invertible. Let f be a continuous function on the compact $\mathsf{Spec}(b)$ and g be a continuous function on the compact $\mathsf{Spec}(c)$. If R_3 holds on the set $(\{I,b,c\},*)$ then R_3 also holds on the set $(\{I,f(b),g(c)\},*)$.*

Proof: Straightforward since $[b,c]=0 \Leftrightarrow [b,c]_*=0$. □

Corollary A.4.8 *Let $b,c \in P(\mathcal{H})$, with b invertible.*

$$[b,c]=0 \Leftrightarrow b*c = b^{\frac{1}{4}} c^{\frac{1}{2}} b^{\frac{1}{4}}.$$

Theorem A.4.9 *Let $a,b,c \in P(\mathcal{H})$, with a,b invertible. Suppose $[b,c]=0$. Then R_3 holds on the set $(\{a,b,b^{\frac{1}{2}},c)\},*)$ is equivalent to say that (a,b,c) commute pairwise.*

Proof: The axiom R_3 holding on the set $(\{a,b,c)\},*)$ implies that, for instance $a*(b*c) = (a*b)*(a*c)$, i.e., since $[b,c]_*=0$, $(a*b)*(a*c) = (a*c)*(a*b)$, i.e., $[a*b, a*c]_* = 0$, i.e., $bac = cab$. Similarly, R_3 holding on the set $(\{a, b^{\frac{1}{2}}, c\},*)$ implies that $b^{\frac{1}{2}}ac = cab^{\frac{1}{2}}$, since $[b^{\frac{1}{2}}, c] = 0$. Therefore, a,b,c commute pairewise. □

Remark: With the assumptions of this Theorem, the impossibility to do the third Reidemeister move on the set $(\{a,b,b^{\frac{1}{2}},c)\},*)$ can be viewed as an obstruction to commutativity.

It is interesting to study all properties above on the density operators set $\mathbb{D}$. For convenience we include 0 in $\mathbb{D}$.

Definition A.4.10 Let $a,b \in \mathbb{D}$ such that $a*b \neq 0$. Define the following law: $a \oslash b = \frac{a*b}{\mathrm{tr}(a*b)}$.

Theorem A.4.11 *Let $\mathcal{T} \subset B(\mathcal{H})$, a set of mutually commuting positive operators. Then $(\mathcal{T},*)$ generates a LDRI system. If all the elements of $\mathcal{T}$ are invertible, then $(\mathcal{T},*)$ generates a quandle algebra. Similarly, if $\mathcal{T}_1 \subset \mathcal{D}$ is a set of mutually commuting density operators, then $(\mathcal{T},\oslash)$ generates a LDRI system. If all the elements of $\mathcal{T}$ are invertible, then $(\mathcal{T},\oslash)$ generates a quandle algebra.*

Proof: Checking the axioms R_1, R_2 and R_3 is straightforward. □

Remark: **[Distributivity]** Let $a,b,c \in \mathcal{T} \subset B(\mathcal{H})$ be a set of mutually commuting positive operators. Then the usual operator product ab is still positive. We can as well study the set generated by $(\mathcal{T},*,m)$, where m denotes the usual operator product in $B(\mathcal{H})$. We have $a(b*c) = (a*b)(a*c)$ and $(aI)*(bc) = a*(bc) = (a*I)(b*c)$. We observe also that $a*I = I*a = a^{\frac{1}{2}}$.

Proposition A.4.12 *Let $a,b \in P(\mathcal{H})_+$. Set $X^{\frac{1}{2}} := a^{-1}*(a*b)$ and $Y^{\frac{1}{2}} := b^{-1}*(b*a)$. Then, $XY = YX = I$.*

Proof: Recall that for any invertible bounded operator Z, the polar decomposition yields $Z = U(Z^\dagger Z)^{\frac{1}{2}}$, where U is a unitary operator. Similarly $UZ^\dagger = (ZZ^\dagger)^{\frac{1}{2}}$, with $Z := a^{\frac{1}{2}}b^{\frac{1}{2}}$. However, $XY = a^{-\frac{1}{2}}(a*b)a^{-\frac{1}{2}}Y = a^{-\frac{1}{2}}(UZ^\dagger)a^{-\frac{1}{2}}b^{-\frac{1}{2}}(U^\dagger Z)b^{-\frac{1}{2}} = I$. Similarly, we obtain the other equality $YX = I$. □

Theorem A.4.13 *[compatibility of the quantum fidelity law $*$ with the order structure in $P(\mathcal{H})$] Let $a, b, x \in P(\mathcal{H})$. Then,*

$$a \le b \Rightarrow x * a \le x * b.$$

Proof: Let $a, b, x \in P(\mathcal{H})$. We get: $0 \le a \le b \Rightarrow 0 \le x^{\frac{1}{2}} a x^{\frac{1}{2}} \le x^{\frac{1}{2}} b x^{\frac{1}{2}} \Rightarrow x * a \le x * b$. □

Theorem A.4.14 *Let $x, y, z \in P(\mathcal{H})$ be three mutually non-orthogonal projectors of rank one. The axiom R_3 holds for the quantum fidelity law $*$ if and only if $x = y = z$. Let $x_1, y_1, z_1 \in P(\mathcal{H})$ be three mutually non-orthogonal (trace-class) operators of rank one, then the axiom R_3 holds for the quantum fidelity law $*$ if and only if the three operators, x_1, y_1, z_1 are proportional.*

Proof: Let $e, f \in P(\mathcal{H})$ be two non-orthogonal projectors of rank one. Then, $e^{\frac{1}{2}} = e$ and $efe = \mathrm{tr}(ef)f$. This remark yields, for $x, y, z \in P(\mathcal{H})$, three mutually non-orthogonal projectors of rank one, $\mathrm{tr}(xy) = \mathrm{tr}(xz) = \mathrm{tr}(yz) = 1$. However, the Schwartz inequality yields $\mathrm{tr}(ef) \le (\mathrm{tr}\, e)(\mathrm{tr}\, f)$, with equality if and only if there exists $\lambda > 0,\ e = \lambda f$. Since, $\mathrm{tr}\ x = \mathrm{tr}\ y = \mathrm{tr}\ z = 1$, we get $x = y = z$. Now, if $e_1 \in P(\mathcal{H})$ is an operator of rank one, then $e_1^2 = \mathrm{tr}(e_1)e_1$, hence the last assertion. □

The Wigner-Yanase information

This Section is an attempt to define left-distributivity up to a commutativity authorised by the trace map. Indeed, in quantum mechanics, the mean of an operator a in a state ρ modelling a quantum system is $\langle\, a \,\rangle_\rho := tr(\rho a)$. It can be interesting to pursue our investigation on left distributive laws, in quantum mechanics, viewed up to a trace. We relate such a point of view to the case of the Wigner-Yanase information.

Definition A.4.15 [The Wigner-Yanase information] Let ρ be a density operator, i.e., a trace one positive operator. Wigner and Yanase [63] define a notion of (quantum) *information* of a self-adjoint operator k with regard to the quantum system ρ by:

$$S_{WY}(k \mid \rho) := \frac{1}{2}\,\mathrm{tr}[\rho^{\frac{1}{2}}, k][\rho^{\frac{1}{2}}, k].$$

Remark: Based on this idea, we define: $S_{WY}(k, l \mid \rho) := \frac{1}{2} tr[\rho^{\frac{1}{2}}, k][\rho^{\frac{1}{2}}, l]$.

Definition A.4.16 Let R be a set of trace-class operators equipped with a product $\star$. This product is said *quasi-distributive* if it verifies:

$$\forall a, b, c \in R, \quad \mathrm{tr}(a \star (b \star c) - (a \star b) \star (a \star c)) = 0.$$

Example A.4.17 We define for all $a, b, c \in P(\mathcal{H})$, $a \star b := \frac{1}{2}\{a^{\frac{1}{2}}, b^{\frac{1}{2}}\}$.

Remark: A priori, $a \star b$ is not positive, though hermitian. If $[a, b] = 0$, then $a \star b = a * b$. The map $\star$ allows us to build an algorithm. Let a, b, c, three positive operators, we define our algorithm by an obvious recurrence: $a \star b := \frac{1}{2}\{a^{\frac{1}{2}}, b^{\frac{1}{2}}\}$, $a \star (b \star c) := \frac{1}{2}\{a^{\frac{1}{2}}, \frac{1}{2}\{b^{\frac{1}{4}}, c^{\frac{1}{4}}\}\}$, and so on.

Theorem A.4.18 *The map $\star$ verifies R_1. Moreover, for all $a, b, c \in P(\mathcal{H})$,*

$$\operatorname{tr}(a \star (b \star c) - (a \star b) \star (a \star c)) = \frac{1}{4} tr[a^{\frac{1}{4}}, b^{\frac{1}{4}}][a^{\frac{1}{4}}, c^{\frac{1}{4}}] = \frac{1}{2} S_{WY}(b^{\frac{1}{4}},\ c^{\frac{1}{4}} \mid a^{\frac{1}{2}}).$$

Proof: Straightforward by tedious calculations. □

Remark: Via the trace map, the Wigner-Yanase information measures the defect of $\star$-homomorphism at point a.

A.5 The Bures distance

In quantum information theory, several distances exist in the space of density operators, one of them is the Bures distance.

Definition A.5.1 [The Bures distance] The *Bures distance* [5, 59, 16] is the map:

$$\mathbb{D} \times \mathbb{D} \xrightarrow{d_B} \mathbb{R}_+ \qquad (a, b) \mapsto d_B(a, b) := (2 - 2\operatorname{tr}(a * b))^{\frac{1}{2}}.$$

Remark: The fact that this distance is symmetric can be viewed as follows. In a Banach algebra, we know that $\mathsf{Spec}(xy) \cup \{0\} = \mathsf{Spec}(yx) \cup \{0\}$. Here $\mathsf{Spec}(a * b)(a * b) = \mathsf{Spec}(b * a)(b * a)$, since if $Y := a^{\frac{1}{2}} b^{\frac{1}{2}}$, we obtain $(a * b)(a * b) = YY^\dagger$ and $(b * a)(b * a) = Y^\dagger Y$ and if Y is not invertible, so is $Y^\dagger$. Since $t \in \mathbb{R}_+ \mapsto \sqrt{t}$ is a continous fonction on the spectra of $(a*b)(a*b)$ and $(b * a)(b * a)$, we get $\mathsf{Spec}(a * b) = \mathsf{Spec}(b * a)$. This implies that if a, b are trace class operators, we get $\operatorname{tr}(a * b) = \operatorname{tr}(b * a)$.

In [55], Rylov defines another concept to replace Riemannian geometry. The key idea is based on a concept used in general relativity called the world function. This function can be obtained from a metric space after removal of some constraints. A *σ-space* [55] is a set $V = (\sigma, \Omega)$ where Ω is a set equipped with a function $\Omega \times \Omega \xrightarrow{\sigma} \mathbb{R}_+$, such that for all points $P, Q \in \Omega$, $\sigma(P, P) = 0$ and $\sigma(P, Q) = \sigma(Q, P)$. For instance, if $V = (\sigma, \Omega)$ is a metric space, with a metric ρ, for all points $P, Q \in \Omega$, the world function can be defined as $\sigma(P, Q) = \frac{1}{2}\rho^2(P, Q)$. The idea behind σ-space and T-geometry, is to reformulate all theorems and concepts from Euclidean or Riemannian geometry only in terms of the world function. In our case, we observe that the Bures distance gives rise, on the set of density operators, to a world function $\mathbb{D} \times \mathbb{D} \xrightarrow{\sigma} [0, 1]$ such that $\sigma_B(\rho_1, \rho_2) := 1 - tr(\rho_1 * \rho_2)$.

Let $\mathcal{P}^n := P_0, \ldots, P_n \subset \mathbb{D}$, a finite set. The basic elements of T-geometry are finite σ_B-subspaces $M_n(\mathcal{P}^n)$ of the σ-space $(\sigma_B, \mathbb{D})$. For example, Rylov defines the squared length $|M_n(\mathcal{P}^n)|^2$ as the real number $|M_n(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n)$, where $F_n(\mathcal{P}^n)$ is the Gram's determinant for the n vectors P_0P_i, $i \in (1, \ldots, n)$, i.e., equals to $F_n(\mathcal{P}^n) = \det \|(P_0P_i.P_0P_j)\|$, with $(P_0P_i.P_0P_j) \equiv \Gamma(P_0, P_i, P_j) \equiv \sigma_B(P_0, P_i) + \sigma_B(P_0, P_j) - \sigma_B(P_i, P_j), \forall\ 1 \leq i, j \leq n$.

Let us establish a relation between commutativity of density operators, quandle 2-cocycle and σ_B-othogonality. For that, let us recall some definitions from quandle (co)-homology developed in [8] [1].

[1] See also the included references.

Definition A.5.2 Let X be a quandle. The set $C_n^R(X)$ will denote the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements of the quandle X. Define a homomorphism $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ by:

$$\partial_n(x_1, \dots, x_n) = \sum_{i=2}^{n} (-1)^i [(x_1, \dots x_{i-1}, x_{i+1}, \dots, x_n) - (x_1 * x_i, \dots x_{i-1} * x_i, x_{i+1}, \dots, x_n)],$$

for $n > 1$ and $\partial_n = 0$ for $n \leq 1$.

By using the axiom R_3, one proves that $C_*^R(X) = \{C_n^R(X), \partial_n\}$ is a chain complex. In fact if X is just a LDI system, $C_*^R(X)$ is still a complex. As $(\mathbb{R}, +)$ is an abelian group we can define what is the chain and cochainquandle complexes $C_*^R(X; (\mathbb{R}, +)) = C_*^R(X) \otimes (\mathbb{R}, +)$, with boundary $\Delta := \partial \otimes id$ and $C_R^*(X; (\mathbb{R}, +)) = \hom(C_*^R(X), (\mathbb{R}, +))$, with coboundary $\delta = \hom(\partial, (\mathbb{R}, +))$.

Example A.5.3 A *quandle 2-cocycle* ϕ satisfies, for a 3-chain (x_1, x_2, x_3), $(\delta_2(\phi))(x_1, x_2, x_3) = \phi(\partial_3(x_1, x_2, x_3)) = 0$, i.e.,

$$\phi(x_1, x_3) + \phi(x_1 * x_3, x_2 * x_3) = \phi(x_1, x_2) + \phi(x_1 * x_2, x_3).$$

Remark: A *LDRI system 2-cocycle* can be defined in the same way.

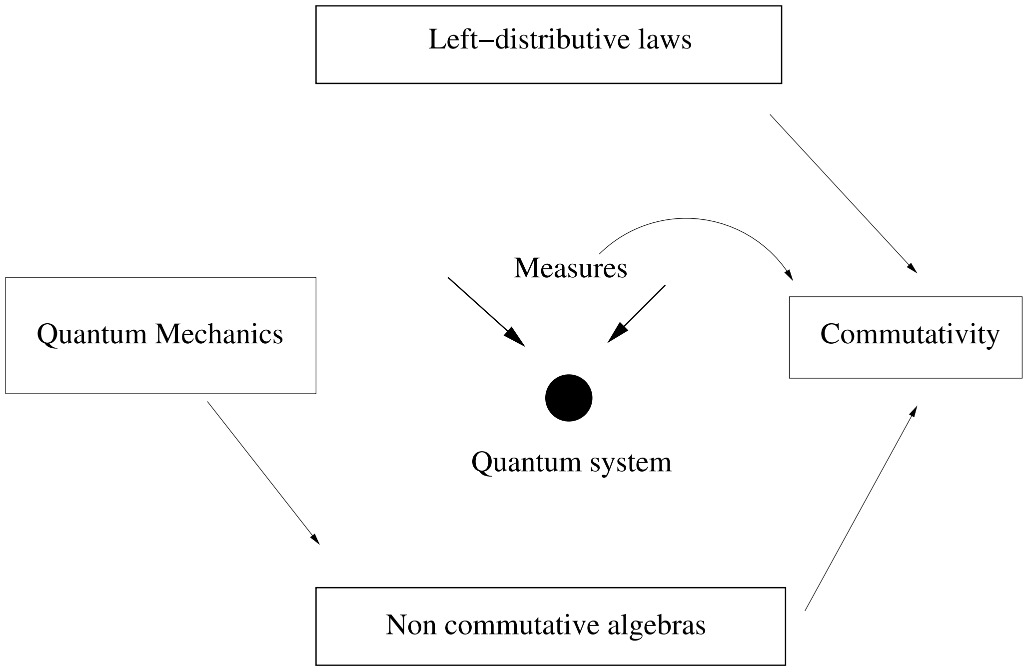
Theorem A.5.4 *Let $\Xi := \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the map such that $(r_1, r_2) \mapsto r_1 + r_2$. Let $(\mathbb{D}_0, *)$ be a LDRI set of commuting density operators, where $*$ still denote the quantum fidelity. The map $\phi := \Xi(tr, tr)$ is a LDRI 2-cocycle if and only if for all $\rho_1, \rho_2, \rho_3 \in \mathbb{D}_0$, $\Gamma(\rho_1, \rho_2, \rho_3) = 0$.*

Proof: Straightforward. □

We have related the quantum fidelity law $*$, left-distributive on a set of pairewise commuting density operators to a geometry defined by Rylov, via the Bures distance. The volumes of this new geometry, which can be calculated via the map Γ can be related to a quandle 2-cocycle when the points of this new geometry, defining volumes, are pairewise commuting density operators.

A.6 Conclusion

In quantum mechanics, a quantum experiment is usually modelled by subspaces of $B(\mathcal{H})$. Commutativity of observables is required to recover information on a quantum system when simultaneous measures are done. In this work, we have produced another way to interpret commutativity and have related it to a left-distributive law, called quantum fidelity in quantum information theory. This allowed us to view commutativity of positive operators as a move known in knot theory as the third Reidemeister move.



Another way to view commutativity in quantum mechanics.

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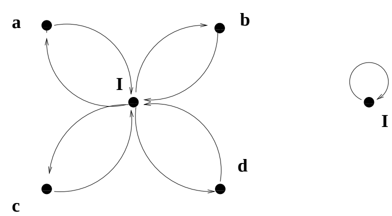
Appendix B

Hochschild complex and Periodic orbits of the flower graph

Hochschild complex and Periodic orbits ... $\longrightarrow$ We advise the reader to have Chapter 3 in mind. The notation come from Chapter 3. Some ideas of this appendix have been exploited in Chapter 4.

Let us recall here a result from Chapter 2. Let A be an unital algebra, with unit I. From the equality $(I \cdot a) \cdot I = I \cdot (a \cdot I)$, A carries a non-trivial finite Markov bi-dialgebra, with coproducts $\delta_f(a) = a \otimes I$ and $\tilde{\delta}_f(a) = I \otimes a$, for all $a \in A$. Suppose $A := kA_0 \oplus kI$ is generated by a set A_0 verifying relations. Its geometric support is then called the flower graph, because it is the concatenation of petals:



Example of geometric support associated with an algebra $k\langle a, b, c, d\rangle \oplus kI$.

Observe that for all $a \in A$ different from I, $a \mapsto \delta_f(a) + \tilde{\delta}_f(a)$ and $I \mapsto I \otimes I$ is a coassociative cocommutative coproduct.

We think that the Markov bi-dialgebra is a fundamental object associated with a unital algebra, Let us start with an unital associative algebra, $A := kA_0 \oplus kI$, generated by a set A_0, viewed as a Markov bi-dialgebra whose geometric support is the flower graph. Let us see how the complex b and b' can be rediscovered.

Definition B.0.1 [pattern] An element $(a_1, \ldots, a_n) \in A^{\otimes n}$, $a_i \in A_0$, defines a periodic orbit on the flower graph, we denote the periodic orbit by $(\ldots, I, a_1, I, a_2, I, a_3, I, \ldots, I, a_n, I, a_1, I, \ldots, I, a_n, \ldots)$, where I stands for the neutral element of A. A *pattern* is denoted by the ordered set

$[I, a_1, I, a_2, I, a_3, I, \ldots, I, a_n]$, repeated infinitely often it generates a periodic orbit. We call left border of the pattern the symbol a_n and the right border the symbol I. It can be represented by $\ldots a_n, [I, a_1, I, a_2, I, a_3, I, \ldots, I, a_n], I \ldots$

To get information contained into a periodic orbit it suffices to read either its pattern or if we want more information, its pattern and its border.

Definition B.0.2 [Reading map] For the unital associative algebra $A := kA_0 \oplus kI$, $\mathcal{L}$ is a *reading map* if it obeys the following conditions:

1. $\mathcal{L}$ starts always after the symbol I.
2. $\mathcal{L} : A^{\otimes 3} \to A$ (the reading is done three by three.). The simplest function we can consider has to use the sole information we have about the algebra that is m. We can choose: $\mathcal{L} = m(m \otimes id) = m(id \otimes m)$.
3. Each shift of the reading map provokes the apparition of a minus sign.
4. $\mathcal{L}$ must explore either all the pattern or the pattern and its border if we want more information.

Remark: The reading must be done three by three because the flower graph is the concatenation of petals and to cover such a petal one must start from let say I, then to a_i for some i then to I. It demands three letters to write down. Recall also that the petal of the flower graph arises naturally when one considers a unital algebra A because every element a from A can be written as: $a = I \cdot (a \cdot I) = (I \cdot a) \cdot I$.

Proposition B.0.3 *Reading the pattern of a periodic orbit produces the complex with boundary b' and reading the pattern and its border produces the complex with boundary b.*

Proof: We proceed by induction. As b and b', the reading map is not defined on periodic orbits of period 1 because there is not enough information to read 3 by 3.
Let $(a_1, a_2) \in A^{\otimes 2}$ be a periodic orbit of period 2. We have the sequence : $\ldots a_2, [I, a_1, I, a_2]$, $I \ldots$. For the moment we focus only on the pattern. We start with reading after I and we find $\ldots a_2, [I, \mathcal{L}(a_1, I, a_2)], I \ldots$ that is $a_1 a_2$. Remark that $b'(a_1, a_2) = a_1 a_2$. Now if we want more information we can read the pattern and the border. Yet a problem appears. We would have to write $\ldots a_2, [I, a_1, I, \mathcal{L}(a_2], I \ldots$ with a minus sign, but then we cannot read three by three any more. The only thing we can do to read is to use the left boarder. By doing so, we shift the pattern too. So we have: $\ldots [\mathcal{L}(a_2, I, a_1)], I, a_2, I \ldots$ and we find $-a_2 a_1$. The complete reading gives: $a_1 a_2 - a_2 a_1$. This is equal to $b(a_1, a_2)$.
Let $(a_1, a_2, a_3) \in A^{\otimes 3}$ be a periodic orbit of period 3, that is we have the sequence:

$$\ldots a_3, [I, a_1, I, a_2, I, a_3], I \ldots$$

We focus on the reading of the pattern.
First step: $\ldots a_3, [I, \mathcal{L}(a_1, I, a_2), I, a_3], I \ldots$ gives $a_1 a_2 \otimes a_3$.

Second step: $\ldots a_3, [I, a_1, I, \mathcal{L}(a_2, I, a_3)], I \ldots$ gives $a_1 \otimes a_2a_3$ with a minus sign. The reading of the pattern is over and we obtain $a_1a_2 \otimes a_3$ $-a_1 \otimes a_2a_3$. This is equal to $b'(a_1, a_2, a_3)$. If we want more information we can read the boarder too.
Third step: $\ldots [\mathcal{L}(a_3, I, a_1), I, a_2], I, a_3, I \ldots$ gives $a_3a_1 \otimes a_2$, with a minus sign. But at the step before we had a minus sign too, so we get a plus sign. The complete reading gives: $a_1a_2 \otimes a_3 - a_1 \otimes a_2a_3 + a_3a_1 \otimes a_2$. This is equal to $b(a_1, a_2, a_3)$. By repeating this process by an obvious recurrence, we recover b' and b. We can interpret the b' and b complexes of a unital algebra only from its Markov L-coalgebra, by reading the periodic orbits of its flower graph. $\square$

Appendix C

Semantics and completely positive semigroups

In the sequel, we will need the following notation and definitions. Let (X, m) be a k-algebra and consider a linear map $\Delta : X \to X^{\otimes 2}$. We define the sequences $(\Delta_1 \equiv \Delta, \Delta_2 = (id \otimes \Delta)\Delta, \Delta_3 = (id \otimes id \otimes \Delta)\Delta_2, \ldots)$ and $(m_1 \equiv m, m_2 = m_1(id \otimes m), m_3 = m_2(id \otimes id \otimes m), \ldots)$. The $\bullet$ product is formally defined by $\Delta_n \bullet \Delta_m = \Delta_{n+m}$.

Let $\mathcal{H}$ be a separable Hilbert space. Denote by $(B(\mathcal{H}), m)$, the space of bounded operators on $\mathcal{H}$ and m its associative product. If A is such an operator, A^* will denote its adjoint. Moreover, we set $AB := m(A \otimes B)$, where $A, B \in (B(\mathcal{H}), m)$. By a Theorem due to Stinespring [56], a completely positive map on $B(\mathcal{H})$ is a linear mapping,

$$B(\mathcal{H}) \xrightarrow{\Phi} B(\mathcal{H}) \quad \text{defined by,}$$

$$\rho \mapsto \Phi(\rho) = \sum_{i=1}^{n} A_i \rho A_i^*, \quad \text{where } A_i \in B(\mathcal{H}).$$

Consider the iterates of Φ, i.e., the sequence $(\Phi, \Phi^2, \Phi^3, \ldots, \Phi^n, \ldots)$, where $\Phi^2 \equiv \Phi \circ \Phi$ and so on. The aim of this appendix is to show that the usual iterates of the completely positive maps come from the combinatorics of n-De Bruijn graphs viewed as Markov L-coalgebras. We generalise this setting by defining a new composition law, coming from a coalgebra. The iterates of a given completely positive map, computed with this new law, will still yield a semigroup of completely positive maps.

C.0.1 The usual $\circ$ operation

Fix $n > 0$. Let G be the n-De Bruijn graph with vertex set $G_0 = (A_1, \ldots, A_n)$, a subset of $(B(\mathcal{H}), m)$. Consider the semigroup generated by G_0, denoted by $(\tilde{G}_0, m)$. Define

$$R : (\tilde{G}_0, m) \to \mathsf{Hom}(B(\mathcal{H}))$$

$$X \mapsto X(\cdot)X^*.$$

Theorem C.0.4 *Let G be the n-De Bruijn graph with vertex set $G_0 = (A_1, \ldots, A_n)$, a subset of $(B(\mathcal{H}), m)$. Embed G into its canonical Markov L-coalgebra $(kG_0, \Delta, \tilde{\Delta})$. We have $\Phi^2(\cdot) = \sum_{i=1}^n R(m\Delta(A_i))(\cdot) = \sum_{i,j=1}^n R(A_iA_j)(\cdot)$.*

Proof: The map R is now naturally extended by linearity. Remark that $\Phi^2(\cdot) = \sum_{i,j=1}^n A_jA_i(\cdot)(A_jA_i)^*$. As $\Delta(A_i) = A_i \otimes \sum_{j=1}^n A_j$, for all $1 \leq i \leq n$, we get $\sum_{j=1}^n A_iA_j = \sum_{i=1}^n m\Delta(A_i)$. Therefore, $\Phi^2(\cdot) = \sum_{i=1}^n R(m\Delta(A_i))(\cdot) = \sum_{i,j=1}^n R(A_iA_j)(\cdot)$. □

Corollary C.0.5 *The set of paths on the n-De Bruijn graph G is encoded into the sequence $(\Phi, \Phi^2, \Phi^3, \ldots, \Phi^k, \ldots)$.*

Proof: Embed the n-De Bruijn graph G into its natural Markov L-coalgebra. We recall that the paths emerging from a given vertex could be obtained by the right coproduct thanks to the sequence $(\Delta_1, \Delta_2, \Delta_3, \ldots)$. Fix $k > 0$. Iterating the previous construction, we observe that Φ^k can be recovered from $\sum_{i=1}^n m_k\Delta_k(A_i)$, since $\Phi^k(\cdot) = \sum_{i=1}^n R(m_k\Delta_k(A_i))(\cdot)$. □

The operation $\circ$ for completely positive map not only generates the completely positive semigroup $(\Phi^k)_{k\in\mathbb{N}}$ (because $\Phi^{k+l} = \Phi^k \circ \Phi^l$) but also generates all the paths on the n-De Bruijn graph via Δ. Let us now replace $\circ$ by $\circ_G$ and generalise this procedure to any coalgebra.

Definition C.0.6 Let (G, Δ_G) be a (not necessary coassociative) coalgebra spanned by $(A_1, \ldots, A_n) \in B(\mathcal{H})$. Let $\Psi(\cdot) : B(\mathcal{H}) \to B(\mathcal{H})$, $\rho \mapsto \Psi(\rho) = \sum_{i=1}^n A_i\rho A_i^*$. We define for all $k > 1$, $\Psi^{\circ_G k}(\cdot) = \sum_{i=1}^n R(m\Delta_{(G,k)}(A_i))(\cdot)$.

Theorem C.0.7 *All the linear mappings $(\Psi, \Psi^{\circ_G 2}, \Psi^{\circ_G 3}, \ldots, \Psi^{\circ_G n}, \ldots)$ are completely positive.*

Proof: Obvious by the Stinespring Theorem. □

Definition C.0.8 As for $\circ$, we wish that the operation $\circ_G$ generates a semigroup. We define $\Psi^{\circ_G k} \circ_G \Psi^{\circ_G l}(\cdot) = \sum_{i=1}^n R(m_{k+l}(\Delta_{(G,k)} \bullet \Delta_{(G,l)}(A_i))(\cdot)$, where the product $\bullet$ is defined in the introduction of this appendix.

Remark: With such a definition, the sequence $(\Psi^{\circ_G k})_{k\in\mathbb{N}}$ is a completely positive semigroup driven by the right part of G. Moreover if G is the n-De Bruijn graph, we recover the usual $\circ$ product.

Example C.0.9 As an example, consider $C = \text{span}\langle 1, X, g \rangle \in B(\mathcal{H})$ equipped with the following coassociative coproduct: $\Delta : C \to C^{\otimes 2}$

$$\Delta X = X \otimes 1 + g \otimes X, \quad \Delta g = g \otimes g, \quad \Delta 1 = 1 \otimes 1.$$

By definition we have:

$$\Psi(\cdot) = 1(\cdot)1 + X(\cdot)X^* + g(\cdot)g^*.$$

Applying Δ to $(1, X, g)$ we get,

$$\Psi^{\circ_G 2}(\cdot) = R(m(1 \otimes 1))(\cdot) + R(m(X \otimes 1 + g \otimes X))(\cdot) + R(m(g \otimes g))(\cdot),$$

that is, $\Psi^{\circ_G 2}(\cdot) = 1(\cdot)1 + X(\cdot)X^* + gX(\cdot)(gX)^* + g^2(\cdot)(g^2)^*$ and so forth by applying the power of Δ.

Example C.0.10 [contractive completely positive n-tuple] A contractive n-tuple [52] is given by n operators $(T_1, \dots, T_n)$ from $B(\mathcal{H})$ such that $\sum_{i=1}^n T_i T_i^* \leq I$. Such a n-tuple generates, for the usual $\circ$ operation, a contractive completely semigroup. Can we generalise the theory of contractive n-tuples for a compositon law coming from any coproduct?

Appendix D

Probabilistic algebraic products and mutation of L-coalgebras

D.1 Introduction

In this final appendix, some warn to the reader is required. This part is "phenomenological" and aims to introduce the idea of a probabilistic algebra and the mutation of algebraic products. This is a natural consequence of graph theory viewed from a L-coalgebra point of view and if we have in mind that such concept can represent a toy model of space time at Planck scale. Viewing weighted directed graphs as L-coalgebras, we can envisage the action of its coproducts on its vertices considered as pointers on a field k or an associative algebra A. The coproduct Δ defines then a product $[\Delta]$ on special vector-space. Now if the space-time is considered as a dynamical object, we have to introduce the notion of dynamical L-coalgebra and thus a dynamical notion of algebraic product opening up the place to the notion of mutation of algebraic product. The reader will surely find some propositions and results quite awkward. It must be confessed that the mathematics used here are far from being rigorous. However, the author can't help thinking they will become one day. Let us mention, that an attempt to deal rigorously with such mutations can be found in the conclusion of Chapter 7 where on a given coassociative manifold, we can "travel smoothly" from an associative product to another one via its coassociative covering. We "travel" from an algebra equipped with a product and go to another one and so on as one goes along the geometric support of the manifold.

D.1.1 Probabilistic (non-deterministic) algebraic products

We define a product coming from a given coproduct of a L-coalgebra and embed any vector space into a polynomial algebra. This observation comes from the graph associated with $\mathcal{F}$ which embeds a vector space into an algebra isomorphic to $M_2(k)$. The generalisation of this idea to any graph, and particularly equipped with a family of probability vectors leads to the natural concept of probabilistic (non-deterministic) algebraic product and the notion of random polynomial algebra.

Let $(G, \Delta_G, \tilde{\Delta}_G)$ be a L-coalgebra generated as a k-vector space by an independent spanning set G_0 with geometric support $Gr(G)$. We will regard the right coproduct $\Delta_G : G \to G \otimes G$, as a product on a particular space.

Fix $n > 0$ and define $G = k\langle X_1, \ldots, X_n \rangle$, the free k-vector space constructed from the X_i. Embed G into a L-coalgebra $(G, \Delta_G, \tilde{\Delta}_G)$. Its geometric support $Gr(G)$ yield a weighted directed graph whose vertex set is $(X_1, \ldots, X_n)$ and arrow set is given by the coproducts as explained in Chapter 2. We now view the X_i as pointers, that is as objects which will act on the scalars from k. Let us see what this means.

Let $(a_1, \ldots, a_n) \in k$, $\sum_{i=1}^n a_i X_i$ will mean: $\sum_{i=1}^n a_i \lhd X_i$, that is X_i points or acts on a_i, thanks to the coproduct, Δ_G, of the graph. For this, we equip $k\langle X_1, \ldots, X_n \rangle$ with a product $[\Delta_G]$, by defining:

$$(\sum_{i=1}^n a_i \lhd X_i)[\Delta_G](\sum_{i=1}^n b_i \lhd X_i) = \sum_{i=1}^n (\sum a_{i,(1)} b_{i,(2)}) \lhd X_i \quad (= \sum_{i=1}^n c_i \lhd \Delta(X_i)),$$

if $\Delta(X_i) := \sum X_{i,(1)} \otimes X_{i,(2)}$ where the $a_i, b_i \in k$, and the $c_i \in k$ are functions of the a_j, b_j. We simply carry the action of $\Delta(X_i)$ on the scalars from k. This means, for example that if $n = 3$ and $G := \mathrm{span}\langle X_1, X_2, X_3 \rangle$ is the L-coalgebra whose right part is defined by $\Delta(X_1) = X_1 \otimes X_3$, $\Delta(X_2) = X_2 \otimes X_2$ and $\Delta(X_3) = X_3 \otimes X_1$, we shall have by definition:

$$\begin{aligned}
E &:= (a_1 \lhd X_1 + a_2 \lhd X_2 + a_3 \lhd X_3)[\Delta](b_1 \lhd X_1 + b_2 \lhd X_2 + b_3 \lhd X_3) \\
&= (c_1 \lhd \Delta(X_1) + c_2 \lhd \Delta(X_2) + c_3 \lhd \Delta(X_3)) \\
&= (m(a_1 \otimes b_3) \lhd X_1 + m(a_2 \otimes b_2) \lhd X_2 + m(a_3 \otimes b_1) \lhd X_3),
\end{aligned}$$

where m is the product of the field k. It is clear with this definition that it is the right (the same notion hold for the left coproduct) coproduct of the L-coalgebra G which gives a product to $k\langle X_1, \ldots, X_n \rangle$ and also which embeds $k^{\times n}$ into an algebra whose product comes from the coproducts. (In this example $n = 3$, the two vectors are $x = (a_1, a_2, a_3) \in k^{\times 3}$ and $y = (b_1, b_2, b_3) \in k^{\times 3}$. The product, still denoted by $[\Delta]$, yields $x[\Delta]y = (a_1 b_3, a_2 b_2, a_3 b_1)$.) Denote by $(k\langle X_1, \ldots, X_n \rangle, [\Delta_G])$ the algebra, induced by the right part of the L-coalgebra G equipped with the product $[\Delta_G]$.

Theorem D.1.1 ***[The matrix product in** $M_2(k)$**]*** *Let $\mathcal{F}$ be our usual coassociative coalgebra and Δ its coproduct. Then,*

$$(k\langle a, b, c, d \rangle, [\Delta]) \simeq M_2(k).$$

Proof: We consider the coassociative coalgebra $\mathcal{F}$ with vertex set (a, b, c, d), (see the introduction or Chapter 2). We define $X_{11} = a, X_{12} = b, X_{21} = c, X_{22} = d$ and compute:

$$\begin{aligned}
E &:= (a_{11} X_{11} + a_{12} X_{12} + a_{21} X_{21} + a_{22} X_{22})[\Delta](b_{11} X_{11} + b_{12} X_{12} + b_{21} X_{21} + b_{22} X_{22}) \\
&= (c_{11} \lhd \Delta(X_{11}) + c_{12} \lhd \Delta(X_{12}) + c_{21} \lhd \Delta(X_{21}) + c_{22} \lhd \Delta(X_{22})) \\
&= ((a_{11} b_{11} + a_{12} b_{21}) X_{11} + (a_{11} b_{12} + a_{12} b_{22}) X_{12} + \ldots
\end{aligned}$$

□

Remark: This theorem can obviously be extended to any dimension.

Remark: Instead of using the field k in $k\langle X_1, \dots, X_n\rangle$, we can also use an associative algebra A with product m and consider $A\langle X_1, \dots, X_n\rangle$. As an example, we get the following Corollary:

Corollary D.1.2 *Consider our usual coassociative coalgebra $(\mathcal{F}, \Delta)$. Then,*

$$(A\langle a, b, c, d\rangle, [\Delta]) \simeq M_2(A).$$

Theorem D.1.3 *Let $(P := k\langle\, X_1, \dots X_n\,\rangle, [\Delta_C])$ be an algebra with product $[\Delta_C]$, where C is the right part of a L-coalgebra. Asserting that C is a coassociative coalgebra is equivalent to the fact that the induced product $[\Delta_C]$ of P is associative.*

Proof: Straightforward. □

Example D.1.4 [The Hadamard product] An important example is to associate with each X_i a loop, i.e., $\Delta_{loop} X_i = X_i \otimes X_i$. The action of this L-coalgebra, the easiest we can envisage, will yield a product $[\Delta_{loop}]$ on the space $k\langle\, X_1, \dots, X_n\,\rangle$. This product is usually called the point by point product or the Hadamard product.

How can we extend the previous concept to enlarge the number of different algebraic products? We now focus on Markov L-coalgebra. The triangle graph of quaternions (Chapter 2) learns us that one has to consider the position of the pointer $\Delta(X_i)$ too. For the moment, the result of the pointer $\Delta(X_i)$ was placed at the position occupied by the pointer X_i. It was a static point of view. The dynamical viewpoint would be to move the result of the pointer $\Delta(X_i)$ along, say, an orbit or a path of a graph.

Example D.1.5 [The wedge product] Consider the directed triangle graph $\equiv \triangle$, equipped with the Markovian coproduct $\Delta_\triangle$ defined in Chapter 2, and the algebra $(k\langle 1, X_0 = i, X_1 = j, X_2 = k\rangle, [\Delta_\triangle])$. We choose to put the result of $\Delta(X_i)$ at the position occupied by the pointer $\lhd X_{i+2 \bmod 3}$ and to fix the result of the operation $\Delta(1)$ at the position occupied by the pointer $\lhd 1$. We have the following, where $x, y, z \in k$:

$$\begin{aligned} E &:= (x \lhd X_0 + y \lhd X_1 + z \lhd X_2)[\Delta_\triangle](x' \lhd X_0 + y' \lhd X_1 + z' \lhd X_2) \\ &= (xy' \lhd X_2 + yz' \lhd X_0 + zx' \lhd X_1). \end{aligned}$$

Remark: Observe for instance, that the result of the operation $\Delta(X_0)$ is no longer in the position occupied by the pointer $\lhd X_0$ but in the position occupied by the pointer $\lhd X_2$.

Theorem D.1.6 *Denote by $\mathcal{A} := (\mathbb{R}\langle 1, X_0 = i, X_1 = j, X_2 = k\rangle, [\Delta_\triangle])$, where $[\Delta_\triangle]$ is the triangle product. Denote the commutator by $[a, b] := a[\Delta_\triangle]b - b[\Delta_\triangle]a$, where $a, b \in \mathcal{A}$. Then $(\mathcal{A},\ [\cdot\ ,\ \cdot])$ is isomorphic to $(\mathbb{R}^3,\ \wedge)$, where $\wedge$ denotes the standard wedge product.*

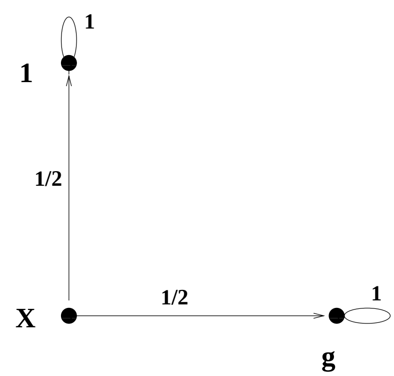
Proof: Let $\vec{a} = (x, y, z),\ \vec{b} = (x', y', z') \in \mathbb{R}^3$. We compute:

$$\begin{aligned} E &:= (x \lhd X_0 + y \lhd X_1 + z \lhd X_2)[\Delta_\triangle](x' \lhd X_0 + y' \lhd X_1 + z' \lhd X_2) \\ &\quad -(x' \lhd X_0 + y' \lhd X_1 + z' \lhd X_2)[\Delta_\triangle](x \lhd X_0 + y \lhd X_1 + z \lhd X_2) \\ &= (xy' \lhd X_2 + yz' \lhd X_0 + zx' \lhd X_1) - (x'y \lhd X_2 + y'z \lhd X_0 + z'x \lhd X_1) \\ &= (xy' - x'y) \lhd X_2 + (yz' - y'z) \lhd X_0 + (zx' - z'x) \lhd X_1 \\ &= \vec{a} \wedge \vec{b}, \end{aligned}$$

since the field $\mathbb{R}$ is commutative. If it was not, this could be a possible generalisation of the wedge product in the non-commutative case. □

For the moment all the graphs and products involved were deterministic. Suppose we consider a more complicated graph with probability measure on it and we choose to place $\Delta(X_i)$ along some well-chosen orbit, depending on the realisation of a stochastic process. We shall obtain a non-deterministic algebraic product on $A\langle X_1, \ldots, X_n\rangle$, with A an associative algebra with product m. Such a mathematical object $(A\langle X_1, \ldots, X_n\rangle, [\Delta_G])$ will be called *a random polynomial algebra.*

As an example we consider the following Markov L-algebra G:



and we choose the convention, called the path convention in the sequel:

$$\lhd\Delta(\cdot) \equiv terminus(\Delta(\cdot)) = t(\Delta(\cdot)).$$

That is, we choose to put the result of the operation say, $\Delta(X_i) := X_i \otimes \sum_j X_j$ into the place occupied by the pointer $\lhd X_j$. For the moment we do not consider probability on G.

Theorem D.1.7 *We have* $(A\langle X_{11} := 1, X_{12} := X, X_{22} := g\rangle, [\Delta_G])^2$ *behaves as* $diag(M_2(A))$ *equipped with its usual product. Here power 2 means that we consider the set* $\{a[\Delta_G]b,\ (a, b) \in (A\langle 1, X, g\rangle, [\Delta_G)])\}$.

Proof: We must only focus on X. Yet, $\Delta(X) = X \otimes g + X \otimes 1$, that is the result of a product will be put on g and 1 that is on X_{11} and X_{22}. For instance the product of two polynomials yields $(a \lhd X + b \lhd g + c \lhd 1)[\Delta_G](a' \lhd X + b' \lhd g + c' \lhd 1) := (ab' + bb') \lhd g + (ac' + cc') \lhd 1$. However $diag(M_2(A)) \simeq (A\langle X_{11}, X_{22}\rangle, [\Delta_H])$ where H is the coassociative coalgebra spanned as a k-vector space by two independent groups like elements indexed by the pointers $\lhd 1$ and $\lhd g$. Therefore the usual product of $diag(M_2(A))$ is recovered. □

Now suppose a probability $\frac{1}{2}$ is associated with each arrow emerging from X and a probability 1 with the two loops. Then we have: $\Delta(X) = \frac{1}{2}X \otimes g + \frac{1}{2}X \otimes 1$. Equipped with this graph, the realisations of random walks on it, turn $(A\langle 1, X, g\rangle, [\Delta_G])\}$ into a random algebra. To study a product in such an algebra we must consider a probability measure $\mathbb{P}$ on $(A\langle 1, X, g\rangle, [\Delta_G])$. In fact the probability measure on paths of the random walk on G is here sufficient. For example the probability to have the walk $w_1 = (X, 1, 1, \ldots)$ is equal to $\frac{1}{2}$ and the probability to have $w_2 = (X, g, g, \ldots)$ is equal to $\frac{1}{2}$. We now assign to $\mathbb{P}(a[\Delta_G]b = a[\text{walk: } w_1]b) = \frac{1}{2}$ and $\mathbb{P}(a[\Delta_G]b = a[\text{walk: } w_2]b) = \frac{1}{2}$. According the realisation of such a stochastic process such or such walk will be chosen.

Theorem D.1.8 $\mathbb{P}\{(A\langle 1, X, g\rangle, [\Delta_G])^2$ *behaves as* $diag(M_2(A))\} = 1$.

Proof: The algebra $(A\langle 1, X, g\rangle, [\Delta_G])$ already contains the sub random polynomial algebra $(A\langle 1, g\rangle, [\Delta_G])$ which is isomorphic to $diag(M_2(A))$, all we do by considering the power 2 of $(A\langle 1, X, g\rangle, [\Delta_G])$ is to eliminate X to restrict ourselves to the attractors generated by the loops 1 and g. □

Remark: This Theorem means that the random polynomial algebra $(A\langle 1, X, g\rangle, [\Delta_G])$ can behave as a deterministic algebra at short term.

Remark: The fact that we recover a deterministic algebra is due to the fact that a loop —the L-coalgebra spanned by a group like element—is a coassociative coalgebra, in addition to the fact that it is placed in an attractor rôle. This remark allows us to generalise the previous Theorem by saying that if the L-coalgebra generated as a k-vector space by an independent spanning set by G has an attractor C, where C is a coassociative coalgebra, and if we decide to choose the path convention, except on C where we choose the static one, then it exists a time n, possibly equal to infinity such that the power of a random polynomial algebra equipped with the product $[\Delta_G]$ converges towards a deterministic algebra.

D.1.2 Mutation of algebraic products

The aim of this part is twofold. The first idea is to consider weighted directed graphs, embbeded into L-coalgebras, as dynamical objects, capable of mutation in order to consider the notion of mutation of algebraic products on polynomial algebras. The second idea is to produce an example of a complex based on random variables.

Definition D.1.9 [Mutation of L-coalgebras] If M and N are L-coalgebras we write:

$$(M, \Delta_M) \looparrowright (N, \Delta_N)$$

to say that the L-coalgebra (M, Δ_M) has undergone a mutation into the L-coalgebra (N, Δ_N).

Definition D.1.10 [Mutation of algebraic product] As we can associate with an L-coalgebra, an algebraic product we define: $[\Delta_M] \looparrowright [\Delta_N]$ to say that the algebraic product $[\Delta_M]$ has undergone an algebraic mutation into $[\Delta_N]$.

So as to be as clear as possible we shall illustrate all these new concepts through an example.

Let H_2 be the graph associated with the coassociative coalgebra $\mathcal{F}$, whose vertex set is still denoted by a, b, c, d. Denote by $\diamondsuit_a$ the coassociative coalgebra represented by a loop at a and $\triangle$, the Markov L-coalgebra associated with the directed triangle graph. For instance, we choose to label the pointers of the triangle graph by $a \to b \to c \to a$. Observe that both are sub-graphs of H_2. As we saw in the previous part, these L-coalgebras define respectively the matrix product on $M_2(A)$, the standard product on A and the wedge product, via commutator, on $A^{\times 3}$, where A is an associative algebra. Let us consider the set $G := \{\{H_2\},\ \{\diamondsuit_a\},\ \{\triangle\}\}$ and define a probability measure on the family of subsets of G, $\mathbb{P} : \mathcal{P}(G) \to [0,1]$ such that:

$$\mathbb{P}(\{\triangle\}) = \frac{\epsilon}{2}, \quad \mathbb{P}(\{\diamondsuit_a\}) = \frac{\epsilon}{2}, \quad \mathbb{P}(\{H_2\}) = 1 - \epsilon, \quad \epsilon \in]0,1[.$$

Naturally associated with G is the dynamical polynomial algebra, $(\mathcal{G}, [\Delta_G])$, whose pointers are labelled by a, b, c, d as well and which behaves for instance as an algebra isomorphic to $M_2(A)$, (respectively to A, the algebra with its associative point by point product or Hadamar product and $(A^{\times 3}, \wedge)$, the algebra $A^{\times 3}$ with the wedge product on it) when G behaves as H_2, (respectively as $\diamondsuit_a$ and $\triangle$) . Suppose now we obtain the following dynamical sequence:

$$H_2 \to H_2 \to H_2 \looparrowright \diamondsuit_a \looparrowright H_2 \to H_2 \dots$$

This dynamic will have repercussions on the algebraic products:

$$[\Delta_{H_2}] \to [\Delta_{H_2}] \to [\Delta_{H_2}] \looparrowright [\Delta_{\diamondsuit_a}] \looparrowright [\Delta_{H_2}] \to [\Delta_{H_2}] \dots$$

this means that if we study the power of a 2 by 2 matrix z we will get:

$$z \xrightarrow{\text{matrix product}} z^2 \xrightarrow{\text{matrix product}} z^3 \xrightarrow{\text{matrix product}} z^4 \looparrowright u \xrightarrow{\text{matrix product}} u^2 \xrightarrow{\text{matrix product}} u^3 \dots$$

where z has undergone a mutation into the 2 by 2 matrix u.

Example D.1.11 This example of mutation looks like a standard projection on the pointer a.

$$\boldsymbol{z} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \boldsymbol{z^2} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \boldsymbol{z^3} = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \to \boldsymbol{z^4} = \begin{pmatrix} 128 & 128 \\ 128 & 128 \end{pmatrix} \looparrowright \boldsymbol{u} = \begin{pmatrix} (128)^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\looparrowright \boldsymbol{u^2} = \begin{pmatrix} (128)^4 & 0 \\ 0 & 0 \end{pmatrix} \to \boldsymbol{u^3} = \begin{pmatrix} (128)^6 & 0 \\ 0 & 0 \end{pmatrix} \dots$$

To explain what follows we need to view the set G as a dynamical L-coalgebra. We start for example with a configuration where G is viewed as the coassociative coalgebra H_2. We ask how such a graph evolves. There is three possibilities. It remains the same, contracts itself into the loop labelled by a or modifies its shape to become the directed triangle graph, support of the L-coalgebra described above (which is a subgraph of the graph associated with H_2). Fix $\epsilon > 0$ and consider the set of dynamical sequences starting with H_2, i.e., $(H_2 \to \dots)$, and denote by $T = \inf\{n > 0, \text{the } \triangle \text{ occurs at time } n\}$. Then, the probability for the following sequence

$$0 \to G \xrightarrow{\Delta_G} G^{\otimes 2} \xrightarrow{\Delta_G \otimes id - id \otimes \Delta_G} G^{\otimes 3} \dots$$

not to be an exact complex at time T is $(1-\frac{\epsilon}{2})^{(T-1)}\frac{\epsilon}{2}$. By reversing the arrows, the probability for the following sequence

$$0 \leftarrow \mathcal{G} \xleftarrow{[\Delta_G]} \mathcal{G}^{\otimes 2} \xleftarrow{[\Delta_G]\otimes id - id\otimes[\Delta_G]} \mathcal{G}^{\otimes 3} \ldots$$

not to be an exact complex at time T is $(1-\frac{\epsilon}{2})^{(T-1)}\frac{\epsilon}{2}$. We recall here that $\mathcal{G}$ is the polynomial algebra over A, equipped with the product $[\Delta_G]$, which is isomorphic to $M_2(A)$ if $[\Delta_G] = [\Delta_{H_2}]$ and so on. Conversely if we fix $\epsilon > 0$ and consider now the set of dynamical sequences starting with the L-coalgebra $\triangle$, i.e., $(\triangle \to \ldots)$, and denote by $T_1 = \inf\{n > 0, \text{a mutation occurs at time } n\}$. Then, the probability for the following sequence

$$0 \to G \xrightarrow{\Delta_G} G^{\otimes 2} \xrightarrow{\Delta_G\otimes id - id\otimes\Delta_G} G^{\otimes 3} \ldots$$

to be an exact complex at time T_1 is $(1-\frac{\epsilon}{2})(\frac{\epsilon}{2})^{(T_1-1)}$. By reversing the arrows, the probability for the following sequence

$$0 \leftarrow \mathcal{G} \xleftarrow{[\Delta_G]} \mathcal{G}^{\otimes 2} \xleftarrow{[\Delta_G]\otimes id - id\otimes[\Delta_G]} \mathcal{G}^{\otimes 3} \ldots$$

to be an exact complex at time T_1 is $(1-\frac{\epsilon}{2})(\frac{\epsilon}{2})^{(T_1-1)}$.

Remark: When G is represented by the loop or the geometric support of $\mathcal{F}$ the previous sequences are exact complexes. Yet if a mutation occurs, i.e., if G becomes $\triangle$, it will break the exactness of such a complex because the usual matrix product will also undergo an algebraic mutation too [1]. Observe also that G and $\mathcal{G}$ play the rôle of random variables.

[1] As an example of application, imagine that some quantum measurements are done on a quantum system which lives on a space-time represented by the dynamical L-coalgebra G, viewed as H_2. Instead of disturbing the quantum system by the measurement, let us suppose that we disturb G and that G undergoes a contraction in the loop labelled by a. This will produce a change of complex for the coassociative coalgebras which will induce a mutation of algebraic product, here a projection on the pointer labelled by a.

Another application of probabilistic algebraic product would be to embed the fundamental biological bricks, i.e., A, C, G, T into a random semigroup ...

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