Hilbertian subspaces, subdualities and applications

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Mathematical Notations

SPACES

SPACE	ELEMENTS		DUAL SPACE	DUAL ELEMENTS
$\mathcal{E}$	ε		$\mathcal{E}'$	ε'
E	e		E'	e'
$\mathfrak{E}$	$\mathfrak{e}$		$\mathfrak{E}'$	$\mathfrak{e}'$
$\mathcal{F}$	φ		$\mathcal{F}'$	φ'
F	f		F'	f'
$\mathfrak{F}$	$\mathfrak{f}$		$\mathfrak{F}'$	$\mathfrak{f}'$
H	h		H'	h'
H_0	h_0		$H_0^{'}$	$h_0^{'}$
$\mathcal{M}$	$\dot{m}$			
$\mathcal{N}$	$\dot{n}$			
$\mathbb{R}^n$	$X = (x_1, x_2, ...)$		$\mathbb{R}^n$	Y
$\mathbb{K} = \mathbb{R}, \mathbb{C}$	α, β, λ			
Ω	t, s			

DUALITY

NAME	NOTATION
Duality product	$(f,e)_{(F,E)}$, $(\varphi,\varepsilon)_{(\mathcal{F},\mathcal{E})}$, $(\varepsilon',\varepsilon)_{(\mathcal{E}',\mathcal{E})}$, $\left(\overline{h_1},h_2\right)_{(\overline{H},H)}$
Inner product	$\langle h_1 \mid h_2\rangle_H = \left(\overline{h_1},h_2\right)_{(\overline{H},H)}$
Norm	$\|h\|_H$

MORPHISMS, FUNCTIONS

FUNCTION NAME	NOTATION
Morphism	u
Canonical injection	$i,\, j$
Kernel	$\varkappa, \chi$
Transpose	${}^t u,\, {}^t i,\, {}^t \varkappa$
Adjoint	$u^*,\, i^*,\, \varkappa^*$
Kernel function	$K(t,s)$
Scalar functions	ϕ, ψ
Other functions	Λ_t, Γ_t
Functional	$\mathcal{J}$
Derivative	$\frac{d}{dt}, \frac{\partial}{\partial t}$

Résumé

L'étude des fonctions de deux variables et des opérateurs intégraux associés, ou l'étude directe des noyaux au sens de L. Schwartz [46] (définis comme opérateurs faiblement continus du dual topologique d'un espace vectoriel localement convexe dans lui-même), est depuis plus d'un demi-siècle une branche des mathématiques en pleine expansion notamment dans le domaine des distributions, des équations différentielles ou dans le domaine des probabilités, avec l'étude des mesures gaussiennes et des processus gaussiens.

Les travaux de Moore, Bergman et Aronszajn ont notamment abouti au résultat fondamental suivant qui concerne les noyaux positifs : il est toujours possible de construire un sous-espace préhilbertien à partir d'un noyau positif et, moyennant une hypothèse (faible) supplémentaire[1], de compléter fonctionnellement cet espace afin d'obtenir un espace de Hilbert. Cet espace possède alors la propriété d'être continûment inclus dans l'espace vectoriel localement convexe de départ. Il existe donc une relation forte entre noyaux positifs et espaces hilbertiens. Dans cette thèse, nous nous sommes posé le problème suivant : que se passe-t-il si l'on lève l'hypothèse de positivité ? d'hermicité ?

Dans cette perspective nous considérons une seconde approche qui consiste à travailler directement sur des espaces vectoriels plutôt que sur les noyaux. Précisément, adoptant une démarche classique en mathématiques, nous étudions les propriétés d'une classe d'espaces vérifiant des hypothèses additionnelles. Partant des espaces de Hilbert continûment inclus dans un espace localement convexe donné, cette approche a conduit aux espaces de Hilbert à noyau reproduisant de N. Aronszajn [5] puis aux sous-espaces hilbertiens de L. Schwartz

[1] la quasi-complétude de l'espace fonctionnel de départ par exemple

[46]. Cette théorie est présentée dans la première partie de la thèse, le résultat majeur de cette théorie étant sans doute l'équivalence entre sous-espaces hilbertiens et noyaux positifs[2], que l'on peut résumer en ces termes :

"Il existe une bijection entre sous-espaces hibertiens et noyaux positifs."

Le principal apport à la théorie existante est l'utilisation intensive de systèmes en dualité et de formes bilinéaires (et pas uniquement sesquilinéaires). De manière surprenante, cela conduit à une certaine perte de symétrie qui porte les germes de la théorie des sous-dualités.

Dans une seconde partie nous suivons encore les travaux de L. Schwartz et étudions la théorie moins connue des sous-espaces de Krein (ou sous-espaces hermitiens). Les espaces de Krein ressemblent aux espaces de Hilbert mais sont munis d'un produit scalaire qui n'est plus nécessairement positif. Les sous-espaces de Krein constituent donc une première généralisation des sous-espaces hilbertiens. Un des principaux intérêt de l'étude de tels espaces réside en la disparition de l'équivalence fondamentale entre les notions de sous-espaces et de noyaux, même si une relation étroite subsiste. Nous étudions plus particulièrement les similitudes et les différences entre ces deux théories, que nous retrouverons dans la théorie des sous-dualités.

La troisième partie généralise la perte de symétrie évoquée dans le chapitre 1. Nous développons les bases d'une théorie non plus fondée sur une structure hilbertienne, mais sur une certaine dualité. Nous développons ainsi le concept de sous-dualité d'un espace vectoriel localement convexe (ou d'un système dual) et de son noyau associé. Une sous-dualité est définie par un système de deux espaces en dualité vérifiant des conditions d'inclusion algébrique (définition 3.2) ou topologique (proposition 3.3). Plus précisément :
un système dual (E, F) est une sous-dualité d'un espace localement convexe $\mathcal{E}$ (ou plus gé-

[2]toujours sous hypothèse de quasi-complétude.

néralement d'un système dual $(\mathcal{E}, \mathcal{F})$) si E et F sont faiblement continûment inclus dans $\mathcal{E}$. Dans ce cas, il est possible d'associer à cette sous-dualité un unique noyau (théorème 3.6) d'image dense dans la sous-dualité (théorème 3.10). Nous étudions également l'effet d'une application linéaire faiblement continue (théorème 3.12). Il devient alors possible (moyennant une relation d'équivalence) de munir l'ensemble des sous-dualités d'une structure d'espace vectoriel qui le rend isomorphe algébriquement à l'espace vectoriels des noyaux (théorème 3.13). Nous exhibons ensuite un représentant canonique de ces classes d'équivalences (théorème 3.20), ce qui permet d'établir une bijection entre sous-dualités canoniques et noyaux.
Nous étudions également le cas particulier des sous-dualités de $\mathbb{R}^{\Omega}$, que nous appelons sous-dualités d'évaluation. Le noyau est alors identifié à une fonction noyau reproduisant (définition 3.34 et lemme 3.35). De telles sous-dualités (et noyaux) apparaissent notamment dans la théorie des espaces de polynômes, des splines de Chebyshev et de "l'épanouissement" (*blossoming*), voir par exemple M-L. Mazure et P-J. Laurent [37].

Une quatrième et dernière partie propose quelques applications. Le premier champ d'application possible est une généralisation du lien entre sous-espaces hilbertiens et mesures gaussiennes. Le second est l'étude d'opérateurs particuliers, les opérateurs dans les sous-dualités d'évaluation (sous-dualités de $\mathbb{K}^{\Omega}$) et les opérateurs différentiels. Enfin, l'étude de l'interpolation dans les sous-dualités d'évaluation est développée.

Ce travail soulève de nombreuses questions au niveau théorique et applicatif. Les principales questions portent sur les sous-dualités canoniques : peut-on les caractériser, sont elles les plus intéressantes, peut-on caractériser directement les noyaux stables ? D'autres portent sur les opérateurs différentiels et leur lien avec les espaces de Sobolev. Enfin, l'interprétation physique des sous-dualités est une question cruciale pour comprendre cette théorie et ses applications.

abstract

Functions of two variables appearing in integral transforms (Bergman, Segal, Carleman), or more generally kernels in the sense of Laurent Schwartz [46] - defined as weakly continuous linear mappings between the dual of a locally convex vector space and itself - have been investigated for half a century, particularly in the field of distributions, differential equations and in the probability field with the study of Gaussian measures or Gaussian processes.

The study of these objects may take various forms, but in case of positive kernels, the study of the properties of the image space initiated by Moore, Bergman and Aronzjan leads to a crucial result: the range of the kernel can be endowed with a natural scalar product that makes it a prehilbertian space and its completion belongs[3] to the locally convex space. Moreover, this injection is continuous. Positive kernels then seem to be deeply related to some particular Hilbert spaces and our aim in this thesis is to study the other kernels. What can we say if the kernel is neither positive, nor Hermitian ?

To do this we actually follow a second path and study directly spaces rather than kernels. Considering Hilbert spaces, some mathematicians have been interested in a particular subset of the set of Hilbert spaces, those Hilbert spaces that are continuously included in a common locally convex vector space. The relative theory is known as the theory of Hilbertian subspaces and is thoroughly investigated in the first chapter. Its main result is that surprisingly the notions of Hilbertian subspaces and positive kernels are equivalent[4], which is generally summarized as follows:

[3] under some weak additional topological conditions on the locally convex space.

[4] under the hypothesis of quasi-completeness of the locally convex space

"there exists a bijective correspondence between positive kernels and Hilbertian subspaces".

The main difference with the existing theory in the first chapter is the use of dual systems and bilinear forms and one of its consequence is the emergence of some loss of symmetry that will lead to our general theory of subdualities.

In the second chapter we study the existing theory of Hermitian (or Krein) subspaces which are indefinite inner product spaces. These spaces actually generalize the previous notion of Hilbertian subspaces and their study is a first step to the global generalization of chapter three. These spaces are deeply connected to Hermitian kernels but interestingly enough the previous fundamental equivalence is lost. Then we focus on the differences between this theory and the Hilbertian one for these differences will of course remain when dealing with subdualities.

In the third chapter we present a new theory of a dual system of vector spaces called subdualities which deals with the previous chapters as particular cases. A topological definition (proposition 3.3) of subdualities is as follows: a duality (E, F) is a subduality of the dual system $(\mathcal{E}, \mathcal{F})$ if and only if both E and F are weakly continuously embedded in $\mathcal{E}$. It appears that we can associate a unique kernel (in the sense of L. Schwarz, theorem 3.6) with any subduality, whose image is dense in the subduality (theorem 3.10). The study of the image of a subduality by a weakly continuous linear operator (theorem 3.12), makes it possible to define a vector space structure upon the set of subdualities (theorem 3.13), but given a certain equivalence relation. A canonical representative entirely defined by the kernel is then given (theorem 3.20), which enables us to state a bijection theorem between canonical subdualities and kernels.

We also study the particular case of subdualities of $\mathbb{R}^{\Omega}$ which we name evaluation subdualities. Their kernel may the be identified with a kernel function (definition 3.34 and lemma 3.35). Such subdualities and kernels appear for instance in the study of polynomial spaces, Chebyshev splines and blossoming, see for instance M-L. Mazure and P-J. Laurent [37].

Finally a fourth chapter is dedicated to applications. We first analyse the link between Hilbertian subspaces and Gaussian measures and try to extend the theory to Krein subspaces and subdualities. Then we focus on some particular operators: operators in evaluation subdualities (subdualities of $\mathbb{K}^{\Omega}$) and differential operators. In a third section, we finally develop an interpolation theory in evaluation subdualities.

This work brings up many questions, with both theoretical and applied insights. The main questions are devoted to canonical subdualities: is there an easy characterization of canonical subdualities, are they interesting enough, can one characterize directly stable kernels ? Other questions deal with differential operators and their link with Sobolev spaces. Finally the physical interpretation of subdualities is a crucial point to understand this theory and its applications.

Foreword

Motivation

The subject of this thesis is outside the mainstream orientations of nowadays mathematics, and may not seem directly connected to actual mathematical questions. That is why before really stating the subject of this thesis, I think I should explain were it stems from. The starting point of this thesis may actually be strange for today's readers: it dealt with the choice of models in approximation theory (based on regularization). The first work I had to do was to understand what the possible models were, and how they worked. It appeared that the framework was clear: we had to work within a Hilbert space so as to minimize an "energy", and the evaluation functionals (the δ_t) had to be continuous so that the values at some points give information about the whole function.

Spaces verifying these two conditions are called Reproducing kernel Hilbert spaces and in this case, regularization is always feasible thanks to the existence of orthogonal projections on convex sets in Hilbert spaces. Moreover, the solution is given in terms of the "reproducing kernel" of the Hilbertian subspace, that is a two variable positive function. Finally, the link between Hilbertian subspaces and Gaussian stochastic processes gives an interpretation of the solution of the regularization problem in terms of conditional expectation.

This framework is however not completely satisfactory:

- on the practical level, many people found empirically that some problems were better solved with non-positive or non hermitian kernels;
- on the conceptual level, if no discussion seems possible about the continuity of the evaluation functionals, the Hilbertian hypothesis seems very strong and only for convenience.

The relaxation of the Hilbertian hypothesis was then the second beginning of this thesis that differs finally very much from its initial idea. We will however go back to the approximation problem in the final chapter and use generic kernels, finally going back to our starting point. Among the multiple possibilities that offer the loss of the Hilbertian hypothesis, we chose one actually hidden in the theory of Hilbertian subspaces. To understand what possibilities were offered, which ones could keep the link with kernels, and what possibility we retain, an historical approach is now given.

Historical approach

Hilbert spaces were originally defined as spaces with the same geometry as the "Hilbert space" i.e. the space of square summable sequences l_2 studied by David Hilbert in [29] (see for instance F. Riesz [44]). They are defined as an algebraic object, a vector space endowed with a positive inner product together with a topological property, the completeness of the space with respect to the norm derived from the inner product.
These spaces are a generalization of the well-known Euclidean spaces and have been widely used during this century for their numerous properties similar to those of Euclidean spaces such as the existence of a orthonormal basis or the existence and uniqueness of orthogonal projections, all deriving from the existence and positivity of the inner product.

So Hilbert spaces were defined at the early beginning of the 20th century. At the same time the notion of "functional" and "operator" came into being. Based on the existing notion of continuity (which meant at that time and till about 1935 transforming convergent sequences into convergent sequences i.e. sequential continuity), the notions of duality and topological duality emerged and in 1908 Frechet and Riesz [44] demonstrated the well-known "Riesz identification theorem" proving that any real Hilbert space is self-dual.

Many roads (illustrated figure 0.1) were then open.
One led to metric functional analysis (notably with the book of S. Banach [11]) and the particular study of normed spaces, Hilbert spaces and Frechet spaces.
Another was the topological road: inspired by Hilbert's axioms of open neighborhoods for the plane Hausdorff defined general topological spaces in 1914 [28]. The notion of uniform space followed but it was only in 1935 with the works of Von Neumann and Kolmogorov that topological spaces extended to topological vector spaces (in short t.v.s.) with the notion of locally convex spaces (l.c.s).
Finally, a general theory of duality was created on the basis of the works of Mackey ([34],[35] and Grothendieck [25] (one of its main consequences in functional analysis being Schwartz's theory of distributions [47]).

Another crucial step in the development of functional analysis is the theory of "Hilbertian subspaces" (L. Schwartz [46]). In terms of foundations ("Grundlagen" in German), this theory is not fundamental since it uses concepts that had already appeared. However it is fundamental in the sense that it links a class of operators (the so-called "positive kernels") and a class of Hilbert spaces (the Hilbertian subspaces) extending the existing results of Aronszajn [6] concerning positive kernel functions and reproducing kernel Hilbert spaces.

But as shown before many different classes of spaces such as topological vector spaces, Banach spaces, dualities, all including Hilbert spaces as a particular case (theses notions are detailed in the Appendix A) have emerged in the 20th century.
That is why it is of prime interest to understand how those different notions (such as a norm, a dual system) are related with the notion of positive inner product and what mathematical objects appear if we weaken some of the hypotheses mainly if one is interested in finding a larger class of spaces than Hilbert spaces. In our particular case we want to refine and extend the theory of Hilbertian subspaces.

An illustrative hierarchy of spaces (precisely of additional structure to be put on a vector space) is given by figure 0.1, where the left part mostly corresponds to algebraic conditions whereas the right part refers to more topological conditions.

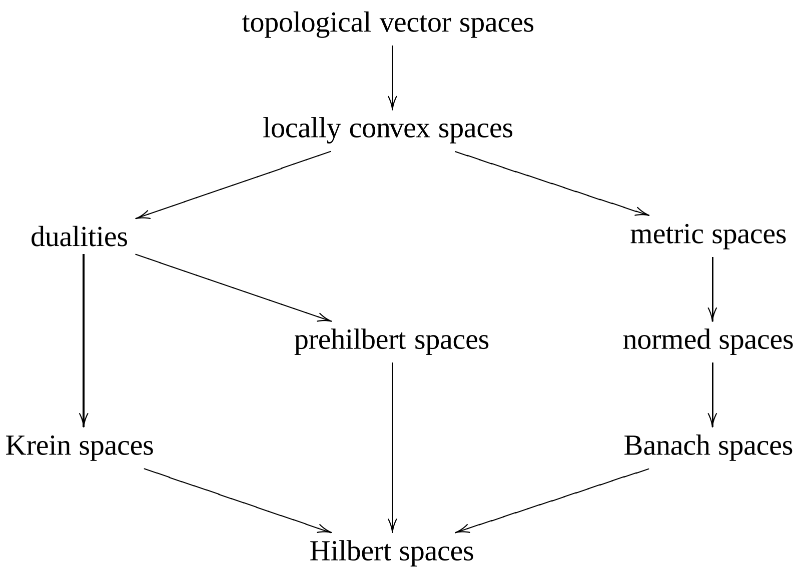

Figure 0.1: Hierarchy of spaces.

In this thesis we mainly investigate the left side of figure 0.1 that is the most natural way to study kernels as the next chapters will show. The right side is also of interest but leads to a rather different theory, a theory of multivariate non-linear applications and subdifferentials of positive semi-continuous functionals called Banachic kernels initiated by M. Attéia [9] that we will not detail here.

Overview of the thesis

Our starting point throughout this thesis will never be kernels, but rather a certain class of spaces (Hilbert spaces, Krein spaces, dualities) verifying additional inclusion properties relative to a common reference space $\mathcal{E}$ (precisely to a common duality $(\mathcal{E}, \mathcal{F})$). Kernels will then naturally appear.

Since the main originality of this work is the generalization of the notion of Hilbertian subspace to subdualities (presented chapter 3) it appears coherent to first restate the theory of Hilbertian subspaces and to introduce two generalizations afterwards. This work is then divided into four chapters:

1. the first is devoted to the study of Hilbertian subspaces of a locally convex space (l.c.s.) $\mathcal{E}$ that are Hilbert spaces continuously embedded in the l.c.s $\mathcal{E}$, or more generally to the study of Hilbertian subspaces of a dual system $(\mathcal{E}, \mathcal{F})$. The intensive use of bilinear rather than sesquilinear form will amazingly lead to a certain loss of symmetry that contains the basis of the theory of subdualities;

2. in the second chapter we generalize to indefinite inner product spaces i.e. we study Krein (or Hermitian) subspaces, which are Krein spaces continuously embedded in the l.c.s $\mathcal{E}$. Most of the results were already contained in L. Schwartz's paper [46] but it is

interesting to see how two different approaches can be followed and what difficulties appear;

3. a new step is made with the generalization of the theory to a dual system of vector spaces both continuously embedded in the l.c.s $\mathcal{E}$, which we call subdualities. This original theory links subdualities with the total set of kernels and gives a coherent and general setting that includes the previous notions. Most of these results have been published in [36];

4. finally some remarks concerning possible applications are given in the fourth chapter.

In order to better understand and follow the path of this work, the study of three general examples will be carried out. The first example is a "toy" one, the simple example of the two dimensional space $\mathbb{R}^2$. The second one, that deals with the theory of differential operators, is the general example of Sobolev spaces and integral (resp. differential) operators.
Finally, we will carry the study of polynomial and Chebyshev spaces (or splines) (i.e. finite dimensional function spaces) in a third example. The study of these spaces and some particular related dualities is very important in the theory of geometric continuity and blossoming (see for instance Mazure and Laurent [37] or Goldman [22]).
These three examples will be referred as **$\mathbb{R}^2$-example**, **Sobolev spaces**, and **Polynomials, splines** afterward.

The theory of Hilbertian subspaces and more generally the theory of subdualities, as its name indicates, relies mainly on the duality theory for topological vector spaces. Therefore we will only consider locally convex (Hausdorff) topological vector spaces or (Hausdorff) dualities[5]. Throughout this study $\mathcal{E}$ will always be a locally convex (Hausdorff) topological vector space (in short l.c.s.) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $(\mathcal{E}, \mathcal{F})$ a dual system of vector spaces.

[5]The link between the two notions is given in the Appendix

For practical reasons any complex vector space $\mathcal{E}$ (i.e. over the scalar field $\mathbb{C}$) will be endowed with an anti-involution (conjugation) $C_j : \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ such that $\overline{\mathcal{E}} \sim \mathcal{E}$. We will then always be able to identify the dual space of a Hilbert space with itself[6] but with respect to a generally asymmetric bilinear form. Moreover, at least for the first two chapters we suppose that the duality $(\mathcal{E}, \mathcal{F})$ verifies:

$$\left(\overline{f}, \overline{e}\right) = \overline{(f, e)}$$

so that for any kernel the following equation will hold: $\forall \varkappa \in \mathbf{L}(\mathcal{F}, \mathcal{E})$,

$$\varkappa^* =^t \overline{\varkappa} = \overline{{}^t\varkappa}$$

Self-adjunction and positivity will then be the classical notions. This is for instance the case of any dual system $(\mathcal{E}, \mathcal{E}')$. These last conditions are however not needed in the chapter dealing with subdualities since we only study bilinear forms (and no positivity or Hermicity is at stake).

[6] It can only be identified with its conjugate space in the most general setting

Chapter 1

Hilbertian subspaces

Introduction

While studying Hilbert spaces of holomorphic functions defined on an open subset of $\mathbb{C}^n$ S. Bergman [13] remarked that those Hilbert spaces were highly linked with some functions of two variables he called kernel functions. The same year N. Aronszajn [6] extended this link to a wider class of spaces, precisely Hilbert spaces continuously included in the product space $\mathbb{C}^\Omega$.

In 1964 L. Schwartz took this definition as a starting point. Rather than studying kernels functions he decided to study the class of Hilbert spaces continuously included in a particular l.c.s. $\mathcal{E}$. This led to the general theory of Hilbertian subspaces and of their associated kernels. This chapter is a presentation of this theory and of some refinements since it gives the foundations of the general theory of subdualities. There are some new ideas as for instance the extensive use of dual systems that open the road to previously unseen interpretations. For readers interested in the foundations of this theory or for precise proofs of the statements in chapter 1 we recommend [46] and [6] for the specific case of reproducing kernel Hilbert spaces.

1.1 Hilbertian subspaces of a locally convex space and Hilbertian subspaces of a duality

1.1.1 Definition of a Hilbertian subspace of a locally convex space

Following the work of L. Schwartz ([46]), we define Hilbertian subspaces of a locally convex space (l.c.s.) $\mathcal{E}$ as Hilbert spaces continuously included in $\mathcal{E}$. To be more precise we define first prehilbertian subspaces of $\mathcal{E}$:

Definition 1.1 (– *prehilbertian subspace of a l.c.s.* –) *Let $\mathcal{E}$ be a l.c.s. Then H_0 is an prehilbertian subspace of $\mathcal{E}$ if and only if H_0 is an algebraic vector subspace of $\mathcal{E}$ endowed with an positive inner product (denoted by $\langle . | . \rangle$) that makes it a prehilbert space and such that the canonical injection is continuous.*

Notice that this last condition is equivalent to:

$$\forall \varepsilon' \in \mathcal{E}',\, \forall h_0 \in H_0,\, \exists M_{\varepsilon'} \in \mathbb{R}^+, \quad |(\varepsilon', h_0)|_{(\mathcal{E}',\mathcal{E})} \leq M_{\varepsilon'} \|h_0\|_{H_0}$$

where $(,)_{(\mathcal{E}',\mathcal{E})}$ denotes the duality product between the l.c.s. $\mathcal{E}$ and its topological dual $\mathcal{E}'$. The definition of Hilbertian subspace follows:

Definition 1.2 (– *Hilbertian subspace of a l.c.s.* –) *Let $\mathcal{E}$ be a l.c.s. Then H is a Hilbertian subspace of $\mathcal{E}$ if and only if H is an algebraic vector subspace of $\mathcal{E}$ endowed with an definite positive inner product that makes it a Hilbert space and such that the canonical injection is continuous.*

At first sight, one could think that the concept of Hilbertian subspaces is purely topological, since the obvious requirement is that the canonical injection is continuous. This is only partially true since one requires the space H to have a Hilbertian structure, which is almost completely an algebraic requirement (except the completeness). In fact, there is a whole algebraic interpretation of Hilbertian subspaces in terms of dual spaces as we will see in the second and third chapters.

In this direction and like many authors we would like to emphasize the fact that the initial topology of $\mathcal{E}$ plays no role in the theory of Hilbertian subspaces, which only depends on the duality $(\mathcal{E}, \mathcal{E}')$. It follows that we can find equivalent conditions for a Hilbert space H to be a Hilbertian subspace of $\mathcal{E}$ based only on the weak topology or the Mackey topology[1]:

Proposition 1.3 *The three following statements are equivalent:*

1. *H is a Hilbertian subspace of $\mathcal{E}$;*
2. *the canonical injection is weakly continuous;*
3. *the canonical injection is Mackey continuous (i.e continuous if H and $\mathcal{E}$ are both endowed with their Mackey topology).*

Proof. – Let us prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$:

$(1) \Rightarrow (2)$ corollary 1 p 106 [26]: if $i : H \mapsto \mathcal{E}$ is continuous, it is weakly continuous.

$(2) \Rightarrow (3)$ We can cite corollary 2 p 111 [26]: if $i : H \mapsto \mathcal{E}$ is weakly continuous, it is continuous if H is endowed with the Mackey topology (and $\mathcal{E}$ with any topology compatible with the duality).

$(3) \Rightarrow (1)$ The topology of the Hilbert space H is the Mackey topology since H is metrizable (corollary p 149 [26] or proposition 6 p 71 [15]) and we use the previous argument (corollary 2 p 111 [26]). □

Finally, it follows from this proposition that the canonical injection, as a weakly continuous application, has a transpose and an adjoint. From now on, we will denote by i the canonical

[1] These topologies are detailed in the Appendix B

injection ${}^t i$ its transpose and i^* its adjoint.

example 1 **$\mathbb{R}^2$-example**

We introduce here for the first time our "toy" example: the space $\mathbb{R}^2$.

If we endow it with the scalar product

$$\langle Y|X\rangle = x_1y_1 + \frac{1}{2}x_2y_2$$

then it is clearly a Hilbertian subspace of $\mathbb{R}^2$ endowed with the supremum norm.

example 2 **Sobolev spaces**

Let $\mathcal{E} = D'$ be the space of distribution on an open set Ω of $\mathbb{R}$ **bounded from the left**.

Then it is a classical result that $\forall p \in \mathbb{N}$ the Sobolev space

$$W^p = \Big\{ f \in D', \frac{d^p}{dt^p}(f) \in L^2,\ f \textit{ and all its derivatives}$$

$$\textit{up to order } p \textit{ null on the left frontier } (\partial\Omega)^- \Big\}$$

endowed with its canonical scalar product

$$\langle f|g\rangle_{W^p} = \int_\Omega \frac{d^p}{dt^p}(f)\frac{d^p}{dt^p}(g)$$

is a Hilbertian subspace of D'.

Equivalently, Sobolev spaces

$$W^{-p} = \Big\{ f \in D', f = \frac{d^p}{dt^p}(\phi),\ \phi \in L^2,\ \phi \textit{ and all its derivatives}$$

$$\textit{up to order } p \textit{ sum to } 0\}$$

endowed with the canonical scalar product

$$\langle f|g\rangle_{W^{-p}} = \int_\Omega \phi\psi$$

are Hilbertian subspaces of D'.

<u>example 3</u> Let $\mathcal{H}$ be a Hilbert space. Then any subspace H_0 of $\mathcal{H}$ is a prehilbert space with the induced inner product and obviously a prehilbertian subspace of $\mathcal{H}$. Any closed subspace H of $\mathcal{H}$ is complete and hence a Hilbertian subspace of $\mathcal{E}$.

<u>example 4</u> Rigged Hilbert space:

Let H be a Hilbert space and $\mathcal{F}$ a topological vector space, algebraic subspace of H such that the inclusion is weakly continuous and $\mathcal{F}$ is dense in H. Then H' is weakly continuously embedded in $\mathcal{F}'$ and by proposition 1.3, H' (generally identified with H) is a Hilbertian subspace of $\mathcal{F}'$[2].

1.1.2 Definition of a Hilbertian subspace of a duality

Proposition 1.3 also allows us to define the Hilbertian subspaces of a dual system $(\mathcal{E}, \mathcal{F})$[3]. We will now follow this perspective all along this thesis. There are three reasons for this: first, since the initial topology of the l.c.s. $\mathcal{E}$ plays no role in the Hilbertian theory that depends only on the duality $(\mathcal{E}, \mathcal{E}')$, it seems natural to show this in the names and notations. Second, for many applications the topological dual $\mathcal{E}'$ is identified with a particular function space. And finally the third chapter precisely deals with this theory.

Definition 1.4 (– *Hilbertian subspace (of a duality)* –) *Let $(\mathcal{E}, \mathcal{F})$ be a duality. Then H is a Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$ if and only if H is an algebraic vector subspace of $\mathcal{E}$ endowed with an definite positive inner product that makes it a Hilbert space and such that the canonical injection is weakly continuous.*

Remark that the canonical injection is weakly continuous if and only if any element of $\mathcal{F}$ (i.e. any continuous linear form on $\mathcal{E}$) restricted to H admits a representative in H.

[2]This is notably the case for Hilbertian subspaces of the space of distributions, $\mathcal{F} = C^\infty$ and $\mathcal{F}' = D'$.
[3]Appendix B

example 5 **$\mathbb{R}^2$-example**

We can identify the dual space of $\mathbb{R}^2$ endowed with the supremum norm with the space $\mathbb{R}^2$, the bilinear form being for instance the Euclidean one. The previous Hilbertian subspace $\mathbb{R}^2$ endowed with the scalar product

$$\langle Y|X\rangle = x_1y_1 + \frac{1}{2}x_2y_2$$

is then a Hilbertian subspace of the Euclidean duality $(\mathbb{R}^2, \mathbb{R}^2)$.

example 6 **Sobolev spaces**

By construction of D', its topology is the weak topology associated with the dual system (D, D'). Any previously defined Hilbertian Sobolev space will then be a Hilbertian subspace of the duality (D', D).

example 7 Let Ω be an open set of $\mathbb{C}^n$, $\mathcal{E} = L^1(\Omega)$, $\mathcal{F} = L^\infty(\Omega)$ put in duality by the bilinear form

$$\begin{array}{rcl} L : L^\infty(\Omega) \times L^1(\Omega) & \longrightarrow & \mathbb{C} \\ (\psi, \phi) & \longmapsto & \int_{t\in\Omega} \phi(t)\psi(t)dt \end{array}$$

Let Ξ be a compact set of Ω and $H = \left\{\phi \in \mathcal{E},\, \phi_{|\Xi} \in L^2(\Xi),\, \phi_{|\Xi^\complement} = 0\right\}$ endowed with the scalar product

$$\begin{array}{rcl} H \times H & \longrightarrow & \mathbb{C} \\ (\psi, \phi) & \longmapsto & \int_{t\in\Xi} \phi(t)\overline{\psi(t)}dt = \int_{t\in\Omega} \phi(t)\overline{\psi(t)}dt \end{array}$$

It is a standard result that $L^\infty(\Xi) \subset L^2(\Xi) \subset L^1(\Xi)$ hence H is a Hilbertian subspace of $\left(L^1(\Omega), L^\infty(\Omega)\right)$.

The set of Hilbertian subspaces of a duality $(\mathcal{E}, \mathcal{F})$ (resp. of a l.c.s. $\mathcal{E}$) is usually denoted by $Hilb\left((\mathcal{E}, \mathcal{F})\right)$ (resp. $Hilb(\mathcal{E})$). We then define the following function: $Hilb : (\mathcal{E}, \mathcal{F}) \mapsto Hilb\left((\mathcal{E}, \mathcal{F})\right)$ (resp. $Hilb : \mathcal{E} \mapsto Hilb\left(\mathcal{E}\right)$) which maps dualities (resp. l.c.s.) to the set of its Hilbertian subspaces.

We will see that this set $Hilb\left((\mathcal{E},\mathcal{F})\right)$ has remarkable features after some remarks on the completion of prehilbertian subspaces that will help understand why one focuses on the set of Hilbertian subspaces rather than prehilbertian ones.

1.1.3 Comments on the completion of a prehilbertian subspace

In this section, we investigate the following problem: is the set of prehilbertian subspaces interesting, or can we restrict our attention to Hilbertian subspaces? This question is actually based on the associated problem of the completion of a prehilbertian subspace and the completion of uniform spaces in general.

This notion of completion is usually well-known in the case of metric spaces, but in fact more general (see [16] for precise statements). Roughly speaking, the notion of Cauchy sequences is generalized to Cauchy filters, that exist on uniform spaces. But topological vector spaces (t.v.s) are naturally endowed with such a structure and the notion of completion arises naturally.Therefore, the comments of this section can be applied to the more general theory of subdualities developed in the third chapter of this part.
The main point is the following: the completion of a prehilbertian space with respect to its norm is always feasible (and that is why we usually consider only Hilbert spaces rather than prehilbert spaces), but in the case of a prehilbertian subspace of a l.c.s $\mathcal{E}$, it may happen that this completion is "bigger" than $\mathcal{E}$. More precisely, it is known that the algebraic dual of $\mathcal{E}'$ is the weak completion of $\mathcal{E}$, which therefore can be seen as a subspace of $\mathcal{E}'^*$. But so does the completion of H_0, $\hat{H}_0$ and it may happen that $\hat{H}_0$ is not included in $\mathcal{E}$ as subspaces of $\mathcal{E}'^*$. Therefore, the Hilbertian subspace completion of a prehilbertian space may not exist[4]. However we will see in proposition 1.18 that for a certain class of "good"

[4] See example below

prehilbertian spaces (those steaming from kernels) and under weak conditions on the space $\mathcal{E}$, the Hilbertian completion always exists. This explains why we can generally restrict our attention to the study of Hilbertian subspaces, although it may mean completing a prehilbert subspace. In [46] for instance L. Schwartz gives a necessary and sufficient condition for such a completion to exist when $\mathcal{E}$ is quasi-complete, which in this case is of course unique.

Proposition 1.5 *Let H_0 be a prehilbertian subspace of a quasi-complete l.c.s. $\mathcal{E}$, i its natural injection. Then it has a Hilbertian subspace completion if and only if the extension of i, $\hat{i} : \hat{H}_0 \to \hat{\mathcal{E}}$, is injective. In this case, the Hilbertian subspace completion is $\hat{H}_0$.*

example 8 This example is based on L. Schwartz's paper [46]. Let $\mathcal{E} = L^2(\mathbb{R})$ and let H_0 be the subspace of continuous functions of $L^2(\mathbb{R})$. We can endow H_0 with the inner product $\langle f|g\rangle = \int fg + f(0)g(0)$ that makes it a prehilbertian subspace of $\mathcal{E}$. We may identify the completion $\hat{H}_0$ of H_0 with $L^2(\mathbb{R}) \times \mathbb{R}$, which is bigger than $L^2(\mathbb{R})$. H_0 has no Hilbertian subspace completion.

1.1.4 The structure of $Hilb((\mathcal{E}, \mathcal{F}))$

The notion of Hilbertian subspace leads to two different paths of investigation: one can be interested in the properties of the whole set $Hilb((\mathcal{E}, \mathcal{F}))$, or one can be interested in the properties of a particular Hilbertian subspace. This second path will be investigated in the next section. **A remarkable fact concerning the set $Hilb((\mathcal{E}, \mathcal{F}))$ that highlights the beauty of the concept is that:**

Theorem 1.6 ***We can endow the set $Hilb((\mathcal{E}, \mathcal{F}))$ with an external multiplication law (on $\mathbb{R}^+$), an intern addition law and an order relation which give $Hilb((\mathcal{E}, \mathcal{F}))$ the structure of a convex cone. Moreover, this cone is salient and regular.***

The definitions (constructions) of the laws and the order relation are thoroughly discussed in details in [46] or [48]. They are partly based on the Hilbertian structure. We give here a brief insight in order to understand the importance of the scalar product. The results of this section are strongly related to the transport of structure via a weakly continuous application, but are best seen in a self-contained material (with remarks concerning the transport of the structure).

addition law

Let $H_1,\ H_2 \in Hilb((\mathcal{E},\mathcal{F}))$. Then the Hilbertian subspace $H_1 + H_2$ is the algebraic subspace of $\mathcal{E}$ $H = \{h = h_1 + h_2,\ h_1 \in H_1,\ \mathrm{h}_2 \in H_2\}$ endowed with the norm

$$\|h\|_{H_1+H_2} = \inf_{h=h_1+h_2} (\|h_1\|^2 + \|h_2\|^2)^{1/2}$$

An easy way to prove that this space is actually a Hilbert space is to remark that it is isomorphic to the Hilbert space $(H_1 \times H_2)/\ker \mathfrak{S}$, where $\mathfrak{S} : (h_1, h_2) \mapsto h_1 + h_2$.
Anticipating the results of the next chapter, we can say that the Hilbertian subspace $H_1 + H_2$ is the image of the Hilbert space $H_1 \times H_2$ by the weakly continuous operator[5] $\mathfrak{S} : H_1 \times H_2 \longrightarrow \mathcal{E}$.
Note that this operation is associative, i.e. a true addition law.

external multiplication law

Let $H \in Hilb((\mathcal{E},\mathcal{F}))$: we want to define for all $\lambda \in \mathbb{R}+$ the Hilbertian subspace λH.
If $\lambda = 0$ then $\lambda H = \{0\}$. If $\lambda > 0$ then we have the algebraic equality $\lambda H = H$, but we endow λH with the inner product:

$$\langle h_1 | h_2 \rangle_{\lambda H} = \frac{1}{\lambda} \langle h_1 | h_2 \rangle_H$$

[5] see corollary 1.40.

Remark that λH may also be defined as the image of the Hilbert space H by the homothecy $h \mapsto (\sqrt{\lambda})h$.

order relation

We define the order relation by $H_1 \leq H_2 \iff H_1 \subseteq H_2$ and the canonical injection has norm less than 1.

Structure of a convex cone

Finally, we verify that these laws and order relation are compatible with the structure of a convex cone, that is $\forall \lambda, \mu \geq 0$:

$$\lambda(H_1 + H_2) = \lambda H_1 + \lambda H_2$$

$$(\lambda + \mu)H = \lambda H + \mu H$$

1.2 Schwartz kernel of a Hilbertian subspace

This section is devoted to the study of a certain class of operators we call kernels. This class of operators has many applications, particularly in the field of partial differential equations or tensor products ([50]) or in the probability theory. In this section we study the subset of positive (self-adjoint) kernels, which is closely linked with the set of Hilbertian subspaces since the two sets are (under weak assumptions) isomorphic. Hilbertian kernels of Hilbertian subspaces have many good properties and for instance Hilbertian kernels may be seen as the generalization of orthogonal projection in Hilbert spaces to arbitrary spaces.
There exist different definitions for kernels related to the particular point of view one has of their relation to Hilbertian subspace. The two main definitions are:

1. Let $\mathcal{E}$ be a l.c.s., $\mathcal{E}'$ its topological dual and $\overline{\mathcal{E}'}$ the conjugate space of its topological dual. Then according to L. Schwartz [46], we call kernel any weakly continuous linear application $\varkappa : \overline{\mathcal{E}'} \longrightarrow \mathcal{E}$.

2. Another interesting definition due to C. Portenier [43] is the following. Let $\mathcal{F}$ be a locally convex space, $\mathcal{F}^{\dagger}$ the space of continuous semilinear forms on $\mathcal{F}$. Then we call kernel any weakly continuous linear application $\varkappa : \mathcal{F} \longrightarrow \mathcal{F}^{\dagger}$.

These two definitions are very convenient when dealing with scalar products and therefore appropriate to the study of Hilbertian subspaces. Here, we however take a third point of view since our principal object of interest in this thesis is a duality rather than a Hilbert space hence a bilinear form rather than a sesquilinear form. Moreover, since the notion of Hilbertian subspace is relative to a duality rather than a locally convex space, we define kernels of a dual system of vector space. **Precisely, we call kernel relative to a duality** $(\mathcal{E}, \mathcal{F})$ **any weakly continuous linear application from** $\mathcal{F}$ **into** $\mathcal{E}$**.** In order to avoid technical difficulties we deal with spaces with an anti-involution i.e. for any space $\mathcal{E}$ $\overline{\mathcal{E}} \sim \mathcal{E}$ but the simple fact that we should distinguish $\mathcal{E}$ and $\overline{\mathcal{E}}$ is crucial. Any Hilbert space will then be in duality with itself thanks to the following bilinear form:

$$\begin{aligned} L : \overline{H} \sim H \times H &\longrightarrow \mathbb{K} \\ \overline{h_1}, h_2 &\longmapsto \langle h_1 | h_2 \rangle \end{aligned}$$

but this bilinear form is asymmetric in general and therefore we should (and will) distinguish them in the sequel.

Transposition is actually defined upon this asymmetric duality and any weakly continuous operator $u : H \longrightarrow \mathcal{E}$ has two transposes whether we deal with $u : \overline{H} \sim H \longrightarrow \mathcal{E}$ or $u : H \longrightarrow \mathcal{E}$.

1.2.1 Kernels

Definition 1.7 (– *kernel (of a duality)* –) *We call kernel relative to a duality $(\mathcal{E},\mathcal{F})$ (and note $\varkappa : \mathcal{F} \longrightarrow \mathcal{E}$) any weakly continuous linear application from $\mathcal{F}$ into $\mathcal{E}$.*

The definition of a kernel relative to a locally convex space follows, since any l.c.s. $\mathcal{E}$ defines a duality $(\mathcal{E},\mathcal{E}')$[6].

Definition 1.8 (– *kernel (of a l.c.s.)* –) *We call kernel relative to a locally convex topological vector space $\mathcal{E}$ any weakly continuous linear application from the topological dual $\mathcal{E}'$ into $\mathcal{E}$, i.e. any kernel relative to the duality $(\mathcal{E},\mathcal{E}')$.*

Since a kernel is weakly continuous, it has a transpose ${}^t\varkappa$ and an adjoint $\varkappa^* = \overline{{}^t\varkappa}$. But from the definition of a kernel its transpose and adjoint are also kernels of the duality $(\mathcal{E},\mathcal{F})$ and we can define the symmetry, self-adjoint and positiveness properties.

Definition 1.9 *We say that a kernel $\varkappa$ is symmetric if ${}^t\varkappa = \varkappa$, self-adjoint (Hermitian) if $\varkappa^* = \varkappa$. It is positive if*

$$\forall \varphi \in \mathcal{F},\ (\varphi, \varkappa(\overline{\varphi}))_{(\mathcal{F},\mathcal{E})} \geq 0$$

(equivalently, it is positive if $\forall \varepsilon' \in \mathcal{E}',\ \left(\varepsilon', \varkappa(\overline{\varepsilon'})\right)_{(\mathcal{E}',\mathcal{E})} \geq 0$.)

One checks easily that

Lemma 1.10 *Any positive kernel is self-adjoint and the positivity condition is equivalent to:*

$$\forall \varphi \in \mathcal{F},\ \left(\varphi, \overline{\varkappa(\varphi)}\right)_{(\mathcal{F},\mathcal{E})} \geq 0$$

A remarkable fact about self-adjoint linear operators from $\mathcal{F}$ into $\mathcal{E}$ is that they are always weakly continuous, i.e. kernels[7]. Moreover, kernels are related to bilinear and sesquilinear forms by the following proposition (see [46]):

[6] Appendix B

[7] proposition 4 p.139 in [46]

Proposition 1.11 *There is a bijective correspondence between separately weakly continuous bilinear forms (res. symmetric) on $\mathcal{F} \times \mathcal{F}$ and kernels (res. symmetric). A separately weakly continuous bilinear form L and a kernel $\varkappa$ are associated thanks to the following identity:*

$$\forall \varphi_1, \varphi_2 \in \mathcal{F},\ L(\varphi_1, \varphi_2) = (\varphi_1, \varkappa(\varphi_2))_{(\mathcal{F},\mathcal{E})}$$

There is a bijective correspondence between separately weakly continuous sesquilinear forms (res. Hermitian, positive) on $\mathcal{F} \times \mathcal{F}$ and kernels (res. Hermitian, positive). It is given by the following identity:

$$\forall \varphi_1, \varphi_2 \in \mathcal{F},\ L(\varphi_2, \varphi_1) = (\varphi_1, \varkappa(\overline{\varphi_2}))_{(\mathcal{F},\mathcal{E})}$$

It is interesting to notice that we can endow the image of a positive kernel with a scalar product that makes it a prehilbertian space the scalar product being the following sesquilinear form:

$$\forall \varepsilon_1, \varepsilon_2 \in \varkappa(\mathcal{F}),\ \langle \varepsilon_2 | \varepsilon_1 \rangle = \left(\varkappa^{-1}(\varepsilon_1), \overline{\varepsilon_2}\right)_{(\mathcal{F},\mathcal{E})}$$

The space of kernels (res. symmetric, self-adjoint, positive) is denoted by $\mathbf{L}(\mathcal{F}, \mathcal{E})$, $\mathbf{L}(\mathcal{E}', \mathcal{E})$ or simply $\mathbf{L}(\mathcal{E})$ (resp. $\mathbf{L}^t$, $\mathbf{L}^*$, $\mathbf{L}^+$). As for the set of Hilbertian subspaces of a given duality $(\mathcal{E}, \mathcal{F})$ $Hilb((\mathcal{E}, \mathcal{F}))$ we can endow the set of positive kernels of this duality $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$ with an external multiplication law, an intern addition law and an order relation which gives to $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$ the structure of a convex cone. Moreover, this cone is salient and regular[8]. We will see in the next section that under very smooth hypothesis the two sets $Hilb((\mathcal{E}, \mathcal{F}))$ and $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$ are isomorphic.

Here are some examples to illustrate this notion:

example 1 **$\mathbb{R}^2$-example**

Let $(\mathcal{E}, \mathcal{F}) = (\mathbb{R}^2, \mathbb{R}^2)$ in Euclidean duality. Any kernel $\varkappa$ may then be identified with

[8] See [48]

a matrix K of $\mathcal{M}_2(\mathbb{R})$ by

$$K(i,j) = (e_i, \varkappa(e_j))_{(\mathcal{F},\mathcal{E})}$$

example 2 **Sobolev spaces** **(- kernel theorem -)**

Let $\mathcal{E} = D'$ be the space of distribution on an open set Ω of $\mathbb{R}$. Then we can identify its dual with the set of test functions $\mathcal{F} = D = C_0^\infty$ and by the kernel theorem of L. Schwartz the set of kernels of D' is isomorphic with the set of distributions on $\Omega \times \Omega$:

$$\varkappa : \phi \mapsto \varkappa(\phi)(.) = \int_\Omega K_{.,s}\phi(s)ds$$

where K is a distribution on $\Omega \times \Omega$.

example 3 Let $\mathcal{E} = \mathcal{F} = H$ be a real Hilbert space. Then by Riesz theorem $(\mathcal{E},\mathcal{F}) = (H,H)$ is a duality (with symmetric bilinear form) and its kernels are the continuous endomorphisms of H. The notions of self-adjointness and positivity are the classical ones.

example 4 Let $b \in L_1(\mathbb{R})$. Then the symmetric separately continuous bilinear form

$$\begin{array}{rcl} L : L_\infty(\mathbb{R}) \times L_\infty(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\psi,\phi) & \longmapsto & \int_\mathbb{R} b(t)\phi(t)\psi(t)dt \end{array}$$

is associated to the symmetric kernel

$$\begin{array}{rcl} \varkappa : L_\infty(\mathbb{R}) & \longrightarrow & L_1(\mathbb{R}) \\ \psi \longmapsto & & b.\psi \end{array}$$

The bilinear form (res. the kernel) is positive if and only if the function b is positive.

Finally as advised by M. Atteia we mention tensor products (see example 2 above), for they are closely related to kernels. The general theory of topological tensor products (and the related nuclear spaces) is due to A. Grothendieck [25]. A comprehensive and clear reference is [50] and we recommend this book for readers interested in this subject. Roughly speaking, we can always identify the tensor product of two locally convex spaces with a particular space of continuous bilinear forms and the completion of this tensor product (with respect to different topologies) will be identified with subsets of separately continuous bilinear forms, i.e. kernels. Moreover, under additional assumptions (mainly of nuclearity), we can identify sometimes identify the completion of the tensor product space with the space of kernels. Precisely, we can state the following theorem:

Theorem 1.12 *Let $(\mathcal{E}, \mathcal{F})$ be a duality such that $\mathcal{E}$ endowed with the Mackey topology is nuclear and complete. Then*

$$\mathbf{L}(\mathcal{F}, \mathcal{E}) = \mathcal{E} \widehat{\otimes} \mathcal{F}' \sim \mathcal{E} \otimes \mathcal{E}$$

where the completion is taken with respect to one of the following equivalent topologies: the projective topology or the equicontinuous topology.

1.2.2 The Hilbertian kernel of a Hilbertian subspace

In this section we precise the link between Hilbertian subspaces (of a given duality $(\mathcal{E}, \mathcal{F})$) and positive kernels (of the same duality).

A first step to understand how positive kernels and Hilbertian subspaces are related is to associate to any Hilbertian subspace of a duality $(\mathcal{E}, \mathcal{F})$ (resp. of a l.c.s $\mathcal{E}$) a (unique) positive definite kernel of $(\mathcal{E}, \mathcal{F})$ (resp. of $\mathcal{E}$). We will later see that this kernel has many interesting properties. The definition of the kernel of a Hilbertian subspace is contained in the next theorem:

Theorem 1.13 *Let H be a Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$. There exist a unique application $\varkappa$ from $\mathcal{F}$ into $\mathcal{E}$ such that*

$$\forall \varphi \in \mathcal{F}, \forall h \in H, \ \big(\varphi, j(\overline{h})\big)_{(\mathcal{F},\mathcal{E})} = \big(\overline{h}, i^{-1} \circ \varkappa(\varphi)\big)_{(\overline{H} \sim H, H)} = \langle h | i^{-1} \circ \varkappa(\varphi) \rangle_H$$

It is the linear application

$$\begin{aligned} \varkappa : \mathcal{F} &\longrightarrow \mathcal{E} \\ \varphi &\longmapsto i \circ \theta_{(\overline{H},H)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi) = i \circ \theta_{(H,H)} \circ^t i \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi) \end{aligned}$$

considering transposition[9] *in the topological dual spaces or simply*

$$\begin{aligned} \varkappa : \mathcal{F} &\longrightarrow \mathcal{E} \\ \varphi &\longmapsto i \circ^t j(\varphi) = i \circ i^*(\varphi) \end{aligned}$$

considering transposition in dual systems where $i : H \longrightarrow \mathcal{E}$ *and* $j : \overline{H} \sim H \longrightarrow \mathcal{E}$ *are the canonical injections*[10]*. This application is a positive kernel called the Hilbertian or Schwartz kernel of* H*.*

Proof. – $\forall h, \in H,\ \varphi \in \mathcal{F}$

$$\begin{aligned} (\varphi, j(\overline{h}))_{(\mathcal{F},\mathcal{E})} &= (\overline{h},^t j(\varphi))_{(\overline{H}\sim H,H)} \\ &= \langle h|^t j(\varphi)\rangle_H \end{aligned}$$

and $\varkappa = i \circ^t j$. We then check that this linear application is weakly continuous by composition of weakly continuous morphisms and positive taking $\overline{h} = j^{-1} \circ \varkappa(\varphi) =^t j(\varphi)$ in the previous equation.

Finally ${}^t j = i^*$ since $\varkappa^* = \overline{j} \circ i^* = j \circ i^*$ is self-adjoint. □

Figure 1.1 illustrates this theorem (considering transposition in topological duals) and figure 1.2 considers transposition in dual systems.

The same theorem may be obviously be given in the context of Hilbertian kernels of locally convex space by taking $\mathcal{F} = \mathcal{E}'$.

Remark 1.14 *It is very important to notice that in this definition of the kernel we first put* j *then* i^{-1}*. Defining the kernel the other way round*

[9] *see Appendix B*

[10] *they are then equal but their transposes are distinct in general*

Z

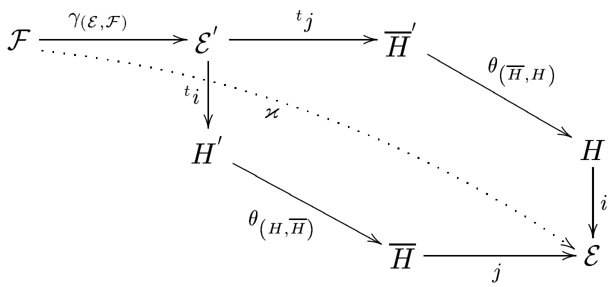

Figure 1.1: Illustration of a subduality, the relative inclusions and its kernel.

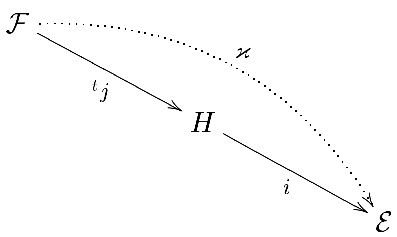

Figure 1.2: Illustration of a Hilbertian subspace and its kernel (transposition in dual systems)

$$\forall \varphi \in \mathcal{F}, \forall h \in H, \ (\varphi, i(h))_{(\mathcal{F},\mathcal{E})} = \left(j^{-1} \circ \chi(\varphi), h\right)_{(\overline{H} \sim H, H)}$$

would have defined an other positive kernel $\chi = j \circ^t i =^t \varkappa = \overline{\varkappa}$*. This is due to the asymmetry of the bilinear form on* $\overline{H} \sim H \times H$ *and* $\chi = \overline{\varkappa}$ *may be seen as the kernel of* $\overline{H}$*. This will be properly explain in the third chapter: subdualities.*

Hence any Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$ is associated with a unique positive kernel of $(\mathcal{E}, \mathcal{F})$ and the *a priori* multivoque application $\Phi : H \in Hilb((\mathcal{E}, \mathcal{F})) \mapsto \varkappa \in \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ is well defined morphism. Before studying the injectivity and surjectivity of this application, we can give some properties of this kernel.

Lemma 1.15 *Let* $\varkappa$ *be the Hilbertian kernel of* H*. Then:*

- $\varkappa =^t j : \mathcal{F} \longrightarrow H$ *is Mackey-continuous.*

•

$$\begin{array}{rcl} p : \mathcal{F} & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \left(\varphi, \overline{\varkappa(\varphi)}\right)_{(\mathcal{F},\mathcal{E})} \end{array}$$

is a lower-semi-continuous semi-norm on $\mathcal{F}$, continuous if $\mathcal{F}$ is endowed with its Mackey topology.

Proof. – These are obvious corollaries of proposition 1.3. □

We will usually note also $\varkappa$ and call kernel of H the application ${}^t j : \mathcal{F} \longrightarrow H$, as in this lemma.

There is an interesting result concerning the image of the kernel:

Lemma 1.16 *The image H_0 of $\mathcal{F}$ by a kernel $\varkappa$ is a prehilbertian subspace of $\mathcal{E}$, dense in H, with scalar product*

$$\langle h_2 | h_1 \rangle_{H_0} = \left(\varkappa^{-1}(h_1), \overline{h_2}\right)_{(\mathcal{F},\mathcal{E})} = \langle h_2 | h_1 \rangle_H$$

entirely defined by the kernel.

Proof. – Corollary p 109 [26]: "If $j : E \longrightarrow \mathcal{E}$ is one to one, its transpose ${}^t j : \mathcal{F} \longrightarrow f$ has weakly dense image". It follows that H_0 is weakly dense in H and finally dense in H for any compatible topology since it is a convex set (theorem 4 p 79 [26]). □

1.2.3 The isomorphism between $Hilb((\mathcal{E}, \mathcal{F}))$ and $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$

The previous morphism $\Phi : H \in Hilb((\mathcal{E}, \mathcal{F})) \mapsto \varkappa \in \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ that associates to any Hilbertian subspace its (unique) kernel has remarkable properties: it is one-to-one (theorem 1.17) and under very mild conditions on the duality $(\mathcal{E}, \mathcal{F})$, it is also onto (theorem 1.19). In this case, it is moreover an isomorphism of convex cones (theorem 1.20).

<u>Φ is one-to-one</u>

There are many ways to prove the injectivity of the morphism Φ, each related to a particular property of the link between the kernel and the Hilbertian subspace[11]. The one chosen here is interesting since it gives a construction of H it terms of its kernel that will be helpful to prove the surjectivity of the morphism Φ.

Theorem 1.17 *Let H be a Hilbertian subspace of $(\mathcal{E},\mathcal{F})$ with Hilbertian kernel $\varkappa$. Then H is the Hilbertian completion of the prehilbertian space $H_0 = \varkappa(\mathcal{F})$ defined in lemma 1.16 and it follows that $\Phi : H \in Hilb((\mathcal{E},\mathcal{F})) \mapsto \varkappa \in \mathbf{L}^+(\mathcal{F},\mathcal{E})$ is one-to-one.*

Proof. – By lemma 1.16, H_0 is dense in H complete, hence H is the completion of H_0 with respect to the topology induced by the scalar product on H; but

$$\forall h_0 \in H_0,\ \|h_0\|^2_{H_0} = \left(\varkappa^{-1}(h_0), h_0\right)_{(\mathcal{F},\mathcal{E})} = \|h_0\|^2_H$$

by definition of the kernel and the norm on H_0 induced by the kernel coincides with the norm on H. □

This result is very important since it gives the injectivity of Φ, but also a construction of H starting from the kernel. But the completion of a prehilbertian subspace may be bigger than $\mathcal{E}$[12] and we need to investigate closely the completion.

<u>Φ is onto</u>

It is widely believed that the previous application

$$\Phi : H \in Hilb((\mathcal{E},\mathcal{F})) \mapsto \varkappa \in \mathbf{L}^+(\mathcal{F},\mathcal{E})$$

is also onto. **The surjectivity of this morphism is however false in general, and we need**

[11] We characterize for instance the elements of the Hilbertian subspace just in terms of its kernel in the section "Other characterizations of the Hilbertian subspace associated to a kernel", which proves the injectivity.

[12] see the section "Comments on the completion of a prehilbertian subspace"

some more properties on the duality $(\mathcal{E},\mathcal{F})$ to ensure the surjectivity.

To be precise, we can state the following theorem due to C. Portenier [43]:

Theorem 1.18 *Let $\varkappa$ be a positive kernel, such that the semi-norm*

$$\begin{array}{rcl} p:\mathcal{F} & \longrightarrow & \mathbb{R} \\ \varphi & \longmapsto & \Big(\varphi,\overline{\varkappa(\varphi)}\Big)_{(\mathcal{F},\mathcal{E})} \end{array}$$

is Mackey-continuous. Then the associated prehilbertian subspace $H_0=\varkappa(\mathcal{F})$ has a unique Hilbertian subspace completion H.

Proof. – The Mackey-continuity of the semi-norm is equivalent to the continuity of ${}^t\varkappa:\mathcal{F}\longrightarrow H_0$ if $\mathcal{F}$ is endowed with the Mackey topology and H_0 with the norm topology. It follows that ${}^t\varkappa:\mathcal{F}\longrightarrow\widehat{H_0}$ is continuous with dense image hence by transposition that $i=\varkappa:\widehat{H_0}\longrightarrow\mathcal{E}$ is injective. $H=\widehat{H_0}$ is the (unique) Hilbertian subspace with kernel $\varkappa$. □

The Mackey continuity of the semi-norm is then a sufficient condition, but it is also necessary by lemma 1.15. It is also obvious that the kernel of the constructed Hilbertian subspace is the one given by theorem 1.13.

It follows that the morphism $\Phi: H\in Hilb((\mathcal{E},\mathcal{F}))\mapsto\varkappa\in\mathbf{L}^+(\mathcal{F},\mathcal{E})$ is onto if and only if any semi-norm defined by a positive kernel is Mackey continuous.

Proposition 1.19 *If $\mathcal{F}$ endowed with its Mackey topology is barreled*[13]*, then any semi-norm is Mackey continuous and in particular any semi-norm defined by a positive kernel is Mackey continuous.*

If $\mathcal{E}$ is quasi-complete for its Mackey topology[14] *then any semi-norm defined by a positive*

[13] *See Appendix B*

[14] *This condition is weaker than the previous since the dual of a barreled space is always weakly (and hence Mackey) quasi-complete. However the barreleness of $\mathcal{F}$ is in general easier to verify.*

kernel is Mackey continuous.

Finally, if $\mathcal{F}$ is barreled or $\mathcal{E}$ Mackey quasi-complete, $\Phi : H \in Hilb((\mathcal{E},\mathcal{F})) \mapsto \varkappa \in \mathbf{L}^{+}(\mathcal{F},\mathcal{E})$ *is onto.*

The isomorphism of convex cones

Finally we can state the most important theorem of this chapter:

Theorem 1.20 ***Suppose $\mathcal{E}$ is quasi-complete (for its Mackey topology). Then there is a bijection between $Hilb((\mathcal{E},\mathcal{F}))$ and $\mathbf{L}^{+}(\mathcal{F},\mathcal{E})$. Moreover, this bijection is an isomorphism of convex cones.***

Figure 1.3 represents the convex cone of positive kernels of the Euclidean space $\mathbb{R}^2$ embedded in the 3 dimensional space of self-adjoint kernels, hence by the isomorphism the convex cone of Hilbertian subspaces of $\mathbb{R}^2$.

We use the matrix representation of kernels (i.e. the kernel function)

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

with $A_{2,1} = A_{1,2}$.

Any reader particularly interested by the isomorphism of convex cone structure can read [46] p 159-161, where the proof is detailed.

We can illustrate this isomorphism by some examples involving the previous kernels seen at the end of the last section:

example 1 **$\mathbb{R}^2$-example**

Let $(\mathcal{E},\mathcal{F}) = (\mathbb{R}^2,\mathbb{R}^2)$ in Euclidean duality. The kernel of the Hilbertian subspace $\mathbb{R}^2$ endowed with the scalar product

$$\langle Y|X\rangle = x_1y_1 + \frac{1}{2}x_2y_2$$

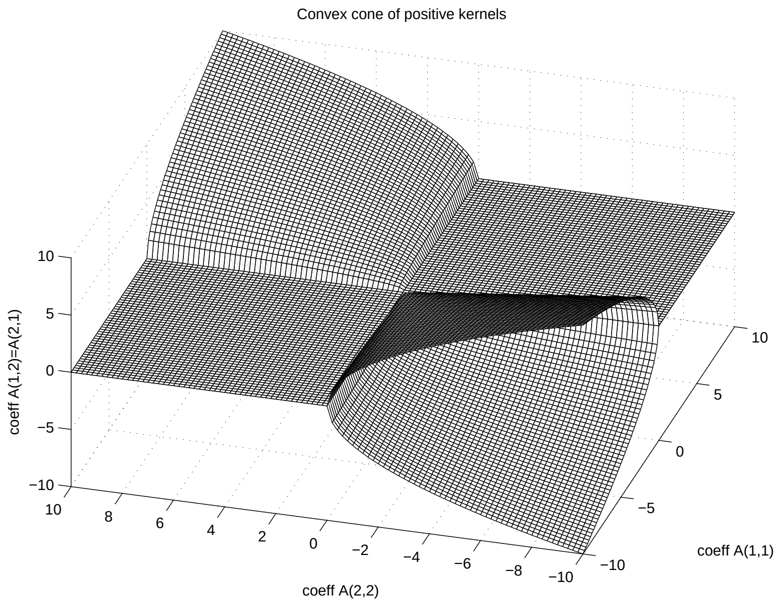

Figure 1.3: Convex cone of positive kernels of $\mathbb{R}^2$ isomorphic to the convex cone of Hilbertian subspaces of $\mathbb{R}^2$

can be identified with the matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

i.e.

$$\begin{aligned} \varkappa : \mathbb{R}^2 & \longrightarrow \mathbb{R}^2 \\ Y = (y_1, y_2) & \longmapsto K.Y = (y_1, 2y_2) \end{aligned}$$

example 2 **Sobolev spaces** **(- Cameron-Martin space -)**

Let $\mathcal{E} = D'$ be the space of distribution on the open set $\Omega =]0, 1[$. The Hilbert space

$$W^1(\Omega) = \left\{ f \in D',\ f(t) = \int_\Omega \mathbb{1}_{s \leq t} \phi(s) ds,\ \phi \in L^2(\Omega) \right\}$$

is a Hilbertian subspace of D' since the canonical injection is continuous. **Its kernel is the opposite of the operator of second order derivation from D in D'**, i.e. the integral operator with kernel $K(t,s) = \min(t,s)$:

$$\forall t \in \Omega, \forall \phi \in D, \ \varkappa(\phi)(t) = \int_{\Omega} \min(t,s)\phi(s)ds$$

This space is sometimes called the **Cameron-Martin space** (of the Wiener measure, [17]), and the kernel function $K(t,s) = \min(t,s)$ is the covariance of the Wiener measure. The link between Gaussian measures and Hilbertian subspaces will be detailed in the last chapter "Applications", section 4.1: From Gaussian measures to Boehmians (generalized distributions) and beyond).

example 3 Suppose $\mathcal{E} = \mathcal{F}$ is a Hilbert space and let $\varkappa$ be a positive homomorphism of $\mathcal{E}$. Then $H_0 = \varkappa(\mathcal{E})$ is a closed subspace of $\mathcal{E}$, it is then a Hilbert space, the Hilbertian subspace of $\mathcal{E}$ associated with $\varkappa$.

example 4 Let $\mathcal{E} = D'$ be the space of distribution on an open set Ω of $\mathbb{K}^n$. The Hilbert space $L^2(\Omega)$ is a Hilbertian subspace of D' since the canonical injection is continuous. Its kernel is the canonical injection of D in D' ($K(t,s) = \delta_t(s)$):

$$\forall t \in \Omega, \forall \phi \in D, \ \varkappa(\phi)(t) = \int_{\Omega} \delta_t(s)\phi(s)ds = \phi(t)$$

example 5 Let $b \in L_1(\mathbb{R}), \ b > 0$. Then the positive symmetric separately continuous bilinear form

$$\begin{array}{rcl} L : L_\infty(\mathbb{R}) \times L_\infty(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ (\psi, \phi) & \longmapsto & \int_{\mathbb{R}} b(t)\phi(t)\psi(t)dt \end{array}$$

is associated to the positive kernel

$$\begin{array}{rcl} \varkappa : L_\infty(\mathbb{R}) & \longrightarrow & L_1(\mathbb{R}) \\ \psi \longmapsto & & b.\psi \end{array}$$

and the Hilbertian subspace of $(L_1(\mathbb{R}), L_\infty(\mathbb{R}))$ associated to $\varkappa$ is $H = \left\{ f, \frac{f}{\sqrt{b}} \in L_2 \right\}$ endowed with the scalar product:

$$\langle \psi | \phi \rangle_H = \int_{\mathbb{R}} \frac{\phi(t)\psi(t)}{b(t)} dt$$

1.2.4 Other characterizations of the Hilbertian subspace associated to a kernel

We have seen in the previous section that the Hilbertian subspace H associated to the kernel $\varkappa$ is the completion (in $\mathcal{E}$) of $H_0 = \varkappa(\mathcal{F})$ with respect to the scalar product induced by the positive kernel. This is a useful abstract result but since $H \subset \mathcal{E}$ it is natural to wonder if there exists other simpler criteria to establish whether a particular vector $f \in \mathcal{E}$ lies in the Hilbert space H or not. It is the aim of this section to study such criteria.

Proposition 1.21 *Let $\varkappa$ be the Hilbertian kernel of a Hilbertian subspace H of $(\mathcal{E}, \mathcal{F})$. Let $BO(1)$ be the open unit ball of the prehilbertian space $H_0 = \varkappa(\mathcal{F})$. Then $H = \bigcup_{\lambda \in \mathbb{R}^+} \lambda \overline{BO(1)}$ where the closure is the weak closure in $\mathcal{E}$.*

Proof. – H is the completion of H_0, $H = \bigcup_{\lambda \in \mathbb{R}^+} \lambda \overline{BO(1)}$ where the closure is taken with respect to the Hilbertian norm in H. However $\overline{BO(1)}$ is weakly compact in H as the unit ball of a Hilbert space, then weakly compact in $\mathcal{E}$ and finally weakly closed in $\mathcal{E}$. The *a priori* two different notions of closure coincides for the open unit ball. □

Proposition 1.22 *Let $\varkappa$ be the Hilbertian kernel of a Hilbertian subspace H of $(\mathcal{E}, \mathcal{F})$. Then for any element $h \in \mathcal{F}^*$ (algebraic dual of $\mathcal{F}$ or weak-completion of $\mathcal{E}$) we have the following equivalence:*

$$h \in H \iff \sup_{\varphi \in \mathcal{F}} \frac{\left(\varphi, \overline{h}\right)_{(\mathcal{F}, \mathcal{F}^*)}}{\left(\varphi, \overline{\varkappa(\varphi)}\right)_{(\mathcal{F}, \mathcal{F}^*)}} \leq \infty$$

In this case, the supremum is the norm of h in H.

Proof. – Evident since the topological dual of $H_0 = \varkappa(\mathcal{F})$ dense in H endowed with the norm-topology is identified with H by the scalar product. □

1.3 Hilbertian functionals and Hilbertian kernels

The material of this section deals with convex analysis. It will not be used afterwards and can be skipped at first reading.

In [8], M. Atteia interprets Hilbertian subspaces in terms of quadratic functionals (he calls them Hilbertian functionals). He follows J.J. Moreau's work [41] who proved that any Hilbertian kernel is the subdifferential of a strictly convex quadratic functional. This will led to the general theory of Banachic kernels [9] we do not investigate here. We refer to [8] for the proofs.

1.3.1 Hilbertian functional of a Hilbertian subspace

Definition 1.23 (– *Hilbertian functional* –) *Let $(\mathcal{E}, \mathcal{F})$ be a duality.*

$$\mathcal{J} : \mathcal{E} \longrightarrow \overline{\mathbb{R}}$$

is a Hilbertian functional (of $(\mathcal{E}, \mathcal{F})$) if:

1. *$dom\mathcal{J} = \{\varepsilon \in \mathcal{E},\ \mathcal{J}(\varepsilon)\}$ is a vector subspace of $\mathcal{E}$;*

2. *$\mathcal{J}$ is quadratic over $dom\mathcal{J}$;*

3. *$\mathcal{J}$ is strictly convex;*

4. *$\{\varepsilon \in \mathcal{E},\ \mathcal{J}(\varepsilon) \leq 1\}$ is weakly compact in $\mathcal{E}$.*

We note the set of Hilbertian functionals of $(\mathcal{E}, \mathcal{F})$ $h\mathcal{F}((\mathcal{E}, \mathcal{F}))$.

To each Hilbertian subspace H of $(\mathcal{E}, \mathcal{F})$ we can associate a functional $\mathcal{J} : \mathcal{E} \longrightarrow \overline{\mathbb{R}}$ defined by:

$$\begin{array}{rcl} \mathcal{J} : \mathcal{E} & \longrightarrow & \overline{\mathbb{R}} \\ \varepsilon & \longmapsto & \frac{1}{2}\sup_{\varphi\in\mathcal{F}} \frac{\left(\varphi,\overline{h}\right)_{(\mathcal{F},\mathcal{F}^*)}}{\left(\varphi,\overline{\varkappa(\varphi)}\right)_{(\mathcal{F},\mathcal{F}^*)}} \end{array}$$

Proposition 1.24 *The previous function is a bijection from $Hilb((\mathcal{E}, \mathcal{F}))$ onto the set of Hilbertian functional of $(\mathcal{E}, \mathcal{F})$ $h\digamma((\mathcal{E}, \mathcal{F}))$.*

Moreover, we can endow $h\digamma((\mathcal{E}, \mathcal{F}))$ with an addition law, external multiplication law by positive real numbers and an order relation that gives to this set the structure of convex cone. Precisely:

addition law: inf-convolution

Let $\mathcal{J}_1,\ \mathcal{J}_2 \in h\digamma((\mathcal{E}, \mathcal{F}))$. Then

$$\forall\varepsilon \in \mathcal{E}, \mathcal{J}_1 + \mathcal{J}_2(\varepsilon) = \inf_{\varepsilon=\varepsilon_1+\varepsilon_2}(\mathcal{J}_1(\varepsilon_1) + \mathcal{J}_2(\varepsilon_2))$$

Remark that this operation is commutative and associative, i.e. a true addition law.

external multiplication law: outer quotient

Let $\mathcal{J} \in h\digamma((\mathcal{E}, \mathcal{F}))$: we define for all $\lambda \in \mathbb{R}+$ the functional $\mathcal{J}_{:\lambda}$ by

$$\forall\varepsilon \in \mathcal{E}, \mathcal{J}_{:\lambda}(\varepsilon) = \lambda\mathcal{J}(\frac{\varepsilon}{\lambda})$$

order relation

We define the order relation by $\mathcal{J}_1 \leq \mathcal{J}_2 \iff \{\forall\varepsilon \in \mathcal{E},\ \mathcal{J}_1(\varepsilon) \leq \mathcal{J}_2(\varepsilon)\}$

Structure of a convex cone

Finally, we verify that these laws and order relation are compatible with the structure of a convex cone, notably

$$(\mathcal{J}_1 + \mathcal{J}_2)_{:\lambda} = (\mathcal{J}_1)_{:\lambda} + (\mathcal{J}_2)_{:\lambda}$$

$$\mathcal{J}_{:(\lambda+\mu)} = \mathcal{J}_{:\lambda} + \mathcal{J}_{:\mu}$$

Theorem 1.25 *The previous function is an isomorphism of convex cones between $Hilb((\mathcal{E},\mathcal{F}))$ and $h\mathcal{F}((\mathcal{E},\mathcal{F}))$. If $\mathcal{E}$ is quasi-complete for its Mackey topology, then $h\mathcal{F}((\mathcal{E},\mathcal{F}))$ is also isomorphic to $\mathbf{L}^+(\mathcal{F},\mathcal{E})$.*

The question is: can we characterize this isomorphism directly (without exhibiting the Hilbertian subspace)?
The answer is positive and investigated in the next section.

1.3.2 Hilbertian kernels as subdifferential of Hilbertian functionals

It is classical in convex analysis do define the dual functional $\mathcal{J}^* : \mathcal{F} \longrightarrow \overline{\mathbb{R}}$:

$$\forall \varphi \in \mathcal{F},\ \mathcal{J}^*(\varphi) = \sup_{\varepsilon\in\mathcal{E}} \Big((\varphi,\varepsilon)_{(\mathcal{F},\mathcal{E})} - \mathcal{J}(\varepsilon) \Big)$$

The dual functional of a Hilbertian functional actually holds remarkable properties:

Proposition 1.26 *Let $\mathcal{J}$ be a Hilbertian functional of $(\mathcal{E},\mathcal{F})$. Then $\mathcal{J}^*$ is a Hilbertian functional of $(\mathcal{F},\mathcal{E})$.*
If $\mathcal{J}$ be the Hilbertian functional associated to H Hilbertian subspace of $(\mathcal{E},\mathcal{F})$, then

1. $\forall \varphi \in \mathcal{F},\ \mathcal{J}^*(\varphi) = \frac{1}{2}(\varphi, \varkappa(\varphi))_{(\mathcal{F},\mathcal{E})}$

2. $\partial \mathcal{J}^* = \varkappa$

where $\partial\mathcal{J}^*$ is the subdifferential of $\mathcal{J}^*$,

$$\partial\mathcal{J}^*(\varphi_0) = \Big\{\varepsilon \in \mathcal{E}, \forall\varphi \in \mathcal{F},\ \mathcal{J}^*(\varphi) - \mathcal{J}^*(\varphi_0) \geq (\varphi - \varphi_0, \varepsilon)_{(\mathcal{F},\mathcal{E})}\Big\}$$

Consequently this section gives another interesting characterization of Hilbertian subspaces and Hilbertian kernels in terms of Hilbertian functionals. The arguments of this section will however collapse in the next chapters since no convex functional can be associated with Krein subspaces or subdualities.

1.4 Reproducing kernel Hilbert spaces

Reproducing kernel Hilbert spaces (in short r.k.h.s.) are the Hilbertian subspaces of $\mathcal{E} = \mathbb{K}^\Omega$ endowed with the product topology, or topology of simple convergence, where Ω is any set. They are thus a special case of Hilbertian subspace and it is natural to wonder why we should treat the case especially. There are many reasons for that, the first being historical: the study of Hilbertian subspaces and their kernels stems from the works of Bergman [13] and Aronszajn [6] in the framework of reproducing kernel Hilbert spaces and reproducing kernel functions in the beginning of the 50s and it is only later (with the work of Schwartz in 1964 [46]) that the notion of Hilbertian subspace emerges. The second reason is that the study of reproducing kernel Hilbert spaces is “universal” in the sense that any Hilbertian subspace may be seen as a r.k.h.s. by the injective mapping:

$$\begin{aligned} \Theta : \mathcal{E} &\longrightarrow \Theta(\mathcal{E}) \subseteq \mathbb{K}^{\mathcal{E}^*} \\ \varepsilon &\longmapsto \{(\varepsilon, \varepsilon')\}_{\{\varepsilon' \in \mathcal{E}^*\}} \end{aligned}$$

(This may be found in [46] or [8] for instance). These spaces are also very interesting for applications such as approximation or estimation since one deals with genuine functions, or for the study of Gaussian stochastic processes. Finally, there is one last reason for paying attention to r.k.h.s.: the locally convex space $\mathcal{E} = \mathbb{K}^\Omega$ and its dual space have good topological and algebraic properties.

1.4.1 The space $\mathbb{K}^\Omega$

Let Ω be any set and $\mathcal{E} = \mathbb{K}^\Omega$ the space of scalar functions on Ω, endowed with the product topology, or topology of simple convergence. Then its dual space $\mathcal{E}' = (\mathbb{K}^\Omega)'$ is the space of measures with finite support on Ω, i.e. $\forall \mu \in (\mathbb{K}^\Omega)', \mu = \sum_{t\in\Omega} a_t\delta_t$ where δ_t is the Dirac measure on Ω and the $a_t \in \mathbb{K}$ are null except a finite number of them. The duality product is

$$(\mu, \phi)_{((\mathbb{K}^\Omega)', \mathbb{K}^\Omega)} = \sum_{t\in\Omega} a_t\phi(t)$$

We may now cite some interesting results about $\mathbb{K}^\Omega$:

Proposition 1.27 *Let Ω be any set, $\mathcal{E} = \mathbb{K}^\Omega$ the space of scalar functions on Ω endowed with the product topology and $(\mathbb{K}^\Omega)'$ its dual space endowed with the Mackey topology.*

1. *The spaces $\mathbb{K}^\Omega$ and $(\mathbb{K}^\Omega)'$ are barreled and nuclear;*

2. *any linear application from $(\mathbb{K}^\Omega)'$ into any topological vector space is weakly continuous;*

3. $\mathbf{L}((\mathbb{K}^\Omega)', \mathbb{K}^\Omega) \cong$ [15] $\mathbb{K}^{\Omega\times\Omega}$.

Proof. –

1. A product of nuclear spaces is nuclear (proposition 50.1 in [50]) and therefore $\mathbb{K}^\Omega$ is nuclear. It is barreled as the product of barreled spaces. But $\mathbb{K}^\Omega$ is also reflexive and it follows that its dual is also barreled and nuclear.

2. It is necessary and sufficient to prove that any linear form is weakly continuous, since $u(\mathbb{K}^\Omega)' \longrightarrow F$ is weakly continuous if and only if $\forall\varphi' \in F',\ \varphi' \circ u$ is a weakly continuous linear form (proposition 24 in [26]). Let u be a linear form. Then u is entirely defined by its action on the δ_t, $t \in \Omega$ and defines a unique function $\tilde{u}(t) = u(\delta_t) \quad t \in \Omega$.

[15] *is isomorph to*

3. The wanted isomorphism is given by $[u(\delta_t)](s) = \tilde{u}(t,s)\,\forall t, s \in \Omega$. □

Remark 1.28 *In some works (for instance [10]), authors identify the space of measure with finite support with the space of functions null except on a finite number of points $\mathcal{F} = \mathbb{K}^{[\Omega]}$:*

$$\mu = \sum_{t\in\Omega} a_t \delta_t \in (\mathbb{K}^{\Omega})' \longmapsto f_\mu(.) = \sum_{t\in\Omega} a_t \delta_{t=s} \in \mathbb{K}^{[\Omega]}$$

The duality of interest is then $(\mathbb{K}^{\Omega}, \mathbb{K}^{[\Omega]})$.

1.4.2 r.k.h.s.

Definition 1.29 *A Hilbert space H is a reproducing kernel Hilbert space (in short r.k.h.s.) if there exists a set Ω, H is a Hilbertian subspace of $\mathcal{E} = \mathbb{K}^{\Omega}$ endowed with the product topology (topology of simple convergence).*

This definition presupposes the knowledge of Hilbertian subspaces. Since r.k.h.s. are anterior to Hilbertian subspaces, this is of course not the way reproducing kernel Hilbert spaces were introduced. The next proposition gives a list of equivalences that may be taken (and have been taken in many works) for definitions.

Proposition 1.30 *Let $H \subset \mathbb{K}^{\Omega}$ be a Hilbert space. The following statements are equivalent:*

1. *H is a r.k.h.s.*
2. *The canonical injection from H into $\mathbb{K}^{\Omega}$ is weakly continuous.*
3. $\forall t \in \Omega, \exists M_t, \forall h \in H, \quad |h(t)| \leq M_t \|h\|_H.$
4. $\forall s \in \Omega, \exists K_s \in H, \forall h \in H, \quad \langle K_s | h \rangle_H = h(s).$
5. $\exists K \in \mathbb{K}^{\Omega\times\Omega}, K(t,s) = \langle K(.,s) | K(t,.)\rangle_H$

The last statement is known as the reproduction property and the word "reproducing kernel" stems from this statement. We have the following relation between the reproducing kernel K and the Schwartz kernel $\varkappa$:

Theorem 1.31 *Let H be a r.k.h.s. in $\mathbb{K}^{\Omega}$. Then its reproducing kernel K is the image of its Schwartz kernel $\varkappa$ under the isomorphism between $\mathbf{L}((\mathbb{K}^{\Omega})', \mathbb{K}^{\Omega})$ and $\mathbb{K}^{\Omega\times\Omega}$ (proposition 1.27):*

$$\forall t, s \in \Omega, \qquad K(t,s) = [\varkappa(\delta_t)](s)$$

example 1 **$\mathbb{R}^2$-example**

Let $\Omega = \{1, 2\}$. Then the Hilbertian space $\mathbb{R}^2$ endowed with the scalar product

$$\langle Y|X\rangle = x_1 y_1 + \frac{1}{2} x_2 y_2$$

is a reproducing kernel Hilbert space of $\mathbb{R}^{\Omega}$ and its kernel function can be identified with the matrix

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

with

$$\forall (i,j) \in \Omega \times \Omega, \quad K(i,j) = K_{i,j}$$

example 2 **Sobolev spaces** **(- Cameron-Martin space -)**

Let $\mathcal{E} = D'$ be the space of distribution on the open set $\Omega =]0, 1[$. The Hilbert space $W^1(\Omega) = \left\{ f \in D',\ f(t) = \int_{\Omega} 1\!\!1_{s \leq t} \phi(s) ds,\ \phi \in L^2(\Omega) \right\}$ is a reproducing kernel Hilbert space on Ω. Its kernel function is $K(t,s) = \min(t,s)$. It verifies the reproducing property:

$$\langle \min(., s) | \min(t, .)\rangle_{W^1(\Omega)} = \int_{\Omega} 1\!\!1_{\{u \leq s\}} 1\!\!1_{\{u \leq t\}} du = \min(t,s)$$

1.4.3 Reproducing kernels

Functions of two variables have been investigated for a long time and the notions of hermicity and positivity have been defined long before the isomorphism between $\mathbf{L}((\mathbb{K}^\Omega)', \mathbb{K}^\Omega)$ and $\mathbb{K}^{\Omega\times\Omega}$ was known. We have that:

Proposition 1.32

1. *Any reproducing kernel is (Hermitian) positive.*
2. $\mathbf{L}^+((\mathbb{K}^\Omega)', \mathbb{K}^\Omega) \cong \mathbb{K}_+^{\Omega\times\Omega}$.

Following proposition 1.16, we have the following:

Proposition 1.33

$$H_0 = Span\,\{K_t,\ t \in \Omega\}$$

$$H = \widehat{H_0}$$

Remark 1.34 *In the r.k.h.s. setting, the fact that* $\widehat{H_0} \subset \mathbb{K}^\Omega$ *is investigated in Aronszajn's paper [6] in a self-contained manner. Prehilbertian subspaces enjoying this property are said to admit a functional completion. It states that a prehilbertian subspace of* $\mathbb{K}^\Omega$ *admits a functional completion (equivalent to the one of Schwartz, theorem 1.5) is: for a Cauchy sequence* $\{h_m,\, m \in \mathbb{N}\}$, $h_m \to 0 \Rightarrow \|h_m\| \to 0.$

example 1 **Polynomials, splines**

The two variable function on $\Omega = \mathbb{R}$ defined by

$$K(t,s) = (1+ts)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^j s^j$$

is positive as a positive linear combination of positive kernels. Its associated r.k.h.s. is the finite -dimensional Hilbert space of univariate polynomials of degree n $H = \mathcal{P}_n$.

example 2 **Polynomials, splines**

The function

$$K(t,s) = (t-s)^n = \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} t^j s^{n-j}$$

is not positive. It is not the reproducing function of a r.k.h.s. According to [23] it is however a classical function associated to $\mathcal{P}_n$. We will see how in our second chapter for n even and in our third chapter ("Subdualities") for any n.

1.5 Transport of structure, categories and construction of Hilbertian subspaces

Previously, the set of Hilbertian subspaces has been shown to have many good properties, one being its structure of a convex cone. A more significant result is that the set of Hilbertian subspaces can be endowed with the structure of a convex cone category isomorphic to the convex cone category of positive kernels.

This result is a consequence of the next theorems concerning the transport of structure by a weakly continuous linear application.

As an application of that result, a construction of Hilbertian subspaces is given. Other constructions of course exist, see for instance [12].

1.5.1 Transport of structure via a weakly continuous linear application

The general problem investigated in this section is as follows: what can we say of the image of an Hilbertian subspace by an operator?

In the case of one-to-one mapping, it is very easy to define a scalar product on the range of the operator. More precisely, let $(\mathcal{E}, \mathcal{F})$ and $(\mathfrak{E}, \mathfrak{F})$ be two dualities, $u : \mathcal{E} \longrightarrow \mathfrak{E}$ a linear application. We suppose moreover that u is one-to-one. Then we can state the following lemma:

Lemma 1.35 *Let H be a Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$ and $\mathbf{H} = u(H)$. We can endow $\mathbf{H}$ with an inner product such that $u_H : H \longrightarrow \mathbf{H}$ is an isometry (and $\mathbf{H}$ is a Hilbert space):*

$$\langle u(h_1)|u(h_2)\rangle_{\mathbf{H}} = \langle h_1|h_2\rangle_H$$

$\mathbf{H}$ is a Hilbertian subspace of $(\mathfrak{E}, \mathfrak{F})$ if and only if u is weakly continuous.

Proof. – Let $j_{\mathbf{H}} : \mathbf{H} \longrightarrow \mathfrak{E}$ be the canonical injection. Then $j_{\mathbf{H}} = u \circ j_H \circ^t u_H$ which is weakly continuous if and only if u is weakly continuous. □

Suppose now that $u : \mathcal{E} \longrightarrow \mathfrak{E}$ a weakly continuous linear application, but without the injectivity property. Does the previous result hold? It holds actually, thanks to orthogonal decompositions in Hilbert spaces. Since u is weakly continuous, $Ker(u)$ is weakly closed and H admits the following orthogonal decomposition:

$$H = M \oplus \ker(u)$$

where M is a Hilbert space isomorphic to $H/\ker(u)$. The restriction of u to M is then one-to-one and we can use the result of theorem 1.35:

Theorem 1.36 *Let $(\mathcal{E}, \mathcal{F})$ and $(\mathfrak{E}, \mathfrak{F})$ be two dualities, $u : \mathcal{E} \longrightarrow \mathfrak{E}$ a weakly continuous linear application. Let H be a Hilbertian subspace of $\mathcal{E}$. Then $\mathbf{H} = u(H)$ may be endowed with the structure of a Hilbert space isomorphic to $H/\ker(u)$ that makes it a Hilbertian subspace of $(\mathfrak{E}, \mathfrak{F})$.*

This theorem appears for the first time in [46] and can be applied directly to construct Hilbertian subspaces.

$u(H)$ is then a Hilbertian subspace of $(\mathfrak{E}, \mathfrak{F})$ and therefore has a kernel (theorem 1.13). What can we say about the image kernel of the Hilbertian subspace $u(H)$? Under the notations of the previous theorem, the following proposition holds:

Proposition 1.37 *Let $\varkappa$ be the Hilbertian kernel of H. Then the Hilbertian kernel of $\mathbf{H} = u(H)$ is $\Phi(\mathbf{H}) = u \circ \varkappa \circ^t u$.*

Proof. – The proof is obvious since $j_{\mathbf{H}} = u \circ j_M \circ^t u_M$ and u_M is an isometry. □

1.5.2 Categories and functors

This section presupposes some knowledge about categories and convex cones and can be skipped at first reading.

Let $\mathcal{C}$ be the category of dual systems $(\mathcal{E}, \mathcal{F})$, $\mathcal{E}$ Mackey quasi-complete, the morphisms being the weakly continuous linear applications. Let $\mathcal{G}$ be the category of salient and regular convex cones, the morphisms being the applications preserving multiplication by positive scalars and addition (hence order). Then **Theorem 1.36 allows us to see** $Hilb : (\mathcal{E}, \mathcal{F}) \mapsto Hilb((\mathcal{E}, \mathcal{F}))$ **as a functor of categories** according that to a morphism $u : \mathcal{E} \longrightarrow \mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl} \tilde{u} : Hilb((\mathcal{E}, \mathcal{F})) & \longrightarrow & Hilb((\mathfrak{E}, \mathfrak{F})) \\ H & \longmapsto & u(H) \end{array}$$

Theorem 1.38 *$Hilb : (\mathcal{E}, \mathcal{F}) \mapsto Hilb((\mathcal{E}, \mathcal{F}))$ is a covariant functor of category $\mathcal{C}$ into category $\mathcal{G}$.*

On the other hand, $\mathbf{L}^+ : (\mathcal{E}, \mathcal{F}) \mapsto \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ is also a covariant functor of category $\mathcal{C}$ into category $\mathcal{G}$, according that to a morphism $u : \mathcal{E} \longrightarrow \mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl} \tilde{u} : \mathbf{L}^+(\mathcal{F}, \mathcal{E}) & \longrightarrow & \mathbf{L}^+(\mathfrak{F}, \mathfrak{E}) \\ \varkappa & \longmapsto & u \circ \varkappa \circ^t u \end{array}$$

and from the isomorphism of the convex cones $Hilb((\mathcal{E},\mathcal{F}))$ and $\mathbf{L}^+(\mathcal{E},\mathcal{F})$ (theorem 1.20)we deduce that:

Theorem 1.39 *The two covariant functors $Hilb$ and $\mathbf{L}^+$ are isomorphic.*

1.5.3 Application to the construction of Hilbertian subspaces

There are many works on the construction of Hilbertian subspaces through linear operators (for instance [45]), but none of them makes reference to the previous theorem of L. Schwartz, whereas they use it implicitly.

Corollary 1.40 (– *construction of Hilbertian subspace* –) *Let $(\mathfrak{E},\mathfrak{F})$ be a duality, H a Hilbert space, $u : H \longrightarrow \mathfrak{E}$ a weakly continuous linear application. H may be seen has a Hilbertian subspace of itself and using theorem 1.36, it follows that $\mathbf{H} = u(H)$ may be endowed with the structure of a Hilbert space isomorphic to $H/\ker(u)$ that makes it a Hilbertian subspace of $(\mathfrak{E},\mathfrak{F})$. Its kernel is then the positive operator $u \circ u^*$.*

Note that any Hilbertian subspace $\mathbf{H}$ may be constructed in this way by taking $H = \mathbf{H}$ and $u = i$ the canonical injection (that is weakly continuous by proposition 1.3).

example 1 **$\mathbb{R}^2$-example**

Let $H = \mathbb{R}^2$ endowed with the Euclidean inner product and

$$\mathcal{E} = \mathbb{R}.\cos(\frac{\pi}{2}t) + \mathbb{R}.\sin(\frac{\pi}{2}t)$$

(subspace of $\mathcal{F}([0,1],\mathbb{R})$) in self-duality with respect to the bilinear form

$$\begin{aligned} L : \mathcal{F} = \mathcal{E}\times\mathcal{E} &\longrightarrow \mathbb{R} \\ (\varphi,\varepsilon) &\longmapsto -\varphi(0)\varepsilon(1) + \varphi(1)\varepsilon(0) \end{aligned}$$

Then the weakly continuous linear mapping

$$\begin{aligned} u : \mathbb{R}^2 &\longrightarrow \mathcal{E} \\ X = (x_1,x_2) &\longmapsto u(X) = x_1\cos(t) + x_2\sin(t) \end{aligned}$$

defines a Hilbertian subspace of $\mathcal{E}\ u(\mathbb{R}^2)$ with kernel

$$\begin{aligned} \varkappa : \mathcal{F} = \mathcal{E} &\longrightarrow \mathcal{E} \\ \varphi &\longmapsto \varkappa(\varphi)(t) = \varphi(1)\cos(\tfrac{\pi}{2}t) - \varphi(0)\sin(\tfrac{\pi}{2}t) = \tfrac{2}{\pi}\varphi'(t) \end{aligned}$$

i.e; the derivation operator since

$$\begin{aligned} u^* : \mathcal{F} &\longrightarrow \mathbb{R}^2 \\ \varphi &\longmapsto (\varphi(1), -\varphi(0)) \end{aligned}$$

example 2 **Sobolev spaces**

Let $\mathcal{E} = D'$ be the space of distribution on an open set Ω of $\mathbb{R}$. The Hilbertian subspace of D' associated to the kernel

$$\begin{aligned} \varkappa : D &\longrightarrow D' \\ \phi &\longmapsto -\tfrac{d^2}{dt^2}\phi \end{aligned}$$

is just the Hilbert space $P(L^2)$, where

$$\begin{aligned} P : L^2 &\longrightarrow D' \\ \phi &\longmapsto \tfrac{d}{dt}\phi \end{aligned}$$

since $P \circ P^* = \varkappa$. It is the Sobolev space W^{-1} (remark that the orthogonal of $\ker(P)$ is the subset of L^2 functions that sum to 0).

example 3 **Sobolev spaces** **(- Dual space of the Cameron-Martin space -)**

There exists an other characterization of the space W^{-1} based on the theory of normal subspaces. Let W^1 be the previously defined Cameron-Martin space (with reproducing kernel $K(t,s) = min(t,s)$). It is a classical result that D is dense in W^1. If we note i the canonical injection, it follows that ${}^t i : W^{1'} \longrightarrow D'$ is injective. The image of the Hilbert space $W^{1'}$ by ${}^t i$ is then a Hilbertian subspace of (D', D). It is precisely the space W^{-1} with kernel $\varkappa = \frac{d^2}{dt^2}$. Remark that $K(t,s)$ is precisely the Green's function associated to $\varkappa$[16].

[16] There exists actually a general result of this type, see chapter 4 "Applications"

<u>example</u> 4 Let $b \in L^1(\mathbb{R}),\ b > 0$ and define

$$\begin{aligned} u : L^2(\mathbb{R}) &\longrightarrow \quad L^1(\mathbb{R}) \\ \phi &\longmapsto \quad \sqrt{b}.\phi \end{aligned}$$

Its adjoint is

$$\begin{aligned} u^* : L^\infty(\mathbb{R}) &\longrightarrow \quad L^2(\mathbb{R}) \\ \psi &\longmapsto \quad \sqrt{b}.\psi \end{aligned}$$

and $H = u(L^2) = \sqrt{b}L^2$ is the Hilbertian subspace of (L^1, L^∞) associated to the positive symmetric kernel $\varkappa = uu^*$ i.e.

$$\begin{aligned} \varkappa : L_\infty(\mathbb{R}) &\longrightarrow \quad L_1(\mathbb{R}) \\ \psi &\longmapsto \quad b.\psi \end{aligned}$$

1.5.4 The special case of r.k.h.s.

When dealing with reproducing kernel Hilbert spaces, it appears that weakly continuous linear mapping have a special form, which allows us to reformulate the previous construction. Weakly continuous linear mapping with range in $\mathbb{K}^\Omega$ hold a very special property, for they may be represented by a family of linear form indexed by a parameter t in Ω:

Theorem 1.41 $u : \mathcal{E} \longrightarrow \mathbb{K}^\Omega$ *is a weakly continuous linear operator if and only if*

$$\exists \left\{\Gamma_t \in \mathcal{F},\ t \in \Omega\right\}, \quad [u(\phi)](.) = (\Gamma_., \phi)_{(\mathcal{F},\mathcal{E})}$$

Proof. – $u : \mathcal{E} \longrightarrow \mathbb{K}^\Omega$ is a weakly continuous linear operator if and only if $\forall t \in \Omega, \quad \delta_t \circ u \in \mathcal{E}'$ (proposition 24 in [26]) and the theorem is proved. □

It follows that weakly continuous linear mapping from a Hilbert space with range in $\mathbb{K}^\Omega$ are represented by a family of functions from the Hilbert space indexed by a parameter t in Ω:

Corollary 1.42 *$u : H \longrightarrow \mathbb{K}^\Omega$ is a weakly continuous linear operator if and only if*

$$\exists \{\Gamma_t \in H,\ t \in \Omega\}, \quad [u(\phi)](.) = \langle \Gamma_. | \phi \rangle_H$$

If $H = L^2$, then such operators are known as Carleman operators (or Carleman integral operators) see for instance [49]. We can now state the main result concerning the construction of r.k.h.s.:

Corollary 1.43 (– *construction of r.k.h.s.* –) ,
Let H be a Hilbert space, $\{\Gamma_t \in H,\ t \in \Omega\}$ and $u : H \longrightarrow \mathbb{K}^\Omega$ the associated operator. Then $\mathbf{H} = u(H)$ may be endowed with the structure of a Hilbert space isomorphic to $H/\ker(u)$ that makes it a reproducing kernel Hilbert space and its kernel function is

$$\forall t, s \in \Omega \times \Omega, \qquad K(t,s) = \langle \Gamma_s | \Gamma_t \rangle_H$$

example 5 **$\mathbb{R}^2$-example**

$\Omega = \{1,2\}$. Define $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{pmatrix}$. Then $H = \Gamma(\mathbb{R}^2)$ is a Hilbertian subspace of $\mathbb{R}^2$ with scalar product

$$\langle X|Y \rangle = x_1 y_1 + \frac{1}{2} x_2 y_2$$

Its kernel function is identified with the matrix $K = \Gamma\Gamma^* = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

example 6 **Sobolev spaces**

Let Ω be an open set of $\mathbb{R}$ bounded from the left. Let Γ be the Carleman operator associated to the family $\Gamma_t(.) = 1\!\!1_{\{. \leq t\}}$. The previously defined Hilbert space $W^1(\Omega) = \left\{ f \in D',\ f(t) = \int_\Omega 1\!\!1_{s \leq t} \phi(s) ds,\ \phi \in L^2(\Omega) \right\}$ is then the reproducing kernel Hilbert space image of L^2 under the mapping Γ. Its kernel is

$$K(t,s) = \langle 1\!\!1_{\{. \leq s\}} | 1\!\!1_{\{. \leq t\}} \rangle_{L^2} = \min(t,s)$$

example 7 **(- Bergman kernel -)**

Let H be the $\mathbb{C}$-reproducing kernel Hilbert space on the unit disk, with kernel $K(z,w) = \frac{1}{1-z\overline{w}}$. Then $H = \Gamma(l^2[\mathbb{C}])$ where Γ is the operator induced by the family $\Gamma_z = \{1, z, z^2, ...\}$ since

$$\sum_{k\in\mathbb{N}} (z\overline{w})^k = \frac{1}{1-z\overline{w}}$$

i.e. the Hardy space $\mathbf{H}_2$.

example 8 **Polynomials, splines**

The finite-dimensional Hilbert space of univariate polynomials of degree n $H = \mathcal{P}_n$ with kernel

$$K(t,s) = (1+ts)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^j s^j$$

is the image of the Hilbert space $H = \mathbb{R}^n$ with scalar product

$$\langle Y|X\rangle_H = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x_j y_j$$

by the operator Γ associated to the family

$$\Gamma_t = \left(1, t, t^2, ..., t^n\right)^T$$

example 9 **Polynomials, splines**

We can construct in the same manner a r.k.h.s of $(k+1)$-multivariate homogeneous polynomials of degree n (see [23]):

Let d be the cardinal of $\{$multi-index $\beta,\ |\beta| = \beta_0 + ... + \beta_k = n\}$ and suppose we order this set $\{\beta^{(1)}, ..., \beta^{(d)}\}$.

The image of the Hilbert space $H = \mathbb{R}^d$ with scalar product

$$\langle Y|X\rangle_H = \sum_{j=0}^{d} \frac{n!}{\beta^{(d)}!} x_j y_j$$

by the operator Γ associated to the family

$$\Gamma_t = \left(1, t^{\beta^{(1)}}, t^{\beta^{(2)}}, ..., t^{\beta^{(d)}}\right)^T$$

where $t^\beta = t_0^{\beta_0}...t_k^{\beta_k}$ is a r.k.h.s, the space $H_n(\mathbb{R}^{k+1})$ of $(k+1)$-multivariate homogeneous polynomials of degree n with kernel

$$K(t,s) = \sum_{j=0}^{d} \frac{n!}{\beta^{(d)}!}(ts)^{\beta^d} = \sum_{|\beta|=n} \frac{n!}{|\beta|!} t^\beta s^\beta$$

Before closing this chapter, it may be interesting to have a brief historical review of some less known results concerning reproducing kernel Hilbert spaces.

The Moore reproducing property Kolmogorov's decomposition and the Kolmogorov's dilation theorem.

It is widely admit that the first results on reproducing kernel Hilbert spaces appeared in 1950 with the article of N. Aronszajn [6]. This is true if we consider it as the first investigation of the set of reproducing kernel Hilbert spaces and their properties. But it is less known that the first ones to notice the correspondence between positive kernels and Hilbert spaces were Moore [40] and Kolmogorov [31]. Their results are contained in the following theorems:

Theorem 1.44 (– Moore's "reproducing" property theorem –) *Let K be a positive definite kernel on a set Ω. Then there exists a functional Hilbert space $H \subset \mathbb{K}^\Omega$ such that*

$$\forall s \in \Omega,\ \forall h \in H, \quad \langle K(.,s)|h\rangle_H = h(s)$$

Theorem 1.45 (– Kolmogorov's decomposition, Kolmogorov's dilation theorem –) *Let K be a positive definite kernel on a set Ω. Then up to a unitary equivalence there exists a* *unique Hilbert space H and a unique embedding $V : \Omega \longrightarrow H$ such that:*

$$\forall (t,s) \in \Omega^2, \quad \langle V_s | V_t \rangle_H = K(t,s)$$

$$Span\{V_t,\ t \in \Omega\} \text{ is dense in } H$$

The pair (H, V) is called a Kolmogorov decomposition of the positive kernel K.

The r.k.h.s. with kernel K is then the image of H by the weakly continuous operator defined by the family $\{V(t) \in H,\ t \in \Omega\}$ (corollary 1.43).

Conclusion and comments

The basis of the Hilbertian subspace theory have been presented. To a great extent the results are drawn from L. Schwartz's paper "Sous espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés" [46]. The main difference is that we choose to deal with a duality $(\mathcal{E}, \mathcal{F})$ rather than with a locally convex space $\mathcal{E}$ and that we introduce the dualities $(E = H, F = \overline{H})$ and $(E = H, F = H)$. We see that one has to be very careful when using this approach but also that two injections are needed. This will be explained in the third chapter.

Some results not appearing in [46], notably those of section "Φ is onto" concerning the Mackey continuity of the semi-norm are due to C. Portenier [43]. The interpretation of Hilbertian kernels in term of Hilbertian (quadratic) functionals is due to M. Atteia [8] after the works of Moreau [41].

The choice made in this chapter is to present only the general theory of Hilbertian subspaces and the properties shared by all Hilbertian subspaces. A lot of papers deal with particular subspaces, Hilbertian subspaces of a specific space $\mathcal{E}$ or with applications of Hilbertian subspaces (in terms of Gaussian measures, approximation, differential equations, system theory, ...). They should all have [46] in their references.

Finally, the next chapters may be introduced as follows: how can we generalize the preceding notion of Hilbertian subspaces and what would the link with kernels be?

Chapter 2

Krein (Hermitian) subspaces, Pontryagin subspaces and admissible prehermitian subspaces

Introduction

This chapter generalizes the previous theory by giving up the positivity of the inner product while retaining its hermicity. Associated inner product spaces are no longer Hilbertian. As expected these inner spaces hold close relations with Hermitian (non necessarily positive) kernels.

We will follow three different paths and study the close relations between the different objects:

- the first is based on a theoretical result that gives the existence of an abstract vector space starting form a regular convex cone;

- the second directly starts from existing spaces related to Hilbert spaces called Krein spaces;

- the third is the most general and deals with the total set of inner product spaces.

Precisely, it has been shown previously that the set of Hilbertian subspace of a dual system can be endowed with the structure of a regular convex cone. Then the construction of the associated vector space of formal difference of two Hilbertian subspaces (noted $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$) is a classical abstract result and L. Schwartz's starting point for the study of what he called Hermitian subspaces. Moreover, the isomorphism of convex cones between Hilbertian subspaces and positive kernels will also extend to an isomorphism of vector spaces (the latter being the vector space of formal difference of two positive kernels $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$). But whereas this second vector space has a clear interpretation in terms of Hermitian kernels, the first had no interpretation in terms of vector subspaces at this time and it was L. Schwartz's purpose to give an interpretation of $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ that lead to Krein subspaces.

On the other hand, some mathematicians but mainly physicists had already seen the necessity of dealing with indefinite metrics to solve some problems outside the standard scheme of Hilbert spaces, notably those spaces that can be seen as the difference of two Hilbert spaces. These indefinite inner product spaces may be seen as the simplest generalization of Hilbert spaces and appear to be a good setting to perform L. Schwartz's program. This was for instance done in [4]. Those spaces got the name Krein spaces after the name of M. Krein [32][1].

We will see that the theory of Hilbertian subspaces extend to Krein subspaces with the latter being now associated to a subset of Hermitian but not necessarily positive kernels. However this extension leads to unexpected difficulties except for the subset of Pontryagin subspaces.

[1] L. Schwartz called them Hermitian spaces

Finally we go further into the abstraction and by following once again the work of L. Schwartz develop the theory of admissible prehermitian subspaces which is isomorphic under a quotient relation to the vector space of self-adjoint kernels $\mathbf{L}^*$. Moreover we interpret the algebraic requirements of L. Schwartz in topological terms making extensive use of dual systems. That will be the second huge step toward subdualities.

2.1 Extension of the isomorphism of convex cones to an isomorphism of (abstract) vector spaces

2.1.1 Construction of the vector spaces

Any regular convex cone C generates a real vector space, which we note $\mathbb{R} \otimes C$ and that is the vector space of formal differences of elements of C. An element of $\mathbb{R} \otimes C$ is then the equivalent class of elements of the form $c_+ - c_-$ with respect to the equivalence relation $\mathcal{R}_{cone}$

$$(c_+^1 - c_-^1)\mathcal{R}_{cone}(c_+^2 - c_-^2) \iff c_+^1 + c_-^2 = c_+^2 + c_-^1$$

Isomorphisms of convex cones extend to isomorphisms of vector spaces (moreover, the functorial character remains).

Example:

Figure 2.1 illustrates this construction. We have drawn four vectors verifying

$$c_+^1 + c_-^2 = c_+^2 + c_-^1$$

Moreover, we may interpret this class of equivalence as the vector

$$c = (c_+^1 - c_-^1) = (c_+^2 - c_-^2)$$

Such an interpretation is however not possible in general.

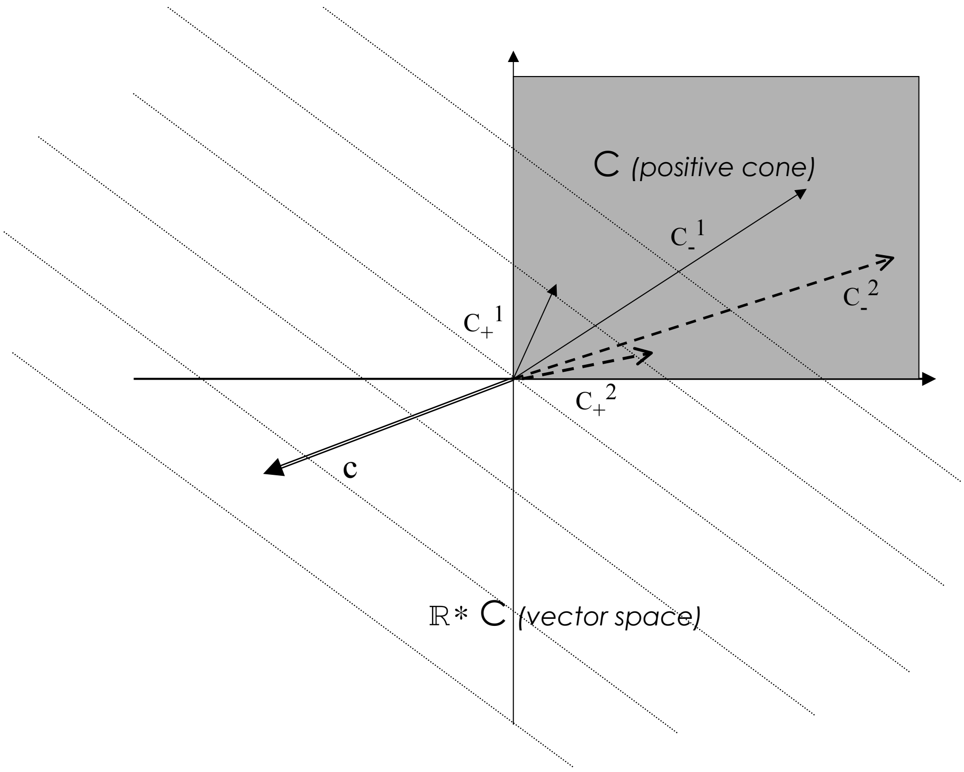

Figure 2.1: Real vector space generated by a regular convex cone

From now on and for the rest of this chapter, $(\mathcal{E}, \mathcal{F})$ is a duality such that $\mathcal{E}$ is Mackey quasi-complete. Then applying these abstract results to the convex cones of the first chapter "Hilbertian subspaces" $Hilb((\mathcal{E}, \mathcal{F}))$ and $\mathbf{L}^+(\mathcal{F}, \mathcal{E})$ we get that:

Theorem 2.1 *$\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ and $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ are two isomorphic vector spaces. We still note this isomorphism Φ*

$$\begin{aligned} \Phi : \mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F})) &\longrightarrow \mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E}) \\ H_+ - H_- &\longmapsto \varkappa_+ - \varkappa_- \end{aligned}$$

The previous theory of Hilbertian subspaces extends naturally to these vector spaces $\mathbb{R} \otimes$

$Hilb((\mathcal{E},\mathcal{F}))$ and $\mathbb{R}\otimes\mathbf{L}^{+}(\mathcal{F},\mathcal{E})$ thanks to this isomorphism. The question of an interpretation of these spaces (other than an abstract equivalence class) is then open.

2.1.2 Interpretation of these abstract vector spaces

The second vector space $\mathbb{R}\otimes\mathbf{L}^{+}(\mathcal{F},\mathcal{E})$ is the easiest to interpret. Two formal differences of positive kernels $\varkappa_{+}^{1}-\varkappa_{-}^{1}$ and $\varkappa_{+}^{2}-\varkappa_{-}^{2}$ are equal if the following equality occurs:

$$\varkappa_{+}^{1}+\varkappa_{-}^{2}=\varkappa_{+}^{2}+\varkappa_{-}^{1}$$

But since the set of kernels is a vector space with respect to the same addition operator, This equivalence relation is exactly the equality of the non-positive kernels

$$\varkappa_{+}^{1}-\varkappa_{-}^{1}=\varkappa_{+}^{2}-\varkappa_{-}^{2}$$

We can interpret $\mathbb{R}\otimes\mathbf{L}^{+}(\mathcal{F},\mathcal{E})$ as the subset of $\mathbf{L}^{*}(\mathcal{F},\mathcal{E})$ (the set of self-adjoint kernels) whose elements admit a decomposition as the difference of two positive kernels. This vector space will be studied further throughout this chapter.

A direct interpretation of the first vector space is far more difficult to give. But a first step to understand this vector space is to define the vector space of Krein subspaces for we will see that these two vector spaces are closely related. It is the aim of the next sections to define Krein subspaces and understand how they are related to the set $\mathbb{R}\otimes Hilb((\mathcal{E},\mathcal{F}))$.

2.2 Krein spaces and Krein subspaces

The previous construction leads to a very important theoretical result, but an interpretation of the elements of $\mathbb{R}\otimes Hilb((\mathcal{E},\mathcal{F}))$ has not been given yet. However, since they are differences of Hilbert spaces quotiented by an equivalence relation, it is natural to wonder whether they are related to Krein spaces, which may be seen as differences of Hilbert spaces (that do not intersect) quotiented by a second equivalence relation.

2.2.1 Krein spaces, Pontryagin spaces

Krein spaces and Pontryagin spaces are special types of inner product spaces, that can be seen as the sum of a Hilbert space and the antispace of a Hilbert space. Before going further, it might be useful to review some prerequisites on inner product spaces and antispaces. We refer for instance to [20] for proofs of these statements.

Definition 2.2 (– *inner product space* –) *An (indefinite) inner product space is a vector space H together with a sesquilinear bilinear form on $H \times H$: $\langle . | . \rangle$, called inner product on H.*

Definition 2.3 (– *antispace* –) *Let $(H, \langle . | . \rangle)$ be an inner product space. Then its antispace $-H$ is the inner product space $(H, -\langle ., . \rangle)$.*

It is then classical to define for an inner product the positivity, negativity, nondegeneracy properties. For instance, a prehilbert space is an inner product space whose inner product is strictly positive (i.e. positive nondegenerate).
As for Hilbert spaces, we can define isomorphisms between inner product spaces and operators may be symmetric, positive for the inner product.

Krein spaces are a special kind a inner product spaces related with Hilbert spaces. There are at least three equivalent definitions of Krein space. The first is about the class of Krein spaces, the second is based on the decomposition of Krein spaces into a sum of two inner product spaces and the third is an operator characterization.

Definition 2.4 (– *class of Krein spaces, Krein spaces* –) *The class of Krein spaces is the smallest class of inner product spaces, closed under orthogonal direct sums, that contains all Hilbert spaces and antispaces of Hilbert spaces. A inner product space is a Krein space if it is in the class of Krein spaces.*

Proposition 2.5 (– *equivalent definitions of Krein spaces* –) *The three following statements are equivalent:*

1. $(H, \langle .|. \rangle)$ *is a Krein space;*

2. *There exists two Hilbert spaces* H_+ *and* H_- *such that* H *has the fundamental decomposition*
$$H = H_+ \ominus H_-$$

3. *There exists a Hilbert space* $(|H|, \langle\langle .|. \rangle\rangle)$ *and a symmetry (unitary, self-adjoint operator)* J *(called the fundamental symmetry or metric operator or Gram operator) such that*
$$\langle .|. \rangle = \langle\langle J(.)|. \rangle\rangle$$

These decompositions are not unique: if $H_+^1, H_-^1, H_+^2, H_-^2$ are four Hilbert spaces such that $H_+^1 \bigcap H_-^1 = \{0\}$ and $H_+^2 \bigcap H_-^2 = \{0\}$, then they define the same Krein space if and only if they verify the following equivalence relation:

$$(H_+^1 - H_-^1)\mathcal{R}_=(H_+^2 - H_-^2) \iff H_+^1 \ominus H_-^1 = H_+^2 \ominus H_-^2$$

Since these decompositions are not unique, it is important to know what fundamental properties of the Krein space do not depend on the particular decomposition chosen. This is the content of the following lemmas:

Lemma 2.6 *Let* $(H, \langle .|. \rangle)$ *be a Krein space. Then for any two fundamental decompositions* $H = H_+^1 \ominus H_-^1 = H_+^2 \ominus H_-^2$, $\dim(H_+^1) = \dim(H_+^2)$ *and* $\dim(H_-^1) = \dim(H_-^2)$.

Lemma 2.7 *Let* $(H, \langle .|. \rangle)$ *be a Krein space and* $(|H|^1, J^1$, $(|H|^2, J^2)$ *two symmetry decompositions. Then the norms on* $|H|^1$ *and* $|H|^2$ *are equivalent. they are also equivalent with the norm* $\|h_+ + h_-\|_H^2 = \|h_+\|_{H_+}^2 + \|h_-\|_{H_-}^2$ *for any fundamental decomposition* $H = H_+ \ominus H_-$.

We can now define some intrinsic features of a Krein space:

Definition 2.8 *Let $H, \langle .|. \rangle$ be a Krein space. We call positive (resp. negative) indice of the Krein space the number $ind_+ H = \dim(H_+)$ (resp. $ind_- H = \dim(H_-)$). We call strong topology on H the topology induces by the norm on $|H|$.*

Remark that from the preceding lemmas, the two quantities are well defined. The negative index is also known as the Pontryagin index. As a consequence, the inner product is continuous with respect to this topology and the Riesz representation theorem holds in Krein spaces:

Theorem 2.9 *Let μ be a linear form on H. Then μ is continuous with respect to its strong topology if and only if it is of the form $\mu = \langle .|h \rangle$ and in this case, h is unique.*

This result may be restated as follows:

Proposition 2.10 *The strong topology on H is the Mackey topology for the dual system (H, H) with (generally asymmetric) bilinear form*

$$\begin{array}{rcl} L : F = H \times E = H & \longrightarrow & \mathbb{K} \\ h_1, h_2 & \longmapsto & \langle \overline{h_1} | h_2 \rangle \end{array}$$

From the preceding proposition and lemmas, we have the following interpretation of Krein spaces in terms of Hilbertian subspaces and their kernels:

Corollary 2.11 *Let H be a Krein space. Then for any of its fundamental decomposition $H = H_+ \ominus H_-$, H_+ and H_- are Hilbertian subspaces of the dual system (H, H) (or of H endowed with its strong topology) and their Hilbertian kernels verify $\varkappa_+ - \varkappa_- = Id_H$.*

Proof. – The fact that H_+ and H_- are Hilbertian subspaces of the dual system (H, H) is a direct application of lemma 2.7. The kernels are exactly the orthogonal projection in the space $|H|$ which gives the equality $\varkappa_+ - \varkappa_- = Id_H$. □

2.2.2 Pontryagin spaces

The class of Pontryagin spaces is the class of Krein spaces with finite negative indice:

Definition 2.12 (– *Pontryagin space* –) *An inner product space H is a Pontryagin space if and only if it is a Krein space H and $ind_{-}H < \infty$.*

There is little interest in studying Pontryagin spaces compared with Krein spaces in general but we will see that when dealing with Hermitian subspaces, they hold a remarkable property.

Example: Minkowski spacetime

The first indefinite metric spaces were probably the finite-dimensional Minkowski spaces of special relativity [21]. They are commonly used nowadays in cosmology for their simple mathematical properties even if they are not curved spaces and hence do not fit the general relativity setting.

We consider here the three dimensional Minkowski space $H = \mathbb{R}^3$ endowed with the indefinite inner product $\langle v_1 | v_2 \rangle_H = x_1 x_2 + x_1' x_2' - y_1 y_2$.
Vectors with positive (resp. negative, zero) length are called positive (resp. negative, neutral). For instance $(1, 1, \sqrt{2})$ is neutral.

Figure 2.2 (taken from [27]) gives a representation of this space, of a positive and negative subspace and of a neutral cone.

2.2.3 Krein (or Hermitian) subspaces

Hermitian subspaces appear for the first time in the work of L. Schwartz [46] who tries to generalize the notion of Hilbertian subspaces. The development of this notion led him to

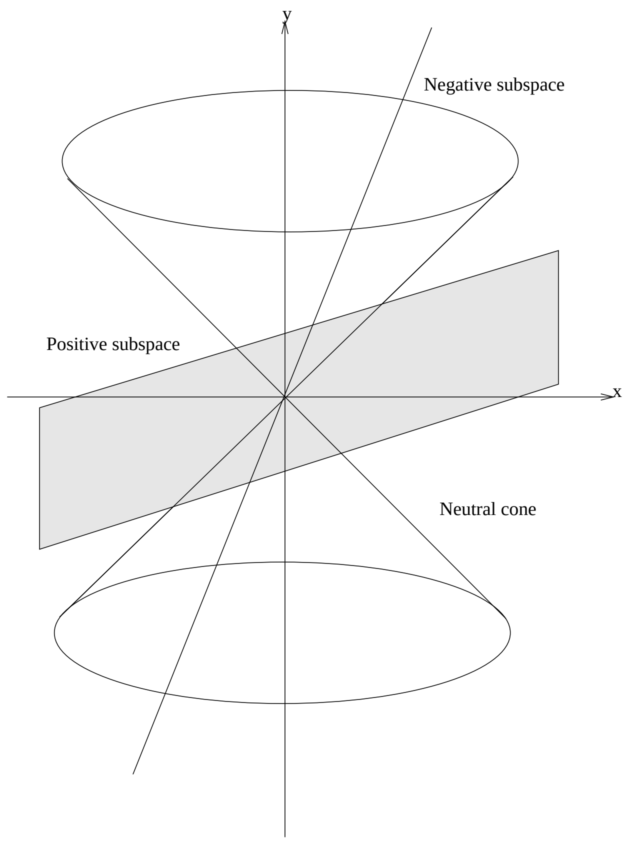

Figure 2.2: 3-D Minkowski spacetime

the particular study of Krein spaces, even if he did not use the word “Krein space” since this term was not used at the time (and that is why he kept the word “Hermitian”). It initiated a new direction in studying Hermitian kernels, notably those that admit a Kolmogorov decomposition, since they may be associated to Krein (Hermitian) subspaces. As for Hilbertian subspaces, Krein subspaces are Krein spaces with the additional property that they are strongly included in a locally convex space $\mathcal{E}$.

Precisely, we have seen in the previous section on Krein spaces that we can endow these spaces with an intrinsic topology for which the inner product is continuous (and the Riesz identification theorem holds). We then characterize Krein (Hermitian) subspaces with respect to this topology (the definition of Krein subspaces then mimics the definition of Hilbertian subspaces):

Definition 2.13 (– *Krein (or Hermitian) subspaces* –) *Let $(\mathcal{E}, \mathcal{F})$ be a duality (resp. $\mathcal{E}$ a l.c.s). A space H is a Krein (Hermitian) subspace of $(\mathcal{E}, \mathcal{F})$ (resp. of $\mathcal{E}$) if it is a Krein space such that:*

1. $H \subset \mathcal{E}$
2. *The canonical injection is continuous if H is endowed with the strong topology and $\mathcal{E}$ with any topology compatible with the duality (resp. with its initial topology).*

We note $Krein((\mathcal{E}, \mathcal{F}))$ the set of Krein (Hermitian) subspaces of the duality $(\mathcal{E}, \mathcal{F})$.

We can equivalently define Pontryagin subspaces:

Definition 2.14 (– *Pontryagin subspaces* –) *Let $(\mathcal{E}, \mathcal{F})$ be a duality (resp. $\mathcal{E}$ a l.c.s). A space H is a Pontryagin subspace of $(\mathcal{E}, \mathcal{F})$ (resp. of $\mathcal{E}$) if and only if it is a Pontryagin space and a Krein subspace of $(\mathcal{E}, \mathcal{F})$ (resp. of $\mathcal{E}$).*

For Krein spaces, the Mackey topology that corresponds to the self-duality (H, H) is exactly the strong topology (proposition 2.10). The strong continuity of the canonical injection can then be interpreted in terms of the weak or Mackey topology:

Proposition 2.15 (– *topological characterization of Krein subspaces* –) *the following statements are equivalent:*

1. *H is Krein subspace of $(\mathcal{E}, \mathcal{F})$;*

2. *The canonical injection* $i : H \mapsto \mathcal{E}$ *is weakly continuous;*

3. $i : H \mapsto \mathcal{E}$ *is continuous with respect to the Mackey topologies on* H *and* $\mathcal{E}$.

Proof. – Let us prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$:

$(1) \Rightarrow (2)$ corollary 1 p 106 [26]: if $j : H \mapsto \mathcal{E}$ is continuous, it is weakly continuous.

$(2) \Rightarrow (3)$ We can cite corollary 2 p 111 [26]: if $j : H \mapsto \mathcal{E}$ is weakly continuous, it is continuous if H is endowed with the Mackey topology (and $\mathcal{E}$ with any topology compatible with the duality).

$(3) \Rightarrow (1)$ The strong topology of the Krein space H is the Mackey topology (proposition 2.10), and we use the previous argument (corollary 2 p 111 [26]). □

What can we say about the Hilbert spaces appearing in any fundamental decomposition of a Krein subspace? Are they Hilbertian subspaces? The answer is positive:

Proposition 2.16 *Let* $(H, \langle .|. \rangle)$ *be a Krein subspace of* $(\mathcal{E}, \mathcal{F})$. *Then for any fundamental decompositions* $H = H_+ \ominus H_-$, H_+ *and* H_- *are Hilbertian subspaces of* $(\mathcal{E}, \mathcal{F})$. *Conversely, If* H_+ *and* H_- *are Hilbertian subspaces of* $(\mathcal{E}, \mathcal{F})$ *in direct sum, then* $H = H_+ \ominus H_-$ *is a Krein subspace of* $(\mathcal{E}, \mathcal{F})$.

Proof. – This proposition follows from lemma 2.7 and the definition of the strong topology. □

example 1 **$\mathbb{R}^2$-example**

Let $H = \mathbb{R}^2$ with inner product

$$\langle Y|X \rangle_H = x_1 y_1 - x_2 y_2$$

It is obviously a Krein subspace of $\mathbb{R}^2$ and for the fundamental decomposition $\mathbb{R}^2 = \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} \ominus \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix}$, $\begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix}$ are Hilbertian subspaces of $\mathbb{R}^2$.

example 2 **Sobolev spaces**

Let Ω be an open set of $\mathbb{R}$ and define

$$H = W^1(\Omega) \ominus \mathbb{R}.\mathbb{1}_\Omega = \left\{ f \in D',\ f(t) = \int_\Omega \mathbb{1}_{s \leq t} \phi(s)ds + \alpha,\ \phi \in L^2(\Omega),\ \alpha \in \mathbb{R} \right\}$$

with inner product ($f(t) = \int_\Omega \mathbb{1}_{s \leq t}\phi(s)ds + \alpha,\ g(t) = \int_\Omega \mathbb{1}_{s \leq t}\psi(s)ds + \beta$)

$$\langle f | g \rangle_H = \langle f - \alpha | g - \beta \rangle_{W^1(\Omega)} - \alpha.\beta = \int_\Omega \phi(s)\psi(s)ds - \alpha.\beta$$

Then H is a Krein subspace of D'.

The set $Krein((\mathcal{E}, \mathcal{F}))$ is then obviously a vector space. In order to understand its link with the vector space of Hermitian kernels, we have to make the link between this vector space and the vector space $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$.

2.3 Hermitian kernels and Krein subspaces

There are two ways to develop the link between Krein subspaces and Hermitian (or self-adjoint) kernels. The first is to interpret the set $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ (isomorphic to $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$) in terms of Krein subspaces and the second is to mimic directly the results (if possible) of the first chapter. The first is sufficient but the second gives the (same) results in a somehow more comprehensible and understandable manner.

2.3.1 $Krein((\mathcal{E}, \mathcal{F}))$ and $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$

This section is the crucial part of this chapter since all the results concerning the links between Krein spaces and Hermitian kernels derive from the similarities and differences between the two sets $Krein((\mathcal{E}, \mathcal{F}))$ and $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$.

The first thing is to understand that these two sets are not isomorphic in general. To be convinced of this, the following example due to L. Schwartz shows that two distinct Krein spaces may be equal if seen as elements of $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$.

<u>example</u> **(- Distinct Krein spaces equal in $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ -)**

Let $\mathcal{E}$ be an infinite dimensional Hilbert space, H_+ and H_- two closed subsets of $\mathcal{E}$ such that $H_+ \cup H_- = 0$ and $H_+ \oplus H_-$ is dense in $\mathcal{E}$ but not equal to $\mathcal{E}$ (we say that H_+ and H_- are in tangent position).

We then define $H_+^\perp$ and $H_-^\perp$: they verify $H_+^\perp \cup H_-^\perp = 0$ and $H_+^\perp \oplus H_-^\perp$ dense in $\mathcal{E}$ but not equal to $\mathcal{E}$ and also $H_+ \oplus H_+^\perp = H_- \oplus H_-^\perp = \mathcal{E}$.

This last equation gives the equality of $H_+ - H_-$ and $H_+^\perp - H_-^\perp$ as elements of $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$, but the spaces $H_+ \oplus H_-$ and $H_+^\perp \oplus H_-^\perp$ are not equal (If they were equal, they would contain $\mathcal{E}$ which is in contradiction with the hypothesis), hence two different Krein spaces.

The two vector spaces are then different, but we can however state a crucial result based on the two following lemmas:

Lemma 2.17 *Let H be a Krein subspace of $(\mathcal{E}, \mathcal{F})$. Then for any two canonical decompositions $H = H_+^1 \ominus H_-^1 = H_+^2 \ominus H_-^2$ we have that $H_+^1 + H_-^2 = H_+^2 + H_-^1$, i.e. they define a unique element of $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$.*

Proof. – Using corollary 2.11, we have that $Id_H = \varkappa_+^1 - \varkappa_-^1 = \varkappa_+^2 - \varkappa_-^2$ hence $\varkappa_+^1 + \varkappa_-^2 = \varkappa_+^2 + \varkappa_-^1$ and finally $H_+^1 + H_-^2 = H_+^2 + H_-^1$. □

Lemma 2.18 *Conversely, let $H_+ - H_- \in \mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$. Then exists H_+^0 and H_-^0 Hilbertian subspaces of $(\mathcal{E}, \mathcal{F})$ such that:*

1. $H_+^0 \bigcap H_-^0 = 0$

2. $H_+ + H_-^0 = H_0^0 + H_0$

Proof. – Let $\varkappa_+$ and $\varkappa_-$ be the Hilbertian kernels of H_+ and H_- and define the positive kernel $\varkappa = \varkappa_+ + \varkappa_-$, H its associated Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$. Then H_+ and H_- are Hilbertian subspaces of H with kernels χ_+ and χ_-. $\chi = \chi_+ - \chi_-$ is a continuous operator on H and its spectral decomposition gives $\chi = \chi_+^0 - \chi_-^0$ with χ_+^0 and χ_-^0 two positive operators verifying $H_+^0 \bigcap H_-^0 = 0$, H_+^0 and H_-^0 being the Hilbertian subspaces of H associated to χ_+^0 and χ_-^0. The equality $H_+ + H_-^0 = H_0^0 + H_0$ then follows from the equality $\chi_+ + \chi_-^0 = \chi_+^0 + \chi_-$. □

It follows from these two lemmas that:

Theorem 2.19 $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ ***is the quotient space of*** $Krein((\mathcal{E}, \mathcal{F}))$ ***with respect to the equivalence relation*** $\mathcal{R}_{cone}$***:***
$H^1 \mathcal{R}_{cone} H^2$ ***if and only if for two canonical decompositions (and then for any two)*** $H^1 = H_+^1 \ominus H_-^1$ ***and*** $H^2 = H_+^2 \ominus H_-^2$ ***we have that*** $H_+^1 + H_-^2 = H_+^2 + H_-^1$***.***

The isomorphism between $Krein((\mathcal{E}, \mathcal{F}))/\mathcal{R}_{cone}$ and $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$ will then extend to an isomorphism between the two vector spaces $Krein((\mathcal{E}, \mathcal{F}))/\mathcal{R}_{cone}$ and $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$. Before stating the main definitions and results based on this isomorphism, we study more closely the set $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$.

2.3.2 Hermitian kernels

We start with some definitions and propositions relative to kernels. At the end of the section on Hilbertian subspaces, the notion of Kolmogorov decomposition has been defined for positive reproducing kernels. We give here an apparently completely different definition for self-adjoint kernels of an arbitrary duality:

Definition 2.20 (– Kolmogorov decomposition (first version) –) *We say that a self-adjoint (Hermitian) kernel $\varkappa \in \mathbf{L}^*(\mathcal{F}, \mathcal{E})$ admits a Kolmogorov decomposition (of the first kind) if there exist two positive kernels $\varkappa_+, \varkappa_- \in \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ such that*

$$\varkappa = \varkappa_+ - \varkappa_-$$

and we note this set $\mathbf{L}^{KD}(\mathcal{F}, \mathcal{E})$.

The equivalence relation is then the equality of the Hermitian kernels:

$$(\varkappa_+^1 - \varkappa_-^1)\mathcal{R}_=(\varkappa_+^2 - \varkappa_-^2) \iff (\varkappa_+^1 - \varkappa_-^1) = (\varkappa_+^2 - \varkappa_-^2)$$

It may not be clear whereas there exists kernels that do not admit a Kolmogorov decomposition. The following example answers this question.

example **(- Kernel without a Kolmogorov decomposition -)**

Let B be a reflexive Banach space (over $\mathbb{R}$) **that cannot be endowed with an Hilbertian structure**. Then the kernel

$$\begin{array}{rcl} \varkappa : B' \times B & \longrightarrow & B \times B' \\ (b', b) & \longmapsto & (b, b') \end{array}$$

does not admit a Kolmogorov decomposition.

If we could write $\varkappa = \chi_+ \ominus \chi_-$, then we would have the set equality $(B, B') = H_+ \oplus H_-$ and B could be endowed with a Hilbertian structure.

Proposition 2.21 *The set of kernels that admit a Kolmogorov decomposition $\mathbf{L}^{KD}(\mathcal{F}, \mathcal{E})$ is also the quotient space of formal difference of positive kernels by the equivalence relation*

$$(\varkappa_+^1 - \varkappa_-^1)\mathcal{R}_{cone}(\varkappa_+^2 - \varkappa_-^2) \iff \varkappa_+^1 + \varkappa_-^2 = \varkappa_+^2 + \varkappa_-^1$$

i.e. the set $\mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$:

$$\mathbf{L}^{KD}(\mathcal{F}, \mathcal{E}) = \mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$$

Remark 2.22 *This proposition is crucial since it gives the equivalence between the two relations $\mathcal{R}_{cone}$ and $\mathcal{R}_{=}$ for kernels whereas they are not equivalent for Krein subspaces.*

However, if such a decomposition exists then there are infinitely many decompositions (we may sum any positive kernel to $\varkappa_+$ and $\varkappa_-$). We need the following concept:

Definition 2.23 (– independent positive kernels –)
Two positive kernels $\varkappa_+, \varkappa_- \in \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ are independent if any positive kernel χ such that $\chi \leq \varkappa_+$ and $\chi \leq \varkappa_-$ is zero.

If $\varkappa_+$ and $\varkappa_-$ are the Hilbertian kernels of H_+ and H_-, this definition is equivalent to $H_+ \bigcap H_- = 0$.

Even with this restriction, the Kolmogorov decomposition of a self-adjoint kernel in two independent positive kernels is not unique in general.

The following lemma gives equivalent conditions for a kernel to admit a Kolmogorov decomposition:

Lemma 2.24 *The following statements are equivalent:*

- *the self-adjoint kernel $\varkappa$ admits a Kolmogorov decomposition;*
- *there exists two independent positive kernels $\varkappa_+, \varkappa_- \in \mathbf{L}^+(\mathcal{F}, \mathcal{E})$ such that*

$$\varkappa = \varkappa_+ - \varkappa_-$$

- *There exists a positive kernel χ dominating $\varkappa$, i.e.*

$$\forall (\varphi_1, \varphi_2) \in \mathcal{F}^2, (\varphi_1, \varkappa(\varphi_2))^2_{(\mathcal{F},\mathcal{E})} \leq (\varphi_1, \chi(\varphi_1))_{(\mathcal{F},\mathcal{E})} (\varphi_2, \chi(\varphi_2))_{(cF,\mathcal{E})}$$

or equivalently

$$\forall \varphi \in \mathcal{F}, |\, (\varphi, \varkappa(\varphi))_{(\mathcal{F},\mathcal{E})} \,| \leq (\varphi, \chi(\varphi))_{(\mathcal{F},\mathcal{E})}$$

Proof. – We refer to the proof in [46] that follows proposition 38 p 242. □

The next two sections will state the main results concerning Krein subspaces and Hermitian kernels. We will give a second definition of the Kolmogorov decomposition of a Hermitian kernel and prove that the two definitions are equivalent.

2.3.3 The fundamental theorem

Combining theorem 2.1 and theorem 2.19, one gets the existence of the following morphism that associates any Krein subspace to a unique element of $\mathbb{R} \otimes \mathbf{L}^{+}(\mathcal{F}, \mathcal{E})$:

Theorem 2.25 *We have the following factorization*

$$Krein((\mathcal{E}, \mathcal{F})) \longrightarrow Krein((\mathcal{E}, \mathcal{F}))/\mathcal{R}_{cone} = \mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F})) \xrightarrow{\Phi} \mathbb{R} \otimes \mathbf{L}^{+}(\mathcal{F}, \mathcal{E})$$

and we still note $\Phi : Krein((\mathcal{E}, \mathcal{F})) \longrightarrow \mathbb{R} \otimes \mathbf{L}^{+}(\mathcal{F}, \mathcal{E})$ *this well defined morphism.*

However this formalism has a drawback since this definition of Φ deals with equivalent classes and is purely abstract. It may then be good to restate this theorem and interpret it directly. The proofs of the following theorems and propositions are omitted since they all are a direct application of theorem 2.25.

2.4 Direct interpretation and application of this theorem

This section is devoted to a precise study of the morphism Φ. We ask for instance the following questions: can we define directly the image by Φ of a Krein subspace? Are there kernels with a unique antecedent?

2.4.1 Hermitian kernel of a Krein subspace

Theorem 2.26 *To any Krein subspace H of $(\mathcal{E},\mathcal{F})$ is associated a unique Hermitian kernel $\varkappa$, verifying*

$$\forall \varphi \in \mathcal{F}, \forall h \in H,\ \big(\varphi, j(\overline{h})\big)_{(\mathcal{F},\mathcal{E})} = \big(\overline{h}, i^{-1} \circ \varkappa(\varphi)\big)_{(\overline{H}\sim H, H)} = \langle h | i^{-1} \circ \varkappa(\varphi)\rangle_H$$

It is the linear application $\varkappa = i \circ^t j$ where $i : H \longrightarrow \mathcal{E}$ and $j : \overline{H} \sim H \longrightarrow \mathcal{E}$ are the canonical injections. This application is called the Hermitian kernel (or Krein kernel) of H.

This kernel is the image of H under the previous morphism Ψ.

The Hermitian kernel of a Krein space is naturally related to the Hilbertian kernels of any of its fundamental decompositions:

Proposition 2.27 *Let $(H, \langle . | . \rangle)$ be a Krein subspace of $(\mathcal{E},\mathcal{F})$. Then for any fundamental decompositions$H = H_+ \ominus H_-$ with Hilbertian kernels $\varkappa_+$ and $\varkappa_-$,*

$$\varkappa = \varkappa_+ - \varkappa_- \tag{2.1}$$

Remark that for any fundamental decomposition $H = H_+ \ominus H_-$, $H_+ \bigcap H_- = 0$ and the Hilbertian kernels are independent.

We can easily find the kernels of the Krein subspaces defined in the previous examples:

example 1 **$\mathbb{R}^2$-example**

The kernel of the Krein subspace of the Euclidean duality $(\mathbb{R}^2, \mathbb{R}^2)$ $H = \mathbb{R}^2$ with inner product

$$\langle Y | X \rangle_H = x_1 y_1 - x_2 y_2$$

is

$$\begin{array}{rcl} \varkappa : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ X = (x_1, x_2) & \longmapsto & \varkappa(X) = (x_1, -x_2) = K.X \end{array}$$

with $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

example 2 **Sobolev spaces**

The kernel of the Krein subspace of (D', D)

$$H = W^1(\Omega)\ominus\mathbb{R}.\mathbb{1}_\Omega = \left\{ f \in D',\ f(t) = \int_\Omega \mathbb{1}_{s\leq t}\phi(s)ds + \alpha,\ \phi \in L^2(\Omega),\ \alpha \in \mathbb{R} \right\}$$

with inner product ($f(t) = \int_\Omega \mathbb{1}_{s\leq t}\phi(s)ds + \alpha,\ g(t) = \int_\Omega \mathbb{1}_{s\leq t}\psi(s)ds + \beta$)

$$\langle f|g\rangle_H = \langle f-\alpha|g-\beta\rangle_{W^1(\Omega)} - \alpha.\beta = \int_\Omega \phi(s)\psi(s)ds - \alpha.\beta$$

is

$$\begin{array}{rcl} \varkappa : D & \longrightarrow & D' \\ \phi & \longmapsto & \varkappa(\phi)(t) = \int_\Omega \min(s,t)\phi(s)ds - \int_\Omega \phi(s)ds = \int_\Omega (\min(s,t)-1)\phi(s)ds \end{array}$$

2.4.2 Krein subspaces associated to a kernel: kernels of unicity and kernels of multiplicity

Theorem 2.28 *Let $(\mathcal{E}, \mathcal{F})$ be a duality, $\mathcal{E}$ Mackey quasi-complete and $\varkappa \in \mathbf{L}(\mathcal{F}, \mathcal{E})$ a self-adjoint kernel. Then the following propositions are equivalent:*

1. *$\varkappa \in \mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$;*
2. *There exists at least one Krein subspace H of $(\mathcal{E}, \mathcal{F})$ with kernel $\varkappa$.*

In this case, for any Kolmogorov decomposition $\varkappa = \varkappa_+ - \varkappa_-$, $H = H_+ \ominus H_-$ is a Krein subspace of $(\mathcal{E}, \mathcal{F})$ with kernel $\varkappa$ and conversely, any fundamental decomposition of a Krein subspace associated to $\varkappa$ gives a Kolmogorov decomposition of $\varkappa$ via its Hilbertian kernels.

Definition 2.29 (– Kolmogorov decomposition (second version) –) *We say that a self-adjoint kernel $\varkappa \in \mathbf{L}(\mathcal{F}, \mathcal{E})$ admits a Kolmogorov decomposition (of the second kind) if there exists a Krein space H and a weakly continuous operator $V : \mathcal{F} \longrightarrow H$ such that:*

1. $\varkappa = V^*V$;

2. $V(\mathcal{F})$ *is dense in* H.

The pair (H, V) *is called a Kolmogorov decomposition of the Hermitian kernel* $\varkappa$.

Theorem 2.30 *If* $\mathcal{E}$ *is quasi-complete, the two definitions of a Kolmogorov decomposition are equivalent.*

Remark 2.31 *The hypothesis of quasi-completeness of the space* $\mathcal{E}$ *(with respect to the Mackey topology) is often passed under silence in many texts, mostly because these texts deal with a particular space* $\mathcal{E}$ *(such as* $\mathbb{R}^\Omega$*) obviously complete for its initial topology. However, the (quasi)-completeness of the space is fundamental to ensure that* V^* *is one-to-one, i.e.* $V(\mathcal{F})$ *dense in* H.

In case $\mathcal{E}$ is not quasi-complete, one can use the results of proposition 1.18. However, there are in general many semi-norms associated to a kernel, each one depending on the particular decomposition we use and the continuity of a particular semi-norm does not imply in general the continuity of one another.
The following proposition gives equivalent criteria for the unicity of the Kolmogorov decomposition:

Proposition 2.32 *Let* $\mathcal{E}$ *be Mackey quasi-complete and* $\varkappa \in \mathbb{R} \otimes \mathbf{L}^+(\mathcal{F}, \mathcal{E})$. *The following statements are equivalent:*

1. *There is only one Krein subspace of* $(\mathcal{E}, \mathcal{F})$ *with kernel* $\varkappa$;

2. *for any two Kolmogorov decompositions of the first kind* $\varkappa = \varkappa_+^1 - \varkappa_-^1 = \varkappa_+^2 - \varkappa_-^2$, $H_+^1 \oplus H_-^2 = H_+^2 \oplus H_-^2$;

3. *for any two Kolmogorov decompositions of the second kind* (H^1, V^1) *and* (H^2, V^2), *there exists a unitary isomorphism* $U : H^1 \longrightarrow H^2$ *such that* $V^2 = UV^1$.

In this case, we say that the Krein kernel $\varkappa$ is a kernel of unicity , or equivalently that it admits a unique Kolmogorov decomposition. If one (any) of these conditions is not fulfilled, we say that $\varkappa$ is a kernel of multiplicity.

Conversely, we say that a Krein subspace is of unicity (resp. multiplicity) if its Krein kernel is of unicity (resp. multiplicity).

The results of this chapter can then be summarized as follows (with $\mathcal{E}$ Mackey quasi-complete):

The set of formal differences of Hilbertian subspaces and the set of formal differences of positive kernels are isomorphic and the results of the previous chapter remain Z
valid. Moreover, we can interpret the first set as the quotient space of Krein subspaces of $(\mathcal{E}, \mathcal{F})$ with respect to an equivalence relation and the second as the subset of self-adjoint kernels that admit a Kolmogorov decomposition.

The next section gives the main results concerning kernels of unicity and multiplicity.

2.4.3 Kernels of unicity, multiplicity and Pontryagin spaces

It is of major importance to know whether a given kernel is of unicity or of multiplicity. This is a rather difficult question in general, but some results based on the rank of the positive kernels appearing in any Kolmogorov decomposition exists, mainly when the space $\mathcal{F}$ is nuclear. Once again we refer to [46] where the proofs are detailed.

Lemma 2.33 *Let $\varkappa = \varkappa_+^1 - \varkappa_-^1$ with $\varkappa_+^1$ and $\varkappa_-^1$ independent and $\varkappa_-^1$ of finite rank r. Then for any other decomposition $\varkappa = \varkappa_+^2 - \varkappa_-^2$, $\varkappa_+^2$ and $\varkappa_-^2$ independent, $rank(\varkappa_-^2) = r$.*

In this case, we say that the kernel is a Pontryagin kernel.

Theorem 2.34 *Let $\varkappa = \varkappa_+ - \varkappa_-$ with $\varkappa_-$ of finite rank. Then the kernel is of unicity and the associated Krein subspace is a Pontryagin space.*
Conversely, the Hermitian kernel associated to any Pontryagin subspace is of unicity.

It follows that any positive kernel is of unicity, which was already known thanks to the Hilbertian subspaces theory.

Corollary 2.35 *The set of Pontryagin subspaces of $(\mathcal{E}, \mathcal{F})$ is a vector space isomorphic to the vector space of Pontryagin kernels (an isomorphism being Φ).*

Corollary 2.36 *Any kernel of multiplicity is of the form $\varkappa = \varkappa_+ - \varkappa_-$ with $\varkappa_+$ and $\varkappa_-$ independent and both of infinite rank.*

Apart this statement, two results are interesting concerning kernels of multiplicity:

Proposition 2.37 *Let $\varkappa$ be a kernel of multiplicity, $H_1 \neq H_2$ two different Krein subspaces associated to $\varkappa$. Then:*

1. $H_1 \bigcap H_2^c \neq 0$
2. $H_2 \bigcap H_1^c \neq 0$

This proposition shows that the different Krein subspaces associated with a kernel of multiplicity do not verify inclusion relations.

There is an other interesting result concerning kernels of multiplicity based on the properties of the space $\mathcal{F}$[2].

Proposition 2.38 *Suppose $\mathcal{F}$ is barreled[3] (for its Mackey topology) and nuclear. Then $\varkappa$ is a kernel of unicity if and only if it admits a Kolmogorov decomposition of the form $\varkappa = \varkappa_+ - \varkappa_-$ with $\varkappa_-$ of finite rank.*

[2] This result will be notably useful in the section that deals with the special case $\mathcal{E} = \mathbb{K}^\Omega$

[3] *Then $\mathcal{E}$ is weakly quasi-complete.*

Corollary 2.39 *If $\mathcal{F}$ is barreled and nuclear, the only Krein subspaces of unicity are the Pontryagin subspaces.*

2.5 Reproducing kernel Krein and Pontryagin spaces

As for Hilbertian subspaces, the case of $\mathcal{E} = \mathbb{K}^{\Omega}$ endowed with the product topology, or topology of simple convergence, where Ω is any set, has a special place in the theory and many papers deal with what are called reproducing kernel Krein spaces (in short r.k.k.s.) or reproducing kernel Pontryagin spaces (in short r.k.p.s.).
Many definitions and properties of r.k.h.s. extend naturally to the Krein setting and will be exposed in the first section whereas the second one deals mainly with the properties of Pontryagin kernels.

2.5.1 Generalities about r.k.k.s.

Definition 2.40 *A Krein space H is a reproducing kernel Krein space if there exists a set Ω, H is a Krein subspace of $\mathcal{E} = \mathbb{K}^{\Omega}$ endowed with the product topology.*

We have seen (theorem 2.9) that in a Krein space, the Riesz representation theorem holds. Characterizations of r.k.h.s. in terms of a reproducing kernel extends then naturally to r.k.k.s. :

Proposition 2.41 *Let $H \subset \mathbb{K}^{\Omega}$ be a Krein space. The following statements are equivalent:*

1. *H is a r.k.k.s.*
2. *The canonical injection from H into $\mathbb{R}^{\Omega}$ is weakly continuous.*
3. $\forall h \in H, \forall t \in \Omega, \exists M_t, \quad |h(t)| \leq M_t \|h\|_{|H|}$.
4. $\forall s \in \Omega, \exists K_s \in H, \forall h \in H, \quad \langle K_s | h \rangle_H = h(s)$.

5. $\exists K \in \mathbb{K}^{\Omega \times \Omega}, K(t,s) = \langle K(.,s) | K(t,.) \rangle_H$

K is still known as the reproducing kernel and:

Theorem 2.42 *Let H be a r.k.k.s. in $\mathbb{K}^{\Omega}$. Then its reproducing kernel K is the image of the Schwartz kernel $\varkappa$ under the isomorphism between $\mathbf{L}((\mathbb{K}^{\Omega})', \mathbb{K}^{\Omega})$ and $\mathbb{K}^{\Omega \times \Omega}$ (proposition 1.27):*

$$\forall t, s \in \Omega, \qquad K(t,s) = [\varkappa(\delta_t)](s)$$

Corollary 2.43 *It follows that K is Hermitian and admits a Kolmogorov decomposition i.e. exist K_+ and K_- independent positive kernels, $K = K_+ - K_-$.*

Following proposition 1.16, we have the following:

Proposition 2.44

$$H_0 = Span\,\{K_t,\ t \in \Omega\}$$

$$H = \widehat{H_0}$$

The previous examples of Krein subspaces fit the r.k.k.s. setting:

example 1 **Sobolev spaces**

The Krein space

$$H = W^1(\Omega) \ominus \mathbb{R}.\mathbb{1}_{\Omega} = \left\{ f \in D',\ f(t) = \int_{\Omega} \mathbb{1}_{s \le t} \phi(s) ds + \alpha,\ \phi \in L^2(\Omega),\ \alpha \in \mathbb{R} \right\}$$

with inner product ($f(t) = \int_{\Omega} \mathbb{1}_{s \le t} \phi(s) ds + \alpha,\ g(t) = \int_{\Omega} \mathbb{1}_{s \le t} \psi(s) ds + \beta$)

$$\langle f | g \rangle_H = \langle f - \alpha | g - \beta \rangle_{W^1(\Omega)} - \alpha.\beta = \int_{\Omega} \phi(s) \psi(s) ds - \alpha.\beta$$

is a r.k.k.s. over Ω with kernel function

$$K(t,s) = \min(s,t) - 1$$

2.5.2 Pontryagin kernels and reproducing kernels of multiplicity

Proposition 2.45 *A reproducing kernel Krein space is of unicity if and only if it is a reproducing kernel Pontryagin space.*

Proof. – By proposition 1.27, $(\mathbb{K}^{\Omega})'$ is barreled and nuclear and we apply proposition 2.38. □

It follows that reproducing kernel Pontryagin spaces (r.k.p.s.) play a special role in this theory for they are the only reproducing kernel Krein spaces of unicity.

example 2 **$\mathbb{R}^2$-example**

$H = \mathbb{R}^2$ with inner product

$$\langle Y|X\rangle_H = x_1y_1 - x_2y_2$$

is a r.k.k.s. (precisely a r.k.p.s.) over $\Omega = \{1,2\}$ with kernel function

$$K(i,j) = K_{i,j},\ K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

example 3 **Polynomials, splines**

In the previous chapter we were interested in the (non-positive) function

$$K(t,s) = ((t-s)^n = \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!(n-j)!} t^j s^{n-j}$$

that, according to [23] is however a classical function associated to $\mathcal{P}_n$ the space of univariate polynomials of degree n. In the special case $n = 2p$ (n even) K is symmetric and spans a finite dimensional space. It follows that it admits a Kolmogorov decomposition and that it is associated with a Pontryagin space. The Kolmogorov decomposition is

$$K(t,s) = ((t-s)^{2p} = K_+(t,s) - K_-(t,s)$$
$$= \sum_{j=0}^{p} \frac{(2p)!}{2j!(2p-2j)!} t^{2j} s^{2(p-j)} - \sum_{j=0}^{p} \frac{(2p)!}{(2j+1)!(2p-2j+1)!} t^{2j+1} s^{2(p-j)-1}$$

However, the case n odd cannot be treated within this formalism. Next chapter will gives us a theory to treat this final case.

2.6 Admissible prehermitian subspaces

In [46] L. Schwartz does more than introduce the notion of Hermitian subspaces entirely defined after the notion of Hilbertian subspaces, since he introduces the notion of admissible prehermitian subspace of a l.c.s. and shows that there exists a surjection from this class of space onto the space of self-adjoint (Hermitian) kernels. Moreover he constructs the image of an admissible prehermitian subspace, but an equivalence relation is still needed to give the set of admissible prehermitian subspaces the structure of a category. This section presents these notions. The formalism will be enlarged in next chapter.

2.6.1 Admissible prehermitian subspaces

Definition 2.46 (– *prehermitian space* –) *A prehermitian space is an inner product space with a Hermitian inner product.*

We can now define admissible prehermitian subspaces of a duality.:

Definition 2.47 (– *admissible prehermitian subspace (of a duality)* –) *Let $(\mathcal{E},\mathcal{F})$ be a duality. A prehermitian space H is an admissible prehermitian subspace of $(\mathcal{E},\mathcal{F})$ if:*

1. *the inner product on H is non-degenerate;*

2. *$H \subset \mathcal{E}$;*

3. *$\forall \varphi \in \mathcal{F},\ \exists h_\varphi \in H,\ \forall h \in H, \quad \left(\varphi, \overline{h}\right)_{(\mathcal{F},\mathcal{E})} = \langle h | h_\varphi \rangle_H$.*

We note this set $Preherm((\mathcal{E}, \mathcal{F}))$

This definition that appears in L. Schwartz's article [46] is purely algebraic and simply states that any linear form on $\mathcal{E}$ restricted to H can be given by the symmetric bilinear form associated to H. To set a topological interpretation of this definition, one needs concepts of dualities (exposed in the Appendix B Algebra).

Proposition 2.48 *Suppose the indefinite inner product on H is separate such that H is in separate duality with itself. Then we can endow H with the weak or Mackey topology with respect to this duality and the following statements are then equivalent:*

1. *H is an admissible prehermitian subspace of $(\mathcal{E}, \mathcal{F})$;*

2. *The canonical injection $i : H \mapsto \mathcal{E}$ is weakly continuous;*

3. *$i : H \mapsto \mathcal{E}$ is continuous with respect to the Mackey topologies on E and $\mathcal{E}$.*

Proof. – Let us prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$:

$(1) \Rightarrow (2)$ $j' : \varphi \in \mathcal{F}, \mapsto \mathrm{h}_\varphi \in H$ is the transpose of the canonical injection and the canonical injection is then weakly continuous.

$(2) \Rightarrow (3)$ We can cite corollary 2 p 111 [26]: if $u : H \mapsto \mathcal{E}$ is weakly continuous, it is continuous if H is endowed with the Mackey topology (and $\mathcal{E}$ with any topology compatible with the duality).

$(3) \Rightarrow (1)$ The strong topology of the Krein space H is the Mackey topology (proposition 2.10, and we use the previous argument (corollary 2 p 111 [26]). □

We can of course derive from this definition the definition of admissible prehermitian subspaces of a l.c.s.:

Definition 2.49 (– admissible prehermitian subspace (of a lcs) –) *Let $\mathcal{E}$ be a l.c.s. A prehermitian space H is an admissible prehermitian subspace of $\mathcal{E}$ if it is an admissible prehermitian subspace of $(\mathcal{E}, \mathcal{E}')$.*

2.6.2 Schwartz kernel of an admissible prehermitian subspace

As for Hilbertian or Krein subspaces, we can associate to any admissible prehermitian subspace of $(\mathcal{E}, \mathcal{F})$ a self-adjoint kernel $\varkappa \in \mathbf{L}^*(\mathcal{F}, \mathcal{E})$.

Proposition 2.50 *To any admissible prehermitian subspace H of $(\mathcal{E}, \mathcal{F})$ is associated a unique self-adjoint (Hermitian) kernel $\varkappa$, verifying*

$$\forall \varphi \in \mathcal{F}, \forall h \in H,\ \big(\varphi, j(\overline{h})\big)_{(\mathcal{F},\mathcal{E})} = \langle h | i^{-1} \varkappa(\varphi) \rangle_H$$

It is the linear application $\varkappa = i \circ \vartheta$, where $\vartheta : \varphi \mapsto \mathrm{h}_\varphi$ thanks to the property of admissible prehermitian subspaces: $\forall \varphi \in \mathcal{F},\ \exists \mathrm{h}_\varphi \in H,\ \forall h \in H,\quad \big(\varphi, \overline{h}\big)_{(\mathcal{F},\mathcal{E})} = \langle h | h_\varphi \rangle_H$ This application is called the Hermitian kernel of H.

Proof. – $\forall h, \in H,\ \varphi \in \mathcal{F}$

$$\begin{aligned} (\varphi, j(\overline{h}))_{(\mathcal{F},\mathcal{E})} &= \langle h | h_\varphi \rangle_H \\ &= \langle h | i^{-1}(i \circ \vartheta(\varphi)) \rangle_H \end{aligned}$$

and $\varkappa = j \circ \vartheta$. Finally, we check that this linear application is weakly continuous by composition of weakly continuous morphisms and self-adjoint. □

2.6.3 Image of an admissible prehermitian subspace by a weakly continuous linear application: the category of admissible prehermitian subspaces

It is interesting to construct directly the image of an admissible prehermitian subspace by a weakly continuous morphism and look at the properties of the image.

Let $u : \mathcal{E} \longrightarrow \mathfrak{E}$ be a weakly continuous morphism and H an admissible prehermitian subspace of $(\mathcal{E}, \mathcal{F})$ with kernel $\varkappa$. Define $\mathfrak{N} = H \bigcap \ker u$ and $\mathcal{K}$ the orthogonal of $\mathfrak{N}$ in H with respect to its indefinite inner product. If $\forall A \subset \mathcal{E},\ u_{|A}$ denotes the restriction of u to the set A, we have $\mathfrak{N} = \ker(u_{|H})$. Finally we define

$$\mathcal{N} = \mathcal{K}/(\mathcal{K} \cap \mathfrak{N})$$

Lemma 2.51 *The linear application $u_{|\mathcal{N}}$ is well defined and injective, and $\forall\,(\dot{n}, \dot{n'}) \in \mathcal{N} \times \mathcal{N}$, the sesquilinear form (indefinite inner product)*

$$B(u_{|\mathcal{N}}(\dot{n}), u_{|\mathcal{N}}(\dot{n'})) = \langle n|n'\rangle_H$$

defines a indefinite inner product space $u_{|\mathcal{N}}(\mathcal{N})$.

Proof. – We have the following factorization

$$u : \ker(u_{|H})^{\perp} \longrightarrow (\ker(u_{|H})^{\perp}/\ker(u_{|H}) \stackrel{u_{|\mathcal{N}}}{\longrightarrow} \mathfrak{E}$$

and $u_{|\mathcal{N}}$ is one-to-one. Moreover the indefinite inner product

$$B : u_{|\mathcal{N}}(\mathcal{N}) \times u_{|\mathcal{N}}(\mathcal{N}) \longrightarrow \mathbb{K}$$

is well defined since:

$\forall(n_1, n_2) \in \dot{n},\ \forall(n'_1, n'_2) \in \dot{n'},\ \langle n_1 - n_2|n'_1 - n'_2\rangle_H = 0.$ □

Theorem 2.52 *The indefinite inner product space $u_{|\mathcal{N}}(\mathcal{N})$ is an admissible prehermitian subspace of $(\mathfrak{E}, \mathfrak{F})$ called the image of H by u and noted $u(H)$. Its kernel is $u \circ \varkappa \circ^t u$.*

Proof. – The algebraic inclusions of definition 2.47 are fulfilled and the space $u_{|\mathcal{N}}(\mathcal{N})$ is an admissible prehermitian subspace of $\mathfrak{E}$. $u \circ \varkappa \circ^t u$ satisfies the requirements of proposition 2.50. □

We can then define the operation of addition and multiplication thanks to the operators

$$\begin{aligned} \mathfrak{S} : \mathcal{E} \times \mathcal{E} &\longrightarrow \mathcal{E} \\ (\varepsilon_1, \varepsilon_2) &\longmapsto \varepsilon_1 + \varepsilon_2 \end{aligned}$$

and

$$\begin{aligned} \mathfrak{P}_\lambda : \mathcal{E} &\longrightarrow \mathcal{E} \\ \varepsilon &\longmapsto \sqrt{\lambda}.\varepsilon \end{aligned}$$

where $\lambda \in \mathbb{R}$ but $\sqrt{\lambda} \in \mathbb{C}$ in general.
It is interesting to note that we cannot do that directly for Krein subspaces since the image of a Krein subspace **needs not** to be a Krein subspace, but only an admissible prehermitian subspace.

However, the operation of addition is not a true addition since it is not associative. To endow the set of admissible prehermitian subspaces $Preherm((\mathcal{E}, \mathcal{F}))$ with a vector space structure, one needs the equivalence relation:

$$H_1 \mathcal{R}_{cone} H_2 \iff H_1 - H_2 = 0$$

Remark that we have seen that equivalence relation before: it is the equality of the Schwartz kernels

$$H_1 \mathcal{R}_{cone} H_2 \iff \varkappa_1 = \varkappa_2$$

hence it induces the previously defined equivalence relation $\mathcal{R}_{cone}$ over Krein spaces.
We can then state the following proposition due to L. Schwartz [46]:

Proposition 2.53 *The three operations* $(\lambda, H) \mapsto \lambda.H$*,* $(H_1, H_2) \mapsto H_1 + H_2$*,* $(u, H) \mapsto u(H)$ *pass to the quotient by the previously defined equivalence relation. The set* $Herm((\mathcal{E}, \mathcal{F}))= Preherm((\mathcal{E}, \mathcal{F}))/\mathcal{R}_{cone}$ *of equivalent classes of admissible prehermitian subspaces endowed with the laws of multiplication by real scalars and addition is a vector space over* $\mathbb{R}$ *isomorphic to the vector space* $\mathbf{L}^*(\mathcal{E}, \mathcal{F})$*.*
Moreover, let $\mathcal{C}$ *be the category of dual systems* $(\mathcal{E}, \mathcal{F})$ *(we do not require here* $\mathcal{E}$ *to be quasi-complete), the morphisms being the weakly continuous linear applications and* $\mathcal{V}$ *be the category of vector spaces, the morphisms being linear applications. Then* $Herm$: $(\mathcal{E}, \mathcal{F}) \mapsto Herm((\mathcal{E}, \mathcal{F}))$ *is a covariant functor of category* $\mathcal{C}$ *into category* $\mathcal{G}$ *isomorphic to the covariant functor* $\mathbf{L}^* : (\mathcal{E}, \mathcal{F}) \mapsto \mathbf{L}^*(\mathcal{F}, \mathcal{E})$

Conclusion and comments

The theory of Krein subspaces may be seen as the pure development of the theory of Hilbertian subspaces with respect to its structure of convex cone. However two problems arise. The first is that we need an equivalence relation to endow the set of Krein subspaces a structure of vector space or study the image of a Krein subspace. Moreover this equivalence relation is necessary to have a bijection between the quotiented set of Krein spaces and the set of Hermitian kernels that admit a Kolmogorov decomposition. The second is that regarding kernels, the set of kernels that admit a Kolmogorov decomposition is in general strictly smaller than the set of Hermitian kernels[4].

One answer is then the study of admissible prehermitian subspaces: we still need an equivalence relation but we deal with the total set of Hermitian (self-adjoint) kernels.

The relations between all these sets is given figure 2.3. $Hilb$ (resp. $\mathbf{L}^+$,...) stands for

[4] However the two sets are equal if $\mathcal{E}$ is finite dimensional or a Hilbert space

$Hilb((\mathcal{E},\mathcal{F}))$ (resp. $\mathbf{L}^+(\mathcal{F},\mathcal{E})$,...) with $(\mathcal{E},\mathcal{F})$ a dual system, $\mathcal{E}$ Mackey quasi-complete.

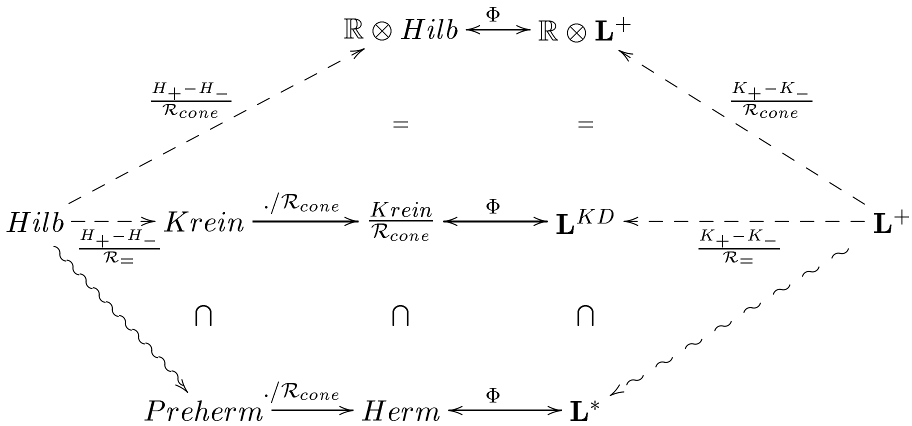

Figure 2.3: Sets of subspaces, sets of Hermitian kernels

Once again only the basic theory of Krein and admissible prehermitian subspaces was presented. Its implication in some other fields of mathematics is very significant (from system theory to quantum mechanics or algebraic curves). A good bibliography on such topics may be found in [3].

So Krein subspaces (or more generally admissible prehermitian subspaces) may be seen as a generalization of Hilbertian subspaces (that are Krein and admissible prehermitian subspaces) as equivalently Hermitian kernels are a generalization of positive kernels.
Looking at kernels, we still have two ways to generalize. The first is based on the following result: any kernel is the sum of a self-adjoint and anti-self-adjoint kernel. Following this idea and in the spirit of the Kolmogorov decomposition, D. Alpay ([2] or [1]) proposes the study of kernels of the form:

$$\varkappa = \varkappa_+ - \varkappa_- + i\chi_+ - i\chi_-$$

where $\varkappa_+, \varkappa_-, \chi_+, \chi_-$ are positive kernels (he precisely deals with reproducing kernels, i.e. positive kernel functions). This leads him to the concept of reproducing kernel Hilbert spaces of pairs and we refer to his papers for further study of these spaces.

The second way is to continue the formalism of admissible prehermitian subspaces. The kernels would however not need to be Hermitian and similarly we will need non Hermitian structures: dualities.

Chapter 3

Subdualities

Introduction

A careful reader of the previous chapters will certainly have noticed that in many theorems and proofs the use of the duality $(H, \overline{H})$ (of Hilbert or Krein spaces) shades a new light notably by introducing two "equal" embeddings (canonical injections) i and j but with distinct transposes in general. This remark is the starting point of this chapter where Hilbertian or Krein subspaces are generalized to dualities (i.e. we introduce a dual system of vector spaces) verifying certain algebraic inclusions (definition 3.2) called subdualities.

These spaces verify the main properties of Hilbertian subspaces (that appear to be particular instances of subdualities) and the set of subdualities may actually be endowed with a vector space structure (given an equivalence relation) isomorphic to the vector space of kernels (theorem 3.13).

This chapter is devoted to the study of these subdualities. A topological definition equivalent to definition 3.2, is that a duality (E, F) is a subduality of the dual system $(\mathcal{E}, \mathcal{F})$ if and only if both E and F are weakly continuously embedded in $\mathcal{E}$ (proposition 3.3). It appears that we can associate to any subduality a unique kernel (in the sense of L. Schwarz, theorem 3.6), whose image is dense in the subduality (theorem 3.10). Figure 3.1 illustrates the different inclusions related to a subduality and its kernel.

Then we study the image of a subduality by a weakly continuous linear operator (theorem 3.12) that makes it possible to define a vector space structure over the (quotiented) set of subdualities (theorem 3.13). A canonical representative entirely defined by the kernel is then given (theorem 3.20). Finally, we study more precisely some particular case of subdualities.

3.1 Subdualities and associated kernels

In this section, we introduce a new mathematical object that we call subduality of a dual system of vector spaces (or equivalently subduality of a locally convex topological vector space). These objects appear to be closely linked with kernels (theorem 3.6 and lemma 3.9) and could therefore be the appropriate setting to study such linear applications.

Hilbertian subspaces and Krein subspaces appear to be subdualities that are therefore a good generalization of the previous concepts. Prehilbertian and prehermitian subspaces are however also subdualities and the class of subdualities may be too general for certain applications. In particular a problem of completion appears. We will address this problem and the choice of a "good" topology to perform the completion in the section 3.3 "canonical subdualities".

3.1.1 Subdualities of a dual system of vector space

Definitions

The definition of subdualities remains heavily on the definition of a duality that therefore is restated below. This definition, the related notations and two basic examples are also given in the Appendix B.

Definition 3.1 (– *dual system of spaces* –) *Two vector spaces E, F are said to be in duality if there exists a bilinear form L on the product space $F \times E$ separate in E and F, i.e.:*

1. $\forall e \neq 0 \in E, \exists f \in F,\ L(f, e) \neq 0$;

2. $\forall f \neq 0 \in F, \exists e \in E,\ L(f, e) \neq 0$.

In this case, (E, F) is said to be a duality (relative to L).

The following morphisms are then well defined:

$$\begin{array}{rcl} \gamma_{(E,F)} : F & \longrightarrow & E^* \text{ algebraic dual of E} \\ y & \longmapsto & L(y, .) \end{array} \qquad \begin{array}{rcl} \theta_{(E,F)} : E' \triangleq \gamma_{(E,F)}(F) & \longrightarrow & F \\ L(y, .) & \longmapsto & y \end{array}$$

We can now give the definition of subdualities. Subdualities may be seen as completely algebraic objects and therefore the first definition is purely algebraic. $\forall A \subset \mathcal{E},\ u_{|A}$ denotes the restriction of u to the set A.

Definition 3.2 (– *subdualities* –) *Let (E, F) and $(\mathcal{E}, \mathcal{F})$ be two dualities.*
(E, F) is a subduality of $(\mathcal{E}, \mathcal{F})$ if (figure 3.1) :

- $E \subseteq \mathcal{E},\ \ F \subseteq \mathcal{E}$;

- $\gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|E)} \subseteq \gamma_{(E,F)}(F),\ \ \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|F}) \subseteq \gamma_{(F,E)}(E)$.

We note $\mathcal{SD}((\mathcal{E}, \mathcal{F}))$ the set of subdualities of $(\mathcal{E}, \mathcal{F})$.

If $\mathcal{E}$ is a locally convex space, we say that (E, F) is a subduality of $\mathcal{E}$ if it is a subduality of $(\mathcal{E}, \mathcal{E}')$ and we denote by $\mathcal{SD}(\mathcal{E})$ the set of subdualities of the l.c.s. $\mathcal{E}$. The second condition only states that every vector of $\mathcal{F}$, as a linear form on $E \subset \mathcal{E}$ (resp. on $F \subset \mathcal{E}$), is in F (respectively in E), *i.e*

$$\forall \varphi \in \mathcal{F},\ \text{there exists}\ f \in F,\ \forall e \in E,\ (\varphi, e)_{(\mathcal{F},\mathcal{E})} = (f, e)_{(F,E)}$$

We can however interpret the previous algebraic inclusions in topological terms, since dualities make a bridge between topological and algebraic properties. An equivalent topological definition of subdualities is then included in the following theorem:

Proposition 3.3 (– ***topological characterization*** **–)** *The following statements are equivalent:*

1. (E, F) *is a subduality of* $(\mathcal{E}, \mathcal{F})$*,*
2. *The canonical injections* $i : E \mapsto \mathcal{E}$ *and* $j : F \mapsto \mathcal{E}$ *are weakly continuous,*
3. $i : E \mapsto \mathcal{E}$ *et* $j : F \mapsto \mathcal{E}$ *are continuous with respect to the Mackey topologies on* E, F *and* $\mathcal{E}$.

The equivalence between (1) and (3) is notably useful in case of metric spaces, since any locally convex metrizable topology is the Mackey topology (corollary p 149 [26] or proposition 6 p 71 [15]).

Proof. – Let us show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$:

$(1) \Rightarrow (2)$ We define the following mappings:

$$i : E \mapsto \mathcal{E}, \qquad j : F \mapsto \mathcal{E},$$
$$i' : \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}) \mapsto \gamma_{(E,F)}(F), \qquad j' : \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}) \mapsto \gamma_{(F,E)}(E)$$

i and i' (resp. j and j') are transposes for the weak topology hence weakly continuous since

$\forall \varepsilon' \in \mathcal{E}' = \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}), \exists i'(\varepsilon') \in E' = \gamma_{(E,F)}(F),\ \forall e \in E :$

$$(\varepsilon', i(e))_{\mathcal{E}',\mathcal{E}} = (i'(\varepsilon'), e)_{E',E}$$

that is exactly the definition of the transpose. It is the classical link between inclusion of the topological dual and weak continuity.

$(2) \Rightarrow (3)$ Since i' (resp.j') is weakly continuous, its transpose is continuous for the Mackey topologies (corollary 3 p 111 [26]). We could also cite corollary 2 p 111 [26]: if $u : E \mapsto \mathcal{E}$ is weakly continuous, then it is continuous if E is endowed with the Mackey topology and $\mathcal{E}$ with any compatible topology).

$(3) \Rightarrow (1)$ Since $i : E \mapsto \mathcal{E}$ and $j : F \mapsto \mathcal{E}$ are continuous for the Mackey topologies, their transposes ${}^t i : \mathcal{E}' \mapsto E'$ and ${}^t j : \mathcal{E}' \mapsto F'$ exist. But $\mathcal{E}' = \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F})$ and $E' = \gamma_{(E,F)}(F)$ (resp. $F' = \gamma_{(F,E)}(E)$) since the Mackey topology is compatible with the duality, that prove the result. □

Remark 3.4 *If (E, F) is a subduality of $(\mathcal{E}, \mathcal{F})$, then (F, E) is also a subduality of $(\mathcal{E}, \mathcal{F})$.*

The crucial part of this definition of subdualities is not the continuity of the inclusion, that is the same requirement than for Hilbertian or Krein subspaces, but the fact that one needs two continuous inclusion, one for each space E and F defining the duality (E, F). The case of inner product spaces is of special interest and we will study them after these 5 examples:

example 1 **$\mathbb{R}^2$-example**

A classical bilinear form over $\mathbb{R}^2$ is the symplectic form that associates to each couple of vectors the oriented area of the parallelogram they define. Precisely the bilinear form is

$$\begin{array}{rcl} L : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (Y, X) & \longmapsto & x_1 y_2 - x_2 y_1 \end{array}$$

and the dual system $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with this bilinear form is a subduality of $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with the canonical Euclidean duality.

<u>example 2</u> **Sobolev spaces**

Let Ω be a bounded open set of $\mathbb{R}$ and $\mathcal{E} = D'(\Omega)$, $\mathcal{F} = D(\Omega)$. Define

$$E = \left\{ e \in D',\ e(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt, \phi \in L^2(\Omega) \right\}$$

and

$$F = \left\{ f \in D',\ f(t) = \int_\Omega 1\!\!1_{t \leq s} \psi(s) ds, \psi \in L^2(\Omega) \right\}$$

in duality with respect to the bilinear form

$$(f, e)_{(F,E)} = \int_\Omega \psi(u) \phi(u) du$$

It is straightforward to see that (E, F) is a subduality of the dual system (D', D). Remark that $E \neq F$.

<u>example 3</u> **Sobolev spaces (-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

let $\Omega =]0,1[$ and define

$$E = F = W^{\frac{3}{2}}(\Omega) = \left\{ g \in D',\ g(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt,\ \phi \in W^{\frac{1}{2}}(\Omega) \right\}$$

where

$$W^{\frac{1}{2}}(\Omega) = \left\{ \phi \in D',\ \int_{\Omega \times \Omega} \left| \frac{\phi(t) - \phi(s)}{(t-s)^2} \right| ds dt < \infty \right\}$$

These spaces are called Sobolev-Slobodeckij spaces[1] or Besov or fractional Sobolev spaces.

E and F can be put in duality with respect to the (separate) bilinear form

$$(f, e)_{(F,E)} = \int_\Omega \psi(u) \frac{d}{du} \phi(u) du$$

($f(t) = \int_\Omega 1\!\!1_{s \leq t} \psi(s) ds, \quad e(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt$).

It is straightforward to see that (E, F) is a subduality of the dual system (D', D).

Remark that

$$\mathfrak{E} = \mathfrak{F} = W^{\frac{1}{2}}(\Omega)$$

[1] See for instance [51]

with bilinear form

$$(\psi, \phi)_{(\mathfrak{F},\mathfrak{E})} = \int_{\Omega} \psi(u) \frac{d}{du} \phi(u) du$$

defines equivalently a subduality of the dual system (D', D).

Remark that this bilinear form has a very interesting property: it is invariant under the action of any diffeomorphism $h : \Omega \longrightarrow \Omega$. It is a reason to prefer this particular bilinear form rather than an Hilbertian one.

example 4 **Polynomials, splines**

In [42] the authors consider the spaces $E = F = \mathcal{P}_n$ of polynomials of degree n and the following bilinear form on $F \times E$

$$\begin{aligned} L : F \times E &\longrightarrow \mathbb{R} \\ (f, e) &\longmapsto \sum_{j=0}^{n} \frac{(-1)^{n-j}}{n!} f^{(j)}(\tau) e^{(n-j)}(\tau) \end{aligned}$$

that does not depend on the particular point τ chosen.

It is straightforward to see that this duality is separate (by using the monomials) and that E and F endowed with the weak-topology are continuously included in the l.c.s. $\mathbb{R}^{\mathbb{R}}$ endowed with the topology of simple convergence (or topology product).

(E, F) is then a subduality of $\mathbb{R}^{\mathbb{R}}$.

example 5 **Polynomials, splines** **(- Piecewise smooth spaces -)**

In [37] the authors study piecewise smooth spaces in duality. A general version of their results is the following:

Let E and F be two piecewise smooth spaces[2] (on a set Ω) of dimension $n + 1$ and $(e_0, ..., e_n)$, $(f_0, ..., f_n)$ two basis of E and F respectively.

Then the bilinear form defined by

$$(f_j, e_i)_{(F,E)} = \delta_{ij}$$

[2] we refer to [37] for the definition of such spaces

puts E and F in duality and E and F endowed with the weak-topology are continuously included in the l.c.s. $\mathbb{R}^\Omega$ endowed with the topology of simple convergence (or topology product). In fact, any element of $\mathbb{R}^{\Omega'}$ admits a representative in E **and** F since

$$(\delta_s, e)_{(\mathbb{R}^{\Omega'},\mathbb{R}^\Omega)} = \left([\sum_{j=0}^{n} f_j(.)e_j(s)], e \right)_{(F,E)}$$

$$(\delta_t, f)_{(\mathbb{R}^{\Omega'},\mathbb{R}^\Omega)} = \left(f, [\sum_{j=0}^{n} f_j(t)e_j(.)] \right)_{(F,E)}$$

where $\sum_{j=0}^{n} f_j(t)e_j(.) \in E$ and $\sum_{j=0}^{n} f_j(.)e_j(s) \in F$.
(E, F) is then a subduality of $\mathbb{R}^\Omega$ (we will study the subdualities of $\mathbb{R}^\Omega$ in detail in the section "evaluation subdualities").

Inner product spaces

Inner product spaces like Hilbert spaces, Krein spaces or generally prehermitian spaces are self-dual[3] *i.e* $E = F$ with the same weak topologies. Then when the previous conditions are fulfilled for E, they are automatically fulfilled for F, hence all prehermitian subspaces are subdualities.
The previous concepts of Hilbertian subspaces, Krein subspaces or prehermitian subspaces are then particular cases of the more general notion of subdualities.

Theorem 3.5 *Let H be an inner product space, (H, H) the (self-)duality induced by the inner product. Then (H, H) is a subduality of the dual system $(\mathcal{E}, \mathcal{F})$ if and only if H is weakly continuously included in $\mathcal{E}$. In this case, we say that the inner product space H is a self-subduality of $(\mathcal{E}, \mathcal{F})$.*

[3] we still suppose that we have an anti-involution

Proof. – Evident since $E = F = H$. □

3.1.2 The kernel of a subduality

Theorem 3.6 (– *kernel of a subduality* –) *Each subduality (E, F) of $(\mathcal{E}, \mathcal{F})$ is associated with a unique kernel $\varkappa$ of $(\mathcal{E}, \mathcal{F})$ verifying*

$$\forall f \in F, \forall \varphi \in \mathcal{F},\ (\varphi, j(f))_{(\mathcal{F},\mathcal{E})} = \big(f, i^{-1}\varkappa(\varphi)\big)_{(F,E)}$$

Z

called kernel of the subduality (E, F) of $(\mathcal{E}, \mathcal{F})$. It is the linear application

$$\begin{aligned} \varkappa : \mathcal{F} &\longrightarrow \mathcal{E} \\ \varphi &\longmapsto i \circ \theta_{(F,E)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi) \end{aligned}$$

considering transposition[4] *in the topological dual spaces or simply*

$$\begin{aligned} \varkappa : \mathcal{F} &\longrightarrow \mathcal{E} \\ \varphi &\longmapsto i \circ^t j(\varphi) \end{aligned}$$

considering transposition in dual systems.

Proof. – If we consider transposition in the topological duals:
$\forall f, \in F,\ \varphi \in \mathcal{F}$

$$\begin{aligned} (\varphi, j(f))_{(\mathcal{F},\mathcal{E})} &= ({}^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi), f)_{(F',F)} \\ &= (f, \theta_{(F,E)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi))_{(F,E)} \\ &= (f, i^{-1}(i \circ \theta_{(F,E)} \circ^t j) \circ \gamma_{(\mathcal{E},\mathcal{F})}(\varphi))_{(F,E)} \end{aligned}$$

The solution is unique since $L(.,.) = (.,.)_{(F,E)}$ separates E and F and

$$\varkappa = i \circ \theta_{(F,E)} \circ^t j \circ \gamma_{(\mathcal{E},\mathcal{F})}$$

[4] *see Appendix B*

If we consider transposition in dual systems, then the proof is direct:

$$(\varphi, j(f))_{(\mathcal{F},\mathcal{E})} = ({}^t j(\varphi), f)_{(E,F)} = L(f, i^{-1} \circ^t j(\varphi))$$

Finally, $\varkappa$ is weakly continuous by composition of weakly continuous linear applications. □

The concept of subduality and of its associated kernel is illustrated by figure 3.1 and figure 3.2. In figure 3.1 we consider transposition in the topological dual spaces and in figure 3.2 transposition in dual systems.

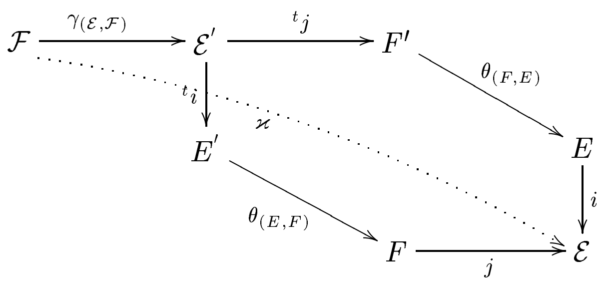

Figure 3.1: Illustration of a subduality, the relative inclusions and its kernel.

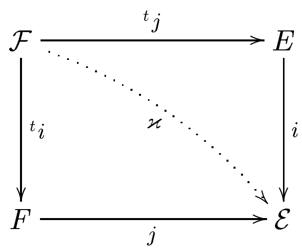

Figure 3.2: Illustration of a subduality and its kernel (transposition in dual systems).

From now on and for the sake of simplicity, we will always consider transposition in dual systems.

We can then define as for Hilbertian subspaces the application

$$\Phi : \mathcal{SD}((\mathcal{E},\mathcal{F})) \longrightarrow \mathbf{L}(\mathcal{F},\mathcal{E})$$

that associates to each subduality its kernel. It is a well defined function.

The following lemma can then be deduced directly from theorem 3.3:

Lemma 3.7 *$\varkappa : \mathcal{F} \longrightarrow E$ is weakly continuous if E and $\mathcal{F}$ are equipped with:*

1. *the weak topologies,*
2. *the Mackey topologies.*

We have seen previously that (F, E) is also a subduality of $(\mathcal{E},\mathcal{F})$. Its kernel is the linear application $\widetilde{\varkappa} = j \circ^t i$ i.e. $\widetilde{\varkappa} =^t \varkappa$.

Examples:

example 1 **$\mathbb{R}^2$-example**

The dual system $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with the symplectic bilinear form

$$\begin{array}{rcl} L : \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (Y, X) & \longmapsto & x_1 y_2 - x_2 y_1 \end{array}$$

is a subduality of $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with the canonical Euclidean duality with kernel

$$\begin{array}{rcl} \varkappa : R^2 & \longrightarrow & \mathbb{R}^2 \\ X & \longmapsto & (x_2, -x_1) \end{array}$$

example 2 **Sobolev spaces**

Suppose $\Omega =]0, 1[$. The kernel of the subduality (E, F) of (D', D) where

$$E = \left\{ e \in D',\ e(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt, \phi \in L^2(\Omega) \right\}$$

and

$$F = \left\{ f \in D',\ f(t) = \int_\Omega 1\!\!1_{t\leq s}\psi(s)ds, \psi \in L^2(\Omega) \right\}$$

are in duality with respect to the bilinear form

$$(f,e)_{(F,E)} = \int_\Omega \psi(u)\phi(u)du$$

is the integral operator

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(.) = \int_\Omega K(t,.)\varphi(t)dt \end{array}$$

where

$$K(t,s) = (s-t)\,1\!\!1_{t\leq s}$$

The kernel of (F,E) is defined by the distribution

$$^tK(t,s) = (t-s)1\!\!1_{s\leq t} = K(s,t)$$

example 3 **Sobolev spaces** **(-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

The previous subduality (E,F) of the dual system (D',D) has for kernel the integral operator:

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(.) = \int_\Omega K(t,.)\varphi(t)dt \end{array}$$

where

$$K(t,s) = \int_0^s \min(u,t)dt = t\min(t,s) - \frac{\min(t,s)^2}{2}$$

Note that this kernel is not self-adjoint.

The subduality $(\mathfrak{E},\mathfrak{F})$ has for kernel

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(.) = \int_\Omega \mathfrak{K}(t,.)\varphi(t)dt \end{array}$$

where

$$\mathfrak{K}(t,s) = 1\!\!1_{t\leq s}$$

example 4 **(- The fundamental example of a Hilbert space -)**

Let H be a Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$ and define the following bilinear form on $\overline{H} \times H$ such that $(H, \overline{H})$ is a duality

$$\begin{aligned} L : \overline{H} \times H &\longrightarrow \mathbb{K} \\ \overline{h_1}, h_2 &\longmapsto \langle h_1 | h_2 \rangle \end{aligned}$$

It is a subduality of $(\mathcal{E}, \mathcal{F})$ with positive kernel $\varkappa = i \circ^t j$ where $i : H \longrightarrow \mathcal{E}$ and $j : \overline{H} \longrightarrow \mathcal{E}$ are the canonical injections. Its transpose ${}^t\varkappa = j \circ^t i = \overline{\varkappa}$ is the kernel of the subduality $(\overline{H}, H)$.

3.1.3 The range of the kernel: the primary subduality

The image (or range) of a positive kernel played a special role in the theory of Hilbertian subspaces: it was a prehilbertian subspace dense in the Hilbertian subspace, that was actually its completion. This latter point cannot be attained for the moment due to the to big generality of subdualities[5]. However, the two other points remain for any kernel as we will see below.

Definition 3.8 (– ***primary subduality*** **–)** *We call primary subduality associated to a kernel $\varkappa$ the subspaces of $\mathcal{E}$ $E_0 = \varkappa(\mathcal{F})$ and $F_0 =^t \varkappa(\mathcal{F})$ put in duality by the following bilinear form L_0:*

$$\begin{aligned} L_0 : F_0 \times E_0 &\longrightarrow \mathbb{K} \\ {}^t\varkappa(\varphi_1), \varkappa(\varphi_2) &\longmapsto (\varphi_1, \varkappa(\varphi_2))_{(\mathcal{F}, \mathcal{E})} = \left({}^t\varkappa(\varphi_1), \varphi_2\right)_{(\mathcal{E}, \mathcal{F})} \end{aligned}$$

Remark that the bilinear form is well defined since the elements of $\ker(\varkappa)$ are orthogonal to ${}^t\varkappa(\mathcal{F})$ and respectively, the elements of $\ker({}^t\varkappa)$ are orthogonal to $\varkappa(\mathcal{F})$.

Lemma 3.9 *The primary subduality is a subduality of $(\mathcal{E}, \mathcal{F})$. Its kernel is $\varkappa$. Any kernel may then be associated to at least one subduality.*

[5] That will however be the crucial point in the section 3.3 “canonical subdualities”

Proof. – From the definition of the primary duality we verify easily that

- $E \subseteq \mathcal{E}, \;\; F \subseteq \mathcal{E}$;
- $\gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|E)} \subseteq \gamma_{(E,F)}(F), \;\; \gamma_{(\mathcal{E},\mathcal{F})}(\mathcal{F}_{|F}) \subseteq \gamma_{(F,E)}(E)$.

and from the definition of L_0 that its kernel is $\varkappa$. □

The following theorem gives an interesting result of denseness:

Theorem 3.10 *Let (E, F) be a subduality with kernel $\varkappa$. Then the primary subduality (E_0, F_0) associated to $\varkappa$ is dense in (E, F) for any topology compatible with the duality.*

Proof. – We use corollary p 109 [26]: "If $u : E \longrightarrow \mathcal{E}$ is one-to-one, its transpose ${}^t u : \mathcal{E}' \longrightarrow E'$ has weakly dense image". Equivalently its transpose considering dual systems ${}^t u : \mathcal{F} \longrightarrow F$ has weakly dense image. Taking $u = j$ gives the desired result since there is an equivalence between closure and weak closure for convex sets (and E_0 is convex), theorem 4 p 79 [26]. □

It follows that the primary subduality associated with $\varkappa$ may be seen as the smallest subduality (in terms of inclusion) of $(\mathcal{E}, \mathcal{F})$ with kernel $\varkappa$.

Examples:

example 1 **$\mathbb{R}^2$-example**

Since we work with finite dimensional spaces, all subdualities with the same kernel are equal (to the primary subduality), in our previous case, to the dual system $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with the symplectic bilinear form

$$\begin{aligned} L : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (Y, X) &\longmapsto x_1 y_2 - x_2 y_1 \end{aligned}$$

<u>example 2</u> **Sobolev spaces**

We see easily that the primary subduality associated to the kernel

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(]0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(t) = \int_\Omega K(t,s)\varphi(s)ds \end{array}$$

where

$$K(t,s) = (s-t)\,1\!\!1_{t\leq s}$$

is

$$E_0 = \left\{\phi \in D(]0,1[),\ \lim_{s\downarrow 0}\phi(s) = 0,\ \lim_{s\downarrow 0}\phi'(s) = 0\right\}$$

and

$$F_0 = \left\{\psi \in D(]0,1[),\ \lim_{t\uparrow 1}\psi(t) = 0,\ \lim_{t\uparrow 1}\phi'(t) = 0\right\}$$

3.2 Effect of a weakly continuous linear application and algebraic structure of $\mathcal{SD}((\mathcal{E},\mathcal{F}))$

We have defined the set of subdualities. It is of prime interest to know what operations one can perform on this set and particularly if one can endow this set with the structure of a vector space. This can be attained by first studying the effect of a weakly continuous linear application.

3.2.1 Effect of a weakly continuous linear application

We suppose now we are given a second pair of spaces in duality $(\mathfrak{E},\mathfrak{F})$. We have seen in the first chapter how a Hilbertian structure can be transported by a weakly continuous linear application thanks to the existence of orthogonal decomposition in Hilbert spaces and that one can extend this construction to admissible prehermitian subspaces if it is carefully done.

Following the same spirit, it is possible to define the image subduality by a weakly continuous linear application $u : \mathcal{E} \to \mathfrak{E}$, of a subduality (E, F) of $(\mathcal{E}, \mathcal{F})$, by using orthogonal relations in the duality (E, F).

$\forall A \subset \mathcal{E},\ u_{|A}$ denotes the restriction of u to the set A. We then define the following quotient spaces:

$$\mathcal{M} = \left(\ker(u_{|F})^{\perp} / \ker(u_{|E})\right)$$

Z

and

$$\mathcal{N} = \left(\ker(u_{|E})^{\perp} / \ker(u_{|F})\right)$$

Lemma 3.11 *The linear applications $u_{|\mathcal{M}}$ and $u_{|\mathcal{N}}$ are well defined and injective, and $\forall\, (\dot{m}, \dot{n}) \in \mathcal{M} \times \mathcal{N}$, the bilinear form $B(u_{|\mathcal{N}}(\dot{n}), u_{|\mathcal{M}}(\dot{m})) = (n, m)_{(F,E)}$ defines a separate duality $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$.*

Proof. – We have the following factorization

$$u : \ker(u_{|F})^{\perp} \longrightarrow (\ker(u_{|F})^{\perp} / \ker(u_{|E}) \stackrel{u_{|\mathcal{M}}}{\longrightarrow} \mathfrak{E}$$

and $u_{|\mathcal{M}}$ (resp. $u_{|\mathcal{N}}$) is one-to-one. Moreover the bilinear form $B : u_{|\mathcal{M}}(\mathcal{M}) \times u_{|\mathcal{N}}(\mathcal{N}) \longrightarrow \mathbb{K}$ is well defined since:

$\forall (m_1, m_2) \in \dot{m},\ \forall (n_1, n_2) \in \dot{n},\ (m_1 - m_2, n_1 - n_2)_{(E,F)} = 0.$ □

The definition of the subduality image of (E, F) by u is then included in the following theorem:

Theorem 3.12 (– *subduality image* –) *The duality $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$ is a subduality of $(\mathfrak{E}, \mathfrak{F})$ called subduality image of (E, F) by u and denoted $u((E, F))$. Its kernel is $u \circ \varkappa \circ {}^t u$.*

Proof. – The algebraic inclusions of definition 3.2 are fulfilled and the dual system $(u_{|\mathcal{M}}(\mathcal{M}), u_{|\mathcal{N}}(\mathcal{N}))$ is a subduality of $\mathfrak{E}$.

Let $\tilde{i} : u_{|\mathcal{M}}(\mathcal{M}) \longrightarrow \mathfrak{E}$ and $\tilde{j} : u_{|\mathcal{N}}(\mathcal{N}) \longrightarrow \mathfrak{E}$ be the canonical inclusions. $u \circ \varkappa \circ^t u$ satisfies the requirements of theorem 3.6 since:

$$\forall n \in u_{|\mathcal{N}}(\mathcal{N}), \forall \mathfrak{f} \in \mathfrak{F},\ \left(\mathfrak{f}, \tilde{j}(n)\right)_{\mathfrak{F},\mathfrak{E}} = B\left(n, \tilde{i}^{-1} \circ u \circ \varkappa \circ^t u(\mathfrak{f})\right)$$

Let f an antecedent by u of n in F. Then:

$$\begin{aligned} B\left(n, \tilde{i}^{-1} \circ u \circ \varkappa \circ^t u(\mathfrak{f})\right) &= (f, \varkappa \circ^t u(\mathfrak{f}))_{(F,E)} \\ &= (f, {}^t u(\mathfrak{f}))_{(\mathcal{E},\mathcal{F})} \\ &= (u(f), \mathfrak{f})_{(\mathfrak{E},\mathfrak{F})} \\ &= \left(\mathfrak{f}, \tilde{j}(n)\right)_{(\mathfrak{F},\mathfrak{E})} \end{aligned}$$

□

Remark that the subduality image $u\left((E, F)\right)$ is included in the set $(u(E), u(F))$ but smaller in general.

Examples:

example 1 **$\mathbb{R}^2$-example**

Let $\mathcal{E} = \mathcal{F} = \mathbb{R}^2$ endowed with the Euclidean inner product and

$$\mathfrak{E} = \mathfrak{F} = \mathbb{R}.\cos(\frac{\pi}{2}t) + \mathbb{R}.\sin(\frac{\pi}{2}t)$$

(subspace of $\mathcal{F}([0,1], \mathbb{R})$) in self-duality with respect to the (positive) bilinear form

$$\begin{aligned} L : \mathfrak{F} = \mathfrak{E} \times \mathfrak{E} &\longrightarrow \mathbb{R} \\ (\mathfrak{f}, \mathfrak{e}) &\longmapsto \mathfrak{f}(0)\mathfrak{e}(0) + \mathfrak{f}(1)\mathfrak{e}(1) \end{aligned}$$

Then the image of the subduality (of $(\mathcal{E},\mathcal{F})$) $(\mathbb{R}^2,\mathbb{R}^2)$ endowed with the symplectic bilinear form

$$\begin{aligned} L:\mathbb{R}^2\times\mathbb{R}^2 &\longrightarrow \mathbb{R}\\ (Y,X) &\longmapsto x_1y_2-x_2y_1 \end{aligned}$$

by the weakly continuous linear mapping

$$\begin{aligned} u:\mathbb{R}^2 &\longrightarrow \mathfrak{E}\\ X=(x_1,x_2) &\longmapsto u(X)=x_1\cos(t)+x_2\sin(t) \end{aligned}$$

defines a subduality of $(\mathfrak{E},\mathfrak{F})$ $u((\mathbb{R}^2,\mathbb{R}^2))$.

It is the system $u((\mathbb{R}^2,\mathbb{R}^2))=(\mathfrak{E},\mathfrak{F})$ in duality with respect to the bilinear form

$$\begin{aligned} L:\mathfrak{F}=\mathfrak{E}\times\mathfrak{E} &\longrightarrow \mathbb{R}\\ (\mathfrak{f},\mathfrak{e}) &\longmapsto -\mathfrak{f}(1)\mathfrak{e}(0)+\mathfrak{f}(0)\mathfrak{e}(1) \end{aligned}$$

and its kernel is

$$\begin{aligned} \varkappa:\mathfrak{F}=\mathfrak{E} &\longrightarrow \mathfrak{E}\\ \mathfrak{f} &\longmapsto \varkappa(\mathfrak{f})(t)=\mathfrak{f}(1)\cos(\tfrac{\pi}{2}t)-\mathfrak{f}(0)\sin(\tfrac{\pi}{2}t)=\tfrac{2}{\pi}\mathfrak{f}'(t) \end{aligned}$$

example 2 **$\mathbb{R}^2$-example**

Let $\mathcal{E}=\mathcal{F}=\mathbb{R}^2$ endowed with the Euclidean inner product. Then the image of the subduality (of $(\mathcal{E},\mathcal{F})$) $(\mathbb{R}^2,\mathbb{R}^2)$ endowed with the symplectic bilinear form

$$\begin{aligned} L:\mathbb{R}^2\times\mathbb{R}^2 &\longrightarrow \mathbb{R}\\ (Y,X) &\longmapsto x_1y_2-x_2y_1 \end{aligned}$$

by the weakly continuous linear mapping

$$\begin{aligned} u:\mathbb{R}^2 &\longrightarrow \mathbb{R}^2\\ X=(x_1,x_2) &\longmapsto u(X)=(x_1,0) \end{aligned}$$

is $u((\mathbb{R}^2,\mathbb{R}^2))=(0,0)$.

Actually we have that

$$\ker(u_{|F})^{\perp}=\ker(u_{|E})$$

and

$$\ker(u_{|E})^{\perp} = \ker(u_{|F})$$

hence

$$\mathcal{M} = \left(\ker(u_{|F})^{\perp} / \ker(u_{|E})\right) = 0$$

and

$$\mathcal{N} = \left(\ker(u_{|E})^{\perp} / \ker(u_{|F})\right) = 0$$

(whereas $u(F) = u(E) = (\mathbb{R}, 0)$).

example 3 **Sobolev spaces** **(-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

For the example of $W^{\frac{3}{2}}(]0,1[)$, it is straightforward to see that the subduality of (D', D) (E, F) where

$$E = F = W^{\frac{3}{2}}(\Omega) = \left\{ g \in D',\ g(s) = \int_{\Omega} 1\!\!1_{t \leq s} \phi(t) dt,\ \phi \in W^{\frac{1}{2}}(\Omega) \right\}$$

are put in duality with respect to the (separate) bilinear form

$$(f, e)_{(F,E)} = \int_{\Omega} \psi(u) \frac{d}{du} \phi(u) du$$

is the image of the subduality $(\mathfrak{E}, \mathfrak{F})$ where

$$\mathfrak{E} = \mathfrak{F} = W^{\frac{1}{2}}(\Omega)$$

with bilinear form

$$(\psi, \phi)_{(\mathfrak{F},\mathfrak{E})} = \int_{\Omega} \psi(u) \frac{d}{du} \phi(u) du$$

under the weakly continuous mapping

$$\begin{array}{rcl} u : D' & \longrightarrow & D' \\ \phi & \longmapsto & u(\phi)(s) = \int_{\Omega} 1\!\!1_{t \leq s} \phi(t) dt \end{array}$$

Conversely, $(\mathfrak{E}, \mathfrak{F})$ can be seen as the subduality image of (E, F) by the mapping

$$\begin{array}{rcl} v : D' & \longrightarrow & D' \\ g & \longmapsto & v(g) = \frac{d}{ds} g \end{array}$$

As for Hilbertian or Krein subspaces, the transport of structure is the basic tool to the construction of subdualities.

3.2.2 The vector space $(\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi),+,*)$

Theorem 3.12 allows us to define the operations of addition and external multiplication on the set $\mathcal{SD}((\mathcal{E},\mathcal{F}))$ by considering the weakly continuous morphisms $+ : \mathcal{E}\times\mathcal{E}\to\mathcal{E}$ and $* : \mathbb{K}\times\mathcal{E}\to\mathcal{E}$. The associated operations for the kernels are then addition and external multiplication on $\mathbf{L}(\mathcal{F},\mathcal{E})$.

However the addition is neither injective nor associative (it is yet not associative upon the subset of admissible prehermitian subspaces):

$$\{(E_1,F_1)-(E_1,F_2)=0\} \not\Rightarrow \{(E_1,F_1)=(E_2,F_2)\}$$

$$((E_1,F_1)+(E_1,F_2))+(E_3,F_3)\neq(E_1,F_1)+((E_1,F_1)+(E_2,F_2)) \text{ in general}$$

and appears the necessity of the following equivalence relation (induced by $\ker(\Phi)$):

$$(E_1,F_1)\mathcal{R}(E_2,F_2) \iff (E_1,F_1)-(E_2,F_2)=0 \iff \varkappa_1=\varkappa_2$$

Theorem 3.13 (– *algebraic structure* –) *The set $(\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi),+,*)$ is a vector space over $\mathbb{K}$ algebraically isomorphic to the vector space of kernels $\mathbf{L}(\mathcal{F},\mathcal{E})$, an isomorphism being $\Phi:\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi)\longrightarrow\mathbf{L}(\mathcal{F},\mathcal{E})$.*

Proof. – The following relation

$$(E_1,F_1)\mathcal{R}(E_2,F_2) \iff (E_1,F_1)-(E_2,F_2)=0 \iff \varkappa_1=\varkappa_2$$

is an equivalence relation and the quotient set $\mathcal{SD}(\mathcal{E})/\ker(\Phi)$ is in bijection with the set of kernels $\mathbf{L}(\mathcal{F},\mathcal{E})$.

One verifies rapidly that the addition and external multiplication are compatible with this bijection, which gives the vector space structure of the set $\mathcal{SD}(\mathcal{E})/\ker(\Phi)$ and the isomorphism of vector space between $\mathcal{SD}(\mathcal{E})/\ker(\Phi)$ and $\mathbf{L}(\mathcal{F},\mathcal{E})$. □

3.2.3 Categories and functors

Let $\mathcal{C}$ the category of dual systems $(\mathcal{E},\mathcal{F})$, (we do not require here $\mathcal{E}$ to be quasi-complete), the morphisms being the weakly continuous linear applications and $\mathcal{V}$ the category of vector spaces, the morphisms being the linear applications. Then according that to a morphism $u:\mathcal{E}\longrightarrow\mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl}\tilde{u}:\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi) & \longrightarrow & \mathcal{SD}((\mathfrak{E},\mathfrak{F}))/\ker(\Phi)\\ (E\dot{,}F) & \longmapsto & u((E\dot{,}F))\end{array}$$

we get

Theorem 3.14 $\frac{\mathcal{SD}}{\ker(\Phi)}:(\mathcal{E},\mathcal{F})\mapsto\mathcal{SD}((\mathcal{E},\mathcal{F}))/\ker(\Phi)$ *is a covariant functor of category $\mathcal{C}$ into category $\mathcal{V}$.*

On the other hand, $\mathbf{L}:(\mathcal{E},\mathcal{F})\mapsto\mathbf{L}(\mathcal{F},\mathcal{E})$ is also a covariant functor of category $\mathcal{C}$ into category $\mathcal{V}$, according that to a morphism $u:\mathcal{E}\longrightarrow\mathfrak{E}$ we associate the morphism

$$\begin{array}{rcl}\tilde{u}:\mathbf{L}(\mathcal{F},\mathcal{E}) & \longrightarrow & \mathbf{L}(\mathfrak{F},\mathfrak{E})\\ \varkappa & \longmapsto & u\circ\varkappa\circ^{t}u\end{array}$$

and

Theorem 3.15 *The two covariant functors $\frac{\mathcal{SD}}{\ker(\Phi)}$ and $\mathbf{L}$ are isomorphic.*

3.3 Canonical subdualities

The classes of equivalences of subdualities with identical kernel are very large and it may be interesting to associate each equivalence class with a canonical representative enjoying good

properties. This section aims at defining this particular set of subdualities that will be called canonical subdualities. The desired good properties (such that the equality with Hilbertian subspaces in case of positive kernels) are listed below.
Actually, before stating the main results of this part, one must ask the following question: what do we mean by canonical representative? And what good properties do we need?
There is probably not a single answer to these questions and there may be many different good ways to define canonical representatives. However, it seems natural to require some properties for a canonical representative. Those chosen here are:

1. the canonical representative must be "representative" of the kernel, i.e. entirely defined by the kernel;

2. the definition of the canonical representative must be "symmetric", i.e. if (E, F) is the canonical subduality associated to $\varkappa$, then (F, E) must be the canonical subduality associated to ${}^t\varkappa$;

3. the definition of the canonical representative must coincide with the definition of the Hilbertian subspace in case of positive kernels.

It is in this spirit that those canonical subdualities have been constructed.

Since Hilbertian subspaces may be seen as the completion of the primary subspace associated to the positive kernel it seems natural to mimic this construction up to a certain extent i.e. do some completion. The first task is then to define "canonical" topologies on the sets E and F.

3.3.1 Definition of the canonical topologies

The definition of a locally convex topology may take various form, one being by means of semi-norms and another by convergence on bounded sets of a dual space. These two are of course closely linked (see for instance [26]) but since we start with a dual system of space we prefer to use the second method. Precisely we aim at defining some "good" bounded sets. Our choice is as follows:
let $\varkappa \in \mathbf{L}(\mathcal{F}, \mathcal{E})$ be a kernel, (E_0, F_0) the associated primary subduality. We define the following sets:

- $\mathcal{T}_{E_0} = \Big\{\sigma$ barrels of $E_0,\ \exists(\lambda, \gamma) \in (\mathbb{R}^+)^2,\ \Re(\big(\varkappa^{-1}(\sigma), \overline{\sigma}\big)_{(\mathcal{F},\mathcal{E})}) \leq \lambda$ and $\Re(\big({}^t\varkappa^{-1}(\sigma^\circ), \overline{\sigma^\circ}\big)_{(\mathcal{F},\mathcal{E})}) \leq \gamma\Big\}$, where σ° is the polar[6] of σ for the duality (E_0, F_0);
- $\mathcal{T}_{F_0} = \{\sigma^\circ,\ \sigma \in \mathcal{T}_{E_0}\}$;

under the following convention:
$\Re(\big(\varkappa^{-1}(\sigma), \overline{\sigma}\big)_{(\mathcal{F},\mathcal{E})}) \leq \lambda$ stands for $\exists\varsigma \in \mathcal{F},\ \varkappa(\varsigma) = \sigma$ and $\Re((\varsigma, \overline{\sigma})_{(\mathcal{F},\mathcal{E})}) \leq \lambda$ (resp. for σ°). Z
Remark that this convention is useless for symmetric, Hermitian or antisymmetric kernels since $\ker(\varkappa)$ (resp. $\ker({}^t\varkappa)$) is orthogonal to ${}^t\varkappa(\mathcal{F})$ (resp. to $\varkappa(\mathcal{F})$) and obviously if the kernel $\varkappa$ is one-to-one.

$\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) is a set of weakly bounded sets of (E_0, F_0) and one can define over F_0 (resp. E_0) the topology of $\mathcal{T}_{E_0}$-convergence, this topology being locally convex and compatible with the vector space structure (proposition 16 p. 86 [26]).

Let us show that $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) is a set of weakly bounded sets:
Let $\sigma \in \mathcal{T}_{E_0}$. It is an equilibrated and absorbing set hence $\forall f \in F, \exists\alpha > 0, \alpha.f \in \mathcal{T}_{E_0}$

[6] Since we deal with barrels, the polar coincide with the absolute polar

and $(\sigma, f)_{(E,F)}$ is bounded. It follows that $\sigma^\circ \in \mathcal{T}_{F_0}$ is a barrel as the absolute polar of an equilibrated weakly bounded set (corollary 3 p 68 [15]) and finally, the elements of $\mathcal{T}_{F_0}$ are also weakly bounded.

3.3.2 Construction of the canonical subdualities

As for hermitian kernels that do not always admit a Kolmogorov decomposition, or for positive kernels that must verify a certain continuity condition when the space $\mathcal{E}$ is not Mackey quasi-complete to be Hilbertian kernels, additional conditions on the kernel are required to be able to construct canonical subdualities.

Definition 3.16 (– *stable kernel* –) *Let $\varkappa \in \mathbf{L}(\mathcal{F}, \mathcal{E})$ a kernel. It is stable if:*

1. *the sets $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$ are non empty;*
2. *$\varkappa : \mathcal{F} \longrightarrow E_0$ (resp. $\varkappa : \mathcal{F} \longrightarrow F_0$) is continuous if $\mathcal{F}$ is endowed with the Mackey topology and E_0 with the topology of $\mathcal{T}_{F_0}$-convergence (resp. F_0 with the $\mathcal{T}_{E_0}$-convergence).*

The first condition is necessary to be able to define the canonical topologies whereas the second condition is needed to perform the completion (see lemma 3.19 below).

Proposition 3.17 *The second condition is equivalent to:*
the elements of $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) are weakly relatively compact in $\mathcal{E}$.
This condition is always fulfilled if $\mathcal{F}$ is (Mackey) barreled.

Proof. – We use proposition 28 p 110 in [26]. The weakly continuous application $\varkappa = {}^t j : \mathcal{F} \longrightarrow E_0$ is continuous if $\mathcal{F}$ is endowed with the Mackey topology and E_0 with the topology of $\mathcal{T}_{F_0}$-convergence if and only if $j(\mathcal{T}_{F_0})$ is a set of weakly relatively compact sets of $\mathcal{E}$ (recall that the Mackey topology on $\mathcal{F}$ is the topology of convergence on the weakly compact sets of $\mathcal{E}$). □

Remark 3.18 *The term "relatively" may be surprising since the elements of* $\mathcal{T}_{E_0}$ *are weakly closed in* E_0 *(as barrels), the weak topology being the topology induced by the weak topology on* $\mathcal{E}$*. But the elements of* $\mathcal{T}_{E_0}$ *are not weakly closed in* $\mathcal{E}$ *in general; they are just of the form* $\sigma = E_0 \bigcap \overline{\sigma}$ *with* $\overline{\sigma}$ *being the weak closure of* σ *in* $\mathcal{E}$ *and being weakly compact.*

Lemma 3.19 *Let* $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ *be a stable kernel,* (E_0, F_0) *the associated primary duality. Let* $E = \widehat{E_0}$ *(resp.* $F = \widehat{F_0}$*) be the completion of* E_0 *endowed with the topology of* $\mathcal{T}_{F_0}$*-convergence (resp. the completion of* F_0 *endowed with the topology of* $\mathcal{T}_{E_0}$*-convergence). Then* E *(resp.* F*) is the vector space generated by the closures (in* $\widehat{E_0}$*, resp.* $\widehat{F_0}$*) of the convex envelopes of finite unions of elements of* $\mathcal{T}_{E_0}$ *(resp.*$\mathcal{T}_{F_0}$*) and* $E \subset \mathcal{E},\ F \subset \mathcal{E}$*.*

Proof. – First, $E = \widehat{E_0}$ is the vector space generated by the closures in $\widehat{E_0}$ of its neighborhoods of zero, i.e. by polarity by the closures of the convex envelopes of finite unions of elements of $\mathcal{T}_{E_0}$.
Second, if we endow $\mathcal{F}$ with the Mackey topology and F_0 with the $\mathcal{T}_{E_0}$-convergence, then $\varkappa : \mathcal{F} \longrightarrow F_0$ is continuous with dense image and $\varkappa : F_0' \longrightarrow \mathcal{E}$ is one-to-one. But F_0' is the vector space generated by the weak closures of the convex envelopes of finite unions of elements of $\mathcal{T}_{E_0}$ in the weak completion of E_0 (corollary 1 p 91 [26]). It follows that $E \subset F_0' \subset \mathcal{E}$ since $\widehat{E_0}$ is continuously included in the weak completion of E_0. □

Theorem 3.20 *Let* $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ *be a stable kernel,* (E_0, F_0) *the associated primary duality,* E *and* F *defined as before. Then the bilinear form* L_0 *defined on the primary duality extends to a unique bilinear form* L *on* $F \times E$ *separate. It defines a duality* (E, F) *called canonical subduality associated to* $\varkappa$*.*

Proof. – We use the extension of bilinear hypocontinuous forms theorem (proposition 8 p 41 [15]). We endow E (resp. F) with the topology of $\mathcal{T}_{F_0}$ (resp. $\mathcal{T}_{E_0}$)-convergence. Then E_0 (resp. F_0) is dense in E (resp. F), every point of E (resp. F) lies in the closure of an element of $\mathcal{T}_{E_0}$ (resp. $\mathcal{T}_{F_0}$) and $L_0 : F_0 \times E_0 \longrightarrow \mathbb{K}$ is hypocontinuous with respect to $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$. The hypothesis of the theorem are then fulfilled and L_0 extends on a unique bilinear form L on $F \times E$. This form is separate by the Hahn-Banach theorem. □

Remark 3.21 *L is hypocontinuous with respect to $\mathcal{T}_{F_0}$ and $\mathcal{T}_{E_0}$.*

The definition of a canonical subduality follows from this theorem:

Definition 3.22 (– ***canonical subduality*** **–)** *A subduality (E, F) of $(\mathcal{E}, \mathcal{F})$ is a canonical subduality if it is the canonical subduality associated to its kernel.*

Corollary 3.23 *Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a kernel such that $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$ are non empty and suppose $\mathcal{F}$ Mackey barreled. Then the previous construction holds.*

Next corollary gives a important result concerning completeness of the spaces:

Corollary 3.24 *If the elements of $\mathcal{T}_{E_0}$ (resp. of $\mathcal{T}_{F_0}$) are weakly relatively compacts in $\widehat{E_0}$ (resp. in $\widehat{F_0}$), then the topology of $\mathcal{T}_{F_0}$-convergence (resp. of $\mathcal{T}_{E_0}$-convergence) is compatible with the duality (E, F) and $E = \widehat{E_0}$ (resp. $F = \widehat{F_0}$) is complete for its Mackey topology.*

We call them weakly locally compact canonical subdualities, since the topologies of $\mathcal{T}_{E_0}$-convergence and of $\mathcal{T}_{F_0}$-convergence are weakly relatively compact. Respectively, a stable kernel verifying such conditions is called a weakly compact kernel.

Proposition 3.25

1. *if (E, F) is the canonical subduality associated to $\varkappa$, then (F, E) is the canonical subduality associated to ${}^t\varkappa$;*

2. *if $\varkappa$ is the Hilbertian kernel of a Hilbertian subspace H, then $\varkappa$ is stable (weakly compact) and the associated canonical subduality is $(H, \overline{H} \sim H)$.*

Proof. – The first statement is obvious by construction and for the second one we have that the elements of $\mathcal{T}_{H_0}$ are the bounded barrels with bounded polars for the Hilbertian norm. □

The notion of canonical subdualities is of course important only for infinite-dimensional vector spaces. As we will see with some examples, it is sometimes relatively hard to represent concretely a canonical subduality, whereas it is easier to know whether a kernel is stable or not.

Examples:

example 1 **Sobolev spaces**

Let $\Omega =]0,1[$. The integral operator

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(.) = \int_\Omega K(t,.)\varphi(t)dt \end{array}$$

where

$$K(t,s) = (s-t)\mathbb{1}_{t \leq s}$$

is associated with the canonical subduality (E, F) where

$$E = \left\{ e \in D',\ e(s) = \int_\Omega \mathbb{1}_{t \leq s}\phi(t)dt, \phi \in L^2(\Omega) \right\}$$

and

$$F = \left\{ f \in D',\ f(t) = \int_\Omega \mathbb{1}_{t \leq s}\psi(s)ds, \psi \in L^2(\Omega) \right\}$$

and the bilinear form is

$$(f,e)_{(F,E)} = \int_\Omega \psi(u)\phi(u)du$$

This is a direct consequence of the following results:

1. Let $\sigma \in \mathcal{T}_{E_0}$, $\Re\big(\big(\varkappa^{-1}(\sigma), \overline{\sigma}\big)_{(\mathcal{F},\mathcal{E})}\big) \leq \lambda$ and $\Re\big(\big({}^t\varkappa^{-1}(\sigma^\circ), \overline{\sigma^\circ}\big)_{(\mathcal{F},\mathcal{E})}\big) \leq \gamma$. Then

$$e(s) = \int_0^s \phi(t)dt \in \sigma \Rightarrow \int_\Omega \phi^2 \leq \lambda$$

and

$$f(t) = \int_t^1 \psi(s)ds \in \sigma^\circ \Rightarrow \int_\Omega \psi^2 \leq \gamma$$

2. By Schwartz inequality

$$\forall (f,e) \in F \times E |\, (f,e)_{(F,E)} \,| \leq \int_\Omega \psi\phi$$

example 2 **Sobolev spaces** **(-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

The following kernels

$$\begin{array}{rcl} \varkappa : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \varkappa(\varphi)(.) = \int_\Omega K(t,.)\varphi(t)dt \end{array}$$

where

$$K(t,s) = \int_0^s \min(u,t)dt = t\min(t,s) - \frac{\min(t,s)^2}{2}$$

and

$(\mathfrak{E},\mathfrak{F})$ has for kernel

$$\begin{array}{rcl} \chi : D(]0,1[) & \longrightarrow & D'(0,1[) \\ \varphi & \longmapsto & \chi(\varphi)(.) = \int_\Omega \mathfrak{K}(t,.)\varphi(t)dt \end{array}$$

where

$$\mathfrak{K}(t,s) = 1\!\!1_{t \leq s}$$

are stable and we conjecture (but it is an open problem) that their canonical subdualities are the previously defined fractional Sobolev subdualities

$$(E,F) = (W^{\frac{3}{2}}, W^{\frac{3}{2}})$$

and

$$((E), \mathfrak{F}) = (W^{\frac{1}{2}}, W^{\frac{1}{2}})$$

This is based on the following result:

$$BF(1) = \left\{ \phi \in \|\phi\|_{W_2^{\frac{1}{2}}} \leq 1 \right\} \cup \mathfrak{E}_0$$

is an element of $\mathcal{T}_{\mathfrak{E}_0}$ from the following majoration [52]

$$|\int \psi\phi'| \leq c\|\phi\|_{W_2^{\frac{1}{2}}} . \|\psi\|_{W_2^{\frac{1}{2}}}$$

However, we did not demonstrate that any element of $\mathcal{T}_{\mathfrak{E}_0}$ is included in such sets.

example 3 **(- Hilbertian subspaces -)**

Let H be a Hilbertian subspace of $(\mathcal{E}, \mathcal{F})$ with positive kernel $\varkappa$. Then the canonical associated subduality is clearly $(H, \overline{H} \sim H)$ with asymmetric bilinear form defined by the scalar product (proposition 3.25).

example 4 **(- Krein spaces -)**

Let $\varkappa$ be an Hermitian kernel that admits a Kolmogorov decomposition. Then the canonical subduality associated to $\varkappa$ is the self-duality intersection of all Krein subspaces with kernel $\varkappa$.

3.3.3 The set of canonical subdualities

In section 3.2 the image of a subduality by a weakly continuous morphism has been defined. It is of prime interest to see whether the image of a canonical subduality is a canonical subduality,. Actually, this set is not stable by the action of a weakly continuous linear application. Hence, the set of canonical subdualities cannot be endowed with the structure of a vector space.

An easy way to see this is to deal with kernels of multiplicity:
Let $\varkappa = \varkappa_+^1 - \varkappa_-^1 = \varkappa_+^2 - \varkappa_-^2$ be a kernel of multiplicity (with two distinct Kolmogorov decompositions leading to two different Krein spaces). The canonical subdualities associated to $\varkappa_+^1, -\varkappa_-^1, \varkappa_+^2, -\varkappa_-^2$ are the Hilbertian and anti-Hilbertian subdualities (H_+^1, H_+^1), etc... and their image by the operator sum $+ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$ are respectively $(H_+^1 + H_-^1, H_+^1 + H_-^1)$ and $(H_+^2 + H_-^2, H_+^2 + H_-^2)$ with $H_1 + H_2 \neq H_3 + H_4$ by hypothesis (the kernel is of multiplicity). These two subdualities are then distinct and cannot be both the canonical subduality associated to $\varkappa$.
The image of a canonical subduality is not a canonical subduality in general. We still need an equivalence relation.

Topological and algebraic properties of canonical subdualities have not been investigated yet (apart from 3.24). It may be interesting to study them in relations with the properties of the kernels (for instance, is there an easy characterization of weakly compact kernels ?)

3.4 Some particular subdualities

To put the framework of subdualities and canonical subdualities at work some instantiations are needed. In particular we study the Banachic case and the case of genuine functions that we call evaluation subdualities. The study of a class of equivalence is also discussed.

3.4.1 Inner and outer subdualities, Banachic subdualities

In the previous section, we considered the topologies induced by the whole set $\mathcal{T}_{E_0}$ and $\mathcal{T}_{F_0}$. However, we can restrict our attention to a particular subset of $\mathcal{T}_{E_0}$ (resp. of $\mathcal{T}_{F_0}$) and apply the previous construction. The constructed subdualities hold a deep link with their kernel and will therefore be called inner subdualities. If the particular subset reduces to one element, it appears that under the hypothesis of theorem 3.24, the constructed subduality is a reflexive Banachic subduality.

Other subdualities (not steaming from the kernel) will be called outer subdualities.

Inner and outer subdualities

Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a stable kernel, (E_0, F_0) the associated primary duality and let Π_{E_0} be a (non-empty)subset of $\mathcal{T}_{E_0}$, stable by homothecy, $\Pi_{F_0} = \Pi^{\circ}_{E_0}$ the subset of $\mathcal{T}_{F_0}$ associated to Π_{E_0} by polarity. Then we can make the previous construction, precisely:

Theorem 3.26 *Let $E_\Pi = \widehat{E_0}$ (resp. $F_\Pi = \widehat{F_0}$) be the completion of E_0 endowed with the topology of Π_{F_0}-convergence (resp. the completion of F_0 endowed with the topology of Π_{E_0}-convergence). Then*

1. $E_\Pi \subset \mathcal{E},\ F_\Pi \subset \mathcal{E}$;
2. *the bilinear form L_0 defined on the primary duality extends in a unique bilinear form L_Π on $F_\Pi \times E_\Pi$ separate.*

The duality (E_Π, F_Π) is a subduality of $(\mathcal{E}, \mathcal{F})$ called inner subduality associated to $(\varkappa, \Pi)$.

Proof. – Since the kernel is stable, the elements of Π_{E_0} are weakly relatively compact and the statements of lemma 3.19 and of theorem 3.20 remain valid. □

Conversely, we will say that a subduality is an inner subduality if it may be constructed in this manner starting from its kernel. Finally, we define the set of outer subdualities to be the complement of the set of inner subdualities in the whole set of subdualities.

Banachic subdualities

Banach spaces hold a place of special interest among the locally convex vector spaces. Hence, it is interesting to confront the theory of subdualities and the theory of Banach spaces.

Definition 3.27 (– *Banachic duality* –) *A Banachic duality (E, F) is a duality such that (E, τ) (i.e. E endowed with the Mackey topology) is a reflexive Banach space.*

This definition is in fact symmetric, if (E, F) is a Banachic duality then (F, E) is also a Banachic duality and (F, τ) is a reflexive Banach space.

Remark 3.28 *The concept of Banachic dualities is actually old, since it has been investigated for instance by N. Aronszajn in [7]. His interest was the Banachic completion of dualities.*

The definition of Banachic subdualities follows (we also use the concept of canonical, inner and outer subdualities):

Definition 3.29 (– *Banachic subduality* –) *We call (resp. inner, outer, canonical) Banachic subduality any (resp. inner, outer canonical) subduality that is a Banachic duality.*

Proposition 3.30 *Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a weakly compact kernel such that:*

$$\exists \sigma_b \in \mathcal{T}_{E_0}, \forall \sigma \in \mathcal{T}_{E_0},\ \exists \lambda_1, \lambda_2 \in \mathbb{R}_*^+,\ \lambda_1 \sigma_b \subseteq \sigma \subseteq \lambda_2 \sigma_b$$

Then the (weakly relatively compact) canonical subduality associated to $\varkappa$ is a Banachic subduality.

Proof. – By polarity, we get $\lambda_2 \sigma_b^\circ \subseteq \sigma^\circ \subseteq \lambda_1 \sigma_b^\circ$ and the neighborhood of 0 in F_0 σ_b° is bounded. The topology of $\mathcal{T}_{E_0}$-convergence in F_0 is then normable (corollary 1 p 33 in [26]). Conversely, the same arguments prove that the topology of $\mathcal{T}_{F_0}$-convergence in E_0 is normable.

Finally, the kernel being weakly compact, the norm-topology is compatible with the duality (E, F) and (E, F) is a Banachic subduality. □

There is an easy asymmetric manner to construct Banachic subdualities:

Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a kernel, (E_0, F_0) the associated primary duality and let $E \subset \mathcal{E}$ be a reflexive Banach space continuously included in $\mathcal{E}$ such that (E_0, τ) is continuously and densely included in E. Then $\varkappa : \mathcal{F} \longrightarrow E$ is continuous with dense image. Defining $F =^t \varkappa(E')$, we have that:

Lemma 3.31 *(E, F) is a Banachic subduality of $(\mathcal{E}, \mathcal{F})$.*

The case of inner Banachic dualities is slightly different:

Let $\varkappa \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ be a stable kernel and choose one $\sigma \in \mathcal{T}_{E_0}$. Let $\widehat{E_0}$ (resp. $\widehat{F_0}$) be the completion of E_0 endowed with the topology of σ°-convergence (resp. the completion of F_0 endowed with the topology of σ-convergence).

Proposition 3.32 *If σ (resp. σ°) is weakly relatively compact in $\widehat{E_0}$ (resp. in $\widehat{F_0}$) then the inner duality $(E = \widehat{E_0}, F = \widehat{F_0})$ associated to $(\varkappa, \sigma)$ is an inner Banachic subduality.*

This construction can for instance be done with any weakly compact kernel.

Examples:

example 1 **Sobolev spaces** **(-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

The previously defined fractional Sobolev subdualities $(E, F) = (W^{\frac{3}{2}}, W^{\frac{3}{2}})$ and $((E), \mathfrak{F}) = (W^{\frac{1}{2}}, W^{\frac{1}{2}})$.

are inner Banachic subdualities.

example 2 **(- Krein spaces -)**

This fundamental example deals with Krein subspaces. Let $(\mathcal{E}, \mathcal{F})$ be a duality and $\varkappa \in \mathbb{R} \otimes \mathbf{L}^{+}(\mathcal{F}, \mathcal{E})$ a kernel of multiplicity. Then any Krein subspace H associated to $\varkappa$ may be seen as a inner Banachic (self-)subduality (H, H) a Banach norm being the norm of the Hilbert space $|H|$.

3.4.2 Example of an equivalence class: the dualities of distributions

The classes of equivalence of subdualities with a common kernel are also very interesting to look at and we had a first example with Hermitian kernels of multiplicity. We deal here with an other example, the dualities associated with the canonical injection from D into the space of distributions D' that is a positive kernel. Remark that the investigation of such subdualities (associated with a positive kernel) is made in [43] under the name of "well-dived dualities".

Let Ω be an open subset of $\mathbb{R}^n$, D' the space of distribution over Ω and D the space of indefinitely differentiable functions with compact support. Let $\varkappa = Id : D \longrightarrow D'$ be the canonical injection of D into D'.

$\varkappa = Id$ is a positive kernel and since D is barreled it is associated to a unique Hilbertian subspace of D', $H = L^2(\mu)$, the Hilbert space of square integrable functions with respect to the Lebesgue measure on $\mathbb{R}^n$. The following classical dualities $(\mathcal{D}', \mathcal{D})$, $(\mathcal{E}', \mathcal{E})$, $(\mathcal{S}', \mathcal{S})$, (L^1, L^∞), (H^k, H^{-k}) are outer subdualities of $(\mathcal{D}', \mathcal{D})$ with kernel $\varkappa = Id$.

Figure 3.3 (taken from [14]) illustrates the main functional spaces in analysis and their relative inclusions, these inclusions being topological (i.e. continuous with respect to the usual topologies).

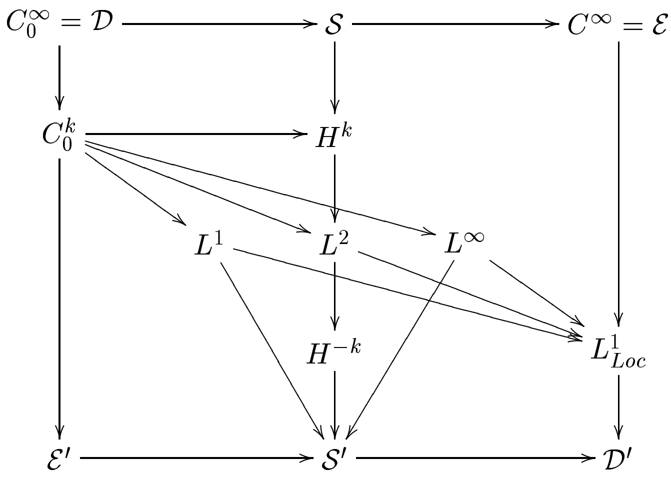

Figure 3.3: Main functional spaces in analysis

3.4.3 Antisymmetric kernels: symplectic subdualities

Other interesting related mathematical objects are symplectic spaces. These spaces are defined as real inner product spaces such that $\forall h \in H, \quad \langle h|h\rangle = 0$. Considering these spaces as subdualities, it follows that their kernels are antisymmetric (or skew-symmetric). We refer to the previous example in $\mathbb{R}^2$ or the following example with kernel function $K(t,s) = (t-s)^n$ (n odd).

3.4.4 Evaluation subdualities

Definition 3.33 (– *evaluation subduality* –) *Let Ω be any set. We call evaluation subduality (or reproducing kernel subduality) on Ω any subduality of $\mathbb{K}^\Omega$ endowed with the product topology.*

Definition 3.34 (– *reproducing kernel* –) *Let (E,F) be an evaluation subduality of Ω*

with kernel $\varkappa$. We call reproducing kernel (function) of (E, F) the function of two variables:

$$\begin{aligned} K : \Omega \times \Omega &\longrightarrow \mathbb{K} \\ t, s &\longmapsto K(t,s) = \left({}^t\varkappa(\delta_s), \varkappa(\delta_t)\right)_{(F,E)} \end{aligned}$$

Lemma 3.35 *Conversely, the kernel $\varkappa$ can be easily deduced from K by the relation $\varkappa(\delta_t) = K(t,.)$ We get $E_0 = Vec\{K(t,.),\ t \in \Omega\}$ (resp. $F_0 = Vec\{K(.,s),\ s \in \Omega\}$).*

Proof. – $K(t,s) = L\left({}^t\varkappa(\delta_s), \varkappa(\delta_t)\right) = \left({}^t i(\delta_s), \varkappa(\delta_t)\right)_{(F,E)} = \varkappa(\delta_t)(s)$.
Formula $E_0 = Vec\{K(t,.),\ t \in \Omega\}$ derives from $(\mathbb{K}^\Omega)' = Vec\{\delta_t,\ t \in \Omega\}$. □

Corollary 3.36 (– ***evaluation, reproduction*** **–)**

1. $\forall s \in \Omega, \forall x \in E,\ x(s) = (K(.,s), x)_{(F,E)}$ *(resp.* $y(t) = (y, K(t,.))_{(F,E)}$*);*

2. $K(t,s) = (K(.,s), K(t,.))_{(F,E)}$.

Proof. – We apply theorem 3.6:
$\forall f \in F, t \in \Omega,$

$$\begin{aligned} f(t) &= (\delta_t, j(f))_{((\mathbb{K}^\Omega)', \mathbb{K}^\Omega)} \\ &= L(f, \varkappa(\delta_t)) \text{ from theorem 3.6} \\ &= L(f, K(t,.)) \end{aligned}$$
□

Examples

example 1 **$\mathbb{R}^2$-example**

The dual system $(\mathbb{R}^2, \mathbb{R}^2)$ endowed with the symplectic bilinear form

$$\begin{aligned} L : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (Y, X) &\longmapsto x_1 y_2 - x_2 y_1 \end{aligned}$$

is an evaluation subduality on $\Omega = \{1, 2\}$ with kernel function

$$K(i,j) = K_{i,j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

example 2 **Sobolev spaces**

Suppose $\Omega =]0,1[$. The duality (E,F) where

$$E = \left\{ e \in D',\ e(s) = \int_\Omega 1\!\!1_{t \leq s} \phi(t) dt, \phi \in L^2(\Omega) \right\}$$

and

$$F = \left\{ f \in D',\ f(t) = \int_\Omega 1\!\!1_{t \leq s} \psi(s) ds, \psi \in L^2(\Omega) \right\}$$

are in duality with respect to the bilinear form

$$(f,e)_{(F,E)} = \int_\Omega \psi(u)\phi(u) du$$

is an evaluation subduality on $\Omega =]0,1[$ with asymmetric kernel function

$$K(t,s) = (s-t) 1\!\!1_{t \leq s}$$

The kernel function of (F,E) is

$${}^t K(t,s) = (t-s) 1\!\!1_{s \leq t} = K(s,t)$$

example 3 **Sobolev spaces** **(-** $W^{\frac{3}{2}}(]0,1[)$ **-)**

The previous subduality (E,F) of the dual system (D',D) with kernel the integral operator:

$$\begin{aligned} \varkappa : D(]0,1[) &\longrightarrow D'(0,1[) \\ \varphi &\longmapsto \varkappa(\varphi)(.) = \textstyle\int_\Omega K(t,.)\varphi(t) dt \end{aligned}$$

is an evaluation subduality on $\Omega =]0,1[$ with asymmetric kernel function

$$K(t,s) = \int_0^s \min(u,t) dt = t \min(t,s) - \frac{\min(t,s)^2}{2}$$

The subduality $(\mathfrak{E}, \mathfrak{F})$ is not an evaluation subduality.

example 4 **Polynomials, splines**

The kernel of the subduality (E, F) of $\mathbb{R}^{\mathbb{R}}$ where $E = F = \mathcal{P}_n$ are in duality with respect to the bilinear form on $F \times E$

$$\begin{array}{rcl} L : F \times E & \longrightarrow & \mathbb{R} \\ (f, e) & \longmapsto & \sum_{j=0}^{n} \frac{(-1)^{n-j}}{n!} f^{(j)}(\tau) e^{(n-j)}(\tau) \end{array}$$

is identified with the kernel function

$$K(t, s) = (t - s)^n$$

remark that when n is odd this kernel is antisymmetric.

example 5 **Polynomials, splines** **(- Piecewise smooth spaces -)**

Consider the previous setting of piecewise smooth spaces in duality. Then the equalities

$$(\delta_s, e)_{(\mathbb{R}^{\Omega'}, \mathbb{R}^{\Omega})} = \left(\sum_{j=0}^{n} f_j(.) e_j(s), e \right)_{(F,E)}$$

$$(\delta_t, f)_{(\mathbb{R}^{\Omega'}, \mathbb{R}^{\Omega})} = \left(f, \sum_{j=0}^{n} f_j(t) e_j(.) \right)_{(F,E)}$$

show that the reproducing kernel function is

$$K(t, s) = \sum_{j=0}^{n} f_j(t) e_j(s)$$

Suppose now we want in addition that

$$\Delta_n(K(s, .))(s) = (0, 0, ..., 0, 1)$$

where $\Delta_n(e)(s) = \big(e(s), e'(s), ..., e^{(n)}(s)\big)$. Then the previous equality

$$K(t, s) = \sum_{j=0}^{n} f_j(t) e_j(s)$$

gives $\forall t \in \Omega$

$$\begin{pmatrix} e_0 & \dots & e_n \\ e_0^{'} & \dots & e_n^{'} \\ \vdots & \dots & \vdots \\ e_0^{(n)} & \dots & e_n^{(n)} \end{pmatrix} \begin{pmatrix} f_0(t) \\ \vdots \\ \vdots \\ f_n(t) \end{pmatrix} = (0, ..., 0, 1)^T$$

that is exactly equation (2.6) defining the dual space of a piecewise smooth W-space in [37].

example 6 Let $\mathcal{E} = l(\mathbb{N}) = \mathbb{K}^{\mathbb{N}}$ be the set of sequences endowed with the pointwise convergence and let $E = \{(e_i) \in l^1(\mathbb{N}),\ e_0 = 0\}$ be the set of absolutely summable sequences starting from zero and $F = \{(f_i) \in l^1(\mathbb{N}),\ \sum_{i=0}^{\infty} f_i = 0\}$ the set of absolutely summable sequences summing to zero.

These two spaces are in separate duality with respect to the following bilinear form

$$\begin{array}{rcl} L : F \times E & \longrightarrow & \mathbb{R} \\ f, e & \longmapsto & \sum_{i=0}^{\infty} f_i(\sum_{j=0}^{i} e_j) = -\sum_{i=0}^{\infty}(\sum_{j=0}^{i} f_j)e_{i+1} \end{array}$$

Their kernel is the two dimensional sequence

$$K(i,j) = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Conclusion and comments

The concept of subduality generalizes the previous concepts of Hilbertian, Krein or admissible prehermitian subspaces (and also D. Alpay's concept of r.k.h.s. of pairs). The set of subduality quotiented by an equivalence relation can the be endowed with the structure of a vector space isomorphic to the set of kernels and one gets a unified theory if one introduces the notions of canonical and inner subdualities.

The example based on a differential operator shows that some existing spaces (like Sobolev-Slobodeckij spaces) seem to be closely linked with kernels.

Symplectic structure or more generally no-symmetric structures (see for instance [24] for an example of use of non-symmetric bilinear form) are more and more used in mathematics. The concept of subdualities gives a new setting to study such objects.

Finally, as L. Schwartz said at the end of [46] after introducing the concept of Hermitian subspaces: “Il serait intéressant d’étendre aux opérateurs différentiels la théorie du potentiel et le problème de Dirichlet”. He was heard beyond his expectations since the theory of Krein subspaces has now many applications. We hope it will be the same for this new theory of subdualities where we can now use kernels that are neither positive nor Hermitian. Next chapter then initiates some directions for applications.

Chapter 4

Applications

Introduction

The previous concepts put an additional structure on locally convex spaces and can therefore be used to extend existing theories related to this particular structure (that exist on Hilbert spaces, Krein spaces or dualities) to locally convex spaces. This is for instance the case for Gaussian measures over locally convex spaces. We moreover go further into the formalism and study also its implication in terms of Krein subspaces and subdualities.

One can also work the other way round: by embedding a duality into a specific locally convex space (or a duality) one can study some objects with the use of the kernel. This is particularly true in the second section that deals with operator theory.

Finally a third section is devoted to the starting point of our investigation: approximation theory.

4.1 From Gaussian measures to Boehmians (generalized distributions) and beyond

Hilbertian subspaces play a great role in the (infinite-dimensional) probability theory since Gaussian measures over a locally convex space may be entirely defined by a positive kernel and its associated Hilbertian subspace. After precisely reviewing the link between Gaussian measures and Hilbertian subspaces we will extend the construction to Krein subspaces which will appear to be strongly linked with some generalization of distributions and finally question the case of subdualities. This will be done through abstract operator algebra theory.

4.1.1 Hilbertian subspaces and Gaussian measures

The Gaussian measures play a fundamental role in probability theory. In infinite-dimensional probability theory at least two approaches are possible, one based after Radon measures theory and the other after cylindrical measures. It is the second we (briefly) study here for we can define Gaussian (cylindrical) measures in terms of Hilbertian subspaces. We refer to [48] for the general theory of Radon, cylindrical and Gaussian measures or to [33] for a more precise study of Gaussian measures.

Gauss measure over a Hilbert space

The Gauss measure over a finite n-dimensional Hilbert space H is defined as follows: Let $dx = dx_1 dx_2 ... dx_n$ be the Lebesgue measure on $\mathbb{K}^n$ and dh its image under the isomorphism $(x_i) \longmapsto h = \sum_1^n x_i e_i$ the Gauss measure γ on H is $d\gamma_H = exp(-\pi \|h\|^2) dh$. Its variance and Fourier transform (characteristic functional) are given by:

$$\int_H \left(\overline{\langle \psi | h \rangle_H} \langle \phi | h \rangle_H \right) \gamma_H(dh) = \frac{1}{2\pi} \langle \psi | \phi \rangle_H$$

$$\mathcal{F}_{\gamma_H}(\overline{h}) = exp(-\pi \|\overline{h}\|_H^2)$$

with the identifications $H' \sim \overline{H} \sim H$.

We can now define the Gauss measure over a arbitrary Hilbert space H. It is the (unique) cylindrical measure γ defined as follows:

Definition 4.1 (– *Gauss measure over a Hilbert space* –) *The Gauss measure is the (unique) cylindrical measure γ such that for any finite-dimensional vector subspace $G \subset H$*

$$p_G(\gamma_H) = \gamma_G$$

where p_G is the orthogonal projection on G and γ_G the previously defined Gauss measure over the finite-dimensional Hilbert space G.

The previous equations regarding the covariance and Fourier transform remain valid.

Gaussian measure over a locally convex space (over a duality)

Based after the definition of the Gauss measure over a Hilbert space, we can define Gaussian measures over a locally convex space (or a duality):

Definition 4.2 (– *Gaussian measure* –) *Let $\mathcal{E}$ be a locally convex space (resp. $(\mathcal{E}, \mathcal{F})$ a duality) and μ a cylindrical measure on $\mathcal{E}$. We say that μ is a Gaussian measure if there exists a Hilbertian subspace H of $\mathcal{E}$ (resp. of $(\mathcal{E}, \mathcal{F})$) such that*

$$\mu = i(\gamma_H)$$

where γ_H is the Gauss measure on H and $i : H \longrightarrow \mathcal{E}$ the canonical injection.

We note $Gauss((\mathcal{E}, \mathcal{F}))$ the set of Gaussian measures over a duality $(\mathcal{E}, \mathcal{F})$.

Covariance operators, kernels and support

The Hilbertian kernel of H is closely linked with the covariance operators and Fourier transform. Precisely

Proposition 4.3 *Let* $(\mathcal{E}, \mathcal{F})$ *be a duality and* μ *the Gaussian measure associated to* H. *then*

$$\int_{\mathcal{E}} \left(\overline{(\psi, \epsilon)_{(\mathcal{F},\mathcal{E})}} \, (\phi, \epsilon)_{(\mathcal{F},\mathcal{E})} \right) \mu(d\epsilon) = \frac{1}{2\pi} \left(\overline{\psi}, \varkappa(\phi) \right)_{(\mathcal{F},\mathcal{E})}$$

$$\mathcal{F}_\mu(\phi) = exp \left(-\pi (\overline{\phi}, \varkappa(\phi))_{(\mathcal{F},\mathcal{E})} \right) = exp \left(-\pi \|\varkappa(\phi)\|_H^2 \right)$$

The Hilbertian subspace itself is related to sets associated to Radon measures (theorem 6p 97 [33]):

Proposition 4.4 *Suppose* μ *is a Radon Gaussian measure on* $(\mathcal{E}, \mathcal{F})$ *associated to* H. *Then its topological support, its linear support and the closure of its kernel (space of admissible directions) coincide with* H.

Cone convex structure and functors

The image of a measure by a weakly continuous application is a classical tool in measure theory and we used it to define Gaussian measures over l.c.s. or dualities. As well it is classical to define the convolution ($*$) of two measures (that stands for an addition law) or the external product (.) of a measure by a positive number. A very significant result concerning Gaussian measures and their Hilbertian subspaces is:

Theorem 4.5 *Let* $(\mathcal{E}, \mathcal{F})$ *be a duality. Then there is a bijection between* $Hilb((\mathcal{E}, \mathcal{F}))$ *and* $Gauss((\mathcal{E}, \mathcal{F}))$. *Moreover, this bijection is compatible with the operations of addition (resp. convolution) and external multiplication over the two sets.* $(Gauss((\mathcal{E}, \mathcal{F})), *, .)$ *is a convex cone isomorphic to the convex cone of Hilbertian subspaces.*

This bijection is moreover compatible with the effect of weakly continuous linear applications and we can state a theorem regarding categories:

Let $\mathcal{C}$ be the category of dual systems $(\mathcal{E}, \mathcal{F})$ the morphisms being the weakly continuous linear applications. Let $\mathcal{G}$ be the category of salient and regular convex cones, the morphisms being the applications preserving multiplication by positive scalars and addition (hence order). Then according that to a morphism $u : \mathcal{E} \longrightarrow \mathfrak{E}$ we associate the morphisms

$$\begin{array}{rcl} \tilde{u} : Hilb((\mathcal{E}, \mathcal{F})) & \longrightarrow & Hilb((\mathfrak{E}, \mathfrak{F})) \\ H & \longmapsto & u(H) \end{array}$$

and

$$\begin{array}{rcl} \tilde{\tilde{u}} : Gauss((\mathcal{E}, \mathcal{F})) & \longrightarrow & Gauss((\mathfrak{E}, \mathfrak{F})) \\ \mu & \longmapsto & u(\mu) \end{array}$$

Then

Theorem 4.6 *$Hilb : (\mathcal{E}, \mathcal{F}) \mapsto Hilb((\mathcal{E}, \mathcal{F}))$ and $Gauss : (\mathcal{E}, \mathcal{F}) \mapsto Gauss((\mathcal{E}, \mathcal{F}))$ are isomorphic covariant functors of category $\mathcal{C}$ into category $\mathcal{G}$.*

4.1.2 Krein subspaces and Boehmians

These last two theorems are very important since the regular convex cone of Gaussian measures will generate a vector space $\mathbb{R} \otimes Gauss((\mathcal{E}, \mathcal{F}))$ isomorphic to $\mathbb{R} \otimes Hilb((\mathcal{E}, \mathcal{F}))$. We will be able to use the theory of Krein subspaces but once again an interpretation of $\mathbb{R} \otimes Gauss((\mathcal{E}, \mathcal{F}))$ is needed.

Boehmians

The name Boehmians is used for all objects obtained by an abstract algebraic construction similar to the one of the field of quotients, but even if the “multiplication” law has divisors of zero (by using “quotients of sequences” instead of “quotients”). We will not deal with sequences here since convolution admits no divisor of zero in the set of Gaussian measures but we keep the name since there has been a great study ([38], [39], [30]) of Boehmians based on function spaces (such as distributions).

A precise definition of Boehmians is given in [19]:
Let G be a vector space and M a subspace of $G, \star$ a binary operation from $G \times H$ into G and Δ a family of sequences of elements of M (the binary operation and the family Δ verifying additional conditions). Then the class of equivalence of quotients sequences $(\frac{g_n}{\phi_n})$ verifying

1. Quotient sequences: $\forall n \in \mathbb{N},\ g_n \in G,\ \phi_n \in \Delta, \quad g_n \star \phi_m = g_m \star \phi_n;$

2. Equivalence: $(\frac{g_n}{\phi_n})\mathcal{R}(\frac{f_n}{\psi_n}) \iff g_n \star \psi_n = f_n \star \phi_n;$

is called a Boehmian and we note the space of Boehmians $\mathcal{B}(G, M, \star, \Delta)$.
In general functional Boehmians are defined after the convolution. For instance it is commonly agreed that by Boehmians one means:

1. $G = C(\mathbb{R}^n \longmapsto \mathbb{C});$

2. $M = D(\mathbb{R}^n \longmapsto \mathbb{C});$

3. $\star = *$ is the standard convolution;

4. Δ is the set of delta sequences.

The obtained space of Boehmians contains Schwartz's space of distributions D', but also hyperdistributions, Mikusinski operators, Roumieu ultradistributions or regular operators ([19]).
Among all properties we may cite this interesting result based on the Fourier transform ([39]):

Theorem 4.7 *The Fourier transform is a one-to-one mapping from the space of tempered Boehmians to the space of distributions over* $\mathbb{R}^n$.

Gauss Boehmians

Let $(\mathcal{E}, \mathcal{F})$ be a duality. Then we define the following space of Boehmians:

Definition 4.8 (– *Gauss Boehmian space* –) *The Gauss Boehmian space (over $(\mathcal{E}, \mathcal{F})$) is the space of Boehmians with*

1. $G = Gauss((\mathcal{E}, \mathcal{F}))$*;*

2. $M = G$*;*

3. $\star = *$ *is the standard convolution;*

4. Δ *is the set of constant sequences.*

A Gauss Boehmian is of the form $\frac{\gamma_{H_1}}{\gamma_{H_2}}$ *and the space of Gauss Boehmians is denoted by* $GB((\mathcal{E}, \mathcal{F}))$ *.*

Theorem 4.9 *The two spaces* $\mathbb{R} \otimes Gauss((\mathcal{E}, \mathcal{F}))$ *and* $GB((\mathcal{E}, \mathcal{F}))$ *are equal.*

Proof. – They both are the vector space extension of the convex cone of Gaussian measures $G((\mathcal{E}, \mathcal{F}))$ with respect to the equivalence relation induced by the cone. □

Fourier transform, covariance and support

We can now define the Fourier transform of a Gauss Boehmian $\frac{\gamma_{H_1}}{\gamma_{H_2}}$:

Proposition 4.10

$$\mathcal{F}\left(\frac{\gamma_{H_1}}{\gamma_{H_2}}\right)(\phi) = exp\left(-\pi(\overline{\phi}, \varkappa(\phi)_{(\mathcal{F},\mathcal{E})}\right) = exp\left(-\pi\|\varkappa(\phi)\|^2_{H=H_1\ominus H_2}\right)$$

where $\varkappa = \varkappa_1 - \varkappa_2$.

From theorem 4.7 it follows that Gauss Boehmians may be seen as ultradistributions, tempered Boehmians, hyperdistributions etc... when $\mathcal{E} = \mathcal{F}$ is a finite-dimensional space.
We can also state a result concerning "covariance":

Proposition 4.11

$$\int_{\mathcal{E}} \left(\overline{(\psi, \epsilon)_{(\mathcal{F},\mathcal{E})}} \, (\phi, \epsilon)_{(\mathcal{F},\mathcal{E})} \right) \frac{\gamma_{H_1}}{\gamma_{H_2}}(d\epsilon) = \frac{1}{2\pi} \left(\overline{\psi}, \varkappa(\phi) \right)_{(\mathcal{F},\mathcal{E})}$$

For instance in [30] p 61 they derive the expression of $\frac{\gamma_{\sigma_1}}{\gamma_{\sigma_2}}$ in a hyperdistribution form where σ_1 and σ_2 are the variances of two Gaussian measures over $\mathbb{R}$:

$$\frac{\gamma_{\sigma_1}}{\gamma_{\sigma_2}} = \sum_{k=0}^{\infty} \frac{(\sigma_1^2 - \sigma_2^2)^k}{k!} \Delta^{2k} \delta$$

In terms of support, we can see that in the Pontryagin kernel case the support will be exactly the (unique) Pontryagin space associated to the kernel. The problem arises when speaking of kernel of multiplicity i.e. in the infinite-dimensional case.
This infinite-dimensional case then seems of very peculiar interest but the existing theory on Boehmians, ultradistributions etc... has not been extended to the infinite-dimensional case so far. The example of Gauss Boehmians would certainly raise interesting questions concerning generalized distributions in infinite dimension.

4.1.3 Interpretation in terms of subdualities: the noncommutative algebra approach ?

The use of symmetry or symmetric structure has always been a crucial tool in mathematics. Symmetry appears to be closely linked with commutativity and the commutativity of the algebra of continuous function over a set M generates Hilbert spaces through the covariance operator of the measure. It then appears "natural" to try to interpret the loss of symmetry dealing with subdualities in terms of non-commutative algebras. The main difficulty is that though the Gelfand transform provides a particularly clever formula to link sets or spaces and commutative algebras (of functions), it is generally assumed that non-commutative algebras cannot be interpreted in terms of functions. A solution may then be the use of subdualities.

Precisely we can interpret Gaussian measures this way: let $(\mathcal{E},\mathcal{F})$ be a duality and $\varkappa$ a positive kernel. Let $\mathcal{A}$ be the algebra of continuous functions over $\mathcal{E}$ and define the following involution over $\mathcal{A}$: $\psi^*(\varepsilon) = \overline{\psi(\varepsilon)}$ (remark that it implies the following identity: $(\psi^*\phi)^* = \phi^*\psi$).

Then we can rewrite the covariance equality of Gaussian measures as:

there exists a unique "Gaussian" linear form μ over the algebra $\mathcal{A}$ (equivalently a Gaussian measure on $\mathcal{E}$) such that $\forall(\phi,\psi) \in \mathcal{F}^2$

$$\mu(\psi^*.\phi) = \int_{\mathcal{E}} \left(\left(\overline{\psi},\overline{\epsilon}\right)_{(\mathcal{F},\mathcal{E})} (\phi,\epsilon)_{(\mathcal{F},\mathcal{E})} \right) \gamma_H(d\epsilon) = \frac{1}{2\pi} \left(\overline{\psi},\varkappa(\phi)\right)_{(\mathcal{F},\mathcal{E})}$$

Remark that by the self-adjoint property of the kernel, we get that

$$\mu((\psi^*.\phi)^*) = \overline{\mu(\phi^*\psi)}$$

or more generally:

$$\forall\psi \in \mathcal{A},\ \mu(\psi^*) = \overline{\mu(\psi)}$$

Regarding subdualities we then would have to define a generally non-commutative algebra $\mathcal{A}$ such that $\mathcal{F} \subset \mathcal{A}$ and a linear form μ on this algebra verifying

$$\mu(\psi^*.\phi) = \frac{1}{2\pi} \left(\psi,\varkappa(\phi)\right)_{(\mathcal{F},\mathcal{E})}$$

4.2 Operator theory

Hilbertian subspaces (and to a lesser extent Pontryagin subspaces) have been widely used in (at least) two directions regarding operator theory. The first concerns operators in reproducing kernel spaces and the second deals with the particular positive differential operators. We then extend these two trends in terms of subdualities and differential kernels of any type. Finally a third application in terms of similarity in Hilbert spaces is given.

4.2.1 Operators in evaluation subdualities

In [3] D. Alpay proves that continuous endomorphisms in reproducing kernel Hilbert spaces are characterized by a function of two variables and up to unitary similarity by actually a function of one single variable called the Berezin symbol (theorem 2.4.1 p 33). This theorem extends naturally to the context of Krein spaces.

In the subduality setting it appears that many morphisms on evaluation subdualities are also characterized by a function of two variables:

Theorem 4.12 *Let (E,F) be an evaluation subduality on the set Ω with reproducing kernel $K(.,.)$. Then any weakly continuous operator $S : F \longrightarrow E$ and $T : E \longrightarrow E$ (resp. from E to F or from F to F) can be written as*

$$S(f)(t) = (f, \mathbf{S}(t,.))_{(F,E)}$$

$$T(e)(s) = (\mathbf{T}(.,s), e)_{(F,E)}$$

where $\mathbf{S}(t,.) =^t S[K(.,t)] \in E$ and $\mathbf{T}(.,s) =^t T[K(.,s)] \in F$

Proof. – For instance for S:

$$S(f)(t) = (K(.,t), S(f))_{(F,E)} = \left(f,^t S[K(.,t)]\right)_{(F,E)} = (f, \mathbf{S}(t,.))_{(F,E)} \qquad \square$$

The following transposition and composition rules follow:

1. ${}^t\mathbf{S}(t,.) = S[K(.,t)] = \mathbf{S}(.,t),\ {}^t\mathbf{T}(s,.) = T[K(s,.)] = \mathbf{T}(s,.)$

2. $T \circ S$ is associated to $[\mathbf{T} \circ \mathbf{S}](t,s) = (\mathbf{T}(.,t), \mathbf{S}(s,.))_{(F,E)}$

3. $T_1 \circ T_2$ is associated to $[\mathbf{T}_1 \circ \mathbf{T}_2](t,s) = (\mathbf{T}_1(.,s), \mathbf{T}_2(t,.))_{(F,E)}$

Example: Consider the previous example of evaluation subduality: $\mathcal{E} = l(\mathbb{N})$ is the set of sequences endowed with the pointwise convergence,

$$E = \left\{(e_i) \in l^1(\mathbb{N}),\ e_0 = 0\right\}$$

the set of absolutely summable sequences starting from zero and

$$F = \left\{(f_i) \in l^1(\mathbb{N}),\ \sum_{i=0}^{\infty} f_i = 0\right\}$$

the set of absolutely summable sequences summing to zero.

These two spaces are in separate duality with respect to the bilinear form

$$\begin{array}{rcl} L : F \times E & \longrightarrow & \mathbb{R} \\ f, e & \longmapsto & \sum_{i=0}^{\infty} f_i (\sum_{j=0}^{i} e_j) = -\sum_{i=0}^{\infty} (\sum_{j=0}^{i} f_j) e_{i+1} \end{array}$$

and their kernel is the two dimensional sequence

$$K(i,j) = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

For the following weakly continuous operator

$$\begin{array}{rcl} S : E & \longrightarrow & F \\ e = (e_i) & \longmapsto & f = \left\{-\sum_{j=0}^{\infty} e_j, e_1, e_2, \cdots\right\} \end{array}$$

a straightforward calculation gives

$$\begin{array}{rcl} {}^tS : E & \longrightarrow & F \\ e = (e_i) & \longmapsto & f = \left\{ e_1 - \sum_{j=0}^{\infty} e_j, e_2, e_3, \cdots \right\} \end{array}$$

and finally

$$\mathbf{S}(i,j) = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

4.2.2 Differential operators and subdualities

Spaces linked with differential theory such as Sobolev spaces are widely used in functional analysis. In particular it is now standard to define Sobolev-Hilbert spaces as Hilbertian subspaces of the space of distributions with a particular differential operator as kernel (see for instance [46]).

Obviously there exist too many useful standard Hilbert spaces (Sobolev spaces, Beppo-Levi spaces, Hardy spaces) to perform a general theory but there exists an interesting result due to L. Schwartz concerning some generalized Sobolev spaces of integer order.

His setting is as follows:

Ω is an open set of $\mathbb{R}^n$ and for any positive integer s we define the space H^s as the equivalent class of functions of $L^2(\Omega)$ such that their derivatives of any order $(p_1, p_2, ..., p_n)$, $|p| = p_1 + p_2 + ... + p_n \leq s$ are in $L^2(\Omega)$. This space is endowed with the scalar product

$$\langle \psi | \phi \rangle_{H^s} = \sum_{|p| \leq s} \int_{\Omega} a_p \Delta^p \psi \Delta^p \phi d\mu$$

that makes it a Hilbert space where the a_p are strictly positive constant coefficients and $\Delta^p = \sum_{k=1}^{n} \frac{\partial^{p_k}}{\partial {t_k}^{p_k}}$.
Moreover we define the Hilbert spaces of distributions H_0^s closure of $D(\Omega)$ in H^s and $H^{-s} = ^t j[(H_0^s)']$ image in $D'(\Omega)$ of $(H_0^s)'$ dual space of H_0^s by the transpose of the canonical dense injection $j : D(\Omega) \longrightarrow H_0^s$.

Proposition 4.13 *The kernel of the Hilbertian subspace H^{-s} of $(D'(\Omega), D(\Omega))$ is the positive differential operator*

$$\begin{array}{rcl} \varkappa D(\Omega) & \longrightarrow & D'(\Omega) \\ \phi & \longmapsto & \sum_{|p|\leq s} (-1)^{|p|} a_p \Delta^{2p} \phi \end{array}$$

The kernel of H_0^s is its Green operator $G_\varkappa$ and the kernel of H^s its Neumann operator $L_\varkappa$. [1]

This theorem naturally extends to the Krein subspaces setting with non-necessarily positive coefficients a_p. We can associate to any differential operator of even order a Krein space constructed after Sobolev spaces of integer order.

The following question then arises naturally: can we associate a subduality of $(D'(\Omega), D(\Omega))$ constructed after (possibly fractional) Sobolev spaces to any differential operator of integer order? The answer is positive and based after the following theorem (see also the examples in chapter 3 concerning Sobolev-Slobodeckij spaces):

Theorem 4.14 *Let $p = (p_1, p_2, ..., p_n)$ be a positive multi-index,*

$$\begin{array}{rcl} \varkappa D(\Omega) & \longrightarrow & D'(\Omega) \\ \phi & \longmapsto & \Delta^p \phi \end{array}$$

[1] *the Green operator of a linear elliptic differential operator is the "inverse" operator that yields the solution as a linear map of the data (Courant and Hilbert [18]).*

and H^p the Sobolev-Slobodeckij space $W^{\frac{p}{2}}$. Let as before H_0^p be the closure of $D(\Omega)$ in H^p and $H^{-p} = {}^t j[(H_0^p)']$ be the image in $D'(\Omega)$ of $(H_0^p)'$ dual space of H_0^p by the transpose of the canonical dense injection $j : D(\Omega) \longrightarrow H_0^p$.
Then (H^{-p}, H^{-p}) is a (inner) subduality of $(D'(\Omega), D(\Omega))$ with kernel $\varkappa$ and The Green's and Neumann's functions are respectively the kernels of the inner subdualities (H_0^p, H_0^p) and (H^p, H^p)

Proof. – This result follows directly from the following majoration (derived from [52]):

$$|\int_\Omega \psi\varkappa(\phi)| \leq c\|\psi\|_{W_2^{\frac{p}{2}}}.\|\varkappa(\phi)\|_{W_2^{\frac{-p}{2}}} \leq c\|\psi\|_{W_2^{\frac{p}{2}}}.\|\phi\|_{W_2^{\frac{p}{2}}}$$

□

Finally any differential operator of integer order can be associated with a generalized Sobolev space via the functor $\mathcal{SD}$ and the previous results (non constant coefficients will also be handled by the image of a continuous morphism).

4.2.3 Similarity in Hilbert spaces

We treat here the problem of similarity for operators in real Hilbert spaces. Let L be on operator on a Hilbert space H. Does there exist a self-adjoint operator A and a isomorphism T such that $L = T^{-1}AT$?
We give here an answer in terms of subdualities. Let (E, F) be the primary subduality associated to L. Then

Proposition 4.15 *The answer to the similarity problem is positive if and only if exists a positive operator Q on H such that:*

1. *$Q(E) = F$;*

2. *$Q : F \longrightarrow E$ is self-adjoint for the duality (E, F) i.e.*

$$(Q(\varepsilon_1), \varepsilon_2)_{(F,E)} = (Q(\varepsilon_2), \varepsilon_1)_{(F,E)}$$

A choice for T and A is then $T = \sqrt(Q)$ and $A = TLT^{-1}$.

Proof. – Suppose the answer is positive. Then one checks easily that

1. ${}^tTT : H \longrightarrow H$ is a positive and self-adjoint isomorphism;

2. ${}^tTT(E) = F$;

3. ${}^tTT : F \longrightarrow E$ is self-adjoint for the duality (E, F)

Conversely the existence of such an operator Q gives $L = \sqrt(Q)^{-1}A\sqrt(Q)$ with $A = \sqrt(Q)A\sqrt(Q)^{-1}$ self-adjoint. □

4.3 Approximation theory: the interpolation problem

Positive reproducing kernels are widely used in the learning community and the domain of application is very large. We are interested here in the approximation problem, or more precisely in the interpolation problem. It appears that this problem can easily be solved using positive kernels. One may then wonder if it is possible to solve it without the positivity requirement.

4.3.1 The problem

A way to state the interpolation problem is as follows:

We are given a data set $\{(s_i, y_i),\ i \in I\}$ (I finite integer set) where the $s_i \in \Omega$ and y_i in $\mathbb{K}$

and we want to find a “good” function ϕ in a suitable space E such that

$$\phi(s_i) = y_i \quad \forall i \in I$$

Obviously the following constraints on E follow:

1. E must be a space of genuine functions on Ω i.e. $E \subset \mathbb{K}^\Omega$
2. The evaluation values must bring some information on the function i.e. the evaluation functionals must be continuous on E which must be continuously embedded in $\mathbb{K}^\Omega$

A classical way to solve the problem is to associate to each function of E an “energy” i.e. to state that E is a Hilbert space. It follows that it is a R.K.H.S. (we note its kernel function K) and a possible choice for the “best” interpolating function would be the one with least energy.

Mathematically, one has then to solve the minimization problem:

Problem 4.16 $\phi = \arg\min_{\phi\in\mathcal{S}} \|\phi\|^2$

where $\mathcal{S} = \{f \in E,\ f(s_i) = y_i \quad \forall i \in I\}$ is the set of interpolating functions.
Remark that this minimization problem always has a unique solution since $\mathcal{S}$ is a closed[2] convex set. If $Q = K(s_i, s_j)$ is the “covariance matrix” we get

$$\phi = \sum_{i\in I} \alpha_i K(s_i, .)$$

with

$$\mathrm{A} = (\alpha) = \mathrm{Q}^{-1}\mathrm{Y}$$

4.3.2 4 equivalent problems: from minimization to projections

It is usual to interpret the previous minimization problem as a projection problem in the Hilbert space E:

[2]by the continuity of the canonical injection

Problem 4.17 $\phi = p_{\mathcal{S}}^{\perp}(0)$

where $p_{\mathcal{S}}^{\perp}$ denotes the orthogonal projection on the closed convex set S. The equivalence of the two problems is clear since by definition, the projection of 0 is the point of S minimizing the distance to 0.

These two equivalent problems rely however heavily on the Hilbertian structure of E and it is interesting to state a third and fourth equivalent problems.

Let L be the vector space spanned by the $K(., s_i),\ i \in I$ (respectively by the $K(s_i, .),\ i \in I$ by the symmetry of the kernel). Then

Problem 4.18 $\forall f \in \mathcal{S}, \phi = \arg\min_{\lambda \in L} \|\mathrm{f} - \lambda\|^2$

But once again, this minimization problem as an interpretation in terms of orthogonal projection:

Problem 4.19

$$\forall f \in \mathcal{S}, \phi = p_L^{\perp}(f) = \sum_{i \in I} \alpha_i K(s_i, .)$$

with $\mathrm{A} = (\alpha) = \mathrm{Q}^{-1}\mathrm{Y}$.

In other terms, all the interpolating functions have the same orthogonal projections on the subspace L.

These results could mean that the finite-dimensional subspace L is a good space to summarized interpolating functions for they all have the same orthogonal projection but we will see below that L defines actually the good direction for projection.

4.3.3 Interpolation in evaluation subdualities

The interest of problem 4.19 is that projections on subspaces in dual systems can be defined naturally whereas we cannot generally define projection on convex sets. The main difference is that we now have to define two subspaces: a support and a direction.

Precisely let (E, F) be a dual system with respect to a bilinear form B and $T \subset E$, $L \subset F$ finite subspaces such that (T, L) are in separate duality with respect to B (and hence P and L have the same dimension). Then it follows that

1. $E = T \oplus L^{\perp}$
2. $F = L \oplus T^{\perp}$

and these decompositions define projections $p_T^{\perp L}$ and $p_L^{\perp T}$ respectively the projection of vectors of E on T orthogonally to L and the projection of vectors of F on L orthogonally to T.

Remark 4.20 *If $E = F$, then any subspace $T = L$ such that (T, T) are in separate duality is called admissible (see for instance [24]).*

Suppose now that (E, F) is an evaluation subduality with kernel K. Fix L the vector subspace of F spanned by the $K(., s_i)$, $i \in I$ and choose a support of the form $T = Vec\{K(t_i, .), i \in I\}$ such that the matrix $Q = K(t_i, s_j)$ is invertible. Then (T, L) is a duality and the projections are well defined.

Theorem 4.21

$$\forall f \in \mathcal{S}, \phi = p_T^{\perp L}(f) = \sum_{i \in I} \alpha_i K(t_i, .)$$

with $\mathrm{A} = (\alpha) = \mathrm{Q}^{-1}\mathrm{Y}$. ϕ *is then independent of the particular interpolating function projected. Moreover* $\phi \in \mathcal{S}$.

Proof. – By the reproduction property the orthogonality of $f-\phi$ with the $K(.,s_i)$ is just $(f-\phi)(s_i)=0$ and the function ϕ is interpolating. Finally, the invertibility of Q gives the desired result. □

We see that the loss of symmetry and of the norm affects the choice of T: there is now no intrinsic reason to choose a particular set of points $\{t_i,\ i\in I\}$ than another. We can however state an interesting result when two data sets are available.

Suppose we are now given two data sets $\{(t_i,x_i),\ i\in I\}$ $\{(s_i,y_i),\ i\in I\}$ (I finite integer set) where the $t_i,s_i\in\Omega$ and x_i,y_i in $\mathbb{K}$ such that the matrix $Q=K(t_i,s_j)$ is invertible. Define $\mathcal{S}=\{f\in E,\ f(s_i)=y_i\quad \forall i\in I\}$ and $\mathcal{G}=\{g\in F,\ f(t_i)=x_i\quad \forall i\in I\}$ and T and L as before. Then

Theorem 4.22

1. $\forall f\in\mathcal{S},\phi=p_T^{\perp L}(f)=\sum_{i\in I}\alpha_i K(t_i,.)\in\mathcal{S}$ *with* $\mathrm{A}=(\alpha)=\mathrm{Q}^{-1}\mathrm{Y}$
2. $\forall g\in\mathcal{G},\psi=p_L^{\perp T}(g)=\sum_{i\in I}\beta_i K(.,s_i)\in\mathcal{G}$ *with* $\mathrm{B}=(\beta)=({}^{\mathrm{t}}\mathrm{Q})^{-1}\mathrm{X}$
3. (ϕ,ψ) *stabilizes the quantity* (g,f) *with* $f\in\mathcal{S},g\in\mathcal{G}$
4. (ϕ,ψ) *stabilizes the quantity* $(g-\lambda,f-\tau)$ *with* $\lambda\in L,\tau\in T$ *for all* $f\in\mathcal{S},g\in\mathcal{G}$

Proof. – A straightforward calculation gives the desired result. □

Figure 4.1 presents the stabilization in the case of two different data sets with asymmetric smooth kernel.

4.3.4 Rupture detection

In the case of (self)-dualities, we can do orthogonal projections on admissible subspaces and then use a single data set when admissible. In the following examples (figures 4.2 and 4.3), we use an asymmetric (discontinuous then smooth) kernel function based on the Heaviside

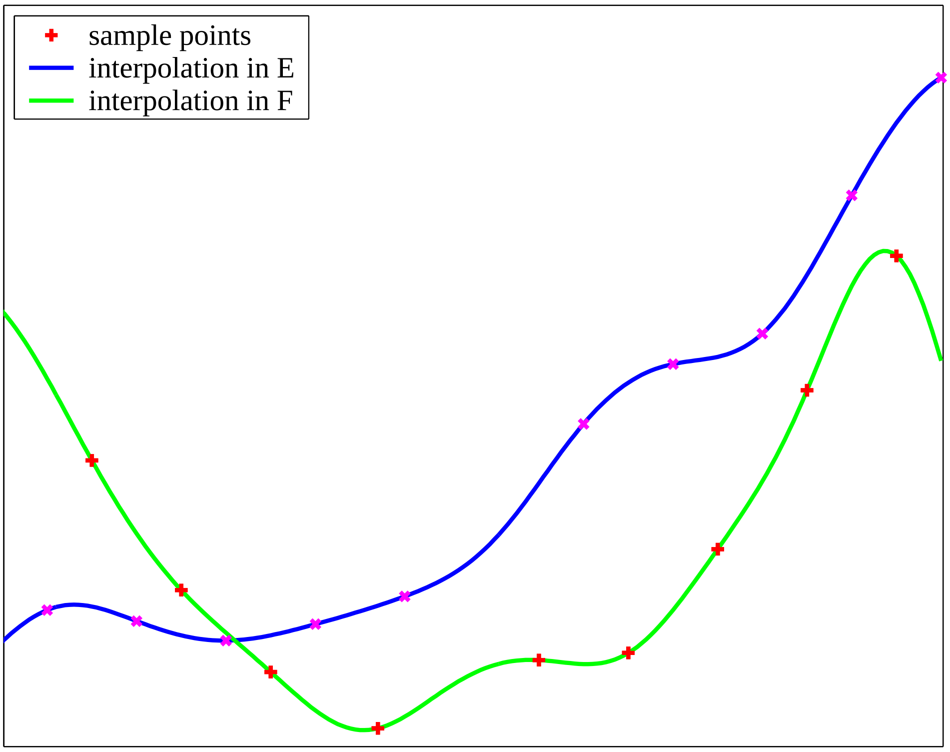

Figure 4.1: stabilization

function to detect discontinuity.

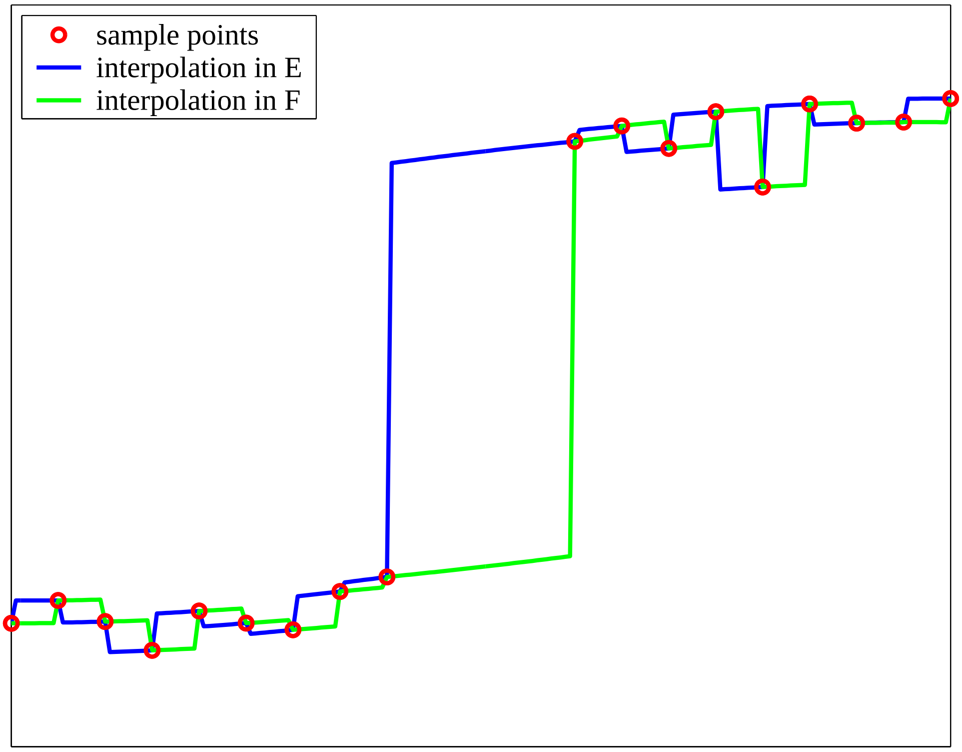

Figure 4.2: Rupture detection, discontinuous kernel

Conclusion and comments

We have initiated here three possible fields of applications that show more insights of the general theory of subdualities. The first one concerning "generalized" measure theory remains widely open. The second concerning operator theory shows some very peculiar uses but there probably are many more problems that could gain something at using subdualities. The case of differential operators gives another perspective on Sobolev spaces. Finally, we solve the interpolation problem by using projection in evaluation subdualities but not without difficulties due to the lack of a norm. A solution is then to solve two different interpolation

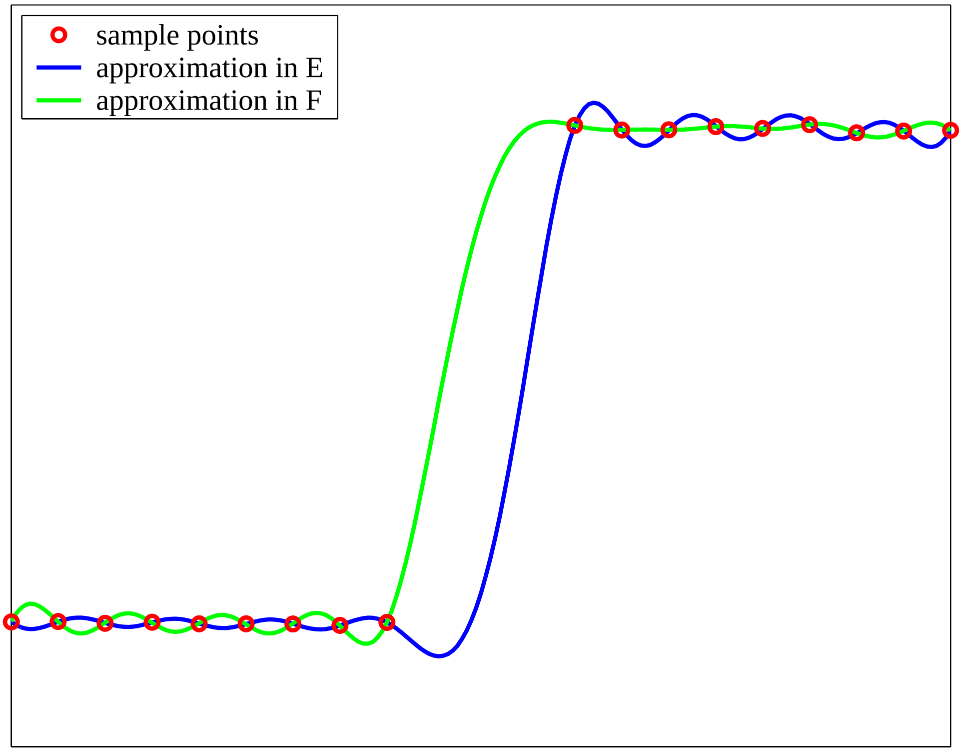

Figure 4.3: Rupture detection, smooth kernel

problems in a single common setting. This can be applied to rupture detection.

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Appendix A

Generalities

A.1 Basic definitions

Definition A.1 (– *topological vector space (t.v.s.)* –) *A topological vector space is a pair $(\mathcal{E}, \mathcal{T})$ where $\mathcal{E}$ is a vector space over a topological field $\mathbb{K}$, and $\mathcal{T}$ is a Hausdorff (separate) topology on $\mathcal{E}$ such that under $\mathcal{T}$, the vector space operations $\varepsilon \mapsto \lambda\varepsilon$ is continuous from $\mathbb{K} \times \mathcal{E}$ to $\mathcal{E}$ and $(\varepsilon_1, \varepsilon_2) \mapsto \varepsilon_1 + \varepsilon_2$ is continuous from $\mathcal{E} \times \mathcal{E}$ to $\mathcal{E}$, where $\mathbb{K} \times \mathcal{E}$ and $\mathcal{E} \times \mathcal{E}$ are given the respective product topologies.*

Proposition A.2 *The topology $\mathcal{T}$ of any t.v.s. $(\mathcal{E}, \mathcal{T})$ defines a uniform structure on $\mathcal{E}$ and the notions of completeness and completion are meaningful for a t.v.s. Moreover, the completion of a t.v.s. remains a t.v.s.*

Here are some particularly interesting classes of topological vector spaces:

Definition A.3 (– *locally convex space (l.c.s.)* –) *A locally convex (vector) space (over any topological field $\mathbb{K}$) is a topological vector space $(\mathcal{E}, \mathcal{T})$ such that the topology is defined by a family of semi-norms. In the special case where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ it is equivalent to say that $\mathcal{T}$ admits a fundamental system of convex neighbourhouds of zero.*

Definition A.4 (– *normed space* –) *A normed space is a locally convex space $(\mathcal{E}, \mathcal{T})$ such that the topology is defined by a unique norm. In the special case where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ it is equivalent to say that $\mathcal{T}$ admits a fundamental system of neighbourhouds of zero that reduces to one convex set.*

Definition A.5 (– *Banach space* –) *A Banach space is a complete normed space.*

A.2 Linear algebra, Hilbert spaces

A.2.1 Linear, semilinear, bilinear and sesquilinear applications

Let E, F, G be three vector spaces. A function $u : E \longrightarrow F$ is linear if:

$$\forall (e_1, e_2) \in E, \forall \lambda \in \mathbb{K},\ u(e_1 + \lambda e_2) = u(e_1) + \lambda u(e_2)$$

It is called semilinear if:

$$\forall (e_1, e_2) \in E, \forall \lambda \in \mathbb{K},\ u(e_1 + \lambda e_2) = u(e_1) + \overline{\lambda} u(e_2)$$

A function $L : F \times E \longrightarrow G$ is bilinear if:

- $\forall f \in F$, $L(f, .)$ is linear;
- $\forall e \in E$, $L(., e)$ is linear.

It is sesquilinear[1] if:

- $\forall e \in E$, $L(., e)$ is semilinear;
- $\forall f \in F$, $L(f, .)$ is linear.

If $G = \mathbb{K}$ we call the application $u : E \longrightarrow \mathbb{K}$ a form.

[1] sesqui means one and a half

A.2.2 Hilbert spaces

Definition A.6 (– *inner product* –) *We call inner product on a vector space H any non-degenerate conjugate symmetric (Hermitian) sesquilinear positive form.*

Definition A.7 (– *prehilbertian space* –) *We call prehilbertian space any vector space H_0 endowed with a non-degenerate conjugate symmetric (Hermitian) sesquilinear positive form, i.e. an inner product on H_0.*

$\|h_0\|_{H_0} = \langle h_0 | h_0 \rangle_{H_0}^{\frac{1}{2}}$ is then a norm that makes H_0 a locally convex space.

Definition A.8 (– *Hilbert space* –) *A Hilbert space is a complete prehilbertian space.*

Appendix B

Dualities

B.1 The algebraic and topological dual spaces

We define the dual space of a t.v.s. upon the linear forms:

Definition B.1 (– *algebraic dual, topological dual* –) *let $\mathcal{E}$ be a vector space over the field $\mathbb{K}$. We call algebraic dual of $\mathcal{E}$ and note $\mathcal{E}^*$ the space of all linear forms on $\mathcal{E}$. If $(\mathcal{E}, \mathcal{T})$ is a t.v.s. over the topological field $\mathbb{K}$, we call topological dual of $\mathcal{E}$ and note $\mathcal{E}^{'}$ the space of continuous linear froms on $\mathcal{E}$ (when $\mathcal{E}$ is endowed with the topolgy $\mathcal{T}$ and $\mathbb{K}$ with its topology).*

B.2 Dualities (dual systems)

Definition B.2 (– *dual system of spaces* –) *Two vector spaces E, F are said to be in duality if there exists a bilinear form L on the product space $F \times E$ separate in E and F, i.e.:*

1. $\forall e \neq 0 \in E, \exists f \in F,\ L(f, e) \neq 0$;

2. $\forall f \neq 0 \in F, \exists e \in E,\ L(f,e) \neq 0$*.*

In this case, (E,F) *is said to be a duality (relative to* L*).*

The following morphisms are then well defined:

$$\gamma_{(E,F)} : F \longrightarrow E^* \text{ algebraic dual of E} \qquad \theta_{(E,F)} : E' \triangleq \gamma_{(E,F)}(F) \longrightarrow F$$
$$y \longmapsto L(y,.) \qquad L(y,.) \longmapsto y$$

<u>Exemples :</u>

1. **- Fundamental exemple -** Let $\mathcal{E}$ be a locally convex space, $\mathcal{E}^*$ its algebraic dual. Then the canonical bilinear form $(\mu,\varepsilon) \mapsto \mu(\varepsilon)$ on $\mathcal{E}^* \times \mathcal{E}$ puts $\mathcal{E}$ and $\mathcal{E}^*$ in duality. $\gamma_{(\mathcal{E},\mathcal{E}*)} : \mathcal{E}^* \longrightarrow \mathcal{E}^*$ is the identity. The same arguments show that any locally convex topological vector space E can be put in duality with its topological dual E'.

2. Let D' be the space of distributions on $\mathbb{C}^n$, C^∞ the space of functions C^∞ with compact support. D' et C^∞ are in duality relative to

$$L : C^\infty \times D' \longrightarrow \mathbb{C}$$
$$(\phi,\psi) \longmapsto \int_{\mathbb{C}^n} \psi(s)\phi(s)ds$$

We give here the fundamental exemple of a Hilbert space H in duality with its conjugate space (that we should not identify with H in general). The inner product induces a bilinear form on the Hilbert space $\overline{H} \times H$:

$$\left(\overline{h_1}, h_2\right)_{(\overline{H},H)} = L(\overline{h_1}, h_2) = \langle h_1 | h_2 \rangle_H \tag{B.1}$$

By Riesz theorem,

$$\gamma_{(H,\overline{H})} : H \longrightarrow H^*$$
$$\overline{h} \longmapsto L(\overline{h},.)$$

is an isomorphism from $\overline{H}$ on the topological dual of H H'.

B.3 Topology and duality theory

Definition B.3 (– *compatible topologies* –) *We call topology on E compatible with the duality (E, F) any locally convex topology on E such that $E^{'} = \gamma_{(E,F)}(F)$.*

The weak (resp. Mackey) topology on E is the coarsest (resp. finest) topology compatible with the dual system (E, F) and we note it $\sigma_{(E,F)}$ (resp. $\tau_{(E,F)}$).
The concept of weak (resp. Mackey) continuity is then entirely defined for morphisms of dualities.

Starting from a duality (E, F) one can then endow E with a locally convex topology (actually any compatible topology will work) such that E becomes a locally convex space with topological dual E isomorph to F.

B.4 Transpose of a weakly continuous morphism

Proposition B.4 *Let (E, F), $(\mathcal{E}, \mathcal{F})$ be two dualites. Then for any weakly continuous linear application $T : E \longrightarrow \mathcal{E}$ there exists a unique application ${}^tT : \mathcal{F} \longrightarrow F$ called the transpose of T verifying:*

$$\forall \phi \in \mathcal{F},\ \forall e \in E\ \ (\phi, T(e))_{(\mathcal{F},\mathcal{E})} = \left({}^tT(\phi), e\right)_{(F,E)}$$

If E and $\mathcal{E}$ are two locally convex space, the transpose is defined upon the dualities (E, E') and $(\mathcal{E}, \mathcal{E}')$.

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