Inegalites de Gagliardo-Nirenberg optimales sur les varietes riemanniennes

Christophe Brouttelande

THÈSE

présentée en vue de l'obtention du

DOCTORAT
DE
L'UNIVERSITÉ PAUL SABATIER
TOULOUSE III

Discipline : Mathématiques
par

Christophe Brouttelande

Inégalités de Gagliardo-Nirenberg optimales sur les variétés Riemanniennes

---

Soutenue le 30 juin 2003 devant le jury composé de
Messieurs les Professeurs :

Dominique Bakry	Université Paul Sabatier	Examinateur
Gilles Carron	Université Nantes	Examinateur
Olivier Druet	ENS Lyon	Examinateur
Emmanuel Hebey	Université Cergy-Pontoise	Examinateur
Michel Ledoux	Université Paul Sabatier	Directeur de thèse
Michel Vaugon	Université Paris VI	Rapporteur

au vu des rapports de messieurs :

William Beckner	University of Texas in Austin
Michel Vaugon	Université Paris VI

Laboratoire de Statistique et Probabilités
UMR CNRS 5883, Université Paul Sabatier, Toulouse III

À tous ceux et celles que j'apprécie
...sans jamais oser leur dire.

Remerciements

Je tiens tout d'abord à remercier Michel Ledoux pour m'avoir dirigé dans mes recherches. Je loue plus particulièrement sa grande gentillesse et sa disponibilité qui m'ont permis d'aborder la thèse en toute sérénité.

Je remercie aussi Emmanuel Hebey pour m'avoir appuyé durant cette période. C'est la clarté de ses cours de maîtrise et de DEA qui m'a fait prendre goût à l'analyse sur les variétés.

Je souhaite exprimer toute ma reconnaissance à William Beckner et Michel Vaugon pour le temps qu'ils ont passé à la lecture de ce manuscrit. Je regrette que la distance qui sépare Toulouse du Texas n'ait pas permis la présence de William Beckner à mon jury. Heureusement, Michel Vaugon aura brillamment représenté les deux rapporteurs à lui seul.

Merci à Dominique Bakry, Gilles Carron et Olivier Druet pour m'avoir fait l'honneur de participer au jury.

J'ai une pensée toute particulière pour tous les thésards et ex-thésards du LSP, en particulier l'équipe des "logsob" (Florent, Djèlil, Greg, Pierre, Ivan, Sébastien, Cyril et Cécile A.). Cette petite troupe m'a accueilli à Toulouse avec beaucoup de gentillesse. Ils ont tous fait preuve de tolérance en acceptant parmi eux un géomètre-EDPiste déguisé qui tentait tant bien que mal de s'incruster dans le monde des probabilistes. Certains savent à quel point il est difficile pour moi d'aller vers les autres.

Je remercie tous mes "cobureautistes" d'hier et d'aujourd'hui, Cécile A., Vincent, Cyril, Clément, Caroline et Renaud pour m'avoir supporté pendant ces quelques années. Clément, quand je pense aux questions que nous nous posions le soir avant de rentrer, je suis rempli de nostalgie. Quand à toi, Caroline, je te dois quelques-uns des quelques rudiments de cuisine que je connais. Ta grande maîtrise du couteau ne sera pas oubliée.

Je pense également aux autres futurs docteurs, Aude, Céline, Abdel, Olivier, Yan, Jérôme, Cécile M. et Elie, avec qui j'ai passé d'agréables moments. Je remercie plus particulièrement Céline, ma correctrice d'orthographe officielle, dont la capacité à corriger mes fautes reste inégalable. Sans elle, mon introduction ne serait, a fortiori, pas si bien accentuée. Je n'oublie pas non plus ma grande amie Lise qui, contrairement aux autres, ne s'est pas engouffrée dans la recherche mathématique. Elle a préféré se consacrer aux autres, dans l'enseignement. Je la remercie pour m'avoir entre autre appris qu'il existait d'autres thés que le Lip... Yel... .

Tous mes voeux de bonheur aux nouveaux mariés Caroline et Jérôme et aux jeunes mariés Lise et Florent qui vont bientôt fêter leurs noces de coton.

Je remercie mes parents pour m'avoir soutenu dans mes choix. Ils ont toujours été là quand j'ai eu besoin d'eux.

Je terminerai enfin avec ma p'tite soeur adorée. Les dix premières années ont été dures, avec des bleus et des bosses. Mais une fois arrivée à un âge plus "communicatif", elle a su montrer combien elle aimait son grand frère. Il faut bien avouer que ce dernier n'était pas très communicatif non plus. Je regrette qu'elle n'ait pas pu venir à ma soutenance. Je l'embrasse bien fort.

Table des matières

Chapitre 1

Introduction

Introduction

Les espaces de Sobolev jouent un rôle central dans la théorie des équations aux dérivées partielles. Les théorèmes de plongement de ces espaces dans les espaces de Lebesgue se traduisent en inégalités dites de Sobolev. Elles sont devenues un outil fondamental en analyse. Ces notions ont été introduites par S. L. Sobolev à la fin des années 30. D'autres mathématiciens se sont intéressés à ce domaine. On peut notamment citer les travaux d'E. Gagliardo et L. Nirenberg dans les années 50. L'étude des inégalités de Sobolev optimales trouve ses origines dans de grands problèmes d'analyse tels que le problème de Yamabe, résolu par T. Aubin et R. Schoen en 1984. Comme nous le verrons par la suite, il existe plusieurs façons d'aborder cette étude. Nous parlerons plus particulièrement de programme AB et de programme BA. Le premier programme a été étudié, entre autre, par T. Aubin, O. Druet, E. Hebey et M. Vaugon. Le second trouve sa source en théorie des semi-groupes de Markov. Il a notamment été étudié par D. Bakry et M. Ledoux.

Les inégalités de Sobolev sont un cas particulier des inégalités de Gagliardo-Nirenberg. Il est donc naturel de se demander si les résultats connus pour les inégalités de Sobolev s'adaptent aux autres inégalités de la famille. Les premiers travaux de ce type se sont portés sur l'inégalité de Nash et les inégalités de Sobolev logarithmique. Dans cette thèse, nous obtenons une généralisation de ces travaux à une famille d'inégalités beaucoup plus large. Plus précisément, nous adaptons les programmes AB et BA à une sous-famille des inégalités de Gagliardo-Nirenberg contenant, entre autres, l'inégalité de Nash.

Dans cette introduction, nous présentons les bases nécessaires à la bonne compréhension de cette thèse. Nous y définissons entre autre les inégalités de Gagliardo-Nirenberg et discutons des relations entre elles. Nous précisons ce que nous entendons par programme AB et programme BA. Nous généralisons aussi quelques résultats simples sur les inégalités de Sobolev et de Nash, en particulier ceux du programme BA. L'adaptation du programme AB est plus délicate et constitue l'essentiel de cette thèse. Cette étude est développée dans les deux parties suivant cette introduction. Il s'agit de travaux de l'auteur, généralisations de ceux d'O. Druet sur les inégalités de Sobolev et d'E. Humbert sur l'inégalité de Nash.

1.1 Généralités

1.1.1 Inégalités de Sobolev

Soit (M, g) une variété riemannienne de classe C^∞, sans bord et de dimension n. On supposera, sauf indication contraire, M complète et $n \geq 2$. Pour $p \geq 1$, on note $L^p(M)$ l'espace des fonctions p-intégrables muni de la norme

$$\|u\|_p = \left(\int_M |u|^p dv_g \right)^{\frac{1}{p}}$$

où dv_g désigne l'élément de volume riemannien. Pour $p \geq 1$, on pose

$$\mathcal{D}^p(M) = \{u \in C^\infty(M) \ / \ \|\nabla u\|_p + \|u\|_p < +\infty\}$$

où $C^\infty(M)$ est l'ensemble des fonctions indéfiniment différentiables sur M. $H_1^p(M)$ est par définition le complété de $\mathcal{D}^p(M)$ pour la norme

$$\|u\|_{H_1^p} = \|\nabla u\|_p + \|u\|_p.$$

La variété étant complète, l'ensemble des fonctions C^∞ à support compact, que l'on notera $C_c^\infty(M)$, est dense dans $H_1^p(M)$.

Si M est à courbure de Ricci minorée et rayon d'injectivité strictement positif, alors, d'après le théorème d'injection de Sobolev, $H_1^p(M)$ s'injecte continûment dans $L^q(M)$ pour $n > p \geq 1$ et $q \leq p^* = \frac{np}{n-p}$. Dans le cas critique $q = p^*$, la continuité de cette injection se traduit par l'existence de deux constantes A et B telles que pour tout $u \in H_1^p(M)$, l'inégalité de Sobolev

$$\|u\|_{p*} \leq A\|\nabla u\|_p + B\|u\|_p.$$

est vérifiée. On préférera travailler sur l'inégalité suivante, appelée aussi inégalité de Sobolev, plus naturelle du point de vue des EDPs,

$$\|u\|_{p*}^p \leq A\|\nabla u\|_p^p + B\|u\|_p^p. \qquad S^p(A, B)$$

De ces inégalités découle toute une famille d'inégalités dites "de type Sobolev". La famille à laquelle nous nous intéresserons plus particulièrement est la famille des inégalités de Gagliardo-Nirenberg.

1.1.2 Inégalités de Gagliardo-Nirenberg

D'après l'inégalité de Hölder, pour toute fonction u à valeurs réelles,

$$\|u\|_r \leq \|u\|_q^\theta \|u\|_s^{1-\theta}$$

où $\theta \in [0, 1]$ et r, q, s sont des nombres réels strictement positifs vérifiant

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}. \qquad (1.1)$$

Soit $n > p \geq 1$. Si on pose $q = p^* = \frac{np}{n-p}$, on peut alors combiner ces inégalités avec les inégalités de Sobolev vues précédemment quand celles-ci sont valides. Dans ce cas, il existe des constantes A et B telles que pour tout $u \in H_1^2(M)$,

$$\|u\|_r \leq \left(A\|\nabla u\|_p^p + B\|u\|_p^p\right)^{\frac{\theta}{p}} \|u\|_s^{1-\theta}.$$

Ces inégalités sont appelées inégalités de Gagliardo-Nirenberg. Le cas $\theta = 0$ étant trivial, on ne considérera que les cas $\theta \neq 0$. On peut alors élever les inégalités à la puissance $\frac{p}{\theta}$, ce qui donne

$$\|u\|_r^{\frac{p}{\theta}} \leq \left(A\|\nabla u\|_p^p + B\|u\|_p^p\right) \|u\|_s^{\frac{p(1-\theta)}{\theta}}. \qquad GN_{r,s,\theta}^p(A,B)$$

D'autre part, on peut inclure dans cette famille l'inégalité de Sobolev logarithmique

$$\int_M |u|^p \ln |u|^p dv_g \leq \frac{n}{p} \ln\left(A\|\nabla u\|_p^p + B\right), \qquad SL^p(A,B)$$

qui est définie pour $\|u\|_p = 1$. Cette inégalité s'obtient en faisant tendre θ vers 0 après avoir fixé $r = p$ dans $GL_{r,s,\theta}^p(A,B)$.

Plusieurs autres inégalités de cette famille ont une place importante dans la littérature. On peut citer en particulier l'inégalité de Nash

$$\left(\int_M |u|^2 dv_g\right)^{1+\frac{2}{n}} \leq \left(A\int_M |\nabla u|^2 dv_g + B\int_M |u|^2 dv_g\right)\left(\int_M |u| dv_g\right)^{\frac{4}{n}}$$

introduite par J. Nash dans [?], qui est obtenue en posant $p = 2$, $r = 2$, $s = 1$ et $\theta = \frac{n}{n+2}$. Cette inégalité fut utilisée dans l'étude de la régularité des solutions de certaines EDPs paraboliques. Par ailleurs, pour $p = 2$, $r = 2 + \frac{4}{n}$, $s = 2$ et $\theta = \frac{n}{n+2}$, on obtient alors l'inégalité de Moser

$$\int_M |u|^{2+\frac{4}{n}} dv_g \leq \left(A\int_M |\nabla u|^2 dv_g + B\int_M |u|^2 dv_g\right)\left(\int_M |u|^2 dv_g\right)^{\frac{2}{n}},$$

qui a été utilisé par J. Moser dans [?], un papier consécutif à celui de J. Nash.

Notons que pour construire ces inégalités, nous avons utilisé les inégalités de Sobolev. Nous avons pour cela supposé $n > p$. On peut pourtant remarquer que si $\theta \neq 1$, la condition (1.1) devient

$$\frac{1}{r} = \frac{\theta(n-p)}{np} + \frac{1-\theta}{s},$$

condition qui ne nécessite pas a priori l'hypothèse $n > p$. Il est par conséquent légitime de se demander si les inégalités $GN_{r,s,\theta}^p(A,B)$ ne peuvent pas être définies pour $p \geq n$. La réponse est affirmative, et ce, malgré la non-validité des inégalités de Sobolev. En particulier, les inégalités de Nash, Moser et Sobolev logarithmiques sont toutes définies pour tout $n \geq 1$. Pour de plus amples détails, on pourra se référer au très complet [?].

1.1.3 Semi-groupe de la chaleur ; Ultracontractivité

Dans la suite, nous utiliserons parfois la notion de semi-groupe. Nous l'introduisons dans cette section. On note Δ_g l'opérateur de Laplace-Beltrami sur (M, g). C'est l'opérateur défini en coordonnées locales par

$$\Delta_g = -\text{div}\nabla = -g^{ij}\left(\frac{\partial^2}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial}{\partial x_k}\right)$$

où Γ_{ij}^k représente les symboles de Christoffel associés à la connexion de Levi-Civita sur (M, g). On appelle solution fondamentale de l'équation de la chaleur

$$\frac{\partial u}{\partial t} + \Delta_g u = 0 \qquad EC$$

toute fonction $u(t, x, y)$ définie sur $(0, +\infty) \times M \times M$ vérifiant, à y fixé et au sens des distributions, EC en les variables (t, x) avec la condition initiale

$$\lim_{t \to 0^+} u(t, \cdot, y) = \delta_y.$$

Le noyau de la chaleur sur M est, par définition, la plus petite solution fondamentale positive de EC. On le note $p_t(x, y)$. Il a été prouvé par J. Dodziuk dans [?] que le noyau de la chaleur existe toujours, même si la variété n'est pas complète, et qu'il est de classe C^∞ en (t, x, y). De plus,

1. pour tout x et y dans M, $p_t(x, y) = p_t(y, x)$,

2. pour tout $0 < s < t$,

$$p_t(x, y) = \int_M p_s(x, z) p_{t-s}(z, y) dv_g(z),$$

3. pour tout $t > 0$ et $x \in M$,

$$\int_M p_t(x, y) dv_g(y) \leq 1.$$

On définit maintenant le semi-groupe de la chaleur $(\mathrm{P}_t)_{t>0}$. Il s'agit de la suite d'opérateurs sur les fonctions mesurables définie par

$$\mathrm{P}_t u(x) = \int_M u(y) p_t(x, y) dv_g(y).$$

Ces opérateurs sont stables sur $L^\infty(M)$, bornés sur $L^2(M)$, et vérifient

1. $\mathrm{P}_0 = Id$,
2. $\forall t, s \geq 0, \ \mathrm{P}_t \circ \mathrm{P}_s = \mathrm{P}_{t+s}$,
3. $\forall u \in L^2(M), \ \lim_{t \to 0_+} \mathrm{P}_t u = u$ dans $L^2(M)$,
4. $\forall t \geq 0, \ \mathrm{P}_t 1 = 1$,
5. $\forall u \geq 0, \ \mathrm{P}_t u \geq 0$.

Enfin, nous introduisons la notion d'ultracontractivité. On dit que $(\mathrm{P}_t)_{t>0}$ est ultracontractif s'il existe $C>0$ tel que pour tout $t>0$

$$\sup_{x,y\in M}|p_t(x,y)|\leq \frac{C}{t^{\frac{n}{2}}}.$$

Pour plus de détails sur le noyau de la chaleur, citons comme référence le livre de E. B. Davies [?].

1.1.4 Equivalence entre les inégalités

Fixons $p\geq 1$. Par construction, l'inégalité de Sobolev $S^p(A,B)$, si elle est définie, est plus forte que les inégalités de Gagliardo-Nirenberg $GN^p_{r,s,\theta}(A,B)$ et, par conséquent, que l'inégalité de Sobolev logarithmique. En fait, elles sont toutes équivalentes entre elles aux constantes A et B près. On peut montrer à partir de [?] que l'inégalité de Sobolev $S^2(A,0)$ est équivalente à l'ultracontractivité. L'analogue de ce résultat pour l'inégalité de Nash fut ensuite montré par E. Carlen, S. Kusuoka et D. Stroock [?]. En fait, dans le cas $p=2$, toutes les inégalités de Gagliardo-Nirenberg vérifiant $B=0$ sont équivalentes à l'ultracontractivité. On peut néanmoins montrer l'équivalence entre elles sans passer par cette propriété. Une telle preuve, utilisant la notion de capacité, se trouve dans le livre de V. Maz'ja [?]. On se place ici dans le cas $p<n$. L'équivalence est encore vraie sans cette hypothèse mais son exposition est plus délicate (voir [?]). Nous allons montrer que $GN^p_{r,s,\theta}(A,B)$ implique l'inégalité de Sobolev. Cette démonstration est due à D. Bakry, T. Coulhon, M. Ledoux et L. Saloff-Coste [?]. Elle fût donnée à l'origine pour l'inégalité de Nash mais nous traitons ici le cas général.

Pour alléger les notations, nous posons

$$W(u)=A\int_M|\nabla u|^p dv_g+B\int_M|u|^p dv_g.$$

Supposons $GN^p_{r,s,\theta}(A,B)$ vraie. Soit u positive dans $H_1^p(M)$. Posons $q=\frac{np}{n-p}$ et pour tout entier relatif k, $u_k=(u-2^k)^+\wedge 2^k$. Alors, $u_k\in H_1^p(M)$ et

$$u_k=\begin{cases}0 & \text{sur } \{u\leq 2^k\}\\ u-2^k & \text{sur } \{2^k\leq u\leq 2^{k+1}\}\\ 2^k & \text{sur } \{u\geq 2^{k+1}\}.\end{cases}$$

Comme

$$2^k\,\mathbf{1}_{\{u\geq 2^{k+1}\}}\leq u_k\leq 2^k\,\mathbf{1}_{\{u\geq 2^k\}},$$

$$\left(\int_M u_k^r dv_g\right)^{\frac{p}{r\theta}}\geq\left(2^{rk}\operatorname{Vol}_g(\{u\geq 2^{k+1}\})\right)^{\frac{p}{r\theta}}$$

et

$$\left(\int_M u_k^s dv_g\right)^{\frac{p(1-\theta)}{s\theta}}\geq\left(2^k\operatorname{Vol}_g(\{u\geq 2^k\})\right)^{\frac{p(1-\theta)}{s\theta}}$$

où Vol_g désigne la mesure riemannienne sur (M,g). Par conséquent, en appliquant $GN^p_{r,s,\theta}(A,B)$ à u_k,

$$\left(2^{rk}\,\mathrm{Vol}_g(\{u \geq 2^{k+1}\})\right)^{\frac{p}{r\theta}} \leq W(u_k)\left(2^{sk}\,\mathrm{Vol}_g(\{u \geq 2^{k}\})\right)^{\frac{p(1-\theta)}{s\theta}}.$$

On pose alors $a_k = 2^{qk}\,\mathrm{Vol}_g(\{u \geq 2^k\})$. En mettant l'inégalité précédente à la puissance $\frac{r\theta}{p}$ puis en multipliant par $2^{q(k+1)-rk}$, on obtient

$$a_{k+1} \leq 2^q\, W(u_k)^{\frac{r\theta}{p}}\, a_k^{\frac{r}{s}(1-\theta)}.$$

Par l'inégalité de Hölder,

$$\sum_k a_k = \sum_k a_{k+1} \quad\leq\quad 2^q\left(\sum_k W(u_k)\right)^{\frac{r\theta}{p}}\left(\sum_k a_k\right)^{\frac{r}{s}(1-\theta)},$$

d'où

$$\sum_k a_k \leq 2^{\frac{q^2}{r\theta}}\left(\sum_k W(u_k)\right)^{\frac{n}{n-p}}.$$

Comme $|\nabla u_k|_g^p = |\nabla u|_g^p\,\mathbf{1}_{\{2^k\leq u\leq 2^{k+1}\}}$,

$$\begin{aligned}\sum_k W(u_k) &= A\sum_k\int_{\{2^k\leq u\leq 2^{k+1}\}}|\nabla u|_g^p dv_g + B\sum_k\int_M u_k^p dv_g\\ &\leq A\int_M|\nabla u|_g^p dv_g + B\sum_k 2^{pk}\,\mathrm{Vol}_g(\{u\geq 2^k\}).\end{aligned}$$

Il est facile de montrer que

$$\sum_k 2^{pk}\,\mathrm{Vol}_g(\{u \geq 2^k\}) \leq \frac{1}{1-2^{-p}}\int_M u^p dv_g \quad\text{et}\quad \sum_k a_k \geq 2^{-q}\int_M u^q dv_g.$$

On obtient par suite

$$\int_M u^q dv_g \leq 2^q 2^{\frac{q^2}{r\theta}}\left(A\int_M|\nabla u|_g^p dv_g + \frac{B}{1-2^{-p}}\int_M u^p dv_g\right)^{\frac{n}{n-p}},$$

ce qui donne l'inégalité de Sobolev

$$\left(\int_M u^q dv_g\right)^{\frac{p}{q}} \leq 2^{p\left(1+\frac{q}{r\theta}\right)}\left(A\int_M|\nabla u|_g^p dv_g + \frac{B}{1-2^{-p}}\int_M u^p dv_g\right).$$

Étudions maintenant le cas de l'inégalité de Sobolev logarithmique. En fixant $r=p$ puis en utilisant (1.1), on peut réécrire $GN^p_{r,s,\theta}(A,B)$ sous la forme

$$\|u\|_p^{p\left(1+\frac{ps}{n(p-s)}\right)} \leq \left(A\|\nabla u\|_p^p + B\|u\|_p^p\right)\|u\|_s^{s\frac{p^2}{n(p-s)}}. \qquad GN^p_s(A,B)$$

Notons que l'on peut toujours trouver A et B indépendantes de s et p. Il suffit pour cela de considérer les constantes de l'inégalité de Sobolev. On pose maintenant $\|u\|_2 = 1$. En passant au logarithme dans $GN_s^p(A, B)$, on obtient

$$-\frac{p^2}{n(p-s)} \ln \int_M |u|^s dv_g \leq \ln \left(A\|\nabla u\|_p^p + B \right).$$

Soit $\phi(s) = \ln \int_M |u|^s dv_g$. D'après l'inégalité de Hölder, ϕ est convexe. Par conséquent, la fonction Φ définie pour $s \in (0, p)$ par

$$\Phi(s) = \frac{\phi(s) - \phi(p)}{s - p}$$

est croissante et se prolonge par continuité en $s = p$ en posant

$$\Phi(p) = \phi'(p) = \int_M |u|^p \ln |u| dv_g.$$

En faisant tendre s vers p, on obtient alors l'inégalité de Sobolev logarithmique

$$\int_M |u|^p \ln |u|^p dv_g \leq \frac{n}{p} \ln \left(A\|\nabla u\|_p^p + B \right).$$

On vient de montrer que les inégalités $GN_s^p(A, B)$ impliquent l'inégalité de Sobolev logarithmique. Il est facile de voir que la croissance de Φ donne la réciproque.

Dans la suite, nous parlerons souvent du lien entre les inégalités de Sobolev logarithmiques et les bornes supérieures du noyau de la chaleur. La section qui suit présente un résultat de D. Bakry sur ce sujet.

1.1.5 Le lien entre $SL^2(A, B)$ et l'ultracontractivité

On s'intéresse dans cette section à la relation entre l'inégalité de Sobolev logarithmique $SL^2(A, B)$ et le contrôle de la norme du semi-groupe de la chaleur. On considère pour cela une inégalité du type

$$\int_M |u|^2 \ln |u|^2 dv_g \leq \Phi \left(\|\nabla u\|_2^2 \right) \qquad SL_\Phi$$

où $\Phi : \mathbb{R}_+^* \to \mathbb{R}$ est concave, strictement croissante de classe C^1 et $\|u\|_2 = 1$. On cherche à estimer $\|\mathrm{P}_t\|_{p,q}$ où $\| \cdot \|_{p,q}$ est la norme sur $\mathcal{L}\left(L^p(M), L^q(M)\right)$ définie par

$$\|H\|_{p,q} = \sup_{u \in L^p(M)} \frac{\|Hu\|_q}{\|u\|_p}.$$

Le théorème qui suit est dû à D. Bakry [?]. Il fût donné dans le cadre, plus général, des processus de diffusion markoviens.

Théorème 1. *Supposons (M,g) compacte. Si SL_Φ est vraie, alors pour tout $1 \leq p < q \leq +\infty$*

$$\|\mathrm{P}_t\|_{p,q} \leq e^m$$

où

$$t = t_{p,q} = \int_p^q \Phi'(v(s)) \frac{ds}{4(s-1)} \quad et \quad m = m_{p,q} = \int_p^q \left(\Phi(v(s)) - v(s)\Phi'(v(s))\right) \frac{ds}{s^2},$$

sous la condition qu'il existe une fonction $v \geq 0$ pour laquelle les deux intégrales précédentes soient convergentes.

Preuve :
Supposons SL_Φ vraie. On a pour tout x et y strictement positifs

$$\Phi(x) \leq \Phi(y) + \Phi'(x)(x-y),$$

ce qui donne pour tout $x > 0$ et tout u,

$$\int_M |u|^2 \ln |u|^2 dv_g \leq \Phi'(x)\|\nabla u\|_2^2 + \varphi(x)\|u\|_2^2$$

où $\varphi(x) = \Phi(x) - x\Phi'(x)$. On peut considérer sans perdre en généralité que $u > 0$. En changeant u en $u^{\frac{s}{2}}$, on obtient que pour tout $u > 0$, $x > 0$ et $s > 1$

$$\int_M u^s \ln u^s dv_g - \int_M u^s dv_g \ln \int_M u^s dv_g \\ \leq -\Phi'(x) \frac{s^2}{4(s-1)} \int_M u^{s-1} \left(\Delta_g u\right) dv_g + \varphi(x) \int_M u^s dv_g. \quad (1.2)$$

Considérons maintenant une fonction $s \to x(s) > 0$, $s > 1$. Posons pour tout $t > 0$

$$V(t) = e^{-m(t)} \|\mathrm{P}_t u\|_{s(t)}.$$

Si l'on choisit m et s tels que

$$\frac{s^2(t)}{s'(t)} = \Phi'(x(s(t))) \frac{s^2(t)}{4(s(t)-1)} \quad \text{et} \quad m'(t) = \varphi(x(s(t))) \frac{s'(t)}{s^2(t)},$$

on constate que l'inégalité (1.2) implique $V' \leq 0$ et donc que V est décroissant. Fixons $1 \leq p < q \leq \infty$ et considérons le système différentiel

$$\begin{cases} dt = \dfrac{\Phi'(x(s))}{4(s-1)} ds, \quad s(0) = p, \\ dm = \dfrac{\varphi(x(s))}{s^2} ds. \end{cases}$$

Alors

$$\|\mathrm{P}_t\|_{p,q} \leq e^m \quad (1.3)$$

où

$$t = t_{p,q} = \int_p^q \Phi'(x(s)) \frac{ds}{4(s-1)} \ \text{ et } \ m = m_{p,q} = \int_p^q \varphi(x(s)) \frac{ds}{s^2}. \tag{1.4}$$

Remarquons que ce résultat donne la borne optimale euclidienne pour chaque p et q. En effet, si l'on pose $x(s) = \frac{\lambda s^2}{s-1}$ avec $\lambda > 0$ un paramètre et si $\Phi = \frac{n}{2} \ln\left(\frac{2}{n\pi e} \cdot\right)$, on obtient par exemple pour $p = 1$ et $q = +\infty$

$$\|\mathrm{P}_t\|_{1,\infty} \leq \frac{1}{(4\pi t)^{\frac{n}{2}}}.$$

Notons qu'il existe une réciproque à ce théorème. Soit $1 \leq p < \infty$. Supposons que pour tout q dans un voisinage de p, on ait (1.3) où t et m vérifient (1.4) pour un certain Φ. On a alors pour tout $x \geq 0$

$$\begin{aligned} &\int_M u^p \ln u^p dv_g - \int_M u^p dv_g \ln \int_M u^p dv_g \\ &\qquad \leq -\Phi'(x) \frac{p^2}{4(p-1)} \int_M u^{p-1} (\Delta_g u)\, dv_g + \varphi(x) \int_M u^p dv_g. \end{aligned}$$

On pose pour cela

$$U(\varepsilon) = e^{-m(\varepsilon)} \|\mathrm{P}_{t(\varepsilon)} u\|_{p+\varepsilon}.$$

avec $t(\varepsilon) = t_{p,p+\varepsilon}$ et $m(\varepsilon) = m_{p,p+\varepsilon}$. On peut alors montrer que $U'(0) \leq 0$, ce qui prouve le résultat.

1.2 Inégalités optimales

De nombreuses questions concernant les inégalités de Gagliardo-Nirenberg se posent naturellement : étude des meilleures constantes, validité des inégalités optimales, existence de fonctions extrémales. Les premières inégalités à être étudiées ont été les inégalités de Sobolev. A ce sujet, on pourra se référer à [?]. Plus récemment, ces travaux ont été généralisés au cas de l'inégalité de Nash (voir [?] et [?]). Le but de cette thèse est l'étude d'une famille plus grande d'inégalités de Gagliardo-Nirenberg. La famille que nous considérerons contient en particulier l'inégalité de Sobolev logarithmique $SL^2(A, B)$.

1.2.1 Le cas euclidien

Comme nous le verrons dans la section suivante, l'étude du cas euclidien est fondamental dans l'étude du cas général. On suppose dans toute cette section que (M, g) est l'espace $(\mathbb{R}^n, \delta)$ où δ désigne la métrique euclidienne standard. Le théorème qui suit fut obtenu indépendamment par T. Aubin [?] et G. Talenti [?].

Théorème 2. *Soient* $1 \leq p < n$ *et* $q = \frac{np}{n-p}$.

1. Pour tout $u \in C^\infty(I\!R^n)$,

$$\|u\|_q \leq K(n,p)\|\nabla u\|_p \tag{1.5}$$

$$\text{où } K(n,p) = \begin{cases} \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}} & \text{si } p = 1 \\ \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-\frac{1}{p}}\left(\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n+1-\frac{n}{p}\right)\omega_{n-1}}\right)^{\frac{1}{n}} & \text{si } p > 1. \end{cases}$$

avec ω_{n-1} *le volume de la sphère unité standard* (S^{n-1}, h) *de* $I\!R^n$.

2. $K(n,p)$ *est la meilleure constante dans (1.5) et si* $p > 1$, *l'égalité dans (1.5) est réalisée par les fonctions*

$$u_\lambda(x) = \left(\lambda + |x|^{\frac{p}{p-1}}\right)^{\frac{n}{p}-1}$$

où λ *est un réel strictement positif quelconque et* $|x|$ *désigne la norme euclidienne de* x.

On entend ici par "*K(n,p) est la meilleure constante*" qu'il n'existe pas de réel $A < K(n,p)$ tel que pour tout $u \in C^\infty(\mathbb{R}^n)$

$$\|u\|_q \leq A\|\nabla u\|_p$$

Il n'existe des équivalents de ce théorème pour les autres inégalités de Gagliardo-Nirenberg que dans peu de cas. On peut toutefois remarquer que l'on peut toujours prendre $B = 0$ dans les inégalités $GN^p_{r,s,\theta}(A,B)$. On ne considère donc que la famille des inégalités suivantes

$$\|u\|_r^{\frac{p}{\theta}} \leq A_{\text{opt}}(p,r,s,\theta)\|\nabla u\|_p^p \|u\|_s^{\frac{p(1-\theta)}{\theta}} \qquad GN^p_{\text{opt},r,s,\theta}$$

où $A_{\text{opt}}(p,r,s,\theta)$ est la meilleure constante dans $GN^p_{\text{opt},r,s,\theta}$. Cette constante a été calculée, de même que les fonctions extrémales, dans quelques cas particuliers autres que les inégalités de Sobolev. E. Carlen et M. Loss [?] ont résolu le cas de l'inégalité de Nash. W. Beckner [?] et E. Carlen [?] ont ensuite respectivement montré le cas de l'inégalité de Sobolev logarithmique pour $p = 1$ et $p = 2$. Plus récemment, M. Del Pino et J. Dolbeault [?] [?] ont calculé les constantes optimales de deux autres sous-familles d'inégalités de Gagliardo-Nirenberg. Pour $n \geq 3$, il s'agit des inégalités où r, s, θ, p vérifient

$$r = p\frac{s-1}{p-1},\ 1 < p < n,\ p < s \leq \frac{p(n-1)}{n-p},\ \theta = \frac{(s-p)n}{(s-1)(np-(n-p)s)}$$

ou bien

$$s = p\frac{r-1}{p-1},\ 1 < p < n,\ 1 < r < p,\ \theta = \frac{(p-r)n}{r(n(p-r)-p(r-1))}.$$

1.2.2 Le cas riemannien

Considérons maintenant le cas d'une variété riemannienne complète (M, g). En général, on ne peut pas prendre $B = 0$ dans $GN^p_{r,s,\theta}(A, B)$. Pour le voir dans le cas compact, il suffit d'appliquer l'inégalité à $u \equiv 1$, ce qui donne $B \geq \mathrm{Vol}_g(M)^{-\frac{p}{n}}$. Deux approches sont donc possibles. La première consiste à d'abord calculer le meilleur B, puis à le fixer et à étudier la constante A. On l'appelle le *programme* BA. L'opération inverse constitue la seconde approche. Il s'agit du *programme* AB. Les inégalités a avoir déjà été étudiées sont les inégalités de Sobolev et de Nash. L'un des objectifs de cette thèse est d'adapter le programme AB à d'autres inégalités, par exemple l'inégalité de Sobolev logarithmique. Toutefois, nous généraliseront aussi quelques résultats classiques du programme BA.

1.2.3 Le programme BA

Posons

$$B_0 = \inf \left\{ B \in \mathbb{R} \ / \ \exists A \in \mathbb{R} \text{ tel que } GN^p_{r,s,\theta}(A, B) \text{ est valide} \right\}.$$

B_0 dépend a priori des coefficients r, s, θ et p ainsi que de la géométrie de (M, g). Le résultat suivant fût d'abord donné dans le cas des inégalités de Sobolev. Sa démonstration se trouve dans [?]. Le cas d'une inégalité de Nash modifiée est traité dans [?]. La preuve que nous donnons ici est une adaptation de ces travaux.

Théorème 3. *Supposons* (M, g) *compacte.*
i) Pour tout $p \in [1, 2]$ *si* $n \geq 2$, *il existe* $A > 0$ *tel que pour tout* $u \in H^p_1(M)$,

$$\|u\|_r^{\frac{p}{\theta}} \leq \left(A \|\nabla u\|_p^p + \mathrm{Vol}_g(M)^{-\frac{p}{n}} \|u\|_p^p \right) \|u\|_s^{\frac{p(1-\theta)}{\theta}}. \tag{1.6}$$

ii) Si p *vérifie* $\frac{r-s}{\theta} + s > p > 2$, *alors pour tout* $A > 0$ *il existe* $u \in H^p_1(M)$ *tel que (1.6) n'est pas valide.*

Preuve :
Il suffit de montrer le théorème pour $\mathrm{Vol}_g(M) = 1$.
i) Si $n = 2$ et $p \in [1, 2)$ ou si $n \geq 3$ et $p \in [1, 2]$, le résultat découle du cas, déjà connu, de l'inégalité de Sobolev (voir [?]). Il suffit alors d'appliquer l'inégalité de Hölder.
Supposons $n = p = 2$. On a pour tout $u \in L^2(M)$

$$\int_M u^2 dv_g = \left(\int_M u dv_g \right)^2 + \int_M (u - \overline{u})^2 dv_g$$

où $\overline{u} = \int_M u dv_g$. L'inégalité de Sobolev-Poincaré pour le plongement de $H^1_1(M)$ dans $L^2(M)$ nous donne de plus l'existence d'un $C > 0$ tel que

$$\int_M (u - \overline{u})^2 dv_g \leq C \left(\int_M |\nabla u| dv_g \right)^2.$$

En combinant les deux, on obtient

$$\int_M u^2 dv_g \le C\left(\int_M |\nabla u| dv_g\right)^2 + \left(\int_M u dv_g\right)^2. \tag{1.7}$$

Pour $u \in C^\infty(M)$ et $\beta \ge 2$, on pose $f = |u|^{\frac{\beta}{2}}$. Alors

$$\begin{aligned}
\int_M |u|^\beta dv_g &= \int_M f^2 dv_g \\
&\le C\left(\int_M |\nabla f| dv_g\right)^2 + \left(\int_M f dv_g\right)^2 \\
&= \frac{\beta^2 C}{4}\left(\int_M |u|^{\frac{\beta}{2}-1}|\nabla u| dv_g\right)^2 + \left(\int_M |u|^{\frac{\beta}{2}} dv_g\right)^2 \\
&= \frac{\beta^2 C}{4}\int_M |\nabla u|^2 dv_g \int_M |u|^{\beta-2} dv_g + \left(\int_M |u|^{\frac{\beta}{2}} dv_g\right)^2.
\end{aligned}$$

Posons $\beta = \max(s+2, r)$. On a $s < r \le \beta$ et par l'inégalité de Hölder,

$$\left(\int_M |u|^r dv_g\right)^{\frac{\beta}{r\alpha}} \le \int_M |u|^\beta dv_g \left(\int_M |u|^s dv_g\right)^{\frac{\beta(1-\alpha)}{s\alpha}},$$

où $\alpha \in (0,1]$ et $\frac{1}{r} = \frac{\alpha}{\beta} + \frac{1-\alpha}{s}$. Un simple calcul donne $\alpha = \frac{\beta}{r}\frac{r-s}{\beta-s}$. En combinant à (1.7), on obtient alors

$$\begin{aligned}
\left(\int_M |u|^r dv_g\right)^{\frac{\beta-s}{r-s}} \le\ & \frac{\beta^2 C}{4}\int_M |\nabla u|^2 dv_g \int_M |u|^{\beta-2} dv_g \left(\int_M |u|^s dv_g\right)^{\frac{r}{s}\frac{\beta-s}{r-s}-\frac{\beta}{s}} \\
& + \left(\int_M |u|^{\frac{\beta}{2}} dv_g\right)^2 \left(\int_M |u|^s dv_g\right)^{\frac{r}{s}\frac{\beta-s}{r-s}-\frac{\beta}{s}}.
\end{aligned} \tag{1.8}$$

Considérons tout d'abord le cas $r \le s+2$. On a $\beta = s+2$ et

$$\begin{aligned}
\left(\int_M |u|^r dv_g\right)^{\frac{2}{r-s}} \le\ & \frac{C(s+2)^2}{4}\int_M |\nabla u|^2 dv_g \int_M |u|^s dv_g \left(\int_M |u|^s dv_g\right)^{\frac{2}{r-s}-1} \\
& + \left(\int_M |u|^{\frac{s+2}{2}} dv_g\right)^2 \left(\int_M |u|^s dv_g\right)^{\frac{2}{r-s}-1}.
\end{aligned}$$

En appliquant une nouvelle fois l'inégalité de Hölder, on obtient (1.6). Plaçons nous maintenant dans le cas $r > s+2$. On a alors $\beta = r$ et

$$\int_M |u|^r dv_g \le \frac{r^2 C}{4}\int_M |\nabla u|^2 dv_g \int_M |u|^{r-2} dv_g + \left(\int_M |u|^{\frac{r}{2}} dv_g\right)^2.$$

Toujours par l'inégalité de Hölder,

$$\int_M |u|^{r-2} dv_g \le \left(\int_M |u|^r dv_g\right)^{1-\frac{2}{r-s}} \left(\int_M |u|^s dv_g\right)^{\frac{2}{r-s}}$$

et

$$\left(\int_M |u|^{\frac{r}{2}} dv_g\right)^2 \leq \int_M |u|^2 dv_g \left(\int_M |u|^r dv_g\right)^{1-\frac{2}{r-s}} \left(\int_M |u|^s dv_g\right)^{\frac{2}{r-s}}.$$

En combinant avec (1.8), on obtient (1.6), ce qui achève la preuve de i).

ii) Soient $p > 2$ et $u \in C^\infty(M)$. Pour $t > 0$ et $\varepsilon > 0$, on vérifie facilement que

$$\int_M |1 + \varepsilon u|^t dv_g = 1 + t \left(\int_M u dv_g\right) \varepsilon + \frac{t(t-1)}{2} \left(\int_M u^2 dv_g\right) \varepsilon^2 + o(\varepsilon^2).$$

Par suite, pour $\gamma > 0$

$$\begin{aligned}\left(\int_M |1 + \varepsilon u|^t dv_g\right)^\gamma &= 1 + \gamma t \left(\int_M u dv_g\right) \varepsilon + \left[\gamma \frac{t(t-1)}{2} \int_M u^2 dv_g \right. \\ &\quad \left. + t^2 \frac{\gamma(\gamma-1)}{2} \left(\int_M u dv_g\right)^2\right] \varepsilon^2 + o(\varepsilon^2).\end{aligned}$$

Supposons (1.6) vraie. En remarquant que

$$\int_M |\nabla(1 + \varepsilon u)|^p dv_g = o(\varepsilon^2),$$

on obtient en appliquant (1.6) à $1 + \varepsilon u$,

$$0 \leq \left(\frac{r-s}{\theta} + s - p\right) \left(\left(\int_M u dv_g\right)^2 - \int_M u^2 dv_g\right).$$

Puisque $\frac{r-s}{\theta} + s - p > 0$, c'est impossible dès que u n'est pas constante.

Posons maintenant

$$A_0 = \inf \left\{A \in \mathbb{R} \ / \ GN^p_{r,s,\theta}(A, B_0) \ \text{valide}\right\}.$$

On connaît peu de choses sur A_0. On peut toutefois donner quelques estimations dans certains cas.

Si $p = 2$ et $\frac{r-s}{\theta} + s - 2 > 0$, il est facile de minorer A_0. Pour cela, supposons $\text{Vol}_g(M) = 1$ et appliquons (1.6) à $1 + \varepsilon u$ où $u \in C^\infty(M)$ vérifie $\int_M u dv_g = 0$. On obtient cette fois-ci

$$\left(\frac{r-s}{\theta} + s - 2\right) \varepsilon^2 \int_M u^2 dv_g \leq \varepsilon^2 A \int_M |\nabla u|^2 dv_g + o(\varepsilon^2).$$

En appliquant cette inégalité à une fonction propre du laplacien associée à la première valeur propre non nulle λ_1, on trouve

$$A_0 \geq \left(\frac{r-s}{\theta} + s - 2\right) \lambda_1^{-1}.$$

Si le volume de (M,g) est quelconque, cela donne

$$A_0 \geq \frac{\left(\frac{r-s}{\theta}+s-2\right)}{\lambda_1 \mathrm{Vol}_g(M)^{\frac{2}{n}}}.$$

Il est facile de généraliser ce calcul à $SL^2(A, \mathrm{Vol}_g(M)^{-\frac{2}{n}})$. Dans ce cas,

$$A_0 \geq \frac{4}{n\lambda_1 \mathrm{Vol}_g(M)^{\frac{2}{n}}}.$$

Nous supposerons jusqu'à la fin de cette section que M est compacte et vérifie $\mathrm{Vol}_g(M) = 1$ et $p = 2$. Passer par l'utilisation des semi-groupes donne alors d'excellents résultats pour l'inégalité de Sobolev $S^2(A,1)$ et l'inégalité de Sobolev logarithmique $SL^2(A,1)$. On pourra trouver la démonstration du théorème qui suit dans [?].

Théorème 4. *Supposons (M,g) compacte. Si $\mathrm{Ric}_g \geq \rho g$ alors pour tout $u \in H_1^2(M)$*
i) Si $n \geq 3$,

$$\|u\|^2_{\frac{2n}{n-2}} \leq \frac{4(n-1)}{n(n-2)\rho}\|\nabla u\|_2^2 + \|u\|_2^2$$

ii) Si $n \geq 2$,

$$\int_M |u|^2 \ln |u|^2 dv_g \leq \frac{n}{2} \ln\left(\frac{4}{n\rho}\|\nabla u\|_2^2 + 1\right).$$

T. Aubin a montré dans [?] que la première des deux inégalités précédentes est optimale si (M,g) est la sphère unité standard (à normalisation du volume près). Dans ce cas, on connaît aussi les fonctions extrémales. A multiplication par un scalaire près, ce sont les fonctions

$$u_{x_0,\beta}(x) = \left(\beta - \cos\left(d_g(x,x_o)\right)\right)^{1-\frac{n}{2}}$$

où $\beta > 1$ et $x_0 \in S^n$.

1.2.4 Le programme AB

Posons

$$A_0 = \inf\left\{A \in \mathbb{R} \ / \ \exists B \in \mathbb{R} \text{ avec } GN^p_{r,s,\theta}(A,B) \text{ valide}\right\}$$

A_0 dépend a priori des coefficients r, s, θ et p ainsi que de la géométrie de (M,g). Plaçons-nous dans le cas $p = 2$. Pour des raisons techniques, nous supposerons dans toute cette section $1 \leq s \leq 2 \leq r < 2 + s\frac{2}{n}$. On a le théorème suivant.

Théorème 5.
i) Supposons qu'il existe deux réels A et B tels que $GN^2_{r,s,\theta}(A,B)$ est valide. Alors $A \geq A_{\mathrm{opt}}(2,r,s,\theta)$.
ii) Supposons que (M,g) est compacte. Alors pour tout $\varepsilon > 0$, il existe $B_\varepsilon \in \mathbb{R}$ tel que $GN^2_{r,s,\theta}(A_{\mathrm{opt}}(2,r,s,\theta)+\varepsilon, B_\varepsilon)$ est valide.

Ce théorème nous dit en fait que si la première meilleure constante A_0 est fini, alors il s'agit nécessairement de la constante optimale du cas euclidien. En particulier, la première meilleure constante ne dépend pas de la géométrie de la variété.

Preuve :
i) Soit (Ω, ϕ) une carte géodésique normale de M. On choisit Ω de la forme $B_x(R)$ avec $x \in M$ et $R > 0$ de sorte que

$$(1-\alpha)\delta_{ij} \leq g_{ij} \leq (1+\alpha)\delta_{ij}$$

comme formes bilinéaires, avec $\alpha > 0$ petit. Notons $B(R)$ la boule euclidienne de centre 0 et de rayon R. Soit $\varepsilon > 0$. Pour s suffisamment petit, on déduit de $GN^2_{r,s,\theta}(A,B)$ que pour tout $u \in C_c^\infty(B(R))$

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}} \leq \left((A+\varepsilon)\int_{\mathbb{R}^n} |\nabla u|^2 dx + \tilde{B}\int_{\mathbb{R}^n} |u|^2 dx\right)\left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Par l'inégalité de Hölder,

$$\int_{\mathbb{R}^n} |u|^2 dx \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}} \leq |B(R)|^{2-\frac{2+s}{r}} \left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}},$$

où $|B(R)|$ représente le volume euclidien de $B(R)$. Sous les hypothèses du théorème, on peut facilement montrer que $2+s < 2r$. Par conséquent, en choisissant R assez petit,

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}} \leq (A+2\varepsilon)\int_{\mathbb{R}^n} |\nabla u|^2 dx \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}}$$

pour tout $u \in C_c^\infty(B(R))$. Remarquons maintenant que pour $u \in C_c^\infty(\mathbb{R}^n)$ et $\lambda > 0$ assez grand, $u_\lambda = u(\lambda\,\cdot)$ est à support compact dans $B(R)$. De plus

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u_\lambda|^r dx\right)^{\frac{2}{r\theta}} &= \lambda^{-n\frac{2}{r\theta}} \left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}}, \\
\int_{\mathbb{R}^n} |\nabla u_\lambda|^2 dx &= \lambda^{2-n}\int_{\mathbb{R}^n} |\nabla u|^2 dx, \\
\left(\int_{\mathbb{R}^n} |u_\lambda|^s dx\right)^{\frac{2(1-\theta)}{s\theta}} &= \lambda^{-n\frac{2(1-\theta)}{s\theta}} \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}}.
\end{aligned}$$

Donc, pour tout $u \in C_c^\infty(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}} \leq (A+2\varepsilon)\int_{\mathbb{R}^n} |\nabla u|_g^2\, dx \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Ceci implique $A+2\varepsilon \geq A_{\text{opt}}(2,r,s,\theta)$ pour tout $\varepsilon > 0$, ce qui achève la démonstration de i).

ii) Supposons que la courbure de Ricci et le rayon d'injectivité i_g vérifient $\mathrm{Ric}_g \geq \lambda$ et $i_g \geq i$ avec $\lambda \in \mathbb{R}$ et $i > 0$. D'après M. T. Anderson et J. Cheeger [?], pour tout $\varepsilon > 0$, il existe $\delta = \delta(n, \varepsilon, \lambda, i)$ tel que pour tout $x \in M$, il existe une carte harmonique $\phi_x : B_x(\delta) \to \mathbb{R}^n$ dans laquelle les composantes de g vérifient

$$(1+\varepsilon)^{-1}\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\delta_{ij}$$

en tant que formes bilinéaires. On obtient alors que pour tout $x \in M$, tout réel $t \geq 1$ et tout $u \in C_c^\infty(B_x(\delta))$,

$$\int_M |\nabla u|_g^2 \, dv_g \geq (1+\varepsilon)^{-\frac{n+2}{2}} \int_{\mathbb{R}^n} |\nabla u \circ \phi_x^{-1}|^2 dx$$

et

$$(1+\varepsilon)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |u \circ \phi_x^{-1}|^t dx \leq \int_M |u|^t dv_g \leq (1+\varepsilon)^{\frac{n}{2}} \int_{\mathbb{R}^n} |u \circ \phi_x^{-1}|^t dx.$$

De l'inégalité optimale euclidienne, on déduit donc que pour tout $\varepsilon > 0$, il existe $\delta = \delta(n, \varepsilon, \lambda, i)$ tel que pour tout $x \in M$ et tout $u \in C_c^\infty(B_x(\delta))$,

$$\left(\int_M |u|^r dv_g\right)^{\frac{2}{r\theta}} \leq \left(A_o + \frac{\varepsilon}{2}\right) \int_M |\nabla u|_g^2 \, dv_g \left(\int_M |u|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Fixons $\varepsilon > 0$ et prenons δ comme ci-dessus. Par des arguments classiques (voir par exemple [?]), il existe une suite finie (x_j) de points de M telle que
- $M = \cup_j B_{x_j}(\frac{\delta}{2})$ et $B_{x_j}(\frac{\delta}{4}) \cap B_{x_{j'}}(\frac{\delta}{4}) = \emptyset$ pour tout $j \neq j'$.
- il existe $N = N(n, \varepsilon, \lambda, i)$ tel que tout point $x \in M$ a un voisinage intersectant au plus N des $B_{x_j}(\delta)$.

Soit (α_j) une suite de fonctions de $C_c^\infty(B_{x_j}(\delta))$ vérifiant

$$0 \leq \alpha_j \leq 1, \, \alpha_j = 1 \text{ sur } B_{x_j}\left(\frac{\delta}{2}\right), \, |\nabla \alpha_j| \leq \frac{4}{\delta}.$$

On pose alors

$$\eta_j = \frac{\alpha_j^2}{\sum_m \alpha_m^2}.$$

On montre facilement que (η_j) est une partition de l'unité de classe C^∞ subordonnée au recouvrement $(B_{x_j}(\delta))$. $\sqrt{\eta_j}$ est C^∞ et il existe $H = H(n, \varepsilon, \lambda, i) > 0$ tel que, pour tout j, $|\nabla\sqrt{\eta_j}| \leq H$. Soit $u \in C^\infty(M)$. On a

$$\begin{aligned}
\left(\int_M |u|^r dv_g\right)^{\frac{2}{r}} &= \left(\int_M \left|\sum_j \eta_j u^2\right|^{\frac{r}{2}} dv_g\right)^{\frac{2}{r}} \\
&\leq \sum_j \left(\int_M \mid \eta_j u^2 \mid^{\frac{r}{2}} dv_g\right)^{\frac{2}{r}} \\
&\leq \left(A_o + \frac{\varepsilon}{2}\right)^\theta \\
&\quad \times \sum_j \left(\int_M \mid \nabla\left(\sqrt{\eta_j} u\right)\mid^2 dv_g\right)^\theta \left(\int_M \mid \sqrt{\eta_j} u \mid^s dv_g\right)^{\frac{2(1-\theta)}{s}}.
\end{aligned}$$

Par l'inégalité de Hölder,

$$\int_M |\sqrt{\eta_j} u|^s dv_g \leq \left(\int_M \eta_j |u|^s dv_g\right)^{\frac{s}{2}} \left(\int_M |u|^s dv_g\right)^{1-\frac{s}{2}},$$

d'où

$$\begin{aligned}
\left(\int_M |u|^r dv_g\right)^{\frac{2}{r}} &\leq \left(A_o + \frac{\varepsilon}{2}\right)^\theta \left(\int_M |u|^s dv_g\right)^{\frac{2-s}{s}(1-\theta)} \\
&\quad \times \sum_j \left(\int_M \mid \nabla\left(\sqrt{\eta_j} u\right)\mid^2 dv_g\right)^\theta \left(\int_M \eta_j |u|^s dv_g\right)^{1-\theta}. \quad (1.9)
\end{aligned}$$

On a

$$\begin{aligned}
&\sum_j \left(\int_M \mid \nabla\left(\sqrt{\eta_j} u\right)\mid^2 dv_g\right)^\theta \left(\int_M \eta_j |u|^s dv_g\right)^{1-\theta} \\
&\leq \left(\sum_j \int_M \mid \nabla\left(\sqrt{\eta_j} u\right)\mid^2 dv_g\right)^\theta \left(\sum_j \int_M \eta_j |u|^s dv_g\right)^{1-\theta} \\
&\leq \left(\sum_j \int_M \left(\eta_j |\nabla u|^2 + u^2 \mid \nabla\sqrt{\eta_j}\mid^2 + 2\eta_j u \left(\nabla\sqrt{\eta_j}, \nabla u\right)\right) dv_g\right)^\theta \\
&\quad \times \left(\int_M |u|^s dv_g\right)^{1-\theta} \quad (1.10)
\end{aligned}$$

ainsi que

$$\begin{aligned}
\sum_j 2\int_M \eta_j u \left(\nabla\sqrt{\eta_j}, \nabla u\right) dv_g &\leq \sum_j 2\int_M \eta_j u \mid \nabla\sqrt{\eta_j}\mid |\nabla u| \, dv_g \\
&\leq 2NH\|\nabla u\|_2 \|u\|_2.
\end{aligned}$$

En remarquant que pour tout $x > 0$, $y > 0$ et $\lambda > 0$,

$$2xy \leq \lambda x^2 + \lambda^{-1} y^2,$$

puis en posant $x = \|\nabla u\|_2$, $y = \|u\|_2$ et $\lambda = \frac{\varepsilon}{NH(2A_0+\varepsilon)}$, on trouve

$$\sum_j 2 \int_M \eta_j u \left(\nabla \sqrt{\eta_j}, \nabla u\right) dv_g \leq \frac{\varepsilon}{2A_0 + \varepsilon} \|\nabla u\|_2^2 + \frac{N^2 H^2 (2A_0 + \varepsilon)}{\varepsilon} \|u\|_2^2.$$

On déduit alors de (1.9) et (1.10)

$$\left(\int_M |u|^r dv_g \right)^{\frac{2}{r}} \leq \left((A_o + \varepsilon) \int_M |\nabla u|^2 dv_g + C \int_M |u|^2 dv_g \right)^{\theta} \left(\int_M |u|^s dv_g \right)^{\frac{2(1-\theta)}{s}}$$

où

$$C = \left(A_0 + \frac{\varepsilon}{2} \right) \left(\frac{N^2 H^2 (2A_0 + \varepsilon)}{\varepsilon} + NH^2 \right).$$

Le théorème est ainsi démontré.

Le réflexe naturel est de se demander si l'on peut prendre $\varepsilon = 0$ dans le théorème précédent. Comme nous l'avons dis précédemment, le cas de l'inégalité de Sobolev fût le premier à être étudiée. Dans [?], T. Aubin conjectura que l'on pouvait poser $\varepsilon = 0$. Cette conjecture fût résolue par E. Hebey et M. Vaugon [?]. Les cas $p \neq 2$ furent plus tard traités par O. Druet [?] puis T. Aubin et Y. Li [?] dans deux travaux indépendants. E. Humbert [?] adapta ensuite les travaux d'O. Druet à l'inégalité de Nash. Il montra que, tout comme pour l'inégalité de Sobolev, la première meilleur constante était atteinte. Compte tenu de ces résultats, il est assez naturel de conjecturer que l'on peut prendre $\varepsilon = 0$ au moins pour une famille contenant les inégalité de Nash et Sobolev. L'un des objectif de cette thèse était d'étudier cette famille. Le résultat suivant, montré dans [?], répond à une partie de la question. La conjecture reste toutefois ouverte dans de nombreux cas.

Théorème 6. *Supposons (M, g) compacte. Supposons aussi que les constantes r, s, θ vérifient*

$$1 \leq s \leq 2 \leq r < 2 + s\frac{2}{n}.$$

Alors il existe une constante B telle que pour tout $u \in C^\infty(M)$,

$$\left(\int_M |u|^r dv_g \right)^{\frac{2}{r\theta}} \leq \left(A_0 \int_M |\nabla u|_g^2 dv_g + B \int_M |u|^2 dv_g \right) \left(\int_M |u|^s dv_g \right)^{\frac{2(1-\theta)}{s\theta}}.$$

On peut remarquer que ce théorème n'inclut malheureusement pas le cas de l'inégalité de Moser. Il contient toutefois le cas de l'inégalité de Nash. Comme nous le montrons dans [?], il contient aussi, par passage à la limite, le cas de l'inégalité de Sobolev logarithmique.

On étudie maintenant la constante B. Posons

$$B_0 = \inf \left\{ B \in \mathbb{R} \ / \ GN^p_{r,s,\theta}(A_{(M,g)}(2,r,s,\theta), B) \ \text{valide} \right\}.$$

On ne connaît que peu de choses sur B_0. Si (M,g) est compacte, on a vu que $B_0 \geq \text{Vol}_g(M)^{-\frac{2}{n}}$. Cette constante n'est malheureusement pas optimale dans la plupart des cas. On peut montrer

$$B_0 \geq \max \left\{ \text{Vol}_g(M)^{-\frac{2}{n}}, C_0 \max_{x\in M} \text{Scal}_g(x) \right\},$$

où C_0 est une constante dépendant de r, s, θ et s'écrivant explicitement à l'aide des fonctions extrémales de l'inégalité euclidienne correspondante. On ne peut en particulier connaître sa valeur exacte que si l'inégalité optimale euclidienne et ses fonctions extrémales sont explicitement connues. On trouvera la résolution des cas de l'inégalité de Sobolev et de l'inégalité de Nash dans [?] et [?]. Pour l'inégalité de Sobolev logarithmique, on montre que

$$C_0 = \frac{1}{2n\pi e}.$$

Ce résultat se prouve très simplement en passant par des estimations locales du noyau de la chaleur (voir [?]). L'étude de B_0 fournit une condition suffisante pour l'existence des fonctions extrémales de $GN^2_{r,s,\theta}(A_0, B_0)$. On montre dans [?] le résultat suivant.

Théorème 7. *Supposons (M,g) compacte. On a au moins une des deux propriétés suivantes :*

i) *Il existe des fonctions extrémales pour l'inégalité* $LS\left(\frac{2}{n\pi e}, B_0\right)$,

ii) $B_0 = \dfrac{\max_M \text{Scal}_g}{2n\pi e}$.

Ce résultat pourrait être montré pour d'autres inégalités que l'inégalité de Sobolev logarithmique, mais C_0 n'est alors plus explicite.

Enfin, on peut trouver des estimations plus précise sur B_0 si l'on restreint l'inégalité aux fonctions à support dans une petite boule. Le résultat qui suit est l'équivalent du résultat de O. Druet [?] pour l'inégalité de Sobolev avec $p = 1$.

Théorème 8. *Supposons que (M,g) est complète. Soit $x_0 \in M$. Pour tout $\varepsilon > 0$, il existe $\delta_\varepsilon > 0$ tel que pour tout $u \in C^\infty_c(B_g(x_0,\delta_\varepsilon))$ avec $\|u\|_2 = 1$,*

$$\int_{B_g(x_0,\delta_\varepsilon)} u^2 \ln u^2 dv_g \leq \frac{n}{2} \ln \left[\frac{2}{n\pi e} \left(\int_{B_g(x_0,\delta_\varepsilon)} |\nabla u|^2_g dv_g + \frac{\text{Scal}_g(x_0)}{4} + \varepsilon \right) \right].$$

Ce théorème permet de retrouver facilement une estimation en temps petit de la borne supérieure du noyau de la chaleur. Il en est question dans [**?**].

Chapitre 2

The best-constant problem for a family of Gagliardo-Nirenberg Inequalities on a compact Riemannian manifold

Proceedings of the Edinburgh Mathematical Society
(2003) 46, 117-146

Proceedings of the Edinburgh Mathematical Society **Submitted Paper**
Paper 10 April 2003

THE BEST CONSTANT PROBLEM FOR A FAMILY OF GAGLIARDO-NIRENBERG INEQUALITIES ON A COMPACT RIEMANNIAN MANIFOLD

CHRISTOPHE BROUTTELANDE

Université Paul Sabatier, Département de Mathématiques, 118 route de Narbonne, 31062 Toulouse cedex 4, France (Christophe.Brouttelande@math.ups-tlse.fr)

(Received 5 April 2001)

Abstract The best constant problem for Nash and Sobolev inequalities on Riemannian manifolds has been intensively studied in the last decades, especially in the compact case. We treat here this problem for a more general family of Gagliardo-Nirenberg inequalities including the previous ones and the limiting case of a particular logarithmic Sobolev inequality. From the last one, we deduce a sharp heat kernel upper bound.

Keywords: Sobolev logarithmic inequality; Gagliardo-Nirenberg inequalities; best-constant problem; optimal inequalities

AMS 2000 *Mathematics subject classification:* Primary 58J05

1. Introduction

(*a*) *The case of the Euclidean space* $\mathbb{R}^n$

Let p be a positive real number. If $n > p$, the $H_1^p(\mathbb{R}^n)$ Sobolev inequality asserts that there exists a constant A such that for all $u \in H_1^p(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}} \leq A \left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

When combining with Hölder's inequality, one gets a new family of inequalities, called Gagliardo-Nirenberg inequalities, asserting that for all $u \in H_1^p(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{1}{r}} \leq \left(A \int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{\theta}{p}} \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{1-\theta}{s}}$$

where $r, s > 0$, $\theta \in [0, 1]$ and

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s}.$$

Actually, according to [3], when p is fixed and $\theta > 0$, these inequalities are all equivalent up to the constant. Some famous particular cases have numerous applications. One may mention Nash's inequality

$$\left(\int_{\mathbb{R}^n} |u|^2 dx\right)^{1+\frac{2}{n}} \leq A \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \left(\int_{\mathbb{R}^n} |u| dx\right)^{\frac{4}{n}}$$

introduced by J.Nash in his celebrated paper [13], which is obtained by setting $r = 2$, $s = 1$ and $\theta = \frac{n}{n+2}$. If $r = 2 + \frac{4}{n}$, $s = 2$ and $\theta = \frac{n}{n+2}$, one then gets the inequality

$$\int_{\mathbb{R}^n} |u|^{2+\frac{4}{n}} dx \leq A \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \left(\int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{2}{n}},$$

which has been used by J.Moser in a subsequent work [12]. Let us note that these inequalities still hold when $n \leq p$ (which implies $\theta \neq 1$) whereas the Sobolev embeddings are not valid in this case. One may see for instance [3] for a more general discussion. In the following, we restrict to $p = 2$ and thus consider, when $\theta \neq 0$, the inequality

$$\left(\int_{\mathbb{R}^n} |u|^r dx\right)^{\frac{2}{r\theta}} \leq A \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \left(\int_{\mathbb{R}^n} |u|^s dx\right)^{\frac{2(1-\theta)}{s\theta}}. \qquad I_{r,s,\theta,n}$$

Let us fix r and assume that $I_{r,s,\theta,n}$ holds with an A independent of θ, which is the case for all $n > 0$ (see [3]). Making θ goes to 0, one gets for all $u > 0$ such that $\|u\|_r = 1$ the logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} u^r \ln u^r dx \leq \left(\frac{2}{n} + \frac{2-r}{r}\right)^{-1} \ln \left(A \int_{\mathbb{R}^n} |\nabla u|^2 dx\right). \qquad SL_{r,n}$$

According to [3], this inequality is again equivalent with the previous ones and we shall thus consider that it represents the $I_{r,r,0,n}$ case.

Let $A_0(r, s, \theta, n)$ be the optimum A such that $I_{r,s,\theta,n}$ is valid. In most cases, its explicit value is unknown. The best constant in Sobolev inequalities was first obtained independently by T. Aubin [1] and G. Talenti [14] when $n \geq 3$. They showed that

$$A_o\left(\frac{2n}{n-2}, s, 1, n\right) = K(n,2)^2 = \frac{4}{n(n-2)\omega_n^{\frac{2}{n}}}$$

where ω_n is the volume of the standard unit sphere of dimension n. Later, the $SL_{2,n}$ case was solved by E. Carlen [4]. More, he computed with M. Loss [5] the best constant for Nash's Inequality. These values are

$$\begin{aligned} A_o(2,2,0,n) &= \frac{2}{n\pi e} \\ A_o\left(2,1,\frac{n}{n+2},n\right) &= \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n \lambda_1(\mathcal{B}) |\mathcal{B}|^{\frac{2}{n}}} \end{aligned}$$

where $\lambda_1(\mathcal{B})$ is the first Neumann eigenvalue of the laplacian for radial functions on the unit ball $\mathcal{B}$ in $\mathbb{R}^n$ and $|\mathcal{B}|$ is the volume of $\mathcal{B}$ in $\mathbb{R}^n$. One may remark that $\lambda_1(\mathcal{B})$ can be numerically computed. A small discussion about this last point can be found in [5].

(b) The Riemannian case

Let (M, g) be a smooth compact Riemannian n-manifold. When $n \geq 3$, the H_1^2 Sobolev inequality on M asserts that there exist constants A and B such that for all $u \in H_1^2(M)$,

$$\left(\int_M |u|^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|_g^2 dv_g + B \int_M |u|^2 dv_g.$$

As in the case of the Euclidean space $\mathbb{R}^n$, one can define all the Gagliardo-Nirenberg inequalities on M by Hölder's inequality. Actually, one gets that for all $u \in H_1^2(M)$,

$$\left(\int_M |u|^r dv_g \right)^{\frac{2}{r\theta}} \leq A \left(\int_M |\nabla u|_g^2 dv_g + B \int_M |u|^2 dv_g \right) \left(\int_M |u|^s dv_g \right)^{\frac{2(1-\theta)}{s\theta}} \qquad I_{r,s,\theta,n}(A, B)$$

where $r, s > 0$, $\theta \in (0, 1)$ and $\frac{1}{r} = \frac{\theta(n-2)}{2n} + \frac{1-\theta}{s}$. Again, these inequalities are all equivalent and can be defined for all $n \geq 1$. For the last assertion, one may see Theorem 1.1 in [8] (which treats the case of a modified Nash inequality) for an easy to adapt proof using a partition of unity argument.

Now, one defines

$$\mathcal{A}(r, s, \theta, n) = \{ A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ for which } I_{r,s,\theta,n}(A, B) \text{ is valid} \}.$$

One may ask if this set is closed and what is its infimum, called the first best constant. This problem has been intensively studied for the Sobolev inequalities (a complete discussion may be found in [10]). Recently, E. Humbert [11] solved the Nash case in a subsequent paper. In both cases, it was shown that the set is closed and that the infimum is the best constant of the corresponding Euclidean inequalities. In these proofs, the explicit value of the best constant was known but not used. Therefore, one may ask if the answer is identical for all the Gagliardo-Nirenberg inequalities. In particular, the explicit value of $A_0(r, s, \theta, n)$ would be useless. The first aim of this paper is to study in which proportion the previous proofs may be generalized to other cases. At the same time, we point out the fact that the knowledge of $A_0(r, s, \theta, n)$ is useless to solve the first best constant problem for the family of inequalities we study.

One may easily check that $\inf \mathcal{A}(r, s, \theta, n) = A_0(r, s, \theta, n)$. To this task, one may again simply follow the proof of Theorem 1.1 in [8]. Our main result in this work is to give conditions on r, s, θ such that $I_{r,s,\theta,n}(A_0(r, s, \theta, n), B)$ holds with some constant B, including the Nash case studied by E. Humbert [11]. The proof we present does not allow us to treat the full range of parameters. It generalizes [11], itself inspired from the paper [7] of O. Druet. While the main ideas of the proof below are already present in these works, the range of parameters r, s, θ under investigation involves a number of new technical difficulties. For the matter of completness, we thus decided to present a self contained proof. Our main result is the following.

Theorem 1. Let (M,g) be a smooth compact Riemannian n-manifold. Let r,s,θ be constants verifying $r \geq 2$, $s \geq 1$, $\theta \in (0,1)$ and $\frac{1}{r} = \frac{\theta(n-2)}{2n} + \frac{1-\theta}{s}$. If $s \leq 2 \leq r < 2 + s\frac{2}{n}$ then there exists a constant B such that for all $u \in C^\infty(M)$,

$$\left(\int_M |u|^r dv_g\right)^{\frac{2}{r\theta}} \leq \left(A_0(r,s,\theta,n)\int_M |\nabla u|_g^2 dv_g + B\int_M |u|^2 dv_g\right)\left(\int_M |u|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Let us now study some interesting particular cases. The Nash inequality is obviously included in our family but one can remark that Moser's inequality only appears as a limiting case. Indeed, one then has $r = 2 + s\frac{2}{n}$. By now, we cannot prove that the B does not explode as A goes to $A_0(r,s,\theta,n)$. Another limiting case can be treated with this theorem: the logarithmic Sobolev inequality $SL_{2,n}(A,B)$. This one is obtained as in Subsection (a), by fixing $r = 2$ and making θ goes to 0. The following result will be proved in Section 3.

Corollary 1. Let (M,g) be a smooth compact Riemannian n-manifold. There exists a constant B such that for all $u \in C^\infty(M)$ verifying $u > 0$ and $\|u\|_2 = 1$

$$\int_M u^2 \ln u^2 dv_g \leq \frac{n}{2}\ln\left(\frac{2}{n\pi e}\int_M |\nabla u|_g^2 dv_g + B\right). \qquad SL_{2,n}(\tfrac{2}{n\pi e}, B)$$

The best constant problem for the Sobolev inequality has many applications as the Yamabe problem. A classical use of the logarithmic Sobolev inequalities is the computation of heat kernel upper bounds (see for instance [6] and [2]). Actually, following a result of D. Bakry [2], the optimal Euclidean inequality can be used to compute the optimal upper bound

$$\|P_t\|_{1,\infty} \leq \frac{1}{(4\pi t)^{\frac{n}{2}}}$$

where $(P_t)_{t>0}$ is the heat semigroup on the Euclidean space $\mathbb{R}^n$. One may ask if a similar argument works on manifolds. In Subsection 3.2, we shall first cite the theorem obtained by D. Bakry [2]. From it and Corollary 1, we will then deduce the following.

Corollary 2. Let (M,g) be a smooth compact Riemannian n-manifold and let $(P_t)_{t>0}$ be the heat semigroup on M. One then has

$$\|P_t\|_{1,\infty} \leq \frac{1}{(4\pi t)^{\frac{n}{2}}}\, e^{\frac{n\pi e B_0}{3} t}$$

where $0 < t \leq (\pi e B_0)^{-1}$ and B_0 is the best constant B in $SL_{2,n}(\frac{2}{n\pi e}, B)$.

2. Proof of Theorem 1

As announced, the proof follows the pattern of the proof of the main result of [11], itself inspired from [7]. As r, s, θ and n are fixed in this section, we shall denote by A_0 and $I(A,B)$ the constant and inequality $A_0(r,s,\theta,n)$ and $I_{r,s,\theta,n}(A,B)$. The case $n=1$ is handled with a partition of unity argument together whith proving that A_0 is the infimum of the suitable A in $I(A,B)$. One can thus assume that $n \geq 2$. Without loss of generality, one can also assume that $Vol_g(M) = 1$. Moreover, let us observe that $\theta \in (0,1)$ implies $s < r$. We proceed here by contradiction. The proof is composed of three steps. The first one is a preliminary step where we introduce the different notations that will be used throughout this section. This part being nearly identical to the one in [11], we keep the notations there to make the comprehension easier. Step 2 is a set of 9 lemmas. The first three are classical ones and deal with concentration point phenomena in EDP whereas the other six give more specific estimates to our problem. We then conclude in the third step.

Step 1: Preliminary

Proceeding by contradiction, we assume that for all $B > 0$ there exists $u \in C^\infty(M)$ such that

$$\left(\int_M |u|^r dv_g\right)^{\frac{2}{r\theta}} > \left(A_0 \int_M |\nabla u|_g^2 dv_g + B \int_M |u|^2 dv_g\right)\left(\int_M |u|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.$$

This is equivalent to

$$\mu_\alpha = \inf_{u \in \mathcal{H}} I_\alpha < A_0^{-1}$$

for all $\alpha > 0$, where

$$I_\alpha = \left(\int_M |\nabla u|_g^2 dv_g + \alpha \int_M |u|^2 dv_g\right)\left(\int_M |u|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}$$

and

$$\mathcal{H} = \left\{u \in C^\infty(M) / \int_M |u|^r dv_g = 1\right\}.$$

We assume throughout the proof that $s > 1$, the case $s = 1$ being handled by replacing s with $1 + \epsilon_\alpha$ in I_α where $(\epsilon_\alpha)_\alpha$ is such that $\lim \epsilon_\alpha = 0$ (see [11] for the particular case $r = 2$ and $s = 1$). Using the same arguments as in [8], one can prove that there exists $u_\alpha \in H_1^2(M)$, $u_\alpha > 0$, such that $I_\alpha(u_\alpha) = \mu_\alpha$. Moreover, in the sense of distributions,

$$2A_\alpha \Delta_g u_\alpha + 2\alpha A_\alpha u_\alpha + \frac{2(1-\theta)}{\theta} B_\alpha u_\alpha^{s-1} = k_\alpha u_\alpha^{r-1} \qquad (E_\alpha)$$

where

$$\begin{aligned}
A_\alpha &= \left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\
B_\alpha &= \left(\int_M |\nabla u_\alpha|_g^2 dv_g + \alpha \int_M u_\alpha^2 dv_g\right)\left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta} - 1} \\
k_\alpha &= \left(\frac{2}{\theta}\right)\mu_\alpha.
\end{aligned}$$

The Sobolev embedding theorems and the standard elliptic theory (see [9]) implies $u_\alpha \in C^2(M)$. From now on, all limits below are taken as $\alpha \to \infty$. Considering subsequences if needed, one can assume that all sequences have limits (finite or infinite).
One has $\mu_\alpha < A_0^{-1}$, hence

$$\lim \left(\int_M u_\alpha^2 dv_g\right)\left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} = 0$$

and

$$\limsup \left(\int_M |\nabla u_\alpha|_g^2 dv_g\right)\left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \leq A_0^{-1}.$$

From $I(A_0+\epsilon, B_\epsilon)(u_\alpha)$ with ϵ small, one gets that

$$(A_0+\epsilon)^{-1} \leq \left(\int_M |\nabla u_\alpha|_g^2 dv_g + \frac{B_\epsilon}{A_0+\epsilon}\int_M |u_\alpha|^2 dv_g\right)\left(\int_M |u_\alpha|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Hence

$$\liminf \left(\int_M |\nabla u_\alpha|_g^2 dv_g\right)\left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \geq A_0^{-1}.$$

As a consequence,

$$\lim A_\alpha \int_M |\nabla u_\alpha|_g^2 dv_g = A_0^{-1} \tag{2.1}$$

$$\lim B_\alpha \int_M u_\alpha^s dv_g = \lim B_\alpha A_\alpha^{\frac{s\theta}{2(1-\theta)}} = A_0^{-1} \tag{2.2}$$

$$\lim k_\alpha = \left(\frac{2}{\theta}\right) A_0^{-1} \tag{2.3}$$

$$\lim \alpha A_\alpha \int_M u_\alpha^2 dv_g = 0. \tag{2.4}$$

Let $x_\alpha \in M$ be such that $u_\alpha(x_\alpha) = \|u_\alpha\|_\infty$. Set $a_\alpha = \left(A_\alpha \|u_\alpha\|_\infty^{2-r}\right)^{\frac{1}{2}}$. Since

$$1 = \int_M u_\alpha^r dv_g \leq \int_M u_\alpha^2 dv_g \|u_\alpha\|_\infty^{r-2}$$

one gets from (2.4) that $a_\alpha \to 0$.

Step 2: Some lemmas

The first three results are now classical. One begins with the following.

Lemma 1. For all $\delta > 0$,

$$\lim \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} > 0.$$

Proof. Let $\delta > 0$. For all $x \in B(0,\delta)$, set

$$\begin{aligned} g_\alpha(x) &= (exp^*_{x_\alpha} g)(a_\alpha x) \\ \varphi_\alpha(x) &= \|u_\alpha\|_\infty^{-1} u_\alpha(exp_{x_\alpha}(a_\alpha x). \end{aligned}$$

It is an easy matter to check that

$$\begin{aligned} \Delta_{g_\alpha}\varphi_\alpha(x) &= \|u_\alpha\|_\infty^{-1} a_\alpha^2 \Delta_g u_\alpha(exp_{x_\alpha}(a_\alpha x)) \\ &= \|u_\alpha\|_\infty^{1-r}\Bigg(\frac{k_\alpha}{2} u_\alpha(exp_{x_\alpha}(a_\alpha x))^{r-1} \\ &\quad -\frac{1-\theta}{\theta} B_\alpha u_\alpha(exp_{x_\alpha}(a_\alpha x))^{s-1} - \alpha A_\alpha u_\alpha(exp_{x_\alpha}(a_\alpha x))\Bigg). \end{aligned}$$

Hence

$$\Delta_{g_\alpha}\varphi_\alpha + \alpha A_\alpha \varphi_\alpha \|u_\alpha\|_\infty^{2-r} + \frac{1-\theta}{\theta}\|u_\alpha\|_\infty^{s-r} B_\alpha \varphi_\alpha^{s-1} = \frac{k_\alpha}{2}\varphi_\alpha^{r-1}.$$

Noticing that $\Delta_g u_\alpha(x_\alpha) \geq 0$, one gets from (E_α) that

$$\alpha A_\alpha + \frac{1-\theta}{\theta} B_\alpha \|u_\alpha\|_\infty^{s-2} \leq \frac{k_\alpha}{2}\|u_\alpha\|_\infty^{r-2}, \tag{2.5}$$

which implies $|\Delta_{g_\alpha}\varphi_\alpha| \leq C$. By standard elliptic arguments (see for instance [9]), one then shows that the sequence (φ_α) is equicontinuous. Hence, by the Ascoli theorem, there exists $\varphi \in C^0(B(0,\delta))$ such that $\varphi_\alpha \to \varphi$ in $C^0(B(0,\delta))$. Moreover

$$\varphi(0) = \lim \varphi_\alpha(0) = 1.$$

Therefore

$$\begin{aligned} \int_{B(0,\delta)} \varphi_\alpha^s dv_{g_\alpha} &= \|u_\alpha\|_\infty^{-s} a_\alpha^{-n} \int_{B_{x_\alpha}(a_\alpha\delta)} u_\alpha^s dv_g \\ &= \|u_\alpha\|_\infty^{-s-(2-r)\frac{n}{2}} A_\alpha^{-\frac{n}{2}+\frac{s\theta}{2(1-\theta)}} \frac{\int_{B_{x_\alpha}(a_\alpha\delta)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}. \end{aligned}$$

Using the relations

$$\frac{2}{n} = 1 - \frac{2}{r\theta} + \frac{2(1-\theta)}{s\theta} \tag{2.6}$$

$$(r-s)\frac{n}{2}\frac{2(1-\theta)}{s\theta} - (2-r)\frac{n}{2} = r, \tag{2.7}$$

one gets that

$$\int_{B(0,\delta)} \varphi_\alpha^s dv_{g_\alpha} = \left(\|u_\alpha\|_\infty^{r-s} A_\alpha^{\frac{s\theta}{2(1-\theta)}}\right)^{1-\frac{n(1-\theta)}{s\theta}} \frac{\int_{B_{x_\alpha}(a_\alpha\delta)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}.$$

One may easily verify that

$$r < 2 + s\frac{2}{n} \Leftrightarrow \frac{2}{r\theta} > 1 \Leftrightarrow 1 - \frac{n(1-\theta)}{s\theta} < 0.$$

Since (2.2) and (2.5) implies $A_\alpha^{-\frac{s\theta}{2(1-\theta)}} \le C\|u_\alpha\|_\infty^{r-s}$, one has

$$\int_{B(0,\delta)} \varphi_\alpha^s dv_{g_\alpha} \le C \frac{\int_{B_{x_\alpha}(a_\alpha \delta)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}.$$

Noticing that $\lim \int_{B(0,\delta)} \varphi_\alpha^s dv_{g_\alpha} > 0$,

$$\frac{\int_{B_{x_\alpha}(a_\alpha \delta)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} \ge C > 0.$$

It ends the proof of Lemma 1. One showed similarly that

$$\|u_\alpha\|_\infty^{r-s} A_\alpha^{\frac{s\theta}{2(1-\theta)}} \to C > 0. \tag{2.8}$$

Let us note that (2.8) leads to $a_\alpha \|u_\alpha\|_\infty^{\frac{r}{n}} \to C > 0$. As a consequence, $\|u_\alpha\|_\infty \to +\infty$ and $A_\alpha \to 0$. Moreover, since $s \le 2$, one also has

$$\int_M u_\alpha^2 dv_g \le \int_M u_\alpha^s dv_g \|u_\alpha\|_\infty^{2-s} = A_\alpha^{\frac{s\theta}{2(1-\theta)}} \|u_\alpha\|_\infty^{2-s}.$$

Consequently, by (2.8) and the inequality $\|u_\alpha\|_\infty^{2-r} \le C \int_M u_\alpha^2 dv_g$, one gets that

$$\|u_\alpha\|_\infty^{r-2} \int_M u_\alpha^2 dv_g \to C > 0. \tag{2.9}$$

Remark. Relations (2.6) and (2.7) are intensively used throughout the proof and we will thus not precise anymore when they are needed.

One can now improve the previous lemma. Actually, one has the following.

Lemma 2. Let $(c_\alpha)_\alpha$ be a sequence of positive real numbers satisfying $\frac{a_\alpha}{c_\alpha} \to 0$. Then

$$\lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} = 1.$$

Proof. Let $\eta \in C_\infty(\mathbb{R})$ be such that

$$\begin{aligned} &i) \quad \eta([0, \tfrac{1}{2}]) = \{1\} \\ &ii) \quad \eta([1, +\infty]) = \{0\} \\ &iii) \quad 0 \le \eta \le 1. \end{aligned}$$

For $k \in \mathbb{N}$, set $\eta_{\alpha,k} = (\eta(c_\alpha^{-1} d_g(x, x_\alpha)))^{2^k}$.
Multiplying (E_α) by $\eta_{\alpha,k}^r u_\alpha$ and integrating over M, one gets that

$$2A_\alpha \int_M \eta_{\alpha,k}^r u_\alpha \Delta_g u_\alpha dv_g + 2\alpha A_\alpha \int_M \eta_{\alpha,k}^r u_\alpha^2 dv_g + \frac{2(1-\theta)}{\theta} B_\alpha \int_M \eta_{\alpha,k}^r u_\alpha^s dv_g$$

$$= k_\alpha \int_M \eta^r_{\alpha,k} u^r_\alpha dv_g.$$

Then, the identity

$$\int_M \eta^r_{\alpha,k} u_\alpha \Delta_g u_\alpha dv_g = \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k} u_\alpha|^2_g dv_g - \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k}|^2_g u^2_\alpha dv_g$$

leads to

$$2A_\alpha \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k} u_\alpha|^2_g dv_g - 2A_\alpha \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k}|^2_g u^2_\alpha dv_g + 2\alpha A_\alpha \int_M \eta^r_{\alpha,k} u^2_\alpha dv_g$$

$$+\frac{2(1-\theta)}{\theta} B_\alpha \int_M \eta^r_{\alpha,k} u^s_\alpha dv_g = \frac{k_\alpha}{2} \int_M \eta^r_{\alpha,k} u^r_\alpha dv_g. \tag{2.10}$$

Moreover, $I(A_0+\epsilon, B_\epsilon)(\eta_{\alpha,k} u_\alpha)$ gives

$$\left(\int_M |\eta_{\alpha,k} u_\alpha|^r dv_g\right)^{\frac{2}{r\theta}} \leq \left((A_0+\epsilon)\int_M |\nabla \eta_{\alpha,k} u_\alpha|^2_g dv_g + B_\epsilon \int_M |\eta_{\alpha,k} u_\alpha|^2 dv_g\right)\left(\int_M |\eta_{\alpha,k} u_\alpha|^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.$$

Set

$$\lambda_k = \lim \frac{\int_M \eta^r_{\alpha,k} u^s_\alpha dv_g}{\int_M u^s_\alpha dv_g}$$

$$\tilde\lambda_k = \lim \frac{\int_M \eta^s_{\alpha,k} u^s_\alpha dv_g}{\int_M u^s_\alpha dv_g}$$

$$X_k = \lim A_\alpha \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k} u_\alpha|^2_g dv_g$$

$$Y_k = \lim A_\alpha \int_M |\nabla \eta_{\alpha,k} u_\alpha|^2_g dv_g$$

$$Z_k = \lim \int_M \eta^r_{\alpha,k} u^r_\alpha dv_g.$$

Let us now search for some relations involving λ_k, $\tilde\lambda_k$, X_k, Y_k and Z_k.
One has the following:
i) Relation (2.9) implies

$$\lim A_\alpha \int_M |\nabla \eta^{\frac{r}{2}}_{\alpha,k}|^2_g u^2_\alpha dv_g \leq \lim C \frac{a^2_\alpha}{c^2_\alpha} = 0.$$

ii) Relation (2.4) implies

$$\lim \alpha A_\alpha \int_M \eta^r_{\alpha,k} u^2_\alpha dv_g = 0.$$

iii) By definition of A_α

$$\lim \left(\int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g\right)\left(\int_M \eta_{\alpha,k}^r u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}$$
$$= \lim A_\alpha \left(\int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g\right) \tilde{\lambda}_k^{\frac{2(1-\theta)}{s\theta}}$$
$$= Y_k \tilde{\lambda}_k^{\frac{2(1-\theta)}{s\theta}}$$

and

$$\lim \left(\int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g\right)\left(\int_M \eta_{\alpha,k}^r u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \leq \lim A_\alpha \int_M u_\alpha^2 dv_g = 0.$$

Therefore, taking the limit in (2.10) and $I(A_0+\epsilon, B_\epsilon)(\eta_{\alpha,k}u_\alpha)$, on gets that

$$X_k + \frac{1-\theta}{\theta}\lambda_k A_0^{-1} = \frac{A_0}{\theta} Z_k$$

$$Z_k^{\frac{2}{r\theta}} \leq (A_0+\epsilon) Y_k \tilde{\lambda}_k^{\frac{2(1-\theta)}{s\theta}}.$$

Set $\tilde{X}_k = A_0 X_k$ and $\tilde{Y}_k = A_0 Y_k$. Noticing that ϵ is arbitrary, one then has

$$\theta \tilde{X}_k + (1-\theta)\lambda_k = Z_k$$

$$Z_k^{\frac{2}{r\theta}} \leq \tilde{Y}_k \tilde{\lambda}_k^{\frac{2(1-\theta)}{s\theta}}.$$

Now, Let us remark that

$$\lambda_k^s = \frac{\lambda_k^s}{\tilde{\lambda}_k^r} \tilde{\lambda}_k^r.$$

After some easy computations, it follows that

$$\lambda_k \leq \frac{1}{1-\theta} \tilde{Y}_k^{\frac{r\theta}{2(1-\theta)}} \left(Z_k^{1-\frac{1}{1-\theta}} - \theta \tilde{X}_k Z_k^{-\frac{1}{1-\theta}}\right) \tilde{\lambda}_k^{\frac{r}{s}}.$$

Set $f(x,z) = z^{1-\frac{1}{1-\theta}} - \theta x z^{-\frac{1}{1-\theta}}$. One has $\frac{\partial f}{\partial z}(x,z) = \frac{\theta}{1-\theta} z^{-\frac{1}{1-\theta}} \left(\frac{x}{z} - 1\right)$. Since $\theta \tilde{X}_k + (1-\theta)\lambda_k = Z_k$, $\lambda_k < Z_k < \tilde{X}_k$ or $\tilde{X}_k < Z_k < \lambda_k$. In both cases, $f(\tilde{X}_k, Z_k) < f(\tilde{X}_k, \tilde{X}_k)$. As a consequence,

$$\lambda_k \leq \left(\tilde{Y}_k^{\frac{r}{2}} \tilde{X}_k^{-1}\right)^{\frac{\theta}{1-\theta}} \tilde{\lambda}_k^r.$$

From Hölder's inequality for the measure $d\mu_\alpha = |\nabla u_\alpha|_g^2 dv_g$ and the equalities

$$\tilde{Y}_k = \lim A_0 A_\alpha \int_M \eta_{\alpha,k}^2 |\nabla u_\alpha|_g^2 dv_g$$

$$\tilde{X}_k = \lim A_0 A_\alpha \int_M \eta_{\alpha,k}^r |\nabla u_\alpha|_g^2 dv_g,$$

it follows that $\tilde{Y}_k^{\frac{r}{2}} \leq \tilde{X}_k$ and $\lambda_k \leq \tilde{\lambda}_k^{\frac{r}{s}}$. Since, by Lemma 1,

$$C \leq \lambda_{k+1} \leq \tilde{\lambda}_{k+1} \leq \lambda_k \leq \tilde{\lambda}_{k+1} \leq \lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g},$$

one then has

$$\forall N \in \mathbb{N}, C \leq \lambda_0^{\frac{Nr}{s}} \leq \lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}.$$

Thereafter, $\lim \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} = 1$ and Lemma 2 is proved.

An important estimates follows.

Lemma 3. There exists $C > 0$ independant of α such that for all $x \in M$ an every α

$$u_\alpha(x) d_g(x, x_\alpha)^{\frac{n}{r}} \leq C.$$

Proof. Let us assume by contradiction that there exists a sequence $(y_\alpha)_\alpha$ of points of M such that

$$u_\alpha(y_\alpha) d_g(y_\alpha, x_\alpha)^{\frac{n}{r}} \to +\infty \qquad (H).$$

From now on, in most cases, we set $r_\alpha = d_g(., x_\alpha)$. Set $v_\alpha = u_\alpha(y_\alpha) d_g(y_\alpha, x_\alpha)^{\frac{n}{r}}$. One can assume without loss of generality that $v_\alpha = \|u_\alpha r_\alpha^{\frac{n}{r}}\|_\infty$.
Let us prove before that for all ν small enough, one has

$$B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}}) \cap B_{x_\alpha}(a_\alpha v_\alpha^\nu) = \emptyset. \qquad (2.11)$$

It is enough to prove $d_g(y_\alpha, x_\alpha) \geq u_\alpha(y_\alpha)^{-\frac{r}{n}} + a_\alpha v_\alpha^\nu$ or, equivalently, $v_\alpha^{\frac{r}{n}-\nu} \geq v_\alpha^{-\nu} + a_\alpha u_\alpha(y_\alpha)^{\frac{r}{n}}$. If $\nu < \frac{r}{n}$, one gets from (H) that $v_\alpha^{\frac{r}{n}-\nu} \to \infty$ and $v_\alpha^{-\nu} \to 0$. One has yet to show that $v_\alpha u_\alpha(y_\alpha)^{\frac{r}{n}} \leq C$. Meanwhile, (2.8) implies

$$\begin{aligned} a_\alpha u_\alpha(y_\alpha)^{\frac{r}{n}} &\leq a_\alpha \|u_\alpha\|_\infty^{\frac{r}{n}} \\ &\leq \left(A_\alpha \|u_\alpha\|_\infty^{2-r} \|u_\alpha\|_\infty^{2\frac{r}{n}}\right)^{\frac{1}{2}} \\ &\leq C, \end{aligned}$$

which proves (2.11).
Let us now set for all $x \in B(0,1)$

$$\begin{aligned} h_\alpha(x) &= \left(exp_{y_\alpha}^* g\right)(l_\alpha x) \\ \psi_\alpha(x) &= u_\alpha(y_\alpha)^{-1} u_\alpha(exp_{y_\alpha}(l_\alpha x)) \end{aligned}$$

where $l_\alpha = \|u_\alpha\|_\infty^{-\left(\frac{1}{2}+\frac{r}{n}\right)} u_\alpha(y_\alpha)^{\frac{1}{2}}$.
One can easily check with (E_α) that

$$\begin{aligned}
\Delta_{h_\alpha}\psi_\alpha(x) &= u_\alpha(y_\alpha)^{-1} l_\alpha^2 \Delta_g u_\alpha(exp_{y_\alpha}(l_\alpha x)) \\
&= \frac{k_\alpha \|u_\alpha\|_\infty^{-1-\frac{2r}{n}} u_\alpha(y_\alpha)^{r-1}}{2A_\alpha}\psi_\alpha(x)^{r-1} \\
&\quad -\alpha\|u_\alpha\|_\infty^{-1-\frac{2r}{n}} u_\alpha(y_\alpha)\psi_\alpha(x) \\
&\quad -\frac{(1-\theta)B_\alpha\|u_\alpha\|_\infty^{-1-\frac{2r}{n}} u_\alpha(y_\alpha)^{s-1}}{\theta A_\alpha}\psi_\alpha(x)^{s-1}.
\end{aligned}$$

Hence, under the assumption $\|\psi_\alpha\|_{L^\infty(B(0,1))} \le C$ and by (2.5),

$$\begin{aligned}
|\Delta_{h_\alpha}\psi_\alpha(x)| &\le C\frac{\|u_\alpha\|_\infty^{-1-\frac{2r}{n}+r-1}}{2A_\alpha} \\
&\le C\frac{\|u_\alpha\|_\infty^{-(r-s)\frac{2(1-\theta)}{s\theta}}}{A_\alpha} \\
&\le C.
\end{aligned}$$

Let us now show that $\|u_\alpha\|_{L_\infty(B_{y_\alpha}(l_\alpha))} \le \|u_\alpha\|_{L_\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}}))} \le Cu_\alpha(y_\alpha)$. By the definition of y_α, one has for all $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})$,

$$u_\alpha(y_\alpha)d_g(y_\alpha,x_\alpha)^{\frac{n}{r}} \ge u_\alpha(x)d_g(x,x_\alpha)^{\frac{n}{r}}. \tag{2.12}$$

Moreover, since $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})$ and $u_\alpha(y_\alpha) \le \|u_\alpha\|_\infty$, one has

$$d_g(x,y_\alpha) \le u_\alpha(y_\alpha)^{-\frac{r}{n}}$$

and by (H), $u_\alpha(y_\alpha)^{-\frac{r}{n}} \le \frac{1}{2}d_g(y_\alpha,x_\alpha)$. Therefore

$$d_g(x_\alpha,x) \ge d_g(y_\alpha,x_\alpha) - d_g(y_\alpha,x) \le d_g(y_\alpha,x_\alpha) - u_\alpha(y_\alpha)^{-\frac{r}{n}} \ge \frac{1}{2}d_g(y_\alpha,x_\alpha),$$

which, combined with (2.12), proves that $\|u_\alpha\|_{L_\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}}))} \le Cu_\alpha(y_\alpha)$. Hence, one has $\|\psi_\alpha\|_{L_\infty(B_{y_\alpha}(l_\alpha))} \le C$ and, as a consequence, $\|\Delta_{h_\alpha}\psi_\alpha\|_{L_\infty(B_{y_\alpha}(l_\alpha))} \le C$. By arguments already used before, there exists $\psi \in C^0(B(0,1))$ such that $\psi_\alpha \to \psi$ in $C^0(B(0,1))$ with $\psi(0) > 0$. One then has

$$\begin{aligned}
\int_{B(0,1)} \psi_\alpha^s dv_{h_\alpha} &= A_\alpha^{\frac{s\theta}{2(1-\theta)}} u_\alpha(y_\alpha)^{-s} l_\alpha^{-n} \frac{\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} \\
&\overset{+\infty}{\sim} C\left(\frac{\|u_\alpha\|_\infty}{u_\alpha(y_\alpha)}\right)^{\frac{n}{2}+s} \frac{\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}.
\end{aligned}$$

Set

$$m_\alpha = \frac{u_\alpha(y_\alpha)}{\|u_\alpha\|_\infty}.$$

One gets that

$$\frac{\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} \overset{+\infty}{\sim} C m_\alpha^{\left(\frac{n}{2}+s\right)}.$$

Lemma 2 and (2.11) imply

$$\lim \frac{\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^s dv_g}{\int_M u_\alpha^s dv_g} = 0.$$

Consequently, $\lim m_\alpha = 0$. Now, let us show that there exists a sequence $(\gamma_k)_{k>0}$ of positive real numbers converging to $+\infty$ such that for all $k > 0$,

$$m_\alpha^{-\gamma_k} \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g \to 0. \tag{H_k}$$

Let us proceed by induction. Since $\|u_\alpha\|_{L_\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}}))} \leq C u_\alpha(y_\alpha)$, one has

$$\begin{aligned}
\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g &\leq C u_\alpha(y_\alpha)^{r-s} \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^s dv_g \\
&\leq C m_\alpha^{r-s} \|u_\alpha\|_\infty^{r-s} \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^s dv_g.
\end{aligned}$$

Therefore, one can set $\gamma_0 = r - s$ by (2.11). Let us assume that we constructed the sequence up to some $k > 0$.
Set $\eta_{\alpha,k}(x) = \eta(2^k u_\alpha(y_\alpha)^{\frac{r}{n}} d_g(y_\alpha, x))$.
Multipliying (E_α) by $\frac{u_\alpha \eta_{\alpha,k}^2}{m_\alpha^{\gamma_k}}$ and integrating over M, one gets that

$$\begin{aligned}
&\frac{2A_\alpha}{m_\alpha^{\gamma_k}} \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g - \frac{2A_\alpha}{m_\alpha^{\gamma_k}} \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 dv_g + \frac{2\alpha A_\alpha}{m_\alpha^{\gamma_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \\
&\quad + \frac{2(1-\theta)}{\theta} \frac{B_\alpha}{m_\alpha^{\gamma_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^s dv_g = \frac{k_\alpha}{m_\alpha^{\gamma_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^r dv_g.
\end{aligned} \tag{2.13}$$

Relation (H_k) and Hölder's inequality imply

$$\begin{aligned}
A_\alpha m_\alpha^{-\gamma_k} \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 dv_g &\leq C A_\alpha u_\alpha(y_\alpha)^{\frac{2r}{n}} m_\alpha^{-\gamma_k} \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^2 dv_g \\
&\leq C \|u_\alpha\|_\infty^{r-2} m_\alpha^{\frac{2r}{n}} m_\alpha^{-\gamma_k} \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^2 dv_g \\
&\leq C \|u_\alpha\|_\infty^{r-2} m_\alpha^{\frac{2r}{n}} m_\alpha^{-\gamma_k} \left(vol_g(B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{r}{n}}))\right)^{1-\frac{2}{r}} \\
&\quad \cdot \left(\int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g\right)^{\frac{2}{r}} \\
&\leq C m_\alpha^{2-r+\frac{2r}{n}-\gamma_k+\frac{2}{r}\gamma_k} \\
&\leq C m_\alpha^{(r-s)\frac{2(1-\theta)}{s\theta}-\gamma_k\left(1-\frac{2}{r}\right)}
\end{aligned}$$

and

$$k_\alpha m_\alpha^{-\gamma_k} \int_M \eta_{\alpha,k}^2 u_\alpha^r dv_g \leq C.$$

From now, there are two possibilities. One is the case

$$(r-s)\frac{2(1-\theta)}{s\theta} - \gamma_k \left(1 - \frac{2}{r}\right) \geq 0.$$

One then has by (2.13)

$$A_\alpha m_\alpha^{-\gamma_k} \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \leq C$$

$$B_\alpha m_\alpha^{-\gamma_k} \int_M \eta_{\alpha,k}^2 u_\alpha^s dv_g \leq C \tag{2.14}$$

$$A_\alpha m_\alpha^{-\gamma_k} \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g \leq C.$$

Moreover, one gets from $I(A,B)(\eta_\alpha u_\alpha)$ that

$$\begin{aligned}\left(\int_M \eta_{\alpha,k}^r u_\alpha^r dv_g\right)^{\frac{2}{r\theta}} &\leq A \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &+ B \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.\end{aligned}$$

Noticing that (2.14) is still valid by changing η into $\eta^{\frac{s}{2}}$, one then has

$$\begin{aligned}&\int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\leq \frac{C}{A_\alpha B_\alpha^{\frac{2(1-\theta)}{s\theta}}} A_\alpha \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g \left(B_\alpha \int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\leq C m_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\gamma_k}\end{aligned}$$

and

$$\begin{aligned}&\int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\leq \frac{C}{B_\alpha^{\frac{2(1-\theta)}{s\theta}}} \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(B_\alpha \int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\leq C m_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\gamma_k}.\end{aligned}$$

Thereafter, by using the relation

$$\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g \le \int_M \eta_{\alpha,k}^r u_\alpha^r dv_g,$$

one gets that

$$\begin{aligned}\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g &\le Cm_\alpha^{\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\gamma_k} \\ &\le Cm_\alpha^{\left(\frac{r\theta}{n}+1\right)\gamma_k}.\end{aligned}$$

Consequently, one can set $\gamma_{k+1} = \left(\frac{r\theta}{2n}+1\right)\gamma_k$.
The other possibility is

$$(r-s)\frac{2(1-\theta)}{s\theta} - \gamma_k\left(1-\frac{2}{r}\right) < 0.$$

The same arguments as above give

$$\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g \le Cm_\alpha^{\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\left((r-s)\frac{2(1-\theta)}{s\theta}+\frac{2}{r}\gamma_k\right)}$$

and

$$m_\alpha^{-\gamma_k}\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g \le Cm_\alpha^{\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\left((r-s)\frac{2(1-\theta)}{s\theta}\right)+\gamma_k\left(\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\frac{2}{r}-1\right)}.$$

Thereafter, the relation

$$\begin{aligned}\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\frac{2}{r}-1 &= \frac{r\theta}{2}\frac{2(1-\theta)}{s\theta}\left(\frac{2}{r}\left(\frac{s\theta}{2(1-\theta)}+1\right)-\frac{2}{r\theta}\frac{s\theta}{2(1-\theta)}\right) \\ &= \frac{r\theta}{2}\frac{2(1-\theta)}{s\theta}\frac{2-s}{r} \ge 0\end{aligned}$$

implies

$$m_\alpha^{-\gamma_k}\int_{B_{y_\alpha}(2^{-(k+1)}u_\alpha(y_\alpha)^{-\frac{r}{n}})} u_\alpha^r dv_g \le Cm_\alpha^{\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right)\left((r-s)\frac{2(1-\theta)}{s\theta}\right)}.$$

Since $\frac{r\theta}{2}\left(1+\frac{2(1-\theta)}{s\theta}\right) > 1$, set $\gamma_{k+1} = \gamma_k + (r-s)\frac{2(1-\theta)}{s\theta}$. One can easily check that the sequence $(\gamma_k)_{k>0}$ converges to $+\infty$. Since $l_\alpha u_\alpha(y_\alpha)^{\frac{r}{n}} \to 0$, one also proved that for all $k > 0$,

$$m_\alpha^{-\gamma_k}\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^r dv_g \to 0.$$

But since

$$\int_{B(0,1)} \psi_\alpha^r dv_{h_\alpha} = u_\alpha(y_\alpha)^{-r} l_\alpha^{-n}\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^r dv_g,$$

one also has

$$\int_{B_{y_\alpha}(l_\alpha)} u_\alpha^r dv_g \quad \overset{+\infty}{\sim} \quad C m_\alpha^{\frac{n}{2}+r}.$$

It leads to a contradiction and this ends the proof of Lemma 3.

Let $c > 0$. Before concluding, we need some sharp estimates. The first one is the following.

Lemma 4. If $r \neq 2$, there exists $C > 0$ independent of α such that

$$A_\alpha^{-\frac{r}{r-2}} \int_{M-B_{x_\alpha}(c)} u_\alpha^r dv_g \leq C. \tag{2.15}$$

If $r = 2$, for all $k > 0$, there exists $C > 0$ independent of α such that

$$A_\alpha^{-k} \int_{M-B_{x_\alpha}(c)} u_\alpha^r dv_g \leq C.$$

Proof. One starts with the case $r \neq 2$.Let $\delta \in]0, \frac{s\theta}{2(1-\theta)}[$. Lemma 3 gives

$$\begin{aligned} A_\alpha^{-\delta} \int_{M-B_{x_\alpha}(c)} u_\alpha^r dv_g &\leq CA_\alpha^{-\delta} \int_{M-B_{x_\alpha}(c)} u_\alpha^s r_\alpha^{\frac{n(r-s)}{r}} dv_g \\ &\leq CA_\alpha^{-\delta} \int_{M-B_{x_\alpha}(c)} u_\alpha^s dv_g \\ &\leq CA_\alpha^{-\delta} A_\alpha^{\frac{s\theta}{2(1-\theta)}}. \end{aligned}$$

Hence, $A_\alpha^{-\delta} \int_{M-B_{x_\alpha}(c)} u_\alpha^r dv_g \to 0$.
Let us show by induction that for all $k_0 + 1 \geq k > 0$,

$$A_\alpha^{-\delta\left(\frac{r\theta}{2n}+1\right)^k} \int_{M-B_{x_\alpha}(2^k c)} u_\alpha^r dv_g \leq C \tag{H'_k}$$

where k_0 is such that $\delta\left(\frac{r\theta}{2n}+1\right)^{k_0} \leq \frac{r}{r-2}$. Set $\eta_{\alpha,k}(x) = 1 - \eta(2^{-k}c^{-1}d_g(x_\alpha, x))$ and $\epsilon_k = \left(\frac{r\theta}{2n}+1\right)^k$. Assume that (H'_k) is true for some $k \leq k_0$. Multiplying (E_α) with $\frac{u_\alpha \eta_{\alpha,k}^2}{A_\alpha^{\delta\epsilon_k}}$ and integrating over M, one then gets that

$$\begin{aligned} &\frac{2A_\alpha}{A_\alpha^{\delta\epsilon_k}} \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g - \frac{2A_\alpha}{A_\alpha^{\delta\epsilon_k}} \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 dv_g + \frac{2\alpha A_\alpha}{A_\alpha^{\delta\epsilon_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \\ &\qquad + \frac{2(1-\theta)}{\theta} \frac{B_\alpha}{A_\alpha^{\delta\epsilon_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^s dv_g = \frac{k_\alpha}{A_\alpha^{\delta\epsilon_k}} \int_M \eta_{\alpha,k}^2 u_\alpha^r dv_g. \end{aligned} \tag{2.16}$$

Since $\delta\epsilon_k \leq \frac{r}{r-2}$, one has by Hölder's inequality and (H'_k)

$$\begin{aligned} A_\alpha^{1-\delta\epsilon_k} \int_M |\nabla \eta_{\alpha,k}|_g^2 u_\alpha^2 dv_g &\leq CA_\alpha^{1-\delta\epsilon_k\left(1-\frac{2}{r}\right)} \left(A_\alpha^{-\delta\epsilon_k} \int_{M-B_{x_\alpha}(2^{-k}c)} u_\alpha^r dv_g\right)^{\frac{2}{r}} \\ &\leq C \end{aligned}$$

and

$$\frac{k_\alpha}{A_\alpha^{\delta\epsilon_k}}\int_M \eta_{\alpha,k}^2 u_\alpha^r dv_g \le C.$$

Hence, by (2.16),

$$\frac{2A_\alpha}{A_\alpha^{\delta\epsilon_k}}\int_M |\nabla \eta_{\alpha,k}u_\alpha|_g^2 dv_g \le C$$

$$\frac{2\alpha A_\alpha}{A_\alpha^{\delta\epsilon_k}}\int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \le C$$

$$\frac{2(1-\theta)}{\theta}\frac{B_\alpha}{A_\alpha^{\delta\epsilon_k}}\int_M \eta_{\alpha,k}^2 u_\alpha^s dv_g \le C. \tag{2.17}$$

Moreover, $I(A,B)(\eta_{\alpha,k}u_\alpha)$ gives

$$\begin{aligned}\left(\int_M \eta_{\alpha,k}^r u_\alpha^r dv_g\right)^{\frac{2}{r\theta}} \quad &\le \quad A\int_M |\nabla \eta_{\alpha,k}u_\alpha|_g^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\quad +B\int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}.\end{aligned}$$

Noticing that (2.17) is still valid after changing η into $\eta^{\frac{s}{2}}$, one then has

$$\begin{aligned}&\int_M |\nabla \eta_{\alpha,k}u_\alpha|_g^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\qquad \le \quad \frac{C}{A_\alpha B_\alpha^{\frac{2(1-\theta)}{s\theta}}} A_\alpha \int_M |\nabla \eta_{\alpha,k}u_\alpha|_g^2 dv_g \left(B_\alpha\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\qquad \le \quad C A_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\delta\epsilon_k}\end{aligned}$$

and

$$\begin{aligned}&\int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\qquad\qquad \le \quad \frac{C}{B_\alpha^{\frac{2(1-\theta)}{s\theta}}} \int_M \eta_{\alpha,k}^2 u_\alpha^2 dv_g \left(B_\alpha\int_M \eta_{\alpha,k}^s u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\qquad\qquad \le \quad C A_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\delta\epsilon_k}.\end{aligned}$$

Thereafter

$$\left(\int_M \eta_{\alpha,k}^r u_\alpha^r dv_g\right)^{\frac{2}{r\theta}} \le C A_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\delta\epsilon_k}.$$

Hence, from the inequality

$$\int_{M-B_{x_\alpha}(2^{k+1}c)} u_\alpha^r dv_g \le \int_M \eta_{\alpha,k}^r u_\alpha^r dv_g,$$

one gets that

$$\int_{M-B_{x_\alpha}(2^{k+1}c)} u_\alpha^r dv_g \le C A_\alpha^{\left(1+\frac{2(1-\theta)}{s\theta}\right)\delta \frac{r\theta}{2}\epsilon_k}.$$

Since

$$\left(1+\frac{2(1-\theta)}{s\theta}\right)\frac{r\theta}{2} = \frac{r\theta}{n} + 1 > \frac{r\theta}{2n} + 1,$$

one deduces (H'_{k+1}). Let us remark that we did not only show (2.15) but similarly, by a last induction,

$$A_\alpha^{-\left(\frac{r}{r-2}+\frac{s\theta}{2(1-\theta)}\right)} \int_{M-B_{x_\alpha}(c)} u_\alpha^s dv_g \le C.$$

The case $r = 2$ is identically handled, except that the induction can be continued forever.

In order to prove Lemma 6, we first have to show the following.

Lemma 5. There exists $t_0 > 0$ such that

$$\forall x \in M - B_{x_\alpha}(t_0 A_\alpha^{\overline{\frac{rs\theta}{2n(r-s)(1-\theta)}}}), \qquad \Delta_g u_\alpha(x) < 0.$$

Proof. Let $x \in M$ be such that $\Delta_g u_\alpha(x) > 0$. One then has by (E_α)

$$\alpha A_\alpha + \frac{1-\theta}{\theta} B_\alpha u_\alpha(x)^{s-2} \le \frac{k_\alpha}{2} u_\alpha(x)^{r-2}.$$

Hence, $C B_\alpha \le u_\alpha(x)^{r-s}$.
Moreover, by (2.2), one has $B_\alpha \ge C A_\alpha^{-\frac{s\theta}{2(1-\theta)}}$. Hence, $u_\alpha(x) \ge C A_\alpha^{-\frac{s\theta}{2(r-s)(1-\theta)}}$. By using Lemma 3, which gives $u_\alpha(x) \le C r_\alpha^{-\frac{n}{r}}$, one gets that

$$d_g(x, x_\alpha) \ge C A_\alpha^{\overline{\frac{rs\theta}{2n(r-s)(1-\theta)}}}.$$

This proves our assertion.

In order to simplify the notations, set $\omega = \frac{rs\theta}{2n(r-s)(1-\theta)}$.
Set $\eta_\alpha = \eta(c^{-1} r_\alpha)$.

One can now prove the following.

Lemma 6. There exists $C >$ independent of α such that

$$\int_M \eta_\alpha^2 r_\alpha^2 |\nabla u_\alpha|_g^2 dv_g \le C \|u_\alpha\|_\infty^{2-r}.$$

Proof. Set $\gamma_\alpha = \int_M \eta_{\alpha,k}^2 r_\alpha^2 |\nabla u_\alpha|_g^2 dv_g$. Integrating by parts, one gets that

$$\gamma_\alpha = \int_M \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta_g u_\alpha dv_g - 2\int_M u_\alpha \eta_\alpha r_\alpha < \nabla u_\alpha, \nabla \eta_\alpha r_\alpha >_g dv_g.$$

Hence, by Lemma 5,

$$\gamma_\alpha \le \int_{B_{x_\alpha}(t_0 A_\alpha^\omega)} \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta_g u_\alpha dv_g + C\int_M u_\alpha \eta_\alpha r_\alpha |\nabla u_\alpha|_g |\nabla \eta_\alpha r_\alpha|_g dv_g.$$

Relations (E_α), (2.5) and (2.8) give

$$\begin{aligned} |u_\alpha \Delta_g u_\alpha| &\leq \frac{1}{2A_\alpha}\left| k_\alpha u_\alpha^r - 2\alpha A_\alpha u_\alpha^2 - \frac{2(1-\theta)}{\theta} B_\alpha u_\alpha^s \right| \\ &\leq C\frac{k_\alpha}{2A_\alpha}\|u_\alpha\|_\infty^r \leq A_\alpha^{-n\omega-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{B_{x_\alpha}(t_0 A_\alpha^\omega)} \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta_g u_\alpha dv_g &\leq CVol_g\left(B_{x_\alpha}(t_0A_\alpha^\omega)\right) A_\alpha^{-n\omega-1}\left(t_0 A_\alpha^\omega\right)^2 \\ &\leq CA_\alpha^{2\omega-1}. \end{aligned}$$

One may easily check that

$$2\omega - 1 = \frac{rs\theta}{n(r-s)(1-\theta)} - 1 = \frac{s\theta}{2(r-s)(1-\theta)}(r-2).$$

Hence

$$\begin{aligned} \int_{B_{x_\alpha}(t_0 A_\alpha^\omega)} \eta_\alpha^2 r_\alpha^2 u_\alpha \Delta_g u_\alpha dv_g &\leq CA_\alpha^{\frac{s\theta}{2(r-s)(1-\theta)}(r-2)} \\ &\leq C\|u_\alpha\|_\infty^{2-r}. \end{aligned}$$

Moreover, Hölder's inequality leads to

$$\int_M u_\alpha \eta_\alpha r_\alpha |\nabla u_\alpha|_g |\nabla \eta_\alpha r_\alpha|_g dv_g \leq \left(\int_M \eta_\alpha^r r_\alpha^r |\nabla u_\alpha|_g^2 dv_g\right)^{\frac{1}{2}} \left(\int_M u_\alpha^2 |\nabla \eta_\alpha r_\alpha|_g^2 dv_g\right)^{\frac{1}{2}}.$$

But

$$|\nabla \eta_\alpha r_\alpha|_g^2 \leq C.$$

Therefore

$$\int_M u_\alpha \eta_\alpha r_\alpha |\nabla u_\alpha|_g |\nabla \eta_\alpha r_\alpha|_g dv_g \leq \left(\gamma_\alpha \|u_\alpha\|_\infty^{2-r}\right)^{\frac{1}{2}}.$$

One then has

$$\frac{\gamma_\alpha}{\|u_\alpha\|_\infty^{2-r}} \leq C + C\left(\frac{\gamma_\alpha}{\|u_\alpha\|_\infty^{2-r}}\right)^{\frac{1}{2}},$$

which proves the lemma.
Changing η into $\eta^{\frac{r}{2}}$, one also gets that

$$\int_M \eta_\alpha^r r_\alpha^2 |\nabla u_\alpha|_g^2 dv_g \leq C\|u_\alpha\|_\infty^{2-r}.$$

We now prove the following main estimate.

Lemma 7. There exists $C > 0$ independant of α such that

$$\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g \leq C\sqrt{\alpha} A_\alpha^{2\omega}.$$

Proof. Assume by contradiction

$$\frac{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}{\sqrt{\alpha} A_\alpha^{2\omega}} \to +\infty. \tag{H''}$$

Multiplying (E_α) by $\frac{u_\alpha \eta_\alpha^r r_\alpha^2}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}$ and integrating over M, one gets that

$$\frac{2A_\alpha \int_M (\Delta_g u_\alpha) u_\alpha \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} + \frac{2\alpha A_\alpha \int_M u_\alpha^2 \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}$$

$$+\frac{2(1-\theta)}{\theta} B_\alpha \frac{\int_M u_\alpha^s \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} = k_\alpha. \tag{2.18}$$

An integration by parts and Lemma 6 lead to

$$\begin{aligned} \left| \int_M (\Delta_g u_\alpha) u_\alpha \eta_\alpha^r r_\alpha^2 dv_g \right| &\le C \left| \int_M \eta_\alpha^r r_\alpha^2 |\nabla u_\alpha|_g dv_g + \int_M u_\alpha^2 |\nabla \eta_\alpha^{\frac{r}{2}} r_\alpha|_g dv_g \right| \\ &\le C \|u_\alpha\|_\infty^{2-r}. \end{aligned}$$

Hence, by (H''),

$$\frac{2A_\alpha \int_M (\Delta_g u_\alpha) u_\alpha \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} \le \frac{C A_\alpha^{1-2\omega} \|u_\alpha\|_\infty^{2-r}}{\sqrt{\alpha}} \le \frac{C}{\sqrt{\alpha}} \to 0.$$

Since

$$\frac{2\alpha A_\alpha \int_M u_\alpha^2 \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} \ge 0,$$

one has by (2.18)

$$B_\alpha \frac{\int_M u_\alpha^s \eta_\alpha^r r_\alpha^2 dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} \le C.$$

Therefore, by (2.2),

$$\frac{\int_M u_\alpha^s \eta_\alpha^r r_\alpha^2 dv_g}{A_\alpha^{\frac{s\theta}{2(1-\theta)}} \int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} \le C. \tag{2.19}$$

Moreover, Lemma 4 gives

$$\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g - \int_M u_\alpha^s \eta_\alpha^r r_\alpha^2 dv_g \le C \int_{M - B_{x_\alpha}(c)} u_\alpha^s dv_g \le C A_\alpha^{2\omega + \frac{s\theta}{2(1-\theta)}}.$$

It follows from (H'') and (2.19) that

$$\frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{A_\alpha^{\frac{s\theta}{2(1-\theta)}} \int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g} \le C. \tag{2.20}$$

Now, let us prove

$$\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g \le C \frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{\left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{2-s}}. \tag{2.21}$$

One has by Lemma 3

$$\begin{aligned}
\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g &= \int_{B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g + \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g \\
&\le \int_{B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g + \frac{C}{\left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{2-s}} \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g.
\end{aligned}$$

Clearly

$$\int_{B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g \le C A_\alpha^{\frac{s\theta}{2(1-\theta)}} \left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^s.$$

Assume by contradiction that

$$A_\alpha^{\frac{s\theta}{2(1-\theta)}} \left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^s \ge \frac{t_\alpha}{\left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{2-s}} \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g \tag{H'''}$$

where $t_\alpha \to +\infty$.
One gets from Lemma 3 that

$$\begin{aligned}
\int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g &\le \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^r \eta_\alpha^s r_\alpha^2 dv_g \\
&\le \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^{2-(r-s)\frac{n}{r}} dv_g \\
&\le C \left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{-(r-s)\frac{n}{r}} \frac{A_\alpha^{\frac{s\theta}{2(1-\theta)}} \left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^2}{t_\alpha} \\
&\le C\sqrt{\alpha} A_\alpha^{2\omega}.
\end{aligned}$$

Moreover, one can easily check that

$$\int_{B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g \le C\sqrt{\alpha} A_\alpha^{2\omega},$$

which contradicts (H''). Hence (H''') is false and one proved

$$\int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g \le \frac{C}{\left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{2-s}} \int_{M - B_{x_\alpha}(A_\alpha^\omega \alpha^{\frac{1}{4}})} u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g.$$

Inequality (2.21) follows.
From $I(A,B)(u_\alpha r_\alpha \eta_\alpha)$, one gets that

$$\begin{aligned} 1 \quad \leq \quad & A\frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} \\ & + B\frac{\int_M u_\alpha^2 \eta_\alpha^2 r_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}. \end{aligned}$$

Let us show that

$$\lim \frac{\int_M u_\alpha^2 \eta_\alpha^2 r_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} = 0. \tag{2.22}$$

One has by Hölder's inequality

$$\begin{aligned} & \frac{\int_M u_\alpha^2 \eta_\alpha^2 r_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} \\ & \qquad \leq \frac{\left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}-\frac{2}{r}}} \\ & \qquad \leq \frac{\left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{1-\theta}{\theta}}} \\ & \qquad \leq \left(\frac{B_\alpha \int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}\right)^{\frac{2(1-\theta)}{s\theta}} \frac{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{2(1-\theta)}{s\theta}-\frac{1-\theta}{\theta}}}{B_\alpha^{\frac{2(1-\theta)}{s\theta}}}. \end{aligned}$$

Then, (2.21), (2.2) and (2.8) lead to

$$\frac{\int_M u_\alpha^2 \eta_\alpha^2 r_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}$$

$$\leq \left(\frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{A_\alpha^{\frac{s\theta}{2(1-\theta)}} \int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}\right)^{\frac{2(1-\theta)}{s\theta}} \frac{CA_\alpha}{\left(A_\alpha^\omega \alpha^{\frac{1}{4}}\right)^{(2-s)\frac{2(1-\theta)}{s\theta}}} \left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{2(1-\theta)}{s\theta}-\frac{1-\theta}{\theta}}.$$

Therefore, one has by (2.20)

$$\frac{\int_M u_\alpha^2 \eta_\alpha^2 r_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} \leq \frac{A_\alpha^{1-\frac{r(2-s)}{n(r-s)}}}{\alpha^{\frac{(2-s)(1-\theta)}{2s\theta}}}.$$

Since

$$\begin{aligned}1-\frac{r(2-s)}{n(r-s)} &= \frac{1}{n(r-s)}(n(r-s)-r(2-s))\\ &= \frac{1}{n(r-s)}((n-2)(r-s)+s(r-2))\\ &\geq 0,\end{aligned}$$

it gives (2.22).
Now, let us prove that

$$\frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} \to 0. \tag{2.23}$$

Using successively (2.21) and (2.20), one gets that

$$\begin{aligned}&\frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}\\
&\leq \frac{C}{\left(A_\alpha^\omega \alpha^{\frac14}\right)^{(2-s)\frac{2(1-\theta)}{s\theta}}} \frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{1}{\theta}}}\\
&\leq \frac{CA_\alpha}{\left(A_\alpha^\omega \alpha^{\frac14}\right)^{(2-s)\frac{2(1-\theta)}{s\theta}}} \left(\frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{A_\alpha^{\frac{s\theta}{2(1-\theta)}} \int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}\right)^{\frac{2(1-\theta)}{s\theta}} \frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{1}{\theta}-\frac{2(1-\theta)}{s\theta}}}\\
&\leq C\frac{A_\alpha^{1-\frac{r(2-s)}{n(r-s)}}}{\alpha^{\frac{(2-s)(1-\theta)}{2s\theta}}} \frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right)^{\frac{1}{\theta}-\frac{2(1-\theta)}{s\theta}}}\\
&\leq \frac{CA_\alpha^{1-\frac{r(2-s)}{n(r-s)}-2\omega\left(\frac{1}{\theta}-\frac{2(1-\theta)}{s\theta}\right)}}{\alpha^{\frac{(2-s)(1-\theta)}{2s\theta}}} \left(\frac{A_\alpha^{2\omega}}{\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g}\right)^{\frac{1}{\theta}-\frac{2(1-\theta)}{s\theta}} \int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g.\end{aligned}$$

Hölder's inequality leads to

$$\begin{aligned}\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g &= \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 r_\alpha^2 dv_g + 2\int_M u_\alpha \eta_\alpha r_\alpha < \nabla u_\alpha, \nabla \eta_\alpha r_\alpha >_g dv_g\\
&\quad + \int_M u_\alpha^2 |\nabla \eta_\alpha r_\alpha|_g^2 dv_g\\
&\leq \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 r_\alpha^2 dv_g + \int_M u_\alpha^2 |\nabla \eta_\alpha r_\alpha|_g^2 dv_g\\
&\quad +2\left(\int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 r_\alpha^2 dv_g\right)^{\frac12} \left(\int_M u_\alpha^2 |\nabla \eta_\alpha r_\alpha|_g^2 dv_g\right)^{\frac12}.\end{aligned}$$

Hence, one has by Lemma 6

$$\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \le C\|u_\alpha\|_\infty^{2-r}.$$

Finally, noticing that

$$\frac{1}{\theta} - \frac{2(1-\theta)}{s\theta} > \frac{2}{r\theta} - \frac{2(1-\theta)}{s\theta} = 1 - \frac{2}{n} > 0,$$

one gets from (H'') that

$$\frac{\int_M |\nabla u_\alpha \eta_\alpha r_\alpha|_g^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s r_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{\left(\int_M u_\alpha^r \eta_\alpha^r r_\alpha^r dv_g\right)^{\frac{2}{r\theta}}} \le C \frac{A_\alpha^{1-\frac{r(2-s)}{n(r-s)}-2\omega\left(\frac{1}{\theta}-\frac{2(1-\theta)}{s\theta}\right)+2\omega-1}}{\alpha^{\frac{(2-s)(1-\theta)}{2s\theta}}}.$$

But

$$\begin{aligned}
&1 - \frac{r(2-s)}{n(r-s)} - 2\omega\left(\frac{1}{\theta} - \frac{2(1-\theta)}{s\theta}\right) + 2\omega - 1 \\
&= 1 - \frac{r(2-s)}{n(r-s)} - \frac{rs}{n(r-s)(1-\theta)} + \frac{2r}{n(r-s)} + \frac{rs\theta}{n(r-s)(1-\theta)} - 1 \\
&= \frac{rs}{n(r-s)}\left(1 - \frac{1}{1-\theta} + \frac{\theta}{1-\theta}\right).
\end{aligned}$$

Relation (2.23) follows. (2.22) and (2.23) contradicts $I(A,B)(u_\alpha r_\alpha \eta_\alpha)$. As a consequence, (H'') is false and Lemma 7 is proved.

The last two estimates are important in the third step.

Lemma 8. There exists $C > 0$ independent of α such that

$$\frac{1 - \left(\int_M u_\alpha^r \eta_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}{\sqrt{\alpha} A_\alpha^{2\omega}} \le C.$$

Proof. Let ξ be the Euclidian metric on M. One then has

$$|\nabla u_\alpha \eta_\alpha|_\xi^2 \le |\nabla u_\alpha \eta_\alpha|_g^2 (1 + C r_\alpha^2)$$

$$(1 - C r_\alpha^2) dv_\xi \le dv_g \le (1 + C r_\alpha^2) dv_\xi$$

$$\int_M |\nabla u_\alpha \eta_\alpha|_\xi^2 dv_\xi \le \int_M |\nabla u_\alpha \eta_\alpha|_g^2 (1 + C r_\alpha^2) dv_g. \tag{2.24}$$

Hence, one gets that

$$\begin{aligned}
1 - \left(\int_M u_\alpha^r \eta_\alpha^r dv_\xi\right)^{\frac{2}{r\theta}} &\le C\left(1 - \int_M u_\alpha^r \eta_\alpha^r dv_\xi\right) \\
&\le C\left(\int_M u_\alpha^r dv_g - \int_M u_\alpha^r \eta_\alpha^r dv_g + C\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right) \\
&\le C\left(\int_M u_\alpha^r (1 - \eta_\alpha^r) dv_g + C\int_M u_\alpha^r \eta_\alpha^r r_\alpha^2 dv_g\right).
\end{aligned}$$

One easily checks that, if $r > 2$, $2\omega < \frac{r}{r-2}$. Therefore, Lemma 6 and 7 lead to Lemma 8. The last lemma we need is the following.

Lemma 9. There exists $C > 0$ independent of α such that

$$\left(\int_M u_\alpha^s \eta_\alpha^s dv_\xi\right)^{\frac{2(1-\theta)}{s\theta}} \leq \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} + CA_\alpha^{1+2\omega}\sqrt{\alpha}.$$

Proof. Multipliying (E_α) by $\frac{u_\alpha r_\alpha^2 \eta_\alpha^r}{A_\alpha^{2\omega}\sqrt{\alpha}}$ and integrating over M, one gets that

$$\frac{2A_\alpha^{1-2\omega}}{\sqrt{\alpha}}\int_M (\Delta_g u_\alpha) u_\alpha r_\alpha^2 \eta_\alpha^r dv_g + 2\sqrt{\alpha}A_\alpha^{1-2\omega}\int_M u_\alpha^2 r_\alpha^2 \eta_\alpha^r dv_g$$
$$+\frac{2(1-\theta)}{\theta}\frac{B_\alpha}{A_\alpha^{2\omega}\sqrt{\alpha}}\int_M u_\alpha^s r_\alpha^2 \eta_\alpha^r dv_g = \frac{k_\alpha}{A_\alpha^{2\omega}\sqrt{\alpha}}\int_M u_\alpha^r r_\alpha^2 \eta_\alpha^r dv_g.$$

One has already shown in the proof of Lemma 6 that

$$\int_M (\Delta_g u_\alpha) u_\alpha r_\alpha^2 \eta_\alpha^r dv_g \leq C\|u_\alpha\|_\infty^{2-r}.$$

Relation (2.2) and Lemma 7 lead then to

$$\int_M u_\alpha^s r_\alpha^2 \eta_\alpha^r dv_g \leq C\frac{A_\alpha^{2\omega}\sqrt{\alpha}}{B_\alpha} \leq CA_\alpha^{2\omega+\frac{s\theta}{2(1-\theta)}}\sqrt{\alpha}.$$

And since this result is also true with $\eta = \eta^{\frac{s}{r}}$,

$$\int_M u_\alpha^s r_\alpha^2 \eta_\alpha^s dv_g \leq CA_\alpha^{2\omega+\frac{s\theta}{2(1-\theta)}}\sqrt{\alpha}. \tag{2.25}$$

Noticing that $dv_\xi \leq \left(1 + Cr_\alpha^2\right) dv_g$, one gets that

$$\begin{aligned}\left(\int_M u_\alpha^s \eta_\alpha^s dv_\xi\right)^{\frac{2(1-\theta)}{s\theta}} &\leq \left(\int_M u_\alpha^s \eta_\alpha^s dv_g + C\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &\leq \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}\left(1 + C\frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{\int_M u_\alpha^s \eta_\alpha^s dv_g}\right)^{\frac{2(1-\theta)}{s\theta}}.\end{aligned}$$

Inequality (2.25) implies

$$\frac{\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g}{\int_M u_\alpha^s \eta_\alpha^s dv_g} \to 0.$$

Consequently,

$$\begin{aligned}\left(\int_M u_\alpha^s \eta_\alpha^s dv_\xi\right)^{\frac{2(1-\theta)}{s\theta}} &\leq \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\ &+ C\left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}-1}\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g.\end{aligned}$$

One deduces from (2.25) and Lemma 2 that

$$
\begin{aligned}
&\left(\int_M u_\alpha^s \eta_\alpha^s dv_\xi\right)^{\frac{2(1-\theta)}{s\theta}} \\
&\quad \le \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\
&\quad\quad + C\left(\int_M u_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}-1}\left(\frac{\int_M u_\alpha^s \eta_\alpha^s dv_g}{\int_M u_\alpha^s dv_g}\right)^{\frac{2(1-\theta)}{s\theta}-1}\int_M u_\alpha^s \eta_\alpha^s r_\alpha^2 dv_g \\
&\quad \le \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} + C A_\alpha^{1+2\omega}\sqrt{\alpha}.
\end{aligned}
$$

This ends the proof of the lemma.

Step 3: Conclusion

One has, by definition of A_0,

$$
\left(\int_M u_\alpha^r \eta_\alpha^r dv_\xi\right)^{\frac{2}{r\theta}} \le A_0 \int_M |\nabla u_\alpha \eta_\alpha|_\xi^2 dv_\xi \left(\int_M u_\alpha^s \eta_\alpha^s dv_\xi\right)^{\frac{2(1-\theta)}{s\theta}}
$$

and, by Lemma 6 and (2.24),

$$
\int_M |\nabla u_\alpha \eta_\alpha|_\xi^2 dv_\xi \le \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 dv_g + C\|u_\alpha\|_\infty^{2-r}.
$$

Hence, one gets from Lemma 9 that

$$
\left(\int_M u_\alpha^r \eta_\alpha^r dv_\xi\right)^{\frac{2}{r\theta}} \le A_0 \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} + C A_\alpha^{4\omega}\sqrt{\alpha}. \tag{2.26}
$$

The definition of u_α leads to

$$
1 = \left(\frac{1}{\mu_\alpha}\int_M |\nabla u_\alpha|_g^2 dv_g + \frac{\alpha}{\mu_\alpha}\int_M u_\alpha^2 dv_g\right) A_\alpha. \tag{2.27}
$$

Computing (2.27)−(2.26), one gets that

$$
\begin{aligned}
1 - \left(\int_M u_\alpha^r \eta_\alpha^r dv_g\right)^{\frac{2}{r\theta}} \ \ge\ & -A_0 \int_M |\nabla u_\alpha|_g^2 \eta_\alpha^2 dv_g \left(\int_M u_\alpha^s \eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}} \\
& + \frac{A_\alpha}{\mu_\alpha}\int_M |\nabla u_\alpha|_g^2 dv_g + \frac{\alpha A_\alpha}{\mu_\alpha}\int_M u_\alpha^2 dv_g - C A_\alpha^{4\omega}\sqrt{\alpha}.
\end{aligned}
$$

Then, noticing that

$$
A_\alpha \int_M u_\alpha^2 dv_g \ge C A_\alpha^{2\omega}
$$

and dividing by $A_\alpha^{2\omega}\sqrt{\alpha}$, it follows that

$$\begin{aligned}\frac{1-\left(\int_M u_\alpha^r\eta_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}{\sqrt{\alpha}A_\alpha^{2\omega}} &\geq -\frac{A_0}{\sqrt{\alpha}}\int_M |\nabla u_\alpha|_g^2\eta_\alpha^2 dv_g \frac{\left(\int_M u_\alpha^s\eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{A_\alpha^{2\omega}} \\ &\quad + \frac{A_\alpha^{1-2\omega}}{\mu_\alpha\sqrt{\alpha}}\int_M |\nabla u_\alpha|_g^2 dv_g + \frac{\sqrt{\alpha}}{\mu_\alpha} - CA_\alpha^{2\omega}.\end{aligned}$$

Finally, since

$$\frac{1}{\mu_\alpha} \geq A_0$$

$$\frac{\left(\int_M u_\alpha^s\eta_\alpha^s dv_g\right)^{\frac{2(1-\theta)}{s\theta}}}{A_\alpha} \leq 1,$$

one finds

$$\frac{1-\left(\int_M u_\alpha^r\eta_\alpha^r dv_g\right)^{\frac{2}{r\theta}}}{\sqrt{\alpha}A_\alpha^{2\omega}} \geq \frac{A_0 A_\alpha^{1-2\omega}}{\sqrt{\alpha}}\int_M |\nabla u_\alpha|_g^2(1-\eta_\alpha^2)dv_g + A_0\sqrt{\alpha} - CA_\alpha^{2\omega}.$$

By Lemma 8, the left member is bounded while the right one converges to $+\infty$. This ends the proof of the theorem.

3. Some applications

(a) The best constant problem for the logarithmic Sobolev inequality

We prove in this subsection Corollary 1. Fix $r = 2$. One then has the following inequalities

$$\left(\int_M |u|^2 dv_g\right)^{1+\frac{2}{n\frac{2-s}{s}}} \leq \left(A\int_M |\nabla u|_g^2 dv_g + B\int_M |u|^2 dv_g\right)\left(\int_M |u|^s dv_g\right)^{\frac{4}{n(2-s)}} \qquad I_s(A,B)$$

where $1 \leq s < 2$. One proved in the section above that all these inequalities hold with their first best constant. Set

$$\begin{aligned} A(s) &= \inf\{A \in \mathbb{R} \text{ s.t. } \exists B \in \mathbb{R} \text{ for which } I_s(A,B) \text{ is valid}\} \\ B(s) &= \inf\{B \in \mathbb{R} \text{ s.t. } I_s(A(s),B) \text{ is valid}\}. \end{aligned}$$

It is clear that $I_{s'}(A,B)$ implies $I_s(A,B)$ when $s' > s$. Therefore, $A(s)$ is increasing. According to [3], $A(s)$ is bounded by a constant independant of s. Hence, $A(s)$ converges to a constant $A(2)$ as $s \to 2$. If $s' > s$, $I_s(A(s'),B(s'))$ holds. One can then set

$$A'(s) = \inf\{A \in \mathbb{R} \text{ s.t. } I_s(A,B(s')) \text{ is valid}\}.$$

Thereafter, by definition of $A'(s)$, for all $\epsilon > 0$, there exists $u \in C^\infty(M)$ such that $\|u\|_s = 1$ and

$$A'(s)\int_M |\nabla u|_g^2 dv_g + B(s')\int_M |u|^2 dv_g \leq \left(\int_M |u|^2 dv_g\right)^{1+\frac{2}{n\frac{2-s}{s}}} + \epsilon.$$

Adding the previous inequality with $I_s(A(s), B(s))(u)$ and noting that $A(s) \leq A'(s)$, one easily gets that $B(s') - B(s) \leq V_g(M)^{\frac{2}{s}-1}\epsilon$. Since ϵ is arbitrary, one proved that $B(s)$ is decreasing and converges to a constant $B(2)$ as $s \to 2$. Now, taking the limit in $I_s(A(s), B(s))$ as $s \to 2$, one gets for all $u > 0$ such that $\|u\|_2 = 1$ the logarithmic Sobolev inequality

$$\int_M u^2 \ln u^2 dv_g \leq \frac{n}{2} \ln \left(A(2) \int_M |\nabla u|_g^2 dv_g + B(2) \right).$$

Clearly, $A(2) = A_0(2,2,0,n) = \frac{2}{n\pi e}$ is optimal and the inequality is optimal in the sense that no constant can be lowered. This proves Corollary 1.

(b) Heat kernel upper bounds estimates

We discuss here one application on the estimates of the heat kernel upper bounds. When M is a complete manifold (not necessarily compact), it is well known (see for instance [6]) that all the previous inequalities are equivalent to

$$\|P_t\|_{1,\infty} \leq \frac{C}{t^{\frac{n}{2}}}$$

where $(P_t)_{t>0}$ is the heat semigroup on M. Moreover, when M is the Euclidean space $\mathbb{R}^n$, one has

$$\|P_t\|_{1,\infty} = \frac{1}{(4\pi t)^{\frac{n}{2}}}.$$

Hence, it is quite intuitive that, on a manifold, we should have the small time estimate

$$\|P_t\|_{1,\infty} \sim \frac{1}{(4\pi t)^{\frac{n}{2}}}.$$

Corollary 2 gives an additional information on this estimate when M is compact. In order to prove it, we need the following theorem from D. Bakry (see [2] for a detailed proof in the more general case of the Markov diffusion generators):

Theorem 2. Let us assume that, for all $u \in C^\infty(M)$ such that $u > 0$ and $\|u\|_2 = 1$,

$$\int_M u^2 \ln u^2 dv_g \leq \phi \left(\int_M |\nabla u|_g^2 dv_g \right)$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is concave, increasing and of class C^1. One then has for all $1 \leq p < q \leq \infty$

$$\|P_t\|_{p,q} \leq e^m$$

where

$$t = \int_p^q \phi'(v(s)) \frac{ds}{4(s-1)} \text{ and } m = \int_p^q \left(\phi(v(s)) - v(s)\phi'(v(s))\right) \frac{ds}{s^2}$$

provided we find a function $v \geq 0$ for which these two integrals are finite.

Set $v(s) = \frac{\lambda s^2}{s-1} - \frac{n\pi e B(2)}{2}$ where $\lambda \geq \frac{n\pi e B(2)}{8}$ is a parameter and $B(2)$ is the constant introduced in the previous subsection. One has

$$\phi(x) = \frac{n}{2} \ln \left(\frac{2}{n\pi e} x + B(2) \right).$$

It is an easy matter to check that

$$\phi'(v(s)) = \frac{n}{2} \frac{s-1}{\lambda s^2}$$

and that

$$\phi(v(s)) - v(s)\phi'(v(s)) = \frac{n}{2} \ln \left(\frac{2\lambda s^2}{n\pi e^2 (s-1)} \right) + \frac{n^2 \pi e B(2)(s-1)}{4\lambda s^2}.$$

Some easy computations lead then to

$$t = \int_1^\infty \frac{n}{8\lambda s^2} ds = \frac{n}{8\lambda}$$

and

$$\begin{aligned} m &= \frac{n}{2} \int_1^\infty \ln \left(\frac{2\lambda s^2}{n\pi e^2 (s-1)} \right) \frac{ds}{s^2} + \frac{n^2 \pi e B(2)}{4\lambda} \int_1^\infty \frac{s-1}{s^4} ds \\ &= \frac{n}{2} \ln \left(\frac{2\lambda}{n\pi e^2} \right) + \frac{n}{2} \int_1^\infty \ln \left(\frac{s^2}{s-1} \right) \frac{ds}{s^2} + \frac{n^2 \pi e B(2)}{24\lambda} \\ &= \frac{n}{2} \ln \left(\frac{2\lambda}{n\pi e^2} \right) + n + \frac{n^2 \pi e B(2)}{24\lambda} \\ &= \frac{n}{2} \ln \left(\frac{2\lambda}{n\pi} \right) + \frac{n^2 \pi e B(2)}{24\lambda}. \end{aligned}$$

Since $\lambda = \frac{n}{8t}$,

$$m = \frac{n}{2} \ln \left(\frac{1}{4\pi t} \right) + \frac{n\pi e B(2) t}{3}.$$

It follows that

$$\|P_t\|_{1,\infty} \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{n\pi e B(2) t}{3}}$$

with $0 < t \leq (\pi e B(2))^{-1}$. It yields Corollary 2.

References

1. T. Aubin, Problmes isoprimtriques et espaces de Sobolev, *Journal of Differential Geometry*, **11** (1976), 573-598.
2. D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes. Ecole d'Eté de Probabilit´s de St-Flour, *Lecture Notes in Math.*, **1581** (Springer, 1994), 1-114.
3. D. Bakry T. Coulhon M. Ledoux L. Saloff-Coste, Sobolev inequalities in disguise, *Indiana J. Math.*, **44** (1995), 1033-1074.

4. E. Carlen, Superadditivity of Fisher's information and logarithmic Sobolev inequalities, *J. Funct. Anal.*, **101** (1991), 194-211.
5. E. Carlen M. Loss Sharp constant in Nash's inequality, *Duke Math. J., International Math. Research Notices,* **7** (1993), 213-215.
6. E. B. Davies, Heat kernels and spectral theory, (Cambridge University Press, 1989).
7. O. Druet, The best constant problem in Sobolev inequalities, *Mathematische Annalen,* **314, No2** (1999), 327-346.
8. O. Druet E. Hebey M. Vaugon, Optimal Nash's inequalities on Riemannian manifolds: the influence of geometry, *Internat. Math. Res. Notices,* **14** (1999), 735-779.
9. D. Gilbarg N. S. Trüdinger, Elliptic Partial Differential Equations of Second Order, *Grundlehren der Mathematischen Wissenschaften,* **224** (Springer Verlag, 1977)
10. E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, *Courant Lecture Notes in Mathematics,* **5** (1999).
11. E. Humbert, Best Constants in the L^2-Nash Inequality, *Procedings of the Royal Society of Edinburgh, to appear*
12. J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.,* **14** (1961), 577-591.
13. J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math,* **80** (1958), 931-954.
14. G. Talenti Best constants in Sobolev inequality, *Annali di Matematica Pura ed Applicata,* **110** (1976), 353-372.

Chapitre 3

On the second best constant in logarithmic Sobolev Inequalities on complete Riemannian manifolds

A paraître dans *Bulletin des Sciences Mathématiques*

On the Second Best Constant in Logarithmic Sobolev Inequalities on Complete Riemannian Manifolds

Christophe Brouttelande*
Université Paul Sabatier, Toulouse†

April 14, 2003

Abstract: We study the second best constant problem for logarithmic Sobolev inequalities on complete Riemannian manifolds and investigate its relationship with optimal heat kernel bounds and the existence of extremal functions.

Résumé: Nous étudions le problème de seconde meilleure constante pour les inégalités de Sobolev logarithmiques sur les variétés riemanniennes complètes et examinons sa relation avec les bornes optimales du noyaux de la chaleur et l'existence de fonctions extrémales.

Mots clés: inégalités de Sobolev, inégalités de Sobolev logarithmiques, inégalités optimales.

Keywords: Sobolev inequalities, logarithmic Sobolev inequalities, optimal inequalities.

AMSclassification: 58J05 (35J20, 46E35, 53C21, 58E35)

1 Introduction

The logarithmic Sobolev inequality for the canonical Gaussian measure

$$d\gamma(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx$$

on $\mathbb{R}^n$ states that

$$\int_{\mathbb{R}^n} g^2 \ln g^2 d\gamma \leq 2 \int_{\mathbb{R}^n} |\nabla g|^2 d\gamma \tag{1.1}$$

for every compactly supported smooth function g on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} g^2 d\gamma = 1$. It has first been introduced by L. Gross in [8]. Following [4], the logarithmic Sobolev inequality (1.1) may be written equivalently with respect to Lebesgue measure. Set indeed

$$f^2(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} g^2(x), \qquad x \in \mathbb{R}^n.$$

*Email: Christophe.Brouttelande@math.ups-tlse.fr
†Université Paul Sabatier, Département de Mathématiques, 118 route de Narbonne, 31062 Toulouse cedex 4, France. Fax Number: (33)0561556089. Telephone: (33)0561557669.

Then $\int_{\mathbb{R}^n} f^2 dx = 1$ and

$$\int_{\mathbb{R}^n} f^2 \ln\left(f^2 (2\pi)^{\frac{n}{2}} e^{\frac{|x|^2}{2}}\right) dx \le 2 \int_{\mathbb{R}^n} |\nabla f + \frac{x}{2} f|^2 dx.$$

An integration by parts easily yields

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le 2 \int_{\mathbb{R}^n} |\nabla f|^2 dx - \frac{n}{2} \ln(2\pi) - n.$$

Changing f into $\lambda^{\frac{n}{2}} f(\lambda x)$, $\lambda > 0$, shows that, for all $\lambda > 0$,

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le 2\lambda^2 \int_{\mathbb{R}^n} |\nabla f|^2 dx - \frac{n}{2} \ln(2\pi) - n - n \ln \lambda.$$

Optimizing in λ, one gets

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le \frac{n}{2} \ln\left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 dx\right). \tag{1.2}$$

Since (1.1) is optimal and exponential functions are extremal, it is an easy matter to check that the constant $\frac{2}{n\pi e}$ is the best constant in (1.2). Indeed, set $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ the best function such that, for every smooth function f on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} f^2 dx = 1$,

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le \Phi\left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right).$$

Applying (1.2) to $f^2(x) = \lambda^n (2\pi)^{-\frac{n}{2}} e^{-\frac{\lambda^2 |x|^2}{2}}$, we get

$$\frac{n}{2} \ln\left(\frac{\lambda^2}{2\pi e}\right) \le \Phi\left(\frac{n\lambda^2}{4}\right).$$

It yields our claim.

The logarithmic Sobolev inequality (1.2) gives rise to a similar inequality on compact Riemannian manifolds. Let (M, g) be such a n-manifold and let $s > 0$. There exist real constants A and B such that for all $u \in H_1^2(M)$, $\|u\|_2 = 1$,

$$\int_M u^2 \ln u^2 dv_g \le \frac{n}{2} \ln\left(A \int_M |\nabla u|_g^2 dv_g + B\right). \tag{$LS(A,B)$}$$

This logarithmic Sobolev inequality may be seen as a limiting case of some family of Gagliardo-Nirenberg inequalities. These state that, for all $s > 0$, there exist $A, B \ge 0$ such that, for all $f \in H_1^2(M)$,

$$\left(\int_M f^2 dv_g\right)^{1+\frac{2s}{n(2-s)}} \le \left(A \int_M |\nabla f|_g^2 dv_g + B \int_M f^2 dv_g\right) \left(\int_M |f|^s dv_g\right)^{\frac{4}{n(2-s)}}. \tag{$I_s(A,B)$}$$

As $s \to 2$, one gets $LS(A,B)$ (more details may be found in [3]). When $n \ge 3$, these inequalities follows from the combination of the classical H_1^2-Sobolev inequality and Hölder's inequality. However, these can actually be proved for all $n \ge 1$. A very good reference for the Euclidean case is [18]. An important property is the equivalence, up

to the constants, between all these inequalities, also described in [18]. For the Riemannian case, one may see [20] in which the authors use a partition of unity argument to prove a modified Nash inequality.
Many famous particular cases in the Gagliardo-Nirenberg family have been independently studied. One may cite the Sobolev inequality (which is defined when $n \geq 3$)

$$\left(\int_M |f|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}} \leq A \int_M |\nabla f|_g^2 dv_g + B \int_M f^2 dv_g,$$

the Moser inequality

$$\int_M |f|^{2+\frac{4}{n}} dv_g \leq \left(A \int_M |\nabla f|_g^2 dv_g + B \int_M f^2 dv_g\right) \left(\int_M f^2 dv_g\right)^{\frac{2}{n}},$$

and the Nash inequality

$$\left(\int_M f^2 dv_g\right)^{1+\frac{2}{n}} \leq \left(A \int_M |\nabla f|_g^2 dv_g + B \int_M f^2 dv_g\right) \left(\int_M |f| dv_g\right)^{\frac{4}{n}},$$

which corresponds to the inequality $I_1(A, B)$. Each of these inequalities has its proper applications. The Sobolev inequality is well known for its important role in many problems in analysis. The Nash inequality was introduced by J. Nash in [17] and is famous in analysis of Hölder continuity of second order parabolic equations. In the subsequent paper [16], Moser's inequality is used instead.

For all these inequalities, the following questions naturally arise:
1. Which are the infima for constants A and B?
2. Do these inequalities hold with these infima?
3. What is the influence of geometry on the optimal inequalities?
4. Are there extremal functions?

Such problems have been studied for the Sobolev and Nash inequalities by O. Druet, E. Hebey, E. Humbert and M. Vaugon (see for instance [6], [9], [20], [11]). These works show the important role played by the Euclidean case. Since one knows the Euclidean optimal constants and extremal functions (see [1] and [14]) for these two inequalities, it is then natural that they had been studied first. One has previously seen that all these informations were known for $LS(A, B)$. Consequently, sharp estimates are made possible and it seems relevant to generalize the preceding works to this case. However, a new difficulty appears since $LS(A, B)$ is a limiting case of $I_s(A, B)$. One may notice that M. Del Pino and J. Dolbeault recently computed Euclidean optimal constants and extremal functions for another family of Gagliardo-Nirenberg inequalities.

Set

$$\mathcal{A} = \{A \in \mathbb{R} \text{ such that } LS(A, B) \text{ holds with some } B \in \mathbb{R}\}, \quad A_0 = \inf \mathcal{A},$$
$$\mathcal{B} = \{B \in \mathbb{R} \text{ such that } LS(A_0, B) \text{ holds}\}, \quad B_0 = \inf \mathcal{B}.$$

As we have seen, in the Euclidean case, $(A_0, B_0) = (\frac{2}{n\pi e}, 0)$. Moreover, it has been proven in [3] that $A_0 = \frac{2}{n\pi e}$ as soon as $\mathcal{A} \neq \emptyset$ and that $\mathcal{A}$ is closed when M is compact. More generally, one may actually prove that $\mathcal{A}$ is closed when M is complete with $|\mathrm{Rm}_g|_g$ and $|\nabla \mathrm{Rm}_g|_g$ bounded, Rm_g standing for the Riemman tensor curvature of (M, g). On the other hand, almost nothing is known about the second constant. Some studies exist for the Sobolev and Nash inequalities ([7], [10]), and this work mainly generalizes these results to our case. First, one has the following.

PROPOSITION 1 *Let (M,g) be a smooth complete Riemannian n-manifold with $n \geq 2$. Assume there exists $B \in I\!R$ such that $LS(\frac{2}{n\pi e}, B)$ holds. Then*

$$B_0 \geq \max\left(\frac{\max_M \mathrm{Scal}_g}{2n\pi e}, \mathrm{Vol}_g(M)^{-\frac{2}{n}}\right).$$

This may be proven by following the idea developed by E. Humbert in [11] for the Nash inequality but a more simple proof shall be given below using heat kernel estimates. More precise results may be obtained when $LS(A,B)$ is restricted to a sufficiently small ball. This is the content of the next statement.

THEOREM 1 *Let (M,g) be a smooth complete Riemannian n-manifold with $n \geq 2$ and let $x_0 \in M$. For any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that for all $u \in C_c^\infty(B_g(x_0,\delta_\varepsilon))$ with $\|u\|_2 = 1$,*

$$\int_{B_g(x_0,\delta_\varepsilon)} u^2 \ln u^2 dv_g \leq \frac{n}{2} \ln\left[\frac{2}{n\pi e}\left(\int_{B_g(x_0,\delta_\varepsilon)} |\nabla u|_g^2 dv_g + \frac{\mathrm{Scal}_g(x_0)}{4} + \varepsilon\right)\right]. \quad (1.3)$$

This theorem is inspired by a work of O. Druet ([5]) in case of the H_1^1-Sobolev inequality on $BV(M)$. Our proof follows the same idea. In his paper, he uses that the optimal H_1^1-Sobolev inequality is the limit as $p \to 1$ of the optimal H_1^p-Sobolev inequalities. In our proof, one uses the similar fact that the optimal $LS(A,B)$ is the limit as $s \to 2$ of the optimal $I_s(A,B)$.
Let us notice that we do not know the explicit value of B_0 except for the standard circle (S^1,h). To get this value, one remarks the second best constant for $I_s(A,B)$ (as defined for $LS(A,B)$) decreases as s increases (see [3]). Since this constant is equal to $\mathrm{Vol}_h(S^1)^{-2} = (2\pi)^{-2}$ when $s = 1$ (see [11]), one gets from (1.6) that $B_0 = (2\pi)^{-2}$.

An important property of $LS(A,B)$ is its close relationship with heat kernel upper bounds. According to a result from D. Bakry [2], such an inequality gives an explicit bound. It can be written as follows.

THEOREM 2 (D. Bakry) *Let Ω be a regular open subset of M. Let $1 \leq p < q \leq +\infty$. Assume that, for all $u \in C_c^\infty(\Omega)$ such that $\|u\|_2 = 1$,*

$$\int_\Omega u^2 \ln u^2 dv_g \leq \phi\left(\int_\Omega |\nabla u|_g^2 dv_g\right)$$

where $\phi : I\!R_+ \to I\!R_+$ is concave, increasing and of class C^1. Assume also that there exists a function $v \geq 0$ such that

$$t(p,q) = \int_p^q \phi'(v(s)) \frac{ds}{4(s-1)} \text{ and } m(p,q) = \int_p^q \left(\phi(v(s)) - v(s)\phi'(v(s))\right) \frac{ds}{s^2}$$

are both finite. One then has

$$\|P_{t(p,q)}\|_{p,q} \leq e^{m(p,q)}$$

where $(P_t)_{t>0}$ is the Dirichlet heat semigroup on Ω.

As described in [3], using the optimal inequality $LS\left(\frac{2}{n\pi e}, B_0\right)$ and the function $v(s) = \frac{\lambda s^2}{s-1} - \frac{n\pi e B_0}{2}$, $\lambda \geq \frac{n\pi e B_0}{8}$ being a parameter, leads to

$$\sup_{x,y \in M} p(t,x,y) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{n\pi e B_0}{3} t} \text{ for all } 0 < t \leq (\pi e B_0)^{-1}. \quad (1.4)$$

In the Euclidean case, it is well known that the equality occurs. One may then ask if the optimality of $LS(\frac{2}{n\pi e}, B_0)$ brings some information on the heat kernel upper bound. Let $x \in M$. One can find in [15] the following small time expansion

$$p(t, x, x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(1 + \frac{\mathrm{Scal}_g(x)}{6} t + o(t)\right).$$

Let us notice that it implies Proposition 1. Actually, from this expansion and (1.4), one deduces that

$$B_0 \geq \frac{\max_M \mathrm{Scal}_g}{2n\pi e}. \tag{1.5}$$

By setting $u = \mathrm{Vol}_g(M)^{-\frac{1}{2}}$ in $LS\left(\frac{2}{n\pi e}, B_0\right)$, one also gets

$$B_0 \geq \mathrm{Vol}_g^{-\frac{2}{n}}, \tag{1.6}$$

which yields the result. One may first remark that there is no reason allowing us to say that (1.4) is optimal. One actually does not know if the function

$$\Phi(x) = \frac{n}{2} \ln \left(\frac{2}{n\pi e} x + B_0\right)$$

is the best possible in the inequality

$$\int_M f^2 \ln f^2 dv_g \leq \Phi\left(\int_M |\nabla f|_g^2 dv_g\right).$$

However, the local inequality (1.3) is optimal in the following sense. Combining Theorem 1 and Theorem 2 implies that for all $x \in M$ and $\varepsilon > 0$, there exists $T > 0$ such that

$$p(t, x, x) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\left(\frac{\mathrm{Scal}_g(x)}{6} + \varepsilon\right) t} \quad \text{for all} \quad 0 < t < T.$$

One then gets

$$p(t, x, x) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \left(1 + \frac{\mathrm{Scal}_g(x)}{6} t + o(t)\right),$$

which is optimal in the sense that the constant $\frac{\mathrm{Scal}_g(x)}{6}$ is the best possible.

One may now ask about the existence of extremal functions for the optimal inequality $(LS(\frac{2}{n\pi e}, B_0))$. The following theorem has been established for the H_1^2-Sobolev and Nash inequalities (see for instance [7] and [10]).

THEOREM 3 *Let (M, g) be a smooth compact Riemannian n-manifold with $n \geq 2$ and let $x_0 \in M$. Then, at least one of the following occurs:*

i) *There exist extremal functions for the inequality* $LS\left(\frac{2}{n\pi e}, B_0\right)$

ii) $B_0 = \frac{\max_M \mathrm{Scal}_g}{2n\pi e}$.

One may notice that, on the n-dimensional standard sphere (S^n, h),

$$\mathrm{Vol}_h(S^n)^{-\frac{2}{n}} > \frac{\max_{S^n} \mathrm{Scal}_h}{2n\pi e}. \tag{1.7}$$

It follows that there exist extremal functions on (S^n, h) for all $n \geq 2$.

2 Proof of Theorem 1

The proof consists of four steps. First, one assumes Theorem 1 is false to obtain a sequence of minimizers for a family of variational problems. To get this family, one uses the fact that the inequality $LS(A,B)$ is the limit of $I_s(A,B)$ as $s \to 2$ and ideas of [6]. Once elementary properties on this sequence are shown, one projects the minimizers on the Euclidean space through the exponential map such that the projected sequence converges to an extremal function of the Euclidean logarithmic Sobolev inequality. Thereafter, one proves some sharp estimates involving the maximum principle and the Cartan expansion of the metric. The conclusion will then follow.

We shall denote by A_0 the constant $\frac{2}{n\pi e}$. For any $\varepsilon, \delta > 0$ and $1 < s < 2$, set

$$\mu_{s,\delta} = \inf_{u \in \mathcal{H}_\delta} \left(\int_{B_g(x_0,\delta)} |\nabla u|_g^2 dv_g + \alpha_\varepsilon \right) \left(\int_{B_g(x_0,\delta)} |u|^s dv_g \right)^{\frac{4}{n(2-s)}}$$

where $\alpha_\varepsilon = \frac{\mathrm{Scal}_g(x_0)}{4} + \varepsilon$ and

$$\mathcal{H}_\delta = \left\{ u \in C_c^\infty(B_g(x_0,\delta)) \text{ such that } \int_{B_g(x_0,\delta)} |u|^2 dv_g = 1 \right\}.$$

We proceed by contradiction. Assume that there exists ε_0 such that for all $\delta > 0$ there exists $u \in C_c^\infty(B_g(x_0,\delta))$ with $\|u\|_2 = 1$ and

$$\int_{B_g(x_0,\delta_\varepsilon)} u^2 \ln u^2 dv_g > \frac{n}{2} \ln \left[A_0 \left(\int_{B_g(x_0,\delta_\varepsilon)} |\nabla u|_g^2 dv_g + \alpha_{\varepsilon_0} \right) \right].$$

The latter is easily seen to be equivalent to the existence of $s_\delta > 1$ such that

$$\forall\, s \in [s_\delta, 2), \quad 1 > A_0 \left(\int_{B_g(x_0,\delta)} |\nabla u|_g^2 dv_g + \alpha_\varepsilon \right) \left(\int_{B_g(x_0,\delta)} |u|^s dv_g \right)^{\frac{4}{n(2-s)}}.$$

Since

$$\lim_{s \to 2} \left(\int_{B_g(x_0,\delta)} |u|^s dv_g \right)^{-\frac{4}{n(2-s)}} = \exp \frac{2}{n} \int_{B_g(x_0,\delta_\varepsilon)} u^2 \ln u^2 dv_g.$$

Hence

$$\mu_\delta = \mu_{s_\delta,\delta} < A_0^{-1}.$$

One may clearly choose s_δ such that $s_\delta \to 2$ as δ goes to 0. To simplify notations, we shall denote by s the constant s_δ. It is an easy matter to check (see for instance [20]) that there exists a minimizer $u_\delta \in H_{0,1}^2(B_g(x_0,\delta))$ satisfying

$$A_\delta \Delta_g u_\delta + \frac{2s}{n(2-s)} B_\delta u_\delta^{s-1} = k_\delta u_\delta \ \text{ on } \ B_g(x_0,\delta) \tag{2.1}$$

in the sense of distributions with

$$u_\delta \in C^\infty(B_g(x_0,\delta))$$
$$u_\delta > 0 \text{ on } B_g(x_0,\delta) \text{ and } u_\delta = 0 \text{ on } M \setminus B_g(x_0,\delta)$$

and where

$$\begin{aligned} A_\delta &= \left(\int_{B_g(x_0,\delta)} u_\delta^s dv_g\right)^{\frac{4}{n(2-s)}} \\ B_\delta &= \left(\int_{B_g(x_0,\delta)} |\nabla u_\delta|_g^2 dv_g + \alpha_\varepsilon\right)\left(\int_{B_g(x_0,\delta)} u_\delta^s dv_g\right)^{\frac{4}{n(2-s)}-1} \\ k_\delta &= \frac{2s}{n(2-s)}\mu_\delta + \left(\int_{B_g(x_0,\delta)} |\nabla u_\delta|_g^2 dv_g\right)\left(\int_{B_g(x_0,\delta)} u_\delta^s dv_g\right)^{\frac{4}{n(2-s)}}. \end{aligned}$$

The limits below are taken as δ goes to 0. Considering subsequences if necessary, one may assume that they all exist (they may be finite or infinite). Since

$$1 = \int_{B_g(x_0,\delta)} u_\delta^2 dv_g \leq \mathrm{Vol}_g(B_g(x_0,\delta))\|u_\delta\|_\infty^2,$$

one has $\lim \|u_\delta\|_\infty = +\infty$. Similarly, from Hölder's inequality,

$$\int_{B_g(x_0,\delta)} u_\delta^p dv_g \leq \mathrm{Vol}_g(B_g(x_0,\delta))^{1-\frac{p}{2}}$$

for all $p < 2$. As a consequence, it follows that $\lim A_\delta = 0$. Moreover, one can easily check that

$$\mu_\delta \to A_0^{-1} \tag{2.2}$$
$$A_\delta \int_{B_g(x_0,\delta)} |\nabla u_\delta|_g^2 dv_g \to A_0^{-1} \tag{2.3}$$
$$B_\delta A_\delta^{\frac{n(2-s)}{4}} \to A_0^{-1}. \tag{2.4}$$

This completes the first step of the proof.

Set $a_\delta = A_\delta^{\frac{1}{2}}$ and let $x_\delta \in B_g(x_0,\delta)$ be such that $u_\delta(x_\delta) = \|u_\delta\|_\infty$. We denote below by $C > 0$ a positive constant independent of δ which may change from line to line.

In the next step, we are going to project the sequence $(u_\delta)_\delta$ on the Euclidean space such that the obtained sequence converges to an extremal function of the Euclidean logarithmic Sobolev inequality (1.2).

Set

$$\Omega_\delta = a_\delta^{-1} \exp_{x_\delta}^{-1}(B_g(x_0,\delta))$$

and for all $x \in \Omega_\delta$,

$$g_\delta(x) = \exp_{x_\delta}^* g(a_\delta x)$$
$$\varphi_\delta(x) = \|u_\delta\|_\infty^{-1} u_\delta(\exp_{x_\delta}(a_\delta x)).$$

Moreover, for all $x \in \mathbb{R}^n \setminus \Omega_\delta$, set $\varphi_\delta(x) = 0$. Clearly,

$$\Delta_{g_\delta} \varphi_\delta + \frac{2s}{n(2-s)} \|u_\delta\|_\infty^{s-2} B_\delta \varphi_\delta^{s-1} = k_\delta \varphi_\delta \ \text{ on } \ \Omega_\delta \tag{2.5}$$

with $\varphi_\delta \in C^\infty(\Omega_\delta)$. One gets from the definition of φ_δ that

$$\int_{\Omega_\delta} \varphi_\delta^s dv_{g_\delta} = \left(\|u_\delta\|_\infty A_\delta^{\frac{n}{4}} \right)^{-s} \tag{2.6}$$

$$\int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta} = \left(\|u_\delta\|_\infty A_\delta^{\frac{n}{4}} \right)^{-2}. \tag{2.7}$$

Now, multiplying (2.5) by φ_δ and integrating over Ω_δ, it follows that

$$\int_{\Omega_\delta} |\nabla \varphi_\delta|_{g_\delta}^2 dv_{g_\delta} + \frac{2s}{n(2-s)} \|u_\delta\|_\infty^{s-2} B_\delta \int_{\Omega_\delta} \varphi_\delta^s dv_{g_\delta} = k_\delta \int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta}.$$

It is then an easy matter to check that

$$\frac{\left(\int_{\Omega_\delta} |\nabla \varphi_\delta|_{g_\delta}^2 dv_{g_\delta} \right) \left(\int_{\Omega_\delta} \varphi_\delta^s dv_{g_\delta} \right)^{\frac{4}{n(2-s)}}}{\left(\int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta} \right)^{1+\frac{2s}{n(2-s)}}} \to A_0^{-1}.$$

Hence, from (2.6) and (2.7),

$$\frac{\int_{\Omega_\delta} |\nabla \varphi_\delta|_{g_\delta}^2 dv_{g_\delta}}{\int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta}} \to A_0^{-1}.$$

Now set $\Phi_\delta = \frac{\varphi_\delta}{\|\varphi_\delta\|_{g_\delta,2}}$ with $\|\varphi_\delta\|_{g_\delta,2}^2 = \int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta}$. From now on, one denotes $\|u\|_{m,p}^p = \int_{\mathbb{R}^n} u^p dv_m$ for any measurable function u, real number p and metric m. One has from (2.5) that

$$\Delta_{g_\delta} \Phi_\delta + \frac{2s}{n(2-s)} \mu_\delta \Phi_\delta^{s-1} = k_\delta \Phi_\delta \ \text{ on } \ \Omega_\delta \tag{2.8}$$

and that

$$\int_{\Omega_\delta} |\nabla \Phi_\delta|_{g_\delta}^2 dv_{g_\delta} \to A_0^{-1}. \tag{2.9}$$

As a consequence of (2.8),

$$\Delta_{g_\delta} \Phi_\delta - (\mu_\delta - \alpha_{\varepsilon_0} A_\alpha) \Phi_\delta = \frac{2s}{n(2-s)} \mu_\delta \Phi_\delta \left(1 - \Phi_\delta^{s-2}\right) \ \text{ on } \ \Omega_\delta,$$

which leads to

$$\frac{2s}{n} \mu_\delta \Phi_\delta^{s-1} \ln \Phi_\delta \leq \Delta_{g_\delta} \Phi_\delta - \mu_\delta \Phi_\delta \leq \frac{2s}{n} \mu_\delta \Phi_\delta \ln \Phi_\delta \ \text{ on } \ \Omega_\delta. \tag{2.10}$$

In particular,

$$\Delta_{g_\delta} \Phi_\delta \leq A_0^{-1} \Phi_\delta^{1+\frac{4}{n}} \ \text{ on } \ \Omega_\delta \tag{2.11}$$

and

$$\Delta_{g_\delta}\Phi_\delta + \mu_\delta \Phi_\delta \le 2\mu_\delta \Phi_\delta^{1+\frac{s}{n}} \ \text{ on } \ \Omega_\delta. \tag{2.12}$$

Let us now prove the convergence of the sequence $(\Phi_\delta)_\delta$ in $C^2(B(0,1))$. The Cartan expansion of the metric g_δ in the exponential chart implies that

$$\begin{aligned} dv_{g_\delta} &\le \left(1 + Ca_\delta^2\right) dv_\xi \\ |\nabla \Phi_\delta|^2_{g_\delta} dv_{g_\delta} &\le \left(1 + Ca_\delta^2\right) |\nabla \Phi_\delta|^2_\xi dv_\xi \end{aligned}$$

where ξ is the Euclidean metric on $\mathbb{R}^n$. Hence, it follows from (2.9) that the sequence $(\Phi_\delta)_\delta$ is bounded in $H_1^2(\mathbb{R}^n)$. Therefore, there exists $\Phi \in H_1^2(\mathbb{R}^n)$ such that

$$\Phi_\delta \to \Phi \ \text{ weakly in } \ H_1^2(\mathbb{R}^n). \tag{2.13}$$

Applying Moser's iterative scheme to (2.11) (see for instance [19]), one gets that

$$\sup_{\Omega_\delta} \Phi_\delta \le C \|\Phi_\delta\|_{g_\delta, 2}. \tag{2.14}$$

One may notice that it is equivalent to

$$C \le \int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta}.$$

Moreover, Jensen's inequality and Cartan's expansion of the metric g_δ leads to

$$\begin{aligned} 0 = \lim -\frac{2}{(2-s)} \ln \int_{\Omega_\delta} \Phi_\delta^s dv_{g_\delta} &\le \lim \int_{\Omega_\delta} \Phi_\delta^2 \ln \Phi_\delta^2 dv_{g_\delta} \\ &\le \lim \int_{\Omega_\delta} \Phi_\delta^2 \ln \Phi_\delta^2 dv_\xi. \end{aligned}$$

Since $\varphi_\delta \le 1$, one may apply Fatou lemma. It follows that

$$0 \le \int_{\mathbb{R}^n} \Phi^2 \ln \Phi^2 dv_\xi$$

where, if it is necessary, one sets $0 \ln 0 = 0$. Therefore, from the Euclidean logarithmic Sobolev inequality, it follows that

$$0 \le \frac{n}{2} \ln A_0 \int_{\mathbb{R}^n} |\nabla \Phi|^2_\xi dv_\xi.$$

Hence

$$A_0^{-1} \le \int_{\mathbb{R}^n} |\nabla \Phi|^2_\xi dv_\xi.$$

Since $\|\nabla \Phi\|_{H_1^2} \le \liminf \|\nabla \Phi_\delta\|_{H_1^2}$, it follows from (2.9) that

$$\Phi_\delta \to \Phi \ \text{ strongly in } \ H_1^2(\mathbb{R}^n). \tag{2.15}$$

One also clearly has that

$$\int_{\mathbb{R}^n} \Phi^2 \ln \Phi^2 dv_\xi = \frac{n}{2} \ln A_0 \int_{\mathbb{R}^n} |\nabla \Phi|^2_\xi dv_\xi.$$

Consequently, Φ is an extremal function for the Euclidean logarithmic Sobolev inequality. E. Carlen proved in [4] that

$$\Phi(x) = ae^{-b|x|^2}$$

where a and b are positive real numbers. One now wants to compute the exact value of b. It is easy to check from (2.10) and (2.14) that $\|\Delta_{g_\delta}\Phi_\delta\|_{C^0(B_\xi(0,1))} \le C$. Hence, by classical elliptic theory (see [19]), the sequence $(\Phi_\delta)_\delta$ is equicontinuous. Then, Ascoli's theorem implies that

$$\Phi_\delta \to \Phi \ \text{ in } \ C^0(B_\xi(0,1)).$$

Moreover, $C^{-1} \le \Phi_\delta \le C$ on $B_\xi(0,1)$. Let $\alpha \in (0,1)$. One then has from (2.10)

$$\begin{aligned} \|\Delta_{g_\delta}\Phi_\delta\|_{C_{0,\alpha}(B(0,1))} &= \max_{\substack{x,y\in B(0,1)\\ x\neq y}} \frac{|\Delta_{g_\delta}\Phi_\delta(x) - \Delta_{g_\delta}\Phi_\delta(y)|}{|x-y|^\alpha} \\ &\le \max_{\substack{x,y\in B(0,1)\\ x\neq y}} \frac{|\Phi_\delta(x) - \Phi_\delta(y)|}{|x-y|^\alpha}\left(C + C\frac{|\ln\Phi_\delta(x) - \ln\Phi_\delta(y)|}{|\Phi_\delta(x) - \Phi_\delta(y)|}\right) \\ &\le C. \end{aligned}$$

Hence, it follows from [19] that $\|\Phi_\delta\|_{C^{2,\alpha}(B(0,1))} \le C$. Ascoli's theorem then gives

$$\Phi_\delta \to \Phi \ \text{ in } \ C^2(B(0,1)).$$

Since $\|\Phi\|_{\xi,2}^2 = 1$ and $\|\nabla\Phi\|_{\xi,2}^2 = A_0^{-1}$, one has $a = \left(\frac{2b}{\pi}\right)^{\frac{n}{4}}$ and $e = a^2\left(\frac{\pi}{2b}\right)^{\frac{n}{2}-1}$. It easily follows

$$\Phi(x) = e^{\frac{n}{4}}e^{-\frac{\pi e}{2}|x|^2}.$$

This ends the second step of the proof.

For our purpose, the convergence in $C^2(B(0,1))$ is not yet sufficient. In order to prove the relations (2.21) and (2.22), one needs a pointwise estimate for Φ_δ.
First, let us show that

$$\Phi_\delta(x) \le C|x|^{-\frac{n}{2}}. \tag{2.16}$$

One proceeds by contradiction and assumes that $\|\Phi_\delta|.|^{\frac{n}{2}}\|_\infty \to \infty$. Let $y_\delta \in \mathbb{R}^n$ be such that $\|\Phi_\delta|.|^{\frac{n}{2}}\|_\infty = \Phi_\delta(y_\delta)|y_\delta|^{\frac{n}{2}}$. Set

$$\begin{aligned} &\widetilde{\Omega}_\delta = \nu_\delta^{-1}\widehat{exp}_{y_\delta}^{-1}(\Omega_\delta) \\ &\nu_\delta^{-\frac{n}{2}} = \Phi_\delta(y_\delta) \\ &h_\delta(x) = \widehat{\exp}_{y_\delta}^{*} g_\delta(\nu_\delta x) \\ &v_\delta(x) = \Phi_\delta(y_\delta)^{-1}\Phi_\delta(\widehat{\exp}_{y_\delta}(\nu_\delta x)) \end{aligned}$$

where $\widehat{\exp}_{y_\delta}$ is the exponential map associated to g_δ at y_δ. It is clear that

$$\frac{|y_\delta|}{\nu_\delta} \to \infty \tag{2.17}$$

$$|y_\delta| \to \infty. \tag{2.18}$$

It is also an easy matter to check with (2.11) that

$$\Delta_{h_\delta} v_\delta \le A_0^{-1} v_\delta^{1+\frac{4}{n}} \text{ on } \widetilde{\Omega}_\delta. \tag{2.19}$$

Let $x \in B_{g_\delta}(y_\delta, \nu_\delta)$. One has

$$|x| = d_{g_\delta}(0,x) \ge d_{g_\delta}(0,y_\delta) - d_{g_\delta}(y_\delta, x) \ge d_{g_\delta}(0,y_\delta) - \Phi_\delta(y_\delta)^{-\frac{2}{n}} \ge |y_\delta| - \frac{1}{2}|y_\delta|$$

and by definition of y_δ,

$$\Phi_\delta(y_\delta)|y_\delta|^{\frac{n}{2}} \ge \Phi_\delta(x)|x|^{\frac{n}{2}}.$$

Hence, $\|\Phi_\delta\|_{L_\infty(B_{g_\delta}(y_\delta,\nu_\delta))} \le C\Phi_\delta(y_\delta)$. One can then apply again Moser's iterative scheme to (2.19) (see [19]) and get that

$$C \le \int_{B_\xi(0,1)\cap\widetilde{\Omega}_\delta} v_\delta^2 dv_{h_\delta} = \int_{B_{g_\delta}(y_\delta,\nu_\delta)\cap\Omega_\delta} \Phi_\delta^2 dv_{g_\delta}.$$

This however contradicts (2.15), (2.17) and (2.18). Hence, $\Phi_\delta(x) \le C|x|^{-\frac{n}{2}}$ for all $x \in \mathbb{R}^n$.
Define then the operator L_δ as

$$L_\delta u = \Delta_{g_\delta} u + \mu_\delta \left(1 - 2\Phi_\delta^{\frac{s}{n}}\right) u.$$

Set

$$H(x) = \left(\frac{R}{|x|}\right)^\omega$$

where $\omega, R > 0$. A direct computation using local maps leads to

$$|x|^2 \frac{L_\delta H(x)}{H(x)} \ge \omega(n-2-\omega) - Ca_\delta^2|x|^2 + \mu_\delta|x|^2 - 2\mu_\delta\Phi_\delta^{\frac{s}{n}}(x)|x|^2.$$

Therefore, by (2.16), for δ small enough and R large enough,

$$L_\delta H \ge 0 \text{ on } \Omega_\delta - B_\xi(0,R).$$

Since $L_\delta\Phi_\delta \le 0$ by (2.12), one may apply the maximum principle as stated in [13] lemma 3.4. It follows

$$\Phi_\delta \le C\left(\frac{R}{|x|}\right)^\omega \text{ on } \Omega_\delta - B_\xi(0,R). \tag{2.20}$$

Since this is obviously true on $B_\xi(0,R)$, it holds on Ω_δ. One ends here the third step.

Before concluding, we need an asymptotic expansion for the quantities $\int_{\Omega_\delta}|\nabla\Phi_\delta|_\xi^2 dv_\xi$ and $\int_{\Omega_\delta}\Phi_\delta^\sigma dv_\xi$ with $1 < \sigma \le 2$ as δ goes to 0. To find these expansions, one needs the following results.

$$\frac{\int_{\Omega_\delta}|\nabla\Phi_\delta|_{g_\delta}^2 Ric_g(x_\delta)_{ij}x^ix^j dv_{g_\delta}}{\int_{\Omega_\delta}|\nabla\Phi_\delta|_{g_\delta}^2 dv_{g_\delta}} \to \frac{\int_{\mathbb{R}^n}|\nabla\Phi|_\xi^2 \mathrm{Ric}_g(x_0)_{ij}x^ix^j dv_\xi}{\int_{\mathbb{R}^n}|\nabla\Phi|_\xi^2 dv_\xi} \tag{2.21}$$

$$\int_{\Omega_\delta}\Phi_\delta^\sigma \mathrm{Ric}_g(x_\delta)_{ij}x^ix^j dv_{g_\delta} \to \int_{\mathbb{R}^n}\Phi^\sigma \mathrm{Ric}_g(x_0)_{ij}x^ix^j dv_\xi \tag{2.22}$$

for all $1 < \sigma \le 2$. The last relation is easily obtained with (2.15) and (2.20). To show the first one, it is sufficient to have

$$\int_{\Omega_\delta - B_\xi(0,c_\delta)} |\nabla\Phi_\delta|^2_{g_\delta}|x|^2 dv_{g_\delta} \to 0$$

where $(c_\delta)_\delta$ is a sequence of real numbers verifying $c_\delta \to \infty$. In order to prove it, one multiplies (2.11) by $\Phi_\delta|x|^2$ and integrates over $\Omega_\delta - B_\xi(0,c_\delta)$. A direct computation using (2.20) and (2.22) gives the result.
Noticing that for any radial function f

$$\int_{\mathbb{R}^n} f x^i x^j dv_\xi = \frac{\delta^{ij}}{n}\int_{\mathbb{R}^n} f|x|^2 dv_\xi,$$

one then gets that

$$\frac{\int_{\Omega_\delta} |\nabla\Phi_\delta|^2_{g_\delta}\mathrm{Ric}_g(x_\delta)_{ij}x^i x^j dv_{g_\delta}}{\int_{\Omega_\delta}|\nabla\Phi_\delta|^2_{g_\delta}dv_{g_\delta}} \to \frac{\mathrm{Scal}_g(x_0)}{n}\frac{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi|x|^2dv_\xi}{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi dv_\xi} \tag{2.23}$$

$$\int_{\Omega_\delta}\Phi_\delta^\sigma \mathrm{Ric}_g(x_\delta)_{ij}x^i x^j dv_{g_\delta} \to \frac{\mathrm{Scal}_g(x_0)}{n}\int_{\mathbb{R}^n}\Phi^\sigma|x|^2 dv_\xi. \tag{2.24}$$

Let us notice that some easy computations leads to

$$\frac{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi|x|^2dv_\xi}{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi dv_\xi} = \frac{n+2}{2\pi e} \tag{2.25}$$

$$\int_{\mathbb{R}^n}\Phi^\sigma|x|^2dv_\xi = \frac{n}{\sigma\pi e}\left(\frac{2}{\sigma}e^{\frac{\sigma}{2}}\right)^{\frac{n}{2}}. \tag{2.26}$$

We now have all the tools required to conclude. The Euclidean Gagliardo-Nirenberg inequalities state that

$$\left(\int_{\mathbb{R}^n}\Phi_\delta^2 dv_\xi\right)^{1+\frac{2\sigma}{n(2-\sigma)}} \le A_0\int_{\mathbb{R}^n}|\nabla\Phi_\delta|^2_\xi dv_\xi\left(\int_{\mathbb{R}^n}\Phi_\delta^\sigma dv_\xi\right)^{\frac{4}{n(2-\sigma)}} \tag{2.27}$$

for all $\sigma \in (1,2)$. To see why the constant A_0 is fitting, one may refer for instance to [3] or [12]. The Cartan expansion of the metric g_δ at 0 gives

$$dv_\xi = \left(1+\frac{a_\delta^2}{6}\mathrm{Ric}_g(x_\delta)_{ij}x^ix^j + o(a_\delta^2|x|^2)\right)dv_{g_\delta}$$

$$|\nabla\Phi_\delta|^2_\xi = |\nabla\Phi_\delta|^2_{g_\delta} + \frac{a_\delta^2}{6}\mathrm{Rm}_g(x_\delta)(\nabla\Phi_\delta,x,x,\nabla\Phi_\delta) + o(|\nabla\Phi_\delta|^2_{g_\delta}a_\delta^2|x|^2).$$

Hence, one gets with (2.23) and (2.24) that

$$\int_{\Omega_\delta}|\nabla\Phi_\delta|^2_\xi dv_\xi = \int_{\Omega_\delta}|\nabla\Phi_\delta|^2_{g_\delta}dv_{g_\delta}\Bigg(1+\frac{a_\delta^2\mathrm{Scal}_g(x_0)}{6n}\frac{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi|x|^2dv_\xi}{\int_{\mathbb{R}^n}|\nabla\Phi|^2_\xi dv_\xi} + \frac{a_\delta^2}{6}\frac{\int_{\Omega_\delta}\mathrm{Rm}_g(x_\delta)(\nabla\Phi_\delta,x,x,\nabla\Phi_\delta)dv_{g_\delta}}{\int_{\Omega_\delta}|\nabla\Phi_\delta|^2_{g_\delta}dv_{g_\delta}} + o(a_\delta^2)\Bigg)$$

$$\int_{\Omega_\delta}\Phi_\delta^\sigma dv_\xi = 1 + \frac{a_\delta^2\mathrm{Scal}_g(x_0)}{6n}\int_{\mathbb{R}^n}\Phi^\sigma|x|^2dv_\xi + o(a_\delta^2). \tag{2.28}$$

Since Φ is radial, $\nabla\Phi$ and x are pointwise colinear vector fields and one has

$$\int_{\Omega_\delta} \mathrm{Rm}_g(x_\delta)(\nabla\Phi_\delta, x, x, \nabla\Phi_\delta)dv_{g_\delta} \to 0.$$

Hence,

$$\int_{\Omega_\delta} |\nabla\Phi_\delta|_\xi^2 dv_\xi = \int_{\Omega_\delta} |\nabla\Phi_\delta|_{g_\delta}^2 dv_{g_\delta} \left(1 + \frac{a_\delta^2 \mathrm{Scal}_g(x_0)}{6n} \frac{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 |x|^2 dv_\xi}{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 dv_\xi} + o(a_\delta^2)\right) \quad (2.29)$$

Moreover, relations (2.6), (2.7) and the definitions of A_δ, B_δ and k_δ lead to

$$\int_{\Omega_\delta} |\nabla\Phi_\delta|_{g_\delta}^2 dv_{g_\delta} = \frac{\int_{\Omega_\delta} |\nabla\varphi_\delta|_{g_\delta}^2 dv_{g_\delta} \left(\int_{\Omega_\delta} \varphi_\delta^s dv_{g_\delta}\right)^{\frac{4}{n(2-s)}}}{\left(\int_{\Omega_\delta} \varphi_\delta^2 dv_{g_\delta}\right)^{1+\frac{2s}{n(2-s)}}} \le \mu_\delta - \alpha_{\varepsilon_0} a_\delta^2. \quad (2.30)$$

Therefore, one may easily get from relations (2.27) to (2.30) that for all $\sigma \in (1,2)$,

$$\left(\alpha_{\varepsilon_0} A_0 - \frac{\mathrm{Scal}_g(x_0)}{6n}\left(\frac{4}{n(2-\sigma)}\int_{\mathbb{R}^n} \Phi^\sigma |x|^2 dv_\xi - \left(1 + \frac{2\sigma}{n(2-\sigma)}\right)\int_{\mathbb{R}^n} \Phi^2 |x|^2 dv_\xi \right.\right.$$
$$\left.\left. + \frac{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 |x|^2 dv_\xi}{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 dv_\xi}\right)\right) a_\delta^2 + o(a_\delta^2) \le 0.$$

Consequently, for all $\sigma \in (1,2)$,

$$\alpha_{\varepsilon_0} A_0 - \frac{\mathrm{Scal}_g(x_0)}{6n}\left(\frac{4}{n(2-\sigma)}\int_{\mathbb{R}^n} \Phi^\sigma |x|^2 dv_\xi - \left(1 + \frac{2\sigma}{n(2-\sigma)}\right)\int_{\mathbb{R}^n} \Phi^2 |x|^2 dv_\xi \right.$$
$$\left. + \frac{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 |x|^2 dv_\xi}{\int_{\mathbb{R}^n} |\nabla\Phi|_\xi^2 dv_\xi}\right) \le 0.$$

Using (2.25) and (2.26) and letting σ go to 2, one finally gets that

$$\alpha_{\varepsilon_0} A_0 - \frac{\mathrm{Scal}_g(x_0)}{2n\pi e} \le 0$$

i.e.

$$\varepsilon_0 \le 0.$$

This contradiction proves the theorem.

3 Proof of Theorem 3

This section has many similarities with the previous one. However, the first step shall be well detailled to make the comprehension easier. Again, we shall denote by A_0 the constant $\frac{2}{n\pi e}$. In order to prove the theorem, we will show that the non-existence of extremal functions for $LS(A_0, B_0)$ implies that $B_0 \le \frac{\max_M \mathrm{Scal}_g}{2n\pi e}$. For any $1 < s < 2$ and $\alpha > 0$, set

$$I_{s,\alpha}(u) = \frac{\left(\int_M |\nabla u|_g^2 dv_g + (\alpha_0 - \alpha)\int_M u^2 dv_g\right)\left(\int_M |u|^s dv_g\right)^{\frac{4}{n(2-s)}}}{\left(\int_M |u|^2 dv_g\right)^{1+\frac{2s}{n(2-s)}}}$$

and

$$\mu_{s,\alpha} = \inf_{u \in \mathcal{H}} I_{s,\alpha}(u)$$

where $\alpha_0 = B_0 A_0^{-1}$ and

$$\mathcal{H} = \left\{ u \in C^\infty(M) \text{ such that } \int_M |u|^2 dv_g = 1 \right\}.$$

Let $(s_\alpha)_\alpha$ be such that $s_\alpha < 2$ and $s_\alpha \to 2$ as α goes to 0. To simplify notations, one shall denote by s, μ_α and $I_\alpha(u)$ the real numbers s_α, $\mu_{s,\alpha}$ and $I_{s,\alpha}(u)$. It is clear that $\mu_\alpha < A_0^{-1}$ (see [3]). It is classical (see for instance [20]) that there exists a minimizer $u_\alpha \in H_1^2(M)$ satisfying

$$A_\alpha \Delta_g u_\alpha + \frac{2s}{n(2-s)} B_\alpha u_\alpha^{s-1} = k_\alpha u_\alpha \tag{3.1}$$

in the sense of distributions with

$$\begin{aligned} &u_\alpha \in \mathcal{H} \\ &u > 0 \end{aligned}$$

and where

$$\begin{aligned} A_\alpha &= \left(\int_M u_\alpha^s dv_g \right)^{\frac{4}{n(2-s)}} \\ B_\alpha &= \left(\int_M |\nabla u_\alpha|_g^2 dv_g + (\alpha_0 - \alpha) \right) \left(\int_M u_\alpha^s dv_g \right)^{\frac{4}{n(2-s)} - 1} \\ k_\alpha &= \frac{2s}{n(2-s)} \mu_\alpha + \left(\int_M |\nabla u_\alpha|_g^2 dv_g \right) \left(\int_M u_\alpha^s dv_g \right)^{\frac{4}{n(2-s)}}. \end{aligned}$$

The limits below are taken as α goes to 0. Considering subsequences if necessary, one may assume that they all exist (they may be finite or infinite). One denotes below by C a constant independent of α which may change from line to line. If we had

$$\left(\int_M u_\alpha^s dv_g \right)^{\frac{4}{n(2-s)}} \geq C,$$

then we would have

$$\int_M |\nabla u_\alpha|_g^2 dv_g \leq C$$

and it would then be easy to prove the existence of extremal functions. Since we assume there is no extremal functions,

$$\left(\int_M u_\alpha^s dv_g \right)^{\frac{4}{n(2-s)}} \to 0.$$

As a consequence, $\lim A_\alpha = 0$. Moreover, one can then easily check that

$$\begin{aligned} &\mu_\alpha \to A_0^{-1} \\ &A_\alpha \int_M |\nabla u_\alpha|_g^2 dv_g \to A_0^{-1} \\ &B_\alpha A_\alpha^{\frac{n(2-s)}{4}} \to A_0^{-1}. \end{aligned}$$

Set $a_\alpha = A_\alpha^{\frac{1}{2}}$ and let $x_\alpha \in M$ be such that $u_\alpha(x_\alpha) = \|u_\alpha\|_\infty$.

As previously, one projects the minimizers on the Euclidean space such to obtain a sequence converging to an extremal function of the optimal Euclidean logarithmic Sobolev inequality. Set

$$\begin{aligned}
&\Omega_\alpha = a_\alpha^{-1} \exp_{x_\alpha}^{-1} (M \setminus Cut(x_\alpha)) \\
&g_\alpha(x) = \exp_{x_\alpha}^* g(a_\alpha x) \\
&\varphi_\alpha(x) = \|u_\alpha\|_\infty^{-1} u_\alpha(\exp_{x_\alpha}(a_\alpha x)) \\
&\varphi_\alpha \equiv 0 \ \text{ on } \ \mathbb{R}^n \setminus \Omega_\alpha
\end{aligned}$$

Most of the results of the previous section hold here. Actually, one has

$$\Delta_{g_\alpha} \varphi_\alpha + \frac{2s}{n(2-s)} \|u_\alpha\|_\infty^{s-2} B_\alpha \varphi_\alpha^{s-1} = k_\alpha \varphi_\alpha \ \text{ on } \ \Omega_\alpha \tag{3.2}$$

with $\varphi_\alpha \in C^\infty(\Omega_\alpha)$. One also has

$$\int_{\Omega_\alpha} \varphi_\alpha^s dv_{g_\alpha} = \left(\|u_\alpha\|_\infty A_\alpha^{\frac{n}{4}} \right)^{-s}$$

$$\int_{\Omega_\alpha} \varphi_\alpha^2 dv_{g_\alpha} = \left(\|u_\alpha\|_\infty A_\alpha^{\frac{n}{4}} \right)^{-2}$$

and

$$\begin{aligned}
&\varphi_\alpha \to \varphi \ \text{ in } \ H_1^2(\mathbb{R}^n) \ \text{ and in } \ C^1(K) \\
&\varphi_\alpha \le C|x|^\omega \ \text{ for all } \ \omega > 0, \ \ C \ \text{ depending on } \ \omega
\end{aligned}$$

where

$$\begin{aligned}
&\varphi(x) = e^{-\frac{\pi e}{2}|x|^2} \\
&K \ \text{ is a compact subset of } \ \mathbb{R}^n.
\end{aligned}$$

Moreover,

$$\frac{\int_{\Omega_\alpha} |\nabla \varphi_\alpha|_{g_\alpha}^2 \text{Ric}_g(x_\alpha)_{ij} x^i x^j dv_{g_\alpha}}{\int_{\Omega_\alpha} |\nabla \varphi_\alpha|_{g_\alpha}^2 dv_{g_\alpha}} \to \frac{(n+2)\text{Scal}_g(x_0)}{2\pi e n} \tag{3.3}$$

$$\frac{\int_{\Omega_\alpha} \varphi_\alpha^2 \text{Ric}_g(x_\alpha)_{ij} x^i x^j dv_{g_\alpha}}{\int_{\Omega_\alpha} \varphi_\alpha^\sigma dv_{g_\alpha}} \to \frac{\text{Scal}_g(x_0)}{2\pi e}. \tag{3.4}$$

It is very easy to see that

$$u_\alpha(x) \le C a_\alpha^{\omega - \frac{n}{2}} d_g(x, x_\alpha)^\omega \ \text{ for all } \ \omega > 0. \tag{3.5}$$

Set then for all positive $u \in H_1^2(M)$

$$I_{g,\alpha}(u) = \frac{\int_M |\nabla u|_g^2 dv_g \left(\int_M u^s dv_g \right)^{\frac{4}{n(2-s)}}}{\left(\int_M u^2 dv_g \right)^{1 + \frac{2s}{n(2-s)}}}.$$

Let us first remark that

$$\frac{A_0^{-1} - I_{g,\alpha}(u_\alpha)}{A_\alpha} \to \alpha_0. \tag{3.6}$$

From the definition of $I_{g,\alpha}$ and the relation $I_\alpha(u_\alpha) < A_0^{-1}$, it follows that

$$\lim_{\alpha\to 0} \frac{A_0^{-1} - I_{g,\alpha}(u_\alpha)}{A_\alpha} \geq \alpha_0.$$

One has

$$\frac{A_0^{-1} - I_{g,\alpha}(u_\alpha)}{A_\alpha} = \frac{A_0^{-1} - (I_{g,\alpha}(u_\alpha) + \alpha_0 A_\alpha) + \alpha_0 A_\alpha}{A_\alpha}.$$

Since, by definition of α_0, $I_{g,\alpha}(u_\alpha) + \alpha_0 A_\alpha \leq A_0^{-1}$, one gets

$$\lim_{\alpha\to 0} \frac{A_0^{-1} - I_{g,\alpha}(u_\alpha)}{A_\alpha} \leq \alpha_0.$$

Hence (3.6) follows.
Let us now show that

$$\lim_{\alpha\to 0} \frac{I_{\xi,\alpha}(u_\alpha) - I_{g,\alpha}(u_\alpha)}{A_\alpha} \leq \frac{\mathrm{Scal}_g(x_0)}{4} \tag{3.7}$$

where ξ represents the Euclidean metric. We can write

$$\begin{aligned} I_{\xi,\alpha}(u_\alpha) &- I_{g,\alpha}(u_\alpha) \\ &= I_{g,\alpha}(u_\alpha) \frac{\frac{\int_M |\nabla u_\alpha|_\xi^2 dv_\xi}{\int_M |\nabla u_\alpha|_g^2 dv_g} \left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g}\right)^{\frac{4}{n(2-s)}}}{\left(\frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g}\right)^{1+\frac{2s}{n(2-s)}}} - I_{g,\alpha}(u_\alpha) \\ &= I_{g,\alpha}(u_\alpha) \frac{\frac{\int_M |\nabla u_\alpha|_\xi^2 dv_\xi}{\int_M |\nabla u_\alpha|_g^2 dv_g} \left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g} \frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\right)^{\frac{4}{n(2-s)}}}{\left(\frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g}\right)^{1-\frac{2}{n}}} - I_{g,\alpha}(u_\alpha). \end{aligned} \tag{3.8}$$

The Cartan expansion of the metric g in the exponential map yields

$$\begin{aligned} dv_\xi &= \left(1 + \frac{1}{6}\mathrm{Ric}_{ij}(x_\alpha)x^i x^j + O(r_\alpha^3)\right) dv_g \\ |\nabla u_\alpha|_\xi^2 &= |\nabla u_\alpha|_g^2 + \frac{1}{6}\mathrm{Rm}_g(x_\alpha)\left(\nabla u_\alpha, x, x, \nabla u_\alpha\right) + |\nabla u_\alpha|_g^2 O(r_\alpha^3) \end{aligned}$$

where $r_\alpha = d_g(x, x_\alpha)$. These relations implies

$$\begin{aligned} \int_M |\nabla u_\alpha|_\xi^2 dv_\xi &= \int_M |\nabla u_\alpha|_g^2 dv_g + \frac{1}{6}\int_M |\nabla u_\alpha|_g^2 \mathrm{Ric}_{ij}(x_\alpha)x^i x^j dv_g \\ &\quad + O(1)\int_M |\nabla u_\alpha|_g^2 r_\alpha^3 dv_g \\ \int_M u_\alpha^2 dv_\xi &= \int_M u_\alpha^2 dv_g + \frac{1}{6}\int_M u_\alpha^2 \mathrm{Ric}_{ij}(x_\alpha)x^i x^j dv_g + O(1)\int_M u_\alpha^2 r_\alpha^3 dv_g \end{aligned}$$

where x^i is the i-th coordonate in the exponential map on x_0. Hence, with the definition of φ_α, one gets that

$$\frac{\int_M |\nabla u_\alpha|^2_\xi dv_\xi}{\int_M |\nabla u_\alpha|^2_g dv_g} = 1 + \frac{a_\alpha^2}{6} \frac{\int_{\Omega_\alpha} |\nabla \varphi_\alpha|^2_{g_\alpha} \mathrm{Ric}_{ij}(x_\alpha) x^i x^j dv_{g_\alpha}}{\int_{\Omega_\alpha} |\nabla \varphi_\alpha|^2_{g_\alpha} dv_{g_\alpha}} + o(a_\alpha^2)$$

$$\frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g} = 1 + \frac{a_\alpha^2}{6} \frac{\int_{\Omega_\alpha} \varphi_\alpha^2 \mathrm{Ric}_{ij}(x_\alpha) x^i x^j dv_{g_\alpha}}{\int_{\Omega_\alpha} \varphi_\alpha^2 dv_{g_\alpha}} + o(a_\alpha^2).$$

Now,

$$\begin{aligned}
&\left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g} \frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\right)^{\frac{4}{n(2-s)}} \\
&= \left(1 + \frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g} - \frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g}\right)\right)^{\frac{4}{n(2-s)}} \\
&\leq \exp\left(\frac{4}{n(2-s)} \frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g} - \frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g}\right)\right) \\
&\leq \exp\left(\frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi} \frac{4}{n(2-s)} \int_M \left(\left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^s - \left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^2\right) A_\alpha^{-\frac{n}{2}} dv_\xi\right).
\end{aligned}$$

From the relation (3.5), one gets

$$\left| \frac{1}{2-s} \int_{\Omega_\alpha} \left(\phi_\alpha^s - \phi_\alpha^2\right) |x|^3 dv_{g_\alpha} \right| < C$$

$$\left| \frac{1}{2-s} \int_{\Omega_\alpha} \left(\phi_\alpha^s - \phi_\alpha^2\right) \mathrm{Ric}_{ij}(x_\alpha) x^i x^j dv_{g_\alpha} \right| < C.$$

Hence

$$\begin{aligned}
&\frac{4}{n(2-s)} \int_M \left(\left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^s - \left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^2\right) A_\alpha^{-\frac{n}{2}} dv_\xi \\
&= \frac{4}{n(2-s)} \int_M \left(\left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^s - \left(\frac{u_\alpha}{A_\alpha^{-\frac{n}{4}}}\right)^2\right) A_\alpha^{-\frac{n}{2}} \left(1 + \frac{1}{6}\mathrm{Ric}_{ij}(x_\alpha) x^i x^j + O(r_\alpha^3)\right) dv_g \\
&= \frac{4}{n(2-s)} \int_{\Omega_\alpha} \left(\phi_\alpha^s - \phi_\alpha^2\right) \left(1 + \frac{a_\alpha^2}{6}\mathrm{Ric}_{ij}(x_\alpha) x^i x^j + O(a_\delta^3 |x|^3)\right) dv_{g_\alpha} \\
&= \frac{4a_\alpha^2}{n6(2-s)} \int_{\Omega_\alpha} \left(\phi_\alpha^s - \phi_\alpha^2\right) \mathrm{Ric}_{ij}(x_\alpha) x^i x^j dv_{g_\alpha} + o(a_\alpha^2)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g}\frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\right)^{\frac{4}{n(2-s)}}\\
&\qquad\qquad \le \exp\left(\frac{4a_\alpha^2}{n6(2-s)}\int_{\Omega_\alpha}\left(\phi_\alpha^s-\phi_\alpha^2\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}+o(a_\alpha^2)\right)\\
&\qquad\qquad \le 1+\frac{4a_\alpha^2}{n6(2-s)}\int_{\Omega_\alpha}\left(\phi_\alpha^s-\phi_\alpha^2\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}+o(a_\alpha^2)\\
&\qquad\qquad +\sum_{k=2}^{+\infty}\frac{1}{k!}\left(\frac{4a_\alpha^2}{n6(2-s)}\int_{\Omega_\alpha}\left(\phi_\alpha^s-\phi_\alpha^2\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}+o(a_\alpha^2)\right)^k\\
&\qquad\qquad \le 1+\frac{4a_\alpha^2}{n6(2-s)}\int_{\Omega_\alpha}\left(\phi_\alpha^s-\phi_\alpha^2\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}+o(a_\alpha^2).
\end{aligned}
$$

Some computations then leads to

$$
\frac{\frac{\int_M |\nabla u_\alpha|_\xi^2 dv_\xi}{\int_M |\nabla u_\alpha|_g^2 dv_g}\left(\frac{\int_M u_\alpha^s dv_\xi}{\int_M u_\alpha^s dv_g}\frac{\int_M u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_\xi}\right)^{\frac{4}{n(2-s)}}}{\left(\frac{\int_M u_\alpha^2 dv_\xi}{\int_M u_\alpha^2 dv_g}\right)^{1-\frac{2}{n}}}\le 1+\frac{a_\alpha^2}{6}X_\alpha+o(a_\alpha^2) \tag{3.9}
$$

where

$$
\begin{aligned}
X_\alpha &= \frac{\int_{\Omega_\alpha}|\nabla\varphi_\alpha|_{g_\alpha}^2\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}}{\int_{\Omega_\alpha}|\nabla\varphi_\alpha|_{g_\alpha}^2 dv_{g_\alpha}}\\
&- \left(1-\frac{2}{n}\right)\frac{\int_{\Omega_\alpha}\varphi_\alpha^2\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}}{\int_{\Omega_\alpha}\varphi_\alpha^2 dv_{g_\alpha}}\\
&+ \frac{4}{n(2-s)}\int_{\Omega_\alpha}\left(\phi_\alpha^s-\phi_\alpha^2\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}.
\end{aligned}
$$

One has

$$
\begin{aligned}
&\left|\frac{2}{2-s}\int_{\Omega_\alpha}\left(\phi_\alpha^2-\phi_\alpha^s\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}-\int_{\Omega_\alpha}\left(\ln\phi_\alpha^2\right)\phi_\alpha^2\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}\right|\\
&\qquad\qquad\le C(2-s)\int_{\Omega_\alpha}\left(\ln\phi_\alpha\right)^2\left(\max_{\beta\in[s,2]}\phi_\alpha^\beta\right)|x|^2 dv_{g_\alpha}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\lim\frac{2}{2-s}\int_{\Omega_\alpha}\left(\phi_\alpha^2-\phi_\alpha^s\right)\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}\\
&\qquad\qquad=\lim\int_{\Omega_\alpha}\left(\ln\phi_\alpha^2\right)\phi_\alpha^2\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha}.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
\lim\int_{\Omega_\alpha}\left(\ln\phi_\alpha^2\right)\phi_\alpha^2\mathrm{Ric}_{ij}(x_\alpha)x^ix^j dv_{g_\alpha} &= \frac{\mathrm{Scal}_g(x_0)}{n}\int_{\mathbb{R}^n}\left(\ln\phi^2\right)\phi^2|x|^2 dx\\
&= -\frac{n}{2\pi e}.
\end{aligned}
$$

One may now easily check with (3.3) and (3.4) that

$$\lim \frac{X_\alpha}{6} = \frac{\mathrm{Scal}_g(x_0)}{2n\pi e}.$$

Hence, by (3.8) and (3.9), it leads to (3.7). On has (see [3]) $I_{\xi,\alpha}(u_\alpha) \geq A_0^{-1}$. Therefore

$$\frac{I_{\xi,\alpha}(u_\alpha) - I_{s,\alpha}(u_\alpha)}{A_\alpha} \geq \frac{A_0^{-1} - I_{s,\alpha}(u_\alpha)}{A_\alpha}.$$

Letting α goes to 0, one gets from (3.6) and (3.7) that

$$\alpha_0 \leq \frac{\mathrm{Scal}_g(x_0)}{4}.$$

This ends the proof of Theorem 3.

References

[1] T. Aubin. Problèmes isopérimétrique et espaces de Sobolev. *Journal of differential geometry*, 11:573–598, 1976.

[2] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Ecole d'Eté de Probabilités de St-Flour*, volume 1581 of *Lecture Notes in math.*, pages 1–114. Springer, 1994.

[3] C. Brouttelande. The best constant problem for a family of Gagliardo-Nirenberg inequalities on a compact Riemannian manifold. To appear in Proc. of the Edinburgh Math. Soc.

[4] E. Carlen. Superadditivity of Fisher's information and logarithmic Sobolev inequalities. *Indiana J. Math.*, 101:194–211, 1991.

[5] O. Druet. Sharp local isoperimetric inequalities involving the scalar curvature. to appear in Proc. of the AMS.

[6] O. Druet. The best constant problem in Sobolev inequalities. *Mathematische Annalen*, 314(2):327–346, 1999.

[7] Z. Djadli O. Druet. Extremal functions for optimal Sobolev inequalities on compact manifolds. *Calculus of Variations and PDE's*, 12(1):59–84, 2001.

[8] L. Gross. Logarithmic Sobolev inequalities. *Amer. J. Math.*, 97:1061–1083, 1975.

[9] E. Hebey. *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, volume 5 of *Courant Lecture Note in Mathematics*. CIMS, 1999.

[10] E. Humbert. *Inégalités optimales de types Nash et Sobolev en géométrie riemannienne*. PhD thesis, Université de Paris VI - Pierre et Marie CURIE, 2000.

[11] E. Humbert. Best constants in the L^2-Nash inequality. *Proc. of the Edinburgh Math. Soc.*, 131(3):621–646, 2001.

[12] M. Ledoux. The geometry of Markov diffusion generators. *Ann. de la Fac. des Sciences de Toulouse*, (3):305–366, 2000.

[13] T. Aubin Y. Y. Li. On the best Sobolev inequality. *J. Math. Pures Appl.*, 78:353–387, 1999.

[14] E. Carlen M. Loss. Sharp constant in Nash's inequality. *Duke Math. J.*, 7:213–215, 1993.

[15] M. Berger P. Gauduchon E. Mazet. *Le spectre d'une variété Riemannienne*, volume 194 of *Lecture Notes in math.* Springer, 1971.

[16] J. Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[17] J. Nash. Continuity of solutions of parabolic equations. *Amer. J. Math.*, 80:931–954, 1958.

[18] D. Bakry T. Coulhon M. Ledoux L. Saloff-Coste. Sobolev inequalities in disguise. *Indiana J. Math.*, 44:1033–1074, 1995.

[19] D. Gilbarg N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften*. Springer Verlag, 1977.

[20] O. Druet E. Hebey M. Vaugon. Optimal Nash's inequalities on Riemannian manifolds: the influence of geometry. *Internat. Math. Re. Notices*, 14:735–779, 1999.

Author : Christophe BROUTTELANDE
Title : Optimal Gagliardo-Nirenberg inequalities on Riemannian manifolds

Abstract

Sobolev spaces are inherent to the theory of PDEs. The embedding theorems of these spaces into the Lebesgue spaces gives the Sobolev inequalities. S. L. Sobolev introduced these notions in the 30s and they are now a fundamental tool in analysis. Some other mathematicians developed this topic.One may cite E. Gagliardo and L. Nirenberg in the 50s.

The study of optimal Sobolev inequalities began with great problems in analysis such as the Yamabe problem. There exist different approaches. We are interested in two of them : the AB program and the BA program. The first one was studied by T. Aubin, O. Druet, E. Hebey and M. Vaugon. D. Bakry and M. Ledoux studied the second one in Markov semi-group theory.

Sobolev inequalities are a sub-family of Gagliardo-Nirenberg inequalities. So it is natural to ask if an extension of the previous works is possible. First, a positive answer was given for Nash and Sobolev logarithmic inequalities. In this thesis, we adapt the AB program and the BA program to a larger sub-family of Gagliardo-Nirenberg inequalities including the Nash inequality.

Auteur : Christophe BROUTTELANDE
Titre de la thèse : Inégalités de Gagliardo-Nirenberg optimales sur les variétés riemanniennes
Directeur de thèse : Michel LEDOUX
Date et lieu de soutenance : Le 30 juin 2003 à l'université Paul Sabatier Toulouse III

Résumé

Les espaces de Sobolev jouent un rôle central dans la théorie des équations aux dérivées partielles. Les théorèmes de plongement de ces espaces dans les espaces de Lebesgue se traduisent en inégalités dites de Sobolev. Elles sont devenues un outil fondamental en analyse. Ces notions ont été introduites par S. L. Sobolev à la fin des années 30. D'autres mathématiciens se sont intéressés à ce domaine. On peut notamment citer les travaux d'E. Gagliardo et L. Nirenberg dans les années 50.

L'étude des inégalités de Sobolev optimales trouve ses origines dans de grands problèmes d'analyse tels que le problème de Yamabe. Il existe plusieurs façons d'aborder cette étude. Nous parlerons plus particulièrement de programme AB et de programme BA. Le premier programme a été étudié, entre autre, par T. Aubin, O. Druet, E. Hebey et M. Vaugon. Le second trouve sa source en théorie des semi-groupes de Markov. Il a notamment été étudié par D. Bakry et M. Ledoux.

Les inégalités de Sobolev sont un cas particulier des inégalités de Gagliardo-Nirenberg. Il est donc naturel de se demander si les résultats connus pour les inégalités de Sobolev s'adaptent aux autres inégalités de la famille. Les premiers travaux de ce type se sont portés sur l'inégalité de Nash et les inégalités de Sobolev logarithmique. Dans cette thèse, nous obtenons une généralisation de ces travaux à une famille d'inégalités plus large. Plus précisément, nous adaptons les programmes AB et BA à une sous-famille des inégalités de Gagliardo-Nirenberg contenant, entre autres, l'inégalité de Nash.

Mots Clés : inégalités de Gagliardo-Nirenberg, inégalités de Sobolev logarithmiques, inégalités optimales.
Discipline : Mathématiques, Probabilités.

Laboratoire de Statistique et Probabilités
UFR MIG - UMR CNRS 5883
Université Paul SABATIER, Toulouse III