Mathematical modelisation of machining process: abrasion and wetting

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par

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Modélisation mathématique de procédés d'usinage: abrasion et mouillage.

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CHAPITRE 1

Introduction

1. Origines du problème et modélisation

Dans ce travail de thèse, on s'est intéressé au procédé d'usinage à très grande vitesse qui consiste à mettre aux côtes désirées un matériau. Plus précisément, les côtes obtenues sont d'autant meilleures que la vitesse de rotation de la fraise est grande. Bien que ce procédé soit utilisé depuis de nombreuses années dans l'industrie, il est loin d'être compris. En effet, pour éviter une casse au niveau du contact, on utilise pendant l'usinage un lubrifiant dont le rôle est de refroidir l'interface fraise–matériau. Ce lubrifiant empêche une élévation trop importante de la température mais il rend impossible toute observation.

Nous avons voulu comprendre ce qui se passe au niveau de cette interface lors de l'usinage à très grande vitesse, sachant que le voisinage fraise–matériau est considéré comme thermoviscoélastique. C'est la raison pour laquelle nous avons modélisé puis étudié des problèmes mathématiques simplifiés qui nous ont permis d'apporter quelques réponses qui seront développées dans cette thèse.

Nous commençons par expliquer la modélisation. Nous supposerons dans cette thèse que le matériau utilisé est homogène et isotrope. S'il y a des mouvements dans le matériau en déformation, sa température n'est pas constante mais elle varie aussi bien en temps que d'un point à l'autre. Cette circonstance complique les équations exactes du mouvement. Néanmoins les choses viennent à se simplifier car la transmission de chaleur d'une région à une autre du matériau est extrêmement limitée. Dans toute la suite, on supposera qu'il n'y a pratiquement pas d'échange thermique entre chaque région du matériau, chaque région sera considérée comme thermoisolée: le mouvement est adiabatique.

Pour établir les équations du mouvement en milieu viscoélastique, il faut égaler les forces des contraintes internes $\partial\sigma^A_{ik}/\partial x_k + \partial\dot\sigma^B_{ik}/\partial x_k + f_i$ au produit de l'accélération $\ddot u_i$ par la densité du matériau ρ:

$$\rho\ddot u_i = \frac{\partial\sigma^A_{ik}}{\partial x_k} + \frac{\partial\dot\sigma^B_{ik}}{\partial x_k} + f_i.$$

Les tenseurs des contraintes σ^A_{ij} et σ^B_{ij} sont définis à l'aide des tenseurs de déformation $\varepsilon_{ij}(u)$ et des tenseurs de Hooke a_{ijkl} et b_{ijkl}. Les conditions aux limites sont des conditions unilatérales appelées également conditions de Signorini.

Les premiers résultats monodimensionnels concernant l'étude de problèmes de vibrations d'un milieu élastique soumis à des contraintes unilatérales sur le bord sont dus à Amerio et Prouse [**3, 4**] et Schatzman [**13**] dans le cas d'un obstacle continu et à Amerio [**1, 2**], Schatzman [**14**], Citrini [**5**], Citrini et Marchionna [**6**] dans le cas d'un obstacle ponctuel. Le problème élastodynamique reste complétement ouvert en plus d'une dimension d'espace. Notons que Lebeau et Schatzman [**7**] ont traité le cas d'une équation des ondes avec contrainte unilatérale au bord dans un demi-espace, ils ont établi l'unicité et la conservation de l'énergie. Par ailleurs, Kim dans [**11**] et Jarušek dans [**9**] et [**10**] ont obtenu des solutions faibles dans un cadre plus général pour des problèmes multidimensionnels avec des contraintes unilatérales mais sans information sur le bilan énergétique, ni sur les traces.

Dans ce travail, nous avons étudié des problèmes viscoélastiques avec contact unilatéral, et on s'est particulièrement intéressé aux espaces fonctionnels qui caractérisent les traces ainsi qu'au bilan énergétique.

La première partie de ce travail, regroupant les deux premiers chapitres, est consacrée à l'étude d'un problème viscoélastique avec contact unilatéral dans un domaine borné en dimension 1. La seconde partie est dédiée à l'étude de problèmes viscoélastiques avec contact unilatéral dans un demi-espace. Dans la suite, on notera par u_0 la position initiale et par u_1 ou v_0 la vitesse initiale et on supposera que $\alpha > 0$.

Dans le chapitre 2, on modélise une barre viscoélastique de longueur L qui vibre longitudinalement entre deux obstales. Le problème mathématique peut être formulé de la manière suivante:

(1.1a) $u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in \Omega, \quad t \geq 0,$

(1.1b) $u(0,\cdot) + a_0 \geq 0,\ -(u_x + \alpha u_{xt})(0,\cdot) \geq 0,\ \big((u+a_0)(u_x + \alpha u_{xt})\big)(0,\cdot) = 0,$

(1.1c) $u(L,\cdot) + a_L \geq 0,\ (u_x + \alpha u_{xt})(L,\cdot) \geq 0,\ \big((u+a_L)(u_x + \alpha u_{xt})\big)(L,\cdot) = 0,$

(1.1d) $u(\cdot,0) = u_0 \quad \text{et} \quad u_t(\cdot,0) = u_1,$

où $\Omega = (0,L)$. L'existence de solutions faibles est prouvée par pénalisation. On détermine, par une analyse de Fourier, les espaces fonctionnels qui caractérisent les traces $u(0,\cdot)$ et $u(L,\cdot)$.

Le chapitre 3 est dédié à l'étude d'un schéma numérique qui approche une solution de (1.1). Il est inspiré du schéma développé par Lebeau et Schatzman [**8**], adapté au cas viscoélastique.

Dans les chapitres 4 et 5, on considère un matériau de Kelvin-Voigt occupant une région $\Omega = (-\infty, 0]$ du plan. On remarque que dans ce cas particulier, le problème peut être découplé et après avoir choisi les unités de manière appropriée, l'équation satisfaite par la première composante est (1.1a) avec des conditions unilatérales:

(1.2) $$u(0,\cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0,\cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0,\cdot) = 0,$$

et des conditions initiales (1.1d). Ce problème peut être réduit à l'inéquation variationnelle suivante:

(1.3) $$\lambda_1 * w = g + b, \quad w \geq 0, \quad b \geq 0, \quad \langle w, b\rangle = 0.$$

Ici $\widehat{\lambda}_1(\omega)$ est la détermination causale de $i\omega\sqrt{1+i\alpha\omega}$. On démontre que ce problème possède une solution et que les pertes d'énergie sont purement visqueuses; ce résultat provient de la relation $\langle \dot{w}, b\rangle = 0$, qui n'est pas triviale puisque, *a priori*, b est une mesure et $\dot{w}$ n'est définie que presque partout.

Le chapitre 6 est consacré à l'étude de l'équation des ondes amorties dans le demi-espace $\Omega = [-\infty, 0) \times \mathbb{R}^{d-1}$ avec des conditions unilatérales en $x_1 = 0$;

(1.4a) $$u_{tt} - \Delta u - \alpha \Delta u_t = f, \quad x \in \Omega, \quad t \geq 0,$$

(1.4b) $$u(0,\cdot) \geq 0, \quad (u_{x_1} + \alpha u_{x_1 t})(0,\cdot) \geq 0, \quad \big(u(u_{x_1} + \alpha u_{x_1 t})\big)(0,\cdot) = 0,$$

(1.4c) $$u(\cdot,0) = u_0 \quad \text{et} \quad u_t(\cdot,0) = u_1.$$

Quant au chapitre 7, il est dédié à l'étude du problème élastodynamique dans le demi-espace $\Omega \in (-\infty, 0] \times \mathbb{R}^{d-1}$ avec des conditions unilatérales en $x_1 = 0$;

(1.5a) $$\rho \ddot{u}_i = \partial_j \sigma^0_{ij}(u) + \partial_j \sigma^1_{ij}(\dot{u}) + f_i, \quad x \in \Omega, \quad t \geq 0,$$

(1.5b) $$\sigma^0_{12}(u) + \sigma^1_{12}(\dot{u}) = 0 \quad \text{et} \quad \sigma^0_{13}(u) + \sigma^1_{13}(\dot{u}) = 0,$$

(1.5c) $$u_1 \leq 0, \quad \sigma^0_{11}(u) + \sigma^1_{11}(\dot{u}) \leq 0, \quad u_1\big(\sigma^0_{11}(u) + \sigma^1_{11}(\dot{u})\big) = 0,$$

(1.5d) $$u(\cdot,0) = u_0 \quad \text{et} \quad \dot{u}(\cdot,0) = v_0.$$

Dans les chapitres 6 et 7, on prouve l'existence d'une solution et on détermine les espaces fonctionnels qui caractérisent les traces. Signalons enfin que l'unicité de la solution reste dans tout les cas un problème ouvert.

Pour terminer nous présentons un résumé détaillé des différents chapitres.

2. Viscoélasticité avec contact unilatéral dans un domaine borné en dimension 1

2.1. Chapitre 2. Pour montrer l'existence d'une solution, on utilise une pénalisation; on approche le problème (1.1) en remplaçant les contraintes rigides (1.1b) et (1.1c) par des réponses très raides: si la contrainte est atteinte, la réponse est linéaire sinon la réponse est nulle. Plus précisément, soit $r^- = -\min(r,0)$ alors le problème approché s'écrit:

$$u^\varepsilon_{tt} - u^\varepsilon_{xx} - \alpha u^\varepsilon_{xxt} = f, \quad x \in \Omega, \quad t \geq 0, \tag{2.1a}$$

$$(u^\varepsilon_x + \alpha u^\varepsilon_{xt})(0,\cdot) = -\frac{\left(u^\varepsilon(0,\cdot) + a_0\right)^-}{\varepsilon}, \tag{2.1b}$$

$$(u^\varepsilon_x + \alpha u^\varepsilon_{xt})(L,\cdot) = \frac{\left(u^\varepsilon(L,\cdot) + a_L\right)^-}{\varepsilon}, \tag{2.1c}$$

$$u^\varepsilon(\cdot,0) = u_0 \quad \text{et} \quad u^\varepsilon_t(\cdot,0) = u_1. \tag{2.1d}$$

On introduit ensuite v qui est solution du problème suivant:

$$\begin{aligned} &v_{tt} - v_{xx} - \alpha v_{xxt} = f, \quad (x,t) \in \Omega \times (0,T),\\ &v(0,\cdot) = e^{-t}\big(u_0(0) - a_0\big) \quad \text{et} \quad v(0,\cdot) = e^{-t}\big(u_0(L) - a_L\big),\\ &v(\cdot,0) = u_0 \quad \text{et} \quad v_t(\cdot,0) = u_1. \end{aligned}$$

Si on pose $h_0 = e^{-t}\big(u_0(0) - a_0\big)$ et $h_L = e^{-t}\big(u_0(L) - a_L\big)$ alors $w^\varepsilon = e^{-\nu t}(u^\varepsilon - v)$, $\nu > 0$, est solution de

$$\big(\nu + \partial_t\big)^2 w^\varepsilon - \big(1 + \alpha\big(\nu + \partial_t\big)\big) w^\varepsilon_{xx} = 0, \quad (x,t) \in \Omega \times (0,T), \tag{2.3a}$$

$$\big(1 + \alpha\big(\nu + \partial_t\big)\big) w^\varepsilon_x(0,\cdot) = e^{-\nu t} g_0 - \big(w^\varepsilon(0,\cdot) + e^{-\nu t} h_0\big)^- / \varepsilon, \tag{2.3b}$$

$$\big(1 + \alpha\big(\nu + \partial_t\big)\big) w^\varepsilon_x(L,\cdot) = e^{-\nu t} g_L + \big(w^\varepsilon(L,\cdot) + e^{-\nu t} h_L\big)^- / \varepsilon, \tag{2.3c}$$

$$w^\varepsilon(\cdot,0) = 0 \quad \text{and} \quad w^\varepsilon_t(\cdot,0) = 0. \tag{2.3d}$$

Soient $\widehat{w}^\varepsilon$ la transformée de Fourier partielle en temps de w^ε et ω la variable duale de t. Par une transformée de Fourier en temps de (2.3a), on s'aperçoit que

$$\widehat{w}^\varepsilon_{xx} = \frac{(i\omega + \nu)^2}{1 + \alpha(\nu + i\omega)}\, \widehat{w}^\varepsilon. \tag{2.4}$$

On montre que $\widehat{w}^\varepsilon$ est une distribution tempérée et si $\widehat{\lambda}$ est la détermination causale de $(i\omega + \nu)/\sqrt{1 + \alpha(\nu + i\omega)}$, alors la solution de (2.4) est

$$\widehat{w}^\varepsilon(x,\omega) = \frac{\widehat{w}^\varepsilon(0,\omega) - e^{\widehat{\lambda} L}\widehat{w}^\varepsilon(L,\omega)}{1 - e^{-2\widehat{\lambda} L}} e^{-\widehat{\lambda} x} + \frac{-e^{\widehat{\lambda} L}\widehat{w}^\varepsilon(0,\omega) + \widehat{w}^\varepsilon(L,\omega)}{1 - e^{-2\widehat{\lambda} L}} e^{\widehat{\lambda}(x-L)} \tag{2.5}$$

Si on pose $\widehat{\eta}(\omega) = e^{-2\widehat{\lambda} L}$ alors on peut déduire de (2.3b), (2.3c) et (2.5) que

$$M * \begin{pmatrix} w^\varepsilon(0,t) \\ w^\varepsilon(L,t) \end{pmatrix} = e^{-\nu t} \begin{pmatrix} -g_0(t) \\ g_L(t) \end{pmatrix} + \begin{pmatrix} (w^\varepsilon(0,t) + e^{-\nu t} h_0(t))^- / \varepsilon \\ (w^\varepsilon(L,t) + e^{-\nu t} h_L(t))^- / \varepsilon \end{pmatrix}, \tag{2.6}$$

où la transformée de Fourier de M est définie par

$$\widehat{M}(\omega) = \frac{\widehat{\lambda}(\omega)(1 + \alpha(\nu + i\omega))}{1 - \widehat{\eta}^2(\omega)} \begin{pmatrix} 1 + \widehat{\eta}^2(\omega) & -2\widehat{\eta}(\omega) \\ -2\widehat{\eta}(\omega) & 1 + \widehat{\eta}^2(\omega) \end{pmatrix}.$$

L'existence et l'unicité d'une solution de (2.6) résulte alors immédiatement du théorème classique des itérations successives de Picard. Après avoir établi des estimations sur la suite $(u^\varepsilon)_{\varepsilon>0}$, le passage à la limite lorsque ε tend vers 0 permet d'obtenir une solution variationnelle du problème étudié. Puis on détermine les espaces fonctionnels qui caractérisent les traces; on multiplie (2.6) par $\big((\nu+\partial_t)w^\varepsilon(0,\cdot),(\nu+\partial_t)w^\varepsilon(L,\cdot)\big)^T$, on intègre sur $\mathbb{R}^+$ puis on applique la formule de Plancherel à l'identité ainsi obtenue. Par ailleurs, on montre en utilisant des estimations d'énergie que $g_0=-(v_x+\alpha v_{xt})(0,\cdot)$ et $g_L=-(v_x+\alpha v_{xt})(L,\cdot)$ sont bornées dans $H^{1/2}(0,T)$. Il en découle que $\big(u^\varepsilon(0,\cdot),u^\varepsilon(L,\cdot)\big)^T$ est borné dans $H^{5/4}_{\text{loc}}(\mathbb{R})^2$.

2.2. Chapitre 3. Les solutions des équations (1.1) peuvent être numériquement approchées de la manière suivante: on approche $u(j\Delta x, n\Delta t)$ par u_j^n. On discrétise les différences en temps et en espace et on applique une formulation variationnelle à notre problème. Soit A l'opérateur obtenu par discrétisation du Laplacien en une dimension avec des conditions de Neumann; pour $i\in[1,J]$,

$$A_{ii}=2/\Delta x^2,\quad A_{i,i+1}=A_{i,i-1}=-1/\Delta x^2,\quad A_{00}=A_{J+1,J+1}=1/\Delta x^2.$$

Posons $f_j^n=(j\Delta x,n\Delta t)$. Alors le schéma numérique est défini par

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}+\left(A\,\frac{u^{n+1}+u^{n-1}}{2}+\alpha A\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)_j=f_j^n.$$

On a le problème complémentaire suivant:

$$\begin{aligned}-a_0\le u_0^n,\quad 0\le X_0^n,\quad u_0^nX_0^n=0,\\ -a_L\le u_{J+1}^n,\quad 0\le X_{J+1}^n,\quad u_{J+1}^nX_{J+1}^n=0,\end{aligned}$$

où pour tout $j=0,\ J+1$,

$$X_j^n=\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}+\left(A\,\frac{u^{n+1}+u^{n-1}}{2}+\alpha A\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)_j-f_j^n.$$

On prouve la convergence de la solution numérique vers une solution de (1.1). On présente ensuite quelques simulations numériques que l'on a effectuées sur un intervalle de temps fini, où nous prenons une ou deux contraintes unilatérales au bord. En pratique, à chaque pas de temps, on regarde si la solution sans contrainte est admissible; si c'est le cas on passe au temps suivant, sinon on résout le problème lorsque une ou deux contraintes sont actives et ainsi de suite.

3. Viscoélasticité avec contact unilatéral dans un demi-espace

3.1. Chapitre 4 et 5. On commence par réduire le problème (1.1a), (1.1d) et (1.2) à un problème au bord. Soit v la solution du problème de Dirichlet pour (1.1) avec des conditions initiales (1.1d). On définit $g=-(\bar u_x+\alpha\bar u_{xt})(0,\cdot)$. Alors $v=u-\bar u$ satisfait le problème suivant:

(3.1a) $$v_{tt}-v_{xx}-\alpha v_{xxt}=0,$$

(3.1b) $$v(0,\cdot)\ge 0,\quad (v_x+\alpha v_{xt})(0,\cdot)\ge g,\quad \big(v(v_x+\alpha v_{xt})\big)(0,\cdot)=0,$$

(3.1c) $$v(\cdot,t)=0\quad\text{if}\quad t\le 0.$$

Par une transformation de Fourier en temps de (3.1a), on a montré que (1.1a), (1.1d) et (1.2) est équivalent à un problème au bord. En effet, $\widehat{v}(\cdot,\omega)$ la transformée de Fourier en temps de v satisfait l'équation différentielle suivante:

(3.2) $$(1+i\alpha\omega)\widehat{v}_{xx}=-\omega^2\widehat{v}.$$

On montre que $\widehat{v}$ est une distribution tempérée et si $\widehat{\lambda}(\omega)=i\omega/\sqrt{1+i\alpha\omega}$ alors la solution de (3.2) est $\widehat{v}(x,\omega)=\widehat{v}(0,\omega)\exp(\widehat{\lambda}(\omega)x)$. En particulier, la transformée de

Fourier de $(v_x + \alpha v_{xt})(0,\cdot)$ est $(1+i\alpha\omega)\widehat{\lambda}(\omega)\widehat{v}(0,\omega)$. Remarquons ensuite que la transformée de Fourier réciproque de $\widehat{\lambda}_1(\omega) = i\omega\sqrt{1+i\alpha\omega}$ est à support dans $\mathbb{R}^+$ et sa partie principale est le produit de $\sqrt{\alpha}$ et de la dérivée d'ordre $3/2$ de la masse de Dirac. Si on pose $w = v(0,\cdot)$ alors le problème (3.1) est équivalent au problème (1.3). On montre que (1.3) possède une solution w en faisant une pénalisation; plus précisément, soit $\varepsilon > 0$ et $(w^\varepsilon)^- = -\min(0, w^\varepsilon)$, ce qui nous conduit au problème approché suivant:

$$\lambda_1 * w^\varepsilon = g + \big(w^\varepsilon\big)^-/\varepsilon. \tag{3.3}$$

Soit μ_1 la convolution inverse de λ_1; sa transformée de Fourier est $\widehat{\mu}_1 = 1/\big(i(\omega - i0)\sqrt{1+i\alpha\omega}\big)$. En particulier, μ_1 est une fonction continue à support dans $\mathbb{R}^+$ et au voisinage de 0, μ_1 peut être approchée par $C\sqrt{\max(t,0)}$. On prouve l'existence d'une solution de (3.3) par une méthode itérative:

$$w_{n+1} = \mu_1 * \big(\big(w_n^\varepsilon\big)^-/\varepsilon + g\big). \tag{3.4}$$

Les propriétés fonctionnelles de μ_1 permettent de déduire que ces itérations convergent vers une solution de (3.3). Pour la démonstration, on a utilisé les itérés de Picard ainsi que la propriété $r \to r^-$ est lipschitzienne.

Une fois que l'on a démontré l'existence d'une solution de (3.3), nous avons prouvé sa convergence lorsque ε tend vers 0. Nous devons pour cela faire des estimations sur w^ε indépendamment de ε. L'idée est de faire des estimations d'énergie, pour cela on a multiplié (3.3) par $\dot{w}^\varepsilon$ puis intégré sur $(0,T)$, on s'aperçoit alors que le membre de gauche de l'équation est non négatif et que le membre de droite peut être estimé. En effet, en utilisant l'identité de Plancherel, on obtient:

$$C\int_{\mathbb{R}} |\omega|^2\big(1+|\omega|\big)\big|\widehat{w}^\varepsilon\big|^2\,d\omega \le \int_{\mathbb{R}} |\omega|\big|\widehat{g}\big|^2\,d\omega. \tag{3.5}$$

Par ailleurs, on montre en faisant des estimations *a priori* que g est bornée dans $H^{1/2}_{\mathrm{loc}}\big([0,\infty)\big)$. Alors on déduit de (3.5) que w^ε est bornée dans $H^{5/4}_{\mathrm{loc}}(\mathbb{R})$. Par conséquent, w^ε converge faiblement vers w et en particulier $\dot{w}^\varepsilon$ et $\big(w^\varepsilon\big)^-/\varepsilon$ convergent faiblement vers leurs limites respectivement dans $H^{1/4}_{\mathrm{loc}}(\mathbb{R})$ et $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$. Le raisonnement effectué pour déterminer la régularité de w^ε est purement formel, une démonstration rigoureuse est donnée dans **[12]**.

Dans les problèmes viscoélastiques standards, les dissipations d'énergie sont dues à la viscosité. D'un point de vue mathématique, on obtient l'énergie cinétique en multipliant (1.1a) par u_t, puis en intégrant sur $(-\infty,0)\times(0,\tau)$, $\tau\in[0,T]$, on trouve l'identité suivante:

$$\frac{1}{2}\int_{-\infty}^0 \big(u_x^2+u_t^2\big)(\cdot,\tau)\,dx + \alpha\int_{-\infty}^0\int_0^\tau u_{xt}^2\,dx\,dt = \frac{1}{2}\int_{-\infty}^0 \big(|u_1|^2+|u_{0,x}|^2\big)\,dx + \int_0^\tau \big((u_x+\alpha u_{xt})u_t\big)(0,\cdot)\,dt + \int_0^\tau\int_{-\infty}^0 f u_t\,dx\,dt.$$

Les pertes d'énergie sont purement visqueuses si l'intégrale de $\big((u_x+\alpha u_{xt})u_t\big)(0,\cdot)$ s'annule. En effet, $u(0,\cdot) = w$ est non négatif alors $u(0,\cdot)$ s'annule presque partout sur l'ensemble des points où $u(0,\cdot)$ s'annule. On s'aperçoit que (1.3) permet de déduire que $\lambda_1 * w - g = (u_x+\alpha u_{xt})(0,\cdot)$ s'annule quand w s'annule. Néanmoins, l'argument donné ci-dessus n'est pas correct car on ne sait pas si $\lambda_1 * w - g$ est une fonction, on sait seulement que c'est une mesure. De plus, on sait que $\dot{w}$ s'annule presque partout, mais ce n'est pas suffisant pour conclure. Par ailleurs, on ne peut passer à la limite dans le produit de dualité $\langle \dot{w}^\varepsilon, (w^\varepsilon)^-/\varepsilon\rangle$ car on sait seulement que $\dot{w}^\varepsilon$ et $(w^\varepsilon)^-/\varepsilon$ sont bornées respectivement dans $H^{1/4}_{\mathrm{loc}}(\mathbb{R})$ et $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$.

Etant donné que les techniques habituelles ne permettent pas de conclure, on a élaboré une nouvelle méthode qui nous permet de déduire que les dissipations d'énergie sont dues essentiellement à la viscosité. On explique dans la suite la démarche que nous avons adoptée.

A priori, on ne sait pas si le support de w est une réunion d'intervalles localement finis et si le support de b est une réunion localement finie de leurs complémentaires. En d'autres termes, on pourrait avoir une accumulation de petits intervalles contenant le support de w encadré par deux accumulations de petits intervalles contenant le support de b. Supposons que le support de w est inclus dans $\cup_{j\geq 0}[\sigma(j),\tau(j))$ et le support de b est inclus dans $\cup_{j\geq 0}[\tau(j),\sigma(j+1))$. Alors on a presque une solution explicite de notre problème si on suppose que g est la convolution de ϕ, une mesure à support dans $\mathbb{R}^+$, et de $\mu(t)=\exp(-t/\alpha)H(t)/\sqrt{\pi\alpha t}$, $H(t)$ étant la fonction de Heaviside. On définit ensuite deux fonctions

$$\omega(t)=\frac{H(t)}{\pi(t+1)\sqrt{t}} \quad \text{et} \quad \Omega(t)=\frac{2}{\pi}\arctan\sqrt{t},$$

et une distribution $\nu(t)=H(t)\exp(-t/\alpha)/\alpha$. Alors la solution est déterminée par récurrence à partir des formules suivantes:

$$\phi_0=\phi, \tag{3.6a}$$

$$\psi_j=\phi_j 1_{[\tau(j),\sigma(j+1))}+\delta(\cdot-\tau(j))e^{-\tau(j)/\alpha}\int_{[\sigma(j),\tau(j))}e^{s/\alpha}\phi_j, \tag{3.6b}$$

$$\begin{aligned}\phi_{j+1}&=1_{[\sigma(j+1),\infty)}\phi_j\\&+1_{[\sigma(j+1),\infty)}\int_{[\tau(j),\sigma(j+1))}e^{-(\cdot-s)/\alpha}\omega\left(\frac{\cdot-\sigma(j+1)}{\sigma(j+1)-s}\right)\frac{\psi_j(s)}{\sigma(j+1)-s},\end{aligned} \tag{3.6c}$$

$$w1_{[\sigma(j),\tau(j)]}=\big(H*\nu*\phi_j\big)1_{[\sigma(j),\tau(j)]}. \tag{3.6d}$$

Par ailleurs, on établit une estimation de la partie négative de la dérivée à droite de w au point t où w est strictement positive:

$$\begin{aligned}\big(\dot w(\sigma-0)\big)^-\leq\frac{1}{\alpha}\int_{[\sigma_j,\sigma)}|\phi_0(s)|+\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)}\left(2\exp\left(\frac{\sigma-\sigma_j}{\alpha}\right)-1\right)\\ \left(\Omega\left(\frac{\sigma-\sigma_i}{\sigma_i-s}\right)-\Omega\left(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\right)\right)\psi_{i-1}^+(s)\psi_{i-1}^+(s).\end{aligned} \tag{3.7}$$

On construit alors une suite approchant les données et les solutions qui a la propriété d'être localement finie. L'idée essentielle est de voir si nos données rendent une solution w telle que b est à support inclus dans une union d'intervalles de taille au moins égale à $1/n$, ce sera alors notre solution. Sinon on prend le plus petit temps pour lequel cette propriété n'est pas vérifiée et on modifie les données de manière à corriger ce défaut: si le problème se produit pour la première fois après l'intervalle $[\sigma(j),\tau(j)]$, on prend un intervalle plus grand $[\tau(j),\sigma^*(j)]$ de longueur au moins égale à $1/n$. Ensuite on met toute la partie négative de la mesure $\psi_j|_{[\tau(j),\sigma^*(j)]}$ dans la masse de Dirac en $\tau(j)$, et toute la partie positive de cette mesure dans la masse de Dirac en $\sigma^*(j)$. Maintenant, la solution modifiée a la propriété désirée sur l'intervalle $[0,\sigma^*(j)]$; on démontre ce résultat par récurrence. Finalement, on obtient ϕ^n, une solution w^n et deux mesures intermédiaires ϕ_j^n et ψ_j^n.

On montre des propriétés d'ordre pour les mesures ϕ_j^n et ψ_j^n qui impliquent que la masse totale de ces mesures est bornée indépendamment de n et j. En particulier, on déduit de (3.6d) que $\dot w^n$ est bornée indépendamment de n. Ce résultat est plus fort que celui obtenu en utilisant la pénalisation et il nous permet de passer à la limite lorsque n tend vers l'infini.

Enfin, *a priori*, la mesure b peut avoir une partie singulière. Puisque $\dot{w}$ est bornée, on a prouvé que b n'a pas d'atome; on ne peut pas exclure la possibilité que b ait une mesure diffuse, c'est-à-dire que b soit une fonction croissante dont les dérivées s'annulent presque partout. Par ailleurs, en utilisant (3.7), on montre que sur le support de b, $\dot{w}$ s'annule sauf en un nombre dénombrable de points. On peut alors déduire que le produit de dualité $\langle b, w\rangle$ a un sens et qu'il s'annule.

3.2. Chapitre 6. On commence par pénaliser le problème (1.4) en utilisant le même procédé que celui employé dans le chapitre 2, on obtient donc

$$u^\varepsilon_{tt} - \Delta u^\varepsilon - \alpha\Delta u^\varepsilon_t = f, \quad x \in \Omega, \quad t \geq 0, \tag{3.8a}$$

$$\big(u^\varepsilon_{x_1} + \alpha u^\varepsilon_{x_1 t}\big)(0,\cdot,\cdot) = \big(u^\varepsilon(0,\cdot,\cdot)\big)^-/\varepsilon, \tag{3.8b}$$

$$u^\varepsilon(\cdot,0) = u_0 \quad \text{et} \quad u^\varepsilon_t(\cdot,0) = u_1. \tag{3.8c}$$

On prouve l'existence et l'unicité de u^ε en utilisant une méthode de Galerkin. Nous établissons l'existence d'une solution de (1.4) à partir d'estimations *a priori* sur le problème pénalisé.

Ensuite on détermine par une analyse de Fourier l'espace fonctionnel qui caractérise la trace $u(0,\cdot,\cdot)$. Supposons que $z \in H^3\big(\Omega\times[0,\infty)\big)$. Soit $\bar u$ la solution du problème suivant:

$$\bar u_{tt} - \Delta\bar u - \alpha\Delta\bar u_t = f, \quad x\in\Omega, \quad t\geq 0, \tag{3.9a}$$

$$\bar u(0,\cdot,\cdot) = z(0,\cdot,\cdot), \tag{3.9b}$$

$$\bar u(\cdot,0) = u_0 \quad \text{et} \quad \bar u_t(\cdot,0) = u_1. \tag{3.9c}$$

Soient $\bar g = -(\bar u_{x_1} + \alpha\bar u_{x_1 t})(0,\cdot,\cdot)$ et $g = e^{-\nu t}\bar g$. Alors $z^\varepsilon = e^{-\nu t}(u^\varepsilon - v)$, $\nu > 0$, satisfait le problème suivant:

$$\big(\nu + \partial_t\big)^2 z^\varepsilon - \big(1 + \alpha(\nu+\partial_t)\big)\Delta z^\varepsilon = 0, \quad x\in\Omega, \quad t\geq 0, \quad \alpha > 0, \tag{3.10a}$$

$$\big(1+\alpha(\nu+\partial_t)\big) z^\varepsilon_{x_1}(0,\cdot,\cdot) = g - \big(z^\varepsilon(0,\cdot,\cdot) + e^{-\nu t}\bar u(0,\cdot,\cdot)\big)^-/\varepsilon, \tag{3.10b}$$

$$z^\varepsilon(\cdot,0) = 0 \quad \text{et} \quad z^\varepsilon_t(\cdot,0) = 0. \tag{3.10c}$$

Soient $\widehat{z^\varepsilon}$ la transformée de Fourier partielle en variables tangentielles t et $x' = (x_2, ..., x_d)$ de z^ε, ω la variable duale de t et ξ la variable duale de x'. Par une transformée de Fourier partielle en variables tangentielles de (3.10a), on s'aperçoit que

$$\widehat{z^\varepsilon}_{x_1x_1} = \left(|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}\right)\widehat{z^\varepsilon}. \tag{3.11}$$

On montre que $\widehat{z^\varepsilon}$ est une distribution tempérée et si $\widehat\lambda$ est la détermination causale de

$$\sqrt{|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}};$$

alors la solution de (3.11) est $\widehat{z^\varepsilon}(x_1,\cdot,\cdot) = \widehat{z^\varepsilon}(0,\cdot,\cdot)e^{\widehat\lambda x_1}$. Si on pose $w^\varepsilon = z^\varepsilon(0,\cdot,\cdot)$ alors le problème (3.8) est équivalent au problème suivant:

$$\lambda_1 * w^\varepsilon = g + \big(w^\varepsilon + e^{-\nu t}\bar u(0,\cdot,\cdot)\big)^-/\varepsilon. \tag{3.12}$$

Si nous multiplions (3.12) par $\alpha\big(\nu w^\varepsilon + w^\varepsilon_t\big) + w^\varepsilon$ puis nous intégrons sur $\mathbb{R}^{d-1}\times[0,\infty)$, nous observons que le membre de gauche de l'équation est non négatif et que le membre de droite peut être estimé. En effet, en utilisant l'identité de Plancherel, on obtient:

$$(3.13)\qquad \begin{aligned} &\Big(C-\frac{\gamma}{2}\Big)\int_{\mathbb{R}^d}|2+i\alpha\omega|^2\big(1+|\xi|+\sqrt{|\omega|}\big)|\widehat{w}^\varepsilon|^2\,d\omega\,d\xi \\ &\qquad\le C_1+\frac{1}{2\gamma}\int_{\mathbb{R}^d}\frac{|\widehat{g}|^2}{1+|\xi|+\sqrt{|\omega|}}\,d\omega\,d\xi. \end{aligned}$$

Par ailleurs, on montre en faisant des estimations *a priori* que g est bornée dans $L^2\big([0,\infty);H^{1/2}(\mathbb{R}^{d-1})\big)$. Alors si on choisit γ tel que $\gamma<2C$, il découle de (3.13) que $w^\varepsilon(0,\cdot,\cdot)$ est bornée dans $H^{5/4}\big([0,\infty);L^2(\Sigma)\big)$. On déduit que $u^\varepsilon(0,\cdot,\cdot)$ est bornée dans $H^{5/4}_{\mathrm{loc}}\big([0,\infty);L^2(\Sigma)\big)$.

3.3. Chapitre 7. Comme précédemment, on pénalise le problème (1.5) de la manière suivante:

$$\begin{aligned} &\rho\ddot{u}^\epsilon-A^0u^\epsilon-A^1\dot{u}^\epsilon=f,\quad x\in\Omega,\quad t\ge 0,\\ &a^0_{11kl}\varepsilon_{kl}(u^\epsilon)+a^1_{11kl}\varepsilon_{kl}(\dot{u}^\epsilon)=-(u^\epsilon_1)^+/\epsilon,\\ &a^0_{12kl}\varepsilon_{kl}(u^\epsilon)+a^1_{12kl}\varepsilon_{kl}(\dot{u}^\epsilon)=0\quad\text{et}\quad a^0_{13kl}\varepsilon_{kl}(u^\epsilon)+a^1_{13kl}\varepsilon_{kl}(\dot{u}^\epsilon)=0,\\ &u^\epsilon(\cdot,0)=u_0\quad\text{et}\quad\dot{u}^\epsilon(\cdot,0)=v_0. \end{aligned}$$

On utilise les mêmes idées que celles développées dans le chapitre 6 pour prouver l'existence d'une solution et pour déterminer les espaces fonctionnels qui caractérisent la trace $u_1(0,\cdot,\cdot)$. Ce qu'il faut retenir ici c'est que la trace $u_1(0,\cdot,\cdot)$ est dans $H^{5/4}_{\mathrm{loc}}\big([0,\infty);L^2_{\mathrm{loc}}(\mathbb{R}^{d-1})\big)$.

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Partie 1

Viscoélasticité avec contact unilatéral dans un domaine borné en dimension 1

CHAPITRE 2

Viscoelastic bar with Signorini conditions

Adrien Petrov and Michelle Schatzman

Abstract. The simplified viscoelastic problem

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad (x,t) \in (0,L) \times (0,T), \quad \alpha > 0,$$

with initial data

$$u(\cdot,0) = u_0 \quad \text{and} \quad u_t(\cdot,0) = u_1,$$

and boundary conditions

$$0 \leq (u(0,t) + a_0) \perp -(u_x + \alpha u_{xt})(0,t) \geq 0,$$
$$0 \leq (u(L,t) + a_L) \perp (u_x + \alpha u_{xt})(L,t) \geq 0.$$

models the longitudinal vibrations of a bar, whose motion is limited by obstacles at both ends. A weak solution is obtained as the limit of penalized of a sequence of problem; the functional properties of the trace are precisely identified.

1. Introduction and notations

The origin of the problem is a model of grinding: the tool is in contact with the metal part to be machined during the rectification process. Usually, the grinding tool is made of small very hard grains held together by a resin–like material. The behavior of the grinding tool is considered as viscoelastic.

The theory of vibrations of continuous media with unilateral conditions at the boundary purports to understand the so-called Signorini problem; when the medium is elastic and satisfies the assumptions of the theory of small deformations, the one-dimensional theory is well understood, starting from the work of Amerio and Prouse [**3, 4**] and of Schatzman [**11**], in the case of a continuous obstacle and the work of Amerio [**1, 2**] of Schatzman [**12**], Citrini [**5**], Citrini and Marchionna [**6**] in the case of a point–like obstacle and the theory can be considered as complete. It was only in the work of Lebeau and Schatzman [**9**] that the equivalence between codimension one obstacle and constraint at the boundary was clearly stated, though it was probably understood before that article appeared.

In multidimensional case, the energy relation is proved only for a wave in the half-space with unilateral boundary conditions [**9**]; weaker results in more general setting have been obtained by Jarušek et al. [**7**] and by Uhn [**8**].

We consider a viscoelastic bar of length L which vibrates longitudinally; each extremity of the bar is free to move as long as it does not hit a material obstacle. Therefore, we may describe this situation mathematically as follows: let the material of the bar be homogeneous, linear and make the approximation of small displacements and deformations. Assume the material can be described as a Kelvin-Voigt material. Let x be the spatial coordinate along the bar with the origin at one of the extremities; let $u(x,t)$ be the displacement at time t of the material point with spatial coordinate x. Let f denote a density of exterior forces, depending on space and time. The displacement u satisfies the following equation:

$$u_{tt} - \sigma_x = f \quad \text{in} \quad (0,L) \times (0,T), \tag{1.1}$$

with the following Kelvin-Voigt constitutive law

$$\sigma = u_x + \alpha u_{xt}, \quad \alpha > 0, \tag{1.2}$$

with initial data

$$u(\cdot,0) = u_0 \quad \text{and} \quad u_t(\cdot,0) = u_1. \tag{1.3}$$

Here we have denoted by u_t, u_x, u_{tt}, u_{xx} and u_{xt} the derivatives of u. Assume that the obstacles at 0 and T have the respective positions: $a_0 \geq 0$ and $a_L \geq 0$. The following boundary conditions are derived from the variational formulation and will be completely justified later on. They are respectively at $x = 0$ and $x = L$,

$$0 \leq (u(0,\cdot) + a_0) \perp -(u_x + \alpha u_{xt})(0,\cdot) \geq 0, \tag{1.4a}$$

$$0 \leq (u(L,\cdot) + a_L) \perp (u_x + \alpha u_{xt})(L,\cdot) \geq 0; \tag{1.4b}$$

here the orthogonality has the natural meaning: an appropriate duality product between the two terms of the relation vanishes. Conditions (1.4a)-(1.4b) are usually termed Signorini conditions. We are given Cauchy initial data u_0 and u_1; it is assumed that the initial position u_0 belongs to the Sobolev space $H^2(0,L)$ and satisfies the compatibility conditions $u_0(0) \geq -a_0$ and $u_0(L) \geq -a_L$. The initial velocity u_1 belongs to $L^2(0,L)$ and the density of forces belongs to $L^2(0,T;L^2(0,L))$.

Let us define for all $\tau \in [0,T]$ the following sets:

$$Q_\tau = (0,L) \times (0,\tau), \quad Q_\tau^- = \mathbb{R}^- \times (0,\tau), \quad \Sigma_\tau = \{0,L\} \times (0,\tau).$$

Let K be a convex set:

$$K = \{v \in H^1(Q_T) : v_{xt} \in L^2(Q_T), v(0,\cdot) \geq -a_0, v(L,\cdot) \geq -a_L\}.$$

The weak formulation of the problem (1.1)-(1.4b) is obtained as follows: multiply (1.1) by $v - u$, for some $v \in K$, integrate formally over Q_τ, we obtain the following formulation:

$$\begin{cases} u \in K \text{ and for all } v \in K \text{ and almost every } \tau \in [0,T], \\ \displaystyle\int_0^L \big(u_t(v-u)\big)\big|_0^\tau \, dx - \int_{Q_\tau} u_t(v_t - u_t)\, dx\, dt \\ \displaystyle + \int_{Q_\tau} (u_x + \alpha u_{xt})(v_x - u_x)\, dx\, dt \geq \int_{Q_\tau} f(v-u)\, dx\, dt. \end{cases} \tag{1.5}$$

The equivalence of this weak formulation with the strong (1.1)-(1.4b) is by no means obvious; it depends on precise information on the trace of u_{xt} on $\{x = 0\}$ and on $\{x = L\}$.

Let us write at least formally an energy relation for (1.1): we multiply this equation by u_t, we integrate by parts over Q_τ, and we get

$$\begin{aligned} &\frac{1}{2}\int_0^L \big(|u_t|^2 + |u_x|^2\big)(\cdot,\tau)\, dx + \alpha \int_{Q_\tau} |u_{xt}|^2\, dx\, dt = \frac{1}{2}\int_0^L \big(|u_1|^2 + |u_{0,x}|^2\big)\, dx \\ &- \int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(0,\cdot)\, dt + \int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(L,\cdot)\, dt + \int_{Q_\tau} f u_t\, dx\, dt. \end{aligned}$$

The energy loss is purely viscous, iff

$$- \int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(0,\cdot)\, dt + \int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(L,\cdot)\, dt = 0. \tag{1.6}$$

REMARK 1.1. *The identity* (1.6) *would be quite easy to prove if we knew that* u *vanishes only on a finite number of intervals, since then* u_t *would vanish on these intervals, and little analysis would be needed to conclude.*

Let us explain now the plan of the article.

We will show in Section 2 that if u_0 and u_1 belong to $H^2(0,L) \cap H_0^1(0,L)$, f and f_t belong to $L^2(Q_T)$ and v is the solution of (1.1)-(1.3) with Dirichlet boundary data at $x = 0$ and $x = L$ then the traces $(v_x + \alpha v_{xt})\big|_{\Sigma_T}$ are bounded in $H^{1/2}(0,T)$.

In Section 3, we define a penalized problem associated to (1.1)-(1.4b) for which we prove the existence and uniqueness of a solution.

In Section 4 and 5, according to estimates on the penalized problem, it is possible to deduce the existence of a weak solution to problem (1.1)-(1.4b).

Once we have obtained the existence of a solution of penalized problem, we prove the convergence in Section 6. Thanks to energy estimates, we establish that the trace $u\big|_{\Sigma_T}$ belongs to the Sobolev space $H^{5/4}_{\mathrm{loc}}(\mathbb{R})$.

2. Regularity results for a viscoelastic problem with Dirichlet boundary conditions

Denote by v the solution of

$$v_{tt} - v_{xx} - \alpha v_{xxt} = f, \quad (x,t) \in (0,L) \times (0,T), \quad \alpha > 0, \tag{2.1}$$

with initial data

$$v(\cdot,0) = u_0 \quad \text{and} \quad v_t(\cdot,0) = u_1, \tag{2.2}$$

and boundary conditions

$$v(0,\cdot) = e^{-t}\big(u_0(0) + a_0\big) \quad \text{and} \quad v(L,\cdot) = e^{-t}\big(u_0(L) + a_L\big). \tag{2.3}$$

THEOREM 2.1. *If* u_0 *and* u_1 *belong to* $H^2(0,L) \cap H^1(0,L)$ *and if* f *and* f_t *belong to* $L^2(Q_T)$, *then* v *has the following functional properties:*

$$v \in W^{2,\infty}(0,T; L^2(0,L)), \tag{2.4a}$$

$$v_x \in W^{2,\infty}(0,T; L^2(0,L)) \cap H^2(0,T; L^2(0,L)), \tag{2.4b}$$

$$v_{xx} \in L^\infty(0,T; L^2(0,L)), \quad v_{xxt} \in L^2(Q_T). \tag{2.4c}$$

PROOF. We sketch here the proof of (2.4), using the straightforward energy inequality. The proof could be easily completed by a Galerkin method, but since it is quite routine, we leave the verification to the reader. Multiply (2.5) by v_t, and integrate by parts in x; we find

$$\frac{1}{2}\int_0^L \big(|v_t(\cdot,\tau)|^2 + |v_x(\cdot,\tau)|^2\big)\,dx + \alpha \int_{Q_\tau} |v_{xt}|^2\,dx\,dt = \int_{Q_\tau} f v_t\,dx\,dt + \frac{1}{2}\int_0^L \big(|u_1|^2 + |u_{0,x}|^2\big)\,dx.$$

A straightforward application of Gronwall's lemma shows that if u_0 belongs to $L^2(0,L) \cap H_0^1(0,L)$, u_1 belongs to $L^2(0,L)$ and f belongs to $L^2(Q_T)$, then v_t and v_x are bounded in $L^\infty(0,T; L^2(0,L))$, v_{xt} is bounded in $L^2(Q_T)$. If we multiply (2.5) by v_{xx}, an application of Cauchy-Schwarz inequality implies that v_{xx} belongs to $L^\infty(0,T; L^2(0,L))$; when performing the calculation one must careful be to integrate by parts $v_{xx}v_{tt}$ first in time, and next in space. Finally, if we multiply (2.5) by v_{xxt}, another application of Cauchy-Schwarz inequality shows that v_{xxt} belongs to $L^2(Q_T)$. Similarly, after differentiating (2.5) with respect to t, and multiplying it by v_{tt}, we find that v_{xtt} belongs to $L^2(Q_T)$ under the assumption that f_t belongs to $L^2(Q_T)$. □

COROLLARY 2.2. *Under the hypotheses of Theorem 2.1, the functions defined by*

(2.5) $$g_0 = -(v_x + \alpha v_{xt})(0,\cdot) \quad \text{and} \quad g_L = -(v_x + \alpha v_{xt})(L,\cdot).$$

belong to the space $H^{1/2}(0,T)$.

PROOF. This proof is a consequence of the classical theory of traces of Sobolev spaces. □

3. Existence and uniqueness of the solution of the penalized equation

We approximate (1.1)-(1.4b) by replacing the rigid constraint (1.4a) by a very stiff response: when the constraint is active, the response is linear, and it vanishes when the constraint is not active. More precisely, letting $r^- = -\min(r,0)$, we replace u by u^ε, which satisfies

(3.1) $$u^\varepsilon_{tt} - u^\varepsilon_{xx} - \alpha u^\varepsilon_{xxt} = f \quad \text{in} \quad (0,L)\times(0,T),$$

with initial data

(3.2) $$u^\varepsilon(\cdot,0) = u_0 \quad \text{and} \quad u^\varepsilon_t(\cdot,0) = u_1,$$

and boundary conditions

(3.3) $$(u^\varepsilon_x + \alpha u^\varepsilon_{xt})(0,\cdot) = -m^\varepsilon(0,\cdot) \quad \text{and} \quad (u^\varepsilon_x + \alpha u^\varepsilon_{xt})(L,\cdot) = m^\varepsilon(L,\cdot),$$

where

(3.4) $$m^\varepsilon(0,\cdot) = \frac{\left(u^\varepsilon(0,\cdot)+a_0\right)^-}{\varepsilon} \quad \text{and} \quad m^\varepsilon(L,\cdot) = \frac{\left(u^\varepsilon(L,\cdot)+a_L\right)^-}{\varepsilon}.$$

Our convention for the Fourier transform is

$$\widehat{z}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} z(t)\,dt.$$

Define

(3.5a) $$h_0(t) = e^{-t}\big(u_0(0) + a_0\big),$$

(3.5b) $$h_L(t) = e^{-t}\big(u_0(L) + a_L\big).$$

We perform a change of unknown function by defining

$$w^\varepsilon = e^{-\nu t}(u^\varepsilon - v);$$

therefore w^ε solves the equation

(3.6) $$\big(\nu+\partial_t\big)^2 w^\varepsilon - \big(1+\alpha\big(\nu+\partial_t\big)\big) w^\varepsilon_{xx} = 0, \quad (x,t)\in(0,L)\times(0,T),$$

together with the boundary conditions

(3.7a) $$-\big(1+\alpha\big(\nu+\partial_t\big)\big) w^\varepsilon_x(0,t) = -e^{-\nu t} g_0(t) + \big(w^\varepsilon(0,t) + e^{-\nu t} h_0(t)\big)^-/\varepsilon,$$

(3.7b) $$\big(1+\alpha\big(\nu+\partial_t\big)\big) w^\varepsilon_x(L,t) = e^{-\nu t} g_L(t) + \big(w^\varepsilon(L,t) + e^{-\nu t} h_L(t)\big)^-/\varepsilon,$$

and the initial data

(3.8) $$w^\varepsilon(\cdot,0) = w^\varepsilon_t(\cdot,0) = 0.$$

We extend g_0 and g_L to all of $\mathbb{R}$ by putting them equal to 0 on $\mathbb{R}^-$ and on $[T,\infty)$; the functions h_0 and h_L are extended to all of $\mathbb{R}$ to all of $\mathbb{R}$ by putting them equal to 0 on $\mathbb{R}^-$. We expect that the solution w^ε vanishes for $t<0$. Since we seek a temperate solution, we apply a partial Fourier transform in time to (3.6), and we see that $\widehat{w}^\varepsilon(x,\omega)$ satisfies

(3.9) $$\widehat{w}^\varepsilon_{xx} = \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}\,\widehat{w}^\varepsilon.$$

Define $\widehat{\lambda}(\omega)$ by the condition

$$\widehat{\lambda}^2(\omega) = \frac{(\nu + i\omega)^2}{1 + \alpha(\nu + i\omega)}, \quad \Re\widehat{\lambda}(\omega) > 0.$$

Let us check that there exists a constant C such that

$$\forall \nu \geq 1, \quad \forall \omega > 0, \quad \Re\widehat{\lambda}(\omega) \geq C\big(1 + \sqrt{|\omega|}\big).$$

It suffices to check the case $\omega \geq 0$, the other case being deduced by complex conjugation. We check that $\widehat{\lambda}^2$ cannot belong to $\mathbb{R}^-$; if there existed a non negative number r such that $\widehat{\lambda}^2(\omega) = -r$, we would have $-r(1+\alpha(\nu+i\omega)) = \nu^2 - \omega^2 + 2i\alpha\nu\omega$. If we take imaginary parts, this gives $-\alpha\omega r = 2\alpha\nu\omega$, which is possible only if ω vanishes; but then, if we take the real parts, we see that $-r(1 + \alpha\nu) = \nu^2$, which is a contradiction. An expression of $\widehat{\lambda}(\omega)$ is

$$\widehat{\lambda}(\omega) = \frac{\nu + i\omega}{\sqrt{1 + \alpha(\nu + i\omega)}},$$

where we take the principal determination of the square root. Write $z = \nu + i\omega$. In the neighborhood of the point at infinity in $\mathbb{C}$, we have

$$\frac{z}{\sqrt{1 + \alpha z}} - \sqrt{\frac{z}{\alpha}} = O\left(\frac{1}{\sqrt{z}}\right), \tag{3.10}$$

where we have taken principal determinations. An elementary calculation shows that $\arg(1 + \alpha z) \geq \arg(z)/2$ iff $\alpha^2(\omega^2 + \nu^2) \geq 1$. On that domain,

$$\arg\left(\frac{z}{\sqrt{1 + \alpha z}}\right) \geq \frac{3\pi}{8} \quad \text{if} \quad \Re z \geq 0.$$

We also observe that $\arg\widehat{\lambda}(\omega) < \pi/2$ if $\omega \geq 0$ and $\nu > 0$; therefore there exists $\theta < \pi/2$ such that

$$\forall \omega \geq 0, \quad \forall \nu \geq 1, \quad \arg\widehat{\lambda}(\omega) \leq \theta.$$

We use the equivalent (3.10) to conclude that there exists $C > 0$ independent of ν such that for all $\omega \in \mathbb{R}$ and all $\nu \geq 1$

$$\Re\widehat{\lambda}(\omega) \geq C\big(1 + \sqrt{\nu} + \sqrt{|\omega|}\big). \tag{3.11}$$

Therefore the real part of $\widehat{\lambda}(\omega)$ is always strictly positive. In particular,

$$\lim_{\omega \to +\infty} \arg\widehat{\lambda}(\omega) = \frac{\pi}{4},$$

which shows that the argument of $\widehat{\lambda}(\omega)$ is bounded from above by some $\theta < \pi/2$. Similarly, it is plain that at $\omega = +\infty$

$$|\widehat{\lambda}(\omega)| \sim \sqrt{\frac{|\omega|}{\alpha}},$$

and since the real part of $\widehat{\lambda}(\omega)$ is strictly positive, we may conclude indeed that there exists $C > 0$ such that

$$\forall \omega \in \mathbb{R}, \quad \Re\widehat{\lambda}(\omega) \geq C\big(1 + \sqrt{|\omega|}\big). \tag{3.12}$$

The solution of (3.9) is a linear combination of $\exp\big(-\widehat{\lambda}x\big)$ and $\exp\big(\widehat{\lambda}x\big)$, with coefficients depending on ω. It is given explicitly as

$$\begin{aligned} \widehat{w}^\varepsilon(x,\omega) = & \frac{\widehat{w}^\varepsilon(0,\omega) - e^{-\widehat{\lambda}L}\widehat{w}^\varepsilon(L,\omega)}{1 - e^{-2\widehat{\lambda}L}} e^{-\widehat{\lambda}x} \\ & + \frac{-e^{-\widehat{\lambda}L}\widehat{w}^\varepsilon(0,\omega) + \widehat{w}^\varepsilon(L,\omega)}{1 - e^{-2\widehat{\lambda}L}} e^{\widehat{\lambda}(x-L)}. \end{aligned} \tag{3.13}$$

We infer from (3.12) that $1-\exp\big(-2\widehat{\lambda}L\big)$ never vanishes. Define now

$$\widehat{\eta}(\omega)=\exp\big(-\widehat{\lambda}(\omega)L\big);$$

with this notation, we calculate $\widehat{w}^\varepsilon_x(0,\omega)$ and $\widehat{w}^\varepsilon_x(L,\omega)$, and we find

$$-\widehat{w}^\varepsilon_x(0,\omega)=\frac{\widehat{\lambda}}{1-\widehat{\eta}^2}\big(\widehat{w}^\varepsilon(0,\omega)-\widehat{\eta}\widehat{w}^\varepsilon(L,\omega)\big)-\frac{\widehat{\lambda}\widehat{\eta}}{1-\widehat{\eta}^2}\big(-\widehat{\eta}\widehat{w}^\varepsilon(0,\omega)+\widehat{w}^\varepsilon(L,\omega)\big),$$

$$\widehat{w}^\varepsilon_x(L,\omega)=-\frac{\widehat{\lambda}\widehat{\eta}}{1-\widehat{\eta}^2}\big(\widehat{w}^\varepsilon(0,\omega)-\widehat{\eta}\widehat{w}^\varepsilon(L,\omega)\big)+\frac{\widehat{\lambda}}{1-\widehat{\eta}^2}\big(-\widehat{\eta}\widehat{w}^\varepsilon(0,\omega)+\widehat{w}^\varepsilon(L,\omega)\big).$$

We define now a matrix-valued function M through its Fourier transform:

$$\widehat{M}(\omega)=\frac{\widehat{\lambda}(1+\alpha(\nu+i\omega))}{1-\widehat{\eta}^2}\begin{pmatrix}1+\widehat{\eta}^2 & -2\widehat{\eta}\\ -2\widehat{\eta} & 1+\widehat{\eta}^2\end{pmatrix}.$$

By comparison with (3.7a) and (3.7b), we see that the vector-valued function

$$W^\varepsilon=\begin{pmatrix}W^\varepsilon_0\\ W^\varepsilon_L\end{pmatrix}=\begin{pmatrix}w^\varepsilon(0,\cdot)\\ w^\varepsilon(L,\cdot)\end{pmatrix}$$

satisfies the integrodifferential equation

$$M*W^\varepsilon=e^{-\nu t}\begin{pmatrix}-g_0\\ g_L\end{pmatrix}+\begin{pmatrix}\big(W^\varepsilon_0+e^{-\nu t}h_0\big)^-/\varepsilon\\ \big(W^\varepsilon_L+e^{-\nu t}h_L\big)^-/\varepsilon\end{pmatrix}. \tag{3.14}$$

Therefore, in order to solve (3.14), it suffices to obtain enough information on L, the convolution inverse of M, whose Fourier transform is

$$\widehat{L}(\omega)=\frac{1}{(1-\widehat{\eta}^2)\big(\widehat{\lambda}(1+\alpha(\nu+i\omega))\big)}\begin{pmatrix}1+\widehat{\eta}^2 & 2\widehat{\eta}\\ 2\widehat{\eta} & 1+\widehat{\eta}^2\end{pmatrix}.$$

Since $\widehat{\lambda}(1+\alpha(\nu+i\omega))=(\nu+i\omega)\sqrt{1+\alpha(\nu+i\omega)}$, giving the inverse Fourier transform of $(1+\alpha(\nu+i\omega))^{-1/2}$ can be obtained by complex integration as in [10]:

$$\mu(t)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{i\omega t}}{1+\alpha(\nu+i\omega)}\,d\omega=\frac{\exp\big(-t(\nu+1/\alpha)\big)}{\sqrt{2\alpha\pi t}}1_{\mathbb{R}^+}(t) \tag{3.15}$$

therefore, μ_1 the inverse Fourier transform of $\widehat{\lambda}(1+\alpha(\nu+i\omega))^{-1}$ is the convolution of μ with $e^{-\nu t}1_{\mathbb{R}^+}$. This is a continuous function which vanishes over $\mathbb{R}^-$, and moreover

$$0\le\mu_1(t)=\int_0^t e^{-\nu(t-s)}\mu(s)\,ds\le\sqrt{\alpha\pi}e^{-\nu t}.$$

Observe now that

$$\frac{1}{1-\widehat{\eta}^2}\begin{pmatrix}1+\widehat{\eta}^2 & 2\widehat{\eta}\\ 2\widehat{\eta} & 1+\widehat{\eta}^2\end{pmatrix}-\begin{pmatrix}1&0\\0&1\end{pmatrix}=\frac{2\widehat{\eta}}{1-\widehat{\eta}^2}\begin{pmatrix}\widehat{\eta}&1\\1&\widehat{\eta}\end{pmatrix}=\widehat{L}_0(\omega).$$

Let us show now that η belong to the Schwartz space $\mathcal{S}(\mathbb{R})$. By induction, it is clear that

$$\frac{d^k\widehat{\eta}}{d\omega^k}=\frac{P_k(\omega)\sqrt{1+\alpha(\nu+i\omega)}+Q_k(\omega)}{(1+\alpha(\nu+i\omega))^{m_k}}\widehat{\eta},$$

where P_k and Q_k are polynomials in ω and m_k is an integer. Therefore, estimate (3.12) implies that all the integrals

$$\int_{\mathbb{R}}|\omega|^m\left|\frac{d^k\widehat{\eta}}{d\omega^k}\right|^2d\omega;$$

are bounded for m and k arbitrary integers; this proves our assertion on $\widehat{\eta}$. Paley-Wiener theorem implies that η is supported in $\mathbb{R}^+$.

Moreover, as ω tends to infinity, $\widehat{\eta}$ tends to zero; since $1-\widehat{\eta}^2$ never vanishes, $1-\widehat{\eta}^2$ is bounded away from 0. Therefore the matrix valued function L_0 belongs to $\mathcal{S}(\mathbb{R})$, and supported in $\mathbb{R}^+$. We may write now

$$L = \mu_1 + \mu_1 * L_0,$$

and, of course, $\mu_1 * L_0$ is a bounded matrix valued function of class C^∞. It is clearly equivalent to solve (3.14) and

$$W^\varepsilon = L * \left(e^{-\nu t} \begin{pmatrix} -g_0 \\ g_L \end{pmatrix} + \begin{pmatrix} \left(W_0^\varepsilon + e^{-\nu t} h_0\right)^- /\varepsilon \\ \left(W_L^\varepsilon + e^{-\nu t} h_L\right)^- /\varepsilon \end{pmatrix} \right). \tag{3.16}$$

It is convenient to define

$$g(t) = e^{-\nu t} \begin{pmatrix} -g_0(t) \\ g_L(t) \end{pmatrix} \quad \text{and} \quad G(W^\varepsilon, t) = \begin{pmatrix} \left(W_0^\varepsilon + e^{-\nu t} h_0\right)^- \\ \left(W_L^\varepsilon + e^{-\nu t} h_L\right)^- \end{pmatrix}.$$

If we equip $\mathbb{R}^2$ with the Euclidean norm, it is plain that

$$|G(W,\cdot) - G(Z,\cdot)| \le |W - Z|. \tag{3.17}$$

THEOREM 3.1. *Let $Z = \{u \in H^1(Q_T) : u_{xt} \in L^2(Q_T)\}$. Then for each $\varepsilon > 0$ there exists a unique weak solution $u^\varepsilon \in Z$ of the problem (3.1)-(3.3); moreover, u^ε satisfies the following functional properties*

$$u^\varepsilon \in L^\infty(0,T; H^2(0,L)), \tag{3.18a}$$

$$u_t^\varepsilon \in L^2(0,T; H^2(0,L)), \tag{3.18b}$$

$$u_{tt}^\varepsilon \in L^2(0,T; L^2(0,L)), \tag{3.18c}$$

and for almost every $\tau \in (0,T)$ and for all $v \in Z$, the following variational equality is satisfied:

$$\begin{aligned} &\int_0^L u_t^\varepsilon(\cdot,\tau) v(\cdot,\tau)\, dx - \int_0^L u_1 v(\cdot,0)\, dx - \int_{Q_\tau} u_t^\varepsilon v_t \, dx\, dt + \int_{Q_\tau} u_x^\varepsilon v_x \, dx\, dt \\ &+\alpha \int_{Q_\tau} u_{xt}^\varepsilon v_x \, dx\, dt - \int_0^\tau (m^\varepsilon v)(L,\cdot)\, dt - \int_0^\tau (m^\varepsilon v)(0,\cdot)\, dt = \int_{Q_\tau} f v \, dx\, dt. \end{aligned} \tag{3.19}$$

Moreover, there exist constant C and C' such that

$$e^{-\nu t}\big(|u^\varepsilon(0,t) - v(0,t)| + |u^\varepsilon(L,t) - v(L,t)|\big) \le C \exp\big(C' t/\varepsilon\big)$$

PROOF. In this proof we drop the superscript ε. We rewrite (3.16) as

$$W = L * \big(g + G(W,\cdot)/\varepsilon\big). \tag{3.20}$$

Define a mapping $\mathcal{T}$ by

$$(\mathcal{T}W)(t) = L * g + L * G(W,\cdot)/\varepsilon.$$

If we let Λ be the maximum of the norm of $L(t)$ over $[0,T]$, we infer from (3.17) the estimate

$$\big|(\mathcal{T}W)(t) - (\mathcal{T}Z)(t)\big| \le \frac{\Lambda}{\varepsilon} \int_0^t \big|W(s) - Z(s)\big|\, ds$$

and by Picard iteration, we see that for p large enough, $\mathcal{T}^p$ is a contraction from $C^0\big([0,T];\mathbb{R}^2\big)$ to itself. Moreover, $L * g$ is of class C^∞; therefore, the standard method for proving Cauchy-Lipschitz theorem works, and moreover, we obtain a pointwise estimate of the form

$$\big|W(t)\big| \le C \exp\big(\Lambda t/\varepsilon\big).$$

We remark that we may apply Picard iterations to (3.20) to obtain the existence of a solution; it suffices to consider the iterations:

$$W^0 = 0, \quad W^{n+1} = L * \big(g + G(W^n,\cdot)\big)/\varepsilon,$$

and apply the standard estimates. Uniqueness is obtained by the very same argument, since a large enough power of the mapping $\mathcal{T} : W^n \mapsto W^{n+1}$ is a strict contraction in the space $C^0([0,T];\mathbb{R}^2)$, since L is bounded on $[0,T]$. □

4. Estimates on the penalized solution

4.1. Estimates up to the boundary.

LEMMA 4.1. *Under assumptions of Theorem 2.1, independently of $\varepsilon > 0$, u_t^ε, u_x^ε are bounded in $L^\infty(0,T;L^2(0,L))$, u_{xt}^ε is bounded in $L^2(Q_T)$, $\varepsilon(m^\varepsilon(0,\cdot))^2$ and $\varepsilon(m^\varepsilon(L,\cdot))^2$ are bounded in $L^\infty(0,T)$.*

PROOF. These estimates are simply an application of Gronwall's lemma to the energy estimate. We multiply (3.1) by u_t^ε; next we integrate this expression over Q_τ to get

$$\int_{Q_\tau} u_{tt}^\varepsilon u_t^\varepsilon \, dx\, dt - \int_{Q_\tau} u_{xx}^\varepsilon u_t^\varepsilon \, dx\, dt - \alpha \int_{Q_\tau} u_{xxt}^\varepsilon u_t^\varepsilon \, dx\, dt = \int_{Q_\tau} f u_t^\varepsilon \, dx\, dt. \tag{4.1}$$

We integrate in time the first integral in (4.1), we integrate by parts in space the second and the third one, and with the help of the boundary conditions (3.3) we obtain

$$\begin{aligned}\frac{1}{2}\int_0^L \left(|u_t^\varepsilon(\cdot,\tau)|^2 + |u_x^\varepsilon(\cdot,\tau)|^2\right) dx + \alpha \int_{Q_\tau} |u_{xt}^\varepsilon|^2 \, dx\, dt - \int_0^\tau (m^\varepsilon u_t^\varepsilon)(L,\cdot)\, dt \\ - \int_0^\tau (m^\varepsilon u_t^\varepsilon)(0,\cdot)\, dt = \int_{Q_\tau} f u_t^\varepsilon \, dx\, dt + \frac{1}{2}\int_0^L \left(|u_{0,x}|^2 + |u_1|^2\right) dx.\end{aligned} \tag{4.2}$$

On the other hand, we observe that for $x_0 = 0$ or $x_0 = L$

$$\int_0^\tau (m^\varepsilon u_t^\varepsilon)(x_0,\cdot)\, dt = -\frac{\varepsilon}{2}\left(m^\varepsilon(x_0,t)\right)^2\Big|_0^\tau. \tag{4.3}$$

Therefore, relation (4.3) implies that

$$\begin{aligned}\frac{1}{2}\int_0^L \left(|u_t^\varepsilon(\cdot,\tau)|^2 + |u_x^\varepsilon(\cdot,\tau)|^2\right) dx + \alpha \int_{Q_\tau} |u_{xt}^\varepsilon|^2 \, dx\, dt + \frac{\varepsilon}{2}\left(m^\varepsilon(0,t)\right)^2\Big|_0^\tau \\ + \frac{\varepsilon}{2}\left(m^\varepsilon(L,t)\right)^2\Big|_0^\tau \le \int_{Q_\tau} |f u_t^\varepsilon| \, dx\, dt + \frac{1}{2}\int_0^L \left(|u_{0,x}|^2 + |u_1|^2\right) dx.\end{aligned} \tag{4.4}$$

We may deduce from a classical Gronwall lemma that u_t^ε, u_x^ε are bounded in $L^\infty(0,T;L^2(0,L))$ and u_{xt}^ε is bounded in $L^2(Q_T)$, $\varepsilon(m^\varepsilon(0,\cdot))^2$ and $\varepsilon(m^\varepsilon(L,\cdot))^2$ are bounded in $L^\infty(0,T)$. □

REMARK 4.2. *If we suppose that f vanishes when $t > T$ then independently of $\varepsilon > 0$, u_t^ε and u_x^ε are bounded in $L^\infty(0,\infty;L^2(0,L))$ and u_{xt}^ε is bounded in $L^2(0,\infty;L^2(0,L))$. These properties can be proved with the help of the arguments given in the proof of Lemma 4.1, with the origin of time moved to T; the important fact is that since the integral involving f vanishes.*

LEMMA 4.3. *Under assumptions of Theorem 2.1, independently of $\varepsilon > 0$, u^ε is bounded in $C^{0,1/2}(\bar{Q}_T)$.*

PROOF. Define

$$\rho = 1_{[0,1]} \quad \text{and} \quad \rho_\beta = \frac{1}{\beta}\rho\left(\frac{\cdot}{\beta}\right).$$

Assume that v belongs to the Hölder space $C^{0,1/2}$. if $\beta \in (0,L)$ and $0 \le x \le L-\beta$, the function

$$v(x) - (\rho_\beta * v)(x) = \frac{1}{\beta}\int_x^{x+\beta} \left(v(x) - v(y)\right) dy,$$

can be estimated as follows:

$$|v(x)-(\rho_\beta * v)(x)| \le \frac{2}{3}\sqrt{\beta}\,\|v\|_{C^{0,1/2}}\,. \tag{4.5}$$

On the other hand, if v belongs to $L^2(0,L)$, and is extended by 0 outside of $(0,L)$, we see immediately that

$$|\rho_\beta * v|_{L^\infty} \le \frac{|v|_{L^2}}{\sqrt{\beta}}. \tag{4.6}$$

For $x \in [0, L-\beta]$, we estimate $u^\varepsilon(x,t)-u^\varepsilon(x,t')$ thanks to the triangle inequality

$$|u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t')| \le |u^\varepsilon(\cdot,t)-(\rho_\beta * u^\varepsilon(\cdot,t'))| + |(\rho_\beta * u^\varepsilon(\cdot,t))-(\rho_\beta * u^\varepsilon(\cdot,t'))| + |u^\varepsilon(\cdot,t')-(\rho_\beta * u^\varepsilon(\cdot,t'))|\,,$$

to which we apply (4.5) and (4.6); we obtain thus

$$|u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t')| \le \frac{4}{3}\sqrt{\beta}|u^\varepsilon|_{L^\infty(0,T;C^{0,1/2}(0,L))} + \frac{1}{\beta}|u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t')|,$$

and thanks to Lemma 4.1, we see that

$$|u^\varepsilon(\cdot,t)-u^\varepsilon(\cdot,t')| \le C\left(\sqrt{\beta}+\frac{|t'-t|}{\sqrt{\beta}}\right).$$

An analogous inequality holds if $\beta \le x \le L$, by choosing $\rho = 1_{[-1,0]}$. The choice $\beta = |t'-t|$ shows that u^ε is Hölder continuous in time with exponent $1/2$; the Hölder continuity in space is classical. □

LEMMA 4.4. *Under assumptions of Theorem 2.1. Let m^ε be the sequence defined by the relations* (3.4). *Then independently of $\varepsilon > 0$, m^ε is bounded in $M^1(\Sigma_T)$, the space of bounded measures on Σ_T.*

PROOF. We integrate (3.1) over Q_τ and we obtain with the help of the boundary conditions

$$\int_0^L u_t^\varepsilon(\cdot,t)\big|_0^\tau\,dx - \int_0^\tau \big(m^\varepsilon(L,\cdot)+m^\varepsilon(0,\cdot)\big)\,dt = \int_{Q_\tau} f\,dx\,dt,$$

which implies immediately

$$\int_0^\tau \big(m^\varepsilon(L,\cdot)+m^\varepsilon(0,\cdot)\big)\,dt \le \frac{1}{2}\int_0^L \big(|u_t^\varepsilon(\cdot,\tau)|^2+|u_1|^2\big)\,dx + L + \int_{Q_\tau}|f|\,dx\,dt.$$

As $f \in L^2(Q_T)$, $u_1 \in L^2(0,L)$ and u_t^ε is bounded in $L^\infty(0,T;L^2(0,L))$, and m^ε is non negative, the conclusion is clear. □

LEMMA 4.5. *Under assumptions of Theorem 2.1, independently of $\varepsilon > 0$, u_{xx}^ε is bounded in $L^\infty(0,T;L^2(0,L))$.*

PROOF. Once again we use energy techniques, but now we multiply relation (3.1) by u_{xx}^ε and we integrate over Q_τ;

$$\int_{Q_\tau} u_{tt}^\varepsilon u_{xx}^\varepsilon\,dx\,dt - \int_{Q_\tau}|u_{xx}^\varepsilon|^2\,dx\,dt - \alpha\int_{Q_\tau} u_{xxt}^\varepsilon u_{xx}^\varepsilon\,dx\,dt = \int_{Q_\tau} f u_{xx}^\varepsilon\,dx\,dt. \tag{4.7}$$

We integrate by parts the first integral in (4.7) once in space and once in time and we observe that the third integral contains a total time derivative; we get thus

$$\begin{aligned}\int_0^L \big((u_t^\varepsilon u_{xx}^\varepsilon)(\cdot,t)\big)\big|_0^\tau dx - \int_0^\tau \big((u_t^\varepsilon u_{xt}^\varepsilon)(x,t)\big)\big|_0^L dt + \int_{Q_\tau}|u_{xt}^\varepsilon|^2\,dx\,dt \\ - \int_{Q_\tau}|u_{xx}^\varepsilon|^2\,dx\,dt - \frac{\alpha}{2}\int_0^L |u_{xx}^\varepsilon(\cdot,t)|^2\big|_0^\tau\,dx = \int_{Q_\tau} f u_{xx}^\varepsilon\,dx\,dt.\end{aligned} \tag{4.8}$$

By application of the boundary conditions (3.3), (4.8) becomes

$$\int_{Q_\tau} |u^\varepsilon_{xx}|^2\, dx\, dt + \frac{\alpha}{2}\int_0^L |u^\varepsilon_{xx}(\cdot,\tau)|^2\, dx = \frac{\varepsilon}{2\alpha}\,|m^\varepsilon(0,t)|^2\Big|_0^\tau + \frac{\varepsilon}{2\alpha}\,|m^\varepsilon(L,t)|^2\Big|_0^\tau$$
$$+\frac{1}{\alpha}\int_0^\tau (u^\varepsilon_t u^\varepsilon_x)(L,\cdot)\, dt - \frac{1}{\alpha}\int_0^\tau (u^\varepsilon_t u^\varepsilon_x)(0,\cdot)\, dt + \int_0^L (u^\varepsilon_t u^\varepsilon_{xx})(\cdot,\tau)\, dx$$
$$-\int_{Q_\tau} f u^\varepsilon_{xx}\, dx\, dt - \int_0^L u_1 u_{0,xx}\, dx + \frac{\alpha}{2}\int_0^L |u_{0,xx}|^2\, dx + \int_{Q_\tau} |u^\varepsilon_{xt}|^2\, dx\, dt.$$

Let us estimate the boundary integrals at $x = 0$ and at $x = L$. As u^ε_t and u^ε_{xt} are bounded in $L^2(Q_\tau)$, we have the trace estimate:

$$\int_0^\tau \left(|u^\varepsilon_t(0,\cdot)|^2 + |u^\varepsilon_t(L,\cdot)|^2\right) dt \le C\left(\|u^\varepsilon_t\|^2_{L^2(Q_\tau)} + \|u^\varepsilon_{xt}\|^2_{L^2(Q_\tau)}\right), \tag{4.9}$$

and similarly

$$\int_0^\tau \left(|u^\varepsilon_x(0,\cdot)|^2 + |u^\varepsilon_x(L,\cdot)|^2\right) dt \le C\left(\|u^\varepsilon_x\|^2_{L^2(Q_\tau)} + \|u^\varepsilon_{xx}\|^2_{L^2(Q_\tau)}\right). \tag{4.10}$$

We estimate the product $|zy|$ by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, choosing different values of $\gamma_i > 0$, $i = 1, 2, 3$, in different terms and we get the following inequality:

$$\int_{Q_\tau} |u^\varepsilon_{xx}|^2\, dx\, dt + \frac{\alpha}{2}\int_0^L |u^\varepsilon_{xx}(\cdot,\tau)|^2\, dx \le \frac{\varepsilon}{2\alpha}\left((m^\varepsilon(0,t))^2 + (m^\varepsilon(L,t))^2\right)\Big|_0^\tau$$
$$+\frac{C\gamma_3}{2\alpha}\,\|u^\varepsilon_{xx}\|^2_{L^2(Q_\tau)} + \frac{\gamma_0}{2}\int_0^L |u^\varepsilon_{xx}(\cdot,\tau)|^2\, dx + \frac{\gamma_L}{2}\int_{Q_\tau} |u^\varepsilon_{xx}|^2\, dx\, dt + E,$$

where

$$E = \frac{C\gamma_3}{2\alpha}\,\|u^\varepsilon_x\|^2_{L^2(Q_\tau)} + \frac{C}{2\alpha\gamma_3}\left(\|u^\varepsilon_t\|^2_{L^2(Q_\tau)} + \|u^\varepsilon_{xt}\|^2_{L^2(Q_\tau)}\right) + \frac{1}{2\gamma_0}\int_0^L |u^\varepsilon_t(x,\tau)|^2\, dx$$
$$+\frac{1}{2\gamma_L}\int_{Q_\tau} |f|^2\, dx\, dt + \int_0^L |u_1 u_{0,xx}|\, dx + \frac{\alpha}{2}\int_0^L |u_{0,xx}|^2\, dx + \int_{Q_\tau} |u^\varepsilon_{xt}|^2\, dx\, dt.$$

We choose $\gamma_i > 0$ such that $\alpha > \gamma_0$ and $\alpha > C\gamma_3/(2 - \gamma_L)$ and we obtain

$$\begin{aligned}\left(1 - \frac{\gamma_L}{2} - \frac{C\gamma_3}{2\alpha}\right)\int_{Q_\tau} |u^\varepsilon_{xx}|^2\, dx\, dt + \left(\frac{\alpha}{2} - \frac{\gamma_0}{2}\right)\int_0^L |u^\varepsilon_{xx}(\cdot,\tau)|^2\, dx \\ \le \frac{\varepsilon}{2\alpha}\left((m^\varepsilon(0,t))^2 + (m^\varepsilon(L,t))^2\right)\Big|_0^\tau + E;\end{aligned} \tag{4.11}$$

as f is bounded in $L^2(Q_T)$, we conclude the proof thanks to Lemma 4.1. □

REMARK 4.6. *If we suppose that f vanishes for $t > T$ then independently of $\varepsilon > 0$, u^ε_{xx} is bounded in $L^2(0,\infty; L^2(0,L))$. These properties can be proved with the argument stated in Remark 4.2.*

REMARK 4.7. *Thanks to Lemmas 4.1 and 4.5, u^ε_x is bounded in $H^1(Q_T)$. Let $u^\varepsilon_x\big|_{\Sigma_T}$ be the trace u^ε_x on Σ_T. Since $u^\varepsilon_x\big|_{\Sigma_T}$ is bounded in $H^{1/2}(\Sigma_T)$, we may extract a subsequence which is still denoted by $u^\varepsilon_x\big|_{\Sigma_T}$ such that*

$$u^\varepsilon_x\big|_{\Sigma_T} \rightharpoonup u_x\big|_{\Sigma_T} \text{ weakly in } H^{1/2}(\Sigma_T).$$

4.2. Interior estimates. Let us turn now to interior estimates.

LEMMA 4.8. *Under assumptions of Theorem 2.1. For all* $\beta \in (0, L/2)$, u^ε_{tt} *and* u^ε_{xxt} *are bounded in* $L^2(0,T; L^2(\beta, L-\beta))$, *independently of* $\varepsilon > 0$.

PROOF. The idea of the proof is double: first, we multiply u^ε by a cutoff function $\omega \in C_0^\infty(\mathbb{R})$, and we define $v^\varepsilon = \omega u^\varepsilon$; next, we observe that $w^\varepsilon = v^\varepsilon_t$ satisfies a heat equation, whose right hand side can be estimated thanks to the previous lemmas. Let us go now into details. The cutoff function is given by

$$\omega(x) = \begin{cases} 1 & \text{if} \quad \beta \le x \le L-\beta, \\ 0 & \text{if} \quad x \le \beta/2 \quad \text{or} \quad x \ge L - \beta/2. \end{cases} \tag{4.12}$$

Define

$$v^\varepsilon(\cdot, t) = \omega u^\varepsilon(\cdot, t). \tag{4.13}$$

The derivatives of v^ε are given by:

$$\begin{aligned} v^\varepsilon_{tt}(\cdot,t) &= \omega u^\varepsilon_{tt}(\cdot,t), \\ v^\varepsilon_{xx}(\cdot,t) &= \omega u^\varepsilon_{xx}(\cdot,t) + 2\,\omega_x u^\varepsilon_x(\cdot,t) + \omega_{xx} u^\varepsilon(\cdot,t), \\ v^\varepsilon_{xxt}(\cdot,t) &= \omega u^\varepsilon_{xxt}(\cdot,t) + 2\,\omega_x u^\varepsilon_{xt}(\cdot,t) + \omega_{xx} u^\varepsilon_t(\cdot,t). \end{aligned}$$

Notice that thanks to these relations and (3.1), v^ε satisfies

$$v^\varepsilon_{tt} - v^\varepsilon_{xx} - \alpha\, v^\varepsilon_{xxt} = \widetilde{g}^\varepsilon, \tag{4.14}$$

where

$$\widetilde{g}^\varepsilon = \omega f - 2\,\omega_x\big(u^\varepsilon_x + \alpha\, u^\varepsilon_{xt}\big) - \omega_{xx}\big(u^\varepsilon + \alpha u^\varepsilon_t\big).$$

We have proved at Lemmas 4.1 and 4.3, that u^ε_t, u^ε_x, u^ε_{xt} are bounded in $L^2(Q_T)$, u^ε is bounded in $C^{0,1/2}(\bar{Q}_T)$; since f belongs to $L^2(Q_T)$, we see that $\widetilde{g}^\varepsilon$ is bounded in $L^2(\mathbb{R} \times (0,T))$. Let us define

$$w^\varepsilon = v^\varepsilon_t \quad \text{and} \quad g^\varepsilon = \widetilde{g}^\varepsilon + v^\varepsilon_{xx}. \tag{4.15}$$

Substituting (4.15) in (4.14), we obtain

$$w^\varepsilon_t - \alpha\, w^\varepsilon_{xx} = g^\varepsilon. \tag{4.16}$$

Since v^ε_{xx} and $\widetilde{g}^\varepsilon$ are bounded in $L^2(\mathbb{R}\times(0,T))$, g^ε is bounded in $L^2\big(\mathbb{R}\times(0,T)\big)$. Let us prove now that w^ε_t is bounded in $L^2(\mathbb{R} \times (0,T))$. For this purpose, we multiply (4.16) by w^ε_t and we integrate the resulting expression over $\mathbb{R}$,

$$\int_{\mathbb{R}} |w^\varepsilon_t|^2\, dx - \alpha \int_{\mathbb{R}} w^\varepsilon_t\, w^\varepsilon_{xx}\, dx = \int_{\mathbb{R}} g^\varepsilon\, w^\varepsilon_t\, dx.$$

We integrate by parts the second term on the left hand side, thanks to relations (4.13), (4.15). Since $\omega \in C_0^\infty(\mathbb{R})$, we may infer that $w^\varepsilon_t w^\varepsilon_x$ vanishes when x tends to infinity, getting thus

$$\int_{\mathbb{R}} |w^\varepsilon_t|^2\, dx + \alpha \int_{\mathbb{R}} w^\varepsilon_{xt}\, w^\varepsilon_x\, dx = \int_{\mathbb{R}} g^\varepsilon\, w^\varepsilon_t\, dx. \tag{4.17}$$

On the other hand, we observe that

$$\int_{\mathbb{R}} g^\varepsilon\, w^\varepsilon_t\, dx \le \frac{1}{2} \int_{\mathbb{R}} |g^\varepsilon|^2\, dx + \frac{1}{2} \int_{\mathbb{R}} |w^\varepsilon_t|^2\, dx. \tag{4.18}$$

Substituting (4.18) into (4.17), and integrating over $(0,\tau)$, we obtain

$$\begin{aligned} &\int_0^\tau \int_{\mathbb{R}} |w^\varepsilon_t|^2\, dx\, dt + \frac{\alpha}{2} \int_{\mathbb{R}} |w^\varepsilon_x(\cdot,\tau)|^2\, dx - \frac{\alpha}{2} \int_{\mathbb{R}} |w^\varepsilon_x(\cdot,0)|^2\, dx \\ &\le \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} |g^\varepsilon|^2\, dx\, dt + \frac{1}{2} \int_0^\tau \int_{\mathbb{R}} |w^\varepsilon_t|^2\, dx\, dt, \end{aligned}$$

which implies immediately that

$$\int_0^\tau \int_{\mathbb{R}} |w_t^\varepsilon|^2\,dx\,dt + \alpha \int_{\mathbb{R}} |w_x^\varepsilon(\cdot,\tau)|^2\,dx \le \alpha \int_{\mathbb{R}} |w_x^\varepsilon(\cdot,0)|^2\,dx + \int_0^\tau \int_{\mathbb{R}} |g^\varepsilon|^2\,dx\,dt. \tag{4.19}$$

As

$$w_x^\varepsilon(\cdot,0) = \omega_x u_1 + \omega u_{1,x}; \tag{4.20}$$

u_1 belongs to $H^1(0,L)$ and ω belongs to $C_0^\infty(\mathbb{R})$, relation (4.20) implies that $w_x^\varepsilon(\cdot,0) \in L^2(\mathbb{R})$. As g^ε belongs to $L^2(\mathbb{R}\times(0,T))$, the left hand side of inequality (4.19) is bounded. Then we infer from (4.13) and (4.15) that u_{tt}^ε is bounded in $L^2((\beta, L-\beta)\times(0,T))$ independently of ε.

We use analogous arguments to show that u_{xxt}^ε is bounded in $L^2((\beta, L-\beta)\times(0,T))$: we multiply (4.16) by w_{xx}^ε, next we integrate over Q_τ and, thanks to an integration by parts, we get

$$-\frac{1}{2}\int_{\mathbb{R}} |w_x^\varepsilon(\cdot,t)|^2 \Big|_0^\tau \,dx - \alpha \int_0^\tau \int_{\mathbb{R}} |w_{xx}^\varepsilon|^2\,dx\,dt = \int_0^\tau \int_{\mathbb{R}} g^\varepsilon w_{xx}^\varepsilon\,dx\,dt. \tag{4.21}$$

We have used the inequality $yz \le y^2/2\gamma + \gamma z^2/2$ where $\gamma \in (0, 2\alpha)$, and therefore

$$\left(\alpha - \frac{\gamma}{2}\right)\int_0^\tau \int_{\mathbb{R}} |w_{xx}^\varepsilon|^2\,dx\,dt \le \frac{1}{2\gamma}\int_0^\tau \int_{\mathbb{R}} |g^\varepsilon|^2\,dx\,dt + \frac{1}{2}\int_{\mathbb{R}} |w_x^\varepsilon(\cdot,0)|^2\,dx.$$

Since g^ε and $w_x^\varepsilon(\cdot,0)$ are respectively bounded in $L^2(\mathbb{R})$ previous inequality, we deduce that w_{xx}^ε is bounded in $L^2(\mathbb{R}\times(0,T))$. Therefore relations (4.13) and (4.15) imply that u_{xxt}^ε is bounded in $L^2(0,T;L^2(\beta, L-\beta))$. □

REMARK 4.9. *If we differentiate* (4.14) *with respect to* t, *multiply it by* v_{tt}^ε *and integrate the resulting expression over* $(0,\tau)\times\mathbb{R}$, *we find that for all* $\beta \in (0, L/2)$ u_{xtt}^ε *belongs to* $L^2(0,T;L^2(\beta, L-\beta))$, *independently of* ε.

5. Passage to the limit in the variational formulation

We have given a variational formulation of the penalized problem. The previous Lemmas enable us to pass to the limit and to deduce the existence of a weak solution (1.1)-(1.4b). We substitute $v - u^\varepsilon$ to v in (3.19) and we get

$$\begin{aligned}&\int_0^L \big(u_t^\varepsilon(v-u^\varepsilon)\big)(\cdot,t)\big|_0^\tau\,dx - \int_{Q_\tau} u_t^\varepsilon\big(v_t - u_t^\varepsilon\big)\,dx\,dt \\ &+ \int_{Q_\tau}\big(u_x^\varepsilon + \alpha u_{xt}^\varepsilon\big)\big(v_x - u_x^\varepsilon\big)dx\,dt - \int_0^\tau \big(m^\varepsilon(v-u^\varepsilon)\big)(L,\cdot)\,dt \\ &- \int_0^\tau \big(m^\varepsilon(v-u^\varepsilon)\big)(0,\cdot)\,dt = \int_{Q_\tau} f\big(v-u^\varepsilon\big)\,dx\,dt.\end{aligned} \tag{5.1}$$

LEMMA 5.1. *Under assumptions of Theorem 2.1. Let* u^ε *be a solution of* (3.1)-(3.3). *There exist subsequences, still denoted by* u^ε, *such that*

$$u^\varepsilon \to u \ \textit{ in } C^{0,1/2}(\bar{Q}_T), \tag{5.2a}$$

$$u_t^\varepsilon \rightharpoonup u_t \ \textit{ weakly } * \textit{ in } L^\infty(0,T;L^2(0,L)), \tag{5.2b}$$

$$u_x^\varepsilon \to u_x \ \textit{ strongly in } L^2(Q_T), \tag{5.2c}$$

$$u_{xx}^\varepsilon \rightharpoonup u_{xx} \ \textit{ weakly } * \textit{ in } L^\infty(0,T;L^2(0,L)), \tag{5.2d}$$

$$u_{xt}^\varepsilon \rightharpoonup u_{xt} \ \textit{ weakly } * \textit{ in } L^2(Q_T), \tag{5.2e}$$

$$m^\varepsilon \rightharpoonup m \ \textit{ weakly } * \textit{ in } M^1(\Sigma_T). \tag{5.2f}$$

Moreover the support of m, *denoted by* $\operatorname{supp} m$ *is included in the following set:*

$$\{t \in \mathbb{R}^+ : u(0,\cdot) = -a_0,\ u(L,\cdot) = -a_L\}.$$

PROOF. Lemma 5.1 is an immediate consequence of Lemmas 4.1-4.4. In fact, we know that u^ε, u^ε_t, u^ε_{xx}, u^ε_{xt} and m^ε are bounded in functional spaces explicitly given in the Lemmas 4.1-4.4. Therefore we may extract subsequences, still denoted by u^ε, u^ε_t, u^ε_{xx}, u^ε_{xt} and m^ε, which converge weakly as in (5.2). On the other hand, u^ε_{xt} is bounded in $L^2(Q_T)$ and u^ε_{xx} is bounded in $L^\infty(0,T;L^2(0,L))$ then by Sobolev injections, we may extract a subsequence, still denoted u^ε_x, such that u^ε_x converges strongly to u_x in $L^2(Q_T)$. □

The convergence of u_t is obtained with the help of interior estimates (Lemma 4.8).

LEMMA 5.2. *Under assumptions of Theorem 2.1. The convergence of* u^ε_t *to its limit is strong in* $L^2(Q_T)$.

PROOF. Write

$$y = |u^\varepsilon_t - u_t|^2.$$

For any $\beta > 0$ we have:

$$\int_{Q_T} y\,dx\,dt = \int_0^T \left(\int_0^\beta y\,dx + \int_\beta^{L-\beta} y\,dx + \int_{L-\beta}^L y\,dx\right) dt. \tag{5.3}$$

Since u^ε_t is bounded in the Sobolev space $H^1(0,L;L^2(0,T))$, u^ε_t is bounded in $C^{0,1/2}(0,L;L^2(0,T))$. Denote by C the supremum of $|u^\varepsilon_t(x,\cdot)|_{L^2(0,T)}$ with respect to x; a Cauchy-Schwarz inequality on the time integral gives

$$\int_0^T \int_0^\beta |u^\varepsilon_t|^2\,dx\,dt + \int_0^T \int_{L-\beta}^L |u^\varepsilon_t|^2\,dx\,dt \le 2C^2\beta. \tag{5.4}$$

Similarly the supremum of $|u_t(x,\cdot)|_{L^2(0,T)}$ is at most equal to C, so that

$$\int_0^T \int_0^\beta |u_t|^2\,dx\,dt + \int_0^T \int_{L-\beta}^L |u_t|^2\,dx\,dt \le 2C^2\beta. \tag{5.5}$$

Let γ be any positive number. Choose β such that $8C^2\beta \le \gamma/2$. Lemma 4.8 implies that u^ε_t is bounded in $H^1\big([\beta, L-\beta]\times[0,T]\big)$ then it is possible to extract a subsequence such that the restriction of u^ε_t to $[\beta, L-\beta]\times[0,T]$ converges strongly to u_t; in particular, for γ small enough,

$$\int_0^T \int_\beta^{L-\beta} |u^\varepsilon_t - u_t|^2\,dx\,dt \le \frac{\gamma}{2}. \tag{5.6}$$

Carrying (5.4)-(5.6) into (5.3), we deduce the Lemma. □

THEOREM 5.3. *Under assumptions of Theorem 2.1, problem* (1.1)-(1.4b) *possesses a weak solution* u *such that* u *is bounded in* $C^{0,1/2}(\bar{Q}_T)$, u_x *and* u_t *are bounded in* $L^\infty(0,T;L^2(0,L))$, u_{xx} *is bounded in* $L^2(Q_T)$.

PROOF. We pass to the limit in (5.1) when ε tends to zero. Since for all $(r,s) \in \mathbb{R}^2$, $(s^- - r^-)(r-s) \ge 0$, if we take $s = u^\varepsilon(0,\cdot) + a_0$ and $r = v(0,\cdot) + a_0$, we may infer that

$$\begin{aligned} &\big(v(0,\cdot)+a_0\big)\big(v(0,\cdot)+a_0\big)^- - \big(u^\varepsilon(0,\cdot)+a_0\big)\big(v(0,\cdot)+a_0\big)^- \\ &\le \big(v(0,\cdot)+a_0\big)\big(u^\varepsilon(0,\cdot)+a_0\big)^- - \big(u^\varepsilon(0,\cdot)+a_0\big)\big(u^\varepsilon(0,\cdot)+a_0\big)^-. \end{aligned} \tag{5.7}$$

Since $v(0,\cdot) \geq -a_0$ for all $t \in [0,\tau]$, the left hand side of inequality (5.7) vanishes. On the other hand, the right hand side of (5.7) is equal to $\big(m^\varepsilon\big(v-u^\varepsilon\big)\big)(0,\cdot)$ which leads to

$$\int_0^\tau \big(m^\varepsilon\big(v-u^\varepsilon\big)\big)(0,\cdot)\,dt \geq 0. \tag{5.8}$$

The similar inequality holds at $x = L$

$$\int_0^\tau \big(m^\varepsilon\big(v-u^\varepsilon\big)\big)(L,\cdot)\,dt \geq 0. \tag{5.9}$$

Thanks to inequalities (5.8), (5.9), Lemmas 5.1 and 5.2, we may pass to the limit in the variational formulation (5.1) and we obtain (1.5), and the conclusion is clear. □

REMARK 5.4. *The question of uniqueness is completely open, and we have no hint whatsoever in this direction.*

6. The trace spaces

We have obtained at 3.1 the existence of a solution u^ε of (3.19) such that

$$e^{-\nu t}\big(|u^\varepsilon(0,t)-v(0,t)| + |u^\varepsilon(L,t)-v(L,t)|\big) \leq C\exp\big(C't/\varepsilon\big)$$

and C' was defined as the maximum of the norm of the matrix $\|L(t)\|$ over $\mathbb{R}$; this matrix depends on ν, as well as C and C' . Pick any $\nu' > 0$, and define

$$\nu = \nu' + \frac{C'(\nu')}{\varepsilon} + 1;$$

then $e^{-\nu t}\big(|u^\varepsilon(0,t)-v(0,t)| + |u^\varepsilon(L,t)-v(L,t)|\big)$ is temperate. Assume from now on that f has compact support in t; we know then from Remark 4.2 that the traces $u_t^\varepsilon(0,\cdot)$ and $u_t^\varepsilon(L,\cdot)$ are bounded in $H^{1/2}(\mathbb{R})$, and therefore $W_{0,t}^\varepsilon = e^{-\nu t}\big(u_t^\varepsilon(0,t) - \nu u^\varepsilon(0,t)\big)$ and $W_{L,t}^\varepsilon = e^{-\nu t}\big(u_t^\varepsilon(L,t) - \nu u^\varepsilon(L,t)\big)$ are square integrable over $\mathbb{R}$. We see immediately that $t\mapsto G\big(W^\varepsilon(t),t\big)$ is also square integrable. We multiply (3.14) scalarly by $W_t^\varepsilon + \nu W^\varepsilon$ and we integrate over $\mathbb{R}^+$, which is permissible according to the above considerations. We observe that

$$\begin{aligned}
&\int_0^\infty \big(G\big(W^\varepsilon(t),t\big)\big)\cdot\big(W_t^\varepsilon + \nu W^\varepsilon\big)\,dt \\
&= \int_0^\infty \big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\big((\partial_t+\nu)\big(W_0^\varepsilon + e^{-t}(u_0(0)+a_0)\big)\big)\,dt \\
&+ \int_0^\infty \big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\big(e^{-t(\nu+1)}(u_0(0)+a_0)\big)\,dt \\
&+ \int_0^\infty \big(W_L^\varepsilon(t) + e^{-t}(u_0(L)+a_L)\big)^-\big((\partial_t+\nu)\big(W_L^\varepsilon + e^{-t}(u_0(L)+a_L)\big)\big)\,dt \\
&+ \int_0^\infty \big(W_L^\varepsilon(t) + e^{-t}(u_0(L)+a_L)\big)^-\big(e^{-t(\nu+1)}(u_0(L)+a_L)\big)\,dt.
\end{aligned}$$

But

$$\begin{aligned}
&\int_0^\infty \big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\big((\partial_t+\nu)\big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)\big)\,dt \\
&= -\frac{1}{2}\Big(\big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\Big)^2\Big|_{t=0}^{t=\infty} \\
&- \nu\int_0^\infty \Big(\big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\Big)^2\,dt,
\end{aligned}$$

and

$$\int_0^\infty \big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^- \big(e^{-t(\nu+1)}(u_0(0)+a_0)\big)\,dt$$
$$\leq \nu \int_0^\infty \big(\big(W_0^\varepsilon(t) + e^{-t}(u_0(0)+a_0)\big)^-\big)^2\,dt + \frac{1}{4\nu}\int_0^\infty e^{-2t(\nu+1)}(u_0(0)+a_0)^2\,dt.$$

Therefore

$$\int_0^\infty \big(G\big(W^\varepsilon(t),t\big)\big)\cdot\big(W_t^\varepsilon + \nu W^\varepsilon\big)\,dt \leq \frac{1}{8\nu(\nu+1)}\big((u_0(0)+a_0)^2 + (u_0(L)+a_L)^2\big).$$

We apply Plancherel's formula to the term involving $M * W^\varepsilon$ in (3.14):

$$\int_0^\infty \big(M * W^\varepsilon\big)\cdot\big((\partial_t+\nu)W^\varepsilon\big)\,dt = \frac{1}{2\pi}\Re\int_{\mathbb{R}} |\nu+i\omega|^2\sqrt{1+\alpha(\nu+i\omega)}\,\big|\widehat{W}^\varepsilon(\omega)\big|^2\,d\omega$$
$$+\frac{1}{2\pi}\Re\int_{\mathbb{R}} |\nu+i\omega|^2\sqrt{1+\alpha(\nu+i\omega)}\,\big(\widehat{W}^\varepsilon\big)^*(\omega)\widehat{M}_0(\omega)\widehat{W}^\varepsilon(\omega)\,d\omega$$

where $\widehat{M_0}$ is defined as

$$\widehat{M}_0(\omega) = \frac{2\widehat{\eta}}{1-\widehat{\eta}^2}\begin{pmatrix} \widehat{\eta} & -1 \\ -1 & \widehat{\eta}\end{pmatrix}.$$

In particular the norm of the matrix

$$|\nu+i\omega|^2\sqrt{1+\alpha(\nu+i\omega)}\,\widehat{M}_0(\omega)$$

is bounded independently of $\omega\in\mathbb{R}$ and $\nu\geq 1$ in consequence of (3.11); we denote by γ an upper bound of this quantity. For $\nu\geq 1$ and $\omega\in\mathbb{R}$, we have

$$\Re\sqrt{1+\alpha(\nu+i\omega)} \geq C_0\big(1+\sqrt{\nu}+\sqrt{|\omega|}\big),$$

and therefore we may write now, with the help of the classical inequality $|ab|\leq |a|^2/2+|b|^2/2$

$$\begin{aligned}
&\frac{C_0}{2\pi}\int_{\mathbb{R}}\Big(\big(1+\sqrt{\nu}+\sqrt{|\omega|}\big)|\nu+i\omega|^2\big|\widehat{W}^\varepsilon(\omega)\big|^2\Big)\,d\omega \\
&\leq \frac{\gamma}{2\pi}\int_{\mathbb{R}}\big|\widehat{W}^\varepsilon(\omega)\big|^2\,d\omega + \frac{1}{\gamma\nu(\nu+1)}\big((u_0(0)+a_0)^2+(u_0(L)+a_L)^2\big) \\
&\quad+\frac{C_0}{4\pi}\int_{\mathbb{R}}\Big(\big(1+\sqrt{\nu}+\sqrt{|\omega|}\big)|\nu+i\omega|^2\big|\widehat{W}^\varepsilon(\omega)\big|^2\Big)\,d\omega \\
&\quad+\frac{1}{4\pi}\int_{\mathbb{R}}\frac{|\widehat{g}(\omega)|^2}{C_0\big(1+\sqrt{\nu}+\sqrt{|\omega|}\big)},
\end{aligned}\tag{6.1}$$

which implies immediately a bound on the L^2 norm of W^ε independently of ε and of $\nu\geq 1$, since W^ε is bounded in $L^2(0,\infty)^2$ thanks to Remark 4.2. There remains to validate (6.1) for all $\nu\geq 1$, while we needed Plancherel formula to obtain it. For this purpose, we let φ_T be a C^∞ function which is equal to 1 for $t\leq T$, to 0 for $t\geq T+2/\nu$, and whose gradient is at most ν over $[T, T+2\nu]$, the approximate problem

$$M * W^{\varepsilon,T} = g + \varphi_T G\big(W^{\varepsilon,T},\cdot\big)/\varepsilon,$$

possesses a unique solution for the same reason that the limiting problem possesses a unique solution. The function $W^{\varepsilon,T}$ is fast decreasing as t tends to infinity. The modifications in estimate (6.1) is straightforward, and we can show easily that

$$\frac{1}{2\pi}\int_{\mathbb{R}}\big(1+\sqrt{\nu}+\sqrt{|\omega|}\big)|\nu+i\omega|^2\big|\widehat{W}^{\varepsilon,T}\big|^2\,d\omega$$

is bounded independently of ε, T and ν. We pass to the limit for $T\to\infty$, and we obtain the conclusion by uniqueness of the solution of (3.20).

THEOREM 6.1. *Under assumptions of Theorem 2.1. For all* $\nu \geq 1$, W^ε *is bounded in* $H^{5/4}(\mathbb{R})^2$ *and therefore we may extract a subsequence such that* $u^\varepsilon\big|_{\Sigma_T}$ *converges to* $u\big|_{\Sigma_T}$ *weakly in* $H^{5/4}_{\mathrm{loc}}(\mathbb{R})$, *and* $\big(u^\varepsilon_x + \alpha u^\varepsilon_{xt}\big)\big|_{\Sigma_T}$ *converges to* $(u_x + \alpha u_{xt})\big|_{\Sigma_T}$ *weakly in* $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$. *Therefore, u is a strong solution of* (1.1)-(1.4).

PROOF. We obtain a strong solution of (1.1)-(1.4), by taking weak limits in appropriate functional spaces and by applying standard methods. Since it is quite routine, we leave the verification of this proof to the reader. □

REMARK 6.2. *We mean by strong solution that all the traces can be defined.*

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CHAPITRE 3

Numerical approximation of a viscoelastic problem with unilateral constraints

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Abstract. The system $u_{tt} - u_{xx} - \alpha u_{xxt} \ni f$, $(x,t) \in (0,L) \times (0,T)$, with initial data $u(\cdot,0) = u_0$, $u_t(\cdot,0) = u_1$ almost everywhere on $(0,L)$ and the unilateral conditions given by

$$0 \le u(0,\cdot) + a_0 \perp -(u_x + \alpha u_{xt})(0,\cdot) \ge 0,$$
$$0 \le u(L,\cdot) + a_L \perp (u_x + \alpha u_{xt})(L,\cdot) \ge 0,$$

models the longitudinal vibrations of a rod, whose motion is limited by obstacles at both ends. We give a numerical scheme, we prove its convergence and we report the results of some numerical experiments.

1. Introduction and notations

Consider the following problem: a viscoelastic rod of length L vibrates longitudinally; each end of the rod is free to move as long as it does not hit a material obstacle. Each obstacle may constrain the displacement of the extremity to be greater than or equal to some number.

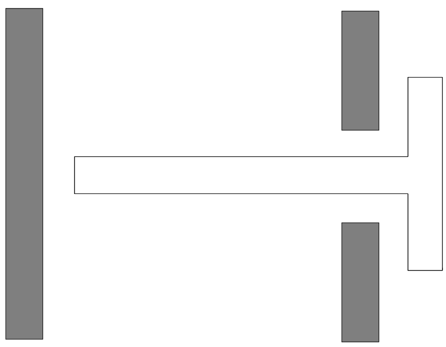

Figure 1. Motion of the rod between two obstacles.

The mathematical situation can be described as follows: assume that the rod is made up of an homogeneous viscoelastic material and satisfies the assumptions of the theory of small deformations. Let x be the spatial coordinate along the rod with the origin at one of the extremities; let $u(x,t)$ be the displacement of the material point with spatial coordinate x at time t. Let f be an exterior force which depends on space and time. The displacement u satisfies the following equation:

(1.1) $$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad (x,t) \in (0,L) \times (0,T), \quad \alpha > 0,$$

with initial data

$$u(\cdot,0)=u_0 \quad \text{and} \quad u_t(\cdot,0)=u_1. \tag{1.2}$$

The boundary conditions are determinated as follows: when the rod touches one of the obstacles, the complementarity conditions are as follows:

$$u_x(0,\cdot)+\alpha u_{xt}(0,\cdot)\le 0 \text{ on the set } \{t:\, u(0,\cdot)=-a_0\}, \tag{1.3}$$

or

$$u_x(L,\cdot)+\alpha u_{xt}(L,\cdot)\ge 0 \text{ on the set } \{t:\, u(L,\cdot)=-a_L\}, \tag{1.4}$$

whereas when the rod does not touch one of the obstacles, each end is free to move

$$u_x(0,\cdot)+\alpha u_{xt}(0,\cdot)=0 \text{ on the set } \{t:\, u(0,\cdot)>-a_0\}, \tag{1.5}$$

or

$$u_x(L,\cdot)+\alpha u_{xt}(L,\cdot)=0 \text{ on the set } \{t:\, u(L,\cdot)>-a_L\}, \tag{1.6}$$

The conditions (1.3)-(1.6) can be summarized as

$$0\le (u(0,\cdot)+a_0)\ \perp\ -(u_x+\alpha u_{xt})(0,\cdot)\ge 0, \tag{1.7a}$$

$$0\le (u(L,\cdot)+a_L)\ \perp\ (u_x+\alpha u_{xt})(L,\cdot)\ge 0, \tag{1.7b}$$

here the orthogonality means that an appropriate duality product between the terms of the relation vanishes. Conditions (1.7)-(1.7a) are usually termed Signorini conditions.

Problem (1.1)-(1.7) has been studied in [**5**]. Under the following assumptions: u_0 belongs to the Sobolev space $H^2(0,L)$, $u_0(0,0)=u_0(0)\ge -a_0$ and $u_0(L,0)=u_0(L)\ge -a_L$, u_1 belongs to the Sobolev space $H^1(0,L)$ and f and f_t belong to $L^2(0,T;L^2(0,L))$, we have proved using penalization the existence of a solution to (1.1)-(1.7). A weak formulation of (1.1)-(1.7b) is given by

$$\begin{cases} u\in K \text{ and for all } v\in K \text{ and for almost every } \tau\in[0,T], \\ \displaystyle\int_0^L u_t(v-u)\Big|_0^\tau dx-\int_0^\tau\int_0^L u_t(v_t-u_t)\,dx\,dt \\ \displaystyle\quad+\int_0^\tau\int_0^L (u_x+\alpha u_{xt})(v_x-u_x)\,dx\,dt\ge\int_0^\tau\int_0^L f(v-u)\,dx\,dt, \end{cases} \tag{1.8}$$

where K is the convex set defined by

$$\{v\in H^1((0,L)\times(0,T)) :\, v_{xt}\in L^2((0,L)\times(0,T)),\, v(0,\cdot)\ge -a_0,\, v(L,\cdot)\ge -a_L\}.$$

Let us explain now the plan of the article.

In Section 2 of this paper, we approximate numerically a solution of (1.1)-(1.7a) though the following numerical method: approximate $u(j\Delta x,n\Delta t)$ by u_j^n. We discretize the time and space differences and apply a variational formulation to our problem. We denote by A the discretization of Laplacian with Neumann conditions. The numerical scheme is defined by

$$X_j^n=\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2}+\left(A\,\frac{u^{n+1}+u^{n-1}}{2}+\alpha A\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)_j-f_j^n=0,$$

with unilateral conditions at the boundary; i.e. for $j=0,\ J+1$ we get

$$0\le u_j^n\perp X_j^n\ge 0.$$

This numerical scheme is the straightforward generalization of the scheme presented in [**6**] to a viscoelastic case.

In Section 3, we prove the convergence of the numerical solution to a solution of (1.1)-(1.7a).

Finally, in Section 4, we display the result of some numerical simulations performed on a finite interval, where we have taken unilateral boundary conditions at the both ends, or one unilateral condition and one Neumann condition.

2. Numerical schemes

We approximate each derivative in (1.1) using the Taylor's formulae. That is to say, we approximate the second derivative in time by a centered finite difference. We approximate the second derivative in space by a centered differences in space at times $t-\Delta t$ and $t+\Delta t$,

$$u_{xx}(x,t) = \frac{u(x+\Delta x,t+\Delta t)-2u(x,t+\Delta t)+u(x-\Delta x,t+\Delta t)}{2\Delta x^2} + \frac{u(x+\Delta x,t-\Delta t)-2u(x,t-\Delta t)+u(x-\Delta x,t-\Delta t)}{2\Delta x^2} + O(\Delta x^2) + O(\Delta t^2),$$

and u_{xxt} by a centered difference in x and t:

$$u_{xxt}(x,t) = \frac{u(x+\Delta x,t+\Delta t)-2u(x,t+\Delta t)+u(x-\Delta x,t+\Delta t)}{2\Delta x^2\Delta t} - \frac{u(x+\Delta x,t-\Delta t)-2u(x,t-\Delta t)+u(x-\Delta x,t-\Delta t)}{2\Delta x^2\Delta t} + O(\Delta x^2) + O(\Delta t^2).$$

It is convenient to denote by A the linear operator obtained by discretizing the Laplace operator in one dimension with Neumann boundary conditions; all its coefficients vanish except the ones on the three main diagonals and they are given by: for $i \in [1,J]$,

$$A_{ii} = 2/\Delta x^2, \quad A_{i,i+1} = A_{i,i-1} = -1/\Delta x^2, \quad A_{00} = A_{J+1,J+1} = 1/\Delta x^2.$$

On the other hand, Δt is the uniform time step, $t_n = t_0 + n\Delta t$ and $n \le N(\Delta t)$ with $N(\Delta t) = \lfloor T/\Delta t \rfloor$ which is the greatest integer at the most equal to $T/\Delta t$; $\Delta x = L/(J+1)$ is the space step. Denote by f_j^n and u_j^n respectively the approximations of $f(j\Delta x, n\Delta t)$ and $u(j\Delta x, n\Delta t)$. We assume that $u_j^0 = u_0(j\Delta x)$ for $j \in [0, J+1]$, $u_0^n \ge -a_0$ and $u_{J+1}^n \ge -a_L$. Then the estimates given above motivate the choice of following scheme: for $j \in [1,J]$,

$$\frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2} + \left(A\,\frac{u^{n+1}+u^{n-1}}{2} + \alpha A\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)_j = f_j^n. \tag{2.1}$$

We have a complementarity problem:

$$-a_0 \le u_0^n \perp X_0^n \ge 0 \quad \text{and} \quad -a_L \le u_{J+1}^n \perp X_{J+1}^n \ge 0, \tag{2.2}$$

where

$$X_l^n = \frac{u_l^{n+1}-2u_l^n+u_l^{n-1}}{\Delta t^2} + \left(A\,\frac{u^{n+1}+u^{n-1}}{2} + \alpha A\,\frac{u^{n+1}-u^{n-1}}{2\Delta t}\right)_l - f_l^n$$
$$\text{for all} \quad l = 0, J+1.$$

REMARK 2.1. *According to standard Fourier techniques, the scheme* (2.1) *is L^2-stable and consistant of second order in time and in space. Then we may deduce that the scheme* (2.1) *converges.*

Let us denote by $|\cdot|$ and $\|\cdot\|$ respectively the norm on $L^2(0,L)$ and $H^1(0,L)$ and let $(\cdot,\cdot)$ denote the scalar product in $L^2(0,L)$. We shall need a number of spaces and sets. Let V_h and H_h be respectively sequences of finite-dimensional subspaces of $H^1(0,L)$ and $L^2(0,L)$ such that

$$L^2(0,L) = \overline{\bigcup_h H_h}^{L^2(0,L)} \quad \text{and} \quad H^1(0,L) = \overline{\bigcup_h V_h}^{H^1(0,L)}.$$

Let K_h be the convex set defined by

$$K_h = \{v \in V_h : v_{xt} \in H_h,\, v(0) \geq -a_0,\, v(L) \geq -a_L\}.$$

Therefore, u_h^{n+1} belongs to K_h and satisfies an evolution variational inequality given by:

$$\begin{cases} u_h^{n+1} \in K_h \text{ for all } w \in K_h, \\ \left(\dfrac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, w - u_h^{n+1}\right) + \left(A\, \dfrac{u_h^{n+1} + u_h^{n-1}}{2}, w - u_h^{n+1}\right) \\ +\alpha\left(A\, \dfrac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, w - u_h^{n+1}\right) \geq \left(f_h^n, w - u_h^{n+1}\right). \end{cases} \tag{2.3}$$

LEMMA 2.2. *The finite difference scheme* (2.1)-(2.2) *is equivalent to the variational inequality* (2.3).

PROOF. We substitute w in (2.3) by $u + \varphi$, with $\varphi(0) = \varphi(L) = 0$, we obtain

$$\begin{aligned} &\left(\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}, \varphi\right) + \left(A\, \frac{u_h^{n+1} + u_h^{n-1}}{2}, \varphi\right) \\ &+\alpha\left(A\, \frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t}, \varphi\right) \geq \left(f_h^n, \varphi\right), \end{aligned}$$

which is equivalent to (2.1). We choose w such that

$$w_j = u_j \quad \text{for} \quad \text{all} \quad j \in [1, J], \quad w_0 \geq a_0, \quad w_{J+1} \geq a_L,$$

then we get

$$\begin{aligned} &\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\Delta t^2} \left(w_0 - u_0^{n+1}\right) \\ &+ \left(A\, \frac{u^{n+1} + u^{n-1}}{2} + \alpha A\, \frac{u^{n+1} - u^{n-1}}{2\Delta t}\right)_0 \left(w_0 - u_0^{n+1}\right) \\ &\geq f_0^n \left(w_0 - u_0^{n+1}\right) \quad \text{for} \quad \text{all} \quad w_0 \geq -a_0, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\frac{u_{J+1}^{n+1} - 2u_{J+1}^n + u_{J+1}^{n-1}}{\Delta t^2} \left(w_{J+1} - u_{J+1}^{n+1}\right) \\ &+ \left(A\, \frac{u^{n+1} + u^{n-1}}{2} + \alpha A\, \frac{u^{n+1} - u^{n-1}}{2\Delta t}\right)_{J+1} \left(w_{J+1} - u_{J+1}^{n+1}\right) \\ &\geq f_{J+1}^n \left(w_{J+1} - u_{J+1}^{n+1}\right) \quad \text{for} \quad \text{all} \quad w_{J+1} \geq -a_L. \end{aligned} \tag{2.5}$$

It is straightforward to check that relations (2.4) and (2.5) are equivalent to (2.2). □

REMARK 2.3. *Relation* (2.3) *can be written in slightly different form: define a maximal monotone operator* $\partial\psi_{K_h}$ (*see* [**1**], [**3**]) *by*

$$\partial\psi_{K_h}(u) = \begin{cases} \{0\} & \text{if} \quad u \in \operatorname{int}(K_h), \\ \{x : (x, v - u),\, \forall v \in K_h\} & \text{if} \quad u \in \partial K_h, \\ \emptyset & \text{otherwise}. \end{cases}$$

Then relation (2.3) *can be rewritten as*

$$\begin{aligned} &\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + A\, \frac{u_h^{n+1} + u_h^{n-1}}{2} + \alpha A\, \frac{u_h^{n+1} - u_h^{n-1}}{2\Delta t} \\ &+\partial\psi_{K_h}\left(\frac{u_h^{n+1} + u_h^{n-1}}{2}\right) \ni f_h^n. \end{aligned} \tag{2.6}$$

It is equivalent to minimize a coercive function and twice differentiable in a convex set. Therefore, for each step u_h^n is unique.

3. Convergence of the scheme

We will prove that the scheme (2.1) with unilateral boundary conditions (2.2) converges to a solution of (1.1)-(1.7a).

THEOREM 3.1. *Under the unilateral conditions* (2.2), *the numerical scheme* (2.1) *converges to a solution of* (1.1)-(1.7b) *when Δx and Δt tend to zero.*

PROOF. Let us first prove the stability: we let $w = u_h^{n-1}$ in (2.3) and we obtain

$$\begin{aligned}&\left(\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\Delta t^2}, u_h^{n-1}-u_h^{n+1}\right)+\left(A\,\frac{u_h^{n+1}+u_h^{n-1}}{2}, u_h^{n-1}-u_h^{n+1}\right)\\ &+\alpha\left(A\,\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}, u_h^{n-1}-u_h^{n+1}\right)\geq (f_h^n, u_h^{n-1}-u_h^{n+1}).\end{aligned}\tag{3.1}$$

The identity

$$\frac{1}{\Delta t^2}\left(u_h^{n+1}-2u_h^n+u_h^{n-1}, u_h^{n-1}-u_h^{n+1}\right)=\left|\frac{u_h^{n-1}-u_h^n}{\Delta t}\right|^2-\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2,$$

implies that

$$\begin{aligned}&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{1}{2}(Au_h^{n+1},u_h^{n+1})+2\alpha\left(A\,\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t},\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}\right)\Delta t\\ &\leq\left|\frac{u_h^{n-1}-u_h^n}{\Delta t}\right|^2+\frac{1}{2}(Au_h^{n-1},u_h^{n-1})+2\left(f_h^n,\frac{u_h^{n+1}-u_h^{n-1}}{2\Delta t}\right)\Delta t.\end{aligned}$$

We perform a discrete time integration of the above expressions and we obtain

$$\begin{aligned}&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{1}{2}(Au_h^{n+1},u_h^{n+1})\\ &+2\alpha\sum_{m=1}^{n}\left(A\,\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t},\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right)\\ &\leq\left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2+\frac{1}{2}(Au_h^0,u_h^0)+2\sum_{m=1}^{n}\left(f_h^m,\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right)\Delta t.\end{aligned}\tag{3.2}$$

Therefore, thanks to Cauchy-Schwarz inequality, we get

$$\sum_{m=1}^{n}\left(f_h^m,\frac{u_h^{m+1}+u_h^{m-1}}{2\Delta t}\right)\Delta t\leq\frac{1}{2}\sum_{m=1}^{n}|f_h^m|^2\Delta t+\frac{1}{2}\sum_{m=1}^{n}\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\Delta t.\tag{3.3}$$

On the other hand, there exist $\gamma>0$ and $\lambda>0$, such that

$$-\lambda|x|^2+\gamma\,\|x\|^2\leq(Ax,x).\tag{3.4}$$

We deduce from (3.3) and the coercivity inequality (3.4):

$$\begin{aligned}&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2+\frac{1}{2}(Au_h^{n+1},u_h^{n+1})+2\alpha\gamma\sum_{m=1}^{n}\left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2\Delta t\\ &\leq\left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2+(1+2\alpha\lambda)\sum_{m=1}^{n}\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\Delta t+\frac{1}{2}(Au_h^0,u_h^0)+\sum_{m=1}^{n}|f_k^m|^2\Delta t.\end{aligned}$$

With the help of the inequality

$$\left|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right|^2\leq\left|\frac{u_h^{m+1}-u_h^m}{\Delta t}\right|^2+\left|\frac{u_h^{m-1}-u_h^m}{\Delta t}\right|^2,$$

we infer then

$$
\begin{aligned}
&\left|\frac{u_h^{n+1}-u_h^n}{\Delta t}\right|^2 + \frac{1}{2}\big(Au_h^{n+1}, u_h^{n+1}\big) + 2\alpha\gamma \sum_{m=1}^{n} \left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2 \Delta t \\
&\le \left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2 + 2(1+2\alpha\lambda)\sum_{m=0}^{n}\left|\frac{u_h^{m+1}-u_h^m}{\Delta t}\right|^2 \Delta t \\
&\quad + \frac{1}{2}\big(Au_h^0, u_h^0\big) + \sum_{m=1}^{n} |f_h^m|^2 \Delta t.
\end{aligned}
\tag{3.5}
$$

We define

$$a^m = \left|\frac{u_h^{m+1}-u_h^m}{\Delta t}\right|^2, \tag{3.6a}$$

$$b^n = \big(Au_h^{n+1}, u_h^{n+1}\big), \tag{3.6b}$$

$$c^n = \sum_{m=1}^{n} \left\|\frac{u_h^{m+1}-u_h^{m-1}}{2\Delta t}\right\|^2, \tag{3.6c}$$

$$\beta = 2(1+2\alpha\lambda), \tag{3.6d}$$

$$g^n = \sum_{m=1}^{n} |f_h^m|^2 \Delta t + \left|\frac{u_h^0-u_h^1}{\Delta t}\right|^2 + \frac{1}{2}\big(Au_k^0, u_k^0\big). \tag{3.6e}$$

Therefore (3.5) can be rewritten as follows:

$$a^n + \frac{1}{2}\, b^n + 2\alpha\gamma c^n \Delta t \le g^n + \beta \sum_{m=0}^{n} a^m \Delta t. \tag{3.7}$$

Define

$$d^n = \sum_{m=0}^{n} a^m, \tag{3.8}$$

Since the matrix A is non negative, relation (3.6), shows that $b^n \ge 0$; moreover it is obvious that $2\alpha\gamma c^n \Delta t \ge 0$, thus we infer from (3.7) that

$$d^n \le \frac{g^n \Delta t}{1-\beta\Delta t} + \frac{d^{n-1}}{1-\beta\Delta t}.$$

The discrete Gronwall lemma enables us to deduce now that

$$d^n \le \sum_{m=1}^{n} \frac{g^m \Delta t}{(1-\beta\Delta t)^{n-m+1}} + \frac{a^0}{(1-\beta\Delta t)^n}. \tag{3.9}$$

Carrying (3.9) into (3.7), we obtain

$$a^n + \frac{1}{2}\, b^n + 2\alpha\gamma c^n \Delta t \le g^n + \sum_{m=1}^{n} \frac{\beta g^m \Delta t}{(1-\beta\Delta t)^{n-m+1}} + \frac{\beta a^0}{(1-\beta\Delta t)^n}. \tag{3.10}$$

The right hand side of the inequality (3.10) is bounded because the elements which composed it are bounded data. Define an interpolation u_h by

$$u_h(x,t) = u_h^n\, \frac{(n+1)\Delta t - t}{\Delta t} + u_h^{n+1}\, \frac{t-n\Delta t}{\Delta t} \quad \text{for} \quad t \in [n\Delta t, (n+1)\,\Delta t].$$

Relation (3.10) implies that we can extract from the sequence u_h, a subsequence, still denoted by u_h, such that

$$u_h \rightharpoonup u \quad \text{in} \quad L^\infty(0,T;H^1(0,L)) \quad \text{weak} \quad *, \tag{3.11a}$$

$$\frac{du_h}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L^\infty(0,T;L^2(0,L)) \quad \text{weak} \quad *, \tag{3.11b}$$

$$u_h \to u \quad \text{in} \quad C^{0,\beta}(Q_T), \quad \forall \beta < 1/2. \tag{3.11c}$$

In order to prove that the limit u satisfies (1.8), it is necessary to take convenient test functions. It is obvious that u_h belongs to K. Thanks to (3.11), we may deduce that u belongs to K. The elements of K are not smooth enough in time, and they have to be approximated before being projected onto V_h. This projection does not conserve the constraints at $x = L$ and $x = 0$, and therefore, the elements of K need another approximation in order to satisfy the constraints strictly. More precisely, let v be an element of K which is equal to u for $t \geq T - \varepsilon$. For $\eta \leq \varepsilon/4$, define

$$v^\eta(x,t) = \begin{cases} u(x,t) & \text{if } t \geq T-\eta, \\ u(x,t) + \dfrac{1}{\eta}\displaystyle\int_t^{t+\eta} (v-u)(x,s)ds + k(\eta)\phi(t) & \text{if } t \leq T-\eta. \end{cases} \tag{3.12}$$

The function ϕ is nonnegative and smooth; it is equal to 1 on $[0, T-\varepsilon/2]$, and it vanishes on $[T-\varepsilon/4, T]$. The parameter $k(\eta)$ is chosen as follows:

$$\begin{aligned} \left| u(L,t) - \frac{1}{\eta}\int_t^{t+\eta} u(L,s)\,ds \right| &\leq \frac{1}{\eta}\int_t^{t+\eta} |u(L,t)-u(L,s)|\,ds \\ &\leq \frac{C}{\eta}\int_0^\eta s^\beta\,ds = \frac{C\eta^\beta}{\beta+1}, \end{aligned}$$

where

$$C = \sup_{s\in]t,t+\eta[} \frac{|u(L,t)-u(L,s)|}{(s-t)^\beta}.$$

We have the inequality, for $t \leq T - \varepsilon/2$,

$$v^\eta(L,t) \geq \frac{1}{\eta}\int_t^{t+\eta} v(L,s)\,ds - \frac{C\eta^\beta}{\beta+1} + k(\eta)\phi(t)\,.$$

If we choose

$$k(\eta) = \frac{2C\eta^\beta}{\beta+1},$$

we will be sure that

$$v^\eta(L,t) \geq -k_2 + \frac{C\eta^\beta}{\beta+1} \text{ for } t \leq T - \frac{\varepsilon}{2}.$$

With the same arguments, we obtain

$$v^\eta(0,t) \geq -k_1 + \frac{C\eta^\beta}{\beta+1} \text{ for } t \leq T - \frac{\varepsilon}{2}.$$

It is not difficult to check that, for t in $[T-\varepsilon/2, T-\eta]$,

$$v^\eta(x,t) = u(x,t) + k(\eta)\phi(t),$$

so that v^η belongs to K. On the other hand, v^η belongs to $L^\infty(0,T;L^2(0,L))$, because time integration has a smoothing effect. We denote by Q_h the projection onto V_h with respect to scalar product of $L^2(0,L)$. The sequence Q_h converges in strong operator topology of $L^2(0,L)$ to the identity, and therefore, thanks to the Sobolev injections, there exists a sequence γ_h converging to zero when h tends to zero such that

$$\|Q_h z - z\|_{C^0(0,L)} \leq \gamma_h \|z\|\,, \quad \forall z \in H^1(0,L).$$

Moreover there exists a positive constant C such that

$$\|Q_h v\| \leq C \|v\|, \quad \forall v \in H^1(0,L).$$

This property is proved by a classical computation. Now, we choose v^η as in (3.12) and we let

$$v_h^n = u_h^{n+1} + Q_h(v^\eta(n\Delta t) - u(n\Delta t)).$$

If we substitute this value for v_h^n in (2.3) and perform a discrete integration, we obtain

$$\begin{aligned}\sum_{n=1}^{N-1}(f_h^n, v_h^n - u_h^{n+1})\Delta t \leq -\left(\frac{u_h^1 - u_h^0}{\Delta t}, v_h^0 - u_h^1\right)\\ - \sum_{n=1}^{N-1}\left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, \frac{v_h^n - u_h^{n+1} - v_h^{n-1} + u_h^n}{\Delta t}\right)\Delta t\\ + \sum_{n=1}^{N-1}\left(A\frac{u_h^{n+1} + u_h^n}{2}, v_h^n - u_h^{n+1}\right)\Delta t\\ + \alpha\sum_{n=1}^{N-1}\left(A\frac{u_h^{n+1} - u_h^n}{2\Delta t}, v_h^n - u_h^{n+1}\right)\Delta t.\end{aligned}$$

The passage to the limit in this expression is obvious. It is enough to show that the total energy of u_h converges to the total energy of u. This is done by a discrete integration of (3.1). □

REMARK 3.2. *Theorem 3.1 is also valid when we replace the Signorini condition at the both ends by a Neumann or a Dirichlet boundary condition at one end and Signorini condition at the other end.*

REMARK 3.3. *Let us define*

(3.13a) $$y_h^n = \frac{u_h^{n+1} + u_h^{n-1}}{2},$$

(3.13b) $$M = 1 + \alpha\frac{\Delta t}{2}A + \frac{\Delta t^2}{2}A,$$

(3.13c) $$\widetilde{f}_h = \frac{\Delta t^2}{2}f_h^n + u_h^n + \alpha\frac{\Delta t}{2}Au_h^{n-1},$$

where 1 *is the identity matrix. Therefore,* (2.6) *can be rewritten as follows:*

(3.14) $$My^n + \partial\psi_{K_h}(y_h^n) \ni \widetilde{f}_h.$$

It is plain that (3.14) *is equivalent to minimize the functional* $J(y_h^n)$ *on* K_h*,*

$$J(y_h^n) = \frac{1}{2}(y_h^n)^t M y_h^n - (\widetilde{f}_h)^t y_h^n.$$

We have used this equivalence for the numerical simulations reported in next section.

4. Numerical experiments

For the following simulations, the space interval is $[0,1]$, the time and space steps are given by $\Delta t = 0.125$, $\Delta x = 1/27$, and initial data are

$$u_0(x) = x(1-x) \quad \text{and} \quad u_1(x) = u_0(x),$$

and $\alpha = 1$. We have performed simulations for $f(x,t) = 0$ and for $f(x,t) = \sin(t\sqrt{2})\cos(2x)$. In practical, resolution is very simple-minded way: at each time step, we check whether the solution of the problem without constraints is admissible; if it is, we advance by one time-step; if it is not, we solve when one or two constraints are active. Thus we have at almost four linear problem to solve per time-step.

In the first experiment, we assume that $f(x,t)$ vanishes. We observe that if the constraint is active, then small oscillations appear at the boundary.

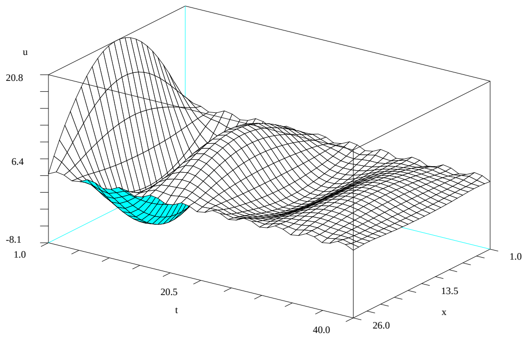

FIGURE 1. A numerical simulation with vanishing right hand side and Signorini condition at both ends.

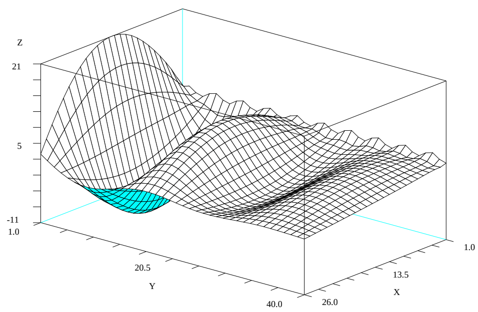

FIGURE 2. A numerical simulation with vanishing right hand side, a Neumann boundary condition at one hand and a Signorini condition at the other.

REMARK 4.1. *The last two simulations concern the case of a distributed constraint, i.e.*

$$K = \{v \in H^1(Q_T) : v_{xt} \in L^2(Q_T),\ v(x,\cdot) \geq a \text{ for all } x \in (0,L)\}.$$

We have not treated the mathematical theory of this problem (see [**2**]*), nor its numerical approximation. We give nevertheless the results of these simulations for the reader's information.*

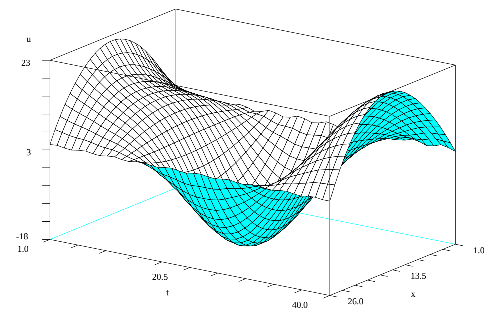

FIGURE 3. A numerical simulation with right hand side $f(x,t) = \sin\left(t\sqrt{2}\right)\cos(2x)$ and a Signorini condition at the both ends.

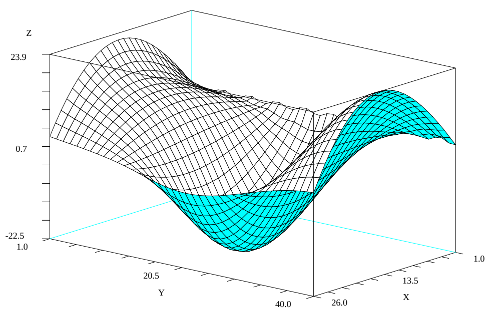

FIGURE 4. A numerical simulation with right hand side $f(x,t) = \sin\left(t\sqrt{2}\right)\cos(2x)$ a Neumann boundary condition at one hand and a Signorini condition at the other.

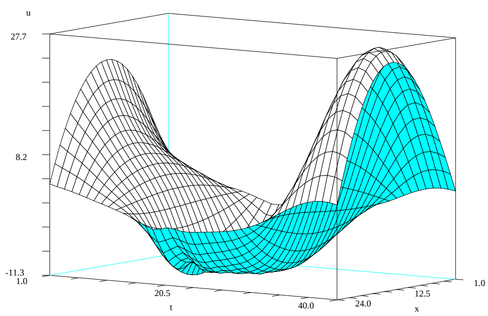

FIGURE 5. A numerical simulation with vanishing right hand side.

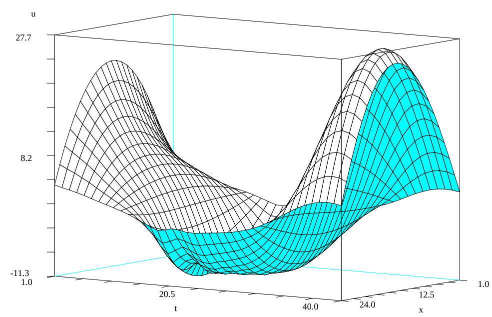

FIGURE 6. A numerical simulation with right hand side $f(x,t) = \sin\big(t\sqrt{2}\big)\cos(2x)$.

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Partie 2

Viscoélasticité avec contact unilatéral dans un demi-espace

CHAPITRE 4

Viscoélastodynamique monodimensionnelle avec conditions de Signorini

Adrien Petrov et Michelle Schatzman

Note aux C.R. Acad. Sci. Paris, Ser. I 334 (2002) 983-988.

Résumé. Soit α un nombre strictement positif. Le problème viscoélastique monodimensionnel

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in (-\infty, 0], \quad t \in [0, +\infty).$$

avec les conditions au bord unilatérales

$$u(0, \cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0, \cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0, \cdot) = 0,$$

peut être réduit à l'inéquation variationnelle suivante:

$$\lambda_1 * w = g + b, \quad w \geq 0, \quad b \geq 0, \quad \langle w, b \rangle = 0.$$

Ici $\widehat{\lambda}_1(\omega)$ est la détermination causale de $i\omega\sqrt{1 + i\alpha\omega}$. On démontre que ce problème possède une solution et que les pertes d'énergie sont purement visqueuses; ce résultat provient de la relation $\langle \dot{w}, b \rangle = 0$, qui n'est pas triviale puisque, *a priori*, b est une mesure et $\dot{w}$ n'est définie que presque partout.

One dimensional viscoelastodynamics with Signorini boundary conditions

Abstract. Let α be a positive number. The one-dimensional viscoelastic problem

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in (-\infty, 0], \quad t \in [0, +\infty).$$

with unilateral boundary conditions

$$u(0, \cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0, \cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0, \cdot) = 0,$$

can be reduced to the following variational inequality:

$$\lambda_1 * w = g + b, \quad w \geq 0, \quad b \geq 0, \quad \langle w, b \rangle = 0.$$

Here $\widehat{\lambda}_1(\omega)$ is the causal determination of $i\omega\sqrt{1 + i\alpha\omega}$. We show that the energy losses are purely viscous; this result is a consequence of the relation $\langle \dot{w}, b \rangle = 0$; since *a priori*, b is a measure and $\dot{w}$ is defined only almost everywhere, this relation is not trivial.

1. Abridged English version

We seek solutions of viscoelastodynamics for a Kelvin-Voigt material with Signorini boundary conditions. In the simple case of a homogeneous isotropic material in a half-space $x \leq 0$ and of a solution depending only on the coordinate x, the only interesting equation is that for the first component of the displacement. After adimensionalization, it reads

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{1.1}$$

with boundary conditions

(1.2) $u(0,\cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0,\cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0,\cdot) = 0,$

and initial data

(1.3) $u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).$

Without loss of generality, we may assume that $u_0(0)$ vanishes; otherwise, we solve the problem with vanishing $\big(u_x + \alpha u_{xt}\big)(0,\cdot)$ until $u(0,t)$ vanishes and we change the origin of times. Define λ_1 as the inverse Fourier transform of the causal determination of $i\omega\sqrt{1+\alpha i\omega}$; the problem (1.1), (1.2) and (1.3) is equivalent to the following problem on the boundary:

(1.4) $\lambda_1 * w = g + b, \quad g \geq 0, \quad b \geq 0, \quad \langle g, b\rangle = 0.$

Here g is equal to $-(\bar u_x + \alpha \bar u_{xt})(0,\cdot)$, where $\bar u$ is the solution of (1.1) and (1.3) with Dirichlet data. If u_0 and u_1 belong to $H^2(-\infty,0) \cap H^1_0(-\infty,0)$, and if f and f_t belong to $L^2_{\rm loc}([0,\infty); L^2(0,\infty))$, then g belongs to $H^{1/2}_{\rm loc}([0,\infty))$.

Observe that the principal term of λ_1 is a constant times the derivative of order $3/2$ of the Dirac mass; one could have thought that the difficulty of this problem is somewhere between the analogous problems

$$Lw = g + b, \quad w \geq 0, \quad b \geq 0, \quad \langle w, b\rangle = 0,$$

with $L = d/dt$ as in [**3**] or with $L = d^2/dt^2$ as in [**7**]. The pseudodifferential character of the convolution with λ_1 makes everything difficult; we thought at the beginning of this study that the viscous term might act as a regularization and make things simpler than for elastodynamics, where the Signorini problem is still wide open. It seems that this hope was not substantiated; nevertheless, we are at least able to provide a few results and proof techniques appropriate to the present frame.

The existence of a solution of (1.4) can be obtained by the penalty method.

The energy balance is obtained by multiplying (1.2) by u_t and integrating over $(-\infty,0) \times (0,\tau)$; see (4.1); the losses are purely viscous iff $\langle \dot w, b\rangle$ vanishes. This statement looks trivial since $\dot w$ vanishes almost everywhere on the support of b; but it is not so, since b is only a measure, and we do not know that its singular part vanishes.

We prove indeed that $\dot w$ vanishes on the set $\{u = 0\}$, except on a countable subset, and that b has no atoms; in consequence, $\langle \dot w, b\rangle$ vanishes. But these results cannot be deduced from functional estimates on the penalized approximation, since they only imply that w is bounded in $H^{5/4}_{\rm loc}([0,\infty))$.

Therefore, we develop another construction: we assume that g is the convolution of a measure ϕ with the kernel $1_{(0,\infty)}(t)/\sqrt{\pi t}$, i.e. g is a half-integral of ϕ; this is true if the data u_0, u_1 and f are smooth enough, as we have seen above. If ϕ is such that the supports of b and u are included in a locally finite union of intervals, we have an almost explicit formula for the solution (see (4.4) and (4.5)); in the general case, we construct an approximation w^n which has this local finiteness property; we are able to estimate $(\dot w^n(\sigma - 0))^-$ for each σ such that $w^n(\sigma) > 0$; together with an order property and L^∞ estimates on w^n, this enables us to conclude.

2. L'origine du problème

Le système de la viscoélastodynamique pour un matériau de Kelvin-Voigt peut s'écrire

$$\rho \mathbf{u}_{tt} = A\mathbf{u} + B\mathbf{u}_t + f, \quad x \in \Omega, \quad t \in (0,T),$$

avec des opérateurs A et B définis au moyen des tenseurs de Hooke; dans ce cas, les conditions de Signorini s'écriront $\sigma^A_T + \dot\sigma^B_T = 0$, $\sigma^A_N + \dot\sigma^B_N \geq 0$, $\mathbf{u}_N \geq 0$, $\mathbf{u}_N\big(\sigma^A_N +$

$\dot{\sigma}_N^B) = 0$. Ici, σ_N^C et σ_T^C désignent respectivement les composantes normales et tangentielles du tenseur des contraintes relativement à $C = A$ ou B.

Si le matériau est homogène et isotrope, et que nous cherchons une solution dans le demi-espace $x \le 0$, ne dépendant que de la variable x, le problème se découple, et après avoir choisi les unités de manière appropriée, l'équation satisfaite par la première composante est (1.1), avec les conditions au bord (1.2) et les conditions initiales (1.3).

Rappelons que le problème de Signorini élastodynamique reste complètement ouvert en plus d'une dimension d'espace; les premiers résultats monodimensionnels sont dus à Amerio [**2**], qui considérait plutôt un problème de vibrations avec obstacle ponctuel; [**6**] a mis en évidence que ces derniers étaient identiques aux problèmes avec contraintes au bord, et surtout a traité le cas d'une équation des ondes avec contrainte unilatérale au bord d'un demi-espace, tout en montrant l'unicité et la conservation de l'énergie. D'autres articles ont abordé des problèmes mulitidimensionnels, notablement [**5**], et [**4**] qui ont obtenu des solutions faibles, sans information sur le bilan énergétique, ni sur les traces.

3. Réduction à un problème au bord

Si $\bar{u}$ est la solution du problème de Cauchy-Dirichlet pour les mêmes données initiales, on prouve par analyse de Fourier que $w(t) = u(0,t)$ vérifie les relations (1.4).

Nous montrons que si u_0 et u_1 appartiennent à $H^2(-\infty,0) \cap H_0^1(-\infty,0)$ et si f et f_t appartiennent à $L^2_{\text{loc}}\big([0,\infty); L^2(-\infty,0)\big)$, alors g appartient à $H^{1/2}_{\text{loc}}\big([0,\infty)\big)$. Remarquons que la partie principale de λ_1 est une dérivée d'ordre $3/2$ de la masse de Dirac. L'existence d'une solution de (1.4) se prouve par pénalisation: la procédure d'itérations de Picard fournit une solution de

$$\lambda_1 * w^\varepsilon = g + \big(w^\varepsilon\big)^-/\varepsilon, \tag{3.1}$$

à support dans $\mathbb{R}^+$. Formellement, si on multiplie (3.1) par $\dot{w}^\varepsilon$ et qu'on intègre, on obtient une estimation sur la norme de w^ε dans $H^{5/4}_{\text{loc}}\big([0,\infty)\big)$; cette estimation peut être rendue rigoureuse, et nous montrons le

Théorème 3.1. *Si g appartient à $L^1_{\text{loc}}(\mathbb{R}) \cap H^{-1/4}_{\text{loc}}(\mathbb{R})$ et est à support dans $\mathbb{R}^+$, il existe une fonction $w \in H^{5/4}_{\text{loc}}(\mathbb{R})$ à support dans $\mathbb{R}^+$ satisfaisant* (1.4)

$$\lambda_1 * w = g + b, \quad w \ge 0, \quad b \ge 0, \quad \langle w, b\rangle = 0. \tag{3.2}$$

Idée de la démonstration. Si l'on peut trouver $T > 0$ tel que pour tout $\varepsilon > 0$, $w^\varepsilon(t) \ge 0$ quand $t \ge T$, alors il est légitime de multiplier (3.1) par $\dot{w}^\varepsilon$ et d'intégrer puisqu'on peut montrer que $\dot{w}^\varepsilon$ décroît exponentiellement à l'infini. Si g est dans $L^2(0,\infty)$, alors $\dot{w}^\varepsilon$ est borné dans $H^{1/4}(0,\infty)$ uniformément en ε. Afin de se ramener à ce cas, on modifie g pour des temps très grands et on conclut par un procédé diagonal.

Remarquons qu'il est difficile d'aller au delà des estimations énoncées dans ce théorème; en effet il est obtenu en mutipliant (3.1) par un facteur p^ε pour lequel $(\lambda_1 * w^\varepsilon, p^\varepsilon)$ est positif ou nul et on sait estimer $((w^\varepsilon)^-, p^\varepsilon)/\varepsilon$. Or le seul que nous ayons trouvé est $p^\varepsilon = \dot{w}^\varepsilon$.

4. Bilan d'énergie

Si nous multiplions (1.1) par u_t et que nous l'intégrons sur $(-\infty, 0] \times [0, \tau]$, nous obtenons formellement la relation

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{0}(u_x^2+u_t^2)(\cdot,\tau)\,dx+\alpha\int_{-\infty}^{0}\int_{0}^{\tau}u_{xt}^2\,dx\,dt=\frac{1}{2}\int_{-\infty}^{0}|u_t(\cdot,0)|^2\,dx\\+\frac{1}{2}\int_{-\infty}^{0}|u_x(\cdot,0)|^2\,dx+\int_0^\tau\int_{-\infty}^{0}fu_t\,dx\,dt+\int_0^\tau\big((u_x+\alpha u_{xt})u_t\big)(0,\cdot)\,dt.\end{aligned} \tag{4.1}$$

Les pertes d'énergies sont purement d'origine visqueuse si et seulement si le dernier terme intégral est nul. Par construction, $u_x + \alpha u_{xt} = \lambda_1 * w - g$; par conséquent, il nous faut montrer que $\langle \dot w, b\rangle$ s'annule si nous voulons comprendre le bilan d'énergie.

Une vision naïve consisterait à dire que comme $\dot w$ est nul sur le support de b, il est clair que $\langle \dot w, b\rangle$ s'annule. Mais, de fait, $\dot w$ n'est nul que presque partout, et nous ne savons pas si b est absolument continue par rapport à la mesure de Lebesgue. Par conséquent l'annulation de $\langle \dot w, b\rangle$ n'est pas triviale, et elle repose sur une construction en plusieurs étapes, demandant une certaine régularité sur g. L'hypothèse fondamentale est que g est une demi-primitive d'une mesure ϕ sur $\mathbb{R}$ à support dans $\mathbb{R}^+$; plus précisément, et en notant $\int_I f\phi$ l'intégrale relativement à la mesure ϕ sur un intervalle I d'une fonction f ϕ-intégrable, nous supposons:

$$g(t)=\int_{[0,t[}\big(\pi(t-s)\big)^{-1/2}\phi(s). \tag{4.2}$$

Soit w une solution de (1.4) possédant une structure localement finie, c'est-à-dire:

$$\operatorname{supp} w \subset \bigcup_{j\in J}[\sigma_j,\tau_j], \quad \operatorname{supp} b \subset \bigcup_{j\in J}[\tau_j,\sigma_{j+1}],$$

avec $0 \le \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \ldots$; notons H la fonction de Heaviside et posons $\omega(t) = H(t)/\big(\pi(t+1)\sqrt{t}\big)$, $\nu(t) = H(t)e^{-t/\alpha}/\alpha$ et $\Omega(t) = \int_0^t \omega(s)\,ds$.

Alors un calcul explicite fournit l'expression suivante de w:

$$w1_{[\sigma_j,\tau_j]}=\big(H*\nu*\phi_j\big)1_{[\sigma_j,\tau_j]}, \tag{4.3}$$

et la suite des ϕ_j est construite à partir de $\phi_0 = \phi$ par récurrence:

$$\psi_j=\phi_j1_{[\tau_j,\sigma_{j+1})}+\delta(\cdot-\tau_j)e^{-\tau_j/\alpha}\int_{[\sigma_j,\tau_j)}e^{s/\alpha}\phi_j, \tag{4.4}$$

$$\begin{aligned}\phi_{j+1}&=1_{[\sigma_{j+1},\infty)}\phi_j\\&\quad+1_{[\sigma_{j+1},\infty)}\int_{[\tau_j,\sigma_{j+1})}\exp\Big(-\frac{\cdot-s}{\alpha}\Big)\omega\Big(\frac{\cdot-\sigma_{j+1}}{\sigma_{j+1}-s}\Big)\frac{\psi_j(s)}{\sigma_{j+1}-s}.\end{aligned} \tag{4.5}$$

Cette formule conduit à l'estimation suivante, en un point σ où $w(\sigma)$ est strictement positif:

$$\begin{aligned}\big(\dot w(\sigma-0)\big)^-\le&\frac{1}{\alpha}\int_{[\sigma_j,\sigma)}|\phi_0(s)|\\&+\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)}\big(2e^{(\sigma-\sigma_j)/\alpha}-1\big)G(\sigma,s,i,j)\psi_{i-1}^+(s),\end{aligned} \tag{4.6}$$

où nous avons posé $G(\sigma,s,i,j) = \Omega\big((\sigma-\sigma_i)/(\sigma_i-s)\big) - \Omega\big((\sigma_j-\sigma_i)/(\sigma_i-s)\big)$.

D'autre part, elle nous permettra ultérieurement de trouver une estimation de la norme de $\dot w$ dans L^∞.

5. Approximation par des solutions à structure localement finie

Cependant nous n'avons aucune garantie d'existence d'une solution à structure localement finie pour des données g quelconques satisfaisant l'hypothèse (4.2). Nous construisons alors récursivement une suite possédant cette structure. Tout d'abord, soit τ_{-1} la borne inférieure du support de ϕ, et soit σ_0 la borne inférieure du support de $(H * \nu * \phi)^+$; si $\sigma_0 = \tau_{-1}$, posons $\phi_0 = \phi$; si $\sigma_0 > \tau_{-1}$, nous définissons ϕ_0 par

$$\phi_0 = 1_{[\sigma_0,\infty)}\left(\phi + \int_{[\tau_{-1},\sigma_0)} \exp\left(-\frac{\cdot - s}{\alpha}\right)\omega\left(\frac{\cdot - \sigma_0}{\sigma_0 - s}\right)\frac{\phi(s)}{\sigma_0 - s}\right);$$

alors $w_0 = w1_{[\sigma_0,\infty)}$ résout le problème

$$\lambda_1 * w_0 = \mu * \phi_0 + g_0, \quad \mu * \phi_0 \geq 0, \quad w_0 \geq 0, \quad \langle \mu * \phi_0, w_0\rangle = 0$$

et la borne inférieure du support de ϕ_0 est égale à σ_0. Posons $\widetilde{\phi}_0^n = \phi_0$ et $\sigma_0^n = \sigma_0$; si la borne inférieure du support de $\left(H * \nu * \widetilde{\phi}_0^n\right)^-$ est strictement supérieure à σ_0^n, nous l'appelons τ_0^n et nous posons $\phi_0^n = \widetilde{\phi}_0^n$. Sinon, nous pouvons trouver un instant τ_0^n dans $]\sigma_0^n, \sigma_0^n + 1/n]$ pour lequel $H * \nu * \widetilde{\phi}_0^n$ s'annule, alors que $(\nu * \widetilde{\phi}_0^n)(\tau_0^n - 0)$ est négatif ou nul, et nous posons

$$\phi_0^n = \delta(\cdot - \sigma_0^n)\int_{[\sigma_0^n,\tau_0^n)} (\widetilde{\phi}_0^n)^+ - \delta(\cdot - \tau_0^n)\int_{[\sigma_0^n,\tau_0^n)} (\widetilde{\phi}_0^n)^- + \phi_0^n 1_{[\tau_0^n,\infty)}.$$

Soit alors ϕ_j^n une mesure dont le support a même borne inférieure que celui de $(H * \nu * \phi_j^n)^+$, et soit

$$\tau_j^n = \inf \operatorname{supp}(H * \nu * \phi_j^n)^-.$$

Si $\tau_j^n = \infty$, la construction s'arrête; par construction $\tau_j^0 > \sigma_j^0$, et il est possible de montrer que $\tau_j^n > \sigma_j^n$ pour tout j. La mesure $\widetilde{\psi}_j^n$ est donnée par (4.4) avec chacune des quantités affectée d'un indice n en haut. Soit $\widetilde{\sigma}_{j+1}^n$ la borne inférieure du support de $\left(\widetilde{\sigma}_j^n\right)^+$; si $\widetilde{\sigma}_{j+1}^n = \infty$, la construction s'arrête; si $\widetilde{\sigma}_{j+1}^n$ est fini, nous prenons pour σ_{j+1}^n n'importe quel instant ne portant pas d'atome de $\widetilde{\psi}_j^n$ et tel que

$$\max\left(\widetilde{\sigma}_{j+1}^n, \tau_j^n + \frac{1}{2n}\right) \leq \sigma_{j+1}^n \leq \max\left(\widetilde{\sigma}_{j+1}^n, \tau_j^n + \frac{1}{2n}\right) + \frac{1}{2n},$$

et nous posons

$$\psi_j^n = -1_{[\tau_j^n,\sigma_{j+1}^n)}\left(\widetilde{\psi}_j^n\right)^- + \delta(\cdot - \sigma_{j+1}^n)\int_{[\tau_j^n,\sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+ + 1_{[\sigma_{j+1}^n,\infty)}\widetilde{\psi}_j^n.$$

Alors ϕ_{j+1}^n peut être déduit de ψ_j^n à partir de (4.5), lui aussi affecté d'indices n en haut. La fonction $w^n = \sum_{j\geq 0} 1_{[\sigma_j^n,\tau_j^n]}\left(H * \nu * \phi_j^n\right)$ est la solution de

$$\lambda_1 * w^n = g^n + b^n, \quad w^n \geq 0, \quad b^n \geq 0, \quad \langle w^n, b^n\rangle = 0$$

avec

$$\begin{aligned} g^n =& \mu * \left(\phi + (\phi_0^n - \phi)1_{[\sigma_0^n,\tau_0^n)}\right) \\ &+ \mu * \left(\sum_{j\geq 0}\left(\delta(\cdot - \sigma_{j+1}^n)\int_{[\tau_j^n,\sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+ - 1_{[\tau_j^n,\sigma_{j+1}^n)}\left(\widetilde{\psi}_j^n\right)^+\right)\right). \end{aligned}$$

Si nous supposons que ϕ est de masse finie, nous montrons que les masses de ϕ_{j+1}^n et de ψ_{j+1}^n sont bornées indépendamment de n et j, et que l'on a des propriétés d'ordre suffisantes pour passer à la limite:

LEMME 5.1. *Les inégalités suivantes ont lieu:*

$$\int |\phi_{j+1}^n| \leq \int |\psi_j^n| \leq \int |\phi_j^n|, \tag{5.1}$$

$$\left(\phi_{j+1}^n\right)^+ \leq 1_{[\sigma_{j+1}^n,\infty)}\left(\phi_j^n\right)^+ + \delta(\cdot - \sigma_{j+1}^n)\int_{[\tau_{j+1}^n,\sigma_{j+1}^n)} (\phi_j^n)^+. \tag{5.2}$$

Nous déduisons de (5.1) que g^n tend vers g dans $L^p(\mathbb{R})$, $p \in [1,2[$, que $\dot{w}^n$ est bornée dans $L^\infty(\mathbb{R})$ et que b^n est d'intégrale bornée sur $\mathbb{R}$. Enfin, w^n converge uniformément vers w sur tout compact, après extraction d'une sous-suite, et la limite vérifie (1.4): les estimations ainsi obtenues sont meilleures que celles obtenues via la pénalisation.

En particulier, comme $\dot{w}$ est essentiellement bornée, b ne peut avoir d'atomes. Par ailleurs, nous utilisons la propriété (4.6) pour montrer le résultat suivant:

THÉORÈME 5.2. *Soit N l'ensemble des atomes de ϕ et soit N_1 l'ensemble des extrémités des composantes connexes de $U = \{t \in \mathbb{R} : w(t) > 0\}$. Alors $w(t)$ est différentiable en tout point qui n'appartient pas à $N \cup N_1 \cup U$, et sa dérivée est nulle.*

Idée de la démonstration. Si $w(t)$ est nul et t n'est pas l'extrémité d'une composante connexe de $U = \{w > 0\}$, le cas où il y a quelque chose à démontrer est celui où t est une valeur d'adhérence de U. Si t est une valeur d'adhérence à droite et s'il n'est pas vrai que $\dot{w}(t)$ est nul, on peut trouver $\beta > 0$ et une suite t_m croissant vers t telle que $w(t_m) \geq \beta(t - t_m)$. Alors pour tout n assez grand, $w^n(t_m)$ est plus grand que $3\beta(t - t_m)/4$; si $]\sigma^n, \tau^n[$ est la composante connexe de t_m dans $\{w^n > 0\}$, on voit qu'il existe un ensemble de mesure non nulle dans $]\sigma^n, \tau^n[$ sur lequel $\dot{w}^n \leq -3\beta/4$; mais en vertu de (5.2) et de (4.6), on obtient une contradiction. Le cas où t est un point d'accumulation à gauche est traité de façon analogue, bien qu'un peu plus simple.

Ce résultat suffit à conclure que $\dot{w}\big|_{U^c}$ est nul sauf sur un ensemble dénombrable de points et par conséquent $\langle b, \dot{w}\rangle$ s'annule.

Remarque 1. *Nous ne savons rien de l'unicité des solutions; nous ne savons même pas si les deux constructions d'une solution de* (1.4) *fournissent le même résultat.*

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CHAPITRE 5

A pseudodifferential linear complementarity problem related to a one dimensional viscoelastic model with Signorini conditions

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Abstract. The simplified viscoelastic problem

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in \mathbb{R}^-, \quad t \in \mathbb{R}^+, \quad \alpha > 0,$$

with boundary condition

$$u(0,\cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0,\cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0,\cdot) = 0,$$

is reduced to pseudodifferential linear complementarity problem (LCP)

$$\lambda_1 * w = g + b, \quad 0 \leq w \perp b \geq 0.$$

where λ_1 is the inverse Fourier transform of the causal determination of $\widehat{\lambda}_1 = i\omega\sqrt{1 + i\alpha\omega}$. We prove the existence of a solution of this LCP; the energy relation for the original problem is equivalent to

$$\langle \dot{w}, b \rangle = 0.$$

This relation is formally and rigorously true, but highly non trivial since *a priori* b is a measure and $\dot{w}$ is defined almost everywhere.

1. Introduction and notations

The theory of vibrations of continuous media with unilateral conditions at the boundary purports to understand the mathematical description of the so-called dynamical Signorini problem; when the medium is elastic and satisfies the assumptions of the theory of small deformations, the one-dimensional medium is fairly well understood, starting from the work of Amerio and Prouse [**3, 4**] and of Schatzman [**10**], in the case of a continuous obstacle and the work of Amerio [**1, 2**] of Schatzman [**11**], Citrini [**5**], Citrini and Marchionna [**6**] and the theory can be considered as complete. In the multidimensional case, the theory is still quite poor, and there are deep functional analytic reasons for our ignorance. The only case where an energy relation is proved is that of is a wave equation in a half-space with unilateral conditions at the boundary [**9**]; see also for weaker results Uhn [**8**] and Jarušek et al. [**2**].

The present work stems from an attempt to replace the elastic case by the viscoelastic case.

In the most general situation, we would like to solve the dynamical evolution system for Kelvin-Voigt material, i.e.

$$\rho u_{tt} = Au + Bu_t + f, \quad x \in \Omega, \quad t \in (0,T),$$

where A and B are elasticity operators defined with the help of Hooke tensors a_{ijkl} and b_{ijkl} and Ω is the part of the space occupied by the material. Define the strain and respective stress tensors:

$$\varepsilon_{ij}(u) = \frac{\partial_i u_j + \partial_j u_i}{2}, \quad \sigma^A_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u), \quad \sigma^B_{ij}(u) = b_{ijkl}\varepsilon_{kl}(u);$$

the normal displacement at the boundary is

$$u_N = u_i \nu_i,$$

where we have chosen ν_i to be unit normal pointing inward; the normal and the tangential components of the stress vectors at the boundary are

$$\begin{aligned} &\sigma^A_N = \sigma^A_{ij}\nu_i\nu_j, \quad \sigma^B_N = \sigma^B_{ij}\nu_i\nu_j, \\ &\left(\sigma^A_T\right)_j = \sigma^A_{ij}\nu_i - \sigma^A_N\nu_j, \quad \left(\sigma^B_T\right)_j = \sigma^B_{ij}\nu_i - \sigma^B_N\nu_j. \end{aligned}$$

With these notations, the boundary condition on that part of the boundary where contact may take place is written:

$$\begin{aligned} &\sigma^A_T + \dot\sigma^B_T = 0, \\ &\sigma^A_N + \dot\sigma^B_N \geq 0, \quad u_N \geq 0, \quad u_N\left(\sigma^A_N + \dot\sigma^B_N\right) = 0. \end{aligned}$$

If we consider the very particular case where $\Omega = \mathbb{R}^- \times \mathbb{R}^{d-1}$ and if we seek a solution u which depends only on x_1 and t, while the material under consideration is homogeneous and isotropic, we are led to the following boundary problem for u_1:

$$\rho\frac{\partial^2 u_1}{\partial t^2} = \left(\lambda^A + 2\mu^A\right)\frac{\partial^2 u_1}{\partial x_1^2} + \left(\lambda^B + 2\mu^B\right)\frac{\partial^3 u_1}{\partial x_1^2\partial t} + f_1, \tag{1.1}$$

with the boundary conditions written in the fashion of a linear complementarity problem (LCP):

$$0 \leq \left(\lambda^A + 2\mu^A\right)\frac{\partial u_1}{\partial x_1} + \left(\lambda^B + 2\mu^B\right)\frac{\partial^2 u_1}{\partial x_1\partial t} \perp u_1 \geq 0;$$

here the orthogonality has the natural meaning: an appropriate duality product between the two terms of the relation vanishes. The problem for the second and third components of u is linear, viz. for $j = 2, 3$,

$$\rho\frac{\partial^2 u_j}{\partial t^2} = \mu^A\frac{\partial^2 u_j}{\partial x_1^2} + \mu^B\frac{\partial^3 u_j}{\partial x_1^2\partial t} + f_j,$$

with boundary conditions given by

$$\mu^A\frac{\partial u_j}{\partial x_1} + \mu^B\frac{\partial^2 u_j}{\partial x_1\partial t} = 0.$$

Therefore we concentrate our efforts on (1.1), which becomes after appropriate adimensionalization the one-dimensional viscously damped wave equation on a half-line:

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{1.2}$$

with initial data

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1, \tag{1.3}$$

and boundary conditions

$$0 \leq u(0,\cdot) \perp (u_x + \alpha u_{xt})(0,\cdot) \geq 0. \tag{1.4}$$

If $u_0(0)$ is strictly positive, we may solve the linear problem (1.2) with initial conditions (1.3) and boundary condition

$$(u_x + \alpha u_{xt})(0,\cdot) = 0. \tag{1.5}$$

Then by energy estimates we conclude easily that if $u_{0,x}$, u_1 and f are square integrable respectively on $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R} \times (0,T)$, u is continuous over $\mathbb{R} \times [0,T]$; it suffices therefore to solve (1.2)-(1.3)-(1.5) on the maximum time interval over which $u_0(0,\cdot)$ is strictly positive, to be reduced to the case

$$u_0(0) = 0.$$

Denote by $\bar{u}$ the solution of

$$\bar{u}_{tt} - \bar{u}_{xx} - \alpha \bar{u}_{xxt} = f, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{1.6}$$

with initial data (1.3) and Dirichlet boundary data at $x = 0$. Define

$$g = -\bar{u}_x(0,\cdot) - \alpha \bar{u}_{xt}(0,\cdot). \tag{1.7}$$

Then $v = u - \bar{u}$ solves

(1.8a) $$v_{tt} - v_{xx} - \alpha v_{xxt} = 0,$$

(1.8b) $$v(0,\cdot) \geq 0, \quad (v_x + \alpha v_{xt})(0,\cdot) \geq g, \quad \big(v(v_x + \alpha v_{xt})\big)(0,\cdot) = 0$$

(1.8c) $$v(\cdot,t) = 0 \quad \text{if} \quad t \leq 0.$$

Call λ_1 the distribution whose Fourier transform in time is the causal determination of $i\omega\sqrt{1 + \alpha i\omega}$; we will show in Section 3 that v solves (1.8) iff $w = v(0,\cdot)$ solves

$$\lambda_1 * w = g + b, \quad 0 \leq w \perp b \geq 0. \tag{1.9}$$

In order to construct a solution of (1.9), we require that g be a half-integral of a measure ϕ with support in $\mathbb{R}^+$, i.e. for almost every $t > 0$:

$$g(t) = \int_{[0,t[} \frac{1}{\sqrt{\pi(t-s)}}\, \phi(s). \tag{1.10}$$

We will show in Section 2 that if u_0 and u_1 belong to $H^2(-\infty,0) \cap H_0^1(-\infty,0)$ and if f and f_t belong to $L^2_{\text{loc}}\big([0,\infty); L^2(-\infty,0)\big)$, then g belongs to $H^{1/2}_{\text{loc}}(\mathbb{R})$ and is supported in $\mathbb{R}^+$, so that our theory can be applied.

In Section 4, we define a penalized problem associated to (1.9), for which we prove the existence and uniqueness of a solution. It is not difficult to extract weakly convergent subsequences, to pass to the limit and to obtain therefore the existence of a solution of (1.9) belonging to $H^{5/4}_{\text{loc}}(\mathbb{R})$.

Let us write at least formally an energy relation for (1.2): we multiply this equation by u_t, we integrate by parts over $(-\infty,0) \times (0,\tau)$, and we get

$$\frac{1}{2}\int_{-\infty}^0 (u_x^2 + u_t^2)(\cdot,\tau)\,dx + \alpha \int_{-\infty}^0 \int_0^\tau u_{xt}^2\,dx\,dt = \frac{1}{2}\int_{-\infty}^0 \big(|u_1|^2 + |u_{0,x}|^2\big)\,dx$$
$$+ \int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(0,\cdot)\,dt + \int_0^\tau \int_{-\infty}^0 f u_t\,dx\,dt.$$

The energy loss is purely viscous, iff

$$\int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(0,\cdot)\,dt = 0. \tag{1.11}$$

By construction, $(u_x + \alpha u_{xt})(0,\cdot)$ is equal to $\lambda_1 * w - g$; therefore, (1.11) is equivalent to

$$\langle b, \dot{w} \rangle = 0. \tag{1.12}$$

A priori, b belongs to $H^{-1/4}_{\text{loc}}(\mathbb{R})$ and is non negative and $\dot{w}$ belongs to $H^{1/4}_{\text{loc}}(\mathbb{R})$: this is not enough to conclude that (1.12) holds.

The standard methods used for variational inequalities break down here: the non local character of the convolution by λ_1 seems to preclude any kind of argument based on the signs of functions. While local estimates using sign cannot work, we construct a global estimate which will work. This is where we need regularity on data, i.e. (1.10).

Let us sketch the principle of the construction of Section 5: define H to be the Heaviside function and

$$\nu(t) = e^{-t/\alpha}\frac{H(t)}{\alpha}.$$

Let ρ be the inverse Fourier transform of the causal determination of $\sqrt{1+\alpha i\omega}$, and μ the inverse Fourier transform of $1/\widehat{\rho}$. Then ν is equal to $\mu * \mu$ and $\lambda_1 = \rho'$; the convolution inverse of λ_1 is $H * \mu = \mu_1$.

More information on these distributions is given in the Appendix.

Assume first that our data ϕ is such that $H * \nu * \phi$ is positive on some interval $[\sigma, \tau)$ and then negative on some interval (τ, σ'); an explicit calculation shows that a good candidate for a solution on $[\sigma, \tau]$ is $1_{[\sigma,\tau]}\big(H * \nu * \phi\big)$; then, we calculate a candidate for b, hoping that the support of b will be included in $[\tau, \sigma']$, and we find $b = -\mu * \psi$, where ψ is obtained from ϕ by a linear operation; however, since after time σ', we expect the solution to be positive again, the really good candidate for b is rather $-\big(\mu * \psi\big)1_{[\tau,\sigma')}$, and, lo and behold, there exists a measure ζ such that

$$-\big(\mu * \psi\big)1_{[\tau,\sigma']} = \mu * \zeta;$$

the calculation of this measure is the object of Lemma 5.1, where we also give the formula for w on the next interval where it is non zero. Once we have a formula for two intervals, we generalize to any number of intervals (Corollary 5.2). Moreover, we are able to give an estimate on the left derivative of w over any interval where w is positive (Lemma 5.5), and this estimate leads to the above mentioned global sign argument.

However, this construction has a very significant defect: we do not know that it is possible to extend it to an interval of finite length; therefore, the next idea is to modify ϕ so as to realize the locally finite construction; this is the recursion of Section 6.

A number of estimates are given in that Section, and they lead easily to the extraction of a convergent subsequence in Section 7; but this is not enough to show the desired energy equality; the function $\dot{w}$ cannot vanish everywhere on the support of b, since w may have a strictly negative left derivative at the right end of intervals where w is strictly positive. The requirement that w vanish almost everywhere on the support of b does not suffice, since we do not know that b is absolutely continuous with respect to Lebesgue's measure. Therefore, we show that $\dot{w}$ vanishes on the support of b except on a countable set; on the other hand, as $\mu * b$ is a locally essentially bounded function, the atomic part of b vanishes; thus $\dot{w}$ is b-integrable and we are able to conclude.

2. Regularity results for the damped wave equation

THEOREM 2.1. *If u_0 and u_1 belong to $H^2(-\infty, 0) \cap H^1_0(-\infty, 0)$ and if f and f_t belong to $L^2_{\text{loc}}\big([0,\infty), L^2(-\infty,0)\big)$, then $\bar{u}$ has the following functional properties:*

$$\begin{cases} \bar{u} \in W^{2,\infty}_{\text{loc}}\big([0,\infty); L^2(-\infty,0)\big), \\ \bar{u}_x \in W^{2,\infty}_{\text{loc}}\big([0,\infty); L^2(-\infty,0)\big) \cap H^2_{loc}\big([0,\infty); L^2(-\infty,0)\big), \\ \bar{u}_{xx} \in L^{\infty}_{\text{loc}}\big([0,\infty); L^2(-\infty,0)\big). \end{cases} \tag{2.1}$$

PROOF. We sketch here the proof of (2.1), using a straightforward energy inequality. The proof could be easily completed by a Galerkin method, but since it is quite routine, we leave the verification to the reader.

Multiply (1.6) by $\bar{u}_t$, and integrate by parts in x; then we find

$$\frac{1}{2}\int_{-\infty}^{0}\big(|\bar{u}_t(\cdot,\tau)|^2+|\bar{u}_x(\cdot,\tau)|^2\big)\,dx+\alpha\int_0^{\tau}\int_{-\infty}^{0}|\bar{u}_{xt}|^2\,dx\,dt$$
$$=\int_0^{\tau}\int_{-\infty}^{0} f\bar{u}_t\,dx\,dt+\frac{1}{2}\int_{-\infty}^{0}\big(|u_1|^2+|u_{0,x}|^2\big)\,dx.$$

A straightforward application of Gronwall's lemma yields: $\bar{u}_t$, $\bar{u}_x$ are bounded in $L^{\infty}_{\text{loc}}\big([0,\infty);L^2(-\infty,0)\big)$, $\bar{u}_{xt}$ is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(-\infty,0)\big)$ under the assumptions u_0 belongs to $L^2(-\infty,0)\cap H^1_0(-\infty,0)$, u_1 belongs to $L^2(-\infty,0)$ and f belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(-\infty,0)\big)$.

If we multiply (1.6) by $\bar{u}_{xxt}$, an application of Cauchy-Schwarz inequality shows that $\bar{u}_{xxt}$ belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(-\infty,0)\big)$, $\bar{u}_{xt}$ and $\bar{u}_{xx}$ belong to the space $L^{\infty}_{\text{loc}}([0,\infty);L^2_{\text{loc}}(-\infty,0))$.

Similarly, after differentiating (1.6) with respect to t, and multiplying it by $\bar{u}_{tt}$, we find that $\bar{u}_{xxt}$ belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(-\infty,0)\big)$. □

COROLLARY 2.1. *Under the hypotheses of Theorem 2.1,* $(\bar{u}_x+\alpha\bar{u}_{xt})(0,\cdot)$ *belongs to the space* $H^{1/2}_{\text{loc}}\big([0,\infty)\big)$.

PROOF. This proof is a consequence of the classical theory of traces of Sobolev spaces. □

REMARK 2.2. *The conclusion of Corollary 2.1 is much stronger than needed; its purpose is only to show that our theory is not empty. Obtaining estimates in Banach spaces for the traces of the solution of the viscoelastodynamic equation* (1.6) *is much more difficult than in Hilbert spaces, which is the reason why we have opted for a simple approach of the regularity theory.*

3. Reduction to a problem at the boundary

Our convention for the Fourier transform is

$$\widehat{z}(\omega)=\int_{\mathbb{R}} e^{-i\omega t}z(t)\,dt.$$

Let us apply a partial Fourier transform in time to (1.8a), calling ω the dual variable of t; then equation (1.8a) becomes

$$\widehat{v}_{xx}=-\frac{\omega^2}{1+i\alpha\omega}\,\widehat{v}. \tag{3.1}$$

The general solution of (3.1) is given by

$$\widehat{v}(x,\omega)=\widehat{a}(\omega)\exp\big(\widehat{\lambda}(\omega)x\big)+\widehat{b}(\omega)\exp\big(-\widehat{\lambda}(\omega)x\big); \tag{3.2}$$

since we performed a Fourier transform on ν, we assumed implicitly that v and $\widehat{v}$ are tempered respectively in t and ω.

Intuitively the term $\widehat{b}\exp\big(-\widehat{\lambda}x\big)$ can be tempered only if $\widehat{b}$ decays at infinity very fast, and since this must be true for all x, it will imply that b vanishes as proved in next lemma:

LEMMA 3.1. *If* v *is of finite energy, the coefficient* b *vanishes.*

PROOF. We eliminate a by performing a linear combination of v and v_x:

$$-\widehat{v}_x(x,\cdot)+\widehat{\lambda}\,\widehat{v}(x,\cdot)=2\widehat{b}\,\widehat{\lambda}\exp\big(-\widehat{\lambda}x\big).$$

The Paley-Wiener-Schwartz theorem implies that λ is a tempered distribution on $\mathbb{R}$, with support included in $\mathbb{R}^+$, i.e. a causal distribution. Let us define

$$\widehat{w}(x,\cdot)=\widehat{b}\,\widehat{\lambda}\exp\big(-\widehat{\lambda}x\big).$$

Since v is of finite energy, it is tempered, and therefore w and $\widehat{w}$ are tempered. Therefore there exists φ_γ, $|\gamma| \le m$, which is continuous and polynomially increasing such that

$$\widehat{w}(x,\cdot) = \sum_{|\gamma|\le m} \partial^\gamma \varphi_\gamma(x,\cdot) \text{ in the distributions sense.}$$

Here γ is multi-index (γ_1, γ_2) and $|\gamma| = \gamma_1 + \gamma_2$.

Let ψ and $\widehat{\varphi}$ belong respectively to $C_0^\infty(\mathbb{R})$ and $C_0^\infty(0,\infty)$; assume that the support of $\widehat{\varphi}$ is included in $\{\omega : \omega_1 \le |\omega| \le \omega_2\}$ with $\omega_1 > 0$, and call $[x_1, x_2]$ an interval containing the support of ψ. The distribution $\widehat{b}$ restricted to $(0,\infty) \times (\mathbb{R} \setminus \{0\})$ is equal to

$$\widehat{w}(x,\cdot) \exp(\widehat{\lambda} x)/\widehat{\lambda}.$$

If we assume that

$$\int_{x_1}^{x_2} \psi(x)\, dx = 1,$$

and if y is an arbitrary negative number, we have the equality in the sense of distributions:

$$(3.3) \qquad \begin{aligned} \langle \widehat{b}, \widehat{\varphi} \rangle &= \langle \widehat{b}, \widehat{\varphi} \otimes \psi \rangle \\ &= \sum_{|\gamma|\le m} \int_{\mathbb{R}} \int_{-\infty}^{0} (-1)^{|\gamma|} \varphi_\gamma(x,\cdot) \partial^\gamma \left(\frac{\exp(\widehat{\lambda} x)}{\widehat{\lambda}} \widehat{\varphi}\psi(x-y) \right) dx\, d\omega. \end{aligned}$$

The reader will check that all the derivatives of $\exp(\widehat{\lambda} x)\widehat{\varphi}\psi(x-y)/\widehat{\lambda}$ are finite sums of expressions of the form

$$\widehat{a}(\omega)\widehat{\varphi}^{(j)}(\omega)\psi^{(k)}(x-y)x^m \exp(\widehat{\lambda}(\omega)x)$$

where $\widehat{a}$ is the quotient of a polynomial in $\widehat{\lambda}$ and a finite number of its derivatives, and of a power of $\widehat{\lambda}$. Since 0 is excluded from the support of ω, we have the estimate

$$|\widehat{a}\widehat{\varphi}^{(j)}| \le C.$$

Let $k > 0$ be a lower bound of $\Re\widehat{\lambda}$ over $\omega_1 \le |\omega| \le \omega_2$. Then there exists C_1 such that for ω verifying $\omega_1 \le |\omega| \le \omega_2$ and for $x < 0$,

$$|\varphi_\gamma(x,\cdot)| \le C_1 \exp(-kx/2).$$

We may estimate each term in the right hand side of (1.4c) by

$$2(\omega_2 - \omega_1) \int_{-\infty}^{0} C C_1 \exp(kx/2) |\psi^{(k)}(x-y)| |x|^m\, dx.$$

As y tends to $-\infty$, it is clear that this integral tends to 0, and therefore the restriction of $\widehat{b}$ to $\mathbb{R} \setminus \{0\}$ vanishes. This means that $\widehat{b}$ can be a finite combination of the derivatives of a Dirac mass at 0; but these terms can be included in $\widehat{a}$, proving thus the lemma. □

We deduce from Lemma 3.1 that if v is tempered in the neighborhood of $x = -\infty$, it must be of the form

$$\widehat{v}(x,\cdot) = \widehat{a} \exp(\widehat{\lambda} x).$$

In particular

$$(3.4) \qquad \widehat{v}_x(0,\cdot) + \alpha \widehat{v_{xt}}(0,\cdot) = \widehat{\lambda}_1 \widehat{v}(0,\cdot).$$

If we let w be the trace $v(0,\cdot)$, (1.8) can be written now

$$(3.5) \qquad \lambda_1 * w = g + b, \quad w \ge 0, \quad b \ge 0, \quad b \perp w.$$

Of course, w vanishes for all negative times, and g has been defined at (1.7). At this point, $b \perp w$ is a formal statement, and a good part of this article aims to turn this formal statement into a *bona fide* mathematical relation — which implies in particular that we are able to assign a coherent sense to all the quantities involved.

4. Existence and uniqueness of the solution of the penalized equation

In (1.7), the rigid constraint is defined by a set of linear complementary conditions. We approximate this constraint by a very stiff response which vanishes when the constraint is not active and is linear when the constraint is active. More precisely, if $r^- = -\min(r, 0)$, we consider the problem

$$\lambda_1 * w^\varepsilon = g + (w^\varepsilon)^-/\varepsilon, \tag{4.1}$$

where w^ε vanishes for all $t < 0$. We recall that μ_1 is the integral of μ which vanishes at 0, where $\widehat{\mu}(\omega)$ is the causal determination of $1/\sqrt{1 + i\alpha\omega}$.

We establish now the existence and uniqueness of the solution of (4.1).

THEOREM 4.1. *Assume that g belongs to $L^1_{\text{loc}}(\mathbb{R}) \cap H^{-1/4}_{\text{loc}}(\mathbb{R})$ and vanishes for $t < 0$. Let h be a uniformly Lipschitz continuous function; then there exists a unique solution vanishing for $t < 0$ of the convolution equation:*

$$w = \mu_1 * (g - h(w)). \tag{4.2}$$

Moreover w is continuous.

PROOF. We see from (7.12) that on any interval $[0, T]$, μ_1 satisfies the estimate

$$0 \le \mu_1(t) \le \frac{2\sqrt{T}}{\sqrt{\pi\alpha}}.$$

Define an integral operator $\mathcal{T}$ by

$$\mathcal{T}w = \mu_1 * (g + h(w)).$$

It is clearly equivalent to find a solution of (4.2) or a fixed point of $\mathcal{T}$. Therefore, it is suffices to find an integer k such that $\mathcal{T}^k$ will be a strict contraction in an appropriate functional space.

Let us check first that $\mathcal{T}$ maps $C^0([0,T])$ to itself: since μ_1 is an integral of the integrable function μ, it is continuous and therefore $\mu_1 * g$ is also continuous. Since the composition $h \circ w$ is continuous, it is plain that $\mathcal{T}w$ is a continuous function.

We estimate now the Lipschitz constant of $\mathcal{T}$ restricted to $C^0([0,T])$, denoting by L the Lipschitz constant of h:

$$\left|\big(\mathcal{T}w_2 - \mathcal{T}w_1\big)(t)\right| \le \int_0^t \mu_1(t - \cdot)L|w_2 - w_1|\,ds \le \frac{2L\sqrt{T}}{\sqrt{\pi\alpha}} \int_0^t |w_2 - w_1|\,ds.$$

Since this estimate is completely analogous to the classical estimate of Picard iterations, we obtain by induction the estimate

$$\left|\big(\mathcal{T}^k w_2 - \mathcal{T}^k w_1\big)(t)\right| \le \left(\frac{2L\sqrt{t}}{\sqrt{\pi\alpha}}\right)^k \frac{\|w_2 - w_1\|_\infty}{k!}.$$

Therefore, for all $T \in (0, \infty)$, we can find an integer k such that the restriction of $\mathcal{T}^k$ to $C^0([0,T])$ is a strict contraction. As T is arbitrary the theorem is proved. □

REMARK 4.2. *We could have obtained the stronger estimate:*

$$\left|\big(\mathcal{T}^k w_2 - \mathcal{T}^k w_1\big)(t)\right| \le \left(\frac{L}{\sqrt{\alpha}}\right)^k \chi^{1+3k/2}(t)\,\|w_2 - w_1\|_\infty,$$

where $\chi^a(t) = (t^+)^{a-1}/\Gamma(a)$, which leads to the same conclusion, but with a smaller k for each T.

REMARK 4.3. *The same proof works if instead of $h(w)$ we introduce a continuous function $h(t,w)$ which is Lipschitz continuous with respect to its second argument.*

REMARK 4.4. *If g_1 and g_2 coincide over $(-\infty,T]$, then the corresponding solutions w_1 and w_2 of $w_1 = \mu_1 * (g_1 - h(w_1))$ and $w_2 = \mu_1 * (g_2 - h(w_2))$ coincide over $(-\infty,T)$, thanks to the causal character of μ_1.*

We would like to estimate w^ε in appropriate functional spaces independently of ε. We will assume that g belongs to $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$. Formally we multiply (4.1) by $\dot w^\varepsilon$, and we estimate the pseudodifferential term in the Fourier variables. We obtain

$$\frac{1}{2\pi}\Re\Big(\int_{\mathbb{R}} \widehat{\lambda}_1 \widehat{w}^\varepsilon \overline{i\omega\widehat{w}^\varepsilon}\,d\omega\Big) + \int_0^\infty \frac{1}{2\varepsilon}\frac{d}{dt}\big((w^\varepsilon)^-\big)^2\,dt = \frac{1}{2\pi}\Re\Big(\int_{\mathbb{R}} \overline{i\omega\widehat{w}^\varepsilon}\widehat{g}\,d\omega\Big). \tag{4.3}$$

We infer from the estimate

$$\big|\widehat{\lambda}_1 i\omega\big| \ge \frac{1}{C}|\omega|^2\big(1+|\omega|\big)^{1/2}$$

that

$$\int_{\mathbb{R}} |\omega|^{5/2}|\widehat{w}^\varepsilon|^2\,d\omega \le C\int_{\mathbb{R}} |\omega|^2\big(1+|\omega|\big)^{1/2}|\widehat{w}^\varepsilon|\big(1+|\omega|\big)^{-1/4}|\widehat{g}|\,d\omega, \tag{4.4}$$

and therefore if $\big(1+|\omega|\big)^{-1/4}|\widehat{g}(\omega)|$ is bounded in $L^2(\mathbb{R})$, we see that $|\omega|\big(1+|\omega|\big)^{1/4}|\widehat{w}^\varepsilon(\omega)|$ is bounded in $L^2(\mathbb{R})$ independently of ε. As it stands, this calculation is insane, and the aim of the present section is turn it into a valid result. The essential idea is to use the causality: it enables us to modify g for large times, to validate the desired result on a time interval for which g has not been modified, and then to conclude for $\mathbb{R}^+$, since the modification time has been arbitrarily chosen.

The first step consists in proving the following lemma:

LEMMA 4.5. *Assume that g belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ and vanishes for $t<0$. Then there exists for all $T>0$ an $S>T$ and a compactly supported function G which coincides with g over $[-\infty,T]$ such that for all $\varepsilon>0$ the solution W^ε of*

$$\lambda_1 * W^\varepsilon = G + (W^\varepsilon)^-/\varepsilon, \tag{4.5}$$

is non negative over $[S,\infty)$.

PROOF. We choose ψ to be a $\mathcal{C}^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$, which takes its values in $[0,1]$, and satisfies

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t<0,\\ 1 & \text{if} \quad t\ge 1.\end{cases}$$

We define then

$$G(t) = \psi(T+1-t)g(t) + \beta\psi(t-T-2)\psi(T+3-t),$$

where β is a number to be defined later. Then, for $t\ge T+3$, we may write

$$W^\varepsilon = \mu_1 * G + \mu_1 * (W^\varepsilon)^-/\varepsilon. \tag{4.6}$$

The first term on the right hand side of (4.6) can be written also as

$$\int_0^{T+1} \mu_1(t-s)g(s)\psi(T+1-s)\,ds + \beta\int_{T+2}^{T+3} \mu_1(t-s)\psi(s-T-2)\psi(T+3-s)\,ds. \tag{4.7}$$

The function $\mu_1(t)$ is increasing and tends to 1 as t tends to infinity, since μ is non negative and $\widehat{\mu}(0) = 1$. Therefore, the limit for t going to infinity of the second term of (4.7) is

$$\lim_{t\to\infty} \beta \int_{T+2}^{T+3} \mu_1(t-s)\psi(s-T-2)\psi(T+3-s)\,ds = \beta \int_0^1 \psi(s)\psi(1-s)\,ds.$$

We estimate from below the first term of (4.7) as follows:

$$\int_0^{T+1} \mu_1(t-s)g(s)\psi(T+1-s)\,ds \geq -\int_0^{T+1} |g(s)|\,ds.$$

We choose β so large that

$$\beta \int_0^1 \psi(t)\psi(1-t)\,dt > \int_0^{T+1} |g(s)|\,ds.$$

Then there exists S such that for all $t \geq S$, $(\mu_1 * G)(t) \geq 0$, and thanks to (4.6), the conclusion is clear. □

This Lemma yields a Corollary:

COROLLARY 4.6. *The function $\lambda_1 * W^\varepsilon$ is compactly supported and $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$; moreover $\dot{W}^\varepsilon$ decays exponentially to* 0 *as time goes to infinity.*

PROOF. As a consequence of Lemma 4.5, the supports of $(W^\varepsilon)^-$ and G are included in the interval $[0, S]$. Define

$$g_1 = G + (W^\varepsilon)^-/\varepsilon.$$

We infer from (7.12) and the identity $\dot{W}^\varepsilon = \mu * g_1$ that for $t > S$

$$|\dot{W}^\varepsilon| \leq \frac{\exp(-t/\alpha)}{\sqrt{\pi\alpha(t-S)}} \int_0^S |g_1(s)| \exp(s/\alpha)\,ds$$

which implies the exponential decay of $\dot{W}^\varepsilon$. On the other hand, G belongs to $H^{-1/4}(\mathbb{R})$ and $(W^\varepsilon)^-$ is square integrable and therefore belongs also to $H^{-1/4}(\mathbb{R})$. Therefore, by Fourier transformation,

$$(\dot{W}^\varepsilon)\widehat{\ }(\omega) = \big(\widehat{\mu}\,\widehat{g}_1\big)(\omega) = \frac{\widehat{g}_1(\omega)}{\sqrt{1+i\alpha\omega}},$$

and it is plain that $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$, thus concluding the proof of corollary. □

LEMMA 4.7. *The following estimate holds:*

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} |\omega|^2 \sqrt{1+|\omega|}\,\big|\widehat{W}^\varepsilon\big|^2\,d\omega < +\infty. \tag{4.8}$$

PROOF. Since G belongs to $H^{-1/4}(\mathbb{R})$ and $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$, we perform the duality product of (4.5) with $\dot{W}^\varepsilon$. By standard properties of the Fourier transform,

$$2\pi\langle \lambda_1 * W^\varepsilon, \dot{W}^\varepsilon\rangle_{H^{-1/4},H^{1/4}} = \Re \int_{\mathbb{R}} \widehat{\lambda}_1 \widehat{W}^\varepsilon \overline{i\omega\widehat{W}^\varepsilon}\,d\omega;$$

thanks to the definition of λ_1, there exists a constant C such that $\Re\widehat{\lambda}_1 \geq C|\omega|\big(1 + |\omega|\big)^{1/2}$, so that

$$2\pi\langle \lambda_1 * W^\varepsilon, \dot{W}^\varepsilon\rangle_{H^{-1/4},H^{1/4}} \geq C\int_{\mathbb{R}} |\omega|^2\big(1+|\omega|\big)^{1/2}\big|\widehat{W}^\varepsilon\big|^2\,d\omega.$$

The duality product of $(W^\varepsilon)^-$ with $\dot{W}^\varepsilon$ can be identified with the L^2 scalar product of these two quantities, since $(W^\varepsilon)^-$ is continuous with compact support. In consequence, as follows from the classical results on the derivative of the negative part of an $H^1_{\rm loc}(\mathbb{R})$ function,

$$\left\langle (W^\varepsilon)^-, \dot W^\varepsilon \right\rangle_{H^{-1/4},H^{1/4}} = -\frac{1}{2}\int_{\mathbb{R}} \frac{d}{dt}\left((W^\varepsilon)^-\right)^2 dt$$

which vanishes since $(W^\varepsilon)^-$ is compactly supported. The last term is

$$\left\langle G, \dot W^\varepsilon \right\rangle_{H^{-1/4},H^{1/4}} = \frac{1}{2\pi}\Re\int_{\mathbb{R}} \widehat{G}\overline{i\omega\widehat{W^\varepsilon}}\,d\omega$$

$$\le \frac{1}{2\pi}\left(\int_{\mathbb{R}} \frac{\left|\widehat{G}\right|^2}{\left(1+|\omega|\right)^{1/2}}\,d\omega\right)^{1/2}\left(\int_{\mathbb{R}} |\omega|^2\left(1+|\omega|\right)^{1/2}\left|\widehat{W^\varepsilon}\right|^2 d\omega\right)^{1/2}.$$

By construction $\left|\widehat{G}\right|\left(1+|\omega|\right)^{-1/4}$ is bounded in $L^2(\mathbb{R})$, and the conclusion is clear. □

THEOREM 4.8. *Assume that g belongs to $L^1_{\rm loc}(\mathbb{R}) \cap H^{-1/4}_{\rm loc}(\mathbb{R})$ and vanishes for $t < 0$. Then there exists a function $w \in H^{5/4}_{\rm loc}(\mathbb{R})$ which vanishes for $t < 0$ and a measure b supported in $\mathbb{R}^+$ such that*

$$\lambda_1 * w = g + b, \quad w \ge 0, \quad b \ge 0, \quad \langle w, b\rangle = 0.$$

PROOF. Define

$$N(T) = \int_0^{T+1} |g(t)|\,dt + \|\psi(T+1-\cdot)g\|_{H^{-1/4}}.$$

The construction of Lemma 4.5 and Corollary 4.6 shows that there exists a constant C such that

$$\|\dot W^\varepsilon\|_{H^{1/4}} \le CN(T).$$

Let us estimate the mass of $(W^\varepsilon)^-/\varepsilon$ over a finite interval: by definition of ρ and λ_1, we have

$$\lambda_1 * W^\varepsilon = \rho * \dot W^\varepsilon;$$

but the distribution ρ can be described precisely since

$$\rho * \rho = \delta + \alpha\delta'$$

and therefore

$$\rho = \mu + \alpha\mu' \text{ in the sense of distributions.}$$

We convolve with H the identity

$$\rho * \dot W^\varepsilon = G + \frac{(W^\varepsilon)^-}{\varepsilon},$$

obtaining therefore

$$H * (\mu + \alpha\mu) * \dot W^\varepsilon = H * \left(G + \frac{(W^\varepsilon)^-}{\varepsilon}\right).$$

As μ is non negative and of integral 1, we know that $0 \le H * \mu \le 1$; therefore for any $T_0 > 0$,

$$|(H * \mu * \dot W^\varepsilon)(T_0)| \le \left(\int_0^{T_0} |\dot W^\varepsilon|^2\,dt\right)^{1/2}\sqrt{T_0} \le \|\dot W^\varepsilon\|_{H^{1/4}}\sqrt{T_0}.$$

On the other hand,

$$(\mu * \dot W^\varepsilon)\widehat{\ }(\omega) = (\dot W^\varepsilon)\widehat{\ }(\omega)/\sqrt{1+i\alpha\omega}$$

and therefore $\mu * \dot W^\varepsilon$ belongs to $H^{3/4}(\mathbb{R})$, with the estimate

$$\|\mu * \dot W^\varepsilon\|_{H^{3/4}} \le C\|\dot W^\varepsilon\|_{H^{1/4}},$$

and by Sobolev injections,

$$\|\mu * \dot{W}^\varepsilon\|_{L^\infty} \le C\|\dot{W}^\varepsilon\|_{H^{1/4}}.$$

Thus, we have obtained the estimate

$$\begin{aligned}\int_0^{T_0} \frac{(W^\varepsilon)^-}{\varepsilon}\, dt &\le \|\dot{W}^\varepsilon\|_{H^{1/4}}\Big(C + \sqrt{T_0}\Big) + \|G\|_{L^1(0,T_0)} \\ &\le CN(T)\Big(1 + \sqrt{T_0}\Big).\end{aligned}$$

We set up now a diagonal process; denote by g_n the function associated to $T = n$, and by S_n the corresponding number constructed at Lemma 4.5. The solution of (4.5) is now called w_n^ε. For each n, let $(\varepsilon_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0 as m tends to infinity. There exists a subset J of $\mathbb{N}$ such that as m tends to infinity in J,

$$\left(\dot{w}_n^{\varepsilon_m}\right)_{m\in J} \rightharpoonup \dot{w}_n \text{ in } H^{1/4}(\mathbb{R}) \text{ weak,} \tag{4.9}$$

and

$$\left((w_n^{\varepsilon_m})^-/\varepsilon_m\right)_{m\in J} \rightharpoonup b_n \text{ in } \mathcal{M}^1(\mathbb{R}) \text{ weak } *. \tag{4.10}$$

Then, in the limit we have

$$\lambda_1 * w_n = g_n + b_n \text{ in the sense of distributions.}$$

It is plain that $b_n \ge 0$. Condition (4.9) implies in particular that $w_n^{\varepsilon_m}$ tends to w_n uniformly on compact subsets of $\mathbb{R}$; condition (4.10) implies that $(w_n^{\varepsilon_m})^-$ tends to 0 strongly in $L^1_{\text{loc}}(\mathbb{R})$ and therefore $w_n \ge 0$. If $w_n(x) > 0$, we can find $\gamma > 0$ such that for all large enough m, and all y such that $|y - x| \le \gamma$,

$$w_n^{\varepsilon_m}(y) \ge \frac{1}{2}\, w_n(x),$$

and therefore, the support of $(w_n^{\varepsilon_m})^-$ does not intersect $(x-\gamma, x+\gamma)$; in the limit, the support of b_n does not intersect $(x-\gamma, x+\gamma)$. Thus we have obtained

$$\operatorname{supp} b_n \subset \{w_n = 0\}.$$

Take now $(\varepsilon_m)_{m\in\mathbb{N}}$ to be any sequence decreasing to 0 which will be fixed henceforth. We define $J_1 \subset \mathbb{N}$ as an infinite set such that $(w_1^{\varepsilon_m})_{m\in J_1}$ converges in the sense (4.9), (4.10). Given J_n, we take $J_{n+1} \subset J_n$ such that $(w_{n+1}^{\varepsilon_m})_{m\in J_{n+1}}$ converges in the sense (4.9), (4.10). Let $\bar{J}$ be the set made out of the first element of J_1, the second of J_2, the n-th of J_n and so forth. Thanks to Remark 4.4, we have also for all $m \in \mathbb{N}$, and all n, all $p \ge n$: $w_p^{\varepsilon_m}\big|_{[0,n]} = w_n^{\varepsilon_m}\big|_{[0,n]}$. As $(w_p^{\varepsilon_m})_{m\in\bar{J}}$ converges to w_p, we see immediately that for all n and all $p \ge n$: $w_p\big|_{[0,n]} = w_n\big|_{[0,n]}$. In particular, we may define w by $w\big|_{[0,n]} = w_n$, and therefore w is the desired solution. □

REMARK 4.9. *We know nothing about uniqueness — alas!*

5. Solutions whose support is included in a locally finite union of intervals

If ψ is a measure over $\mathbb{R}$ and h a function which is ψ-measurable, we shall write either

$$\int h\psi = \int h(s)\psi(s)$$

or

$$\langle \psi, h\rangle$$

for the integral of h against ψ. If h is ψ-measurable, for all interval I, the function $h1_I$ is also ψ-measurable, and its integral against ψ can be written

$$\int h1_I\psi = \int_I h\psi.$$

We shall keep the traditional notation dt for the Lebesgue measure, though it is not entirely coherent with the above notation; nevertheless, the meaning of these notations will always be clear from the context.

We denote by $\mathcal{M}$ the space of Radon measures on $\mathbb{R}$ and by $\mathcal{M}^1(\mathbb{R})$ the space of bounded measures on $\mathbb{R}$ with norm given by

$$\|\lambda\|_{\mathcal{M}^1} = \langle|\lambda|, 1\rangle.$$

We assume from now on that there exists $\phi \in \mathcal{M}^1(\mathbb{R})$ with support in $\mathbb{R}^+$ such that

$$g = \mu * \phi. \tag{5.1}$$

We recall that a measure ψ on $\mathbb{R}$ has a positive and a negative part, denoted respectively by ψ^+ and ψ^-; the following identities hold:

$$\psi^+ = \max(\psi, 0), \quad \psi^- = -\min(\psi, 0), \quad \psi = \psi^+ - \psi^-, \quad |\psi| = \psi^+ + \psi^-.$$

The first step is to prove an identity for which we need the function:

$$\omega(t) = \frac{1_{(0,\infty)}(t)}{\pi(t+1)\sqrt{t}}. \tag{5.2}$$

Later, we shall need an integral of ω:

$$\Omega(t) = \int_0^t \omega(s)\,ds = \frac{2}{\pi}1_{(0,\infty)}(t)\arctan\sqrt{t}. \tag{5.3}$$

In the following proof and in the remainder of this section, the prime symbol will never denote a derivative; the distributions μ, ρ, μ_1, λ, λ_1 have been defined previously and the reader is referred to the Appendix for formulas.

LEMMA 5.1. *Assume that g satisfies the hypothesis* (5.1) *and that w satisfies the relation*

$$\lambda_1 * w = g + b; \tag{5.4}$$

assume moreover that b is a measure belonging to $\mathcal{M}(\mathbb{R}^+)$ and that there exist four numbers $\sigma < \tau < \sigma' < \tau'$ for which w and b satisfy the support conditions:

$$\operatorname{supp} w \subset [\sigma, \tau] \cup [\sigma', \tau'], \tag{5.5a}$$

$$\operatorname{supp} b \subset [\tau, \sigma'] \cup [\tau', \infty]. \tag{5.5b}$$

Then the following identities hold:

$$w1_{[\sigma',\tau']} = \big(H * \nu * \phi'\big)1_{[\sigma',\tau']}, \tag{5.6a}$$

where ϕ' is a measure given by

$$\begin{aligned}\phi' 1_{[\sigma',\infty)} &= \phi 1_{[\sigma',\infty)} \\ &+ 1_{[\sigma',\infty)}\int_{[\tau,\sigma']} \exp\Big(-\frac{\cdot - s}{\alpha}\Big)\omega\left(\frac{\cdot - \sigma'}{\sigma' - s}\right)\frac{\psi(s)}{\sigma' - s}.\end{aligned} \tag{5.6b}$$

PROOF. Thanks to the support condition (5.5a) and to equation (5.4), w satisfies the relation

$$w1_{[\sigma,\sigma']} = \big(H * \nu * \phi\big)1_{[\sigma,\tau]}. \tag{5.7}$$

Relation (7.11) enables us to observe that $\big((H*\nu*\phi)1_{]\tau,\infty)}\big)(t)$ can be decomposed as

$$\Big[\int_{[\sigma,\tau]}\Big(1-\exp\Big(-\frac{\tau-s}{\alpha}\Big)\Big)\phi(s)$$
$$+\int_{[\sigma,\tau]}\Big(\exp\Big(-\frac{\tau-s}{\alpha}\Big)-\exp\Big(-\frac{t-s}{\alpha}\Big)\Big)\phi(s)$$
$$+\int_{]\tau,t]}\Big(1-\exp\Big(-\frac{t-s}{\alpha}\Big)\Big)\phi(s)\Big]1_{]\tau,\infty)}.$$

But

$$\int_{[\sigma,\tau]}\Big(1-\exp\Big(-\frac{\tau-s}{\alpha}\Big)\Big)\phi(s)=w(\tau) \tag{5.8}$$

which vanishes, and

$$\begin{aligned}&\int_{[\sigma,\tau]}\Big(\exp\Big(-\frac{\tau-s}{\alpha}\Big)-\exp\Big(-\frac{t-s}{\alpha}\Big)\Big)\phi(s)\\&=\Big(1-\exp\Big(-\frac{t-\tau}{\alpha}\Big)\Big)\int_{[\sigma,\tau]}\exp\Big(-\frac{\tau-s}{\alpha}\Big)\phi(s);\end{aligned}$$

therefore

$$\begin{aligned}&(H*\nu*\phi)1_{]\tau,\infty)}\\&=H*\nu*\Big(\delta(\cdot-\tau)\int_{[\sigma,\tau]}\exp\Big(-\frac{\tau-s}{\alpha}\Big)\phi(s)+\phi 1_{]\tau,\infty)}\Big),\end{aligned}$$

which implies that

$$\begin{aligned}w1_{[\sigma,\sigma']}&=(H*\nu*\phi)1_{[\sigma,\tau]}\\&=H*\nu*\Big(\phi 1_{[\sigma,\tau]}-\delta(\cdot-\tau)\int_{[\sigma,\tau]}\exp\Big(-\frac{\tau-s}{\alpha}\Big)\phi(s)\Big).\end{aligned} \tag{5.9}$$

We infer from this identity that

$$\begin{aligned}b1_{[\tau,\sigma')}&=-\big(\dot\rho*\big(w1_{[\sigma,\tau)}\big)-g\big)1_{[\tau,\sigma')}\\&=-\Big(\rho*\frac{d}{dt}\big(w1_{[\sigma,\tau)}\big)-g\Big)1_{[\tau,\sigma')},\end{aligned}$$

and therefore

$$b1_{[\tau,\sigma')}=-\big(\mu*\psi\big)1_{[\tau,\sigma')},$$

with ψ given by

$$\psi=\phi 1_{(\tau,\infty)}+\delta(\cdot-\tau)\exp\Big(-\frac{\tau}{\alpha}\Big)\int_{[\sigma,\tau]}\exp\Big(\frac{s}{\alpha}\Big)\phi(s).$$

Define now

$$w'=w1_{[\sigma',\infty)},\quad g'=\big(g-\lambda_1*(w1_{[\sigma,\tau]})\big)1_{[\sigma',\infty)},\quad b'=b1_{[\sigma',\infty)}.$$

Then it is immediate that w' satisfies

$$\lambda_1*w'=g'+b'.$$

Our purpose now is to identify a measure ϕ' such that

$$\big(\mu_1*g'\big)1_{[\sigma',\tau']}=\big(H*\nu*\phi'\big)1_{[\sigma',\tau']}.$$

If this identity holds and ϕ' is supported in $[\sigma',\infty)$, we must have over $[\sigma',\tau']$

$$\mu_1*g'=H*\nu*\phi',$$

or in other words

$$\phi' 1_{[\sigma',\tau']} = \big(\rho * g'\big) 1_{[\sigma',\tau']}.$$

We proceed now to calculate

$$\rho * g' = \rho * \big(1_{[\sigma',\infty)}\big(g - \lambda_1 * \big(w 1_{[\sigma,\tau]}\big)\big)\big)\,;$$

relation (5.9) implies

$$1_{[\sigma',\infty)}\big(g - \lambda_1 * \big(w 1_{[\sigma,\tau]}\big)\big) = 1_{[\sigma',\infty)}(\mu * \psi).$$

Therefore, we need the value of the expression

$$\rho * \big(1_{[\sigma',\infty)}(\mu * \psi)\big),$$

which is equal to $\rho * \rho * \mu * \big(1_{[\sigma',\infty)}(\mu * \psi)\big)$, or in other words to

$$\left(1 + \alpha \frac{d}{dt}\right)\big(\mu * \big(1_{[\sigma',\infty)}(\mu * \psi)\big)\big).$$

But, for $t > \sigma'$, we have the formula

$$\mu * \big(1_{[\sigma',\infty)}(\mu * \psi)\big)(t) = \int_{[\sigma',t]} \mu(t-s)\left(\int_{[\tau,s]} \mu(s-r)\psi(r)\right) ds.$$

We exchange the order of the integrations and we write $r \vee \sigma' = \max(r, \sigma')$, obtaining thus

$$\mu * \big(1_{[\sigma',\infty)}(\mu * \psi)\big)(t) = \int_{[\tau,t]} \psi(r) \int_{r\vee\sigma'}^{t} \mu(t-s)\mu(s-r)\, ds.$$

It is plain that

$$\begin{aligned}&\int_{r\vee\sigma'}^{t} \mu(t-s)\mu(s-r)\, ds \\ &= \frac{1}{\pi\alpha} \exp\left(-\frac{t-r}{\alpha}\right)\left(\frac{\pi}{2} - \arcsin \frac{2(r \vee \sigma') - t - r}{t-r}\right),\end{aligned}$$

so that

$$\begin{aligned}\mu * \big(1_{[\sigma',\infty)}(\mu * \psi)\big)(t) &= \frac{1}{\alpha} \int_{]\sigma',t]} \exp\left(-\frac{t-r}{\alpha}\right)\psi(r) \\ &+ \frac{1}{\alpha\pi} \int_{[\tau,\sigma')} \exp\left(-\frac{t-r}{\alpha}\right)\left(\frac{\pi}{2} - \arcsin \frac{2\sigma' - t - r}{t-r}\right)\psi(r);\end{aligned}$$

now, we have to prove that we can exchange differentiation with respect to t and integration with respect to the measure $\psi(r)$. For $\sigma' \in]r, t[$, the function $t \mapsto \arcsin\big((2\sigma' - t - r)/(t - r)\big)$ is analytic in t; moreover, its derivative with respect to t is equal to

$$-\frac{\pi}{\sigma' - r}\,\omega\left(\frac{t - \sigma'}{\sigma' - r}\right),$$

as can be immediately checked; it is a bounded function with respect to r when t is bounded away from σ'; therefore, we have the pointwise equality for $t > \sigma'$

$$\begin{aligned}&\left(1 + \alpha \frac{d}{dt}\right) \int_{[\tau,\sigma')} \frac{\psi(r)}{\pi\alpha}\left(\frac{\pi}{2} - \arcsin \frac{2\sigma' - t - r}{t-r}\right) \exp\left(-\frac{t-r}{\alpha}\right) \\ &= \int_{[\tau,\sigma')} \omega\left(\frac{t-\sigma'}{\sigma'-r}\right) \frac{\psi(r)}{\sigma'-r} \exp\left(-\frac{t-r}{\alpha}\right).\end{aligned} \tag{5.10}$$

The right hand side of (5.10) is integrable with respect to Lebesgue measure on every compact subinterval of $[\sigma', \infty)$: it is clearly measurable for $t > \sigma'$, and, exchanging the order of integrations, we have the following estimate

$$\int_{[\sigma',\infty]} \left| \int_{[\tau,\sigma']} \omega\left(\frac{t-\sigma'}{\sigma'-r}\right) \frac{\psi(r)}{\sigma'-r} \exp\left(-\frac{t-r}{\alpha}\right) \right| dt \le \int_{[\tau,\sigma')} |\psi(r)|,$$

since $\Omega(+\infty)$ is equal to 1. Then a plain application of general theorems on the differentiation of integral expressions shows indeed that the expression on the right hand side of (5.6b) is a measure on $[\sigma', \tau']$. □

This identity can be made recursive:

COROLLARY 5.2. *Assume that g satisfies assumption* (5.1)*, that w, g and the measure b are related by condition* (5.4)*; assume moreover that there exist two sequences $(\tau_j)_{j\in J}$, $(\sigma_j)_{j\in J}$ where J is a finite or infinite interval of $\mathbb{N}$ starting at* 0*, satisfying*

$$0 \le \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \dots, \tag{5.11}$$

such that w and b satisfy the following support conditions:

$$\operatorname{supp} w \subset \bigcup_{j\in J} [\sigma_j, \tau_j], \quad \operatorname{supp} b \subset \bigcup_{j\in J} [\tau_j, \sigma_{j+1}].$$

Then, if we define

$$\phi_0 = \phi$$

and for all $j \in J \setminus \{\max J\}$

$$\psi_j = \phi_j 1_{[\tau_j,\sigma_{j+1})} + \delta(\cdot - \tau_j) \exp\left(-\frac{\tau_j}{\alpha}\right) \int_{[\sigma_j,\tau_j)} \exp\left(\frac{s}{\alpha}\right) \phi_j, \tag{5.12}$$

$$\begin{aligned} \phi_{j+1} &= 1_{[\sigma_{j+1},\infty)} \phi_j \\ &+ 1_{[\sigma_{j+1},\infty)} \int_{[\tau_j,\sigma_{j+1})} \exp\left(-\frac{\cdot - s}{\alpha}\right) \omega\left(\frac{\cdot - \sigma_{j+1}}{\sigma_{j+1} - s}\right) \frac{\psi_j(s)}{\sigma_{j+1} - s}, \end{aligned} \tag{5.13}$$

then w is given by

$$w 1_{[\sigma_j,\tau_j]} = \bigl(H * \nu * \phi_j\bigr) 1_{[\sigma_j,\tau_j]}. \tag{5.14}$$

Moreover, the expression for ϕ_j can be rewritten

$$\begin{aligned} \phi_j &= 1_{[\sigma_j,\infty)} \phi_0 \\ &+ 1_{[\sigma_j,\infty)} \sum_{i=1}^{j} \int_{[\tau_{i-1},\sigma_i)} \exp\left(-\frac{\cdot - s}{\alpha}\right) \omega\left(\frac{\cdot - \sigma_i}{\sigma_i - s}\right) \frac{\psi_{i-1}(s)}{\sigma_i - s}. \end{aligned} \tag{5.15}$$

PROOF. The proof of this Corollary is a simple induction which is left to the reader. □

Let us obtain now estimates on the negative part of the derivative $\dot w(t)$, assuming now that w is non negative. These estimates depend on some elementary inequalities relative to Ω and to ω, which were defined at (5.2) and (5.3).

LEMMA 5.3. *Let H and H_1 be defined for $\sigma > \sigma' > 0$ and $r > 0$ by*

$$H(r, \sigma, \sigma') = \frac{\sigma - \sigma'}{\alpha} \exp\left(-\frac{\sigma r}{\alpha}\right) \bigl(\Omega(\sigma) - \Omega(\sigma')\bigr),$$

and

$$H_1(r,\sigma,\sigma') = \frac{1}{\alpha}\int_{\sigma'}^{\sigma} \exp\Big(-\frac{rt}{\alpha}\Big)\Omega(t)\,dt + \frac{1}{r}\Big(\exp\Big(-\frac{r\sigma}{\alpha}\Big) - \exp\Big(-\frac{r\sigma'}{\alpha}\Big)\Big)\Omega(\sigma').$$

Then the following inequalities hold:

$$\frac{1}{2}H(r,\sigma,\sigma') \le H_1(r,\sigma,\sigma') \le \exp\Big(r\,\frac{\sigma-\sigma'}{\alpha}\Big)H(r,\sigma,\sigma'). \tag{5.16}$$

PROOF. We rewrite H_1 with the help of an integration by parts:

$$H_1(r,\sigma,\sigma') = \frac{1}{r}\int_{\sigma'}^{\sigma} \omega(t)\Big(\exp\Big(-\frac{rt}{\alpha}\Big) - \exp\Big(-\frac{r\sigma}{\alpha}\Big)\Big)\,dt.$$

Then the first inequality in (5.16) is equivalent to

$$\int_{\sigma'}^{\sigma} \omega(t)\Big[\frac{K}{\rho}\Big(\exp\Big(-\frac{rt}{\alpha}\Big) - \exp\Big(-\frac{r\sigma}{\alpha}\Big)\Big) - \frac{\sigma-\sigma'}{\alpha}\exp\Big(-\frac{r\sigma}{\alpha}\Big)\Big]\,dt \ge 0, \tag{5.17}$$

with $K \le 2$. We choose $K > 0$ so that the factor of ω in the above integral vanishes at $t = (\sigma+\sigma')/2$; hence

$$K = r(\sigma-\sigma')\Big(\alpha\exp\Big(r\,\frac{\sigma-\sigma'}{2\alpha}\Big) - \alpha\Big)^{-1},$$

and clearly $K \le 2$. The factor of ω in (5.17) is decreasing function of t which is positive in the first half of the interval $[\sigma',\sigma]$ and negative in its second. As ω is a positive and decreasing function, it is clear that (5.17) holds. For the second inequality in (5.16), we use the inequality $\exp(-rt/\alpha) - \exp(-rt/\alpha) \le r\alpha^{-}(\sigma - \sigma')\exp(-r\sigma'/\alpha)$ so that

$$\begin{aligned} H_1(r,\sigma,\sigma') &\le \frac{\sigma-\sigma'}{\alpha}\exp\Big(-\frac{r\sigma'}{\alpha}\Big)\int_{\sigma'}^{\sigma}\omega(t)\,dt \\ &\le \frac{\sigma-\sigma'}{\alpha}\exp\Big(-\frac{r\sigma'}{\alpha}\Big)\exp\Big(-r\,\frac{\sigma-\sigma'}{\alpha}\Big)\big(\Omega(\sigma)-\Omega(\sigma')\big) \\ &\le \exp\Big(-r\,\frac{\sigma-\sigma'}{\alpha}\Big)H(r,\sigma,\sigma'). \end{aligned}$$

□

In consequence, we have the

COROLLARY 5.4. *Let*

$$F(\sigma,s,i,j) = \frac{1}{\alpha}\exp\Big(-\frac{\sigma-s}{\alpha}\Big)\Big(\Omega\Big(\frac{\sigma-\sigma_i}{\sigma_i-s}\Big) - \Omega\Big(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\Big)\Big), \tag{5.18}$$

and

$$F_1(\sigma,s,i,j) = \frac{1}{\alpha(\sigma-\sigma_j)}\int_{[\sigma_j,\sigma)}\exp\Big(-\frac{t-s}{\alpha}\Big)\Omega\Big(\frac{t-\sigma_i}{\sigma_i-s}\Big)\,dt + \frac{1}{\sigma-\sigma_j}\Big(\exp\Big(-\frac{\sigma-s}{\alpha}\Big) - \exp\Big(-\frac{\sigma_j-s}{\alpha}\Big)\Big)\Omega\Big(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\Big). \tag{5.19}$$

Then for all $i \le j$, all $\sigma \in (\sigma_j,\tau_j]$ and all $s \in [\tau_{i-1},\sigma_i)$, the following inequalities hold:

$$\frac{1}{2}F(\sigma,s,i,j) \le F_1(\sigma,s,i,j) \le \exp\Big(\frac{\sigma-\sigma_j}{\alpha}\Big)F(\sigma,s,i,j). \tag{5.20}$$

PROOF. After multiplying them by

$$\exp\left(\frac{\sigma_i - s}{\alpha}\right)\frac{\sigma - \sigma_j}{\sigma_i - s},$$

the inequalities (5.20) are equivalent to

$$\frac{1}{2}H\left(\sigma_i - s, \frac{\sigma - \sigma_i}{\sigma_i - s}, \frac{\sigma_j - \sigma_j}{\sigma_i - s}\right) \le H_1\left(\sigma_i - s, \frac{\sigma - \sigma_i}{\sigma_i - s}, \frac{\sigma_j - \sigma_j}{\sigma_i - s}\right)$$
$$\le \exp\left(\frac{\sigma - \sigma_j}{\alpha}\right)H\left(\sigma_i - s, \frac{\sigma - \sigma_i}{\sigma_i - s}, \frac{\sigma_j - \sigma_j}{\sigma_i - s}\right).$$

The conclusion is immediate. □

These inequalities will allow us to estimate the negative part of the left derivatives $\dot{w}(\sigma - 0)$, for $\sigma \in (\sigma_j, \sigma]$ as is explained in next Lemma:

LEMMA 5.5. *Assume that the condition of Corollary 5.2 are satisfied. For all $j \in J$, the following estimate holds if σ belongs to $(\sigma_j, \tau_j]$ and $w(\sigma) \ge 0$:*

$$(5.21)\qquad \big(\dot{w}(\sigma - 0)\big)^- \le \frac{1}{\alpha}\int_{[\sigma_j,\sigma)} |\phi_0(s)|$$
$$+ \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)}\left(2\exp\left(\frac{\sigma - \sigma_j}{\alpha}\right) - 1\right)G(\sigma, s, i, j)\psi_{i-1}^+(s),$$

where

$$G(\sigma, s, i, j) = \left(\Omega\left(\frac{\sigma - \sigma_i}{\sigma_i - s}\right) - \Omega\left(\frac{\sigma_j - \sigma_i}{\sigma_i - s}\right)\right)\psi_{i-1}^+(s).$$

PROOF. In virtue of (5.14), for $\sigma \in [\sigma_j, \tau_j]$, $w(\sigma)$ can be rewritten as

$$w(\sigma) = \int_{[\sigma_j,\sigma)}\left(1 - \exp\left(-\frac{\sigma - s}{\alpha}\right)\right)\phi_0(s)$$
$$+ \sum_{i=1}^{j}\int_{[\sigma_j,\sigma)}\left[\int_{[\tau_{i-1},\sigma_i)}\left(\exp\left(-\frac{t - s}{\alpha}\right)\right.\right.$$
$$\left.\left.- \exp\left(-\frac{\sigma - s}{\alpha}\right)\right)\omega\left(\frac{t - \sigma_i}{\sigma_i - s}\right)dt\right]\frac{\psi_{i-1}(s)}{\sigma_i - s}.$$

We exchange the order of the integrations in the double integral, we divide by $\sigma - \sigma_j$ and we get with the help of (5.19):

$$\frac{w(\sigma)}{\sigma - \sigma_j} = \int_{[\sigma_j,\sigma)}\frac{1}{\sigma - \sigma_j}\left(1 - \exp\left(-\frac{\sigma - s}{\alpha}\right)\right)\phi_0(s)$$
$$+ \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma, s, i, j)\psi_{i-1}(s).$$

Under the assumption $w(\sigma) \ge 0$, we obtain the following estimate:

$$(5.22)\qquad \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma, s, i, j)\psi_{i-1}^-(s) \le \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma, s, i, j)\psi_{i-1}^+(s)$$
$$+ \int_{[\sigma_j,\sigma)}\frac{1}{\sigma - \sigma_j}\left(1 - \exp\left(-\frac{\sigma - s}{\alpha}\right)\right)\phi_0(s).$$

According to (5.18), we consider the expression for $\dot{w}(\sigma - 0)$, which is given by

$$\dot{w}(\sigma - 0) = \frac{1}{\alpha}\int_{[\sigma_j,\sigma)}\exp\left(-\frac{\sigma - s}{\alpha}\right)\phi_0(s) + \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma, s, i, j)\psi_{i-1}(s).$$

We have now the inequality

$$(5.23)\qquad \big(\dot w(\sigma-0)\big)^- \le \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\psi_{i-1}^-(s) \\ -\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\psi_{i-1}^+(s) - \frac{1}{\alpha}\int_{[\sigma_j,\sigma)} \exp\Big(-\frac{\sigma-s}{\alpha}\Big)\phi_0(s).$$

We substitute the inequality (5.20) into the factors of ψ_{i-1}^- in (5.23), and then, thanks to (5.22), we get

$$(5.24)\qquad \big(\dot w(\sigma-0)\big)^- \le \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} \big(2F_1(\sigma,s,i,j)-F(\sigma,s,i,j)\big)\psi_{i-1}^+(s) \\ +\int_{[\sigma_j,\sigma)}\bigg(\frac{2}{\sigma-\sigma_j}\Big(1-\exp\Big(-\frac{\sigma-s}{\alpha}\Big)\Big)-\frac{1}{\alpha}\exp\Big(-\frac{\sigma-s}{\alpha}\Big)\bigg)\phi_0(s).$$

Since the value of the factor of ϕ_0 in (5.24) is comprised between $-1/\alpha$ and $1/\alpha$ and according to inequality (5.20), we infer that

$$\big(\dot w(\sigma-0)\big)^- \le \frac{1}{\alpha}\int_{[\sigma_j,\sigma)}|\phi_0(s)| \\ +\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\bigg(2\exp\Big(\frac{\sigma-\sigma_j}{\alpha}\Big)-1\bigg)\psi_{i-1}^+(s),$$

which is exactly relation (5.21). □

6. Construction of the approximate solution

The principle of the construction of an approximate solution is the following: we do not know *a priori* whether for a given g satisfying (5.1), there is a solution of (5.4) which has the locally finite structure determined by the conditions of Corollary 5.2, and most probably there is no such solution; however, we shall choose a parameter $n \gg 1$ and construct a slightly different ϕ^n and a solution w^n which has the structure described at Corollary 5.2 and which approximates well a solution of (5.4). The construction is recursive.

Let us start by a lemma which tells us that the lower bound of the support of ϕ can be taken equal to the lower bound of the support of w:

LEMMA 6.1. *Let τ_{-1} be the lower bound of the support of ϕ and let w be a solution of* (5.4)*; assume that the lower bound of the support of the positive part of $H*\rho*\phi$ is $\sigma_0>\tau_{-1}$; define*

$$(6.1)\qquad \phi_0 = 1_{[\sigma_0,\infty)}\bigg(\phi+\int_{[\tau_{-1},\sigma_0)}\exp\Big(-\frac{\cdot-s}{\alpha}\Big)\omega\Big(\frac{\cdot-\sigma_0}{\sigma_0-s}\Big)\frac{\phi(s)}{\sigma_0-s}\bigg);$$

then the functions

$$w_0 = w1_{[\sigma_0,\infty)},\quad g_0=\mu*\phi_0,\quad b_0=b1_{[\sigma_0,\infty)}$$

solve the problem

$$(6.2)\qquad \lambda_1 * w_0 = g_0+b_0,\quad 0\le b_0\perp w_0\ge 0.$$

Moreover, the lower bound of the support of ϕ_0 is equal to the lower bound σ_0 of the support of w_0.

PROOF. Relation (6.2) holds as a corollary of the calculation performed at Lemma 5.2. The definition of σ_0 implies immediately that the lower bound of the support of w_0 is indeed σ_0. There remains to prove that the lower bound of the support of ϕ_0 is also σ_0. Definition (6.1) implies that this lower bound of the support of ϕ_0 is at least equal to σ_0. If the lower bound of the support of ϕ_0 were strictly larger than the lower bound of the support of w_0, $\mu * \phi_0$ would vanish identically on some interval $[\sigma_0, \sigma_1]$. Then, w_0 would coincide on $[\sigma_0, \sigma_1]$ with $H * \mu * b_0$ which is strictly positive at every point of $(\sigma_0, \sigma_1]$. But w_0 could not be orthogonal to b_0 unless it vanished on that interval, which is a contradiction. □

6.1. Initialization of the recursion. Let us describe now formally the construction of the approximate solution. If $\tau_{-1} < \sigma_0$, we start with ϕ_0 defined by (6.1); the number σ_0 is the lower bound of the support of $(H * \nu * \phi_0)^+$. If $\tau_{-1} = \sigma_0$, we let $\phi_0 = \phi$. We let

$$\widetilde{\phi}_0^n = \phi_0, \quad \sigma_0^n = \sigma_0.$$

Call $\widetilde{\tau}_0^n$ the lower bound of the support of the negative part of $H * \nu * \widetilde{\phi}_0^n$; if $\widetilde{\tau}_0^n > \sigma_0^n$, we let $\phi_0^n = \widetilde{\phi}_0^n$ and $\tau_0^n = \widetilde{\tau}_0^n$.

If the lower bound of the support of the negative part of $H * \nu * \widetilde{\phi}_0^n$ is equal to σ_0^n, this means that we can find, arbitrarily close to σ_0^n, times t for which $H * \nu * \widetilde{\phi}_0^n$ is of either sign. In particular, we can find, arbitrarily close to σ_0^n, times t for which $\big(H * \nu * \widetilde{\phi}_0^n\big)(t)$ vanishes, while $\big(\nu * \widetilde{\phi}_0^n\big)(t-0)$ is less than or equal to 0. We choose any time t in the interval $(\sigma_0^n, \sigma_0^n + 1/n]$ which satisfies all these conditions, we call it τ_0^n, we let

$$a = \int_{[\sigma_0^n, \tau_0^n)} (\widetilde{\phi}_0^n)^+ \quad \text{and} \quad b = \int_{[\sigma_0^n, \tau_0^n)} (\widetilde{\phi}_0^n)^-, \tag{6.3}$$

and we define

$$\phi_0^n = a\delta(\cdot - \sigma_0^n) - b\delta(\cdot - \tau_0^n) + \phi_0^n 1_{[\tau_0^n, \infty)}. \tag{6.4}$$

It is plain that the lower bound of the support of the negative part of $H * \nu * \phi_0^n$ is equal to τ_0^n, and that the respective mass of the positive and the negative parts of ϕ_0^n on $[\sigma_0^n, \tau_0^n)$ coincide with their counterparts for $\widetilde{\phi}_0^n$, in particular, $b > 0$.

6.2. The recursion. The construction will now be described inductively.

We start from a measure ϕ_j^n such that

$$\inf\operatorname{supp}\phi_j^n = \inf\operatorname{supp}\big(H * \nu * \phi_j^n\big)^+. \tag{6.5}$$

It is important to observe that if ϕ is a measure, with support bounded on the left, $H * \nu * \phi$ is a continuous function. Let

$$\tau_j^n = \inf\operatorname{supp}\big(H * \nu * \phi_j^n\big)^-.$$

If $\tau_j^n = \infty$, the construction stops; we shall show later that τ_j^n is always strictly larger than σ_j^n for $j \ge 1$; it is already true by construction for $j = 0$.

With reference to (5.12), we define

$$\widetilde{\psi}_j^n = 1_{[\tau_j^n, \infty)}\phi_j^n + \delta(\cdot - \tau_j^n)\exp\left(-\frac{\tau_j^n}{\alpha}\right)\int_{[\sigma_j^n, \tau_j^n)} \exp\left(\frac{s}{\alpha}\right)\phi_j^n(s), \tag{6.6}$$

and

$$\widetilde{\sigma}_{j+1}^n = \inf\operatorname{supp}\big(\widetilde{\psi}_j^n\big)^+. \tag{6.7}$$

We have two cases to consider.

Case 1: If $\widetilde{\sigma}_{j+1}^n = \infty$, the construction stops.

Case 2: If $\infty > \widetilde{\sigma}^n_{j+1} \geq \tau^n_j$, we choose any time σ^n_{j+1} which does not carry an atom of $\widetilde{\psi}^n_j$ and which satisfies

$$\max\left(\widetilde{\sigma}^n_{j+1}, \tau^n_j + \frac{1}{2n}\right) \leq \sigma^n_{j+1} \leq \max\left(\widetilde{\sigma}^n_{j+1}, \tau^n_j + \frac{1}{2n}\right) + \frac{1}{2n}, \tag{6.8}$$

and we define

$$\begin{aligned} \psi^n_j &= -1_{[\tau^n_j, \sigma^n_{j+1})} \left(\widetilde{\psi}^n_j\right)^- \\ &+ \delta(\cdot - \sigma^n_{j+1}) \int_{[\tau^n_j, \sigma^n_{j+1})} \left(\widetilde{\psi}^n_j\right)^+ + 1_{[\sigma^n_{j+1}, \infty)} \widetilde{\psi}^n_j. \end{aligned} \tag{6.9}$$

Since the lower bound of the support of $\left(\widetilde{\psi}^n_j\right)^+$ is equal to $\widetilde{\sigma}^n_{j+1}$, the mass of the atom of ψ^n_j at σ^n_{j+1} is strictly positive.

The last step of the construction is the construction of ϕ^n_{j+1}; with reference to (5.13), it is given by

$$\begin{aligned} \phi^n_{j+1} &= 1_{[\sigma^n_{j+1}, \infty)} \psi^n_j \\ &+ 1_{[\sigma^n_{j+1}, \infty)} \int_{[\tau^n_j, \sigma^n_{j+1})} \exp\left(-\frac{\cdot - s}{\alpha}\right) \omega\left(\frac{\cdot - \sigma^n_{j+1}}{\sigma^n_{j+1} - s}\right) \frac{\psi^n_j(s)}{\sigma^n_{j+1} - s}. \end{aligned} \tag{6.10}$$

In order to validate this process, we have to prove the

Lemma 6.2. *For all $j \geq 0$, there exists a non empty interval $(\sigma^n_{j+1}, \tau^n_{j+1})$ on which $H * \nu * \phi^n_{j+1}$ is strictly positive.*

Proof. Write for simplicity

$$\begin{aligned} &\tau = \tau^n_j, \quad \sigma' = \sigma^n_{j+1}, \quad \psi = \psi^n_j, \\ &\phi' = 1_{[\sigma', \infty)} \left(\psi + \int_{[\tau, \sigma')} \exp\left(-\frac{\cdot - s}{\alpha}\right) \omega\left(\frac{\cdot - \sigma'}{\sigma' - s}\right) \frac{\psi(s)}{\sigma' - s}\right). \end{aligned}$$

By construction, $\psi\big|_{[\tau, \sigma')} \leq 0$, and ψ has a positive atom at σ', whose measure will be denoted by $\beta > 0$. We have the following identity for $t > \sigma'$:

$$\begin{aligned} (H * \nu * \phi')(t) &= \left(1 - \exp\left(-\frac{t - \sigma'}{\alpha}\right)\right)\beta \\ &+ \int_{(\sigma', t)} \left(1 - \exp\left(-\frac{t - s}{\alpha}\right)\right)\psi(s) \\ &+ \int_{[\sigma', t)} \int_{[\tau, \sigma')} \exp\left(-\frac{s - r}{\alpha}\right) \omega\left(\frac{s - \sigma'}{\sigma' - r}\right) \frac{\psi(r)}{\sigma' - r}\, ds \\ &- \int_{(\sigma', t)} \exp\left(-\frac{t - s}{\alpha}\right) \int_{[\tau, \sigma')} \exp\left(-\frac{s - r}{\alpha}\right) \omega\left(\frac{s - \sigma'}{\sigma' - r}\right) \frac{\psi(r)}{\sigma' - r}\, ds. \end{aligned}$$

We can find $t_1 > \sigma'$ such that

$$\int_{(\sigma', t_1)} |\psi(s)| \leq \frac{\beta}{4};$$

then, for $t \in [\sigma', t_1]$, we will have

$$\left| \int_{(\sigma', t)} \left(1 - \exp\left(-\frac{t - s}{\alpha}\right)\right)\psi(s) \right| \leq \left(1 - \exp\left(-\frac{t - \sigma'}{\alpha}\right)\right)\frac{\beta}{4};$$

we cut the interval $[\tau, \sigma')$ into two pieces, $[\tau, \sigma' - \varepsilon)$ and $[\sigma' - \varepsilon, \sigma')$, thus, we have the estimate

$$\left|\int_{[\sigma',t)} \left(1 - \exp\left(-\frac{t-s}{\alpha}\right)\right) \int_{[\tau,\sigma')} \exp\left(-\frac{s-r}{\alpha}\right) \omega\left(\frac{s-\sigma'}{\sigma'-r}\right) \frac{\psi(r)}{\sigma'-r}\, ds\right|$$
$$\leq \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right) \left(\int_{[\tau,\sigma'-\varepsilon)} |\psi(r)| + \int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)|\right).$$

We choose ε so small that

$$\int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)| \leq \frac{\beta}{4};$$

then

$$\left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right) \int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)| \leq \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right) \frac{\beta}{4};$$

we fix ε and we choose $t_2 \in (\sigma', t_1]$ so small that

$$\left(1 - \exp\left(-\frac{t_2-\sigma'}{\alpha}\right)\right) \int_{[\tau,\sigma'-\varepsilon)} |\psi(r)| \leq \left(1 - \exp\left(-\frac{t_2-\sigma'}{\alpha}\right)\right) \frac{\beta}{4};$$

then, for $t \in (\sigma', t_2)$, $(H * \nu * \phi')(t) \geq \big(1 - \exp(-(t-\sigma')/\alpha)\big)\beta/4 > 0$ and the lemma is proved. □

6.3. Mass and order properties of the measures ϕ_j^n and ψ_j^n. The measures defined in this part have some important order properties, which are summarized in next lemma:

LEMMA 6.3. *The following inequalities hold:*

$$(\widetilde{\psi}_j^n)^+ \leq (\phi_j^n)^+, \tag{6.11}$$

$$\phi_{j+1}^n \leq 1_{[\sigma_{j+1}^n,\infty)} \psi_j^n, \tag{6.12}$$

$$\int |\phi_{j+1}^n| \leq \int |\psi_j^n| \leq \int |\phi_j^n|, \tag{6.13}$$

$$\int_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ \leq \int_{[\tau_j^n,\sigma_{j+1}^n)} \phi^+, \tag{6.14}$$

$$(\phi_{j+1}^n)^+ \leq 1_{[\sigma_{j+1}^n,\infty)} (\phi_j^n)^+ + \delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n,\sigma_{j+1}^n)} (\phi_j^n)^+. \tag{6.15}$$

PROOF. For relation (6.11), we just take the definition (6.9) of $\widetilde{\psi}_j^n$, observing that the quantity

$$\frac{1}{\alpha} \exp\left(-\frac{\tau_j^n}{\alpha}\right) \int_{[\sigma_j^n,\tau_j^n)} \exp\left(\frac{s}{\alpha}\right) \phi_j^n(s)$$

is simply equal to the velocity $\dot{w}(\tau_j^n - 0)$ and hence less than or equal to 0. Relation (6.12) can be read on formula (6.10) with upper indices n thrown in and the sign condition $\psi_j^n \leq 0$ over $[\tau_j^n, \sigma_{j+1}^n)$ coming from the construction. In order to obtain estimate (6.13), we integrate

$$\int_{[\tau_j^n,\sigma_{j+1}^n)} \exp\left(-\frac{t-s}{\alpha}\right) \omega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n - s}\right) \frac{|\psi_j^n(s)|}{\sigma_{j+1}^n - s} \tag{6.16}$$

over $[\sigma_{j+1}^n, \infty)$, we exchange the order of the integrations, we find that

$$\int_{[\sigma_{j+1}^n,\infty)} \left(\int_{[\tau_j^n,\sigma_{j+1}^n)} \exp\left(-\frac{t-s}{\alpha}\right) \omega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n - s}\right) \frac{|\psi_j^n(s)|}{\sigma_{j+1}^n - s} \right) dt$$
$$\leq \int_{[\tau_j^n,\sigma_{j+1}^n)} \Omega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n - s}\right) \Big|_{t=\sigma_{j+1}^n}^{t=\infty} \left|\psi_j^n(s)\right| = \int_{[\tau_j^n,\sigma_{j+1}^n)} \left|\psi_j^n(s)\right|,$$

where we have estimated $\exp(-(t-s)/\alpha)$ by 1; the conclusion is clear. Relation (6.14) is a consequence of (6.6) and of the relations (6.11) and (6.12) with an induction on j; finally (6.15) is an immediate consequence of (6.9)-(6.11). □

6.4. The approximate problem. We define now

$$\begin{aligned} \bar{\phi}^n &= \phi + (\phi_0^n - \phi) 1_{[\sigma_0^n,\tau_0^n)} \\ &+ \sum_{j\geq 0} \left(\delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ - 1_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ \right). \end{aligned} \tag{6.17}$$

The above sum must be understood as extended to the set of indices for which the recursion is defined: it is a finite sum if one of the τ_j^n or σ_j^n is infinite. As the length of the intervals $[\tau_j^n, \sigma_{j+1}^n)$ is at least equal to $1/(2n)$, we know that this sum is locally finite.

THEOREM 6.4. *Let $g^n = \mu * \bar{\phi}^n$. Then, the function w^n given by*

$$w^n = \sum_{j\geq 0} 1_{[\sigma_j^n,\tau_j^n]} (H * \nu * \phi_j^n) \tag{6.18}$$

is continuous, non negative, and it is a solution of

$$\lambda_1 * w^n = g^n + b^n, \tag{6.19a}$$
$$w^n \geq 0 \tag{6.19b}$$
$$b^n \geq 0 \tag{6.19c}$$
$$\langle w^n, b^n \rangle = 0. \tag{6.19d}$$

PROOF. The function w^n is non negative on the intervals $[\sigma_j^n, \tau_j^n]$ by construction, i.e. (6.19b) holds.

The definition of g^n comes from the fact that we modify the data in each interval $[\tau_j^n, \sigma_{j+1}^n)$, according to (6.9); in particular,

$$\psi_j^n - \widetilde{\psi}_j^n = \delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ - 1_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+; \tag{6.20}$$

therefore w^n satisfies (6.19a), with b^n given by

$$b^n = -\sum_{j\geq 0} 1_{[\tau_j^n,\sigma_{j+1}^n)} (\mu * \psi_j^n),$$

and we just have to check (6.19c): but it is a result of the construction performed in Subsection 6.2 that the measure ψ_j^n is negative on $[\tau_j^n, \sigma_{j+1}^n)$. Moreover, $\mu * \psi_j^n$ belongs to the space L_{loc}^p for all $p \in [1, 2)$, and therefore, the duality product $\langle w^n, b^n \rangle$ makes sense locally as a Lebesgue integral of the product of two functions, and therefore (6.19d) is true. □

6.5. Estimates on the approximation. We prove first a result on the convergence of the measure $\bar\phi^n$:

LEMMA 6.5. *The norm of the measure $\bar\phi^n$ is bounded by $3\|\phi\|_{\mathcal{M}^1}$ and converges weakly $*$ to ϕ as n tends to infinity.*

PROOF. We estimate the norm in $\mathcal{M}^1$ of all the terms in (6.17): thanks to (6.3) and (6.4), we have the estimate

$$\|(\phi_0^n - \phi)1_{[\sigma_0^n,\tau_0^n)}\|_{\mathcal{M}^1} \leq 2\|\phi 1_{[\sigma_0^n,\tau_0^n)}\|_{\mathcal{M}^1}.$$

Similarly thanks to (6.20)

$$\|(\psi_j^n - \widetilde{\psi}_j^n)1_{[\tau_j^n,\sigma_{j+1}^n)}\|_{\mathcal{M}^1} \leq 2\int_{[\tau_j^n,\sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+$$

and in virtue of (6.11) and (6.15)

$$\|(\psi_j^n - \widetilde{\psi}_j^n)1_{[\tau_j^n,\sigma_{j+1}^n)}\|_{\mathcal{M}^1} \leq 2\int_{[\tau_j^n,\sigma_{j+1}^n)} \phi^+;$$

therefore, we find the inequality

$$\|\bar\phi^n\|_{\mathcal{M}^1} \leq 3\|\phi\|_{\mathcal{M}^1}. \tag{6.21}$$

Let h be a continuous function on $\mathbb{R}$ with compact support and let ρ be its modulus of continuity: ρ is a continuous increasing function from $\mathbb{R}^+$ to itself such that

$$\forall x, y \in \mathbb{R}, \quad |h(x) - h(y)| \leq \rho|x-y|;$$

moreover, ρ vanishes at 0. According to (6.9), we rewrite as follows the duality product between h and $\psi_j^n - \widetilde{\psi}_j^n$:

$$\langle \psi_j^n - \widetilde{\psi}_j^n, h\rangle = \int_{[\tau_j^n,\sigma_{j+1}^n)} \big(h(\sigma_{j+1}^n) - h(s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+;$$

but we have the straightforward estimate

$$\left|\int_{[\tau_j^n,\sigma_{j+1}^n)} \big(h(\sigma_{j+1}^n) - h(s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+\right| \leq \rho\big(\sigma_{j+1}^n - \widetilde{\sigma}_{j+1}^n\big)\int_{[\tau_j^n,\sigma_{j+1}^n)} \big(\widetilde{\psi}_j^n(s)\big)^+;$$

thanks to the choice (6.8) of σ_{j+1}^n, we know that

$$\sigma_{j+1}^n - \widetilde{\sigma}_{j+1}^n \leq \frac{1}{n},$$

and therefore in virtue of (6.9) and (6.14),

$$\big|\langle \psi_j^n - \widetilde{\psi}_j^n, h\rangle\big| \leq \frac{\rho}{n}\int_{[\tau_j^n,\sigma_{j+1}^n)} \phi^+;$$

we have an analogous estimate for the initial term, and the lemma is proved. □

The next result gives an estimate of $g^n - g$:

LEMMA 6.6. *For all $p \in [1,2)$, there exists a constant C such that the following inequality holds:*

$$\|g^n - g\|_{L^p} \leq Cn^{1/2-1/p}\exp\Big(\frac{1}{2\alpha n}\Big)\Big(\int_{[\sigma_0^n,\tau_0^n)} |\phi| + \sum_{j\geq 0}\int_{[\tau_j^n,\sigma_{j+1}^n)} (\phi)^+\Big).$$

PROOF. The difference $g^n - g$ a sum of terms of the form

$$\mu * \left(\delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n, \sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ - 1_{[\tau_j^n, \sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ \right), \tag{6.22}$$

and possibly of an initial term given by

$$\mu * \left(\delta(\cdot - \sigma_0^n) \int_{[\sigma_0^n, \tau_0^n)} \phi^+ + \delta(\cdot - \bar{\tau}) \int_{[\sigma_0^n, \tau_0^n)} \phi^- - \phi 1_{[\sigma_0^n, \tau_0^n)} \right). \tag{6.23}$$

Let us start by the terms of the form (6.22): they can be rewritten as

$$\int_{[\tau_j^n, \sigma_{j+1}^n)} \left(\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right) \left(\widetilde{\psi}_j^n(s)\right)^+,$$

which we estimate in $L^p(0,T)$ by appealing to Minkowski inequality for integrals:

$$\left(\int_{\tau_j^n}^T \left| \int_{[\tau_j^n, \sigma_{j+1}^n)} \left(\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right) \left(\widetilde{\psi}_j^n(s)\right)^+ \right|^p dt \right)^{1/p}$$
$$\leq \int_{[\tau_j^n, \sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n(s)\right)^+ \left(\int_{\tau_j^n}^T \left|\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right|^p dt \right)^{1/p}.$$

But we observe the following inequality

$$\int_{[\tau_j^n, \sigma_{j+1}^n)} \left|\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right|^p dt$$
$$\leq \frac{(\sigma_{j+1}^n - \tau_j^n)^{1-p/2}}{(\alpha\pi)^{p/2}(1-p/2)} \exp\left(p\, \frac{\sigma_{j+1}^n + \tau_j^n}{\alpha}\right);$$

We cut the interval $[\sigma_{j+1}^n, T]$ into two pieces, one from σ_{j+1}^n to $\sigma_{j+1}^n + \varepsilon$ and the second one on the remainder of the interval, and we will adjust ε so as to obtain the best possible result. On the first piece, we have the estimate

$$\int_{\sigma_{j+1}^n}^{\sigma_{j+1}^n + \varepsilon} \left|\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right|^p dt$$
$$\leq \int_{\sigma_{j+1}^n}^{\sigma_{j+1}^n + \varepsilon} \left|\mu(t - \sigma_{j+1}^n)\right|^p dt = \frac{\varepsilon^{1-p/2}}{(\alpha\pi)^{p/2}(1-p/2)}.$$

On the third piece, we use the derivative of μ, this derivative is equal to $-(1/(2t^{3/2} \sqrt{\alpha\pi}) + 1/(\alpha t^{1/2}\sqrt{\alpha\pi})) \exp(-t/\alpha)$ for $t > 0$, and we obtain the estimate

$$\int_{\sigma_{j+1}^n + \varepsilon}^T \left|\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right|^p dt$$
$$\leq \int_{\sigma_{j+1}^n + \varepsilon}^T \left| \frac{(s - \sigma_{j+1}^n)(\alpha + 2(t - \sigma_{j+1}^n))}{2\alpha\sqrt{\alpha\pi}\,(t - \sigma_{j+1}^n)^{3/2}} \right|^p \exp\left(-\frac{p\varepsilon}{\alpha}\right) dt.$$

Since $s \in [\tau_j^n, \sigma_{j+1}^n)$, we estimate this integral by

$$\frac{(\sigma_{j+1}^n - \tau_j^n)^p (\alpha + 2(T - \sigma_{j+1}^n))^p}{\varepsilon^{3p/2-1}(3p/2-1)(2\alpha)^p(\alpha\pi)^{p/2}} \exp\left(-\frac{p\varepsilon}{\alpha}\right).$$

We choose $\varepsilon = \sigma_{j+1}^n - \tau_j^n$, and we see that there is a constant C such that for all n and all j:

$$\left(\int_{\tau_j^n}^T \left|\mu(t - \sigma_{j+1}^n) - \mu(t-s)\right|^p \right)^{1/p} dt$$
$$\leq C\left(\sigma_{j+1}^n - \tau_j^n\right)^{1/p-1/2} \exp\left(\frac{\sigma_{j+1}^n - \tau_j^n}{\alpha}\right).$$

But $\sigma^n_{j+1} - \tau^n_j$ is at most equal to $1/(2n)$, and we obtain finally the estimate

$$\|\mu * (\widetilde{\psi}^n_j - \psi^n_j)\|_{L^p} \le C n^{1/2-1/p} \exp\left(\frac{1}{2\alpha n}\right) \int_{[\tau^n_j, \sigma^n_{j+1})} (\widetilde{\psi}^n_j)^+.$$

Inequality (6.14) enables us to estimate the integral of $\left(\psi^n_j\right)^+$ over $[\tau^n_j, \sigma^n_{j+1}]$, by the integral of ϕ^+ over the same interval. Let us pass now to estimates on (6.23). Arguing as above, we observe that

$$\left(\int_{[\sigma^n_0, T]} \left|\mu * (\phi^n_0 - \phi)\right|^p \, dt\right)^{1/p} \le C n^{1/2-1/p} \exp\left(\frac{1}{2\alpha n}\right) \int_{[\sigma^n_0, \tau^n_0]} |\phi|.$$

The assertion of the lemma is proved. □

Let us obtain now some estimates on w^n and its derivatives:

LEMMA 6.7. *The time derivative $\dot{w}^n$ belongs to $L^\infty(\mathbb{R})$ and the bound on w^n is independent of n; the measure b^n is a function which is locally integrable on $\mathbb{R}^+$, with bound independent of n.*

PROOF. By the definition (6.18) of w^n on the interval $[\sigma^n_j, \tau^n_j)$ we have the estimate

$$\left\|\dot{w}^n|_{[\sigma^n_j, \tau^n_j)}\right\|_{L^\infty} \le \frac{1}{\alpha} \int_{[\sigma^n_j, \tau^n_j)} |\phi^n_j|,$$

and thanks to (6.13), we obtain immediately the estimate

$$\|\dot{w}^n\|_{L^\infty} \le \frac{1}{\alpha} \|\phi\|_{\mathcal{M}^1}. \tag{6.24}$$

Since $\lambda_1 = \dot{\rho}$, the convolution of (6.19a) with H yields the identity

$$\rho * w^n = H * (g^n + b^n). \tag{6.25}$$

But $\rho = \mu + \alpha\dot{\mu}$, hence (6.25) can be rewritten

$$\left(\mu * w^n + \alpha\mu * \dot{w}^n\right)(t+0) = \int_{[0,t]} g^n(s)\, ds + \int_{[0,t]} b^n(s)\, ds.$$

Relation (6.24) implies that the left hand side of the above relation is bounded by $(1+t)\|\phi\|_{\mathcal{M}^1}/\alpha$; similarly, we use (6.21) to find that the integral of g^n over $[0, T]$ is bounded by $3\|\phi\|_{\mathcal{M}^1}$. This shows the desired estimate. □

7. Passage to the limit

We start by an easy convergence result:

LEMMA 7.1. *There exists a subsequence, still denoted by w^n which has the following convergence properties:*

w^n converges to w uniformly on compact sets;

$\dot{w}^n$ converges to $\dot{w}$ in $L^\infty(0, T)$ weakly $*$ for all positive t;

b^n converges to b in $\mathcal{M}^1(0, T)$ weakly $*$ for all positive t.

Moreover, w and b satisfy (5.4).

PROOF. The possibility of extracting a subsequence is an immediate consequence of Lemma 6.7. It is clear that w and b are non negative; the duality product $\langle w^n, b^n\rangle$ converges to its limit which is $\langle w, b\rangle$. Thus we have constructed a solution of (3.5). □

We infer from this result an important information on the measure b:

LEMMA 7.2. *The measure b has no atoms.*

PROOF. The derivative $\dot{w}$ is equal in the sense of distributions to $\nu*\phi+\mu*b$. As ϕ is a measure, $\nu*\phi$ is locally essentially bounded; since $\dot{w}$ is essentially bounded, this means that $\mu*b$ is essentially bounded. Denoting by b^a its atomic part, we infer from the positivity of b that $\mu*b^a$ is also essentially bounded; but this means clearly that b^a must vanish, which concludes the proof. □

Now comes the essential result of this article:

THEOREM 7.3. *Let N be the set of atoms of ϕ: $1_N\phi$ is a purely atomic measure and $(1-1_N)\phi$ is a diffuse measure. For any solution of (3.5) defined by the above convergence process, let U be the open set*

$$U=\{t\in\mathbb{R}:w(t)>0\}, \tag{7.1}$$

which is a countable union of connected components:

$$U=\bigcup_{\kappa\in K}(\sigma_\kappa,\tau_\kappa).$$

The set composed of all the points σ_κ and τ_κ is a countable set called N_1. Then for all $t\notin N\cup N_1\cup U$, w is differentiable at t and its derivative vanishes.

PROOF. Assume thus that $w(t)$ vanishes, that t is not an end point σ_κ or τ_κ, and that t does not belong to N. We have to deal with derivatives on the left and on the right, and we use different strategies for each of them. If there exist respectively a non empty interval $[t,t+\varepsilon)$ or $(t-\varepsilon,t]$ included in the complement of U, it is clear that the derivative on the right or on the left of w at t vanishes.

Assume that there is no interval of the form $[t,t+\varepsilon)$ included in the complement of U. Since t is not an end point σ_κ or τ_κ, this means that there exists decreasing subsequences $\sigma_{\kappa(m)}$ and $\tau_{\kappa(m)}$ converging to t.

If it is not true that the right derivative of w at t vanishes, we can find a subsequence t_m decreasing to t and a number $\beta>0$ such that

$$\frac{w(t_m)}{t_m-t}\geq\beta>0, \tag{7.2}$$

and in particular, for all m, $w(t_m)$ is strictly positive. Possibly extracting subsequences, we may assume that the following situation holds for all large enough m:

$$\sigma_{\kappa(m)}<t_m<\tau_{\kappa(m)}<\sigma_{\kappa(m-1)}$$

For all m, there exists a number $n(m)$ such that for all $n\geq n(m)$, $w^n(t_m)$ is strictly positive, thanks to the uniform convergence of w^n to its limit. Let $(\sigma^n_{j(n)},\tau^n_{j(n)})$ be the connected component of t_m in the open set

$$U^n=\{t\in\mathbb{R}:w^n(t)>0\}.$$

We infer from (5.14) the identity

$$w^n(t_m)=\int_{[\sigma^n_{j(n)},t_m)}\left(1-\exp\left(-\frac{t_m-s}{\alpha}\right)\right)\phi^n_{j(n)}(s).$$

At this point, we observe that relation (6.15) implies

$$\int_{[\sigma^n_{j(n)},t_m)}\phi^n_{j(n)}\leq\int_{[\tau^n_{j(n)-1},t_m)}\phi^+;$$

therefore, we have the estimate

$$\frac{w^n(t_m)-w(t)}{t_m-t}\leq\frac{1}{t_m-t}\int_{[\tau^n_{j(n)-1},t_m)}\phi^+. \tag{7.3}$$

If the inferior limit of $\tau^n_{j(n)-1}$ as n tends to infinity is $\bar\sigma < t$, this means that there exists a subsequence of w^n, still denoted by w^n, such that for n large enough

$$\forall s \in [(t+\bar\sigma)/2, t_m], \quad w^n(s) > 0,$$

and hence the support of b does not meet $((t+\bar\sigma)/2, t_m)$; in particular, on this interval, $\dot w$ is of bounded variation, and as t does not carry an atom of ϕ, $\dot w$ is continuous at t; the sign condition implies then that $\dot w(t)$ vanishes. If the inferior limit of $\tau^n_{j(n)-1}$ is at least equal to t, then we estimate for n large enough the right hand side of (7.3) by

$$\frac{1}{t_m - t}\int_{(t-\varepsilon, t_m)} \phi^+(s),$$

with ε an arbitrary positive number. We pass to the limit in n and then in ε and we obtain the inequality

$$\frac{w(t_m) - w(t)}{t_m - t} \le \int_{[t,t_m)} \phi^+(s); \tag{7.4}$$

since t is not an atom of ϕ, we may choose m so large that the right hand side of (7.4) is less than or equal to $\beta/2$, contradicting thus the assumption (7.2). Therefore, the derivative of w on the right at t exists and vanishes.

Let us turn now to the other side of the estimates. This is where estimate (5.21) will prove useful. Assume then that there is no interval of the form $(t-\varepsilon, t]$ included in the complement of U. Since t is not an end point σ_κ or τ_κ, this means that there exists increasing subsequences $\sigma_{\kappa(m)}$ and $\tau_{\kappa(m)}$ converging to t.

If it is not true that the derivative on the left of w at t vanishes, we can find a number $\beta > 0$ and a sequence of times t_m increasing to t such that

$$\frac{w(t_m) - w(t)}{t - t_m} \ge \beta. \tag{7.5}$$

We assume also that for all m, b charges a neighborhood of $\sigma_{\kappa(m)}$ and a neighborhood of $\tau_{\kappa(m)}$: if this were not true, we could always take a smaller $\sigma_{\kappa(m)}$ and a larger $\tau_{\kappa(m)}$.

As t does not carry an atom of ϕ, we may choose m and $\varepsilon > 0$ such that

$$\frac{1}{\alpha}\int_{[\sigma_{\kappa(m)}-\varepsilon, \tau_{\kappa(m)}]} |\phi| \le \frac{\beta}{4};$$

as above, we denote by $(\sigma^n_{j(n)}, \tau^n_{j(n)})$ the connected component of t_m in U^n; relation (7.5) implies that for all large enough n,

$$w^n(t_m) - w^n(\tau^n_{j(n)}) \ge \frac{3\beta}{4}(t - t_m),$$

and therefore

$$w^n(t_m) - w^n(\tau^n_{j(n)}) \ge \frac{3\beta}{4}(t - \tau^n_{j(n)});$$

therefore, there exists in $[\sigma^n_{j(n)}, \tau^n_{j(n)}]$ a set M of positive measure on which the derivative of w is negative enough:

$$\dot w^n(\sigma) \le -\frac{3\beta}{4}, \quad \forall \sigma \in M.$$

We apply inequality (5.21), observing that the terms ψ^n_i are all non positive on $[\tau^n_{i-1}, \sigma^n_i)$ and therefore

$$\big(\dot w^n(\sigma - 0)\big)^- \le \frac{1}{\alpha}\int_{[\sigma^n_{j(n)}, \sigma]} |\phi(s)|,$$

which is at most equal to

$$\frac{1}{\alpha}\int_{[\sigma_{\kappa(m)}-\varepsilon,\tau_{\kappa(m)}]}|\phi|,$$

since $\sigma^n_{j(n)}$ tends to $\sigma_{\kappa(m)}$ under the assumption that b charges a neighborhood of $\sigma_{\kappa(m)}$. Therefore, on the set M, we have the estimate

$$\big(\dot w^n(\sigma)\big)^- \le \frac{\beta}{4},$$

which is clearly a contradiction. □

We have another expression for the derivative of w in the sense of distributions:

$$\dot w = \mu * (g+b).$$

Under assumption (5.1), this relation can be rewritten

(7.6) $\dot w = \nu * \phi + \mu * b$ in the sense of distributions.

Except at the atoms of ϕ, $\nu * \phi$ is a continuous function. On the other hand, $\mu * b$ is defined everywhere on $\mathbb{R}$, as proved in next Lemma:

Lemma 7.4. *If $\mu * b$ is locally essentially bounded, the function $\mu * b$ is defined for all $t \in \mathbb{R}$, lower semi-continuous and locally bounded on $\mathbb{R}$.*

Proof. The function $\mu(t-\cdot)$ is continuous except at 0; therefore, it is b-measurable, since b has no atoms, thanks to Lemma 7.2. Therefore, the expression

$$\int \mu(t-s)b(s)$$

is defined as an element of $[0,\infty]$ and can be obtained as a limit of integral of continuous functions with respect to the measure b. Take for instance

$$\rho_h(t) = \begin{cases} 0 & \text{if } 0 \le t \le h,\\ (t-h)/h & \text{if } h \le t \le 2h,\\ 1 & \text{if } 2h \le t;\end{cases}$$

then it is plain that

$$\big(\mu * b\big)(t) = \lim_{h\downarrow 0}\int \mu(t-s)\rho_h(t-s)b(s).$$

Moreover, $\mu * b$ is lower semi-continuous: if t_n is a sequence converging to t, the inferior limit of $\mu(t_n-\cdot)$ is greater than or equal to $\mu(t-\cdot)$ and thanks to Fatou's lemma

$$\liminf \int \mu(t_n-s)b(s) \ge \int \mu(t-s)b(s).$$

Finally, the function $\mu * b$ is locally bounded if it is locally essentially bounded: suppose indeed that there exists a time t such that

$$\int \mu(t-s)b(s) = \infty.$$

Let M be the essential bound of $\mu * b$ over $[0,T]$; for all $M' > M$ there exists $\varepsilon > 0$ such that

$$\int_0^{t-\varepsilon} \mu(t-s)b(s) \ge M'.$$

Then, for all $t' \in (t-\varepsilon, t]$, and for all $s \in [0, t-\varepsilon]$ we have the inequality

$$\mu(t'-s) \ge \mu(t-s)$$

which we integrate over $[0, t-\varepsilon]$ with respect to b, obtaining thus

$$\int_{[0,t-\varepsilon]} \mu(t'-s)b(s) \geq \int_{[0,t-\varepsilon]} \mu(t-s)b(s),$$

so that there is a set of measure ε on which $\mu * b$ is at least equal to M', which contradicts the assumption on the essential bound of $\mu * b$ over $[0,T]$. □

We have now two expressions for the derivative of w, which are known to coincide in the sense of distributions and therefore almost everywhere. We wish to show that they coincide everywhere, except at a countable number of point; this will be a consequence of next Lemma. Write $\Phi = \nu * \phi$ and observe that *a priori*, thanks to the lower semi-continuity, we expect the inequality

$$\dot{w}(t) \geq \Phi(t) + \big(\mu * b\big)(t),$$

if we are able to prove that $\dot{w}$ is continuous at the points t which are not atoms of ϕ.

LEMMA 7.5. *At all the points where w is differentiable and Φ continuous we have the relation*

$$\dot{w}(t) = \Phi(t) + \big(\mu * b\big)(t).$$

PROOF. Let t be a point which is not an atom of ϕ and at which w is differentiable. Let us examine the differentiation of $H * \mu * b$ at t from either side. For $t + h > 0$ and $t > 0$, we use the identity

$$\frac{1}{h}\left(2\sqrt{t+h} - 2\sqrt{t} - h\frac{1}{\sqrt{t}}\right) = -\frac{1}{\sqrt{t}}\frac{\sqrt{t+h}-\sqrt{t}}{\sqrt{t+h}+\sqrt{t}}; \tag{7.7}$$

We decompose the expression used for defining the derivative as follows for $h > 0$:

$$\begin{aligned}
&\frac{w(t+h)-w(t)}{h} - (\mu * b)(t) - \Phi(t) \\
=&\frac{1}{h}\int_0^{t+h} \big((H*\nu)(t+h-s) - (H*\nu)(t-s) - h\nu(t-s)\big)\phi(s) \\
&+ \frac{1}{h}\int_0^{t} \big((H*\mu)(t+h-s) - (H*\mu)(t-s) - h\mu(t-s)\big)b(s) \\
&+ \frac{1}{h}\int_{[t,t+h]} \big((H*\nu)(t-s) - h\nu(t-s)\big)\phi(s) \\
&+ \frac{1}{h}\int_{[t,t+h]} (H*\nu)(t+h-s)b(s).
\end{aligned}$$

Passing to the limit and using Lebesgue's dominated convergence theorem we obtain, thanks to (7.7)

$$\begin{aligned}
\dot{w}(t+0) =&\Phi(t) + (\mu * b)(t) - \lim_{h\downarrow 0}\frac{1}{h}\int_{[t,t+h]}(H*\nu)(t+h-s)b(s) \\
&- \lim_{h\downarrow 0}\frac{1}{h}\int_{[t,t+h]}\big((H*\nu)(t-s) - h\nu(t-s)\big)\phi(s).
\end{aligned} \tag{7.8}$$

Similarly, for $h < 0$, we write

$$\begin{aligned}&\frac{w(t+h)-w(t)}{h}-(\mu * b)(t)-\Phi(t)\\=&\frac{1}{h}\int_0^{t+h}\big((H*\nu)(t+h-s)-(H*\nu)(t-s)-h\nu(t-s)\big)\phi(s)\\&+\frac{1}{h}\int_0^{t+h}\big((H*\mu)(t+h-s)-(H*\mu)(t-s)-h\mu(t-s)\big)b(s)\\&-\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\nu)(t-s)\phi(s)-\lim_{h\uparrow 0}\int_{t+h}^{t}\nu(t-s)\phi(s)\\&-\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\mu)(t-s)b(s)-\lim_{h\uparrow 0}\int_{t+h}^{t}\mu(t-s)b(s).\end{aligned}$$

Using as above (7.7) and Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned}\dot w(t-0)=&\Phi(t)+\int_0^t\mu(t-s)b(s)\\&+\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\nu)(t-s)\phi(s)+\lim_{h\uparrow 0}\int_{t+h}^{t}\nu(t-s)\phi(s)\\&+\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\mu)(t-s)b(s)+\lim_{h\uparrow 0}\int_{t+h}^{t}\mu(t-s)b(s).\end{aligned}\tag{7.9}$$

Relation (7.8) implies that

$$\dot w(t)\geq \Phi(t)+\big(\mu*b\big)(t),$$

and symmetrically, relation (7.9) implies that

$$\dot w(t)\leq \Phi(t)+\big(\mu*b\big)(t).$$

These two inequalities enable us to conclude the proof. □

We are able now to conclude the article by proving the last result:

PROPOSITION 7.6. *Let w be the solution constructed at Lemma 7.1; then, for all $T > 0$, we have the identity*

$$\int_0^T\big(\dot\mu*w\big)\dot w\,dt=\int_0^T g\dot w\,dt.$$

PROOF. The convolution $\dot\mu * w$ is the sum of the measure b and the function g; therefore, the duality product $\langle 1_{[0,T]}, g+b\rangle$ is well defined; the function $\dot w(t)$ is bounded, and it vanishes b-almost everywhere on the support of b; therefore, for all $T > 0$, the integral $\int_0^T b\dot w$ vanishes, which proves the proposition. □

Now, we can drop the requirement that ϕ be a bounded measure:

COROLLARY 7.7. *Let ϕ be a Radon measure with support included in $\mathbb{R}^+$; then there exists a function w such that* (6.2) *and hold.*

PROOF. For each m, the measure $\phi 1_{[0,m]}$ is bounded and we may construct $w^{n,m}(t)$ so that it coincides with $w^{n,k}(t)$ for $t \leq \min(m,k)(t)$. Therefore, a diagonal process lets us extract a solution possessing the required properties. □

Appendix

Define the following complex-valued functions:

$$\begin{aligned}
\widehat{\nu}(\omega) &= \frac{1}{1+i\alpha\omega},\\
\widehat{\mu}(\omega) &= \sqrt{\widehat{\nu}(\omega)} = \frac{1}{\sqrt{1+i\alpha\omega}}, \quad \Re\widehat{\mu} \geq 0,\\
\widehat{\rho}(\omega) &= \frac{1}{\widehat{\mu}(\omega)} = \sqrt{1+i\alpha\omega}, \quad \Re\widehat{\rho} \geq 0,\\
\widehat{\lambda}_1(\omega) &= i\omega\widehat{\rho}(\omega) = i\omega\sqrt{1+i\alpha\omega},\\
\widehat{\lambda}(\omega) &= i\omega\widehat{\mu}(\omega) = \frac{i\omega}{\sqrt{1+i\alpha\omega}},\\
\widehat{\mu}_1(\omega) &= \frac{\widehat{\mu}(\omega)}{i(\omega - i0)},
\end{aligned}$$

where we have used the notation

$$\frac{1}{\omega - i0} = \lim_{\varepsilon\downarrow 0} \frac{1}{\omega - i\varepsilon}.$$

The inverse Fourier transform of $\widehat{\nu}$ is given by

$$\nu(t) = \frac{\exp(-t/\alpha)}{\alpha} 1_{\mathbb{R}^+}(t). \tag{7.10}$$

Let H denote the Heaviside function. We observe that

$$H * \nu = \big(1 - \exp(-\cdot/\alpha)\big)H. \tag{7.11}$$

The functions $\widehat{\mu}$, $\widehat{\rho}$, μ_1 and $\widehat{\lambda}_1$ are analytic in the upper half-plane, so that their inverse Fourier transform, μ, ρ and λ_1 are supported in $\mathbb{R}^+$.

We have an explicit expression of μ:

LEMMA 7.8. *The inverse Fourier transform of $\widehat{\mu}$ is*

$$\mu(t) = \frac{\exp(-t/\alpha)}{\sqrt{2\pi\alpha t}} 1_{(0,\infty)}(t). \tag{7.12}$$

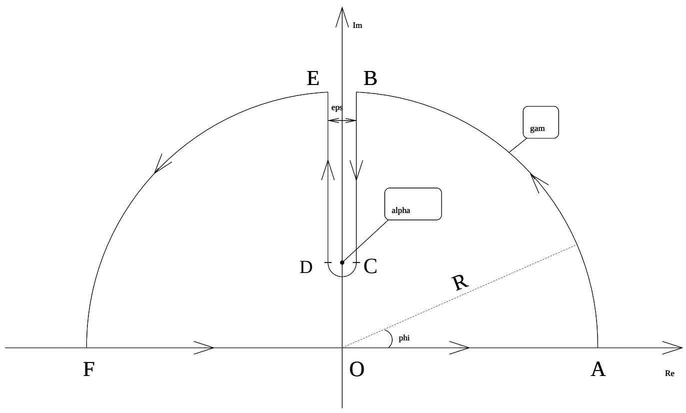

FIGURE 1. The path Γ in the complex plane.

PROOF. It is plain that $\widehat{\mu}$ is holomorphic in $\mathbb{C}\setminus i[1/\alpha,+\infty)$; thanks to Paley-Wiener-Schwartz theorem, the support of μ is included in $[0,+\infty)$. In order to calculate the inverse Fourier transform of $\widehat{\mu}$, choose the integration path Γ pictured at Figure 1; the part on the arcs of circle AB and EF converges to 0 as R tends to infinity; thanks to Cauchy's theorem and an obvious passage to the limit

$$\int_{\mathbb{R}} \exp(i\omega t)\widehat{\mu}\, d\omega = 2\int_{1/\alpha}^{+\infty} \frac{\exp(-rt)}{\sqrt{\alpha r - 1}}\, dr.$$

The change of variable $s = (r - 1/\alpha)t$ yields the desired conclusion. □

The function μ_1 is given in physical variables by

$$\mu_1(t) = \int_0^t \mu(s)\, ds = \int_0^t \frac{\exp(-s/\alpha)}{\sqrt{\pi\alpha s}}\, ds. \tag{7.13}$$

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CHAPITRE 6

The damped wave equation with unilateral boundary conditions in a half-space

Adrien Petrov and Michelle Schatzman

Abstract. Let α be a positive number. We consider a damped wave equation

$$u_{tt} - \Delta u - \alpha \Delta u_t = f, \quad x \in (-\infty, 0] \times \mathbb{R}^{d-1}, \quad t > 0,$$

with unilateral boundary conditions

$$u(0, \cdot) \geq 0, \quad (u_{x_1} + \alpha u_{x_1 t})(0, \cdot) \geq 0, \quad \big(u(u_{x_1} + \alpha u_{x_1 t})\big)(0, \cdot) = 0,$$

and Cauchy initial data u_0 and u_1; we suppose that the initial position u_0 and the initial velocity belong respectively to $H^2\big((-\infty, 0] \times \mathbb{R}^{d-1}\big)$ and $H^1\big((-\infty, 0] \times \mathbb{R}^{d-1}\big)$, the density of forces f belongs to $L^2_{\text{loc}}\big([0, \infty); L^2\big((-\infty, 0] \times \mathbb{R}^{d-1}\big)\big)$. A weak solution is obtained as the limit of penalized problem, the functional properties of all the traces are precisely identified which enables us to infer the existence of a strong solution.

1. Introduction and notations

The present article is a generalization to dimension d of results established in dimension 1. Indeed, we have proved the existence of a solution on a half-line of the damped wave equation:

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in (-\infty, 0] \times \mathbb{R}^{d-1}, \quad t > 0,$$

with Cauchy initial conditions and unilateral conditions at $x_1 = 0$:

$$u \geq 0, \quad (u_{x_1} + \alpha u_{x_1 t}) \geq 0, \quad u(u_{x_1} + \alpha u_{x_1 t}) = 0.$$

We have proved in this case that the energy losses are purely viscous [**3**].

We treat the case of a damped wave equation as a preparation for the more difficult case of full viscoelasticity. We are quite aware that this is an easier problem than, the analogous problem for the full system of viscoelasticity. The main result of this article is a characterization of the trace spaces, which enables us to prove that the weak solution is indeed strong.

Define the following sets: $\Omega = (-\infty, 0] \times \mathbb{R}^{d-1}$, $\Sigma = \{0\} \times \mathbb{R}^{d-1}$ is the boundary of Ω and $Q_R = \big\{x : x_1 < 0, |x'| \leq R\big\} \times [0, R]$. Let $u(x, t)$ be the displacement at time t of the material point of spatial coordinate $x = (x_1, x') \in \Omega$ at rest. Let f denote a density of exterior forces, depending on space and time. We denote by $\xi = (\xi_2, ..., \xi_d)^T$ and ω respectively the dual variable to $x' = (x_2, ..., x_d)^T$ and t, $\widehat{u}(0, \xi, \omega)$ is the Fourier transform of $u(0, x', t)$. The convention for the Fourier transform is

$$\widehat{u}(0, \xi, \omega) = \int_{\mathbb{R}^d} e^{-i(\xi x' + \omega t)} u(0, x', t)\, dx'\, dt.$$

In this article, we consider a d-dimensional wave equation

$$u_{tt} - \Delta u - \alpha \Delta u_t = f, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

with Cauchy initial data

$$u(\cdot,0)=u_0 \quad \text{and} \quad u_t(\cdot,0)=u_1, \tag{1.2}$$

and unilateral boundary conditions at $x_1=0$

$$u\geq 0,\quad (u_{x_1}+\alpha u_{x_1t})\geq 0,\quad u(u_{x_1}+\alpha u_{x_1t})=0. \tag{1.3}$$

The existence of a weak solution of this problem is easily established by the penalty method, and was already known Jarušek et al [**2**] in the case of distributed constraints. The main result of this article is a characterization of the trace space, which enables us to prove that the weak solution is indeed strong. We also get an energy inequality; unfortunately, we have been unable to establish an energy equality, and we know nothing about uniqueness.

We are given Cauchy initial data u_0 and u_1; we suppose that the initial position u_0 belongs to the Sobolev space $H^2(\Omega)$ and satisfies the compatibility condition $u_0(0,\cdot,\cdot)\geq 0$. The initial velocity u_1 belongs to $H^1(\Omega)$ and the density of forces f belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. Here the choice of a function f is defined for all time is justified by the use of a Fourier transform in the latter part of the article. Let K be the convex set:

$$K=\big\{v\in H^1_{\text{loc}}\big(\Omega\times[0,\infty)\big) : \nabla v_t\in L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big),\ v\big|_\Sigma\geq 0\big\}.$$

This unusual convex set has been devised in order to write a weak formulation of our problem. Since we expect to find a scalar product $(\nabla u_t,\nabla w)$, we require ∇u_t to be square integrable. Thus, the weak formulation (1.1)-(1.3) is obtained by multiplying (1.1) by $v-u$, $v\in K$ and by integrating formally over $\Omega\times(0,\tau)$, we obtain therefore:

$$\begin{cases} u\in K \text{ and for all } v\in K \text{ and for every } \tau\in[0,T], \\ \displaystyle\int_\Omega \big(u_t(v-u)\big)\big|_0^\tau\,dx-\int_0^\tau\int_\Omega u_t(v_t-u_t)\,dx\,dt \\ \displaystyle +\int_0^\tau\int_\Omega(\nabla u+\alpha\nabla u_t)(\nabla v-\nabla u)\,dx\,dt\geq\int_0^\tau\int_\Omega f(v-u)\,dx\,dt. \end{cases} \tag{1.4}$$

The equivalence between the weak formulation (1.4) and the strong formulation (1.1)-(1.3) is not obvious; it depends on precise information on the trace of u_{x_1t} on the boundary $x_1=0$.

Let us explain the plan of this paper.

In Section 2, we define a penalized problem related to (1.1)-(1.4b) for which we prove the existence and uniqueness of a solution.

In Section 3 and 4, we infer the existence of a weak solution to problem (1.1)-(1.4b) from *a priori* estimates.

We will establish a preliminary result in Section 2, which is needed in the following Section. More precisely, if u_0 belongs to $H^2(\Omega)$ and u_1 belongs to $H^1(\Omega)$, if f and f_t belong to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$ and if $\bar u$ is the solution of (1.1)-(1.3) with Dirichlet boundary data on Σ, then the trace $(\bar u_{x_1}+\alpha\bar u_{x_1t})\big|_\Sigma$ is bounded in $L^2_{\text{loc}}\big([0,\infty);H^{1/2}(\Sigma)\big)$.

In Section 6, we reduce the penalized problem to a problem on the boundary and thanks to energy estimates, we show that the trace $u(0,\cdot,\cdot)$ belongs to the Sobolev space $H^{5/4}_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)\cap H^1_{\text{loc}}\big([0,\infty),H^{1/2}(\Sigma)\big)$ and $u_{x_1t}(0,\cdot,\cdot)$ belongs to $L^2_{\text{loc}}\big([0,\infty);H^{1/2}(\mathbb{R}^{d-1})\big)$.

2. The penalized problem

We approximate (1.1)-(1.3) by the penalty method. This means that we replace the rigid constraint (1.3) by a very stiff response: when the constraint is active, the response is linear, and it vanishes when the constraint is not active. More precisely, letting $r^- = -\min(r, 0)$, we replace u by u^ε, which satisfies

$$u^\varepsilon_{tt} - \Delta u^\varepsilon - \alpha \Delta u^\varepsilon_t = f, \quad x \in \Omega, \quad t > 0, \tag{2.1}$$

with initial data

$$u^\varepsilon(\cdot, 0) = u_0 \quad \text{and} \quad u^\varepsilon_t(\cdot, 0) = u_1, \tag{2.2}$$

and boundary condition

$$\big(u^\varepsilon_{x_1} + \alpha u^\varepsilon_{x_1 t}\big)(0, \cdot, \cdot) = \big(u^\varepsilon(0, \cdot, \cdot)\big)^- / \varepsilon. \tag{2.3}$$

THEOREM 2.1. *Define*

$$W_{\text{loc}} = \big\{u \in H^1_{\text{loc}}\big([0, \infty) \times \Omega\big) : \nabla u_t \in L^2_{\text{loc}}\big([0, \infty); L^2(\Omega)\big)\big\}.$$

Assume that u_0 belongs to $H^1(\Omega)$, that u_1 belongs to $H^1(\Omega)$, and f belongs to $L^2_{\text{loc}}\big([0, \infty); L^2(\Omega)\big)$; then for every $\varepsilon > 0$ there exists a unique weak solution $u^\varepsilon \in W_{\text{loc}}$ of the problem (2.1)-(2.3) *such that*

$$u^\varepsilon \in L^\infty_{\text{loc}}\big([0, \infty); H^1(\Omega)\big), \tag{2.4a}$$

$$u^\varepsilon_t \in L^2_{\text{loc}}\big([0, \infty); H^1(\Omega)\big), \tag{2.4b}$$

$$u^\varepsilon_{tt} \in L^2_{\text{loc}}\big([0, \infty); L^2(\Omega)\big), \tag{2.4c}$$

and for every $\tau \in (0, T)$ and for all $v \in W_{\text{loc}}$, the following variational equality is satisfied:

$$\begin{aligned} &\int_\Omega \big((u^\varepsilon_t v)(\cdot, \tau) - (u_1 v)(\cdot, 0)\big)\, dx - \int_0^\tau \int_\Omega u^\varepsilon_t v_t \, dx\, dt + \int_0^\tau \int_\Omega \nabla u^\varepsilon \nabla v \, dx\, dt \\ &+\alpha \int_0^\tau \int_\Omega \nabla u^\varepsilon_t \nabla v \, dx\, dt - \frac{1}{\varepsilon} \int_0^\tau \int_\Sigma \big((u^\varepsilon)^- v\big)(0, \cdot)\, dx'\, dt = \int_0^\tau \int_\Omega f v \, dx\, dt. \end{aligned} \tag{2.5}$$

PROOF. We drop the superscript ε in this proof. Let us first prove the uniqueness: let u, $\widetilde{u}$ be two solutions satisfying (2.4)-(2.5) and let $w = u - \widetilde{u}$. Denote by $\|\cdot\|$ and $(\cdot, \cdot)$ denote the norm and the scalar product in $L^2(\Omega)$. Then $w(\cdot, 0) = 0$, $w_t(\cdot, 0) = 0$ and for every $v \in W_{\text{loc}}$,

$$\begin{aligned} &\int_\Omega w_t(\cdot, \tau) v(\cdot, \tau)\, dx - \int_{Q_\tau} w_t v_t \, dx\, dt + \int_{Q_\tau} \nabla w \nabla v \, dx\, dt \\ &+\alpha \int_{Q_\tau} \nabla w_t \nabla v \, dx\, dt - \frac{1}{\varepsilon} \int_{I_\tau} \big((u^- - \widetilde{u}^-) v\big)(0, \cdot)\, dx'\, dt = 0. \end{aligned} \tag{2.6}$$

Taking $v = w_t$ as test function in (2.6), thanks to the inequality $|yz| \le y^2/2\gamma + \gamma z^2/2$, $\gamma > 0$, we may infer that

$$\begin{aligned} &\frac{1}{2} \int_\Omega \big(|w_t(\cdot, \tau)|^2 + |\nabla w(\cdot, \tau)|^2\big)\, dx + \alpha \int_0^\tau \int_\Omega |\nabla w_t|^2 \, dx\, dt \\ &\le \frac{1}{2\gamma\varepsilon} \int_0^\tau \int_\Sigma |w(0, \cdot)|^2 \, dx'\, dt + \frac{\gamma}{2\varepsilon} \int_0^\tau \int_\Sigma |w_t(0, \cdot)|^2 \, dx'\, dt. \end{aligned}$$

Let us observe that

$$\int_0^\tau \int_\Sigma |z(0, \cdot)|^2 \, dx'\, dt \le C \int_0^\tau \int_\Omega \big(|z|^2 + |\nabla z|^2\big)\, dx\, dt,$$

implies

$$\begin{aligned} &\frac{1}{2} \int_\Omega \big(|w_t(\cdot, \tau)|^2 + |\nabla w(\cdot, \tau)|^2\big)\, dx + \left(\alpha - \frac{C\gamma}{2\varepsilon}\right) \int_0^\tau \int_\Omega |\nabla w_t|^2 \, dx\, dt \\ &\le \frac{C}{\varepsilon} \int_0^\tau \int_\Omega \left(\frac{1}{2\gamma}\big(|w|^2 + |\nabla w|^2\big) + \frac{\gamma}{2}|w_t|^2\right) dx\, dt. \end{aligned} \tag{2.7}$$

If we choose γ such that $\alpha > C\gamma/2\varepsilon$, then (2.7) is a Gronwall inequality for

$$\chi(\tau) = \int_0^\tau \int_\Omega \big(|w_t|^2 + |\nabla w|^2\big)\, dx\, dt;$$

indeed, since $\tau \in [0,T]$, we deduce from

$$\int_0^\tau \int_\Omega |w|^2\, dx\, dt \le \int_0^\tau t \int_0^t \|w_t(\cdot,s)\|^2\, ds\, dt \le T^2 \int_0^\tau \|w_t(\cdot,s)\|^2\, ds,$$

that χ satisfies the differential inequality:

$$\dot\chi(\tau) \le \frac{C}{2\varepsilon} \max\big(1/\gamma, T^2/\gamma + \gamma\big)\chi(\tau),$$

and the uniqueness is now clear.

The existence is proved by the Galerkin method. Let $\{wj\}_{j=1}^\infty$ is a complete orthonormal sequence in $L^2(\Omega)$ whose elements belong to $H^2(\Omega)$. Let $u^k(x,t) = \sum_{i=1}^k \xi_i(t) w_i(x)$ satisfying the variational equality:

$$\begin{cases} \text{for all } v(x,t) = \displaystyle\sum_{i=1}^k \zeta_i(t) w_i(x), \\ \displaystyle\int_0^\tau (\ddot u^k, v)\, dt + \int_0^\tau (\nabla u^k, \nabla v)\, dt + \alpha \int_0^\tau (\nabla \dot u^k, \nabla v)\, dt \\ \quad -\displaystyle\frac{1}{\varepsilon}\int_0^\tau \int_\Sigma \big((u^k)^- v\big)(0,\cdot)\, dx'\, dt = \int_0^\tau (f,v)\, dt, \end{cases}$$

where $\dot u^k$ and $\ddot u^k$ are respectively the first and the second time derivative of u^k. Therefore we deduce that for almost every $t \in (0,T)$, the system ($j = 1, 2, ..., k$) of ordinary differential equations

$$\begin{aligned} \ddot\xi_j + \sum_{i=1}^k \xi_i (\nabla w_i, \nabla w_j) + \alpha \sum_{i=1}^k \dot\xi_i (\nabla w_i, \nabla w_j) \\ - \frac{1}{\varepsilon} \int_\Sigma \big((u^k)^- w_j\big)(0,\cdot)\, dx' = (f, w_j), \end{aligned} \tag{2.8}$$

possesses a unique solution on $[0,T]$ since the non linear term

$$(\xi_j)_{1\le j\le k} \mapsto -\frac{1}{\varepsilon}\Big(\sum_{j=1}^k \xi_j w_j(0,\cdot)\Big)^-,$$

is Lipschitz continuous. We multiply the j-th equation in (2.8) by $\dot\xi_j$, we sum with respect to j and we integrate over $(0,\tau)$; thanks to the L^2-orthonormality of the w_j, we get the energy identity:

$$\begin{aligned} \int_0^\tau \sum_{j=1}^k \ddot\xi_j \dot\xi_j\, dt + \int_0^\tau \sum_{i,j=1}^k \xi_i \dot\xi_j (\nabla w_i, \nabla w_j)\, dt + \alpha \int_0^\tau \sum_{i,j=1}^k \dot\xi_i \dot\xi_j (\nabla w_i, \nabla w_j)\, dt \\ - \frac{1}{\varepsilon} \int_0^\tau \int_\Sigma \big((u^k)^- u_t^k\big)(0,\cdot)\, dx'\, dt = \int_0^\tau (f, u_t^k)\, dt. \end{aligned} \tag{2.9}$$

A time integration of (2.9) implies that

$$\begin{aligned} \frac{1}{2}\big(\|u_t^k(\cdot,\tau)\|^2 + \|\nabla u^k(\cdot,\tau)\|^2\big) + \alpha \int_0^\tau \|\nabla u_t^k\|^2\, dt \\ + \frac{1}{2\varepsilon} \int_\Sigma \big((u^k(0,\cdot,\tau))^-\big)^2\, dx' \le \frac{1}{2}\int_\Omega \big(|\nabla u_0|^2 + |u_1|^2\big)\, dx + \int_0^\tau \big|(f, u_t^k)\big|\, dt. \end{aligned} \tag{2.10}$$

A classical Gronwall lemma enables us to infer that u_t^k, ∇u^k are bounded in $L^\infty(0,T;L^2(\Omega))$, ∇u_t^k is bounded in $L^2(0,T;L^2(\Omega))$, $\big(u^k(0,\cdot)\big)^-/\sqrt{\varepsilon}$ is bounded in $L^\infty(0,T;L^2(\Sigma))$.

We want to prove now an estimate on u_{tt}^k; for this purpose we multiply the j-th equation in (2.8) by $\ddot{\xi}_j$, we sum with respect to j and we integrate over $(0,\tau)$; thanks to the L^2-orthonormality of the w_j, we get the identity:

$$\begin{aligned}&\int_0^\tau \sum_{j=1}^k |\ddot{\xi}_j|^2\,dt + \int_0^\tau \sum_{i,j=1}^k \big(\xi_i\ddot{\xi}_j(\nabla w_i,\nabla w_j) + \alpha\ddot{\xi}_j(\nabla w_i,\nabla w_j)\big)\,dt \\ &\qquad -\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma \big((u^k)^- u_{tt}^k\big)(0,\cdot)\,dx'\,dt = \int_0^\tau (f,u_{tt}^k)\,dt.\end{aligned}\tag{2.11}$$

We observe that for all $\gamma_1>0$, we have the following inequality:

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma \big|\big((u^k)^- u_{tt}^k\big)(0,\cdot)\big|\,dx'\,dt \\ &\leq \frac{1}{2\varepsilon\gamma_1}\int_0^\tau\int_\Sigma \big|\big(u^k(0,\cdot)\big)^-\big|^2\,dx'\,dt + \frac{\gamma_1}{2\varepsilon}\int_0^\tau \sum_{j=1}^k |\ddot{\xi}_j|^2\,dt.\end{aligned}\tag{2.12}$$

Since $\big(u^k(0,\cdot)\big)^-/\sqrt{\varepsilon}$ is bounded in $L^\infty(0,T;L^2(\Sigma))$, there exists $M>0$ such that we infer from (2.12) that

$$\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma \big|\big((u^k)^- u_{tt}^k\big)(0,\cdot)\big|\,dx'\,dt \leq \frac{MT}{\gamma_1} + \frac{\gamma_1}{2\varepsilon}\int_0^\tau \sum_{j=1}^k |\ddot{\xi}_j|^2\,dt.\tag{2.13}$$

On the other hand, we integrate in time the second and the third term on the left hand side of (2.11); we remark that the product $|zy|$ can be estimated by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, for $i=2,3$ and we will choose different γ_i. Carrying (2.11) into (2.13) and choosing γ_i such that $\alpha-\gamma_2>0$ and $1-\gamma_1/2\varepsilon-\gamma_3/2>0$, we obtain

$$\begin{aligned}&\left(1-\frac{\gamma_1}{2\varepsilon}-\frac{\gamma_3}{2}\right)\int_0^\tau \|u_{tt}^k\|^2\,dt + \frac{1}{2}(\alpha-\gamma_2)\,\|\nabla u_t^k(\cdot,\tau)\|^2 \leq \frac{1}{2\gamma_2}\,\|\nabla u^k(\cdot,\tau)\|^2 \\ &\qquad + \frac{1}{2}\,\|\nabla u_0\|^2 + \frac{1}{2}\,\|\nabla u_1\|^2 + \int_0^\tau \|\nabla u_t^k\|^2\,dt + \frac{MT}{\gamma_1} + \frac{1}{2\gamma_3}\int_0^\tau \|f\|^2\,dt.\end{aligned}$$

Hence u_{tt}^k is bounded in $L^2(0,T;L^2(\Omega))$ and ∇u_t^k is bounded in $L^\infty(0,T;L^2(\Omega))$. So, there exists a subsequence, still denoted by u^k, such that

$$\begin{aligned}&u^k \rightharpoonup u \quad \text{weakly in} \quad L^\infty_{\text{loc}}\big([0,\infty);H^1(\Omega)\big),\\ &u_t^k \rightharpoonup u_t \quad \text{weakly in} \quad L^2_{\text{loc}}\big([0,\infty);H^1(\Omega)\big),\\ &u_{tt}^k \rightharpoonup u_{tt} \quad \text{weakly in} \quad L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big),\end{aligned}$$

which proves the Theorem. □

3. A priori estimates

In this Section, we establish estimates up to the boundary and interior estimates which, later, will enable us to infer the existence of a weak solution to (1.1)-(1.3).

LEMMA 3.1. *Assume that f belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, u_0 to $H^1(\Omega)$ and u_1 to $L^2(\Omega)$. Then independently of $\varepsilon>0$, u_t^ε, ∇u^ε are bounded in the space $L^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, ∇u_t^ε is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$ and $(u^\varepsilon(0,\cdot,\cdot))^-/\sqrt{\varepsilon}$ is bounded in the space $L^\infty_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)$.*

PROOF. These estimates are simply an application of Gronwall lemma to the energy identity. We multiply (2.1) by u_t^ε and we integrate this expression over $\Omega \times (0,\tau)$ to get

$$\int_0^\tau \int_\Omega u_{tt}^\varepsilon u_t^\varepsilon \, dx\, dt - \int_0^\tau \int_\Omega \Delta u^\varepsilon u_t^\varepsilon \, dx\, dt$$
$$-\alpha \int_0^\tau \int_\Omega \Delta u_t^\varepsilon u_t^\varepsilon \, dx\, dt = \int_0^\tau \int_\Omega f u_t^\varepsilon \, dx\, dt.$$

We integrate in time the first integral in the above relation and we use Green's formula for the second and the third one, and with the help of the boundary conditions (2.3), we obtain

$$\frac{1}{2}\int_\Omega \big(|u_t^\varepsilon(\cdot,\tau)|^2 + |\nabla u^\varepsilon(\cdot,\tau)|^2\big)\, dx + \alpha \int_0^\tau \int_\Omega |\nabla u_t^\varepsilon|^2 \, dx\, dt$$
$$-\frac{1}{\varepsilon}\int_0^\tau \int_\Sigma \big((u^\varepsilon)^- u_t^\varepsilon\big)(0,\cdot,\cdot)\, dx'\, dt = \int_0^\tau \int_\Omega f u_t^\varepsilon \, dx\, dt + \frac{1}{2}\int_\Omega \big(|\nabla u_0|^2 + |u_1|^2\big)\, dx,$$

which implies

$$\frac{1}{2}\int_\Omega \big(|u_t^\varepsilon(\cdot,\tau)|^2 + |\nabla u^\varepsilon(\cdot,\tau)|^2\big)\, dx + \alpha \int_0^\tau \int_\Omega |\nabla u_t^\varepsilon|^2 \, dx\, dt$$
$$+\frac{1}{2\varepsilon}\int_\Sigma \big((u^\varepsilon(0,\cdot,\cdot))^-\big)^2\Big|_0^\tau \, dx' = \int_0^\tau \int_\Omega f u_t^\varepsilon \, dx\, dt + \frac{1}{2}\int_\Omega \big(|\nabla u_0|^2 + |u_1|^2\big)\, dx.$$

We may deduce from a classical Gronwall lemma that u_t^ε, ∇u^ε are bounded in $L^\infty_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$, ∇u_t^ε is bounded in $L^2_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$ and $(u^\varepsilon(0,\cdot,\cdot))^-/\sqrt{\varepsilon}$ is bounded in $L^\infty_{\text{loc}}\big([0,\infty); L^2(\Sigma)\big)$. □

REMARK 3.2. *If we suppose that f vanishes for t large then, independently of $\varepsilon > 0$, u_t^ε and ∇u^ε are bounded in $L^\infty\big([0,\infty); L^2(\Omega)\big)$ and ∇u_t^ε is bounded in $L^2\big([0,\infty); L^2(\Omega)\big)$. These properties can be proved using the arguments given in the proof of lemma 3.1, with the origin of time moved to T if $f(\cdot,t)$ vanishes for $t \geq T$; since the integral involving f vanishes, the conclusion is clear.*

LEMMA 3.3. *Assume the hypotheses of Lemma 3.1. Then $(u^\varepsilon(0,\cdot,\cdot))^-/\varepsilon$ is bounded in the space of measures on $\Sigma \times (0,T)$; more precisely, for all $\varphi \geq 0$, $\varphi \in C_0^1(\mathbb{R}^{d-1})$, for all $\tau \in [0,T]$,*

$$\int_0^\tau \int_\Sigma \frac{(u^\varepsilon(0,\cdot,\cdot))^-}{\varepsilon}\varphi \, dx'\, dt$$

is bounded independently of ε.

PROOF. Let ϕ belong to $C_0^1(\Omega)$; we multiply (2.1) by ϕ and we integrate over $\Omega \times (0,\tau)$; thanks to the boundary conditions (2.3) and Green's formula, we obtain

$$\int_\Omega \phi u_t^\varepsilon(\cdot,t)\big|_0^\tau \, dx + \int_0^\tau \int_\Omega \nabla\phi \nabla u^\varepsilon \, dx\, dt + \alpha \int_0^\tau \int_\Omega \nabla\phi\nabla u_t^\varepsilon \, dx\, dt$$
$$-\frac{1}{\varepsilon}\int_0^\tau \int_\Sigma (u^\varepsilon(0,\cdot,\cdot))^- \phi \, dx'\, dt = \int_0^\tau \int_\Omega \phi f \, dx\, dt.$$

Since the product $|zy|$ can be estimated by $|z|^2/2 + |y|^2/2$, we get the following inequality:

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\tau \int_\Sigma (u^\varepsilon(0,\cdot,\cdot))^- \phi \, dx'\, dt &\leq \frac{1}{2}\int_\Omega \big(|u_t^\varepsilon(\cdot,\tau)|^2 + |u_1|^2\big)\, dx + \int_\Omega |\phi|^2 \, dx \\ &+ \int_0^\tau \int_\Omega |\nabla\phi\nabla u^\varepsilon| \, dx\, dt + \alpha\int_0^\tau \int_\Omega |\nabla\phi\nabla u_t^\varepsilon| \, dx\, dt + \int_0^\tau \int_\Omega |\phi f| \, dx\, dt.\end{aligned} \tag{3.1}$$

The right hand side of (3.1) is bounded since f belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, u_1 to $L^2(\Omega)$ and u^ε_t, ∇u^ε and ∇u^ε_t belong to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. Moreover $(u^\varepsilon(0,\cdot,\cdot))^-$ is non negative; if the trace of ϕ over Σ is $\varphi\ge 0$, the conclusion is clear. □

LEMMA 3.4. *Assume the hypotheses of Lemma 3.1, and suppose moreover that u_0 belongs to $H^2(\Omega)$. Then independently of $\varepsilon>0$, Δu^ε is bounded in the space $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$.*

PROOF. Once again we use energy techniques, but now we multiply relation (2.1) by Δu^ε and we integrate over the result $\Omega\times(0,\tau)$:

$$\begin{aligned}\int_0^\tau\int_\Omega u^\varepsilon_{tt}\Delta u^\varepsilon\,dx\,dt-\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt\\ -\alpha\int_0^\tau\int_\Omega\Delta u^\varepsilon_t\Delta u^\varepsilon\,dx\,dt=\int_0^\tau\int_\Omega f\Delta u^\varepsilon\,dx\,dt.\end{aligned}\tag{3.2}$$

We integrate by parts the first integral in (3.2) first in time, then in space; we use Green's formula several times, and since the third integral in the left hand side of (3.2) contains a total time derivative, we obtain

$$\begin{aligned}\int_\Omega\big((u^\varepsilon_t\Delta u^\varepsilon)(\cdot,t)\big)\big|_0^\tau dx-\int_0^\tau\int_\Sigma\big(u^\varepsilon_t u^\varepsilon_{x_1t}\big)(0,\cdot)\,dx'\,dt+\int_0^\tau\int_\Omega|\nabla u^\varepsilon_t|^2\,dx\,dt\\ -\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt-\frac{\alpha}{2}\int_\Omega|\Delta u^\varepsilon(\cdot,t)|^2\big|_0^\tau\,dx=\int_0^\tau\int_\Omega f\Delta u^\varepsilon\,dx\,dt.\end{aligned}\tag{3.3}$$

According to the boundary condition (2.3), (3.3) becomes

$$\begin{aligned}\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt+\frac{\alpha}{2}\int_\Omega|\Delta u^\varepsilon(\cdot,\tau)|^2\,dx=\frac{\alpha}{2}\int_\Omega|\Delta u_0|^2\,dx\\ +\frac{1}{2\alpha\varepsilon}\int_0^\tau\int_\Sigma\big((u^\varepsilon(0,\cdot,\cdot))^-\big)^2\,dx'\,dt-\frac{1}{\alpha}\int_0^\tau\int_\Sigma\big(u^\varepsilon_t u^\varepsilon_{x_1}\big)(0,\cdot,\cdot)\,dx'\,dt\\ +\int_\Omega\big(u^\varepsilon_t\Delta u^\varepsilon\big)(\cdot,\tau)\,dx-\int_0^\tau\int_\Omega f\Delta u^\varepsilon\,dx\,dt\\ -\int_\Omega u_1\Delta u_0\,dx+\int_0^\tau\int_\Omega|\nabla u^\varepsilon_t|^2\,dx\,dt.\end{aligned}\tag{3.4}$$

In order to estimate the left hand side of (3.4), we organize the terms of its right hand side into different groups. The initial data terms

$$\frac{\alpha}{2}\int_\Omega|\Delta u_0|^2\,dx\quad\text{and}\quad-\int_\Omega u_1\Delta u_0\,dx$$

are bounded thanks to our assumptions on u_0 and u_1. The terms

$$\frac{1}{2\alpha\varepsilon}\int_0^\tau\int_\Sigma\big((u^\varepsilon(0,\cdot,\cdot))^-\big)^2\,dx'\,dt\quad\text{and}\quad\int_0^\tau\int_\Omega|\nabla u^\varepsilon_t|^2\,dx\,dt$$

are bounded independently of ε thanks to Lemma 3.1. We estimate the remaining terms with the help of the inequality

$$xy\le\gamma|x|^2/2+|y|^2/(2\gamma),$$

for all $\gamma>0$ and all real x and y. Therefore,

$$\begin{aligned}\int_\Omega\big(u^\varepsilon_t\Delta u^\varepsilon\big)(\cdot,\tau)\,dx&\le\frac{\gamma_1}{2}\int_\Omega|\Delta u^\varepsilon(\cdot,\tau)|^2\,dx+\frac{1}{2\gamma_1}\int_\Omega|u^\varepsilon_t(\cdot,\tau)|^2\,dx\\ \int_0^\tau\int_\Omega f\Delta u^\varepsilon\,dx\,dt&\le\frac{1}{2\gamma_2}\int_0^\tau\int_\Omega|f|^2\,dx\,dt+\frac{\gamma_2}{2}\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt\end{aligned}$$

and we will choose γ_1 and γ_2 later. The boundary term is estimated as

$$\begin{aligned}\int_0^\tau\int_\Sigma |(u_t^\varepsilon u_{x_1}^\varepsilon)(0,\cdot,\cdot)|\,dx'\,dt \le \frac{1}{2\gamma_3}\int_0^\tau\int_\Sigma |u_t^\varepsilon(0,\cdot,\cdot)|^2\,dx'\,dt\\ +\frac{\gamma_3}{2}\int_0^\tau\int_\Sigma |u_{x_1}^\varepsilon(0,\cdot,\cdot)|^2\,dx'\,dt.\end{aligned} \tag{3.5}$$

If w and w_{x_1} belong to $L^2(\Omega)$, we have the obvious estimate

$$\int_\Sigma |w(0,\cdot)|^2\,dx' \le C\int_\Omega \left(|w|^2+|w_{x_1}|^2\right)dx,$$

which we apply to the right hand side of (3.5), getting thus

$$\begin{aligned}\int_0^\tau\int_\Sigma |(u_t^\varepsilon u_{x_1}^\varepsilon)(0,\cdot,\cdot)|\,dx'\,dt \le &\frac{C}{2\gamma_3}\int_0^\tau\int_\Omega \left(|u_t^\varepsilon|^2+|u_{x_1t}|^2\right)dx\,dt\\ &+\frac{C\gamma_3}{2}\int_0^\tau\int_\Omega\left(|u_{x_1}^\varepsilon|^2+|u_{x_1x_1}^\varepsilon|^2\right)dx\,dt.\end{aligned}$$

We use now the ellipticity of Δ: there exists a constant C_1 such that for all w in $H^2(\Omega)$,

$$\int_\Omega |w_{x_1x_1}|^2\,dx \le C_1\int_\Omega\left(|w|^2+|\Delta w|^2\right)dx.$$

Now, we see that inequality (3.4) implies:

$$\begin{aligned}\int_0^\tau\int_\Omega |\Delta u^\varepsilon|^2\,dx\,dt+\frac{\alpha}{2}\int_\Omega|\Delta u^\varepsilon(\cdot,\tau)|^2\,dx \le C_0+\frac{\gamma_1}{2}\int_\Omega |\Delta u^\varepsilon(\cdot,\tau)|^2\,dx\\ +\frac{1}{2\gamma_1}\int_\Omega |u_t^\varepsilon(\cdot,\tau)|^2\,dx+\frac{1}{2\gamma_2}\int_0^\tau\int_\Omega |f|^2\,dx\,dt+\frac{\gamma_2}{2}\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt\\ +\frac{C}{2\gamma_3}\int_0^\tau\int_\Omega|u_t^\varepsilon|^2\,dx\,dt+\frac{C}{2\gamma_3}\int_0^\tau\int_\Omega|u_{x_1t}^\varepsilon|^2\,dx\,dt+\frac{C\gamma_3}{2}\int_0^\tau\int_\Omega|u_{x_1}^\varepsilon|^2\,dx\,dt\\ +\frac{CC_1\gamma_3}{2}\int_0^\tau\int_\Omega|u^\varepsilon|^2\,dx\,dt+\frac{CC_1\gamma_3}{2}\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt.\end{aligned} \tag{3.6}$$

We can choose now the γ_i's: it suffice to have the inequalities

$$\gamma_1<\alpha \quad\text{and}\quad \gamma_2/2+CC_1\gamma_3/2<1,$$

and the conclusion is clear. □

REMARK 3.5. *If we suppose that f vanishes for $t \ge T$, then, independently of $\varepsilon>0$, we have the estimate*

$$\int_0^\tau\int_\Omega|\Delta u^\varepsilon|^2\,dx\,dt \le C(1+\tau).$$

This property is proved by moving the origin of times to T, and by looking carefully at (3.6) *with the help of Remark 3.2.*

Let us turn now to interior estimates.

LEMMA 3.6. *Assume the hypotheses of 3.4. Then for all $\beta>0$, u_{tt}^ε and Δu_t^ε are bounded in the space $L^2_{\mathrm{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$, independently of $\varepsilon>0$.*

PROOF. The idea of the proof is twofold: first, we multiply u^ε by a truncation function $\varphi\in C_0^\infty(\mathbb{R})$, and we define $v^\varepsilon=\varphi u^\varepsilon$; we observe that $w^\varepsilon=v_t^\varepsilon$ satisfies a heat equation, whose right hand side can be estimated thanks to the previous lemmas. Let us go now into details. We multiply u^ε by a truncation function

$$\varphi(x)=\begin{cases}1 & \text{if} \quad x\le-\beta,\\ 0 & \text{if}\quad x\ge-\beta/2,\end{cases} \tag{3.7}$$

which enables us to forget about the strongly non linear boundary conditions. Define

$$v^\varepsilon = \varphi(x_1)u^\varepsilon. \tag{3.8}$$

The derivatives of v^ε are given by:

$$v^\varepsilon_{tt} = \varphi u^\varepsilon_{tt}, \tag{3.9a}$$

$$\Delta v^\varepsilon = \varphi \Delta u^\varepsilon + 2\varphi_{x_1}\nabla u^\varepsilon + \varphi_{x_1x_1}u^\varepsilon, \tag{3.9b}$$

$$\Delta v^\varepsilon_t = \varphi \Delta u^\varepsilon_t + 2\varphi_{x_1}\nabla u^\varepsilon_t + \varphi_{x_1x_1}u^\varepsilon_t. \tag{3.9c}$$

Notice that thanks to relations (2.1) and (3.9), we have

$$v^\varepsilon_{tt} - \Delta v^\varepsilon - \alpha \Delta v^\varepsilon_t = \widetilde{g}^\varepsilon, \tag{3.10}$$

where

$$\widetilde{g}^\varepsilon = \varphi f - 2\varphi_{x_1}(x_1)\big(\nabla u^\varepsilon + \alpha \nabla u^\varepsilon_t\big) - \varphi_{x_1x_1}(x_1)\big(u^\varepsilon + \alpha u^\varepsilon_t\big).$$

Since f, u^ε_t, ∇u^ε, ∇u^ε_t and u^ε are bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, $\widetilde{g}^\varepsilon$ is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. Let us define

$$w^\varepsilon = v^\varepsilon_t \quad \text{and} \quad g^\varepsilon = \widetilde{g}^\varepsilon + \Delta v^\varepsilon. \tag{3.11}$$

Substituting (3.11) in (3.10), we obtain

$$w^\varepsilon_t - \alpha \Delta w^\varepsilon = g^\varepsilon. \tag{3.12}$$

Let us prove now that w^ε_t is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. For this purpose, we multiply (3.12) by w^ε_t; we integrate this expression over Ω,

$$\int_\Omega |w^\varepsilon_t|^2\,dx - \alpha \int_\Omega \Delta w^\varepsilon w^\varepsilon_t\,dx = \int_\Omega g^\varepsilon\, w^\varepsilon_t\,dx.$$

We use Green's formula in the second term on the left hand side of the above expression, getting thus the following inequality:

$$\int_\Omega |w^\varepsilon_t|^2\,dx + \alpha \int_\Omega \nabla w^\varepsilon_t \nabla w^\varepsilon\,dx = \int_\Omega g^\varepsilon w^\varepsilon_t\,dx. \tag{3.13}$$

We integrate (3.13) over $(0,\tau)$ and we observe that the product $|g^\varepsilon w^\varepsilon_t|$ can be estimated by $|g^\varepsilon|^2/2 + |w^\varepsilon_t|^2/2$ and we obtain

$$\begin{aligned}&\int_0^\tau \int_\Omega |w^\varepsilon_t|^2\,dx\,dt + \alpha \int_\Omega |\nabla w^\varepsilon(\cdot,\tau)|^2\,dx \\ &\le \alpha \int_\Omega |\nabla w^\varepsilon(\cdot,0)|^2\,dx + \int_0^\tau \int_\Omega |g^\varepsilon|^2\,dx\,dt.\end{aligned} \tag{3.14}$$

Since u_1 belongs to $H^1(\Omega)$ and φ belongs to $C_0^\infty(\mathbb{R})$, $\nabla w^\varepsilon(\cdot,0) = \varphi_{x_1}u_1 + \varphi \nabla u_1$ is bounded in $L^2(\Omega)$. Moreover g^ε is bounded in the space $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$ because Δv^ε and $\widetilde{g}^\varepsilon$ are bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. Therefore (3.8), (3.11) and (3.14) enable us to deduce that u^ε_{tt} is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$. We use analogous arguments to show that Δu^ε_t is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$. We multiply (3.12) by Δw^ε, we integrate over $\Omega\times(0,\tau)$ and thanks to Green's formula, we obtain

$$-\frac{1}{2}\int_\Omega |\nabla w^\varepsilon|^2\Big|_0^\tau\,dx - \alpha \int_0^\tau \int_\Omega |\Delta w^\varepsilon|^2\,dx\,dt = \int_0^\tau \int_\Omega g^\varepsilon \Delta w^\varepsilon\,dx\,dt. \tag{3.15}$$

Therefore the product $|g^\varepsilon \Delta w^\varepsilon|$ can be estimated by $|g^\varepsilon|^2/2\gamma + \gamma|\Delta w^\varepsilon|^2/2$, and if we choose $\gamma \in (0,2\alpha)$, we obtain the following inequality:

$$\Big(\alpha - \frac{\gamma}{2}\Big)\int_0^\tau \int_\Omega |\Delta w^\varepsilon|^2\,dx\,dt \le \frac{1}{2\gamma}\int_0^\tau \int_\Omega |g^\varepsilon|^2\,dx\,dt + \frac{1}{2}\int_\Omega |\nabla w^\varepsilon(\cdot,0)|^2\,dx. \tag{3.16}$$

Since g^ε and $\nabla w^\varepsilon(\cdot,0)$ are respectively bounded in $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$ and $L^2(\Omega)$, according to (3.8), (3.11) and (3.16), we infer that Δu^ε_t is bounded in the space $L^2_{\text{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$. □

4. Existence of a weak solution

In this Section, we shall show that it is possible to pass to the limit in the variational formulation of the penalized problem, and to obtain a weak solution of (1.1)-(1.3). There is a small subtlety due to unboundedness of Ω.

THEOREM 4.1. *Assume the hypotheses of Lemma 3.4. Then there exists a solution of the variational inequality* (1.4)*; this solution can be obtained as a limit of a subsequence of the penalty approximation defined by* (2.1)-(2.3).

PROOF. Let v belong to K, and let φ a function belonging to $C_0^\infty\big(\bar\Omega\times[0,\infty)\big)$ which takes its values in $[0,1]$. Multiplying (2.1) by $(v-u^\varepsilon)\varphi$ and integrating over $(0,\tau)\times\Omega$ and then observing that

$$\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma\big((u^\varepsilon)^-\varphi(v-u^\varepsilon)\big)(0,\cdot,\cdot)\,dx'\,dt$$
$$=\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma\big(\big((u^\varepsilon)^-\big)^2\varphi\big)(0,\cdot,\cdot)\,dx'\,dt+\frac{1}{\varepsilon}\int_0^\tau\int_\Sigma\big((u^\varepsilon)^-\varphi v\big)(0,\cdot,\cdot)\,dx'\,dt$$

is non negative, we may deduce the following inequality:

$$\begin{aligned}&\int_\Omega u^\varepsilon_t\varphi(v-u^\varepsilon)\big|_0^\tau\,dx-\int_0^\tau\int_\Omega u^\varepsilon_t\big(\varphi(v-u^\varepsilon)\big)_t\,dx\,dt\\&+\int_0^\tau\int_\Omega\big(\nabla u^\varepsilon+\alpha\nabla u^\varepsilon_t\big)\nabla\big(\varphi(v-u^\varepsilon)\big)dx\,dt\ge\int_0^\tau\int_\Omega f\varphi(v-u^\varepsilon)\,dx\,dt.\end{aligned}\tag{4.1}$$

We infer from Lemmas 3.1, 3.3 and 3.4 that it is possible to extract a subsequence, still denoted by u^ε, such that

$$u^\varepsilon\rightharpoonup u\quad\text{in}\quad L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak }*,\tag{4.2a}$$

$$u^\varepsilon_t\rightharpoonup u_t\quad\text{in}\quad L^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak }*,\tag{4.2b}$$

$$\nabla u^\varepsilon\rightharpoonup\nabla u\quad\text{in}\quad L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak }*,\tag{4.2c}$$

$$\Delta u^\varepsilon\rightharpoonup\Delta u\quad\text{in}\quad L^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak }*,\tag{4.2d}$$

$$\nabla u^\varepsilon_t\rightharpoonup\nabla u_t\quad\text{in}\quad L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak }*.\tag{4.2e}$$

Thanks to the classical compactness properties of injection of Sobolev spaces on bounded open sets, we see for all $R>0$, the restrictions of u^ε and ∇u^ε to Q_R converge strongly to their respective limits in $L^2(Q_R)$; therefore, we can pass to the limit in all the terms of (4.1) except possibly the first two terms.

Let us prove that u_t is continuous from $[0,\infty)$ to $L^2(\Omega)$ equipped with the weak topology: we infer from the estimates of Lemma 3.6 that for all $\beta>0$, u^ε_{tt} restricted to $x_1<-\beta$ is bounded in $L^2_{\text{loc}}\big([0,\infty);L^2\big((-\infty,-\beta]\times\Sigma\big)\big)$; therefore it is plain that u^ε_t converges to a function u_t whose restriction to $x_1<-\beta$ is continuous from $[0,\infty)$ to $L^2\big((-\infty,-\beta]\times\Sigma\big)$. Let $t_j\in[0,\infty)$ be a sequence converging to $t_\infty<\infty$; as u_t belongs to $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, we may extract a subsequence, still denoted by t_j, such that

$$u^\varepsilon_t(\cdot,t_j)\rightharpoonup z\quad\text{in}\quad L^2(\Omega)\quad\text{weak.}$$

But since for all $\beta>0$,

$$u^\varepsilon_t(\cdot,t_j)1_{\{x_1<-\beta\}}\to u_t(\cdot,t_\infty)1_{\{x_1<-\beta\}}\quad\text{in}\quad L^2(\Omega)\quad\text{weak,}$$

we see that z must coincide with $u_t(\cdot,t_\infty)$, and that all the sequence converges strongly to $u_t(\cdot,t_\infty)$; this proves that u_t is continuous from $[0,\infty)$ to $L^2(\Omega)$ weak.

Let us prove now that $u_t^\varepsilon(\cdot,t)$ converges weakly to $u_t(\cdot,t)$ for all $t>0$: let γ be an arbitrary positive number; let z belong to $L^2(\Omega)$; denote by C_1 is an upper bound for $\left|u_t^\varepsilon\right|_{L^\infty([0,T];L^2(\Omega))}$ with T fixed. We choose β so small that

$$\left(\int_{-\beta<x_1<0}|z|^2\,dx\right)^{1/2}\le\frac{\gamma}{4C_1};$$

then, for $t\in[0,T]$

$$\begin{aligned}\left|\int_\Omega\big(u_t^\varepsilon(\cdot,t)-u_t(\cdot,t)\big)z\,dx\right|&\le\left|\int_{x_1<-\beta}\big(u_t^\varepsilon(\cdot,t)-u_t(\cdot,t)\big)z\,dx\right|\\&+\left(\int_{-\beta<x_1<0}|z|^2\,dx\right)^{1/2}\left(\int_{-\beta<x_1<0}\left|u_t^\varepsilon(\cdot,t)-u_t(\cdot,t)\right|^2dx\right)^{1/2}.\end{aligned}\tag{4.3}$$

By definition of C_1, the second term on the right hand side of (4.3) is estimated by $C_1\gamma/4C_1=\gamma/2$. As $u_t^\varepsilon\big|_{(-\infty,-\beta)\times\Sigma\times(0,T)}$ is bounded in $H^1\big((-\infty,-\beta)\times\Sigma\times(0,T)\big)$, we see that

$$\int_{-\infty}^{-\beta}\int_\Sigma u_t^\varepsilon z\,dx\quad\text{converges}\quad\text{to}\quad\int_{-\infty}^{-\beta}\int_\Sigma u_tz\,dx$$

uniformly with respect to $t\in[0,T]$. It suffices therefore to choose ε so small that the first term on the right hand side of (4.3) is estimated by $\gamma/2$. This proves that the convergence of $\int_\Omega u_t^\varepsilon z\,dx$ to $\int_\Omega u_tz\,dx$ is uniform on compact sets in time. In particular, as ε tends to 0, it is plain that for all $\tau>0$,

$$\int_\Omega u_t^\varepsilon\varphi(v-u^\varepsilon)\,dx\to\int_\Omega u_t\varphi(v-u)\,dx.$$

Let us turn now to the term

$$\int_0^\tau\int_\Omega u_t^\varepsilon\big(\varphi_t(v-u^\varepsilon)+\varphi(v_t-u_t^\varepsilon)\big)\,dx\,dt.$$

It is plain that

$$\int_0^\tau\int_\Omega u_t^\varepsilon\big(\varphi_t(v-u^\varepsilon)+\varphi v_t\big)\,dx\,dt\to\int_0^\tau\int_\Omega u_t\big(\varphi_t(v-u)+\varphi v_t\big)\,dx\,dt.$$

There remains to prove the convergence

$$\int_0^\tau\int_\Omega|u_t^\varepsilon|^2\varphi\,dx\,dt\to\int_0^\tau\int_\Omega|u_t|^2\varphi\,dx\,dt.$$

We observe that

$$\begin{aligned}&\int_0^\tau\int_\Omega\left|u_t^\varepsilon-u_t\right|^2\varphi\,dx\,dt\\\le&\int_0^\tau\int_{x_1\le-\beta}\left|u_t^\varepsilon-u_t\right|^2\varphi\,dx\,dt+\int_0^\tau\int_{-\beta\le x_1\le0}\left|u_t^\varepsilon-u_t\right|^2\varphi\,dx\,dt.\end{aligned}$$

Let γ be any positive number. We infer from the estimates over $\left|u_t^\varepsilon\right|_{L^2(\Omega\times(0,\tau))}$ and $\left|\nabla u_t^\varepsilon\right|_{L^2(\Omega\times(0,\tau))}$ that there exists a constant C_1 independent from ε such that

$$\left|u^\varepsilon(x_1,\cdot,\cdot)\right|_{L^2(\Sigma\times(0,\tau))}\le C_1.$$

Therefore,

$$\int_0^\tau\int_{-\beta\le x_1\le0}\left|u_t^\varepsilon-u_t\right|^2\varphi\,dx\,dt\le C_1^2\beta.$$

We choose β so small that

$$C_1^2\beta\le\gamma/2;$$

then we know from the estimates of Lemmas 3.4 and 3.6 that the restriction of u^ε to $\{x_1 < -\beta\}$ intersected with a ball containing the support of φ is bounded in H^2 of that set; therefore, for ε small enough,

$$\int_0^\tau \int_\Omega \left|u_t^\varepsilon - u_t\right|^2 \varphi \, dx \, dt \le \frac{\gamma}{2},$$

and the convergence of the first two terms of (4.1) is proved.

We observe now that since u, u_t, ∇u and ∇u_t belong to $L^2\big([0,\infty); L^2(\Omega)\big)$, we may replace φ by φ_R in the variational inequality where φ_R is equal to 1 over the set Q_R and vanishes outside of Q_{R+1}. It is plain that as $R \to \infty$ all the terms in (4.1) converge to their limit; thus we have proved the existence of the desired weak solution. □

REMARK 4.2. *Nothing is known about uniqueness.*

5. Auxiliary results on the damped wave equation with Dirichlet boundary conditions

LEMMA 5.1. *Assume u_0 belongs to $H^{5/2}(\Omega)$; then, there exists a function $z \in H^3\big(\Omega \times [0,\infty)\big)$ with compact support in t such that the trace of z on $\Omega \times \{0\}$ is equal to u_0.*

PROOF. We extend u_0 into a function belonging to $H^{5/2}(\mathbb{R}^d)$: as the boundary of Ω is smooth, this is a consequence of classical results on Sobolev spaces. Then there exists a function Z belonging to $H^3\big(\mathbb{R}^d \times [0,\infty)\big)$ whose trace is u_0. It suffices now to select a cutoff function $\varphi \in C^\infty\big([0,\infty)\big)$ which is equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$, and to define z as the restriction of φZ to $\Omega \times [0,\infty)$. □

LEMMA 5.2. *Assume u_0 belongs to $H^{5/2}(\Omega)$, u_1 belongs to $H^1(\Omega)$ and f belongs to $L^2_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$. Define z as in Lemma 5.1 and let $\bar u$ be the solution of*

$$\bar u_{tt} - \Delta \bar u - \alpha \Delta \bar u_t = f, \quad x \in \Omega, \quad t > 0,$$

with the initial data

$$\bar u(\cdot, 0) = u_0 \quad \text{and} \quad \bar u_t(\cdot, 0) = u_1,$$

and boundary condition

$$\bar u(0, \cdot, \cdot) = z(0, \cdot, \cdot).$$

Then the trace $\bar g = -(\bar u_{x_1} + \alpha \bar u_{x_1 t})(0, \cdot, \cdot)$ is well defined and belongs to the space $L^2_{\text{loc}}\big([0,\infty); L^2(\Sigma)\big)$. Moreover, if f is compactly supported in time,

$$\int_0^\tau |\bar g(\cdot, t)|^2_{L^2(\Sigma)} \, dt$$

increases at most polynomially.

PROOF. The function $\zeta = \bar u - z$ satisfies the equation

$$\zeta_{tt} - \Delta \zeta - \alpha \Delta \zeta_t = F, \quad x \in \Omega, \quad t > 0, \tag{5.1}$$

where $F = f - z_{tt} + \Delta z + \alpha \Delta z_t$, with the initial data

$$\zeta(\cdot, 0) = 0 \quad \text{and} \quad \zeta_t(\cdot, 0) = u_1,$$

and the boundary condition

$$\zeta(0, \cdot, \cdot) = 0. \tag{5.2}$$

We multiply (5.1) by ζ_t and integrate over $\Omega \times (0, \tau)$, if we suppose that f is compactly supported in time, we remark that F is also compactly supported in time. Then ζ_t and $\nabla \zeta$ are bounded in $L^\infty_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$ and $\nabla \zeta_t$ is bounded in

$L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. In order to get more information, we multiply (5.1) by $\Delta\zeta_t$; we observe that

$$\int_0^\tau\int_\Omega \zeta_{tt}\Delta\zeta_t\,dx\,dt = \int_0^\tau\int_\Sigma \zeta_t\zeta_{x_1t}\,dx'\,dt - \int_0^\tau\int_\Omega \nabla\zeta_{tt}\nabla\zeta_t\,dx\,dt,$$

and since the boundary term vanishes thanks to (5.2), we have

$$\int_0^\tau\int_\Omega \zeta_{tt}\Delta\zeta_t\,dx\,dt = \frac{1}{2}\int_\Omega |\nabla\zeta_t(\cdot,0)|^2\,dx - \frac{1}{2}\int_\Omega |\nabla\zeta_t(\cdot,\tau)|^2\,dx.$$

Therefore, we have the identity

$$\alpha\int_0^\tau\int_\Omega |\Delta\zeta_t|^2\,dx\,dt + \frac{1}{2}\int_\Omega |\Delta\zeta(\cdot,\tau)|^2\,dx + \int_\Omega |\nabla\zeta_t(\cdot,\tau)|^2\,dx$$
$$= \int_\Omega |\nabla\zeta_t(\cdot,0)|^2\,dx - \int_0^\tau\int_\Omega F\Delta\zeta_t\,dx\,dt.$$

We infer from the inequality

$$F\Delta\zeta_t \le \frac{\alpha}{2}|\Delta\zeta_t|^2 + \frac{|F|^2}{2\alpha},$$

that $\Delta\zeta_t$ is bounded in the space $L^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$ and $\Delta\zeta$ and $\nabla\zeta_t$ are bounded in the space $L^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$. In particular, if the support in time of F is bounded, $\Delta\zeta_t$ is bounded in $L^2\big([0,\infty);L^2(\Omega)\big)$ and $\Delta\zeta$ and $\nabla\zeta_t$ are bounded in $L^\infty\big([0,\infty);L^2(\Omega)\big)$. Therefore ζ_{x_1t} belongs to $L^2_{\text{loc}}\big([0,\infty);H^{1/2}(\Sigma)\big)$ and ζ_{x_1} belongs to $L^\infty_{\text{loc}}\big([0,\infty);H^{1/2}(\Sigma)\big)$, and if the support in time of f is bounded, the local feature of these spaces may be removed. □

6. Regularity of the trace

We characterize the trace spaces using the Fourier analysis and we prove that u is a strong solution of (1.1)-(1.3). Here, we mean by strong solution that all the traces can be defined. Let ν be a positive number. Denote by $v^\varepsilon = \exp(-\nu t)(u^\varepsilon - \bar u)$ a solution of

(6.1a) $\quad \big(\nu+\partial/\partial t\big)^2 v^\varepsilon - \big(1+\alpha\big(\nu+\partial/\partial t\big)\big)\Delta v^\varepsilon = 0, \quad x\in\Omega, \quad t>0,$

(6.1b) $\quad \big(1+\alpha\big(\nu+\partial/\partial t\big)\big)v^\varepsilon_{x_1}(0,\cdot,\cdot) = e^{-\nu t}\bar g - \big(v^\varepsilon(0,\cdot,\cdot)+e^{-\nu t}\bar u(0,\cdot,\cdot)\big)^-/\varepsilon,$

(6.1c) $\quad v^\varepsilon(\cdot,t)=0 \quad\text{and}\quad v^\varepsilon_t(\cdot,t)=0.$

We apply a partial Fourier transform in the tangential variable to (6.1a), calling ω the dual variable to t and ξ the dual variable to x'; we obtain the following differential equation:

$$\widehat{v}^\varepsilon_{x_1x_1} = \left(|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}\right)\widehat{v}^\varepsilon. \tag{6.2}$$

Define $\widehat\lambda$ to be the causal determination of the square root of $|\xi|^2+(\nu+i\omega)^2/(1+\alpha(\nu+i\omega))$:

$$\widehat\lambda(\xi,\omega) = \sqrt{|\xi|^2 + \frac{(\nu+i\omega)^2}{1+\alpha(\nu+i\omega)}};$$

thus $\widehat\lambda$ is holomorphic in the lower half-plane $\Im(\omega)<0$ and $\Re\widehat\lambda \ge 0$ for $\Im(\omega)=0$. The general solution of (6.2) is given by

$$\widehat{v}^\varepsilon(x_1,\xi,\omega) = \widehat{a}^\varepsilon e^{\widehat\lambda x_1} + \widehat{b}^\varepsilon e^{-\widehat\lambda x_1}; \tag{6.3}$$

since we performed a Fourier transform on v^ε, we assumed implicitly that v^ε and $\widehat{v}^\varepsilon$ are tempered respectively in (x',t) and (ξ,ω).

The term $\widehat{b}^\varepsilon e^{-\widehat{\lambda}x_1}$ can be tempered only if $\widehat{b}^\varepsilon$ decays at infinity very fast, and since this must be true for all x_1, it implies that $\widehat{b}^\varepsilon$ vanishes, the proof is similar as this one given in [**3**], we get

$$\widehat{v}^\varepsilon(x_1, \cdot, \cdot) = \widehat{a}^\varepsilon e^{\widehat{\lambda}x_1}.$$

In particular

$$\big((1+\alpha(\nu+\partial/\partial t))v^\varepsilon_{x_1}\big)\widehat{\ }(0,\xi,\omega) = \widehat{\lambda}_1\widehat{v}^\varepsilon(0,\xi,\omega), \tag{6.4}$$

where $\widehat{\lambda}_1 = (1+\alpha(\nu+i\omega))\widehat{\lambda}$. Define

$$g(x',t) = e^{-\nu t}\bar{g}(x',t) \quad \text{and} \quad h(x',t) = e^{-\nu t}\bar{u}(0,x',t)$$

If we let $w^\varepsilon(x',t)$ be the trace $v^\varepsilon(0,x',t)$, (6.1) can be written now

$$\lambda_1 * w^\varepsilon = g + \big(w^\varepsilon + h\big)^-/\varepsilon, \tag{6.5}$$

where w^ε vanishes for all $t \leq 0$.

REMARK 6.1. *We observe that if $\widehat{\lambda}$ vanishes then $\Im(\omega) > 0$. Therefore it is clear that $\widehat{\lambda}$ is a holomorphic function in $\Im(\omega) < 0$ and next we may deduce that λ_1 is a causal distribution.*

LEMMA 6.2. *Let w^ε be the solution of* (6.5). *Then we may extract a subsequence, still denoted by w^ε such that*

$$w^\varepsilon \rightharpoonup w \quad \text{weakly} \quad \text{in} \quad H^{5/4}\big([0,\infty); L^2(\Sigma)\big) \cap H^1\big([0,\infty); H^{1/2}(\Sigma)\big).$$

PROOF. Formally, we multiply (6.5) by $\alpha\big(\nu w^\varepsilon + w^\varepsilon_t\big) + w^\varepsilon$, and we estimate the pseudodifferential term in the Fourier variable, we obtain

$$\begin{aligned} &\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{\lambda}_1\widehat{w}^\varepsilon\,\overline{\big(1+\alpha(\nu+i\omega)\big)\widehat{w}^\varepsilon}\,d\omega\,d\xi \\ &= \frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\,\overline{\big(1+\alpha(\nu+i\omega)\big)\widehat{w}^\varepsilon}\,d\omega\,d\xi \\ &+\frac{1}{\varepsilon}\int_0^\infty\int_{\mathbb{R}^{d-1}}\big(w^\varepsilon+h\big)^-\big(1+\alpha(\nu+\partial/\partial t)\big)w^\varepsilon\,dx'\,dt. \end{aligned} \tag{6.6}$$

Since $(u^\varepsilon(0,\cdot,\cdot))^-/\sqrt{\varepsilon}$ is bounded in the space $L^\infty_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)$, the absolute value of the second integral in the right hand side of (6.6) is bounded and we infer that

$$\Re\int_{\mathbb{R}^d}\widehat{\lambda}_1|\widehat{w}^\varepsilon|^2\overline{\big(1+\alpha(\nu+i\omega)\big)}\,d\omega\,d\xi \leq C_1 + \Re\int_{\mathbb{R}^d}\widehat{g}\,\overline{\big(1+\alpha(\nu+i\omega)\big)\widehat{w}^\varepsilon}\,d\omega\,d\xi. \tag{6.7}$$

On the other hand, it is plain that

$$\Re\widehat{\lambda}^2 = |\xi|^2 + \frac{\nu^2(1+\alpha\nu)+(-1+\alpha\nu)\omega^2}{|1+\alpha(\nu+i\omega)|^2} \quad \text{and} \quad \Im\widehat{\lambda}^2 = \frac{2\nu\omega+\alpha\omega(\nu^2+\omega^2)}{|1+\alpha(\nu+i\omega)|^2}.$$

We may choose ν such that $\nu\alpha = 1$, we get

$$\Re\widehat{\lambda}^2 = |\xi|^2 + \frac{2}{\alpha^2|2+i\alpha\omega|^2} \quad \text{and} \quad \Im\widehat{\lambda}^2 = \frac{\omega(3+\alpha^2\omega^2)}{\alpha|2+i\alpha\omega|^2}. \tag{6.8}$$

Therefore we infer that

$$\arg\widehat{\lambda} = \frac{1}{2}\arctan\left(\frac{|\xi|^2|2+i\alpha\omega|^2+2}{\alpha\omega(3+\alpha^2\omega^2)}\right).$$

According to (6.8), it is plain that $\arg\widehat{\lambda}$ belongs to $[0,\pi/4]$ and since $\widehat{\lambda}$ is never equal to zero, we get for $|\xi|+|\omega| \gg 1$ the following inequality:

$$\Re\widehat{\lambda} \geq C\big(1+|\xi|+\sqrt{|\omega|}\big). \tag{6.9}$$

Therefore, we obtain

$$C\int_{\mathbb{R}^d}|2+i\alpha\omega|^2\big(1+|\xi|+\sqrt{|\omega|}\big)|\widehat{w}^\varepsilon|^2\,d\omega\,d\xi\le C_1+\int_{\mathbb{R}^d}|2+i\alpha\omega||\widehat{g}||\widehat{w}^\varepsilon|\,d\omega\,d\xi.$$

We estimate the product $|zy|$ by $|z|^2/(2\gamma)+\gamma|y|^2/2$, $\gamma>0$, we see that

$$\begin{aligned}\Big(C-\frac{\gamma}{2}\Big)\int_{\mathbb{R}^d}&|2+i\alpha\omega|^2\big(1+|\xi|+\sqrt{|\omega|}\big)|\widehat{w}^\varepsilon|^2\,d\omega\,d\xi\\ &\le C_1+\frac{1}{2\gamma}\int_{\mathbb{R}^d}\frac{|\widehat{g}|^2}{1+|\xi|+\sqrt{|\omega|}}\,d\omega\,d\xi.\end{aligned}\tag{6.10}$$

We choose γ such that $\gamma<2C$, since g belongs to $L^2\big([0,\infty);H^{1/2}(\mathbb{R}^{d-1})\big)$ then the right hand side of (6.10) is bounded independently of ε and the conclusion is clear. □

THEOREM 6.3. *Let $u^\varepsilon\big|_\Sigma$ be the trace of u^ε which is the solution of* (2.1)-(2.3). *Then we may extract a subsequence, still denoted by $u^\varepsilon\big|_\Sigma$ such that*

$$u^\varepsilon\big|_\Sigma\rightharpoonup u\big|_\Sigma\quad\text{weakly}\quad\text{in}\quad H^{5/4}_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big).$$

PROOF. Since w^ε is bounded in $H^{5/4}\big([0,\infty);L^2(\Sigma)\big)\cap H^1\big([0,\infty);H^{1/2}(\Sigma)\big)$, we may deduce that $v^\varepsilon(0,\cdot,\cdot)=w^\varepsilon e^{-\nu t}$ is bounded in the space $H^{5/4}_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)\cap H^1_{\text{loc}}\big([0,\infty);H^{1/2}(\Sigma)\big)$. On the other hand, using the Lemma 5.2, we may deduce that $\bar{u}$ is bounded in $H^{5/4}_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)$ because $\bar{u}=\zeta-z$, $z\in H^3\big(\Omega\times[0,\infty)\big)$ and $\zeta(0,\cdot,\cdot)$ is bounded in $H^{5/4}_{\text{loc}}\big([0,\infty);L^2(\Sigma)\big)$. Moreover it is clear that $(u_{x_1}+\alpha u_{x_1t})(0,\cdot,\cdot)$ is bounded in $H^{-1/4}\big([0,\infty);L^2(\Sigma)\big)\cap H^1\big([0,\infty);H^{-1/2}(\Sigma)\big)$.
□

REMARK 6.4. *Jiři Jarušek has treated a more general and complicated problem in* [**1**], *since it includes possibly nonlinear constitutive laws in viscoelasticity, contact and given friction at the boundary; however, his result do not define a trace* $(u_{x_1}+\alpha u_{x_1t})(0,\cdot,\cdot)$.

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CHAPITRE 7

Viscoelastodynamic with Signorini conditions

Adrien Petrov and Michelle Schatzman

Abstract. In this paper, we consider the evolution of Kelvin-Voigt material:

$$\rho\ddot{u}_i = \partial_j \sigma^0_{ij}(u) + \partial_j \sigma^1_{ij}(\dot{u}) + f_i, \quad x \in (-\infty, 0] \times \mathbb{R}^{d-1} = \Omega, \quad t > 0,$$

with boundary conditions

$$\sigma^0_{12}(u) + \sigma^1_{12}(\dot{u}) = 0 \quad \text{and} \quad \sigma^0_{13}(u) + \sigma^1_{13}(\dot{u}) = 0,$$
$$u_1 \leq 0, \quad \sigma^0_{11}(u) + \sigma^1_{11}(\dot{u}) \leq 0, \quad u_1\big(\sigma^0_{11}(u) + \sigma^1_{11}(\dot{u})\big) = 0,$$

and the initial position and velocity are respectively $u_0 \in H^{5/2}(\Omega)^3$ and $v_0 \in H^{3/2}(\Omega)^3$. A weak solution is obtained as the limit of penalized problem and the functional space of the trace is precisely identified.

1. Introduction

In this article, we consider the evolution of a Kelvin-Voigt material [**1**] occupying a half-space, satisfying Signorini conditions at the boundary and Cauchy data at $t = 0$, under the assumption of small deformations; we will show that there exists a weak solution of this problem, and that under regularity assumptions on the data, the solution we construct through the penalty approximation has traces of large enough order to be a strong solution.

The mathematical model is written classically as follows: let $u(x,t) \in \mathbb{R}^3$ be the displacement at time t of the material point which was at x initially; We use the notation $\dot{u} = \partial u/\partial t$, $\ddot{u} = \partial^2 u/\partial t^2$, $\partial_j u = \partial u/\partial x_j$. The spatial domain is $\Omega = (-\infty, 0) \times \mathbb{R}^{d-1}$ and $\Sigma = \{0\} \times \mathbb{R}^{d-1}$. The spatial coordinate is $x = (x_1, x') \in [-\infty, 0) \times \mathbb{R}^{d-1}$. The strain tensor is $\varepsilon_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2$; we are given two Hooke tensors, a^n_{ijkl}, $n = 0, 1$, and we define the respective stress tensors σ^n_{ij}:

$$\sigma^n_{ij}(u) = a^n_{ijkl}\varepsilon_{kl}(u); \tag{1.1}$$

here, we have used the summation convention on repeated indices.

The displacement field u satisfies the system

$$\rho\ddot{u}_i = \partial_j \sigma^0_{ij}(u) + \partial_j \sigma^1_{ij}(\dot{u}) + f_i, \quad x \in \Omega, \quad t > 0. \tag{1.2}$$

Initial data are given by

$$u(\cdot, 0) = u_0 \quad \text{and} \quad \dot{u}(\cdot, 0) = v_0. \tag{1.3}$$

The components of the unit exterior normal are δ_{1j}, and a basis of tangential vectors can be taken as $\tau_j = \delta_{2j}$ and $\tau'_j = \delta_{3j}$. Therefore, the boundary conditions are written

$$\sigma^0_{12}(u) + \sigma^1_{12}(\dot{u}) = 0, \tag{1.4a}$$

$$\sigma^0_{13}(u) + \sigma^1_{13}(\dot{u}) = 0, \tag{1.4b}$$

$$0 \geq u_1 \perp \sigma^0_{11}(u) + \sigma^1_{11}(\dot{u}) \leq 0. \tag{1.4c}$$

Here the orthogonality has the natural meaning: an appropriate duality product between two terms of relation vanishes. It is in fact one of the main results of this article to give a precise sense to (1.4c), and to justify the duality which is used in this assertion.

In order to simplify the problem, we have considered a homogeneous and isotropic material; then, the Hooke tensor a^n_{ijkl} is defined with the help of Lamé constants λ^n and μ^n:

$$a^n_{ijkl} = \lambda^n \delta_{ij}\delta_{kl} + 2\mu^n \delta_{ik}\delta_{jl}$$

for $n = 0, 1$. It is convenient to define two elasticity operators A^n by

$$A^n u = \partial_j a^n_{ijkl}\varepsilon_{kl}(u).$$

With these notations, (1.2)-(1.4) can be rewritten as

(1.5a) $\rho\ddot{u} - A^0 u - A^1 \dot{u} = f, \quad x \in \Omega, \quad t \geq 0,$

(1.5b) $\sigma^0_{12}(u) + \sigma^1_{12}(\dot{u}) = 0 \quad \text{and} \quad \sigma^0_{13}(u) + \sigma^1_{13}(\dot{u}) = 0 \text{ on } \Sigma \times [0, \infty),$

(1.5c) $0 \geq u_1\big|_{\Sigma\times[0,\infty)} \perp \big(\sigma^0_{11}(u) + \sigma^1_{11}(\dot{u})\big)\big|_{\Sigma\times[0,\infty)} \leq 0,$

(1.5d) $u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x) \text{ for } x \in \Omega.$

Let us describe now the functional hypotheses on the data; if X is a space of scalar functions, the bold-face notation $\mathbf{X}$ denotes systematically the space X^d. for the final result, we require u_0 to belong to $\mathbf{H}^{5/2}(\Omega)$, v_0 to $\mathbf{H}^{3/2}(\Omega)$ and f to $H^1_{\text{loc}}([0, \infty); \mathbf{L}^2(\Omega))$. The initial data must satisfy the compatibility condition $(u_0)_1(0, x') \leq 0$ for all $x' \in \Sigma$. Let K be the convex set defined by:

$$K = \big\{v \in \mathbf{H}^1(\Omega \times (0, T)) : \nabla v_t \in \mathbf{L}^2(0, T; L^2(\Omega)), v(0, \cdot) \leq 0\big\}.$$

We need the two bilinear forms defined by

$$a^0(u, v) = \int_\Omega a^0_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,dx \quad \text{and} \quad a^1(u, v) = \int_\Omega a^1_{ijkl}\varepsilon_{ij}(u)\varepsilon_{kl}(v)\,dx.$$

We obtain a weak formulation of the problem (1.5) as follows: we multiply (1.5a) by $v - u$, $v \in K$ and we integrate formally the result over $\Omega \times (0, T)$; we obtain then the variational inequality:

$$(1.6)\qquad \begin{cases} u \in K \text{ and for all } v \in K \text{ and for every } \tau \in [0, T], \\ \displaystyle\int_0^\tau \int_\Omega \rho\ddot{u}\cdot(v - u)\,dx\,dt + \int_0^\tau a^0(u, u - v)\,dt \\ \displaystyle\quad + \int_0^\tau a^1(\dot{u}, v - u)\,dt \geq \int_0^\tau \int_\Omega f\cdot(v - u)\,dx\,dt. \end{cases}$$

The equivalence between the weak formulation (1.6) and the strong formulation (1.5) is not obvious; it depends on precise information on the trace of $\partial_1\dot{u}_1$ on the boundary $x_1 = 0$. The main result of this article is that it is indeed possible to obtain regularity results on the trace of the solution of (1.2), (1.1) and (1.3).

2. The penalized problem

We approximate (1.5) by the penality method. This means that we replace the rigid constraint (1.5c) by a very stiff response: when the constraint is active, the response is linear, and it vanishes when the constraint is not active. More precisely, letting $r^+ = \max(r, 0)$, we replace u by u^ϵ, which satisfies

(2.1) $\rho\ddot{u}^\epsilon - A^0 u^\epsilon - A^1\dot{u}^\epsilon = f, \quad x \in \Omega, \quad t \geq 0,$

with initial data

(2.2) $u^\epsilon(\cdot, 0) = u_0 \quad \text{and} \quad \dot{u}^\epsilon(\cdot, 0) = v_0.$

and boundary conditions

(2.3a) $\quad a^0_{11kl}\varepsilon_{kl}(u^\epsilon) + a^1_{11kl}\varepsilon_{kl}(\dot u^\epsilon) = -(u^\epsilon_1)^+/\epsilon,$

(2.3b) $\quad a^0_{12kl}\varepsilon_{kl}(u^\epsilon) + a^1_{12kl}\varepsilon_{kl}(\dot u^\epsilon) = 0 \quad \text{and} \quad a^0_{13kl}\varepsilon_{kl}(u^\epsilon) + a^1_{13kl}\varepsilon_{kl}(\dot u^\epsilon) = 0,$

THEOREM 2.1. *Let* $W = \{u \in \mathbf{H}^1_{\mathrm{loc}}([0,\infty)\times\Omega) : \nabla\dot u \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))\}$. *Then for each* $\epsilon > 0$ *there exists a unique weak solution* $u^\epsilon \in W$ *of the problem* (2.1)-(2.3) *such that*

$$u^\epsilon \in \mathbf{L}^\infty_{\mathrm{loc}}([0,\infty); H^1(\Omega)),$$
$$\dot u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty); H^1(\Omega)),$$
$$\ddot u^\epsilon \in \mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)),$$

and for every $\tau \in (0,T)$ *and for all* $v \in W$, *the following variational equality is satisfied:*

$$\begin{aligned}\int_0^\tau\int_\Omega \rho\ddot u^\epsilon\cdot v\,dx\,dt &+ \int_0^\tau \big(a^0(u^\epsilon,v) + a^1(\dot u, v)\big)\,dt \\ &+ \int_\Sigma \frac{(u^\epsilon_1)^+}{\epsilon} v^\epsilon_1\,dx' \ge \int_0^\tau\int_\Omega f\cdot v\,dx\,dt,\end{aligned} \tag{2.4}$$

PROOF. We leave the verification of this proof to the reader since we use an analogous method to this one developed in [**4**]. □

3. Estimates on the penalized solution

LEMMA 3.1. *Assume that* f *belongs to* $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, u_0 *to* $\mathbf{H}^1(\Omega)$ *and* v_0 *to* $\mathbf{L}^2(\Omega)$. *Then independently of* ϵ , $\dot u^\epsilon$, ∇u^ϵ *are bounded in* $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, ∇u^ϵ_t *is bounded in* $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ *and* $(u^\epsilon_1(0,\cdot,\cdot))^+/\sqrt{\epsilon}$ *is bounded in the space* $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Sigma))$.

PROOF. These estimates are simply an application of Gronwall lemma to the energy estimate. We multiply (2.1) by $\dot u^\varepsilon$ and we integrate this expression over $\Omega\times(0,\tau)$ to get

$$\begin{aligned}\frac12\int_\Omega \big(\rho|\dot u^\epsilon|^2 + a^0_{ijkl}\varepsilon_{ij}(u^\epsilon)\varepsilon_{kl}(u^\epsilon)\big)\big|_0^\tau\,dx &+ \int_0^\tau\int_\Omega a^1_{ijkl}\varepsilon_{ij}(\dot u^\epsilon)\varepsilon_{kl}(\dot u^\epsilon)\,dx\,dt \\ &+ \frac{1}{2\epsilon}\int_\Sigma \big((u^\epsilon_1)^+\big)^2\big|_0^\tau\,dx' = \int_0^\tau\int_\Omega f\cdot u^\epsilon\,dx\,dt.\end{aligned} \tag{3.1}$$

According to the Korn inequality, it is possible to infer that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\int_\Omega a^n_{ijkl}\varepsilon_{kl}(z)\varepsilon_{ij}(z) \ge C_1\int_\Omega |\nabla z|^2\,dx - C_2\int_\Omega |z|^2\,dx, \quad n = 0,1.$$

Since $f\cdot\dot u^\epsilon$ can be estimated by $|f|^2/(2\gamma) + \gamma|\dot u^\epsilon|^2/2$, $\gamma > 0$, and using the above inequality, we deduce from (3.1) that

$$\begin{aligned}\frac12\int_\Omega \big(\rho|\dot u^\epsilon|^2 + C_1|\nabla u^\epsilon|^2\big)(\cdot,\tau)\,dx &+ C\int_0^\tau\int_\Omega |\nabla u^\epsilon|^2\,dx\,dt \\ +\frac{1}{2\epsilon}\int_\Sigma \big((u^\epsilon_1)^+\big)\big|_0^\tau\,dx' &\le \frac{C_2}{2}\int_\Omega |u^\epsilon(\cdot,\tau)|^2\,dx + \Big(C_2 + \frac{\gamma}{2}\Big)\int_0^\tau\int_\Omega |\dot u^\epsilon|^2\,dx\,dt \\ &+\frac{1}{2\gamma}\int_0^\tau\int_\Omega |f|^2\,dx\,dt + \frac12\int_\Omega \big(\rho|v_0|^2 + a^0_{ijkl}\varepsilon_{ij}(u_0)\varepsilon_{kl}(u_0)\big)\,dx.\end{aligned}$$

A classical Gronwall lemma enables us to deduce that $\dot u^\epsilon$, ∇u^ϵ are bounded in the space $\mathbf{L}^\infty_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, $\nabla\dot u^\epsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}([0,\infty); L^2(\Omega))$, $(u^\epsilon_1(0,\cdot,\cdot))^+/\sqrt{\epsilon}$ is bounded in $L^\infty_{\mathrm{loc}}([0,\infty); L^2(\Sigma))$. □

REMARK 3.2. *If we suppose that f vanishes for large t then independently of $\epsilon > 0$,*

$$\operatorname{ess}\sup_{0\le t\le T} |u^\epsilon(\cdot,t)|_{H^1} \le C(1+T) \quad \text{and} \quad \left(\int_0^T |\dot{u}^\epsilon(\cdot,t)|^2_{H^1}\,dt\right)^{1/2} \le C(1+T).$$

These properties can be proved using the arguments given in the proof of lemma 3.1, with the origin of time moved to T if $f(\cdot,t)$ vanishes for $t \ge T$; since the integral involving f vanishes, the conclusion is clear.

LEMMA 3.3. *Assume that f belongs to $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$, u_0 to $\mathbf{H}^1(\Omega)$ and u_1 to $\mathbf{L}^2(\Omega)$. Then independently of ϵ, the trace $(u_1^\epsilon(0,\cdot,\cdot))^-/\epsilon$ is bounded in the space of measures on $\Sigma\times(0,T)$.*

PROOF. Define the cutoff function φ; φ belongs to $\mathbf{C}^1\big(\mathbb{R}^{d-1}\big)$ and is equal to 1 in the sphere of center 0 and radius $R>0$ and vanishes outside of a sphere of radius $R+1$. We multiply (2.1) by φ and we integrate over $\Omega\times(0,\tau)$, thanks to the boundary conditions (2.3), we obtain

$$\begin{aligned}\int_\Omega \rho\dot{u}^\epsilon\cdot\varphi\big|_0^\tau\,dx + \frac{1}{\epsilon}\int_0^\tau\int_\Sigma (u_1^\epsilon)^+\varphi_1\,dx'\,dt + \int_0^\tau\int_\Omega \sigma^0_{ij}(u^\epsilon)\varepsilon_{ij}(\varphi)\,dx\,dt\\ + \int_0^\tau\int_\Omega \sigma^1_{ij}(\dot{u}^\epsilon)\varepsilon_{ij}(\varphi)\,dx\,dt = \int_0^\tau\int_\Omega f\cdot\varphi\,dx\,dt.\end{aligned}$$

Since the product $|zy|$ can be estimated by $|z|^2/2+|y|^2/2$, we get the following inequality:

$$\begin{aligned}\frac{1}{\epsilon}\int_0^\tau\int_\Sigma (u^\epsilon)^+\varphi_1\,dx'\,dt \le \frac{\rho}{2}\int_\Omega\big(|\dot{u}^\epsilon(\cdot,\tau)|^2+|v_0|^2\big)\,dx + \rho\int_\Omega|\varphi|^2\,dx\\ + \int_0^\tau\int_\Omega\big|\big(\sigma^0_{ij}(u^\epsilon)+\sigma^1_{ij}(\dot{u}^\epsilon)\big)\varepsilon_{ij}(\varphi)\big|\,dx\,dt + \int_0^\tau\int_\Omega|f\cdot\varphi|\,dx\,dt.\end{aligned} \tag{3.2}$$

We may deduce that the right hand side of (3.2) is bounded using the Lemma 3.1. Since $(u^\epsilon(0,\cdot,\cdot))^+$ is non negative, the conclusion is clear. □

LEMMA 3.4. *Assume that f belongs to $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$, u_0 to $\mathbf{H}^2(\Omega)$, v_0 to $\mathbf{L}^2(\Omega)$. Then independently of $\epsilon>0$, A^0u^ϵ and A^1u^ϵ are is bounded in the space $\mathbf{L}^\infty_{\mathrm{loc}}\big([0,\infty),L^2(\Omega)\big)$.*

PROOF. Once again we use energy techniques, but now we multiply relation (2.1) by A^1u^ϵ; next we integrate over $\Omega\times(0,\tau)$;

$$\begin{aligned}\frac{1}{2}\int_\Omega|A^1u^\epsilon(\cdot,\tau)|^2\,dx = \frac{1}{2}\int_\Omega|A^1u_0|^2\,dx + \int_0^\tau\int_\Omega(\rho\ddot{u}^\epsilon)\cdot(A^1u^\epsilon)\,dx\,dt\\ -\int_0^\tau\int_\Omega(A^0u^\epsilon)\cdot(A^1u^\epsilon)\,dx\,dt - \int_0^\tau\int_\Omega f\cdot(A^1u^\epsilon)\,dx\,dt.\end{aligned} \tag{3.3}$$

We observe that

$$\begin{aligned}\int_0^\tau\int_\Omega(\rho\ddot{u}^\epsilon)\cdot(A^1u^\epsilon)\,dx\,dt = \rho\int_\Omega\dot{u}^\epsilon\cdot(A^1u^\epsilon)\big|_0^\tau\,dx\\ -\rho\int_0^\tau\int_\Sigma\dot{u}_1^\epsilon\sigma^1_{1j}\,dx'\,dt + \int_0^\tau\int_\Omega a^1_{ijkl}\varepsilon_{ij}(\dot{u}^\epsilon)\varepsilon_{kl}(\dot{u}^\epsilon)\,dx\,dt.\end{aligned} \tag{3.4}$$

Carrying (3.4) into (3.3) and using the boundary conditions (2.3), we obtain

$$\frac{1}{2}\int_\Omega |A^1u^\epsilon(\cdot,\tau)|^2\,dx = \frac{1}{2}\int_\Omega |A^1u_0|^2\,dx - \int_0^\tau\int_\Omega (A^0u^\epsilon)\cdot(A^1u^\epsilon)\,dx\,dt$$

$$(3.5)\qquad -\int_0^\tau\int_\Omega f\cdot(A^1u^\epsilon)\,dx\,dt + \rho\int_\Omega \dot u^\epsilon\cdot(A^1u^\epsilon)\big|_0^\tau\,dx + \frac{\rho}{\epsilon}\int_0^\tau\int_\Sigma \dot u_1^\epsilon(u_1^\epsilon)^+\,dx'\,dt$$

$$+\rho\int_0^\tau\int_\Sigma \dot u_1^\epsilon\sigma_{1j}^0(u^\epsilon)\,dx'\,dt + \int_0^\tau\int_\Omega a_{ijkl}^1\varepsilon_{ij}(\dot u^\epsilon)\varepsilon_{kl}(\dot u^\epsilon)\,dx\,dt.$$

On the other hand, we observe that

$$(3.6)\qquad \int_0^\tau\int_\Sigma |\sigma_{1j}^0(u^\epsilon)|^2\,dx'\,dt \le C\left(\int_0^\tau\int_\Omega |u^\epsilon|^2\,dx\,dt + \int_0^\tau\int_\Omega |A^1u^\epsilon|^2\,dx\,dt\right),$$

and for all v belonging to $\mathbf{H}^1(\Omega)$ and A^1v belonging to $\mathbf{L}^2(\Omega)$, we get

$$(3.7)\qquad |A^0v|_{L^2(\Omega)} \le C|v|_{L^2(\Omega)} + |A^1v|_{L^2(\Omega)}.$$

Define

$$(3.8)\qquad F(t) = \int_\Omega |A^1u^\epsilon(\cdot,t)|^2\,dx.$$

According to (3.6)-(3.8) and since $\dot u^\epsilon\cdot(A^1u^\epsilon)$ can be estimated by $|\dot u^\epsilon|^2/(2\gamma) + \gamma|A^1u^\epsilon|^2/2$, $\gamma > 0$, it is possible to infer from (3.5) the following inequality:

$$\left(\frac{1}{2}-\frac{\rho\gamma}{2}\right)F(\tau) \le \frac{1}{2}F(0) + (2+C)\int_0^\tau F(t)\,dt + \frac{1}{2}\int_0^\tau\int_\Omega |f|^2\,dx\,dt$$

$$+\frac{\rho}{2\gamma}\int_\Omega |\dot u^\epsilon(\cdot,\tau)|^2\,dx + \rho\int_\Omega |v_0\cdot(A^1u_0)|\,dx + \int_0^\tau\int_\Omega a_{ijkl}^1\varepsilon_{ij}(\dot u^\epsilon)\varepsilon_{kl}(\dot u^\epsilon)\,dx\,dt$$

$$+\frac{\rho}{2\varepsilon}\int_\Sigma \big(u_1^\epsilon(\cdot,\tau)\big)^+\,dx' + (1+C)\int_0^\tau\int_\Omega |u^\epsilon|^2\,dx\,dt + \int_0^\tau\int_\Sigma |\dot u^\epsilon|^2\,dx'\,dt.$$

If we choose γ such that $\rho\gamma < 1$, we may infer using Lemma 3.1 and a classical Gronwall inequality that F is bounded in $L^\infty_{\text{loc}}\big([0,\infty)\big)$. Therefore, we may deduce the Lemma. □

REMARK 3.5. *If we suppose that f vanishes for t large, then, independently of ε, A^0u^ϵ and A^1u^ϵ are almost polynomilly increasing. These properties can be proved using the arguments given in Remark 3.2.*

Let us turn now to interior estimates.

LEMMA 3.6. *Assume that f belongs to $\mathbf{L}^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, u_0 to $\mathbf{H}^2(\Omega)$, v_0 to $\mathbf{L}^2(\Omega)$. Then for all $\beta > 0$, $\ddot u^\epsilon$, $A^1\dot u^\epsilon$ are bounded in $\mathbf{L}^2\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$, independently of ε.*

PROOF. We multiply u^ε by a truncation function

$$(3.9)\qquad \varphi(x) = \begin{cases} 1 & \text{if} \quad x \le -\beta,\\ 0 & \text{if} \quad x \ge -\beta/2,\end{cases}$$

which enables us to forget about the strongly non linear boundary conditions. Define

$$(3.10)\qquad v^\varepsilon = \varphi(x_1)u^\varepsilon.$$

The derivatives of v^ε are given by:

$$(3.11a)\qquad \ddot v^\varepsilon = \varphi\ddot u^\varepsilon,$$

$$(3.11b)\qquad \partial_j\varepsilon_{kl}(\varphi v^\epsilon) = \varphi\partial_j\varepsilon_{kl}(u^\epsilon) + 2\varphi_{x_1}\varepsilon_{kl}(u^\epsilon) + \varphi_{x_1x_1}u_k^\epsilon,$$

$$(3.11c)\qquad \partial_j\varepsilon_{kl}(\varphi\dot v^\epsilon) = \varphi\partial_j\varepsilon_{kl}(\dot u^\epsilon) + 2\varphi_{x_1}\varepsilon_{kl}(\dot u^\epsilon) + \varphi_{x_1x_1}\dot u_k^\epsilon.$$

Notice that thanks to relations (2.1) and (3.11), we have

$$\ddot{v}^\epsilon - A^0 v^\epsilon - A^1 v^\epsilon = \widetilde{g}^\epsilon, \tag{3.12}$$

where

$$\widetilde{g}_i^\varepsilon = \varphi f_i - 2\varphi_{x_1}\big(a^0_{ijkl}\varepsilon_{kl}(u^\epsilon) + a^1_{ijkl}\varepsilon_{kl}(\dot{u}^\epsilon)\big) - \varphi_{x_1x_1}\big(a^0_{ijkl}u_k^\epsilon + a^1_{ijkl}\dot{u}_k^\epsilon\big).$$

Thanks to Lemma 3.1, we may deduce that $\widetilde{g}^\varepsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$. Define

$$w^\epsilon = \dot{v}^\epsilon \quad \text{and} \quad g^\epsilon = \widetilde{g}^\epsilon + A^0 v^\epsilon. \tag{3.13}$$

We substitute (3.13) in (3.12), we obtain

$$\dot{w}^\epsilon - A^1 w^\epsilon = g^\epsilon. \tag{3.14}$$

We will prove that $\dot{w}^\varepsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$. For this purpose, we multiply (3.14) by $\dot{w}^\varepsilon$; we integrate this expression over $\Omega\times(0,\tau)$, we obtain

$$\int_0^\tau\int_\Omega |\dot{w}^\epsilon|^2\,dx\,dt - \int_0^\tau\int_\Omega (A^1 w^\epsilon)\cdot\dot{w}^\epsilon = \int_0^\tau\int_\Omega g^\epsilon\cdot\dot{w}^\epsilon\,dx\,dt.$$

Since

$$\int_0^\tau\int_\Omega (A^1 w^\epsilon)\cdot\dot{w}^\epsilon\,dx\,dt = -\frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\big|_0^\tau\,dx, \tag{3.15}$$

we infer that

$$\begin{aligned}\int_0^\tau\int_\Omega |\dot{w}^\epsilon|^2\,dx\,dt &+ \frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\big|_{t=\tau}\,dx\\ &= \frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\big|_{t=0}\,dx + \int_0^\tau\int_\Omega g^\epsilon\cdot\dot{w}^\epsilon\,dx\,dt.\end{aligned} \tag{3.16}$$

According to the Korn inequality, we infer that there exists C_1 and C_2 such that

$$\int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\,dx \ge C_1\int_\Omega |\nabla w^\epsilon|^2\,dx - C_2\int_\Omega |w^\epsilon|^2\,dx. \tag{3.17}$$

Carrying the above inequality into (3.16) and observing that $g^\epsilon\cdot\dot{w}^\epsilon$ can be estimated by $|g^\epsilon|^2/2 + |\dot{w}^\epsilon|^2/2$, we get

$$\begin{aligned}\int_0^\tau\int_\Omega |\dot{w}^\epsilon|^2\,dx\,dt + C_1\int_\Omega |\nabla w^\epsilon(\cdot,\tau)|^2\,dx &\le \int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\big|_{t=0}\,dx\\ &+ C_2\int_\Omega |w^\epsilon(\cdot,\tau)|^2\,dx + \int_0^\tau\int_\Omega |g^\epsilon|^2\,dx\,dt.\end{aligned} \tag{3.18}$$

Since u_0 belongs to $\mathbf{H}^2(\Omega)$, v_0 belongs to $\mathbf{H}^1(\Omega)$, φ belongs to $C_0^\infty(\mathbb{R})$, g^ε is bounded in $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$, we infer that the right hand side of (3.18) is bounded. Therefore using the identities (3.10) and (3.13), it is possible to deduce that $\ddot{u}^\varepsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$.

We will show that $A^1\dot{u}^\epsilon$ is bounded in $\mathbf{L}^2_{\mathrm{loc}}\big([0,\infty);L^2((-\infty,-\beta)\times\Sigma)\big)$ using an analogous method as this one developed above. We multiply (3.14) by $A^1\dot{u}^\epsilon$, we integrate over $\Omega\times(0,\tau)$, we obtain

$$\int_0^\tau\int_\Omega \dot{w}^\epsilon\cdot(A^1 w^\epsilon)\,dx\,dt - \int_0^\tau\int_\Omega |A^1 w^\epsilon|^2\,dx\,dt = \int_0^\tau\int_\Omega g^\epsilon\cdot(A^1 w^\epsilon)\,dx\,dt. \tag{3.19}$$

Carrying (3.15) and (3.17) into (3.19) and since $g^\epsilon\cdot(A^1 w^\epsilon)$ can be estimated by $|g^\epsilon|^2/2 + |A^1 w^\epsilon|^2/2$, we obtain

$$\begin{aligned}\int_0^\tau\int_\Omega |A^1 w^\epsilon|^2\,dx\,dt + C_1\int_\Omega |\nabla w^\epsilon(\cdot,\tau)|^2\,dx &\le \int_\Omega a^1_{ijkl}\varepsilon_{ij}(w^\epsilon)\varepsilon_{kl}(w^\epsilon)\big|_{t=0}\,dx\\ &+ C_2\int_\Omega |w^\epsilon(\cdot,\tau)|^2\,dx + \int_0^\tau\int_\Omega |g^\epsilon|^2\,dx\,dt.\end{aligned} \tag{3.20}$$

Thanks to (3.10) and (3.13), we may deduce from (3.20) that $A^1\dot{u}^\epsilon$ is bounded in $\mathbf{L}^2_{\text{loc}}\big([0,\infty); L^2((-\infty,-\beta)\times\Sigma)\big)$. □

4. Existence of a weak solution

In this Section, we prove that it is possible to pass to the limit in the variational formulation of the penalized problem and therefore to deduce that there exists a weak solution of (1.2)-(1.4). Since Ω is unbounded set, the proof shall be technical but similar to this one developed in [**4**].

THEOREM 4.1. *Assume that f belongs to $\mathbf{L}^2_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$, u_0 to $\mathbf{H}^2(\Omega)$, v_0 to $\mathbf{L}^2(\Omega)$. Then there exists a solution of the variational inequality* (1.6)*; this solution is a limit of a subsequence of the penalty approximation defined by* (2.1)-(2.3).

PROOF. Let $\varphi \in C_0^\infty\big(\bar{\Omega}\times[0,\infty)\big)$ be a function which takes its values between 0 and 1. We suppose here that v belongs to K. We multiply (2.1) by $(v-u^\epsilon)\varphi$ and we integrate by parts, we obtain the identity

$$\begin{aligned}
&\int_\Omega \rho\dot{u}^\epsilon\cdot\big(\varphi(v-u^\epsilon)\big)\big|_0^\tau\,dx - \int_0^\tau\int_\Omega \rho\dot{u}^\epsilon\cdot\frac{\partial}{\partial t}\big(\varphi(v-u^\epsilon)\big)\,dx\,dt\\
&+\int_0^\tau\int_\Omega \big(a^0_{ijkl}\varepsilon_{kl}(u^\epsilon)\varepsilon_{ij}(u^\epsilon)+a^1_{ijkl}\varepsilon_{kl}(\dot{u}^\epsilon)\varepsilon_{ij}(u^\epsilon)\big)\big(\varphi(v_i-u^\epsilon_i)\big)\,dx\,dt\\
&+\frac{1}{\epsilon}\int_0^\tau\int_\Sigma (u^\epsilon_1)^+\big(\varphi(v_1-u^\epsilon_1)\big)\,dx'\,dt = \int_0^\tau\int_\Omega f\cdot\big(\varphi(v-u^\epsilon)\big)\,dx\,dt.
\end{aligned}\tag{4.21}$$

We observe that

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^\tau\int_\Sigma (u^\epsilon_1)^+\big(\varphi(v_1-u^\epsilon_1)\big)\,dx'\,dt\\
&= -\frac{1}{\epsilon}\int_0^\tau\int_\Sigma \big((u^\epsilon_1)^+\big)^2\varphi\,dx'\,dt + \frac{1}{\epsilon}\int_0^\tau\int_\Sigma (u^\epsilon_1)^+\varphi v_1\,dx'\,dt,
\end{aligned}$$

is negative. Therefore, (4.22) implies that

$$\begin{aligned}
&\int_\Omega \rho\dot{u}^\epsilon\cdot\big(\varphi(v-u^\epsilon)\big)\big|_0^\tau\,dx - \int_0^\tau\int_\Omega \rho\dot{u}^\epsilon\cdot\frac{\partial}{\partial t}\big(\varphi(v-u^\epsilon)\big)\,dx\,dt\\
&+\int_0^\tau\int_\Omega \big(a^0_{ijkl}\varepsilon_{kl}(u^\epsilon)\varepsilon_{ij}(u^\epsilon)+a^1_{ijkl}\varepsilon_{kl}(\dot{u}^\epsilon)\varepsilon_{ij}(u^\epsilon)\big)\big(\varphi(v_i-u^\epsilon_i)\big)\,dx\,dt\\
&\geq \int_0^\tau\int_\Omega f\cdot\big(\varphi(v-u^\epsilon)\big)\,dx\,dt.
\end{aligned}\tag{4.22}$$

We may deduce from Lemmas 3.1 and 3.4 that there exists a subsequence, still denoted by u^ϵ, such that

$$u^\varepsilon \rightharpoonup u \quad\text{in}\quad \mathbf{L}^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak } *,\tag{4.23a}$$

$$\dot{u}^\varepsilon \rightharpoonup \dot{u} \quad\text{in}\quad \mathbf{L}^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak } *,\tag{4.23b}$$

$$\nabla u^\varepsilon \rightharpoonup \nabla u \quad\text{in}\quad \mathbf{L}^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak } *,\tag{4.23c}$$

$$A^n u^\varepsilon \rightharpoonup A^n u \quad\text{in}\quad \mathbf{L}^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak } *,\quad n=0,1,\tag{4.23d}$$

$$\nabla\dot{u}^\varepsilon \rightharpoonup \nabla u_t \quad\text{in}\quad \mathbf{L}^2_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)\quad\text{weak } *.\tag{4.23e}$$

Thanks to the classical compactness properties of injection of Sobolev spaces on bounded open sets, we see for all $R>0$, the restrictions of u^ε, ∇u^ε and $a^n_{ijkl}\varepsilon_{kl}(u^\epsilon)$, $n=0,1$, to $Q_R=\big\{x : x_1<0, |x|\leq R\big\}\times[0,R]$ converge strongly to their respective limits in $\mathbf{L}^2(Q_R)$; therefore, we can pass to the limit in all the terms of (4.22) except possibly the first two terms.

Let us prove now that $\dot u^\epsilon(\cdot,t)$ converges weakly to $\dot u(\cdot,t)$ for all $t>0$: let γ be an arbitrary positive number; let z belong to $\mathbf{L}^2(\Omega)$; denote by C_1 is an upper bound for $|\dot u^\epsilon|_{L^\infty([0,T];L^2(\Omega))}$ with T fixed. We choose β so small that

$$\left(\int_{-\beta<x_1<0}|z|^2\,dx\right)^{1/2}\le\frac{\gamma}{4C_1};$$

then, for $t\in[0,T]$

$$\begin{aligned}\left|\int_\Omega\big(\dot u^\epsilon(\cdot,t)-\dot u(\cdot,t)\big)\cdot z\,dx\right|&\le\left|\int_{x_1<-\beta}\big(\dot u^\epsilon(\cdot,t)-\dot u(\cdot,t)\big)\cdot z\,dx\right|\\&+\left(\int_{-\beta<x_1<0}|z|^2\,dx\right)^{1/2}\left(\int_{-\beta<x_1<0}\left|\dot u^\epsilon(\cdot,t)-\dot u(\cdot,t)\right|^2dx\right)^{1/2}.\end{aligned}\tag{4.24}$$

By definition of C_1, the second term on the right hand side of (4.24) is estimated by $C_1\gamma/4C_1=\gamma/2$. As $\dot u^\epsilon\big|_{(-\infty,-\beta)\times\Sigma\times(0,T)}$ is bounded in $\mathbf{H}^1\big((-\infty,-\beta)\times\Sigma\times(0,T)\big)$, we see that

$$\int_{-\infty}^{-\beta}\int_\Sigma\dot u^\epsilon\cdot z\,dx\quad\text{converges}\quad\text{to}\quad\int_{-\infty}^{-\beta}\int_\Sigma\dot u\cdot z\,dx$$

uniformly with respect to $t\in[0,T]$. It suffices therefore to choose ϵ so small that the first term on the right hand side of (4.24) is estimated by $\gamma/2$. This proves that the convergence of $\int_\Omega\dot u^\epsilon\cdot z\,dx$ to $\int_\Omega\dot u\cdot z\,dx$ is uniform on compact sets in time. In particular, as ϵ tends to 0, it is plain that for all $\tau>0$,

$$\int_\Omega\dot u^\epsilon\cdot\big(\varphi(v-u^\epsilon)\big)\,dx\to\int_\Omega\dot u\cdot\big(\varphi(v-u)\big)\,dx.$$

Let us turn now to the term

$$\int_0^\tau\int_\Omega\dot u^\epsilon\cdot\big(\dot\varphi(v-u^\epsilon)+\varphi(\dot v-\dot u^\epsilon)\big)\,dx\,dt.$$

It is plain that

$$\int_0^\tau\int_\Omega\dot u^\epsilon\cdot\big(\dot\varphi(v-u^\epsilon)+\varphi\dot v\big)\,dx\,dt\to\int_0^\tau\int_\Omega\dot u\cdot\big(\dot\varphi(v-u)+\varphi\dot v\big)\,dx\,dt.$$

There remains to prove the convergence

$$\int_0^\tau\int_\Omega|\dot u^\epsilon|^2\varphi\,dx\,dt\to\int_0^\tau\int_\Omega|\dot u|^2\varphi\,dx\,dt.$$

We observe that

$$\begin{aligned}&\int_0^\tau\int_\Omega\left|\dot u^\epsilon-\dot u\right|^2\varphi\,dx\,dt\\&\le\int_0^\tau\int_{x_1\le-\beta}\left|\dot u^\epsilon-\dot u\right|^2\varphi\,dx\,dt+\int_0^\tau\int_{-\beta\le x_1\le0}\left|\dot u^\epsilon-\dot u\right|^2\varphi\,dx\,dt.\end{aligned}$$

Let γ be any positive number. We infer from the estimates over $|\dot u^\epsilon|_{L^2(\Omega\times(0,\tau))}$ and $|\nabla\dot u^\epsilon|_{L^2(\Omega\times(0,\tau))}$ that there exists a constant C_1 independent from ϵ such that

$$\left|u^\epsilon(x_1,\cdot,\cdot)\right|_{L^2(\Sigma\times(0,\tau))}\le C_1.$$

Therefore,

$$\int_0^\tau\int_{-\beta\le x_1\le0}\left|\dot u^\epsilon-\dot u\right|^2\varphi\,dx\,dt\le C_1^2\beta.$$

We choose β so small that

$$C_1^2\beta\le\gamma/2;$$

then we know from the estimates of Lemmas 3.4 and 3.6 that the restriction of u^ϵ to $\{x_1 < -\beta\}$ intersected with a ball containing the support of φ is bounded in H^2 of that set; therefore, for ϵ small enough,

$$\int_0^\tau \int_\Omega \left|\dot{u}^\epsilon - \dot{u}\right|^2 \varphi \, dx \, dt \le \frac{\gamma}{2},$$

and the convergence of the first two terms of (4.1) is proved.

We observe now that since u, $\dot{u}$, ∇u and $\dot{\nabla} u$ belong to $\mathbf{L}^2\big([0,\infty); L^2(\Omega)\big)$, we may replace φ by φ_R in the variational inequality where φ_R is equal to 1 over the set $K_R = [-R, 0] \times \{|y| \le R\} \times [0, R]$ and vanishes outside of K_{R+1}. It is plain that as $R \to \infty$ all the terms in (4.22) converge to their limit; thus we have proved the existence of the desired weak solution. □

5. Preliminary results

Lemma 5.1. *Assume u_0 belongs to $\mathbf{H}^{5/2}(\Omega)$; then, there exists a function $z \in \mathbf{H}^3\big(\Omega \times [0,\infty)\big)$ with compact support in t such that the trace of z on $\Omega \times \{0\}$ is equal to u_0.*

Proof. We extend u_0 into a function belonging to $\mathbf{H}^{5/2}(\mathbb{R}^d)$: as the boundary of Ω is smooth, this is a consequence of classical results on Sobolev spaces. Then there exists a function Z belonging to $\mathbf{H}^3\big(\mathbb{R}^d \times [0,\infty)\big)$ whose trace is u_0. It suffices now to select a cutoff function $\varphi \in C^\infty\big([0,\infty)\big)$ which is equal to 1 on $[0,1]$ and to 0 on $[2,\infty)$, and to define z as the restriction of φZ to $\Omega \times [0,\infty)$. □

Lemma 5.2. *Assume u_0 belongs to $\mathbf{H}^{5/2}(\Omega)$, u_1 belongs to $\mathbf{H}^1(\Omega)$ and f belongs to $\mathbf{L}^2_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$. Define z as in Lemma 5.1 and let $\bar{u}$ be the solution of*

$$\rho \frac{\partial^2 \bar{u}}{\partial t^2} - A^0 \bar{u} - A^1 \frac{\partial \bar{u}}{\partial t} = f, \quad x \in \Omega, \quad t \ge 0,$$

with the initial data

$$\bar{u}(\cdot, 0) = u_0 \quad \text{and} \quad \frac{\partial \bar{u}}{\partial t}(\cdot, 0) = v_0,$$

and boundary condition

$$\bar{u}(0, \cdot, \cdot) = z(0, \cdot, \cdot).$$

Then the trace $\bar{g} = -\big(a^0_{11kl}\varepsilon_{kl}(\bar{u}) + a^1_{11kl}\varepsilon_{kl}(\dot{\bar{u}})\big)\big|_{\Sigma \times [0,\infty)}$ is well defined and belongs to the space $L^2_{\text{loc}}\big([0,\infty); L^2(\Sigma)\big)$. Moreover, there exists K such that $e^{-Kt}\bar{g} \in L^2\big(\Sigma \times [0,\infty)\big)$.

Proof. It is convenient to define

$$F = f - \rho \ddot{z} + A^0 z + A^1 \dot{z}.$$

The function $\zeta = \bar{u} - z$ satisfies the equation

$$\rho \ddot{\zeta} - A^0 \zeta - A^1 \dot{\zeta} = F, \quad x \in \Omega, \quad t \ge 0, \tag{5.1}$$

with the initial data

$$\zeta(\cdot, 0) = 0 \quad \text{and} \quad \dot{\zeta}(\cdot, 0) = v_0,$$

and the boundary condition

$$\zeta(0, \cdot, \cdot) = 0. \tag{5.2}$$

Multiplying (5.1) by $\dot{\zeta}$ and integrating over $\Omega \times (0, \tau)$, thanks to Korn inequality we may deduce that

$$\dot{\zeta} \in \mathbf{L}^\infty_{\text{loc}}\big([0,\infty); L^2(\Omega)\big),$$
$$\nabla \zeta \in \mathbf{L}^\infty_{\text{loc}}\big([0,\infty); L^2(\Omega)\big),$$
$$\nabla \dot{\zeta} \in \mathbf{L}^2_{\text{loc}}\big([0,\infty); L^2(\Omega)\big).$$

Let us observe that if f is compactly supported in time, F is also compactly supported in time, and therefore it is plain that the above estimates hold in the spaces $\mathbf{L}^\infty\big([0,\infty);L^2(\Omega)\big)$ and $\mathbf{L}^2\big([0,\infty);L^2(\Omega)\big)$ respectively. If we multiply (5.1) by $A^1\zeta$, we may deduce that $A^0\zeta$ and $A^1\zeta$ are bounded in $\mathbf{L}^\infty_{\text{loc}}\big([0,\infty);L^2(\Omega)\big)$, using the same method as the one developed in the Lemma 3.4. Multiplying (5.1) by $A^1\dot\zeta$ and integrating over $\Omega\times(0,\tau)$, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^\tau\int_\Omega (A^0\zeta)\cdot(A^1\zeta)\,dx\,dt + \int_0^\tau\int_\Omega |A^1\dot\zeta|^2\,dx\,dt\\ = \int_0^\tau\int_\Omega \ddot\zeta\cdot(A^1\dot\zeta)\,dx\,dt - \int_0^\tau\int_\Omega F\cdot(A^1\dot\zeta)\,dx\,dt.\end{aligned}$$

According to (5.2), it is clear that

$$\int_0^\tau\int_\Omega \ddot\zeta\cdot(A^1\dot\zeta)\,dx\,dt = -\frac{1}{2}\int_\Omega a^1_{ijkl}\varepsilon_{kl}(\dot\zeta)\varepsilon_{ij}(\dot\zeta)\Big|_0^\tau\,dx.$$

Therefore, we get

$$\begin{aligned}a^1_{ijkl}\varepsilon_{kl}(\dot\zeta)\varepsilon_{ij}(\dot\zeta)\Big|_{t=\tau}\,dx + \int_0^\tau\int_\Omega |A^1\dot\zeta|^2\,dx\,dt + \frac{1}{2}\int_\Omega (A^0\zeta)\cdot(A^1\zeta)\Big|_0^\tau\,dx\\ = \int_\Omega a^1_{ijkl}\varepsilon_{kl}(\dot\zeta)\varepsilon_{ij}(\dot\zeta)\Big|_{t=0}\,dx - \int_0^\tau\int_\Omega F\cdot(A^1\dot\zeta)\,dx\,dt.\end{aligned}\tag{5.3}$$

According to a Gronwall lemma, there exists K such that

$$\int_0^\tau\int_\Omega |A^1\dot\zeta|^2\,dx\,dt \le Ce^{K\tau}\left(|F|^2_{L^2(0,\tau;L^2(\Omega))} + |\dot\xi(\cdot,0)|^2_{H^1(\Omega)} + |\xi|^2_{L^2(0,\tau;L^2(\Omega))}\right).$$

Therefore we deduce the Lemma. □

6. The trace spaces

The convention for the Fourier transform is

$$\widehat{z}(\xi,\omega) = \int_{\mathbb{R}^d} e^{-i(\xi\cdot x+\omega t)} z(x',t)\,dx'\,dt,$$

where $\xi = (\xi_1,\dots,\xi_{d-1})^t$. Denote by $v^\epsilon = u^\epsilon - \bar u$ a solution of

$$\rho\ddot v^\epsilon_i - (\lambda^0+\mu^0)\partial_i \operatorname{div} v^\epsilon - \mu^0\Delta v^\epsilon_i - (\lambda^1+\mu^1)\partial_i\operatorname{div}\dot v^\epsilon - \mu^1\Delta\dot v^\epsilon_i = 0,\tag{6.1a}$$

$$\mu^0\left(\partial_1 v^\epsilon_j + \partial_j v^\epsilon_1\right) + \mu_1\left(\partial_1\dot v^\epsilon_j + \partial_j\dot v^\epsilon_1\right) = 0 \quad \text{for} \quad j=2,3,\tag{6.1b}$$

$$\left(\lambda^0\operatorname{div} v^\epsilon + 2\mu^0\partial_1 v^\epsilon_1 + \lambda^1\operatorname{div}\dot v^\epsilon + 2\mu^1\partial_1\dot v^\epsilon_1\right)(0,\cdot) = \bar g - \frac{\left((v^\epsilon_1-\bar u_1)(0,\cdot)\right)^+}{\epsilon},\tag{6.1c}$$

$$v^\epsilon(\cdot,0) = u_0 \quad \text{and} \quad \dot v^\epsilon(\cdot,0) = v_0.\tag{6.1d}$$

Define $v^\epsilon = \big(v^\epsilon_1,(v^\epsilon)'\big)$. We perform a Fourier transform in tangential variable $x' = (x_2,\dots,x_d), t$ and a Laplace transform in x_1. Denoting by ξ' and ω the dual variables of x' and t and η the dual variable of x_1, we are led to the system:

$$\begin{aligned}-\omega^2\widehat{v}^\epsilon - (\lambda^0+\mu^0)\begin{pmatrix}\eta\\ i\xi\end{pmatrix}(\eta,i\xi^T)\widehat{v}^\epsilon + \mu^0\left(|\xi|^2-\eta^2\right)\widehat{v}^\epsilon\\ -(\lambda^1+\mu^1)\begin{pmatrix}\eta\\ i\xi\end{pmatrix}(\eta,i\xi^T)i\omega\widehat{v}^\epsilon + \mu^1\left(|\xi|^2-\eta^2\right)i\omega\widehat{v}^\epsilon = 0.\end{aligned}\tag{6.2}$$

The eigenvalues η_i and eigenvectors ϕ_i, $i = 1, 2, 3$, of the equation (6.2) are given by

(6.3a) $$\eta_1^2 = |\xi|^2 - \frac{\rho\omega^2}{i\omega\mu^1 + \mu^0} \quad \text{and} \quad \phi_1 = \begin{pmatrix} 0 \\ i\xi^\perp \end{pmatrix},$$

(6.3b) $$\eta_2^2 = |\xi|^2 - \frac{\rho\omega^2}{i\omega\mu^1 + \mu^0} \quad \text{and} \quad \phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(6.3c) $$\eta_3^2 = |\xi|^2 - \frac{\rho\omega^2}{i\omega(\lambda^1 + 2\mu^1) + \lambda^0 + 2\mu^0} \quad \text{and} \quad \phi_3 = \begin{pmatrix} \eta_3 \\ i\xi \end{pmatrix},$$

where $\xi^\perp$ is obtained from ξ by a rotation of $\pi/2$. Here η_i, $i = 1, 2, 3$, is the causal determination of the square root of η_i; therefore η_i is holomorphic in the lower half-plane $\Im(\omega) < 0$ and $\Re\eta_i \geq 0$ for $\Im(\omega) = 0$. We may establish using the method of [**4**] that the solution of (6.2) is given by

(6.4) $$\widehat{v}^\epsilon(x_1, \xi, \omega) = C_1(\xi, \omega)\phi_1 e^{\eta_2 x_1} + C_2(\xi, \omega)\phi_2 e^{\eta_2 x_1} + C_3(\xi, \omega)\phi_3 e^{\eta_3 x_1}.$$

We determine now C_i. We apply a partial Fourier transform in the tangential variable to the boundary condition (6.1b), we obtain

(6.5) $$\partial_1(\widehat{v}^\epsilon)'(0, \xi, \omega) = -i\xi\widehat{v}_1^\epsilon(0, \xi, \omega).$$

Carrying (6.4) into (6.5), we infer that at $x_1 = 0$, we get

$$i\xi^\perp\eta_2 C_1 + i\xi\eta_3 C_3 = -i\xi(C_2 + \eta_3 C_3),$$

thus it is clear that $C_1 = 0$. We deduce also from the above equality that $C_2 = -2\eta_3 C_3$. On the other hand, we infer from (6.4) that $C_3 = -\widehat{v}_1(0, \xi, \omega)/\eta_3$. Then

(6.6) $$\widehat{v}^\epsilon(x_1, \xi, \omega) = 2\widehat{v}_1^\epsilon(0, \xi, \omega)\phi_2 e^{\eta_2 x_1} - \widehat{v}_1^\epsilon(0, \xi, \omega)\phi_3 e^{\eta_3 x_1}/\eta_3.$$

In particular, we observe that

$$\begin{aligned} &\big(\lambda^0 \operatorname{div} v^\epsilon + 2\mu^0\partial_1 v_1^\epsilon + \lambda^1 \operatorname{div} \dot{v}^\epsilon + 2\mu^1\partial_1\dot{v}_1^\epsilon\widehat{\big)}(0, \xi, \omega) \\ &= \big(\lambda^0 + 2\mu^0 + i\omega(\lambda^1 + 2\mu^1)\big)\partial_1\widehat{v}_1^\epsilon(0, \xi, \omega) + (\lambda^0 + i\omega\lambda^1)i\xi^T(\widehat{v}^\epsilon)'(0, \xi, \omega). \end{aligned}$$

Since

$$(\widehat{v}^\epsilon)'(0, \xi, \omega) = -i\xi\widehat{v}_1^\epsilon(0, \xi, \omega)/\eta_3 \quad \text{and} \quad \partial_1\widehat{v}_1^\epsilon(0, \xi, \omega) = (2\eta_2 - \eta_3)\widehat{v}_1^\epsilon(0, \xi, \omega),$$

we may infer that

$$\begin{aligned} &\big(\lambda^0 \operatorname{div} v^\epsilon + 2\mu^0\partial_1 v_1^\epsilon + \lambda^1 \operatorname{div} \dot{v}^\epsilon + 2\mu^1\partial_1\dot{v}_1^\epsilon\widehat{\big)}(0, \xi, \omega) \\ &= \left(\frac{(\lambda^0 + i\omega\lambda^1)|\xi|^2}{\eta_3} + \big(\lambda^0 + 2\mu^0 + i\omega(\lambda^1 + 2\mu^1)\big)(2\eta_2 - \eta_3)\right)\widehat{v}_1^\epsilon(0, \xi, \omega). \end{aligned}$$

Define

$$\widehat{b} = \frac{(\lambda^0 + i\omega\lambda^1)|\xi|^2}{\eta_3} + \big(\lambda^0 + 2\mu^0 + i\omega(\lambda^1 + 2\mu^1)\big)(2\eta_2 - \eta_3).$$

If we let w^ϵ be the trace $v^\epsilon(0, x', t)$, (6.1c) can be written now

(6.7) $$b * w_1^\epsilon = \bar{g} - \frac{\big(w_1^\epsilon - \bar{u}_1(0, \cdot, \cdot)\big)^+}{\epsilon}.$$

Let $V^\epsilon = e^{-\nu t}v^\epsilon$, $W^\epsilon = e^{-\nu t}w^\epsilon$, $g = e^{-\nu t}\bar{g}$ and $h = e^{-\nu t}\bar{u}(0, \cdot, \cdot)$. We deduce from (6.7) that

(6.8) $$\beta * W_1^\epsilon = g - \frac{\big(W_1^\epsilon - h_1\big)^+}{\epsilon},$$

where

$$\widehat{\beta} = \frac{(\lambda^0 + (\nu + i\omega)\lambda^1)|\xi|^2}{\eta_3} + \big(\lambda^0 + 2\mu^0 + (\nu + i\omega)(\lambda^1 + 2\mu^1)\big)(2\eta_2 - \eta_3).$$

LEMMA 6.1. *Let W^ϵ be the solution of* (6.8). *Then we may extract a subsequence, still denoted by W_1^ϵ such that*

$$W_1^\epsilon \rightharpoonup W_1 \text{ weakly in } H^{5/4}_{\text{loc}}\big([0,\infty); L^2_{\text{loc}}(\mathbb{R}^{d-1})\big) \cap H^1_{\text{loc}}\big([0,\infty); H^{1/2}_{\text{loc}}(\mathbb{R}^{d-1})\big).$$

PROOF. Define $\psi = \lambda^0 + 2\mu^0 + (\lambda^1 + 2\mu^1)\big(\nu + \partial/\partial t\big)$ and $\widehat{\psi} = \lambda^0 + 2\mu^0 + (\lambda^1 + 2\mu^1)(\nu + i\omega)$. We multiply (6.8) by ψW_1^ϵ, thanks to Plancherel identity, we get

$$\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\overline{\widehat{\psi}}\,\widehat{\beta}|\widehat{W}_1^\epsilon|^2\,d\xi\,d\omega = \frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\overline{\widehat{\psi}\widehat{W}_1^\epsilon}\,d\xi\,d\omega$$
$$-\int_0^\infty\int_{\mathbb{R}^{d-1}}\frac{\big(W_1^\epsilon - h_1\big)^+}{\epsilon}\psi W_1\,dx'\,dt.$$

Since $\big(u_1^\epsilon(0,\cdot,\cdot)\big)^+/\sqrt{\epsilon}$ is bounded in $L^\infty_{\text{loc}}\big([0,\infty); L^2(\Sigma)\big)$, the absolute value of the second integral in the right hand side of the above inequality is bounded, we deduce that there exists $C_1 > 0$ such that

$$\frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\overline{\widehat{\psi}}\,\widehat{\beta}|\widehat{W}_1^\epsilon|^2\,d\xi\,d\omega \le C_1 + \frac{1}{(2\pi)^d}\Re\int_{\mathbb{R}^d}\widehat{g}\overline{\widehat{\psi}\widehat{W}_1^\epsilon}\,d\xi\,d\omega. \tag{6.9}$$

Define

$$\kappa = \frac{-\rho(\nu + i\omega)^2 - 2|\xi|^2(\mu^0 + (\nu + i\omega)\mu^1)}{-\rho(\nu + i\omega)^2 - |\xi|^2\big(\lambda^0 + 2\mu^0 + (\nu + i\omega)(\lambda^1 + 2\mu^1)\big)},$$
$$x_0 = \sqrt{\frac{2\rho(\lambda^0 + \nu\lambda^1)}{4\rho\lambda^1\nu + 4\lambda^1\mu^1 + (\lambda^1)^2}}.$$

We observe that

$$\widehat{\beta} = \big(\lambda^0 + 2\mu^0 + (\nu + i\omega)(\lambda^1 + 2\mu^1)\big)(2\eta_2 - \kappa\eta_3).$$

Therefore, it is sufficient to find a function h which depends on ξ and ω such that $\Re(2\eta_2 + \kappa\eta_3) \ge |h|$. It is possible to establish that

$$|\kappa|^2 \le 1 + \frac{C}{|\omega|^2}1_{\{|\xi|\le x_0\}}. \tag{6.10}$$

If we suppose that $|\xi|^2 \ge 2\mu^0/(\mu^1)^2$, η_2 can be approximate by $\widetilde{\eta}_2$ defined as follows:

$$|\widetilde{\eta}_2^2|^2 = \left(|\xi|^2 - \frac{\mu^0}{(\mu^1)^2}\right) + \frac{\omega^2}{(\mu^1)^2},$$

which implies that

$$|\widetilde{\eta}_2^2|^2 \ge \frac{\omega^2}{(\mu^1)^2} + C|\xi|^4. \tag{6.11}$$

On the other hand, we observe that

$$|\xi|^2 - \frac{\mu^0}{(\mu^1)^2} \ge \sqrt{C}|\xi|^2,$$

since $|\xi|^2 \ge 2\mu^0(\mu^1)^2$, we may deduce that $C = 1/4$. Therefore, we infer from (6.11) that

$$\Re\eta_2 \ge \cos(\pi/4)|\eta_2| \ge \frac{1}{\sqrt{2}}\left(\omega^2 + |\xi|^4/4\right)^{1/4} \quad \text{if} \quad |\xi|^2 \ge 2\mu^0/(\mu^1)^2. \tag{6.12}$$

We suppose now that $|\xi|^2 \le 2\mu^0/(\mu^1)^2$, we see that

$$|\Re\eta_2^2| \le \frac{3\mu^0}{(\mu^1)^2} \quad \text{and} \quad |\Im\eta_2^2| \ge \frac{\mu^1|\omega|}{(\mu^1)^2 + (\mu^0)^2},$$

which imply that there exists $C > 0$ such that

$$|\operatorname{arc}\cot \eta_2^2| \leq \frac{3\mu^0\left((\mu^1)^2 + (\mu^0)^2\right)}{(\mu^1)^3|\omega|} \leq \frac{C}{|\omega|}.$$

We deduce from the above inequality that

$$|\arg \eta_2^2| \leq \frac{\pi}{2} + \frac{C}{|\omega|},$$

thus, we get $\cos\arg\eta_2 \geq 1/2$, moreover $|\eta_2|^2 \geq C|\omega|$, we get

$$\Re\eta_2 \geq \frac{1}{2}\sqrt{|\omega|} \quad \text{if} \quad |\xi|^2 \leq 2\mu^0/(\mu^1)^2. \tag{6.13}$$

Therefore we infer from (6.12) and (6.13) that

$$\Re\eta_2 \geq \frac{1}{\sqrt{2}}\left(\omega^2 + |\xi|^4/4\right)^{1/4}. \tag{6.14}$$

On the other hand, for $|\xi|$ great enough, we get $|\eta_3| \geq |\eta_2|$ and thanks to (6.14), we get

$$\widehat{\beta} \geq \frac{1}{\sqrt{2}}\widehat{\psi}\left(\omega^2 + |\xi|^4/4\right)^{1/4}. \tag{6.15}$$

Carrying (6.15) into (6.9), we obtain

$$C\int_{\mathbb{R}^d}|\widehat{\psi}|^2\left(\omega^2 + |\xi|^4/4\right)^{1/4}|\widehat{w}_1^\varepsilon|^2\,d\xi\,d\omega \leq C_1 + \int_{\mathbb{R}^d}|\widehat{g}||\widehat{\psi}||\widehat{w}_1^\varepsilon|)\,d\xi\,d\omega. \tag{6.16}$$

We estimate the product zy by $|z|^2/(2\gamma) + \gamma|y|^2/2$, $\gamma > 0$, we see that

$$\begin{aligned}\left(C - \frac{\gamma}{2}\right)&\int_{\mathbb{R}^d}|\omega|^2\left(\omega^2 + |\xi|^4/4\right)^{1/4}|\widehat{w}_1^\varepsilon|^2\,d\xi\,d\omega \\ &\leq C_1 + \frac{1}{2\gamma}\int_{\mathbb{R}^d}\frac{|\widehat{g}|^2}{\left(\omega^2 + |\xi|^4/4\right)^{1/4}}\,d\xi\,d\omega.\end{aligned}$$

We choose γ such that $\gamma < 2C$ and since $|\bar{g}(\cdot,t)|_{H^{1/2}} \leq Ce^{Kt}$, if we choose $\nu > K$ then the conclusion is clear. □

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MODÉLISATION MATHÉMATIQUE DE PROCÉDÉS D'USINAGE:

ABRASION ET MOUILLAGE

Ce travail de thèse est consacré à l'étude d'un modèle viscoélastique avec des contraintes unilatérales, modélisé comme un matériau de Kelvin-Voigt.

Le chapitre un est consacré au cas monodimensionnel: on approche la solution du problème par pénalisation, ce qui conduit à un théorème d'existence d'une solution faible. Un résultat de régularité des traces permet de montrer que la solution est forte.

Le chapitre deux comporte un schéma numérique dont on montre la convergence vers une solution faible.

Les chapitres trois et quatre permettent de construire une solution forte dans un milieu monodimensionnel semi-infini, pour laquelle on sait établir un bilan d'énergie: les pertes sont purement visqueuses. Le problème est réduit à une inégalité variationnelle au bord faisant intervenir un opérateur pseudodifférentiel dont le terme principal est une dérivation d'ordre 3/2.

Les chapitres cinq et six comportent des théorèmes de trace pour une équation des ondes amorties et pour un opérateur de viscoélasticité dans un demi-espace, avec application aux solutions fortes.

MATHEMATICAL MODELISATION OF MACHINING PROCESS:

ABRASION AND WETTING

This PhD work is dedicated to a study of a viscoelastic model with unilateral constraints modeled as a Kelvin-Voigt material.

The chapter one is dedicated to the monodimensional case: we approximate the solution of the problem by penalization, which leads to a theorem of existence of a weak solution. A result of regularity of the traces enables to show that the solution is strong.

The chapter two includes a numerical scheme, we prove the convergence of this scheme to a weak solution.

The chapters three and four enable to construct a strong solution in a semi-infinite monodimensional domain. We establish an energy relation for this solution: the only losses are purely viscous. The problem is reduced to a variational inequality at the boundary involving a pseudodifferential operator which principal term is a derivation of order 3/2.

The chapters five and six include the trace theorems for a damped waves equation and an operator of viscoelasticity in a half-space, with application to strong solutions.