On the stability of Spherically symmetric travelling waves and the motion of a fluid between two infinte plates

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Introduction

De nombreux phénomènes naturels peuvent être appréhendés par les mathématiques grâce aux *équations aux dérivées partielles*. Les équations d'évolution traduisent notamment des phénomènes de propagation combinés ou non à d'autres d'interactions. Deux exemples sont traités dans cette thèse. Tout d'abord, *l'équation scalaire de réaction-diffusion*

$$\partial_t u = \Delta u + F(u)\,, \quad t > 0\,, \quad x \in \mathbf{R}^2\,, \quad F(0) = F(1) = 0\,, \tag{1}$$

où $u(x,t)$ modélise, par exemple, l'évolution de la densité d'un gène dominant au sein d'une population. Ensuite, *l'équation de Navier-Stokes*

$$\partial_t u + (u \cdot \nabla)u = \nu\Delta u - \nabla p\,, \text{div } u = 0\,, \quad t > 0\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1) \tag{2}$$

traduit l'écoulement d'un fluide incompressible, de vitesse $u(x,z,t)$ et de pression $p(x,z,t)$, entre deux plaques infinies. En ajoutant à (2) un terme exprimant une force extérieure de Coriolis, on traitera aussi l'équation de Navier-Stokes Coriolis

$$\partial_t u + (u \cdot \nabla)u + \Omega(e_3 \wedge u) = \nu\Delta u - \nabla p\,, \text{div } u = 0\,, \quad t > 0\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1) \tag{3}$$

qui décrit le mouvement des océans.

Au delà du problème majeur de l'existence de solutions, une grande part de la compréhension de ces équations vient de leur étude qualitative. Comprendre la géométrie du flot, décrire le comportement asymptotique d'une solution lorsque t tend vers $+\infty$, prouver l'existence de solutions particulières (point d'équilibre, onde progressive, solution autosimilaire) ou d'un ensemble de solutions (variétés invariantes), étudier leur stabilité face à une petite perturbation, tel est le sujet de la *théorie qualitative* dans laquelle s'inscrit cette thèse.

Cette démarche a été initiée sur les Equations Différentielles Ordinaires dès H. Poincaré et A. Lyapunov et comporte déjà en soi de nombreuses difficultés. Sa généralisation aux EDP paraboliques semi-linéaires utilise des outils fondamentaux tels que la formule de variation des constantes, l'approximation par linéarisation, la décomposition de l'espace d'étude en sous-espaces invariants par l'équation linéarisée et l'obtention de bornes exponentielles pour cette dernière équation [48]. Elle comporte également d'autres outils tels que le principe du maximum ou les fonctionnelles de Lyapunov.

Les équations (1-3) constituent en effet des *systèmes gradients*, c'est-à-dire associés à des fonctionnelles de Lyapunov strictes. Pour (1), cette fonctionnelle s'écrit

$$E_1(u) = \int_{\mathbf{R}^2} |\nabla u|^2 dx + \int_{\mathbf{R}^2} G(u(x)) dx$$

où $G(s) = -\int_0^s F(y)dy$, $s \in \mathbf{R}$. Pour les équations (2) et (3), cette fonctionnelle est l'énergie cinétique du fluide

$$E_2(u) = \|u\|^2_{L^2(\mathbf{R}^2\times(0,1))} = \int_{\mathbf{R}^2\times(0,1)} |u(x,z,t)|^2 dxdz\,.$$

L'existence de telles fonctionnelles oriente l'étude qualitative des solutions. En effet, si l'on regarde les équations (1) et (2) en domaine borné et si l'on impose des conditions aux bords adéquates, alors les solutions convergent, lorsque t tend vers l'infini, vers les points d'équilibre du système. Par contre, en domaine non borné, d'autres comportements asymptotiques peuvent apparaître.

L'équation (1) est *invariante par translation*. Il peut donc exister, en dimension 1 d'espace, des solutions particulières de la forme $u(x,t) = w_0(x-ct)$ se déplaçant en translation uniforme à la célérité c. L'allure du front $w_0(\xi) = u(\xi, 0)$, $\xi \in \mathbf{R}$, est alors donnée par une équation elliptique

$$\begin{aligned} &w_0''(\xi) + cw_0'(\xi) + F(w_0(\xi)) = 0\,, \quad \xi \in \mathbf{R} \\ &w_0(-\infty) = 1\,, \quad w_0(+\infty) = 0\,. \end{aligned} \tag{4}$$

Ces solutions appelées *ondes progressives* modélisent le transport d'information et représentent mathématiquement l'invasion du point d'équilibre $u = 0$ par un autre point d'équilibre $u = 1$.

En dimension 1 d'espace, de nombreuses études [16], [31], [53], [81] ont montré l'existence et la stabilité de ces ondes pour différentes non-linéarités F. En dimension supérieure $n = 2$, la généralisation aux ondes planes de la forme $u(x,t) = w_0(x\cdot k - ct)$ où $k \in S^1$ a été traitée par T. Kapitula [56], J.X Xin et C.D Levermore [61], [95] ou D.G Aronson et H.F Weinberger [2]. Par contre, le cas des ondes sphériques ne semble pas avoir donné lieu à de nombreuses recherches. C.K.R.T Jones [52] a effectué quelques approches géométriques, tandis que K. Uchiyama [92] a prouvé des résultats plus analytiques sur les équations de réaction-diffusion avec cœfficients variables. La première partie de cette thèse (chapitre I) va au-delà de ces travaux en étudiant tout à la fois la stabilité des ondes progressives à symétrie sphérique pour des perturbations radiales (chapitre I.2) et l'instabilité de ces mêmes ondes pour des perturbations quelconques (chapitre I.3).

D'autres invariances peuvent également avoir lieu. L'équation de Navier-Stokes (2) dans l'espace tout entier $\mathbf{R}^3$ est *invariante par le changement d'échelle*

$$u(x,t) \mapsto \lambda u(\lambda x, \lambda^2 t)\,, \quad p(x,t) \mapsto \lambda^2 p(\lambda x, \lambda^2 t)\,.$$

Il peut donc exister des solutions particulières auto-similaires de la forme $u(x,t) = \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right)$ où $\phi(\xi) = u(\xi, 1)$ est le *profil* de la solution. Ces solutions sont invariantes par le changement d'échelle ci-dessus. Bien sûr, une telle invariance n'est possible dans la bande

$\mathbf{R}^2 \times (0,1)$ que dans les directions non-bornées de $\mathbf{R}^2$. On est donc amené à chercher les solutions auto-similaires de l'équation de Navier-Stokes bidimensionnelle

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p \,, \quad \text{div } u = 0 \,, \quad t > 0 \,, \quad x \in \mathbf{R}^2 \,.$$

En utilisant la formulation en tourbillon $\omega = \text{rot } u$, on met en évidence une solution $u^G(\frac{x}{\sqrt{t}})$ appelée Vortex d'Oseen, dont le tourbillon associé G est solution de l'équation elliptique scalaire

$$\Delta_\xi G + \frac{1}{2}(\xi \cdot \nabla_\xi) G + G = 0 \,, \quad \text{div } G = 0 \,, \quad \xi \in \mathbf{R}^2 \,. \tag{5}$$

En dimension 2 d'espace, la stabilité de ce tourbillon a été étudiée par Y. Giga et T. Kambe [42], par A. Carpio [20] et par Th. Gallay et C.E Wayne [39]. La deuxième partie de cette thèse (chapitre II) montre comment le tourbillon d'Oseen intervient dans le développement asymptotique des solutions de l'équation de Navier-stokes (2) ou de Navier-Stokes Coriolis (3) dans la bande $\mathbf{R}^2 \times (0,1)$. Le comportement en grands temps de (2) et de (3) est en fait régi par l'équation de Navier-Stokes bidimensionnelle et toute solution globale uniformément bornée en temps converge vers le tourbillon d'Oseen.

Le propos de cette thèse est donc de développer, à travers deux exemples d'*équations d'évolution paraboliques semi-linéaires*, une étude qualitative de solutions. En particulier, on s'intéresse à la *stabilité* de solutions particulières (ondes progressives, solutions auto-similaires) données par des équations elliptiques comme (4) et (5). On utilise à ce propos les méthodes mentionnées précédemment: approximation par linéarisation, décomposition de l'espace, bornes exponentielles...

La suite de cette introduction passe successivement en revue l'état d'avancement des recherches actuelles sur les équations (1), (2) et (3) et présente les travaux de cette thèse dans leur contexte. On propose aussi pour chacune des équations quelques perspectives de recherche.

Equation de réaction-diffusion

La première partie de cette thèse s'inscrit dans un cadre plus général d'étude des équations de réaction-diffusion-advection

$$\partial_t u - \text{div } (A(x) \nabla_x u) + q(x) \cdot \nabla u = F(x,u) \,, \quad t > 0 \,, \quad x \in \Omega$$

où $u(x,t) \in \mathbf{R}$ est l'inconnue du problème tandis que $A(x) \in \mathcal{M}_n(\mathbf{R})$, $q(x) \in \mathbf{R}^n$ et $F \in \mathcal{C}^0(\Omega \times \mathbf{R}, \mathbf{R})$ sont donnés. L'ensemble Ω est l'espace $\mathbf{R}^n$ tout entier ou un cylindre infini de $\mathbf{R}^n$, $n \geq 2$. Ces équations ont été utilisées à de multiples reprises pour modéliser des phénomènes de propagation tels que la dynamique des populations, les systèmes chimiques ou la combustion. On donne tout d'abord un aperçu de ces modélisations avant de montrer comment les fronts progressifs jouent un rôle important dans la description qualitative de ces équations.

Dans le cas $n = 2$ et $\Omega = \mathbf{R}^2$, considérons une population composée d'individus répartis dans le plan. On suppose qu'un des gènes de ces individus existe sous deux formes alléliques a et A. On peut donc classer la population en trois catégories selon leur génotype: les homozygotes aa ou AA et les hétérozygotes aA. Supposons que cette population se mélange au hasard dans le plan avec une constante de diffusion égale à 1, connaît un taux de natalité r et un taux de mortalité τ_1, τ_2 ou τ_3 selon le genotype aa, aA ou AA, certains étant plus viables que d'autres. Soit $u(x,t) \in [0,1]$ la densité relative de l'allèle A à l'instant t et au point $x \in \mathbf{R}^2$ du plan. Alors sous certaines hypothèses [1], une bonne approximation de u est donnée par l'équation

$$\partial_t u = \Delta u + F(u)\,, \quad t > 0\,, \quad x \in \mathbf{R}^2$$

où $F(u) = u(1-u)(\tau_1 - \tau_2 - (\tau_1 - 2\tau_2 + \tau_3)u)$. La non-linéarité F possède alors les caractéristiques suivantes:

$$F \in \mathcal{C}^1(\mathbf{R})\,, \quad F(0) = F(1) = 0\,.$$

Selon les valeurs des taux τ_j, F possède une troisième propriété qui intervient de manière fondamentale dans la suite du propos.

Cas hétérozygote intermédiaire: Si $\tau_3 < \tau_2 < \tau_1$, alors F vérifie

$$F'(0) > 0\,,\ F'(1) < 0\,, \quad F(u) > 0 \text{ pour } u \in (0,1)\,.$$

Ce cas est autrement appelé “KPP” du nom des auteurs A.N Kolmogorov, I.G Petrovsky and N.S Piskunov qui l'ont étudié [59], ou *monostable* car $u \equiv 0$ est un état d'équilibre instable du système tandis que $u \equiv 1$ est un état stable.

Cas hétérozygote supérieur: Si $\tau_2 < \tau_3 \leq \tau_1$ alors F satisfait

$$\exists \theta \in (0,1) \,|\, F(u) > 0 \text{ sur } (0,\theta) \text{ et } F(u) < 0 \text{ sur } (\theta,1)\,, \quad F'(0) > 0\,,\ F'(1) > 0\,.$$

Ce cas n'est pas fréquemment étudié puisque les deux états d'équilibre 0 et 1 sont instables.

Cas hétérozygote inférieur: Si $\tau_3 \leq \tau_1 < \tau_2$ alors F vérifie

$$\exists \theta \in (0,1) \,|\, F(u) < 0 \text{ sur } (0,\theta) \text{ et } F(u) > 0 \text{ sur } (\theta,1)\,, \quad F'(0) < 0\,,\ F'(1) < 0\,.$$

Ce cas est aussi appele *bistable* puisque les deux états d'équilibre sont stables. C'est cette configuration qui est traitée dans cette thèse, et notamment dans l'article de recherche [78] dont est constitué le chapitre I.

Cette petite modélisation génétique a surtout le mérite de mettre en lumière les différentes propriétés que peut vérifier la non-linéarité F. D'autres cas importants sont obtenus dans le modèle de la combustion.

On considère une réaction chimique gazeuse ayant lieu dans un cylindre infini $\Omega = \mathbf{R}\times\omega$ où ω est un domaine connexe borné et régulier de $\mathbf{R}^{n-1}$, $n \geq 2$. On suppose que la réaction est irréversible, totale et élémentaire du type $A \to B$. Le taux de transformation ϕ du réactif A en produit B est alors donné par la loi d'action de masse

$$\phi = k(T)[A]$$

où T est la température, $k(T)$ le cœfficient thermique de la réaction donné par la loi d'Arrhenius et $[A]$ la concentration de la substance A. Si on suppose de plus que les concentrations sont uniformément réparties en espace, que la réaction s'accompagne d'une diffusion des produits et d'un échange de chaleur, on obtient alors le système d'EDP couplées

$$\begin{aligned} &\partial_t T + q(x)\cdot\nabla T = \Delta T + k(T)[A]\,, && t>0\,, \quad x\in \mathbf{R}\times\omega \\ &\partial_t [A] + q(x)\cdot\nabla [A] = (Le)^{-1}\Delta [A] - k(T)[A]\,, && t>0\,, \quad x\in \mathbf{R}\times\omega \end{aligned}$$

où $q(x)$ est le champ de vitesse donné par la mécanique des fluides dans le cylindre infini, indépendant de T et de $[A]$. Le nombre de Lewis Le est le rapport des cœfficients de conduction thermique et de diffusion moléculaire des espèces réactives . S'il est quelconque, l'étude de ce système est difficile et de nombreuses questions restent ouvertes [45]. Par contre, si $Le = 1$, on peut se ramener [9] à l'étude d'une seule équation pour la température renormalisée $u \in [0, 1]$

$$\partial_t u = \Delta u - q(x)\cdot\nabla u + F(u)\,, \quad t>0\,, \quad x\in\mathbf{R}\times\omega \tag{6}$$

où $F(u) = (1-u)e^{-E/u}$. Cette équation s'inscrit dans le cadre des équations de réaction-diffusion-advection énoncé précédemment et la non-linéarité F peut être classée dans le cas “hétérozygote intermédiaire” bien que $F'(0) = 0$. Cette variante est aussi appelée cas “ZFK” [98] et induit des conclusions semblables au cas “KPP”.

On distingue enfin un autre type de non-linéarité, souvent appelé “combustion”. C'est une approximation par cut-off du cas “ZFK”. Soit θ une température d'ignition normalisée en dessous de laquelle la réaction ne se produit pas. Alors, la température est donnée par l'équation (6) où F vérifie les propriétés caractéristiques

$$\exists\theta\in(0,1)\,|\,F\equiv 0 \text{ sur } [0,\theta]\,, \quad F>0 \text{ sur } (\theta,1)\,, \quad F'(1)<0\,.$$

Les trois types de non-linéarités fréquemment étudiés dans la littérature sont les cas “KPP” ou “ZFK” [1], [16], [59], [98], “combustion” [9], [45], [54] et “bistable” [2], [31], [32], [78].

Un des intérêts de ces équations réside dans le comportement en grands temps des solutions et notamment l'existence et la stabilité de fronts progressifs. Ces fronts sont des solutions d'évolution qui relient les deux états stationnaires $u\equiv 0$ et $u\equiv 1$ en gardant leur forme (ou profil) et en se propageant à vitesse constante [6], [93]. Essentiellement, ce sont des solutions de la forme $u(x,t) = w_0(x\cdot k - ct)$ où k est un vecteur unité qui détermine la direction de propagation (k peut dépendre de x comme le montre l'exemple des fronts sphériques traité ultérieurement).

Une première étude concerne les solutions comprises entre 0 et 1 de l'équation

$$\partial_t u = \Delta u + F(u)\,, \quad t>0\,, \quad x\in\mathbf{R}^n\,, \tag{7}$$

en dimension $n = 1$ d'espace. Les fronts sont des solutions particulières de la forme $u(x,t) = w_0(x-ct)$ et leur existence comme leur stabilité dépend beaucoup de la non-linéarité F considérée. Dans les cas “combustion” et “bistable”, il existe une unique vitesse c^* et un profil w_0 strictement décroissant, unique à translation près, tandis que dans le cas “KPP”, il existe une demi-droite $[c^*, +\infty)$, $c^* > 0$, de vitesses solutions et pour tout $c \geq c^*$, il existe un profil w_0 strictement décroissant, unique à translation près. Nous renvoyons pour plus de détails à [1], [10], [33], [54], [59] et [98]. En dimension supérieure $n \geq 2$, une généralisation peut être faite en définissant des fronts plans de la forme

$$u(x,t) = w_0(x \cdot k - c^* t), \quad k \in S^{n-1}$$

ou des fronts à symétrie sphérique de la forme

$$u(x,t) = w_0(|x| - c^* t).$$

Ces fronts ne sont pas toujours des solutions exactes des équations de réaction-diffusion dans $\mathbf{R}^n$ mais jouent un rôle important dans le comportement asymptotique des solutions.

En effet, la dynamique de ces fronts et notamment la convergence des solutions de (7) vers ces ondes progressives, en fonction des conditions initiales, a été très largement étudiée. Pour le cas "KPP" ou "ZFK", nous renvoyons à [80] pour une monographie complète. Nous nous focalisons désormais sur le cas "bistable".

Soit une donnée initiale $u_0 : \mathbf{R}^n \to [0,1]$, continue et qui tend vers 0 à l'infini. Il existe alors une unique solution $u(x,t)$ à l'équation (7) dans $\mathbf{R}^n$ où F est "bistable", possédant les mêmes propriétés que u_0. YA.I Kanel [55] puis D.G Aronson et H.F Weinberger [2] ont étudié un effet de seuil régissant les comportements de u en fonction de u_0. En effet, si u_0 est "suffisamment petite", alors u converge vers 0 lorsque t tend vers $+\infty$, uniformément en x et exponentiellement en t. Par contre, si u_0 est "suffisamment grande", alors u converge vers 1 uniformément en x sur tout compact et exponentiellement en temps. C'est ce deuxième comportement que nous nous proposons d'approfondir. D.G Aronson et H.F Weinberger montrent en fait que si u_0 est à support compact "assez grand" alors l'état stationnaire $u \equiv 1$ envahit l'état $u \equiv 0$ à la vitesse asymptotique c^* des ondes progressives associées à (7) avec $n = 1$. Il reste donc à préciser l'allure de cette invasion.

En dimension $n = 1$ d'espace, P.C Fife et J.B Mc Leod [31] montrent que si u converge vers 1 uniformément sur tout compact quand t tend vers $+\infty$ alors u se comporte en fait comme une paire de fronts divergents. En dimension supérieure, la généralisation de ces résultats passe par les fronts sphériques. C.K.R.T Jones [51], [52] et K. Uchiyama [92] ont montré la convergence des solutions radiales vers une onde progressive sphérique de rayon

$$R(t) = c^* t - \frac{n-1}{c^*} \log t + L + \epsilon(t)$$

où L est une constante et $\epsilon(t) \to 0$ lorsque $t \to +\infty$. Cette expression montre que l'onde progresse à la vitesse c^* des ondes unidimensionnelles atténuée par un terme dû à la courbure de l'interface entre les états $u \equiv 1$ et $u \equiv 0$. Dans la première partie de cette thèse (chapitre I.2), nous montrons que la famille des ondes progressives à symétrie sphérique est asymptotiquement stable pour toute petite perturbation radiale. Nous précisons le taux de convergence des solutions u radiales vers les ondes progressives sphériques: toute petite perturbation converge vers 0 en $O(\log t/t^2)$ et la convergence établie par K. Uchiyama et C.K.R.T Jones est en $O(\log t/t)$ lorsque t tend vers $+\infty$.

Dans le cas plus général des solutions quelconques, C.K.R.T Jones [52] a montré que si u converge vers 1 uniformément en x sur tout compact lorsque t tend vers $+\infty$, alors la solution u converge vers une onde progressive unidimensionnelle dans chaque direction radiale. Il a aussi prouvé que pour tout $t > 0$ et pour tout $\beta \in (0, 1)$, les normales à la surface de niveau $\{x \in \mathbf{R}^n \,|\, u(x, t) = \beta\}$ coupent le support de u_0. L'interface entre les deux états $u \equiv 0$ et $u \equiv 1$ devient donc localement sphérique. Ces résultats montrent que

$$W(x, t) = w_0(|x| - c^* t + \frac{n-1}{c^*} \log t)$$

est une solution asymptotique de (7) et que toute solution à symétrie sphérique qui s'annule à l'infini et qui converge vers 1 uniformément sur tout compact converge en fait vers un translaté temporel de W. Par contre, ce résultat est faux pour des solutions quelconques. En effet, on montre au chapitre I.3 que la famille des ondes progressives à symétrie sphérique n'est pas asymptotiquement stable pour des perturbations quelconques. On met en évidence une classe de solutions qui ne convergent pas vers un front sphérique et l'on montre que dans cette classe de solutions, certaines ont d'autres comportements asymptotiques comme des fronts plissés. Ces résultats font l'objet d'un article de recherche, à paraître dans les Annales de l'IHP, analyse non-linéaire. Ces travaux soulèvent naturellement d'autres questions: quel est le taux de convergence optimal dans les résultats précédents? Peut-on caractériser la classe des conditions initiales telles que les solutions associées convergent vers un front plissé? Peut-on obtenir la stabilité asymptotique de ces fronts plissés?

Le cas de la non-linéarité "combustion" fait également l'objet de nombreuses recherches, notamment dans des cylindres infinis. Soit l'équation

$$\partial_t u + q(x) \cdot \nabla u = \Delta u + F(u)\,, \quad x = (x_1, y) \in \mathbf{R} \times \omega$$

avec les conditions aux limites

$$u(t, -\infty, y) = 1\,,\ u(t, +\infty, y) = 0\,,\ \text{uniformément par rapport à } y \in \bar{\omega}\,,$$
$$\partial_\nu u = 0 \text{ sur } \partial\omega\,.$$

Si on considère des écoulements parallèles où q s'écrit $q(x) = (\alpha(y), 0, .., 0)$, on peut rechercher des fronts progressifs de la forme

$$u(x, t) = w_0(x_1 + ct, y)$$

solution de l'équation elliptique

$$\Delta w_0 - (c + \alpha(y))\partial_{x_1} w_0 + F(w_0) = 0\,, \quad x \in \mathbf{R} \times \omega\,.$$

Les résultats sur ce problème sont plus récents [11], [12], [14] et semblables au cas unidimensionnel. Si F est une non-linéarité de type "combustion", alors il existe une unique vitesse c^* et un profil w_0 décroissant en x_1 et unique aux translations près dans la variable x_1. Ces problèmes posés dans des cylindres infinis ont également donné lieu à des recherches pour les autres types de non-linéarités. Si F est une non-linéarité "KPP", il existe une demi-droite de vitesses solutions $[c^*, +\infty)$ et pour toute vitesse $c \geq c^*$, il existe un profil w_0 décroissant en x_1 et unique aux translations près dans la variable x_1. Si F

est une non-linéarité "bistable", les résultats varient selon la géométrie de la section du cylindre: si α est constante sur ω quelconque ou continue sur $\bar{\omega}$ convexe, alors la solution est celle du problème vu précédemment. Par contre, il existe des domaines non convexes ω et des fonctions non constantes α pour lesquels il n'y a pas de solutions qui connectent les états 0 et 1 sous la forme de fronts progressifs [7]. Enfin, quelle que soit la non-linéarité considérée, la stabilité de ces fronts a été également étudiée dans [13], [64], [75], [76], [77].

Enfin, si on ne considère plus des écoulements parallèles mais périodiques (par exemple dans des grilles), q est périodique et on obtient une généralisation des fronts progressifs à travers la théorie des fronts pulsatoires progressifs [8], [96], [97]. D'autres développements récents généralisent ces théories d'ondes progressives au cas de la combustion dans les becs Bunsen, c'est-à-dire dans des flammes de formes coniques [15], [46], [47].

Equation de Navier-Stokes

La deuxième partie de cette thèse traite quant à elle des équations de Navier-Stokes. Présenter ces équations de la mécanique des fluides n'est pas chose aisée tant la littérature et les recherches abondent depuis l'article pionnier de Jean Leray [60]. Comme pour l'équation de la chaleur, nous commencerons donc par quelques remarques sur l'origine de ces équations.

On considère un fluide dans un domaine Ω de $\mathbf{R}^3$ supposé indépendent du temps. La mécanique des fluides peut décrire son mouvement de deux manières différentes. Soit, on utilise la description Lagrangienne du mouvement qui à chaque particule de fluide associe une trajectoire $x = \phi(x_0, t)$ où $x_0 \in \Omega$ est la position initiale ($t = 0$) de cette particule et x celle au temps t. Soit, comme pour les équations de Navier-Stokes, on considère la description Eulérienne du mouvement et on s'attache à décrire la vitesse $u(x,t)$ du fluide qui est en $x \in \Omega$ au temps $t \geq 0$. Ces deux visions sont bien sûr reliées par les équations différentielles suivantes

$$\begin{aligned} u(x,t) &= \partial_t \phi(x_0, t) \\ \text{et } \partial_t \phi(x_0,t) &= u(\phi(x_0,t),t)\,, \quad \phi(x_0,0) = x_0\,. \end{aligned}$$

On cherche maintenant à évaluer l'évolution de u. Le fluide est soumis tout à la fois à des forces extérieures notées f et à un tenseur de contraintes appelé tenseur de Cauchy et noté $\sigma(x,t) \in \mathcal{M}_3(\mathbf{R})$. La relation fondamentale de la dynamique s'écrit alors pour une particule de fluide [89]

$$\rho(x,t)\gamma(x,t) = f(x,t) + \nabla_x \cdot \sigma(x,t)\,, \quad x \in \Omega$$

où $\gamma = \partial_t u + (u \cdot \nabla)u$ est l'accélération du fluide et ρ sa densité. De plus, pour les fluides dits Newtoniens, le tenseur des contraintes σ est donné pour tout $(i,j) \in \{1,..,3\}^2$,

$$\sigma_{ij}(x,t) = \mu(\partial_{x_j} u_i(x,t) + \partial_{x_i} u_j(x,t)) + \lambda(\text{div } u)\delta_{ij} - p(x,t)\delta_{ij}$$

où $p(x,t)$ est la pression, $\mu > 0$, $\lambda \in \mathbf{R}$. A cela s'ajoute la loi de conservation de la masse

$$\partial_t \rho + \text{div } (\rho u) = 0\,.$$

On ne considère dans notre propos que des fluides incompressibles, donc u est de divergence nulle. Ceci se traduit dans la loi de conservation de masse par une densité ρ constante au cours du temps. Les équations de Navier-Stokes pour les fluides incompressibles homogènes s'écrivent alors

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u = f - \nabla p \,, \quad \text{div } u = 0 \tag{8}$$

où ν est la viscosité cinématique. Le cas $\nu = 0$ correspond aux fluides parfaits pour lesquels le terme de diffusion est négligé et dont l'écoulement est modélisé par les équations d'Euler.

Afin de traiter le problème de Cauchy associé à (8), on considère également la condition initiale suivante

$$u(x,0) = u_0(x) \,,\; x \in \Omega$$

et, si Ω possède une frontière, des conditions de bord. Les conditions de Dirichlet s'écrivent

$$u \equiv 0 \text{ sur } \partial\Omega$$

et correspondent à une vitesse nulle au bord, c'est-à-dire à une condition de non-glissement du fluide sur les parois. Elles peuvent donner lieu, lorsque la viscosité est petite, à des couches limites, c'est-à-dire des zones minces près du bord où la vitesse varie rapidement, [85]. Moins réalistes physiquement, les conditions de bord périodiques dans une ou plusieurs directions s'inscrivent dans le cadre de l'étude des turbulences et permettent d'un point de vue mathématique l'utilisation d'outils aussi efficaces que la transformée de Fourier. On pourra aussi noter l'existence de conditions "stress-free" pour les écoulements sans cisaillement qui se traduisent mathématiquement par

$$u \cdot \mathbf{n} = 0 \text{ et } (\text{rot } u) \wedge \mathbf{n} = 0 \text{ sur } \partial\Omega$$

où $\mathbf{n}$ est la normale extérieure de Ω.

Il convient enfin de rappeler que les inconnues u et p de (8) ne jouent pas le même rôle. En effet, la pression p peut être obtenue comme fonction de u en résolvant l'équation suivante

$$\Delta p = \text{div } f - \Delta(u \otimes u)$$

avec les conditions aux limites périodiques ou de Neuman

$$\frac{\partial p}{\partial \mathbf{n}} = f \cdot \mathbf{n} + \nu \Delta u \cdot \mathbf{n} \text{ sur } \partial\Omega$$

selon que l'on a choisi les conditions périodiques ou de Dirichlet pour (8). Ces équations sur p sont obtenues en prenant soit la divergence soit le produit scalaire avec $\mathbf{n}$ de (8).

Ces équations de Navier-Stokes peuvent être vues sous deux angles différents et complémentaires: celui des solutions faibles développé par J. Leray [60] ou celui des solutions fortes initié par H. Fujita et T. Kato [34]. Si ces deux théories se rejoignent en dimension 2 d'espace, leur divergence en dimension 3 reste l'un des grands problèmes ouverts de ce début de siècle. Développons maintenant chacune des deux approches dans le cas simplifié où les forces extérieures au fluide f sont nulles.

La vision de J. Leray considère des solutions d'énergie finie. Pour une condition initiale $u_0 \in L^2(\mathbf{R}^n)$, $n = 2$ ou 3, à divergence nulle, J. Leray [60] construit par régularisation du terme non-linéaire des solutions faibles globales à divergence nulle

$$u \in L^\infty(\mathbf{R}^+, L^2(\mathbf{R}^n)) \cap L^2(\mathbf{R}^+, \dot{H}^1(\mathbf{R}^n))$$

appelées aujourd'hui solutions "à la Leray" associées à la condition initiale u_0. Elles vérifient également l'inégalité d'énergie

$$\|u(t)\|^2_{L^2(\mathbf{R}^n)} + 2\nu \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbf{R}^n)} ds \leq \|u_0\|^2_{L^2(\mathbf{R}^n)} \,.$$

De plus, en dimension $n = 2$ d'espace, la solution u est unique et vérifie l'égalité d'énergie. En dimension $n = 3$ d'espace, l'unicité de ces solutions reste une inconnue depuis 70 ans. La régularité de ces solutions dites "turbulentes" a été largement étudiée, notamment par J. Leray lui-même qui montre qu'elles sont indéfiniment différentiables sur $O \times \mathbf{R}^3$ où O est le complémentaire d'un fermé de mesure nulle. Un travail plus récent de L. Caffarelli, R. Kohn et L. Nirenberg [17] vient préciser cette étude des singularités des solutions "à la Leray". Pour tous ceux que la lecture de l'article "ancien" de J. Leray rebuterait, nous conseillons vivement la revue qu'en a faite J.Y. Chemin [23].

La vision de H. Fujita et T. Kato [34] considère, pour sa part, les solutions fortes et régulières de (8). Pour une condition initiale $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$ à divergence nulle, H. Fujita et T. Kato montrent à l'aide de la théorie des semi-groupes qu'il existe un unique temps maximal $T^* > 0$ et une unique solution u de (8) avec $u(0) = u_0$ tels que pour tout temps $T < T^*$, on ait

$$u \in \mathcal{C}^0([0,T], \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)) \cap L^2((0,T), \dot{H}^{\frac{3}{2}}(\mathbf{R}^3)) \,.$$

De plus, il existe une constante $C > 0$ telle que si $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq C\nu$, alors la solution u est globale dans $\mathcal{C}^0(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}(\mathbf{R}^3))$. Un autre résultat de T. Kato [57] montre un théorème équivalent pour une donnée initiale dans $L^3(\mathbf{R}^3)$. Bien sûr, l'existence globale de telles solutions pour des conditions initiales arbitrairement grandes est un problème ouvert intimement lié au précédent.

Il est important de noter que ces espaces $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$ ou $L^3(\mathbf{R}^3)$ sont des espaces respectant le scaling de l'équation de Navier-Stokes tridimensionnelle, c'est-à-dire que les normes associées sont invariantes par le changement d'échelle

$$\phi(x) \to \lambda\phi(\lambda x) \,, \quad x \in \mathbf{R}^3 \,.$$

Rappelons que si $u(x,t)$ est une solution de (8) associée à la donnée initiale u_0, alors pour tout $\lambda \in \mathbf{R}$, $u_\lambda(x,t)$ est une solution associée à $u_{0,\lambda}$ où

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \text{ et } u_{0,\lambda} = \lambda u_0(\lambda x) \,.$$

La condition de petitesse imposée par le théorème de T. Kato ne dépend donc pas de l'échelle choisie. De plus, notons que $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3) \hookrightarrow L^3(\mathbf{R}^3)$. La recherche d'espaces de plus en plus gros invariants par ce changement d'échelle et dans lesquels l'équation de Navier-Stokes tridimensionnelle peut être résolue localement en temps et globalement pour des données initiales petites a fait l'objet de nombreux travaux. De $\dot{H}^{\frac{1}{2}}$ à ∂BMO, citons les travaux de H. Fujita et T. Kato [34], [57], de G. Furioli, P.G Lemarié et E. Terraneo [35], de M. Cannone [19], de F. Planchon [69] et de H. Koch et D. Tataru [58]. Enfin, pour qui hésiterait à lire l'article de Tosio Kato, nous recommandons fortement l'introduction du livre de Henry [48] qui ne manque pas d'éloges sur la clarté de ce travail.

Pour récapituler, notons que la théorie des solutions faibles de Navier-Stokes est liée à la structure spécifique de ces équations et en particulier à l'inégalité d'énergie. Par contre, l'approche de T. Kato est plus générale et pourrait s'appliquer à d'autres équations paraboliques semi-linéaires. La théorie des solutions faibles est donc liée à l'espace d'énergie L^2 qui est sous-critique par rapport à l'espace de Lebesgue L^3 invariant par changement d'échelle et lié quant à lui aux solutions fortes [38]. Cette remarque semble dire que l'utilisation de l'estimation d'énergie pour obtenir des informations sur les solutions fortes mène d'emblée à une impasse. Enfin, pour combler le fossé entre les solutions faibles L^2 et fortes L^3, C. Calderòn [18] a montré l'existence de solutions faibles globales dans $L^p(\mathbf{R}^3)$ pour $p \in [2, 3]$.

L'article de J. Leray laisse d'autres questions en suspens. Notamment, comment se comporte une solution régulière sur l'intervalle de temps maximal $[0, T^*)$ lorsque t tend vers T^*? Dans le cas où T^* est fini, J. Leray donne quelques éléments de réponse en minorant les normes $L^p(\mathbf{R}^3)$ des solutions par $C/(T^* - t)^{\frac{1}{2}(1-\frac{3}{p})}$ pour $p \in \{2\} \cup (3, +\infty)$. Mais la question de savoir si cette rupture de régularité a effectivement lieu n'est toujours pas claire. J. Leray propose de chercher du côté des fonctions auto-similaires de la forme

$$u(x,t) = \frac{1}{\sqrt{T^* - t}} U\left(\frac{x}{\sqrt{T^* - t}}\right),$$

mais il a été récemment démontré par J. Neças, M. Ruczicka et V. Sverák [66] qu'il n'existe pas de solutions non-triviales de ce type avec $U \in L^3(\mathbf{R}^3)$. Un autre travail de L. Escauriaza, G. Seregin et V. Sverák [30] montre que si u est une solution régulière sur $(0, T^*)$ avec $T^* < +\infty$ alors $\lim_{t \to T^*} \|u(t)\|_{L^3(\mathbf{R}^3)} = +\infty$.

Si maintenant T^* est infini (ce qui est le cas pour des conditions initiales petites par exemple), ce problème revient à s'intéresser au comportement asymptotique des solutions globales. J. Leray donnait déjà quelques majorations de $\|\nabla u\|_{L^2}$ ou de $\|u(t)\|_{L^\infty}$, mais l'engouement pour le sujet vient incontestablement des résultats de T. Kato [57] qui montrent que si $T^* = +\infty$, $\|u(t)\|_{L^p(\mathbf{R}^3)}$ décroît comme $t^{-\frac{1}{2}(1-\frac{3}{p})}$ lorsque t tend vers $+\infty$, pour $p \in (3, +\infty]$ et $\|\nabla u(t)\|_{L^p(\mathbf{R}^3)}$ comme $t^{-(1-\frac{3}{2p})}$ pour $p \in [3, +\infty)$. Ce résultat a été récemment complété par I. Gallagher, D. Iftimié et F. Planchon [38] qui montrent que si $u(t)$ est une solution forte globale de (8) alors la norme L^3 de $u(t)$ tend vers 0 quand t tend vers $+\infty$. Une première estimation de décroissance pour la norme L^2 est donnée par M. Wiegner [94] qui montre que $\|u(t)\|_{L^2(\mathbf{R}^3)}$ décroît comme $t^{-5/4}$ quand t tend vers l'infini. Une série de papiers de M. Schonbek [82], [83], [84] étudie également la décroissance en temps des solutions de Navier-Stokes pour différentes normes avec taux optimaux. Plus récemment, Th. Gallay et C.E Wayne [39], [40] ont revisité ces résultats à l'aide des variétés invariantes. Utilisant l'idée que ces variétés contrôlent le comportement en grands temps des solutions des équations de Navier-Stokes bi ou tri-dimensionnelles, ils ont été en mesure de calculer le comportement asymptotique des solutions, en théorie à tout ordre, en pratique jusqu'à l'ordre 2.

Les résultats présentés jusqu'à présent sont ceux de l'espace $\mathbf{R}^n$ tout entier pour $n = 2$ ou 3. Une présentation très complète des équations de Navier-Stokes en domaine borné Ω peut être trouvée dans [88]. Nous nous intéressons maintenant au cas de la bande tridimensionnelle. On trouve à ce sujet une série de travaux dans le cas d'une bande fine de type $\mathbf{R}^2 \times (0, \epsilon)$ où ϵ est "suffisamment petit". Dans ce cas, des résultats d'existence globale de solutions fortes pour une classe de données initiales éventuellement grandes sont donnés dans [70], [71], [72], [90]. Les résultats dans une bande de largeur constante $\mathbf{R}^2 \times (0, 1)$ sont moins exhaustifs. On montre dans le deuxième chapitre de cette thèse (chapitre II.2) que le comportement asympotique des solutions des équations de Navier-Stokes (8) dans la bande $\mathbf{R}^2 \times (0, 1)$ avec conditions de bord périodiques ou "stress-free" peut être déduit de celui de l'équation de Navier-Stokes bidimensionnelle. Qu'il s'agisse de solutions à données initiales petites ou de solutions a priori régulières et globales (sans condition sur la donnée initiale), on montre leur convergence vers le tourbillon d'Oseen. On calcule également à l'aide d'une méthode générique le comportement asymptotique des solutions jusqu'au second ordre. Ces résultats font l'objet d'un deuxième article de recherche soumis [79].

Equation pour les fluides tournants

Dans la dernière partie de cette thèse (chapitre II.3), on s'intéresse aux mouvements des océans. Les équations de Navier-Stokes sont insuffisantes pour modéliser ce type d'évolution car elles négligent trop de phénomènes importants comme la rotation de la Terre. Pour donner un ordre de grandeur, une particule de fluide met environ 50 jours à traverser l'Océan Atlantique (en dehors de forts courants comme le Gulf Stream). La Terre a donc effectué 50 rotations pendant ce même laps de temps et l'effet de la force de Coriolis sur l'étude du mouvement des océans pour des temps grands ne peut être négligé. On renvoie à la lecture des monographies de H.P Greenspan [43] et de J. Pedlosky [68] pour une présentation physique exhaustive du modèle.

Dans une première approximation, on peut donc considérer que la vitesse d'un océan est donnée localement par l'équation de Navier-Stokes Coriolis

$$\partial_t u + (u \cdot \nabla) u + \Omega(e_3 \wedge u) = \nu \Delta u - \nabla p \, , \; \mathrm{div}\; u = 0 \, , \quad (x, z) \in \mathbf{R}^2 \times (0, 1) \, , \, t > 0 \qquad (9)$$

où $\Omega \in \mathbf{R}$ est la rotation de la Terre et e_3 le vecteur unité de l'axe de rotation vertical. Bien sûr, ce modèle est assez simpliste, ne serait-ce que parce qu'il suppose que la Terre est plate et qu'il ne prend en compte ni la géographie du fond des océans, ni des paramètres importants en océanographie comme la température, la salinité ou les courants. D'autres modèles ont été envisagés pour mieux rendre compte de cette réalité plus complexe. Citons notamment le modèle primitif étudié par J.L Lions, R. Temam et S. Wang [62] [63], par T. Beale et A. Bourgeois [5], par I. Gallagher [36], par D. Iftimié [50] ou par F. Charve [22] ou le modèle des vents par B. Desjardins et E. Grenier [28]. Des efforts sur la géométrie du domaine ont également été faits par R. Temam et M. Ziane [91] en considérant des domaines compris entre deux sphères proches.

Revenons à l'équation de Navier-Stokes Coriolis, appelée également équation de Navier-Stokes pour les fluides tournants . Mathématiquement, l'équation (9) est une équation parabolique pénalisée par l'opérateur antisymétrique L défini par $Lw = \Omega(e_3 \wedge w)$. Le caractère antisymétrique de L assure que les équations (8) et (9) vérifient la même estimation d'énergie. En particulier, on peut développer pour l'équation de Navier-Stokes Coriolis les mêmes approches que pour l'équation de Navier-Stokes sur les solutions fortes ou faibles. Notons que le temps maximal d'existence des solutions fortes est alors indépendant de Ω. Notamment, ce temps ne tend pas vers 0 lorsque Ω tend vers l'infini. Cependant, les mêmes questions se posent: les solutions fortes sont-elles globales, et les solutions faibles, uniques?

Le terme de Coriolis, s'il ne change pas l'estimation d'énergie, apporte tout de même des contributions intéressantes pour répondre à ces questions, notamment dans la limite des rotations rapides. (Physiquement, cette limite est justifiée puisque le nombre de Rossby $(1/\Omega)$ est habituellement d'un ordre de grandeur de 10^{-1} à 10^{-3}s.) La force de Coriolis crée alors une dissymétrie entre les mouvements verticaux et horizontaux, l'évolution verticale du fluide étant freinée. Cette propriété est parfois prise en compte pour l'étude de la turbulence en remplaçant la viscosité isotrope ν par une viscosité anisotrope. On distingue alors le terme de diffusion horizontale $-\nu_h(\partial_{x_1}^2 + \partial_{x_2}^2)$ où $\nu_h > 0$, du terme de diffusion verticale $-\nu_v \partial_z^2$ où $\nu_v \geq 0$ et $\nu_v \neq \nu_h$. En général, ν_v dépend alors de Ω et tend vers 0 lorsque Ω tend vers l'infini. Ceci traduit le fait que les turbulences verticales sont pénalisées par rapport aux turbulences horizontales. En conséquence de cette anisotropie, le fluide adopte, sous l'influence d'une rotation rapide, un comportement bidimensionnel et horizontal, c'est-à-dire que $u(x,z,t) = (u_1(x,t), u_2(x,t), 0)$ dans la limite où Ω tend vers l'infini. Ce phénomène se traduit physiquement par l'apparition de colonnes de Taylor Proudman et cette contrainte sur le mouvement du fluide provoque également d'autres phénomènes.

Supposons par exemple que le fluide évolue entre deux plaques infinies fixées. Les conditions au bord de Dirichlet imposent que u soit nulle au bord, en $z = 0$ et en $z = 1$. Pour ne pas contredire les explications précédentes selon lesquelles la vitesse du fluide est asymptotiquement bidimensionnelle et horizontale, la vitesse doit donc soit être uniformément nulle dans la bande (et alors son étude est donc moins intéressante), soit varier très rapidement au niveau du bord: c'est le phénomène des couches limites d'Ekman [24], [29], [44].

Revenons à un cas plus général. Une question naturelle se pose face à cette contrainte de mouvement asymptotique bidimensionnel lorsque Ω tend vers l'infini: comment évolue la composante tridimensionnelle de la vitesse initiale? Notons que l'on peut décomposer toute donnée initiale u_0 en une partie 2D notée $\bar{u}_0(x)$ et une partie 3D $\tilde{u}_0(x,z)$ vérifiant les mêmes conditions aux limites que u_0 (voir chapitre II.4.1). Alors, aux termes non-linéaires de (9) près, la partie 3D $\tilde{u}$ de la solution u vérifie une équation du type

$$\partial_t \tilde{u} - \nu \Delta \tilde{u} + \Omega e_3 \wedge \tilde{u} = -\nabla \tilde{p}\,, \quad \text{div } \tilde{u} = 0\,, \quad \tilde{u}(0) = \tilde{u}_0$$

qui décrit la création d'ondes appelées ondes de Poincaré se propageant très vite dans le domaine (vitesse de l'ordre de Ω). Avec des conditions de bord périodiques dans toutes les directions, cela produit des phénomènes de résonance [4]. En revanche, dans les cas de la bande $\mathbf{R}^2 \times (0,1)$ avec conditions périodiques ou de l'espace tout entier $\mathbf{R}^3$, ces ondes créent un effet de dispersion et la composante 3D de la solution vérifie sous certaines hypothèses des inégalités de Strichartz. De plus, ces ondes deviennent négligeables sous l'influence d'une rotation rapide.

Cette dernière propriété permet notamment d'établir l'existence globale de solutions fortes de Navier-Stokes Coriolis (9) pour des données initiales quelconques et une rotation suffisamment rapide (chapitre II.3.4 et [25]). L'existence de solutions globales débouche naturellement sur la question de leur comportement asymptotique en temps. On montre dans la deuxième partie de cette thèse (voir chapitre II.3) que celui-ci est identique au cas sans rotation. Rappelons en effet que l'on se place ici dans le cas d'une viscosité isotrope. Les estimations sur les semi-groupes avec ou sans terme de rotation sont identiques et les développements asymptotiques peuvent être calculés à tout ordre. On montre notamment que toute solution globale régulière issue de donnée initiale petite converge vers le Vortex d'Oseen quand t tend vers l'infini, quelle que soit la rotation. L'étape suivante de ce travail serait bien sûr de joindre les résultats d'existence globale à rotation rapide (chapitre II.3.4) à ceux de convergence globale pour l'équation de Navier-Stokes (chapitre II.2.4) afin d'obtenir la stabilité des Vortex d'Oseen pour des rotations rapides dans l'équation de Navier-Stokes Coriolis. Une autre perspective de recherche serait de reprendre ces résultats de convergence avec une viscosité verticale ν_v de la forme $\beta\Omega^{-1}$ avec $\beta \geq 0$.

Part I

Stability of radially symmetric travelling waves in reaction-diffusion equations

Cet article est à paraître dans les annales de l'IHP, analyse non-linéaire

Résumé: On étudie le comportement pour les grands temps des solutions $u(x,t)$ de l'équation parabolique $u_t = \Delta u + F(u)$ dans le cas "bistable" et dans tout l'espace, en dimension supérieure. Plus précisément, on s'intéresse à la stabilité d'ondes progressives à symétrie sphérique pour de petites perturbations. Dans un premier temps, on montre que cette famille d'ondes est stable pour des perturbations à symétrie sphérique et que cette perturbation décroît comme $(\log t)/t^2$ quand t tend vers l'infini. On montre ensuite que cette stabilité est mise en défaut pour des perturbations quelconques. En effet, on met en évidence des perturbations pour lesquelles la solution ne tend pas vers une onde à symétrie sphérique: dans chaque direction $k \in S^{n-1}$, la restriction de $u(x,t)$ au rayon $\{x = kr, r \geq 0\}$ converge vers un translaté de l'onde progressive unidimensionnelle dépendant de k.

Abstract: The asymptotic behaviour as t goes to infinity of solutions $u(x,t)$ of the multidimensional parabolic equation $u_t = \Delta u + F(u)$ is studied in the "bistable" case. More precisely, we consider the stability of spherically symmetric travelling waves with respect to small perturbations. First, we show that such waves are stable against spherically symmetric perturbations, and that the perturbations decay like $(\log t)/t^2$ as t goes to infinity. Next, we observe that this stability result cannot hold for arbitrary (i.e. non-symmetric) perturbations. Indeed, we prove that there exist small perturbations such that the solution $u(x,t)$ does not converge to a spherically symmetric profile as t goes to infinity. More precisely, for any direction $k \in S^{n-1}$, the restriction of $u(x,t)$ to the ray $\{x = kr | r \geq 0\}$ converges to a k-dependent translate of the one-dimensional travelling wave.

Keywords: Semilinear parabolic equation, long-time asymptotics, stability, travelling waves.

AMS classification codes (2000): 35B40, 35K57, 35B35

Chapitre 1

Introduction

We consider the initial value problem for the semilinear parabolic equation

$$\begin{cases} u_t(x,t) &= \Delta u(x,t) + F(u(x,t)) \quad x \in \mathbf{R}^n, t > 0, \\ u(x,0) &= u_0(x) \quad\quad\quad\quad\quad\quad\quad x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $u \in \mathbf{R}$ and $n \geq 2$. Throughout this first part, it is assumed that the nonlinearity F is a continuously differentiable function on $\mathbf{R}$ satisfying the following assumptions:

`i)` $F(0) = F(1) = 0$;

`ii)` $F'(0) < 0, \quad F'(1) < 0$;

`iii)` There exists $\mu \in (0,1)$ such that $F(u) < 0$ if $u \in (0,\mu)$ and $F(u) > 0$ if $u \in (\mu, 1)$;

`iv)` $\int_0^1 F(u)du > 0$.

A typical example is the cubic nonlinearity

$$F(u) = 2u(1-u)(u-\mu) \text{ where } 0 < \mu < 1/2. \tag{1.2}$$

Equation (1.1) is a classical model for spreading and interacting particles, which has been often used in biology (population dynamics, propagation of nerves pulses), in physics (shock waves), or in chemistry (chemical reactions, flame propagation). Fisher [33] first proposed a genetical context in which the spread of advantageous genetical traits in a population was modeled by equation (1.1). At the same time, Kolmogorov, Petrovskii and Piskunov [59] gave a mathematical treatment of this equation for a slightly different nonlinearity. Later on, Aronson and Weinberger [1] also discussed the genetical background in some details. In their terminology, the nonlinearity satisfying `i`) to `iv`) is referred to as the "heterozygote inferior" case. In mathematical terms, this is called the "bistable" case as, by `i`) and `ii`), $u \equiv 0$ and $u \equiv 1$ are both stable steady states.

As far as the initial value problem is concerned, if u_0 is a continuous function from $\mathbf{R}^n$ to $(0,1)$ which goes to 0 as $|x|$ goes to infinity, then there exists a unique solution $u(x,t)$ of equation (1.1) with the same properties as u_0 for any $t \geq 0$.

One question of interest for this reaction-diffusion equation is the behaviour, as t goes to infinity, of the solutions $u(x,t)$ of (1.1). In one space dimension, a prominent role is played by a family of particular solutions of (1.1), called travelling waves. These are uniformly translating solutions of the form

$$u(x,t) = w_0(x-ct),$$

where $c \in \mathbf{R}$ is the speed of the wave. The profile w_0 satisfies the ordinary differential equation:

$$w_0'' + cw_0' + F(w_0) = 0, \qquad x \in \mathbf{R}, \tag{1.3}$$

together with the boundary conditions at infinity

$$\lim_{x\to-\infty} w_0(x) = 1 \text{ and } \lim_{x\to+\infty} w_0(x) = 0. \tag{1.4}$$

These waves are characterized by their time independent profile and usually represent the transport of information in the above models. They also often describe the long-time behaviour of many solutions.

Since Fisher and KPP, there has been an extensive literature on the subject. In the one dimensional bistable case, Kanel [53] proved that there exist a unique speed $c > 0$ and a unique (up to translations) monotone profile w_0, satisfying (1.3, 1.4). Moreover, $|w_0|$ (resp. $|1-w_0|$) decays exponentially fast as x goes to $+\infty$ (resp. $-\infty$). From now on, we fix w_0 by choosing $w_0(0) = 1/2$. For example, if F is given by (1.2), one finds $c = 1-2\mu \in (0,1)$ and $w_0(x) = (1+e^x)^{-1}$.

Afterwards, Sattinger [81] was interested in the local stability of travelling waves. He proved that the family $\{w_0(.-\gamma), \gamma \in \mathbf{R}\}$ is normally attracting. More precisely, for any initial data u_0 of the form

$$u_0(x) = w_0(x) + \epsilon v_0(x),$$

where $\epsilon > 0$ is sufficiently small and v_0 bounded in a weighted space, Sattinger proved that there exist a $\mathcal{C}^1$ function $\rho(\epsilon)$ and positive constants K and γ such that the solution $u(x,t)$ of (1.1) satisfies

$$\|u(x+ct,t) - w_0(x+\rho(\epsilon))\| \le Ke^{-\gamma t}, \quad t \ge 0,$$

in an appropriate weighted norm. This is the local stability of travelling waves in one dimension. Sattinger's proof uses the spectral properties of the linearised operator $L_0 = \partial_y^2 + c\partial_y + F'(w_0)$ around the travelling wave w_0 in the c-moving frame. These properties can be summarized as follows:

Let $\phi_0 = \bar{\alpha} w_0'$ and $\psi_0 = e^{cx}\phi_0$ where $\bar{\alpha} > 0$ is chosen so that

$$\int_{\mathbf{R}} \phi_0(x)\psi_0(x)dx = 1. \tag{1.5}$$

Then, ϕ_0 is an eigenfunction of L_0 (associated with the eigenvalue 0), and ψ_0 is the corresponding eigenfunction of the adjoint operator L_0^*:

$$\phi_0'' + c\phi_0' + F'(w_0)\phi_0 = 0,$$
$$\psi_0'' - c\psi_0' + F'(w_0)\psi_0 = 0.$$

Moreover, there exists some $\gamma > 0$ such that the real part of the spectrum of L_0 in $L^2(\mathbf{R})$ is included in $]-\infty, -\gamma] \cup \{0\}$, see [48, 81]. Since the eigenvalue 0 is isolated, there exists a projection operator P onto the null space of L_0. This operator is given by

$$Pu = \frac{1}{2\pi i}\int_\Gamma R(\lambda, L_0)u\,d\lambda$$

where $R(\lambda, L_0) = (\lambda - L_0)^{-1}$ and Γ is a simple closed curve in the complex plane enclosing the eigenvalue 0, see [81, 87]. Define the complementary spectral projection $Q = I - P$ where I is the identity operator in $L^2(\mathbf{R})$. These projection operators P and Q are also given by

$$Pu = \left(\int_{\mathbf{R}} u(x)\psi_0(x)dx\right)\phi_0, \quad Qu = (I - P)u,$$

see for instance [56, 81]. The spectral subspace corresponding to the eigenvalue 0 is defined by $\{u \in L^2(\mathbf{R}) \mid u = Pu\}$ and its supplementary by

$$\mathcal{R} = \{u \in L^2(\mathbf{R}) \mid u = Qu\} = \{u \in L^2(\mathbf{R}) \mid Pu = 0\}.$$

Then $\mathcal{R}$, equipped with the L^2 norm, is a Banach space and $L_0|_{\mathcal{R}}$ generates an analytic semi-group which satisfies $\|e^{tL_0}\|_{\mathcal{L}(\mathcal{R})} \le c_0 e^{-\gamma t}$ for all $t \ge 0$.

On the other hand, Fife and McLeod [31] proved the global stability of travelling waves: they showed, using comparison theorems, that if u_0 satisfies $0 \le u_0 \le 1$ and $\liminf_{-\infty} u_0(x) > \mu$, $\limsup_{+\infty} u_0(x) < \mu$, then the solution $u(x,t)$ of (1.1) approaches exponentially fast in time a translate of the travelling wave in the supremum norm. Fife [32] also highlighted other possible types of asymptotic behaviour: if u_0 vanishes at infinity in x and if the solution converges uniformly to 1 on compact sets, then $u(x,t)$ behaves as a pair of diverging fronts where a wave goes off in each direction.

In higher dimensions, Aronson and Weinberger [2], Xin and Levermore [95, 61] and Kapitula [56] were interested in planar travelling waves. These are particular solutions of equation (1.1) of the form $u(x,t) = w_0(x \cdot k - ct)$ where $k \in S^{n-1}$. Existence of such solutions can be proved as in the one-dimensional case, but the stability analysis is quite different: unlike in the one-dimensional case, the gap in the spectrum of the linearised operator around the travelling wave disappears. Instead, there exists continuous spectrum all the way up to zero which is due, intuitively, to the effects of the transverse diffusion. To overcome this difficulty, Kapitula decomposed the solution $u(x,t)$ as

$$u(x,t) = w_0(x \cdot k - ct + \rho(x,t)) + v(x,t)$$

where $\rho(x,t)$ represents a local shift of the travelling wave and $v(x,t)$ a transverse perturbation in $\mathcal{R}$. The equation for ρ can be analyzed by the one-dimensional result and

Fourier transform, while the transverse perturbation v satisfies a semilinear heat equation in $\mathbf{R}^{n-1}$. Therefore, Kapitula proved that the perturbation decays to zero with a rate of $O\left(t^{-\frac{n-1}{4}}\right)$ in $H^k(\mathbf{R}^n)$, $k \geq [\frac{n+1}{2}]$.

Apart from this particular planar case, Aronson and Weinberger [2] also studied the asymptotic behaviour of other solutions in higher dimensions. They proved that the state $u \equiv 0$ is stable with respect to perturbations which are not too large on too large a set, but is unstable with respect to some perturbations with bounded support. Moreover, assuming u_0 vanishes at infinity in x and u converges to 1 as t goes to infinity, they showed that the disturbance is propagated with asymptotic speed c.

Finally, Uchiyama [92] and Jones [51] were interested in spherically symmetric solutions. If u_0 is spherically symmetric with $\limsup_{|x|\to+\infty} u_0(x) < \mu$, and if the solution $u(x,t)$ of (1.1) with initial data u_0 converges to 1 uniformly on compact sets as t goes to infinity, they proved that there exists a function $g(t)$ such that

$$\lim_{t\to+\infty} \sup_{x\in\mathbf{R}^n} |u(x,t) - w_0(|x| - ct + g(t))| = 0. \tag{1.6}$$

Jones proved with dynamical systems considerations that $\lim_{t\to+\infty} g(t)/t = 0$ and Uchiyama precised, using energy methods and comparison theorems, that there exists some $L \in \mathbf{R}$ such that

$$\lim_{t\to+\infty} \left(g(t) - \frac{n-1}{c}\log t\right) = L. \tag{1.7}$$

This important result establishes the existence of a family of asymptotic solutions of (1.1), which we call spherically symmetric travelling waves: $W(x,t) = w_0(|x| - ct + \frac{n-1}{c}\log t)$ and its translates in time. It also shows that this family is asymptotically stable with respect to spherically symmetric perturbations.

We give in the second chapter of this part another method, based on Kapitula's decomposition, which enables us to get more information on how fast the solution $u(x,t)$ of (1.1) converges to a travelling wave and on the asymptotic behaviour of the function $g(t)$. To do that, we introduce the following Banach spaces:

$$\begin{aligned}
&Y = H^1(\mathbf{R}^+),\\
&X = \{u : \mathbf{R}^n \to \mathbf{R} \mid \exists \tilde{u} \in Y \text{ so that } u(x) = \tilde{u}(|x|) \text{ for } x \in \mathbf{R}^n\},\\
&\|u\|_X = \|\tilde{u}\|_Y = \left(\int_0^\infty |\tilde{u}(r)|^2 + |\tilde{u}_r(r)|^2 dr\right)^{\frac{1}{2}}.
\end{aligned}$$

Note that X is included in $H^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ and contains spherically symmetric functions. Then, we prove in the second chapter the following theorem:

Theorem 1 *Assume F is a "bistable" non-linearity. There exist positive constants $R_0, \delta_0, c_1, c_2, \gamma_0$ such that, if $u_0 : \boldsymbol{R}^n \to \boldsymbol{R}$ is a spherically symmetric function satisfying*

$$\|u_0(x) - w_0(|x| - R)\|_X \leq \delta$$

for some $R \geq R_0$ and some $\delta \leq \delta_0$, then equation (1.1) has a unique solution $u \in C^0([0,+\infty), X)$ with initial data u_0. Moreover, there exists $\rho \in C^1([0,+\infty))$ such that

$$\|u(x,t) - w_0(|x| - s(t))\|_X + |\rho'(t)| \leq c_1 \delta e^{-\gamma_0 t} + c_2 \frac{\log(R+ct)}{(R+ct)^2}$$

for all $t \geq 0$, where

$$s(t) = R + ct - \frac{n-1}{c} \log\left(\frac{R+ct}{R}\right) + \rho(t). \tag{1.8}$$

This first theorem shows that the family of spherically symmetric travelling waves is asymptotically stable for small symmetric perturbations. Indeed, any small perturbation tends to zero with a rate of $O(\log t/t^2)$. Moreover, as $|\rho'(t)|$ is bounded by an integrable function of time, the function $\rho(t)$ converges to a constant ρ_∞ as t goes to infinity, which corresponds to L in (1.7) and, with our hypothesis on u_0, the convergence (1.6) satisfies:

$$|u(x,t) - w_0(|x| - ct + \frac{n-1}{c}\log t + L)| \leq c_0 \frac{\log t}{t}.$$

In a third section, we are interested in non spherically symmetric perturbations of travelling waves in higher dimensions. Based on Uchiyama's work and a comparison theorem, a corollary on the Lyapunov stability of travelling waves against general small perturbations is first stated.

The only result so far concerning the long-time behaviour of non spherically symmetric solutions is due to Jones [52]. He considered solutions $u(x,t)$ whose initial data u_0 have compact support, and he also assumed that $u(x,t)$ converges to 1 uniformly on compact sets as t goes to infinity. He then showed that, if followed out in a radial direction at the correct speed c, the solution approaches the one-dimensional travelling wave, at least in shape. Moreover, for any $l \in (0,1)$ and any sufficiently large $t > 0$, he proved that, for all point P of the level surface $S_l(t) = \{x \in \mathbf{R}^n \mid u(x,t) = l\}$, the normal to $S_l(t)$ at P must intersect the support of u_0. Obviously, this result implies that the surface $S_l(t)$ becomes rounder and rounder as t goes to infinity. It is thus natural to expect spherically symmetric travelling waves to be asymptotically stable against any small non-symmetric perturbations. However, we prove in the third chapter of this part that this is not the case. In the two-dimensional case, we give an example of non-spherically symmetric function u_0 close to a spherically symmetric wave such that the solution $u(x,t)$ of (1.1) with initial data u_0 never approaches the family of spherically symmetric travelling waves. Indeed, the translate of the wave which is approached depends on the radial direction.

Subsequently, we require some more technical assumptions. For convenience, we choose to work in $\mathbf{R}^2$ so that polar coordinates are easier to handle. We assume that F is in $\mathcal{C}^3(\mathbf{R})$ and satisfies the condition: $F^{(3)}(u) \leq 0$ for $u \in [0,1]$. In this case, we prove in appendix 4.3 that ϕ_0 is log-concave, i.e. $(\phi_0'/\phi_0)' < 0$. Finally, we also assume that every solution of the ODE, $u_t = F(u)$, is bounded uniformly in time. By the maximum principle, this easily means that for any bounded initial condition, the solution $u(x,t)$ is uniformly bounded in time. Example (1.2) for F satisfies both conditions.

Precisely, we prove in the third chapter the following theorem:

Theorem 2 *Assume F is a "bistable" nonlinearity satisfying both above conditions. There exist positive constants $R_0', \delta_0', \eta, c_0$ such that if $u_0 \in H^1(\boldsymbol{R}^2)$ satisfies*

$$\|u_0(x) - w_0(|x| - R)\|_{H^1(\boldsymbol{R}^2)} \leq \delta$$

for some $\delta \leq \delta_0'$ and some $R \geq R_0'$ such that $R^{1/4}\delta \leq \eta$, then equation (1.1) has a unique solution $u \in C^0(\boldsymbol{R}^+, H^1(\boldsymbol{R}^2))$ with initial data u_0. Moreover, there exist $\rho \in C^0(\boldsymbol{R}^+, H^1(0, 2\pi))$ and $\rho_\infty \in L^2(0, 2\pi)$ such that

$$\begin{aligned} &\|u(r, \theta, t) - w_0(r - s(\theta, t))\|_{H^1(\boldsymbol{R}^2)} \leq \frac{c_0}{(R + ct)^{\frac{1}{4}}}, \\ &s(\theta, t) = R + ct - \frac{1}{c}\log\left(\frac{R + ct}{R}\right) + \rho(\theta, t), \\ &\lim_{t \to +\infty} \|\rho(\theta, t) - \rho_\infty(\theta)\|_{L^2(0, 2\pi)} = 0 \end{aligned} \tag{1.9}$$

where $(r, \theta) \in \boldsymbol{R}^+ \times (0, 2\pi)$ are the polar coordinates in $\boldsymbol{R}^2$.

This second theorem first illustrates Jones' theorem. Indeed, there exists a class of initial data for which solutions converge to a creased profile as t goes to infinity. And, if followed out in a radial direction (i.e. for $\theta =$constant), the solutions behave asymptotically as a one-dimensional travelling wave whose position $s(\theta, t)$ depends on the radial direction. Precisely, we show that $s(\theta, t)$ is given by (1.9), that $\rho(\theta, t)$ converges in the $L^2(0, 2\pi)$ norm to a function $\rho_\infty(\theta)$ and we give an example of initial data for which the solution does not converge to a spherically symmetric travelling wave, i.e. the corresponding function $\rho_\infty(\theta)$ is not constant. Moreover, we show that the set of all functions ρ_∞ that can be constructed in that way, is dense in a ball of $H^1(0, 2\pi)$. Therefore, there exist a lot of asymptotic behaviours which look like a creased travelling front which never becomes round.

Finally, this theorem shows that the family of spherically symmetric travelling waves is not asymptotically stable for arbitrary perturbations: this means that the higher dimensional case $n \geq 2$ is very different from the one-dimensional case $n = 1$ where the asymptotical stability of travelling waves has been widely proved.

Let us now make a few technical remarks on the statement of theorem 2. We assume that the initial condition u_0 is close to a travelling wave ($\delta \leq \delta_0'$ small) whose interface $\{w_0 = \frac{1}{2}\}$ is large enough ($R \geq R_0'$ large). The relation $R^{\frac{1}{4}}\delta \leq \eta$ should be a technical assumption and we do believe that it can be relaxed by changing the function spaces we use. Actually, we prove in this paper a stronger theorem (theorem 3.2.3) where this constraint only appears on one part of the perturbation. We also show in this theorem that the perturbation decreases like $1/(R + ct)^{\frac{1}{4}}$. This rate may not be optimal but shows the convergence of the solutions towards travelling fronts. Once more, we prove in theorem 3.2.3 a more precise result where the dependance of the initial condition on the convergence rate is emphasized.

Notations: Throughout the first part, we use the following notations: $\|.\|_Z$ is a norm in the Banach space Z, $|.|$ is the usual euclidean norm in $\mathbf{R}$ and x is a vector of $\mathbf{R}^n$ while (r, θ) are the polar coordinates in $\mathbf{R}^2$ where $r \geq 0, \theta \in [0, 2\pi)$. We also denote by c_i generic positive constants which may differ from place to place, even in the same chain of inequalities.

Chapitre 2

Radial Solutions

The aim of this chapter is to prove theorem 1, i.e. the stability of travelling waves against radial perturbations. Hence, we only work with spherically symmetric functions and we always use, for convenience, the notation $u(r,t)$ instead of $\tilde{u}(r,t)$ defined in the introduction.

For spherically symmetric solutions, equation (1.1) reduces to the following Cauchy problem:

$$\begin{cases} u_t(r,t) & = u_{rr}(r,t) + \frac{n-1}{r} u_r(r,t) + F(u(r,t)) & r > 0, t > 0, \\ u(r,0) & = u_0(r) & r > 0, \\ u_r|_{r=0} & = 0 & t \geq 0. \end{cases}$$

The Neumann boundary condition at zero is due to the regularity of the function $u(x,t)$, $x \in \mathbf{R}^n$. In this chapter, we first write a decomposition of the solution $u(r,t)$ as Kapitula [56] did. Then, we study the new evolution equations in a moving frame to take advantage of spectral properties of the operator L_0 defined in the introduction.

2.1 A coordinate system

We first need to define more precisely a spherically symmetric travelling wave in higher dimension. Since the function $x \in \mathbf{R}^n \mapsto W(x,t) = w_0(|x| - R - ct + \frac{n-1}{c} \log\left(\frac{R+ct}{R}\right))$ is not smooth at $x = 0$, we have to modify w_0 in a function w called also travelling wave or "modified wave".

Let $\chi \in C^\infty(\mathbf{R}^+)$ so that $\chi(r) \equiv 0$ if $r \leq 1$ and $\chi(r) \equiv 1$ if $r \geq 2$, and define

$$w(y,r) = 1 + \chi(r)(w_0(y) - 1), \ (y,r) \in \mathbf{R} \times \mathbf{R}^+.$$

Then, $w(y,r)$ is identically equal to 1 if $r \leq 1$ and $w(y,r) = w_0(y)$ if $r \geq 2$. Note that r is a positive parameter which flattens the wave around the origin. Then, for any $s \in \mathbf{R}$, $r \in \mathbf{R}^+ \mapsto w(r-s,r)$ is a function of $Y = H^1(\mathbf{R}^+)$, equal to 1 near the origin and decreasing like the wave w_0 at infinity. In a similar way, $x \in \mathbf{R}^n \mapsto w(|x| - s, |x|)$ is a spherically symmetric function of X, equal to 1 near the origin and decreasing like the wave w_0 at infinity in all directions. We also define $\psi(y,r) = \bar{\alpha}\chi(r)\psi_0(y)$ where $\bar{\alpha}$ has been chosen in (1.5).

In a neighborhood of the wave w, it will be convenient to use a coordinate system given by $(v,s) \in Y \times \mathbf{R}$ with perturbations of the wave being given at any time by

$$u(r) = w(r-s,r) + v(r) \quad r \geq 0,$$

where s is chosen so that $\int_0^\infty v(r)\psi(r-s,r)dr = 0$. We have decomposed the solution u as a translate of the wave w and a transversal perturbation v. The following lemma shows that this decomposition is always possible:

Lemma 2.1.1 *There exist positive constants R_1, δ_1, K such that for any $R \geq R_1$ and any $\xi \in Y$ with $\|\xi\|_Y \leq \delta_1$, threre exists a unique pair $(v,\rho) \in Y \times \boldsymbol{R}$ such that*

i) $\|v\|_Y + |\rho| \leq K\|\xi\|_Y$,

ii) $w(r-R,r) + \xi(r) = w(r-R-\rho,r) + v(r)$ *for all* $r \geq 0$,

iii) $\int_0^\infty v(r)\psi(r-R-\rho,r)dr = 0$.

Proof: Define the operator $A : \mathbf{R} \times Y \to \mathbf{R}$ by

$$A(\rho,\xi) = \int_0^\infty \xi(r)\psi(r-R-\rho,r)dr + \rho\int_0^\infty \psi(r-R-\rho,r)\int_0^1 w_y(r-R-\rho h,r)dhdr.$$

Since $A(0,0) = 0$ and the derivative $A_\rho(0,0) = \int_{-R}^{+\infty} \phi_0\psi_0(y)\chi^2(y+R)dy \neq 0$ for $R \geq R_1$, by the implicit function theorem on Banach spaces, there exist a small neighborhood $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ of $(0,0)$ in $\mathbf{R} \times Y$ a function $\rho(\xi) : \mathcal{V}_2 \mapsto \mathcal{V}_1$ such that $A(\rho(\xi),\xi) = 0$ and $|\rho| \leq K\|\xi\|_Y$ for some $K > 0$. This yields the spatial translational component ρ. Let $v(.) = \xi(.) + w(.-R,.) - w(.-R-\rho(\xi),.)$ a function of Y. Then, $\|v\|_Y + |\rho| \leq K\|\xi\|_Y$ for some $K > 0$. As $A(\rho(\xi),\xi) = 0$, and by Taylor's theorem, $\int_0^\infty v(r)\psi(r-R-\rho,r)dr = 0$. Then, (v,ρ) satisfies the lemma if $\|\xi\|_Y \leq \delta_1$ where $\delta_1 > 0$ is sufficiently small so that $B_Y(0,\delta_1) \subset \mathcal{V}_2$. ∎

Using the result of lemma 2.1.1, we can write for any $t \geq 0$ and some $R \geq R_1$,

$$u(r,t) = w(r-s(t),r) + v(r,t), \qquad r \geq 0, \tag{2.1}$$

$$s(t) = R + ct - \frac{n-1}{c}\log\left(\frac{R+ct}{R}\right) + \rho(t),$$

$$\int_0^\infty v(r,t)\psi(r-s(t),r)dr = 0. \tag{2.2}$$

By lemma 2.1.1, such a decomposition exists if, for all $t \geq 0$, the solution $u(r,t)$ is close to the wave, namely if $\|u(r,t) - w(r-s(t),r)\|_Y \leq \delta_1$. This assumption will be validated later by the proof of theorem 1. We are now going to work with these new variables v and ρ which are more convenient than u. We first give the equations they satisfy.

Substitute the decomposition (2.1) of the solution into equation (1.1) and use equation (1.3) satisfied by w_0 to get the evolution equation satisfied by v:

$$\begin{aligned} v_t &= v_{rr} + \frac{n-1}{r} v_r + F'(w_0(r-s(t)))v && && (2.3)\\ &+ \left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t) \right) w_y(r-s(t),r) + N + S, && r \geq 0, \quad t > 0, \\ v(r,0) &= v_0(r), && r \geq 0, \\ v_r|_{r=0} &= 0, && t > 0, \end{aligned}$$

where

$$\begin{aligned} N &= F(w+v) - F(w_0)\chi(r) - F'(w_0)v \quad \text{is the nonlinear term,} \\ S &= w_{rr} + 2w_{ry} + \frac{n-1}{r} w_r. \end{aligned}$$

The functions w, w_0, ψ and their derivatives are taken at $(r-s(t),r)$ or $(r-s(t))$, depending if the wave is modified or not. Note the Neumann condition at zero $v_r|_{r=0} = 0$. Indeed, if $u(x) = \tilde{u}(|x|)$, $u \in \mathcal{C}^1(\mathbf{R}^2)$ is equivalent to $\tilde{u} \in \mathcal{C}^1(\mathbf{R}^+)$ and $\tilde{u}'(0) = 0$. As $u = w + v$ and w is identically zero near the origin, the regularity of u is forwarded to v and $v_r|_{r=0} = 0$.

Derivating equation (2.2) with respect to t and using equations (1.8) and (2.3) satisfied by s and v, we get the evolution equation satisfied by ρ:

$$\begin{aligned} \rho'(t) \int_0^\infty (\psi w_y - v\psi_y)\, dr &= \int_0^\infty [v\Lambda - (N+S)\psi]\, dr, \qquad t > 0, && (2.4)\\ \rho(0) &= \rho_0, \end{aligned}$$

where

$$\Lambda = \left(\frac{n-1}{R+ct} - \frac{n-1}{r} \right) \psi_y + \frac{n-1}{r^2}\psi + (\psi_{rr} + 2\psi_{yr} - \frac{n-1}{r}\psi_r) + (\psi_{yy} - c\psi_y + F'(w_0)\psi).$$

The functions ψ, w, w_0 and their derivatives are taken at $(r-s(t),r)$ or $(r-s(t))$.

We first consider the initial value problem for equations (2.3, 2.4):

Lemma 2.1.2 *Fix $R > 0$. There exist $\delta_4 > 0, T > 0$ such that for any initial data $(v_0, \rho_0) \in Y \times \boldsymbol{R}$ with $\|v_0\|_Y \leq \delta \leq \delta_4$ and $|\rho_0| \leq \frac{1}{2}$, the integral equations corresponding to (2.3, 2.4) have a unique solution $(v,\rho) \in C^0([0,T], Y \times \boldsymbol{R})$. In addition, $(v,\rho) \in C^1((0,T], Y \times \boldsymbol{R})$, and equations (2.3, 2.4) are satisfied for $0 < t \leq T$.*

Proof: If $\|v_0\|_Y \leq \delta$ and $\delta \leq \delta_4$ is sufficiently small, then $\int_0^\infty \psi w_y - v\psi_y dr \neq 0$ and $\rho'(t)$ can be expressed easily as a function of v and ρ. Then, equations (2.3, 2.4) can be written as follows:

$$\begin{aligned} \partial_t(v,\rho) &= \bar{L}(v,\rho) + f(v,\rho,t), \\ v_r|_{r=0} &= 0, \\ (v,\rho)(0) &= (v_0,\rho_0), \end{aligned}$$

where

$$\bar{L}(v,\rho) = (Lv, 0) = (\partial_r^2 v + \frac{n-1}{r}\partial_r v, 0).$$

As $\bar{L}$ generates a semigroup on $Y \times \mathbf{R}$ (see lemma 2.2.2 for a detailed proof) and $f \in \mathcal{C}^1(Y \times \mathbf{R} \times \mathbf{R}^+)$, the integral equations corresponding to (2.3, 2.4) have a unique solution $(v,\rho) \in \mathcal{C}^0([0,T], Y \times \mathbf{R})$, see for instance [67]. In addition, this mild solution is classical and $(v,\rho) \in \mathcal{C}^1((0,T], Y \times \mathbf{R})$. ■

We now work on the two evolution equations (2.3, 2.4) to get information on the asymptotic behaviours of v and ρ. Before stating our result, let us explain its content in a heuristic way. Consider first equation (2.3) for v. The leading term in the right-hand side is

$$\left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right) w_y(r - s(t), r),$$

which decays exponentially in time for any fixed $r > 0$, but only like $\frac{\log(R+ct)}{(R+ct)^2}$ for $r = s(t)$. On the other hand, as we shall show in section 2.2.3, the evolution operator generated by the time-dependent operator $\partial_r^2 + \frac{n-1}{r}\partial_r + F'(w_0(r - s(t)))$ is exponentially contracting in the space of functions v satisfying (2.2). Therefore, we expect the solution v of (2.3) to decay like $(\log t/t^2)$ as t goes to infinity. As for ρ, we observe that equation (2.4) is close for large times to

$$\rho'(t) = \int_0^\infty \left[\left(\frac{n-1}{R+ct} - \frac{n-1}{r}\right)\psi_y + \frac{n-1}{r^2}\psi\right] v(r,t)dr,$$

since $\int_0^\infty \psi w_y dr$ is close to $\int_{\mathbf{R}} \psi_0 \phi_0 dx = 1$. Thus, we also expect $\rho'(t)$ to decrease at least like $(\log t/t^2)$ as t goes to infinity. The following result shows that these heuristic considerations are indeed correct:

Theorem 2.1.3 *There exist positive constants $R_2, \delta_2, c_1, c_2, \gamma_0$ such that, if $R \geq R_2$ and $(v_0, \rho_0) \in Y \times \boldsymbol{R}$ satisfy $\|v_0\|_Y \leq \delta_2$, $|\rho_0| \leq \frac{1}{2}$, then equations (2.3, 2.4) have a unique solution $(v,\rho) \in C^0([0,+\infty), Y \times \boldsymbol{R})$ with initial data (v_0, ρ_0). In addition, $\rho \in C^1([0,+\infty), \boldsymbol{R})$ and*

$$\|v(t)\|_Y + |\rho'(t)| \leq c_1 \|v_0\|_Y e^{-\gamma_0 t} + c_2 \frac{\log(R+ct)}{(R+ct)^2}, \qquad t \geq 0.$$

Theorem 2.1.3 is a new version of theorem 1 in the variables v and ρ. We give right now the proof of theorem 1 under the assumption that theorem 2.1.3 is proved.

Proof of theorem 1: Let $R_2, \delta_2, c_1, c_2, \gamma_0$ be as in theorem 2.1.3 and R_1, δ_1, K be as in lemma 2.1.1. Choose R_0 and δ_0 so that:

$$2\delta_0 \leq \delta_1, \quad 2K\delta_0 \leq \min(\delta_2, \frac{1}{2}), \quad R_0 \geq \max(R_2, R_1), \quad c_0 e^{-\gamma_1 R_0} \leq \delta_0,$$

where $c_0 > 0$ and $\gamma_1 > 0$ are chosen so that for any $R \geq 0$,

$$\|w_0(r - R) - w(r - R, r)\|_Y \leq c_0 e^{-\gamma_1 R}. \tag{2.5}$$

Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a spherically symmetric function satisfying

$$\|u_0(r) - w_0(r - R)\|_Y \leq \delta$$

for some $R \geq R_0$ and $\delta \leq \delta_0$. Let $\xi(r) = u_0(r) - w(r - R, r)$, $r \geq 0$. Then, $\xi \in Y$ and $\|\xi\|_Y \leq \delta + c_0 e^{-\gamma_1 R} \leq 2\delta_0 \leq \delta_1$. Then, by lemma 2.1.1, there exists a unique pair $(v_0, \rho_0) \in Y \times \mathbf{R}$ such that:

i) $\|v_0\|_Y + |\rho_0| \leq K\|\xi\|_Y$,

ii) $u_0(r) = w(r - R, r) + \xi(r) = w(r - R - \rho_0, r) + v_0(r)$ for all $r \geq 0$,

iii) $\int_0^\infty v_0(r)\psi(r - R - \rho_0, r)dr = 0$.

As $R \geq R_2$ and $(v_0, \rho_0) \in Y \times \mathbf{R}$ satisfy $\|v_0\|_Y \leq \delta_2$ and $|\rho_0| \leq \frac{1}{2}$, it follows from theorem 2.1.3 that equations (2.3, 2.4) have a unique solution $(v, \rho) \in C^0([0, +\infty), Y \times \mathbf{R})$ with initial data (v_0, ρ_0). In addition, $\rho \in C^1([0, +\infty), \mathbf{R})$ and

$$\|v(t)\|_Y + |\rho'(t)| \leq c_1\|v_0\|_Y e^{-\gamma_0 t} + c_2 \frac{\log(R + ct)}{(R + ct)^2}, \qquad t \geq 0.$$

Let $u(x, t) = w(|x| - s(t), |x|) + v(|x|, t)$, $x \in \mathbf{R}^n$ where $s(t)$ is given by (1.8). Then, $u \in C^0([0, +\infty), X)$ is the unique solution of equation (1.1) with initial data u_0 and

$$\begin{aligned}
&\|u(x,t) - w_0(|x| - s(t))\|_X + |\rho'(t)| \\
&\quad \leq \|u(x,t) - w(|x| - s(t), |x|)\|_X + \|w(r - s(t), r) - w_0(r - s(t))\|_Y + |\rho'(t)| \\
&\quad \leq c_1\|v_0\|_Y e^{-\gamma_0 t} + c_2 \frac{\log(R + ct)}{(R + ct)^2} + c_0 e^{-\gamma_1 s(t)} \\
&\quad \leq c_1 K \delta e^{-\gamma_0 t} + c_2 \frac{\log(R + ct)}{(R + ct)^2} + c_1 K c_0 e^{-\gamma_1 R - \gamma_0 t} + c_0 e^{-\gamma_1 s(t)}.
\end{aligned}$$

Define $c_1' = Kc_1$ and c_2' so that for any $t \geq 0$, any $R \geq 0$,

$$c_2 + c_1 c_0 K \frac{e^{-\gamma_1 R - \gamma_0 t}}{\frac{\log(R+ct)}{(R+ct)^2}} + c_0 \frac{e^{-\gamma_1 s(t)}}{\frac{\log(R+ct)}{(R+ct)^2}} \leq c_2'.$$

Then,

$$\|u(x,t) - w_0(|x| - s(t))\|_X + |\rho'(t)| \leq c_1' \delta e^{-\gamma_0 t} + c_2' \frac{\log(R + ct)}{(R + ct)^2}.$$

This ends the proof of theorem 1. ■

2.2 Estimates on the solutions v and ρ

Let us now prove theorem 2.1.3. We begin with a proposition close to this theorem but local in time. We then show how theorem 2.1.3 follows from this proposition.

Proposition 2.2.1 *There exist positive constants $R_3, \delta_3, c_1, c_2, \gamma_0$ such that, if $R \geq R_3, T > 0$ and $(v, \rho) \in C^0([0,T], Y \times \boldsymbol{R})$ is any solution of (2.3, 2.4) satisfying*

$$\|v(t)\|_Y \leq \delta_3, \quad |\rho(t)| \leq 1, \quad 0 \leq t \leq T,$$

then

$$\|v(t)\|_Y + |\rho'(t)| \leq c_1 \|v_0\|_Y e^{-\gamma_0 t} + c_2 \frac{\log(R+ct)}{(R+ct)^2}, \quad 0 \leq t \leq T.$$

Proof of theorem 2.1.3: Let $R_3, \delta_3, c_1, c_2, \gamma_0$ be as in Proposition 2.2.1 and choose positive constants R_2, δ_2 so that $R_2 \geq R_3$ and

$$c_1 \delta_2 < \min\left(\frac{\delta_3}{2}, \frac{\gamma_0}{4}\right), \quad \delta_2 \leq \min\left(\frac{\delta_3}{2}, \delta_4\right), \quad c_2 \frac{\log R_2}{R_2^2} < \frac{\delta_3}{2}, \quad \frac{c_2}{c} \frac{1+\log R_2}{R_2} < \frac{1}{4}.$$

Take $R \geq R_2$ and $(v_0, \rho_0) \in Y \times \mathbf{R}$ so that $\|v_0\|_Y \leq \delta_2, |\rho_0| \leq \frac{1}{2}$. By lemma 2.1.2, let $(v, \rho) \in C^0([0, T^*), Y \times \mathbf{R})$ be the maximal solution of (2.3, 2.4) with initial data (v_0, ρ_0). Define

$$T = \sup\{\tilde{T} \in [0, T^*) \mid \|v(t)\|_Y \leq \delta_3 \text{ and } |\rho(t)| \leq 1 \text{ for any } t \in [0, \tilde{T}]\}.$$

Since $\delta_2 < \delta_3$, it is clear that $T > 0$. We claim that $T = T^*$, which also implies $T = T^* = +\infty$. Indeed, if $T < T^*$, it follows from proposition 2.2.1 that for $t \in [0, T]$,

$$\|v(t)\|_Y \leq c_1 \|v_0\|_Y e^{-\gamma_0 t} + c_2 \frac{\log(R+ct)}{(R+ct)^2} \leq c_1 \delta_2 + c_2 \frac{\log R_2}{R_2^2} < \delta_3,$$

$$|\rho(t)| \leq |\rho_0| + \int_0^t |\rho'(s)| ds \leq \frac{1}{2} + \frac{c_1 \delta_2}{\gamma_0} + \frac{c_2}{c} \frac{1+\log R_2}{R_2} < 1,$$

which contradicts the definition of T. Thus $T = T^* = +\infty$. Since $\delta_2 < \delta_3$, the inequality satisfied by $\|v(t)\|_Y + |\rho'(t)|$ is true for all $t \geq 0$ and theorem 2.1.3 follows immediately from proposition 2.2.1. ■

Let us now prove proposition 2.2.1. We are first interested in the behaviour of v which satisfies equation (2.3). The main idea is to work, as in one dimension, in the moving frame at speed $s(t)$ to get, in equation (2.3), a time independent-operator instead of $\partial_r^2 + \frac{n-1}{r}\partial_r + F'(w_0(r - s(t)))$. Therefore, we need to work on the whole real line which is invariant by translation. That is why we first extend v to $\mathbf{R}$ by a function z which is convenient, i.e. which decreases exponentially fast in time in the H^1-norm. Precisely, we already explained in a heuristic way that v decreases exponentially fast as t goes to infinity near $r = 0$. Therefore, we first define a function z equal to v near the origin and then extend v to $\mathbf{R}$ by z. We can then use theorems on spectral perturbations of operators, energy estimates and spectral decomposition to highlight the behaviour of v in X. As equations (2.3) and (2.4), satisfied by v and ρ, are coupled, we need at the end to study the behaviour of ρ as we explained before.

From now on, we fix $R > 0$ (large), $0 < \delta \leq \delta_4$ (small), and we assume that $(v, \rho) \in C^0([0,T], Y \times \mathbf{R})$ is a solution of (2.3, 2.4) satisfying

$$\|v(t)\|_Y \leq \delta, \qquad |\rho(t)| \leq 1, \qquad 0 \leq t \leq T,$$

for some $T > 0$. We call these assumptions (H).

2.2.1 Localisation near $r = 0$

Let $\xi \in C^\infty(\mathbf{R}^+)$, $R_4 \geq 2$ and $\beta > 0$ so that $\xi \equiv 1$ on $[0, R_4]$ and $\xi(r) \sim e^{-\beta r}$ as r goes to infinity. Let

$$z(r,t) = \xi(r)v(r,t) \tag{2.6}$$

for all $r \in \mathbf{R}^+$ and $t \geq 0$. Then, z is equal to v near $r = 0$ and satisfies

$$\begin{aligned} z_t(r,t) &= L_1 z(r,t) + G_1(r,t), && r \geq 0, t > 0, \\ z_r|_{r=0} &= 0, && t > 0, \end{aligned}$$

where

$$\begin{aligned} L_1 &= \partial_r^2 + \left(\frac{n-1}{r} + a(r)\right)\partial_r + b(r), \\ G_1(r,t) &= (F'(w_0(r - s(t))) - h_-)\xi(r)v(r,t) + (S+N)\xi(r) \\ &\quad + \left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right) w_y(r - s(t), r)\xi(r), \\ a(r) &= -2\xi'(r)/\xi(r), \\ b(r) &= 2(\xi'(r)/\xi(r))^2 - (\xi''(r)/\xi(r)) - \frac{n-1}{r}(\xi'(r)/\xi(r)) + h_-, \\ h_- &= \inf\left[\lim_{y\to+\infty} F'(w_0(y)), \lim_{y\to-\infty} F'(w_0(y))\right] \\ &= \inf(F'(0), F'(1)). \end{aligned}$$

Note that $h_- < 0$ and b equals h_- near $r = 0$. Therefore, by choice of appropriate β, $a(r)$ can be chosen small and $b(r) \leq -b_0 < 0$ for all $r \in \mathbf{R}^+$.

Lemma 2.2.2 *Under assumptions (H) for any $R \geq R_4$, L_1 generates an analytic semi-group on Y and there exist positive constants c_0, c_1, c_2, γ_2 such that for any $t \in (0,T)$,*

$$\begin{aligned} &\|e^{tL_1}\|_{\mathcal{L}(Y)} \leq c_0 e^{-\gamma_2 t}, \\ &\|G_1(t)\|_Y \leq c_1(1+\delta)e^{-\gamma_2(R+ct)} + c_2\delta\|v(t)\|_Y. \end{aligned}$$

Proof: We first study the behaviour of $\|G_1(t)\|_Y$: it is a standard result that w_0, ϕ_0 and ψ_0 decrease exponentially fast at infinity. Then, it comes that

$$\begin{aligned} &\|(F'(w_0(r - s(t))) - h_-)\xi(r)v(r,t)\|_Y \leq c_0\delta e^{-\gamma_2(R+ct)}, \\ &\|S\|_Y \leq c_0 e^{-\gamma_2(R+ct)}. \end{aligned}$$

In addition, $N = [F(w+v) - F(w_0+v)] + [F(w_0+v) - F(w_0) - F'(w_0)v] + F(w_0)(1-\chi(r))$ and

$$\|N\|_Y \leq c_0 e^{-\gamma_2(R+ct)} + c_0\|v\|_Y^2 \leq c_0 e^{-\gamma_2(R+ct)} + c_0\delta\|v\|_Y.$$

Finally, we want to bound $\| \left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right) w_0'(r-s(t))\chi(r)\xi(r)\|_Y$. As $R \geq R_4$, $s(t) \geq R_4$ and the particular case $r = s(t)$ explained in a heuristic way does not occur as $\xi(r)$ decays exponentially fast as r goes to infinity. To conclude, we have to explain the bound of $|\rho'(t)|$. Indeed, by equation (2.4),

$$|\rho'(t)| \leq c_0 \left((1+\delta)e^{-\gamma_0(R+ct)} + \delta\frac{\log(R+ct)}{(R+ct)^2} + \frac{\delta}{(R+ct)^2} + \delta\|v\|_Y \right), \qquad (2.7)$$

and

$$\| \left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right) w_0'(r-s(t))\chi(r)\xi(r)\|_Y \leq c_2(1+\delta)e^{-\gamma_2(R+ct)}.$$

This ends the proof for $\|G_1\|_Y$.

On the other hand, the semi-group generated by L_1 on Y is studied by energy estimates. Let u be a solution of

$$\begin{cases} u_t = L_1 u & r > 0, \quad t > 0, \\ u_r|_{r=0} = 0 & \qquad\quad t > 0, \\ u(r,0) = u_0(r) & r > 0. \end{cases}$$

Let $I_1(t) = \frac{1}{2}\int_0^\infty u^2 dr$ and $I_2(t) = \frac{1}{2}\int_0^\infty u_r^2 dr$. Then, the derivatives with respect to t of I_1 and I_2 satisfy

$$\begin{aligned} \dot{I}_1(t) &= -2I_2 + (n-1)\int_0^\infty \frac{uu_r}{r}dr + \int_0^\infty \left(b - \frac{a'}{2}\right) u^2 dr, \\ \dot{I}_2(t) &= -\int_0^\infty u_{rr}^2 dr - \frac{n-1}{2}\int_0^\infty \left(\frac{u_r}{r}\right)^2 dr + \int_0^\infty \left(b + \frac{a'}{2}\right) u_r^2 dr - \int_0^\infty \frac{b''}{2} u^2 dr. \end{aligned}$$

Let introduce $e > 0, \epsilon > 0, I(t) = I_1(t) + eI_2(t)$, then

$$\begin{aligned} \dot{I}(t) \leq & \int_0^\infty \left(\left(b - \frac{a'}{2}\right) - e\frac{b''}{2} + \frac{(n-1)\epsilon}{2}\right) u^2 dr \\ & + \int_0^\infty \left(-1 + e\left(b + \frac{a'}{2}\right)\right) u_r^2 dr + \frac{n-1}{2}\left(\frac{1}{\epsilon} - e\right)\int_0^\infty \left(\frac{u_r}{r}\right)^2 dr. \qquad (2.8) \end{aligned}$$

Choosing first $\epsilon << 1$, then $e >> 1$ depending on ϵ and $\beta << 1$ depending on e, we obtain

$$\begin{aligned} &\left(b - \frac{a'}{2}\right) - e\frac{b''}{2} + \frac{(n-1)\epsilon}{2} \leq \frac{-\gamma_2}{2} < 0, \\ &-1 + e\left(b + \frac{a'}{2}\right) \leq \frac{-\gamma_2}{2}e < 0, \\ &\frac{1}{\epsilon} - e \leq -1, \end{aligned}$$

where $\gamma_2 = |b_0|$. It follows that $\dot{I}(t) \leq -\gamma_2 I(t)$ and $\|u(t)\|_Y \leq c_0 e^{-\gamma_2 t}\|u_0\|_Y$. This proves the lemma. ■

We shall use these calculations to get some further information on the behaviour of the semigroup generated by L_1 which are useful in the following sections. Let $\alpha(t) = \int_0^\infty (\frac{u_r}{r})^2 dr$. Then, according to (2.8),

$$\frac{d}{dt}(e^{\gamma_2 t} I(t)) + \frac{n-1}{2} e^{\gamma_2 t}\alpha(t) \leq 0.$$

Integrating the latter inequality between σ and t and using Hölder's inequality, we obtain the following result for γ defined in the introduction and any $(\sigma, t) \in (0, T)$ such that $\sigma \leq t$:

$$\int_\sigma^t e^{-\gamma(t-s)} \|\frac{u_r}{r}(s)\|_{L^2(\mathbf{R}^+)} ds \leq c_3 \|u(\sigma)\|_Y e^{-\frac{\gamma_2}{2}(t-\sigma)}. \tag{2.9}$$

In the same way, using convolution inequality $\|f * g\|_{L^1(\mathbf{R})} \leq \|f\|_{L^1(\mathbf{R})}\|g\|_{L^1(\mathbf{R})}$, we obtain for $\gamma' < \gamma_2$,

$$\int_\sigma^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}} \|\frac{u_r}{r}(s)\|_{L^2(\mathbf{R}^+)} ds \leq c_3 \|u(\sigma)\|_Y e^{-\gamma'(t-\sigma)}.$$

The next lemma is a corollary of these calculations and will be used in the following to compute asymptotics of the solutions (v, ρ).

Lemma 2.2.3 *Under assumptions (H) for any $R \geq R_4$, there exist positive constants c_0, c_1, c_2, γ_3 such that for any $t \in (0, T)$,*

$$\begin{aligned}\int_0^t e^{-\gamma(t-s)} \|\frac{z_r}{r}(s)\|_{L^2(\mathbf{R}^+)} ds \leq & c_0 \|v_0\|_Y e^{-\gamma_3 t} + c_1(1+\delta) e^{-\gamma_3(R+ct)} \\ & + c_2 \delta \int_0^t e^{-\frac{\gamma_2}{2}(t-s)} \|v(s)\|_Y ds, \\ \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}} \|\frac{z_r}{r}(s)\|_{L^2(\mathbf{R}^+)} ds \leq & c_0 \|v_0\|_Y e^{-\gamma_3 t} + c_1(1+\delta) e^{-\gamma_3(R+ct)} \\ & + c_2 \delta \int_0^t e^{-\gamma'(t-s)} \|v(s)\|_Y ds,\end{aligned}$$

where z is defined in (2.6).

Proof: The proofs of these two inequalities are very similar. Therefore, we only prove the first one. We recall that $z(r,s) = e^{sL_1} z_0 + \int_0^s e^{(s-\sigma)L_1} G_1(r,\sigma) d\sigma$ for any $r \geq 0$, $s \geq 0$. Then,

$$\begin{aligned}\int_0^t e^{-\gamma(t-s)} \|\frac{z_r}{r}(s)\|_{L^2(\mathbf{R}^+)} ds \leq & \int_0^t e^{-\gamma(t-s)} \|\frac{\partial_r}{r} e^{sL_1} z_0\|_{L^2(\mathbf{R}^+)} ds \\ & + \int_0^t e^{-\gamma(t-s)} \int_0^s \|\frac{\partial_r}{r} e^{(s-\sigma)L_1} G_1(r,\sigma)\|_{L^2(\mathbf{R}^+)} d\sigma ds.\end{aligned}$$

The first term of the right hand side is bounded by (2.9):

$$\int_0^t e^{-\gamma(t-s)} \|\frac{\partial_r}{r} e^{sL_1} z_0\|_{L^2(\mathbf{R}^+)} ds \leq c_3 e^{-\frac{\gamma_2}{2}t} \|z_0\|_Y.$$

The second term is bounded by Fubini's theorem, (2.9) and lemma 2.2.2:

$$\int_0^t \int_0^s e^{-\gamma(t-s)} \|\frac{\partial_r}{r} e^{(s-\sigma)L_1} G_1(r,\sigma)\|_{L^2(\mathbf{R}^+)} d\sigma ds \leq \int_0^t c_3 e^{-\frac{\gamma_2}{2}(t-\sigma)} \|G_1(r,\sigma)\|_Y d\sigma$$
$$\leq c_1(1+\delta)e^{-\gamma_3(R+ct)} + c_2\delta \int_0^t e^{-\frac{\gamma_2}{2}(t-\sigma)} \|v(\sigma)\|_Y d\sigma.$$

This ends the proof of lemma 2.2.3. ■

Corollary 2.2.4 *Under assumptions (H) for any $R \geq R_4$, the behaviour of z is a result of lemma 2.2.2. Indeed, there exist positive constants c_1, c_2, c_3 such that for any $t \in (0,T)$,*

$$\|z(t)\|_Y \leq c_1 \|v_0\|_Y e^{-\gamma_2 t} + c_2(1+\delta)e^{-\gamma_2(R+ct)} + c_3\delta \int_0^t e^{-\gamma_2(t-s)} \|v(s)\|_Y ds.$$

2.2.2 Extension to the real line

As we said before, we need to work on the whole real line and therefore to extend v for $r < 0$. Let

$$\tilde{z}(r,t) = \begin{cases} z(-r,t) & if \quad r < 0, \\ v(r,t) & if \quad r \geq 0. \end{cases}$$

Then, $\tilde{z}$ is smooth in $\mathbf{R}$ and satisfies for any $r \in \mathbf{R}$,

$$\begin{aligned} \tilde{z}_t(r,t) =& \tilde{z}_{rr}(r,t) + \frac{n-1}{r}\tilde{z}_r(r,t) + F'(w_0(r-s(t)))\tilde{z}(r,t) \\ &+ \left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right) w_0'(r-s(t))\chi(r) + \tilde{N} + G_2(r,t), \end{aligned} \tag{2.10}$$

where

$$\tilde{N} = \begin{cases} N & if \quad r \geq 0, \\ N\xi(|r|) & if \quad r < 0, \end{cases}$$

$$G_2(r,t) = \begin{cases} S \quad if \quad r \geq 0, \\ az_r + (b-h_-)z + S\xi(|r|) \\ +(F'(w_0(|r|-s(t))) - F'(w_0(r-s(t))))z(|r|,t) \\ +(\frac{n-1}{|r|} - \frac{n-1}{R+ct} + \rho'(t))w_0'(|r|-s(t))\chi(|r|)\xi(|r|) \\ -(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t))w_0'(r-s(t))\chi(r)\xi(r) \qquad if \quad r \leq 0. \end{cases}$$

Using lemma 2.2.2 and corollary 2.2.4, we have the following lemma:

Lemma 2.2.5 *Under assumptions (H) with $R \geq R_4$, there exist positive constants c_1, c_2, c_3 such that for any $t \in (0,T)$,*

$$\|G_2(t)\|_{L^2(\mathbf{R})} \leq c_1 \|v_0\|_Y e^{-\gamma_2 t} + c_2(1+\delta)e^{-\gamma_2(R+ct)} + c_3\delta \int_0^t e^{-\gamma_2(t-s)} \|v(s)\|_Y ds.$$

2.2.3 Moving Frame

In order to take advantage of spectral properties of the time independent operator L_0, it is convenient to work in the moving frame with speed $s(t)$. So let $\bar{z}(r-s(t),t)=\tilde{z}(r,t)$ and $G_3(r-s(t),t)=G_2(r,t)$. Then, $\bar{z}$ satisfies an equation similar to (2.10). As $\eta(t)=\int_{\mathbf{R}}\bar{z}(y,t)\psi_0(y)dy=\int_{\mathbf{R}}\tilde{z}(r,t)\psi_0(r-s(t))dr$ is non zero in general, $\bar{z}$ does not belong to $\mathcal{R}$. We recall that $\mathcal{R}$ has been defined in the introduction as the supplementary of the spectral subspace corresponding to the eigenvalue 0 of the operator L_0 in $L^2(\mathbf{R})$. As $L_0=\partial_y^2+c\partial_y+F'(w_0)$ has interesting spectral properties in $\mathcal{R}$, it is convenient to use the following spectral decomposition:

$$\bar{z}(y,t)=\eta(t)\phi_0(y)+r(y,t),\text{ where } r\in\mathcal{R}. \tag{2.11}$$

Note that this $r\in\mathcal{R}$ is different from the $r\in\mathbf{R}^+$ used so far. Before going on, notice that $\eta(t)$ decreases exponentially fast in time: $|\eta(t)|\leq c_0e^{-\gamma_4(R+ct)}$ for $\gamma_4>0$, and let introduce a few notations. Let $\zeta\in C_0^\infty(\mathbf{R})$, positive, even, which satisfies $\zeta\equiv 1$ on $[-R_4,R_4]$ and $\zeta\equiv 0$ on $[-R_4-1,R_4+1]^c$.

We decompose the nonlinear terms as follows: $\tilde{N}=N_1+N_2$ where

$$N_1=F(w+r)-F(w_0)\chi(y+s(t))-F'(w_0)r\text{ and } N_2=\tilde{N}-N_1.$$

Then,

$$\begin{aligned}||N_1||_{L^2}&\leq c_0||r||_Y^2+c_0e^{-\gamma_2(R+ct)},\\ ||N_2||_{L^2}&\leq c_0|\eta(t)|.\end{aligned} \tag{2.12}$$

Substitute the decomposition (2.11) into equation (2.10) to get:

$$\begin{aligned}&r_t(y,t)=L_2r(y,t)+Q(G_4)(y,t) && t\geq 0,\quad y\in\mathbf{R},\\ &\int_{\mathbf{R}}r(y,t)\psi_0(y)dy=0 && t>0,\end{aligned}$$

where

$$\begin{aligned}L_2&=\partial_y^2+c\partial_y+F'(w_0)+Q(N_1+(1-\zeta)G_5)\\ G_5&=\left(\frac{n-1}{y+s(t)}-\frac{n-1}{R+ct}+\rho'(t)\right)r_y(y,t)\\ G_4&=G_3(y,t)+N_2+\zeta G_5(y,t)\\ &\quad+\left(\frac{n-1}{y+s(t)}-\frac{n-1}{R+ct}+\rho'(t)\right)(\eta(t)\phi_0'(y)+\xi(y+s(t))\phi_0(y)).\end{aligned} \tag{2.13}$$

We recall that Q is a projector onto $\mathcal{R}$ defined in the introduction.

Lemma 2.2.6 *There exist positive constants R_5,δ_5 such that under assumptions (H) with $R\geq R_5$ and $\delta\leq\delta_5$, L_2 generates a family of evolution operators $A(t,s)$ on $\mathcal{R}$ which satisfies*

$$||A(t,s)||_{\mathcal{L}(\mathcal{R})}\leq c_0e^{-\gamma(t-s)},\qquad 0\leq s\leq t.$$

Proof: Let $L_0 = \partial_y^2 + c\partial_y + F'(w_0)$ defined on $\mathcal{R}$. Then $\sigma(L_{0|\mathcal{R}}) \subset]-\infty; -\gamma]$, $\gamma > 0$ and L_0 generates an analytic semi-group on $\mathcal{R}$ which satisfies $\|e^{tL_0}\|_{\mathcal{L}(\mathcal{R})} \leq c_0 e^{-\gamma t}$ and $\mathcal{R}^{1/2} \equiv D(L_0^{1/2}) = H^1(\mathbf{R})$, see for instance [67]. Let

$$\begin{aligned} B: \quad \mathbf{R}^+ &\longrightarrow \mathcal{L}(H^1(\mathbf{R}), L^2(\mathbf{R})) \\ t &\longmapsto B(t) : H^1(\mathbf{R}) \to L^2(\mathbf{R}) \\ &\qquad\qquad r \mapsto Q(N_1 + (1-\zeta)G_5). \end{aligned}$$

We want to prove that B is a small perturbation of the operator L_0 which does not affect its exponential decrease. As $\|B(t)\|_{\mathcal{L}(H^1,L^2)} \leq c_0(\frac{n-1}{R} + \delta)$, appendix 4.1 ends the proof, namely there exist some $R_5 >> 1$ and some $\delta_5 > 0$ so that for all $R \geq R_5$ and $\delta \leq \delta_5$, L_2 generates a family of evolution operators $A(t,s)$ on $\mathcal{R}$ which satisfies lemma 2.2.6 for a slightly different γ. ∎

Lemma 2.2.7 *Under hypothesis (H) with $R \geq R_4$, there exist positive constants c_i, $i = 0, .., 5$ and γ_5 such that for any $t \in (0, T)$,*

$$\begin{aligned} \|Q(G_4)(t)\|_{L^2(\mathbf{R})} \leq & c_0\|v_0\|_Y e^{-\gamma_5 t} + c_1(1+\delta)e^{-\gamma_5(R+ct)} + c_3\frac{\log(R+ct)}{(R+ct)^2} \\ & + c_2\delta\int_0^t e^{-\gamma_2(t-s)}\|v(s)\|_Y ds + c_4|\rho'(t)| + c_5\|\frac{z_r}{r}(t)\|_{L^2(\mathbf{R}^+)}. \end{aligned}$$

Proof: As G_4 is given by (2.13), the first two terms have already been studied in lemma 2.2.5 and (2.12):

$$\begin{aligned} \|Q(G_3)(t)\|_{L^2(\mathbf{R})} &\leq \|G_2(t)\|_{L^2(\mathbf{R}^+)} \\ &\leq c_1\|v_0\|_Y e^{-\gamma_2 t} + c_2(1+\delta)e^{-\gamma_2(R+ct)} + c_3\delta\int_0^t e^{-\gamma_2(t-s)}\|v(s)\|_Y ds, \\ \|Q(N_2)(t)\|_{L^2} &\leq c_0|\eta(t)| \leq c_0 e^{-\gamma_4(R+ct)}. \end{aligned}$$

The last terms will be cut into four parts with the cut-off ζ. As

$$\left(\frac{n-1}{y+s(t)} - \frac{n-1}{R+ct} + \rho'(t)\right)\xi(y+s(t))\phi_0(y)\zeta(y+s(t)) = 0$$

by definition of ξ and ζ, we obtain

$$\|\left(\frac{n-1}{y+s(t)} - \frac{n-1}{R+ct} + \rho'(t)\right)\xi(y+s(t))\phi_0(y)\|_{L^2} \leq c_0\frac{\log(R+ct)}{(R+ct)^2} + c_1|\rho'(t)|.$$

In the same way, we get

$$\|\left(\frac{n-1}{y+s(t)} - \frac{n-1}{R+ct} + \rho'(t)\right)\eta(t)\phi_0'(y)(1-\zeta(y+s(t)))\|_{L^2} \leq c_2(1+\delta)e^{-\gamma_4(R+ct)}.$$

Finally, we join the last two terms:

$$\begin{aligned}\|\zeta G_5 + \left(\frac{n-1}{y+s(t)} - \frac{n-1}{R+ct} + \rho'(t)\right)\eta(t)\phi_0'(y)\zeta(y+s(t))\|_{L^2(\mathbf{R})}\\ \leq \|\left(\frac{n-1}{r} - \frac{n-1}{R+ct} + \rho'(t)\right)\zeta(r)\tilde{z}_r(r,t)\|_{L^2(\mathbf{R})}\\ \leq c_2\|\frac{z_r(r,t)}{r}\|_{L^2(\mathbf{R}^+)} + (\frac{n-1}{R} + c_0\delta + c_1)\|z(r,t)\|_Y\end{aligned}$$

as $\tilde{z} = z = v$ on $[0, R_4]$. By corollary 2.2.4, we conclude that:

$$\begin{aligned}\|\zeta G_5 + \left(\frac{n-1}{y+s(t)} - \frac{n-1}{R+ct} + \rho'(t)\right)\eta(t)\phi_0'(y)\zeta(y+s(t))\|_{L^2(\mathbf{R})}\\ \leq c_0\|v_0\|_Y e^{-\gamma_2 t} + c_1(1+\delta)e^{-\gamma_2(R+ct)}\\ + c_2\|\frac{z_r(r,t)}{r}\|_{L^2(\mathbf{R}^+)} + c_3\delta\int_0^t e^{-\gamma_2(t-s)}\|v(s)\|_Y ds.\end{aligned}$$

Define $\gamma_5 = \inf\{\gamma_2, \gamma_4\}$. This ends the proof. ■

Corollary 2.2.8 *Under assumptions (H) with* $R \geq \max(R_4, R_5)$ *and* $\delta \leq \delta_5$*, there exist positive constants* c_i*,* $i = 1, .., 4$ *and* γ_7, γ' *such that for any* $t \in (0, T)$*,*

$$\begin{aligned}\|r(t)\|_{H^1(\mathbf{R})} \leq & c_1\|r_0\|_{H^1(\mathbf{R})}e^{-\gamma_7 t} + c_2(1+\delta)e^{-\gamma_7(R+ct)} + c_3\frac{\log(R+ct)}{(R+ct)^2}\\ & + c_4\int_0^t \frac{e^{-\gamma'(t-s)}}{\sqrt{(t-s)}}|\rho'(s)|ds.\end{aligned}$$

Proof: We first want to bound the L^2 norm of r. As a consequence of lemmas 2.2.6, 2.2.7 and 2.2.3, we get for any $t \in (0, T)$,

$$\begin{aligned}\|r(t)\|_{L^2(\mathbf{R})} \leq & c_1\|r_0\|_{H^1(\mathbf{R})}e^{-\gamma_6 t} + c_2(1+\delta)e^{-\gamma_6(R+ct)} + c_3\frac{\log(R+ct)}{(R+ct)^2}\\ & + c_4\int_0^t e^{-\gamma(t-s)}|\rho'(s)|ds + c_5\delta\int_0^t e^{-\gamma_6(t-s)}\|r(s)\|_{H^1(\mathbf{R})}ds. \qquad (2.14)\end{aligned}$$

In order to bound the H^1 norm of r, we recall that $r_t = L_2 r + Q(G_4)$ and $L_2 = L_0 + B(t)$. According to lemma 2.2.6, operator $B(t)$ is a small perturbation of L_0. Then, the Banach space $\mathcal{R}^{1/2}$ can be defined by $D(A^{1/2})$ as well as $D(L_0^{1/2})$, and the graph norms are equivalent . Thus, $\|\partial_x A(t,s)\|_{\mathcal{L}(\mathcal{R})} \leq c_0\frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}$. In addition, $r(y,t) = A(t,0)r_0(y) + \int_0^t A(t,s)Q(G_4)(y,s)ds$. Derivating this last expression with respect to y and bounding the L^2 norm, we get:

$$\|\partial_y r(t)\|_{L^2(\mathbf{R})} \leq c_0\|r_0\|_{H^1(\mathbf{R})}e^{-\gamma t} + \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}\|Q(G_4)(s)\|_{L^2}ds.$$

Finally, by (2.14) and lemmas 2.2.7 and 2.2.3, we get

$$\begin{aligned}\|r(t)\|_{H^1(\mathbf{R})} \leq & c_1\|r_0\|_{H^1(\mathbf{R})}e^{-\gamma_6 t} + c_2(1+\delta)e^{-\gamma_6(R+ct)} + c_3\frac{\log(R+ct)}{(R+ct)^2} \\ & + c_4\int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}|\rho'(s)|ds + c_5\int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}\|r(s)\|_{H^1}ds.\end{aligned}$$

Indeed, by Fubini's theorem and one integration by parts,

$$\int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{t-s}}\int_0^s e^{-\gamma_2(s-\sigma)}\|v(\sigma)\|_Y d\sigma ds \leq c_0\int_0^t e^{-\gamma'(t-s)}\|v(s)\|_Y ds.$$

Gronwall's lemma ends the proof. ∎

Corollary 2.2.9 *Under the same assumptions (H) with* $R \geq \max(R_4, R_5)$ *and* $\delta \leq \delta_5$*, there exist positive constants* c_i*,* $i = 1, .., 3$ *such that for any* $t \in (0, T)$*,*

$$\begin{aligned}\|v(t)\|_Y \leq c_1\|v_0\|_Y e^{-\gamma_8 t} & + c_2(1+\delta)e^{-\gamma_8(R+ct)} \\ & + c_3\frac{\log(R+ct)}{(R+ct)^2} + \int_0^t \frac{e^{-\gamma(t-s)}}{\sqrt{(t-s)}}|\rho'(s)|ds.\end{aligned}$$

2.2.4 Conclusion

Proof of proposition 2.2.1: Take $R_3 = \max\{R_4, R_5\}$ and $\delta_3 = \inf\{\delta_4, \delta_5\}$. Let $T > 0$, $\delta \leq \delta_3$ and $R \geq R_3$. Consider $(v, \rho) \in \mathcal{C}^0([0, T], Y \times \mathbf{R})$ any solution of (2.3, 2.4) satisfying

$$\|v\|_Y \leq \delta, \quad |\rho(t)| \leq 1, \quad 0 \leq t \leq T.$$

Then, assumptions (H) are valid and by inequality (2.7), corollary 2.2.9 and Gronwall's lemma, there exist positive constants c_1, c_2, γ_0 such that

$$\|v(t)\|_Y + |\rho'(t)| \leq c_1\|v_0\|_Y e^{-\gamma_0 t} + c_2\frac{\log(R+ct)}{(R+ct)^2}, \quad 0 \leq t \leq T.$$

This ends the proof of proposition 2.2.1. ∎

Chapitre 3

Nonradial Solutions

In this chapter, we deal with non radial solutions of equation (1.1). We prove, in this case, that travelling waves are Lyapunov stable but not necessarily asymptotically stable for general (i.e. non necessarily spherically symmetric) perturbations. In the first section of this chapter, we explain how the Lyapunov stability follows from Uchiyama's proposition and the maximum principle. In the second section, we prove theorem 2. To this end, we introduce some energy functionals which enable us to rule out the asymptotic stability of travelling waves against arbitrary small perturbations. In particular, we give an example in $\mathbf{R}^2$ of an initial data u_0 close to a travelling wave which converges to a non-radial profile as t goes to infinity.

3.1 Lyapunov Stability

In chapter I.2, we proved in theorem 2.1.3 the local stability of travelling waves in X, i.e. among radial perturbations. Note that Uchiyama [92] proved a similar result in the L^∞ norm in her lemma 4.5 without any information on the decay rate of the perturbation. Using comparison theorem, we show easily the Lyapunov stability of travelling waves against arbitrary small perturbations.

Proposition 3.1.1 *For any $\epsilon > 0$, there exist positive constants R_0, δ such that if $u_0 : \boldsymbol{R}^n \to \boldsymbol{R}$ is a spherically symmetric function satisfying*

$$\|u_0(x) - w_0(|x| - R)\|_{L^\infty(\boldsymbol{R}^n)} \leq \delta$$

for some $R \geq R_0$, then equation (1.1) has a unique solution $u \in C^0(\boldsymbol{R}^+, L^\infty(\boldsymbol{R}^n))$ with initial data u_0 and for all $t \in \boldsymbol{R}^+$,

$$\|u(x,t) - w_0(|x| - \bar{s}(t))\|_{L^\infty(\boldsymbol{R}^n)} \leq \epsilon$$

where $\bar{s}(t) = R + ct - \frac{n-1}{c} \log\left(\frac{R+ct}{R}\right)$.

Proof: See Uchiyama [92], lemma 4.5. ■

Corollary 3.1.2 *For any $\epsilon > 0$, there exist positive constants R_0, δ such that if $u_0 : \boldsymbol{R}^n \to \boldsymbol{R}$ satisfies*

$$\|u_0(x) - w_0(|x| - R)\|_{L^\infty(\boldsymbol{R}^n)} \leq \delta$$

for some $R \geq R_0$, then equation (1.1) has a unique solution $u \in \mathcal{C}^0(\boldsymbol{R}^+, L^\infty(\boldsymbol{R}^n))$ with initial data u_0 and for all $t \in \boldsymbol{R}^+$,

$$\|u(x,t) - w_0(|x| - \bar{s}(t))\|_{L^\infty(\boldsymbol{R}^n)} \leq \epsilon$$

where $\bar{s}(t) = R + ct - \frac{n-1}{c}\log\left(\frac{R+ct}{R}\right)$.

Proof: Let $u(x,t), u_1(x,t), u_2(x,t)$ be the solutions of equation (1.1) with initial data $u_0, w_0(|x| - R) - \delta, w_0(|x| - R) + \delta$ respectively. Then, combining the maximum principle and proposition 3.1.1, we have $u_1(x,t) \leq u(x,t) \leq u_2(x,t)$ on $\mathbf{R}^n \times \mathbf{R}^+$ and $\|u(x,t) - w_0(|x| - \bar{s}(t))\|_{L^\infty(\mathbf{R}^n)} \leq \epsilon$. This ends the proof. ■

3.2 Energy Estimates

In order to prove theorem 2 about non radial profiles, we need to control the perturbation of the wave and in particular the shape of the interface. We proceed as in the previous chapter: we decompose the solution $u(x,t)$ as a translate of the wave and a transversal perturbation. We use the same notations as in chapter I.2. As is explained in the introduction, we restrict ourselves for convenience in the two-dimensional case, and we use polar coordinates $(r, \theta) \in \mathbf{R}^+ \times [0, 2\pi)$ in $\mathbf{R}^2$. Define the open set $\Omega = \mathbf{R}^{+*} \times (0, 2\pi)$ and the measure $d\nu = r dr d\theta$. We need to introduce some Banach spaces adapted to these new variables:

$$W = \{v(r,\theta) \in H^1_{loc}(\Omega) \,|\, v, v_r, \frac{v_\theta}{r} \in L^2(\Omega, d\nu) \text{ and } v(r,0) = v(r,2\pi) \text{ in } L^2_{loc}(\mathbf{R}^+, dr)\}$$
$$Z = \{\rho(\theta) \in H^1(0,2\pi) \mid \rho(0) = \rho(2\pi)\}.$$

We also define the associated norms:

$$\|v\|_W = \left(\int_\Omega \left(v^2 + v_r^2 + \frac{v_\theta^2}{r^2}\right) d\nu\right)^{\frac{1}{2}}$$
$$\|\rho\|_Z = \left(\int_0^{2\pi} \left(\rho^2 + \rho_\theta^2\right) d\theta\right)^{\frac{1}{2}} = \|\rho\|_{H^1(0,2\pi)}.$$

The space W does not seem to be very suitable to our problem as the measure $d\nu$ induces a linear grow in time of the norm due to the expansion of the front. However, it is convenient for energy estimates as we shall see below. In those spaces, the coordinate system developed in the first section is still valid. More precisely, we have the following lemma:

Lemma 3.2.1 *There exist positive constants R_1', δ_1', K' such that for any $R \geq R_1'$ and any $\xi \in W$ with $\|\xi\|_W \leq \delta_1'$, there exists a unique pair $(v, \rho) \in W \times Z$ with*

i) $\|v\|_W + \|\rho\|_Z \leq K'\|\xi\|_W$,

ii) $w(r-R,r)+\xi(r,\theta)=w(r-R-\rho(\theta),r)+v(r,\theta)$ *for all* $(r,\theta)\in\bar{\Omega}$,

iii) $\int_0^\infty v(r,\theta)\psi(r-R-\rho(\theta),r)dr=0$ *for any* $\theta\in[0,2\pi)$.

Proof: The proof is very similar to the one of lemma 2.1.1 and we may omit it. ■

Using lemma 3.2.1, assuming the solution $u(x,t)$ is close to a travelling wave, we have for any $t\geq 0, \theta\in[0,2\pi)$, and some $R\geq 0$,

$$u(r,\theta,t)=w(r-s(\theta,t),r)+v(r,\theta,t), \qquad r\geq 0, \tag{3.1}$$
$$s(\theta,t)=R+ct-\frac{1}{c}\log\left(\frac{R+ct}{R}\right)+\rho(\theta,t),$$
$$\int_0^\infty v(r,\theta,t)\psi(r-s(\theta,t),r)dr=0. \tag{3.2}$$

Note that according to Jones [52], the solution $u(r,\theta,t)$ is close to a travelling wave in every radial direction of $\mathbf{R}^2$. Therefore, in (3.2), v is transversal to $\psi(r-s(\theta,t),r)$ for all $\theta\in[0,2\pi)$.

Then, we get two new evolution equations. The one satisfied by v is obtained by equations (1.1) and (3.1):

$$\begin{aligned} v_t(r,\theta,t)=&\Delta v(r,\theta,t)+F'(w(r-s(\theta,t),r))v(r,\theta,t)+N+S \\ &+w_y(r-s(\theta,t),r)\rho_t(\theta,t)-\frac{1}{r^2}\partial_\theta(w_y(r-s(\theta,t),r)\rho_\theta(\theta,t)), \\ v(r,\theta,0)=&v_0(r,\theta), \end{aligned} \tag{3.3}$$

where

$$\Delta=\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2,$$
$$N=F(w+v)-F(w)-F'(w)v,$$
$$S=\left(\frac{1}{r}-\frac{1}{R+ct}\right)w_y+\left(w_{rr}+2w_{ry}+\frac{1}{r}w_r\right)+w_{yy}+cw_y+F(w).$$

Differentiating equation (3.2) with respect to t and integrating by parts, we get as in the previous chapter, the equation satisfied by ρ:

$$\rho_t(\theta,t)\lambda(\infty,\theta,t)=-\int_0^\infty g(r,\theta,t)dr, \tag{3.4}$$
$$\rho(\theta,0)=\rho_0(\theta),$$

where

$$\lambda(r,\theta,t) = \int_0^r \left(\psi(z-s(\theta,t),z)w_y - \psi_y v\right) dz,$$
$$g(z,\theta,t) = g_1(z,\theta,t) + g_2(z,\theta,t),$$
$$g_1(z,\theta,t) = v\Lambda + \psi(z-s(\theta,t),z)\,(N+S)\,,$$
$$g_2(z,\theta,t) = -\frac{1}{z^2}\psi\partial_\theta(w_y\rho_\theta) + \frac{1}{z^2}\psi v_{\theta\theta},$$
$$\Lambda(z,\theta,t) = \left(\frac{1}{R+ct} - \frac{1}{z}\right)\psi_y + \frac{1}{z^2}\psi + \left(\psi_{rr} + 2\psi_{ry} - \frac{1}{z}\psi_r\right) + \left(\psi_{yy} - c\psi_y + F'(w)\psi\right).$$

As in chapter I.2, we consider the initial value problem for equations (3.3, 3.4).

Lemma 3.2.2 *There exist $R_0 > 0$, $\epsilon_0 > 0$ and $T > 0$ such that, for any $R \geq R_0$ and for all initial data $(v_0, \rho_0) \in W \times Z$ with $\|v_0\|_W \leq \epsilon_0$ and $\|\rho_0\|_Z \leq \epsilon_0$, the integral equations corresponding to (3.3, 3.4) have a unique solution $(v,\rho) \in C^0([0,T], W \times Z)$. In addition, $(v,\rho) \in C^1((0,T],W) \times C^1((0,T],Z)$, and equations (3.3, 3.4) are satisfied for $0 < t \leq T$.*

Proof: Define $\epsilon = \delta_1'$ and let δ be as in corollary 3.1.2. Choose $0 < \epsilon_0 \leq \delta(1+c_0 e^{-\gamma_1 R_0})^{-1}$ for some fixed $R_0 > 0$ large enough. Let $(v_0,\rho_0) \in W \times Z$ such that $\|v_0\|_W \leq \epsilon_0$ and $\|\rho_0\|_Z \leq \epsilon_0$. Finally, define $u_0(r,\theta) = w(r - R - \rho_0(\theta), r) + v_0(r,\theta)$. Then, $u_0 \in H^1(\mathbf{R}^2)$ and it is a standard result that there exists a unique solution $u(x,t) \in C^0([0,T], H^1(\mathbf{R}^2)) \cap C^1((0,T], H^1(\mathbf{R}^2))$ to equation (1.1) with initial data u_0. According to corollary 3.1.2, $u(x,t)$ stay close to a travelling wave in the L^∞-norm for all $t > 0$. By energy estimates, we show in sections 2.2.1 and 2.2.2 that this is also the case in the H^1 norm. Thus, lemma 3.2.1 is still valid and there exists a unique pair $(v,\rho) \in W \times Z$ such that (3.1, 1.9, 3.2) hold and (v,ρ) satisfy equations (3.3, 3.4). ■

These two equations are very similar to those found in the second chapter. We choose here to deal with energy estimates. We study the behaviour of $\|v(t)\|_W$ and $\|\rho(t)\|_Z$ under the assumption that the initial data are small. We have the following theorem:

Theorem 3.2.3 *There exist positive constants R_1, ϵ_1, n such that if $(v_0,\rho_0) \in W \times Z$ satisfy*

$$R^{\frac{1}{2}}\|v_0\|_W^2 + \|\rho_0\|_Z^2 \leq \epsilon$$

for some $R \geq R_1$ and some $\epsilon \leq \epsilon_1$, then equations (3.3, 3.4) have a unique solution $(v,\rho) \in C^0([0,+\infty), W \times Z)$ with initial data (v_0,ρ_0), and

$$(R+ct)^{\frac{1}{2}}\|v(t)\|_W^2 + \|\rho(t)\|_Z^2 \leq n\left(\epsilon + \frac{1}{R}\right)$$

for all $t \geq 0$.

These estimates will be useful to prove theorem 2. We now give the proof of the first part of theorem 2:

Proof of theorem 2: Let R'_1, δ'_1, K' be as in lemma 3.2.2, R_1, ϵ_1, n as in theorem 3.2.3 and c_0, γ_1 as in (2.5). Choose R'_0, δ'_0 and η such that:

$$R'_0 \geq \max(R_1; R'_1) \qquad \eta = \frac{\sqrt{\epsilon_1}}{2\sqrt{2}K'}$$
$$\delta'_0 + c_0 e^{-\gamma_1 R'_0} \leq \min\left(\delta'_1; 2\eta\right) \qquad (R'_0)^{\frac{1}{4}} c_0 e^{-\gamma_1 R'_0} \leq \eta.$$

Let now $u_0 \in H^1(\mathbf{R}^2)$ such that $\|u_0(x) - w_0(|x| - R)\|_{H^1(\mathbf{R}^2)} \leq \delta$ for some $\delta \leq \delta'_0$, $R \geq R'_0$ and $R^{\frac{1}{4}}\delta \leq \eta$. Let $\xi(r,\theta) = u_0(r,\theta) - w(r-R,r)$. Then, by (2.5), $\|\xi\|_W \leq \delta + c_0 e^{-\gamma_1 R} \leq \delta'_1$ and $R \geq R'_1$. Thus, by lemma 3.2.1, there exists a unique pair $(v_0, \rho_0) \in W \times Z$ such that

i) $\|v_0\|_W + \|\rho_0\|_Z \leq K'\|\xi\|_W$,

ii) $w(r-R,r) + \xi(r,\theta) = w(r - R - \rho_0(\theta), r) + v_0(r,\theta)$ for all $(r,\theta) \in \bar{\Omega}$,

iii) $\int_0^\infty v_0(r,\theta)\psi(r - R - \rho_0(\theta), r)dr = 0$ for any $\theta \in [0, 2\pi)$.

Then, with the above conditions on R and ϵ,

$$R^{\frac{1}{2}}\|v_0\|_W^2 + \|\rho_0\|_Z^2 \leq \epsilon_1; \quad R \geq R_1.$$

Then, by theorem 3.2.3, equations (3.3, 3.4) have a unique solution (v,ρ) in $C^0([0,+\infty), W \times Z)$ and

$$(R+ct)^{\frac{1}{2}}\|v(t)\|_W^2 + \|\rho(t)\|_Z^2 \leq n\left(\epsilon + \frac{1}{R}\right) \text{ for all } t \geq 0.$$

Let $u(r,\theta,t) = w(r - s(\theta,t), r) + v(r,\theta,t)$ where $s(\theta,t)$ is given by (1.9). Then, by (2.5), u is a solution of (1.1) satisfying

$$\|u(r,\theta,t) - w_0(r - s(\theta,t))\|_W \leq \frac{c_0}{(R+ct)^{\frac{1}{4}}}.$$

This ends the proof of the first part of theorem 2. ■

We now prove theorem 3.2.3. Therefore, we introduce a few functionals linked with the norms of v and ρ in W and Z respectively.

3.2.1 Definition of primitive and functionals

If $T>0$ and $(v,\rho)\in C^1((0,T],W\times Z)$ is any solution of (3.3, 3.4), we first introduce functionals for the functions v and ρ:

$$\begin{aligned}
E_1(t) &= \frac{1}{2}\|v(t)\|^2_{L^2(\mathbf{R}^2)} = \frac{1}{2}\int_0^{2\pi}\int_0^\infty v^2(r,\theta,t)rdrd\theta = \frac{1}{2}\int_\Omega v^2 d\nu\\
E_2(t) &= \frac{1}{2}\|\nabla v(t)\|^2_{L^2(\mathbf{R}^2)} = \frac{1}{2}\int_\Omega \left(v_r^2+\frac{v_\theta^2}{r^2}\right)d\nu\\
E_3(t) &= \frac{1}{2}\|\Delta v(t)\|^2_{L^2(\mathbf{R}^2)} = \frac{1}{2}\int_\Omega \left(v_{rr}+\frac{v_r}{r}+\frac{v_{\theta\theta}}{r^2}\right)^2 d\nu\\
E_4(t) &= \frac{1}{2}\|\rho(t)\|^2_{L^2(0,2\pi)} = \frac{1}{2}\int_0^{2\pi}\rho^2(\theta,t)d\theta\\
E_5(t) &= \frac{1}{2}\|\rho_\theta(t)\|^2_{L^2(0,2\pi)} = \frac{1}{2}\int_0^{2\pi}\rho_\theta^2(\theta,t)d\theta\\
E_6(t) &= \frac{1}{2}\|\rho_{\theta\theta}(t)\|^2_{L^2(0,2\pi)} = \frac{1}{2}\int_0^{2\pi}\rho_{\theta\theta}^2(\theta,t)d\theta.
\end{aligned}$$

It will be useful to consider also the weighted primitive V of v:

$$V(r,\theta,t) = \int_0^r v(z,\theta,t)\psi(z-s(\theta,t),z)dz = -\int_r^\infty v(z,\theta,t)\psi(z-s(\theta,t),z)dz.$$

Note that $V(0,\theta,t)=V(\infty,\theta,t)=0$ as v is a transversal perturbation for any $\theta\in(0,2\pi)$, see (3.2). Under the above assumptions on v and ρ, $V\in C^1((0,T],W)$ and it satisfies an evolution equation easily computed by integrations by parts from (3.3, 3.4):

$$V_t = V_{rr} - \omega_1(r,\theta,t)V_r + G_5(r,\theta,t), \tag{3.5}$$

where

$$\begin{aligned}
\omega_1(r,\theta,t) &= 2\frac{\psi_0'(r-s(\theta,t))}{\psi_0(r-s(\theta,t))} + 2\frac{\chi'(r)}{\chi(r)} - \frac{1}{r},\\
G_5(r,\theta,t) &= \left(1-\frac{\lambda(r,\theta,t)}{\lambda(\infty,\theta,t)}\right)\int_0^r g(z,\theta,t)dz - \frac{\lambda(r,\theta,t)}{\lambda(\infty,\theta,t)}\int_r^\infty g(z,\theta,t)dz.
\end{aligned}$$

We also consider the last functional E_0 for V:

$$E_0(t) = \frac{1}{2}\|V(t)\|^2_{L^2(\mathbf{R}^2)} = \frac{1}{2}\int_\Omega V^2(r,\theta,t)d\nu.$$

Note that there exist two positive constants l_1 and l_2 such that for any $t\in(0,T)$, (see appendix 4.2),

$$l_1E_1(t)\le E_0(t)\le l_2E_1(t). \tag{3.6}$$

We first give the equations satisfied by these functionals and then find the inequalities involving E_0 to E_6 which are useful for the next calculations.

Lemma 3.2.4 *If $T > 0$ and $(v,\rho) \in C^1((0,T], W \times Z)$ is any solution of (3.3, 3.4), then $E_i \in C^1((0,T])$ for $i = 0,..,6$. E_0 satisfies the equation:*

$$\begin{aligned}\dot{E}_0(t) = &- \int_\Omega V_r^2 d\nu + \int_\Omega \left(\frac{\psi_0'(r - s(\theta,t))}{\psi_0(r-s(\theta,t))} \right)' V^2 d\nu \\ &+ \int_\Omega \omega_2(r,\theta,t) V^2 dr d\theta + \int_\Omega V(r,\theta,t) G_5(r,\theta,t) d\nu \end{aligned} \tag{3.7}$$

where $\omega_2(r,\theta,t) = \frac{\psi_0'(r-s(\theta,t))}{\psi_0(r-s(\theta,t))} + \frac{\chi'(r)}{\chi(r)} + \left(\frac{\chi'(r)}{\chi(r)}\right)' r$.

Moreover, the functions E_1, E_2, E_4 and E_5 satisfy:

$$\dot{E}_1(t) = -2E_2 + \int_\Omega F'(w) v^2 d\nu + \int_\Omega v \left(w_y \rho_t - \frac{1}{r^2} \partial_\theta (w_y \rho_\theta) + N + S \right) d\nu \tag{3.8}$$

$$\dot{E}_2(t) = -2E_3 - \int_\Omega \Delta v \left(F'(w) v + w_y \rho_t - \frac{1}{r^2} \partial_\theta (w_y \rho_\theta) + N + S \right) d\nu$$

$$\begin{aligned}\dot{E}_4(t) = &- \int_\Omega \frac{\rho_\theta^2}{r^2} \frac{\psi w_y}{\lambda(\infty,\theta,t)} dr d\theta + \int_\Omega \frac{\rho \rho_\theta^2}{r^2} \frac{\psi_y w_y}{\lambda(\infty,\theta,t)} dr d\theta \\ &+ \int_\Omega \frac{\rho \rho_\theta}{r^2} \psi w_y \frac{\lambda_\theta(\infty)}{\lambda^2(\infty)} dr d\theta - \int_\Omega \frac{\rho}{\lambda(\infty,\theta,t)} \left(g_1 + \frac{1}{r^2} \psi v_{\theta\theta} \right) dr d\theta \end{aligned} \tag{3.9}$$

$$\begin{aligned}\dot{E}_5(t) = &- \int_\Omega \frac{\rho_{\theta\theta}^2}{r^2} \frac{\psi w_y}{\lambda(\infty,\theta,t)} dr d\theta + \int_\Omega \frac{\rho_{\theta\theta} \rho_\theta^2}{r^2} \frac{\psi w_{yy}}{\lambda(\infty,\theta,t)} dr d\theta \\ &+ \int_\Omega \frac{\rho_{\theta\theta}}{\lambda(\infty,\theta,t)} \left(g_1 + \frac{1}{r^2} \psi v_{\theta\theta} \right) dr d\theta. \end{aligned} \tag{3.10}$$

Proof: Obviously, $\dot{E}_0(t) = \int_\Omega V V_t d\nu$. Equation (3.5) and integrations by parts yield to the desired expression for $\dot{E}_0$. The derivatives with respect to t of E_1 and E_2 are more easily computed by analogy with the heat equation in $\mathbf{R}^2$ with usual coordinates $x \in \mathbf{R}^2$ instead of polar coordinates. As far as the functionals for ρ are concerned, $\dot{E}_4$ and $\dot{E}_5$ are computed by a few integrations by parts. Note that all the functions depending on θ are 2π periodic. The expressions of $\dot{E}_4$ and $\dot{E}_5$ have been put in that way to highlight the first terms. Indeed, as we shall see below, $\int_\Omega \frac{\rho_\theta^2}{r^2} \frac{\psi w_y}{\lambda(\infty)} dr d\theta$ behaves essentially like $\frac{E_5(t)}{(R+ct)^2}$ and $\int_\Omega \frac{\rho_{\theta\theta}^2}{r^2} \frac{\psi w_y}{\lambda(\infty)} dr d\theta$ like $\frac{E_6(t)}{(R+ct)^2}$. These quantities are going to play an important role in the next energy estimates. Finally, we do not mind about $\dot{E}_3$ and $\dot{E}_6$ as we are only interested in the H^1 norms of v and ρ. ■

3.2.2 Bounds on the functionals and proof of theorem 3.2.3

Proposition 3.2.5 *There exist positive constants $R_2, \epsilon_2, k, c_0, d, e_6$ and e_{ij} for $(i,j) \in \{0,..,6\}^2$ such that if $T > 0$ and $(v,\rho) \in C^0([0,T], W \times Z)$ is any solution of (3.3, 3.4) satisfying for all $t \in [0,T]$,*

$$\|v(t)\|_W^2 + \|\rho(t)\|_Z^2 \leq \epsilon$$

for some $R \geq R_2$ and some $\epsilon \leq \epsilon_2$, then the following inequalities hold:

$$\begin{aligned}
\dot{E}_0(t) \leq & -\int_\Omega \psi^2 v^2 d\nu + e_{01}E_1 + e_{02}E_2 + \frac{e_6}{\sqrt{R+ct}}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^2} \\
\dot{E}_1(t) \leq & -2E_2 + \int_\Omega F'(w)v^2 d\nu + e_{11}E_1 + e_{12}E_2 + e_{13}E_3 \\
& + e_{15}\frac{E_5}{(R+ct)^2} + \frac{e_6}{\sqrt{R+ct}}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^3} \\
\dot{E}_2(t) \leq & -2E_3 + (e_{21} + (dk)^2)E_1 + e_{22}E_2 + (e_{23}+1)E_3 \\
& + e_{25}\frac{E_5}{(R+ct)^2} + \frac{e_6}{\sqrt{R+ct}}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^3} \\
\dot{E}_4(t) \leq & -d\frac{E_5}{(R+ct)^2} + e_{41}E_1 + e_{42}E_2 + e_{43}E_3 + e_{45}\frac{E_5}{(R+ct)^2} \\
& + e_{46}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^2} \\
\dot{E}_5(t) \leq & -d\frac{E_6}{(R+ct)^2} + e_{51}E_1 + e_{52}E_2 + e_{53}E_3 + (e_{56} + \frac{d}{4})\frac{E_6}{(R+ct)^2} \\
& + \frac{c_0}{(R+ct)^2} + \frac{2}{d}(R+ct)(E_1+E_2)^2
\end{aligned} \quad (3.11),\ (3.12)$$

and

$$\sup_{x\in\mathbf{R}} \left(F'(w_0(x)) - k\psi_0^2(x)\right) \leq -2.$$

Moreover, constants e_{ij} can be chosen as small as we want by choosing R_2 large enough and ϵ_2 small enough.

We prove right now how theorem 3.2.3 follows from proposition 3.2.5.

Proof of theorem 3.2.3: Let $R_2, \epsilon_2, k, c_0, d, e_6$ and e_{ij} be as in proposition 3.2.5, R_0, ϵ_0 be as in lemma 3.2.2 and l_1, l_2 as in (3.6). Choose $m > 0, R_1 > 0, \epsilon_1 > 0, l = \frac{1}{a} > 0$ such that

$$R_1 \geq \max(1, R_0, R_2, a^2, ac), \quad \epsilon_1 \leq \min(\epsilon_0^2, \epsilon_2), \quad \frac{\sqrt{\epsilon_1}}{R_1^{\frac{1}{4}}} \leq \epsilon_0, \quad m(dk)^2 \leq \frac{1}{2},$$

where $a = \max(1 + kl_2, m)$ and $b = \min(1 + kl_1, m)$. We also request that for any $R \geq R_1$, and any $0 < \epsilon \leq \epsilon_1$, the following inequalities hold for any $t \geq 0$:

$$\begin{aligned}
& ke_{01} + e_{11} + m(e_{21} + (dk)^2) + e_{41} + e_{51} \leq 1 \\
-2 \; & + ke_{02} + e_{12} + me_{22} + e_{42} + e_{52} \leq -1 \\
-2m \; & + e_{13} + m(e_{23}+1) + e_{43} + e_{53} \leq -\frac{m}{2} \\
-d \; & + e_{15} + me_{25} + e_{45} \leq 0 \\
-d \; & + k\frac{e_6}{\sqrt{R+ct}} + \frac{e_6}{\sqrt{R+ct}} + m\frac{e_6}{\sqrt{R+ct}} + e_{46} + e_{56} + \frac{d}{4} \leq -\frac{d}{2}.
\end{aligned} \quad (3.13)$$

This is possible by first choosing $m > 0$, then ϵ_1 small enough and R_1 large enough. Take $R \geq R_1, \epsilon \leq \epsilon_1$ and $(v_0, \rho_0) \in W \times Z$ satisfying

$$R^{\frac{1}{2}}\|v_0\|_W^2 + \|\rho_0\|_Z^2 \leq \epsilon.$$

By lemma 3.2.2, let $(v, \rho) \in C^0([0, T^*), W \times Z)$ be the maximal solution of (3.3,3.4) with initial data (v_0, ρ_0). Define, for some $n \in \mathbf{N}^*$,

$$T = \sup\Big\{\tilde{T} \in [0, T^*) \mid (R+ct)^{\frac{1}{2}}\|v(t)\|_W^2 + \|\rho(t)\|_Z^2 \leq n\left(\epsilon + \frac{1}{R}\right)$$
$$\text{and } \int_0^t (R+cs)\mathcal{U}^2(s)ds \leq 2\left(\epsilon + \frac{1}{R}\right) \text{ for } 0 \leq t \leq \tilde{T}\Big\}.$$

where

$$\mathcal{U}(t) = kE_0 + E_1 + mE_2$$
$$\text{and} \quad b(E_1+E_2) \leq \mathcal{U}(t) \leq a(E_1+E_2).$$

We also give some conditions on n: we assume that

$$\frac{a}{b} + \left(\frac{2(k+1+m)e_6}{db}(1+\sqrt{2})+1\right)\left(a + \frac{4}{db^2}\right) \leq n-1 \tag{3.14}$$

$$\left(\frac{2(k+1+m)e_6}{db}(1+\sqrt{2})+1\right)\left(\frac{\bar{c}}{c} + \frac{4}{db^2}\right) + \frac{\bar{c}}{2b} + \frac{\bar{c}\sqrt{2}}{bl} \leq n-1 \tag{3.15}$$

$$\frac{a^2\epsilon_1}{l} + \frac{2an}{l}\left(\frac{2(k+1+m)e_6\tilde{\epsilon}_1}{d} + \frac{2\bar{c}}{cR_1^{\frac{1}{2}}}\right) \leq 1 \tag{3.16}$$

$$n\left(\epsilon_1 + \frac{1}{R_1}\right) \leq \epsilon_2 \tag{3.17}$$

where $\tilde{\epsilon}_1 = a\epsilon_1 + \frac{\bar{c}}{cR_1} + \frac{4}{db^2}\left(\epsilon_1 + \frac{1}{R_1}\right)$ and $\bar{c}$ is definied by (3.18, 3.21) and (3.22). This is possible by first choosing n large enough such that the first two inequalities are valid and finally ϵ_1 small enough and R_1 large enough such that the last two inequalities hold.

By continuity of v and ρ, it is clear that $T > 0$. We claim that $T = T^*$, which also implies $T = T^* = +\infty$. Then, the inequalities satisfied by v and ρ are true for all $t \geq 0$ and theorem 3.2.3 follows immediately. Indeed, if $T < T^*$, it follows from proposition 3.2.5 and inequality (3.17) that for $t \in [0, T]$, inequalities (3.11) are satisfied. To get a contradiction on the definition of T, we must judiciously bound the expressions $(R+ct)^{\frac{1}{2}}\|v(t)\|_W^2 + \|\rho(t)\|_Z^2$ and $\int_0^t (R+cs)\mathcal{U}^2(s)ds$. Therefore, define

$$\mathcal{E}(t) = kE_0 + E_1 + mE_2 + E_4 + E_5 = \mathcal{U}(t) + E_4 + E_5.$$

Using (3.11) and (3.13), there exists $\bar{c} > 0$ such that

$$\dot{\mathcal{E}}(t) \leq -E_1 - E_2 - \frac{m}{2}E_3(t) - \frac{dE_6(t)}{2(R+ct)^2} + \frac{\bar{c}}{(R+ct)^2} + \frac{2}{d}(R+ct)(E_1+E_2)^2. \tag{3.18}$$

Integrating this inequality between 0 and $t \leq T$, we get

$$\mathcal{E}(t) + \int_0^t (E_1 + E_2)(s)ds + \int_0^t \frac{m}{2} E_3(s)ds + \int_0^t \frac{dE_6(s)}{2(R+cs)^2}ds$$
$$\leq \mathcal{E}(0) + \int_0^t \frac{\bar{c}}{(R+cs)^2}ds + \int_0^t \frac{2}{d}(R+cs)(E_1+E_2)^2(s)ds \leq \tilde{\epsilon} \qquad (3.19)$$

where $\tilde{\epsilon} = a\epsilon + \frac{\bar{c}}{cR} + \frac{4}{db^2}\left(\epsilon + \frac{1}{R}\right)$. Moreover, we also get from inequalities (3.11) that

$$\dot{\mathcal{U}}(t) \leq -E_1(t) - E_2(t) + f(t) \leq -l\mathcal{U}(t) + f(t) \qquad (3.20)$$

where

$$f(t) = \frac{(k+1+m)e_6}{\sqrt{R+ct}} \frac{E_6(t)}{(R+ct)^2} + \frac{(e_{15}+me_{25})E_5(t)}{(R+ct)^2} + \frac{c_0}{(R+ct)^2}.$$

Then, $\mathcal{U}(t) \leq \mathcal{U}(0)e^{-lt} + \int_0^t e^{-l(t-s)}f(s)ds$. Finally,

$$E_1(t) + E_2(t) \leq \frac{a\epsilon}{b\sqrt{R}}e^{-lt} + \int_0^t \frac{f(s)}{b}e^{-l(t-s)}ds.$$

To evaluate this last integral, we cut it into two parts and use inequality (3.19) and the fact that $E_5(t) \leq n\left(\epsilon + \frac{1}{R}\right) \leq \epsilon_2$:

$$\int_0^{\frac{t}{2}} e^{-l(t-s)}f(s)ds \leq e^{-\frac{lt}{2}}\left(\frac{2(k+1+m)e_6\tilde{\epsilon}}{d\sqrt{R}} + \frac{c_0 t}{2R^2}\right),$$

$$\int_{\frac{t}{2}}^t e^{-l(t-s)}f(s)ds \leq \frac{e_6(k+1+m)}{\sqrt{R+c\frac{t}{2}}}\int_{\frac{t}{2}}^t \frac{E_6(s)}{(R+cs)^2}ds + \frac{c_0}{l(R+c\frac{t}{2})^2}$$
$$\leq \frac{2(k+1+m)e_6\tilde{\epsilon}}{d\sqrt{R+c\frac{t}{2}}} + \frac{c_0}{l(R+c\frac{t}{2})^2}.$$

Finally, using (3.14, 3.15, 3.19) and the above inequalities, there exists $\bar{c} > 0$ such that

$$(R+ct)^{\frac{1}{2}}(E_1+E_2)(t) + E_4(t) + E_5(t)$$
$$\leq \frac{a\epsilon}{b} + \frac{2(k+1+m)e_6\tilde{\epsilon}}{db} + \frac{\bar{c}}{2bR^{\frac{3}{2}}} + \frac{\sqrt{2}}{b}\left(\frac{2(k+1+m)e_6\tilde{\epsilon}}{d} + \frac{\bar{c}}{lR^{\frac{3}{2}}}\right) + \tilde{\epsilon}$$
$$\leq (n-1)\left(\epsilon + \frac{1}{R}\right). \qquad (3.21)$$

We now want to evaluate the integral $\int_0^t (R+cs)\mathcal{U}^2(s)ds$. Therefore, define

$$\mathcal{G}(t) = (R+ct)\mathcal{U}^2(t).$$

Then, using (3.20) and $R_1 \geq ac$,

$$\frac{d\mathcal{G}}{dt} \leq -l\mathcal{G} + 2an\left(\epsilon + \frac{1}{R}\right)\left(\frac{(k+1+m)e_6E_6(t)}{(R+ct)^2} + \frac{c_0}{(R+ct)^{\frac{3}{2}}}\right).$$

By Gronwall's lemma, we get a bound on $\mathcal{G}$ and by integrating between 0 and t,

$$\int_0^t \mathcal{G}(s)ds \leq \frac{a^2\epsilon^2}{l} + 2an\left(\epsilon + \frac{1}{R}\right) \times$$
$$\int_0^t\int_0^s e^{-l(s-\tau)}\left(\frac{(k+1+m)e_6E_6(\tau)}{(R+c\tau)^2} + \frac{c_0}{(R+c\tau)^{\frac{3}{2}}}\right)d\tau ds.$$

Finally, by Fubini's theorem, (3.19) and (3.16), there exists $\bar{c} > 0$ such that

$$\begin{aligned}\int_0^t (R+cs)\mathcal{U}^2(s)ds &\leq \frac{a^2\epsilon^2}{l} + \frac{2an}{l}\left(\epsilon + \frac{1}{R}\right)\left(\frac{2(k+1+m)e_6\tilde{\epsilon}}{d} + \frac{2\bar{c}}{cR^{\frac{1}{2}}}\right)\\ &\leq \left(\epsilon + \frac{1}{R}\right).\end{aligned} \tag{3.22}$$

Then, by (3.21) and (3.22), we get for any $\epsilon \leq \epsilon_1$ and any $R \geq R_1$,

$$(R+ct)^{\frac{1}{2}}(E_1+E_2) + E_4 + E_5 \leq (n-1)\left(\epsilon + \frac{1}{R}\right)$$
$$\int_0^t (R+cs)\mathcal{U}^2 ds \leq \left(\epsilon + \frac{1}{R}\right)$$

for all $0 \leq t \leq T$. This contradicts the definition of T and concludes the proof. ■

3.2.3 Proof of proposition 3.2.5

The proof of proposition 3.2.5 is technical and we need a few intermediate lemmas to prove inequalities (3.11). We only use a few fundamental ideas: Cauchy-Schwartz' inequality, Jensen's inequality, Schur's lemma and the fact that $\psi_0(r-R-ct)$ and $\phi_0(r-R-ct)$ are localized around $r = R+ct$. We encourage the reader to refer to appendix 4.2 where we explain in detail the way those fundamental ideas are used in the following lemmas. For the whole section 3.2.3, we call (H) the following assumptions:

Fix ϵ, R, T positive constants.

Let $(v,\rho) \in C^0([0,T], W \times Z)$ be any solution of (3.3,3.4) satisfying

$$\|v(t)\|_W^2 + \|\rho(t)\|_Z^2 \leq \epsilon,\ t \in [0,T]. \tag{3.23}$$

In the following six lemmas, we prove that inequalities (3.11) follow from equations (3.7) to (3.10) of $\dot{E}_0$ to $\dot{E}_6$ and inequality (3.23).

Lemma 3.2.6 *Under assumptions (H), there exist positive constants R_2, ϵ_2, c_0 such that for any $t \in (0,T]$ and any $R \geq R_2$, $\epsilon \leq \epsilon_2$,*

$$||\rho_t||_{L^2(0,2\pi)} \leq c_0 \left(\frac{E_6^{\frac{1}{2}}}{(R+ct)^2} + \frac{E_5^{\frac{3}{4}} E_6^{\frac{1}{4}}}{(R+ct)^2} + A + B \right)$$

where

$$A = \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{2}} \left[\frac{E_1}{(R+ct)^2} + \frac{(E_1+E_2)^{\frac{1}{2}} E_5^{\frac{1}{4}}}{(R+ct)^{\frac{5}{4}}} + \frac{(E_1E_2)^{\frac{1}{4}}}{R+ct} + \frac{(E_1E_5)^{\frac{1}{2}}}{(R+ct)^{\frac{3}{2}}} \right] + \frac{E_3^{\frac{1}{2}}}{(R+ct)^{\frac{3}{2}}} + \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{4}} \frac{E_2^{\frac{1}{2}} E_5^{\frac{1}{4}}}{R+ct},$$

$$B = \frac{2E_1^{\frac{1}{2}}}{(R+ct)^{\frac{5}{2}}} + \frac{E_1+E_2}{\sqrt{R+ct}} + \frac{1}{(R+ct)^2}.$$

Proof: ρ is a solution of equation (3.4) and we want to bound the L^2 norm of ρ_t. Therefore, we need to bound $\lambda(\infty, \theta, t)$ from below and $|\int_0^\infty g(r,\theta,t)dr|$ from above. Using Jensen's and Cauchy-Schwartz' inequalities and the Sobolev's embedding $H^1(\mathbf{R}^2) \hookrightarrow L^4(\mathbf{R}^2)$, we first have

$$\begin{aligned} \sup_{\theta\in(0,2\pi)} \left| \int_0^\infty \psi_y v dr \right|^2 &\leq \sup_{\theta\in(0,2\pi)} \int_0^\infty |\psi_y v^2| dr \text{ (Jensen)} \\ &\leq \int_0^{2\pi}\int_0^\infty |v^2\psi_y| dr d\theta + \int_0^{2\pi}\int_0^\infty |2vv_\theta\psi_y| dr d\theta + \int_0^{2\pi}\int_0^\infty |v^2\rho_\theta\psi_{yy}| dr d\theta \\ &\leq c_0 \left(\frac{E_1}{R+ct} + +(E_1E_2)^{\frac{1}{2}} + E_5^{\frac{1}{2}} \frac{(E_1+E_2)}{(R+ct)^{\frac{1}{2}}} \right) \leq c_0\epsilon \end{aligned}$$

as for any function f such that $\int_0^{2\pi} f(\theta)d\theta = 0$, $\sup |f| \leq \int_0^{2\pi} |f_\theta| d\theta$.

As $\lambda(\infty,\theta,t) = \int_0^\infty \psi w_y dr - \int_0^\infty \psi_y v dr$ and $\int_0^\infty \psi w_y dr = 1 - O(e^{-(R+ct)})$, we have

$$1 - c_0 \left(\epsilon^{\frac{1}{2}} + e^{-R} \right) \leq \lambda(\infty, \theta, t)$$

for any $\theta \in (0,2\pi)$ and any $t > 0$. Then, for convenient ϵ_2 and R_2, $\lambda(\infty,\theta,t)^{-1} \leq 2$ for any $\theta \in (0,2\pi), t > 0, R \geq R_2$ and $\epsilon \leq \epsilon_2$.

Moreover, using Schur's lemma (see appendix 4.2), we have

$$||\rho_{\theta\theta} \int_0^\infty \frac{\psi w_y}{r^2 \lambda(\infty,\theta,t)} dr||_{L^2(0,2\pi)} \leq \frac{c_0 E_6^{\frac{1}{2}}}{(R+ct)^2}$$

and

$$||\rho_\theta^2 \int_0^\infty \frac{\psi w_{yy}}{r^2\lambda(\infty,\theta,t)} dr||_{L^2(0,2\pi)} \leq \frac{c_0}{(R+ct)^2} ||\rho_\theta||_{L^2(0,2\pi)} ||\rho_\theta||_{L^\infty(0,2\pi)} \leq \frac{c_0 E_5^{\frac{3}{4}} E_6^{\frac{1}{4}}}{(R+ct)^2}$$

as $\|\rho_\theta\|_{L^\infty(0,2\pi)} \leq (E_5E_6)^{\frac{1}{4}}$. To bound the norm of $\int_0^\infty \frac{\psi v_{\theta\theta}}{r^2\lambda(\infty)}dr$, we introduce the difference $\frac{1}{r^2} - \frac{1}{(R+ct)^2}$:

$$\int_0^\infty \frac{\psi v_{\theta\theta}}{r^2\lambda(\infty,\theta,t)}dr = \int_0^\infty \left(\frac{1}{(R+ct)^2} - \frac{1}{r^2}\right)\frac{\psi v_{\theta\theta}}{\lambda(\infty)}dr + \frac{1}{(R+ct)^2}\int_0^\infty \frac{\psi v_{\theta\theta}}{\lambda(\infty)}dr$$

The first term is bounded in the $L^2(0,2\pi)$ norm by $E_3^{\frac{1}{2}}/(R+ct)^{\frac{3}{2}}$. For the second one, we write $\int_0^\infty \psi v_{\theta\theta}dr$ with derivatives of ρ and v by derivating twice identity (2.2) with respect to θ:

$$\int_0^\infty \psi v_{\theta\theta}dr = \rho_{\theta\theta}\int_0^\infty \psi_y v dr - \rho_\theta^2\int_0^\infty \psi_{yy}v dr + 2\rho_\theta\int_0^\infty \psi_y v_\theta dr.$$

Finally, by Jensen's and Cauchy-Schwartz' inequalities, Schur's lemma and the Sobolev's embedding $H^1(\mathbf{R}^2) \hookrightarrow L^4(\mathbf{R}^2)$, we get

$$\begin{aligned}
\|\int_0^\infty \psi v_{\theta\theta}dr\|_{L^2(0,2\pi)} &\leq c_0\left(\int_0^{2\pi}\int_0^\infty \rho_{\theta\theta}^2\psi_y v^2 dr d\theta\right)^{\frac{1}{2}} + c_0\left(\int_0^{2\pi}\int_0^\infty \rho_\theta^4\psi_{yy}v^2 dr d\theta\right)^{\frac{1}{2}}\\
&\quad + c_0\left(\int_0^{2\pi}\int_0^\infty \rho_\theta^2\psi_y v_\theta^2 dr d\theta\right)^{\frac{1}{2}}\\
&\leq c_0\|\rho_{\theta\theta}\|_{L^2(0,2\pi)}\|\int_0^\infty \psi_y v^2 dr\|_{L^\infty(0,2\pi)}^{\frac{1}{2}} + c_0\|\rho_\theta\|_{L^\infty(0,2\pi)}^2\left(\int_0^{2\pi}\int_0^\infty \frac{v^2}{r}\psi_{yy}d\nu\right)^{\frac{1}{2}}\\
&\quad + c_0\|\rho_\theta\|_{L^\infty(0,2\pi)}\left(\int_0^{2\pi}\int_0^\infty \frac{v_\theta^2}{r^2}\psi_y r d\nu\right)^{\frac{1}{2}}\\
&\leq c_0 E_6^{\frac{1}{2}}\left(\frac{E_1}{R+ct} + (E_1E_2)^{\frac{1}{4}} + E_5^{\frac{1}{4}}\frac{(E_1+E_2)^{\frac{1}{2}}}{(R+ct)^{\frac{1}{4}}}\right)\\
&\quad + c_0\left(\frac{E_1E_5E_6}{R+ct}\right)^{\frac{1}{2}} + c_0(E_5E_6)^{\frac{1}{4}}\sqrt{R+ct}E_2^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$\|\int_0^\infty \frac{\psi v_{\theta\theta}}{r^2\lambda(\infty,\theta,t)}dr\|_{L^2(0,2\pi)} \leq c_0 A.$$

The last term $\int_0^\infty v\Lambda + \psi(N+S)dr$ is bounded by Jensen's inequality, Schur's lemma (see appendix 4.2) and the Sobolev's embedding $H^1(\mathbf{R}^2) \hookrightarrow L^4(\mathbf{R}^2)$. Then, $\|\int_0^\infty v\Lambda + \psi(N+S)dr\|_{L^2(0,2\pi)} \leq c_0 B$. Notice that as $H^1(\mathbf{R}^2)$ is not an algebra, we need some more assumptions to bound the norm of N. We assumed in the introduction that every solution of $u_t = F(u)$ is uniformly bounded in time. Therefore, v is bounded and Taylor's theorem and Sobolev's embedding enable us to bound $\|N\|_{L^2(\mathbf{R}^2)}$. This concludes the proof of lemma 3.2.6. ■

Lemma 3.2.7 *Under assumptions (H), there exist positive constants c_0, R_2, ϵ_2 such that for any $t \in [0,T]$ and any $R \geq R_2$, $\epsilon \leq \epsilon_2$,*

$$\dot{E}_0(t) \leq -\int_\Omega \psi^2 v^2 d\nu + c_0\left(\frac{E_1}{R+ct} + E_0^{\frac{1}{2}}B + C\right)$$

where

$$C = \frac{E_2}{R+ct} + \left(\frac{E_6}{(R+ct)^2}\right)^{\frac{1}{4}} \left(\frac{(E_1E_2)^{\frac{1}{2}}E_5^{\frac{1}{4}}}{\sqrt{R+ct}} + \frac{(E_0E_2)^{\frac{1}{2}}E_5^{\frac{1}{4}}}{\sqrt{R+ct}}\right)$$
$$+ \left(\frac{E_6}{(R+ct)^2}\right)^{\frac{1}{2}} \left(\frac{(E_0E_5)^{\frac{1}{2}}}{R+ct} + \frac{(E_1E_5)^{\frac{1}{2}}}{R+ct}\right) + \frac{(E_2E_5)^{\frac{1}{2}}}{R+ct}.$$

Consequently, there exist positive constants e_{01}, e_{02}, e_6 such that

$$\dot{E}_0(t) \leq -\int_\Omega \psi^2 v^2 d\nu + e_{01}E_1 + e_{02}E_2 + \frac{e_6}{\sqrt{R+ct}}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^2}$$

where e_{01} and e_{02} can be chosen small with appropriate R_2 and ϵ_2.

Proof: We know that $V_r = \psi v$; by appendix 4.3, we have $\left(\frac{\psi_0'}{\psi_0}\right)' = \left(\frac{\phi_0'}{\phi_0}\right)' < 0$ and there exists some constant $c_0 > 0$ such that $|\omega_2| < c_0$. Then, by equation (3.7), the only difficulty in $\dot{E}_0$ comes from $\int_\Omega VG_5 d\nu$. If $r << R+ct$, the main term in G_5 is $\int_0^r gdz$ and if $r >> R+ct$, $\int_r^\infty gdz$. We bound separately the term with g_1 and the one with g_2.

The term with g_1 is bounded by $E_0^{\frac{1}{2}}B$ as in lemma 3.2.6. The term with g_2 is bounded after one integration by parts in θ, Cauchy-Schwartz' and Jensen's inequalities by C. Indeed, if $r << R+ct$, as

$$V_\theta = \int_0^r (\psi v_\theta - \rho_\theta \psi_y v)\, dz,$$

$$\int_0^{R+ct}\int_0^{2\pi} V(r,\theta,t)\int_0^r g_2(z,\theta,t)dzd\nu = -\int_0^{R+ct}\int_0^{2\pi}\frac{1}{r^2}\left(\int_0^r \psi v_\theta dz\right)^2 d\nu$$
$$+\int_0^{R+ct}\int_0^{2\pi}\left(\int_0^r \psi v_\theta dz\right)\left(\int_0^r \psi v_\theta\left(\frac{1}{r^2}-\frac{1}{z^2}\right)dz\right)d\nu$$
$$+\int_0^{R+ct}\int_0^{2\pi}\rho_\theta\left(\int_0^r \psi_y v dz\right)\left(\int_0^r \psi\frac{v_\theta}{z^2}dz\right)d\nu$$
$$+\int_0^{R+ct}\int_0^{2\pi}\rho_\theta V\left(\int_0^r \psi_y\frac{v_\theta}{z^2}dz\right)d\nu - \int_0^{R+ct}\int_0^{2\pi}\rho_\theta^2 V\left(\int_0^r \frac{\psi_y w_y}{z^2}dz\right)d\nu$$
$$+\int_0^{R+ct}\int_0^{2\pi}\rho_\theta\left(\int_0^r \frac{\psi w_y}{z^2}dz\right)\left(\int_0^r \psi v_\theta dz\right)d\nu$$
$$-\int_0^{R+ct}\int_0^{2\pi}\rho_\theta^2\left(\int_0^r \frac{\psi w_y}{z^2}dz\right)\left(\int \psi_y v dz\right)d\nu.$$

Notice that the first term is negative and can be omitted. The following terms can be treated as described before.

Inequality (3.11) for $\dot{E}_0$ is easily computed from this result using inequalities such as $ab \leq \frac{a^2+b^2}{2}$. Then,

$$e_{01} = c_0 \left(\frac{1}{R+ct} + \frac{\sqrt{\epsilon}}{\sqrt{R+ct}} + \frac{\sqrt{\epsilon}}{R+ct} \right)$$
$$e_{02} = c_0 \left(\frac{1}{R+ct} + \frac{\sqrt{\epsilon}}{\sqrt{R+ct}} + \sqrt{\epsilon} \right)$$
$$e_6 = c_0 \left(\frac{\sqrt{\epsilon}}{\sqrt{R+ct}} + \sqrt{\epsilon} \right).$$

We easily notice that e_{01} and e_{02} can be chosen very small with appropriate R_2 and ϵ_2.

Lemma 3.2.8 *Under assumptions (H), there exist positive constants c_0, R_2, ϵ_2 such that for any $t \in (0,T]$ and any $R \geq R_2$, $\epsilon \leq \epsilon_2$,*

$$\dot{E}_1(t) \leq -2E_2 + \int_\Omega F'(w)v^2 d\nu + c_0 \left(E_1^{\frac{1}{2}} D + (E_1+E_2)^{\frac{3}{2}} \right)$$

where

$$D = \sqrt{R+ct}\|\rho_t\|_{L^2} + \frac{1}{\sqrt{R+ct}} \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{2}} + \frac{E_5^{\frac{3}{4}}}{R+ct} \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{4}} + \frac{1}{(R+ct)^{\frac{3}{2}}}.$$

Consequently, there exist positive constants $e_{11}, e_{12}, e_{13}, e_{15}$ and e_6 such that

$$\begin{aligned} \dot{E}_1(t) \leq & -2E_2 + \int_\Omega F'(w)v^2 d\nu + e_{11}E_1 + e_{12}E_2 + e_{13}E_3 \\ & + e_{15}\frac{E_5}{(R+ct)^2} + \frac{e_6}{\sqrt{R+ct}}\frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^3} \end{aligned}$$

where $\{e_{1j}\}_{j=1..5}$ can be chosen small with appropriate R_2 and ϵ_2.

Proof: From equation (3.8), we bound $\dot{E}_1(t)$ term by term: $\|v\rho_t w_y\|_{L^2(\mathbf{R}^2)}$ is bounded with Cauchy-Schwartz' inequality by $\sqrt{R+ct}\|\rho_t\|_{L^2(0,2\pi)}E_1^{\frac{1}{2}}$. The three other terms are bounded as explained in appendix 4.2:

$$\|\frac{v}{r^2}\rho_{\theta\theta}w_y\|_{L^2(\mathbf{R}^2)} \leq c_0 \frac{E_6^{\frac{1}{2}}E_1^{\frac{1}{2}}}{(R+ct)^{\frac{3}{2}}}$$
$$\|\frac{v}{r^2}\rho_\theta^2 w_{yy}\|_{L^2(\mathbf{R}^2)} \leq c_0 \frac{E_1^{\frac{1}{2}}E_5^{\frac{3}{4}}E_6^{\frac{1}{4}}}{(R+ct)^{\frac{3}{2}}}$$
$$\|v(N+S)\|_{L^2(\mathbf{R}^2)} \leq c_0 \left((E_1+E_2)^{\frac{3}{2}} + \frac{E_1^{\frac{1}{2}}}{(R+ct)^{\frac{3}{2}}} \right).$$

This last inequality is also obtained by Sobolev's embedding $H^1(\mathbf{R}^2) \hookrightarrow L^3(\mathbf{R}^2)$. We then get inequality (3.11) for $\dot{E}_1$ using inequalities such as $ab \le \frac{a^2+b^2}{2}$. ■

Lemma 3.2.9 *Under assumptions (H), there exist positive constants c_0, d, R_2, ϵ_2 and $k>1$ such that for any $t \in (0,T]$ and $R \ge R_2$, $\epsilon \le \epsilon_2$,*

$$\dot{E}_2(t) \le -2E_3 + c_0 E_3^{\frac{1}{2}} \left((dk) E_1^{\frac{1}{2}} + E_1 + E_2 + D \right).$$

Consequently, there exist positive constants $e_{21}, e_{22}, e_{23}, e_{25}$ and e_6 such that

$$\begin{aligned} \dot{E}_2(t) \le & -2E_3 + (e_{21} + (dk)^2) E_1 + e_{22} E_2 + (e_{23}+1) E_3 \\ & + e_{25} \frac{E_5}{(R+ct)^2} + \frac{e_6}{\sqrt{R+ct}} \frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^3} \end{aligned}$$

where $\{e_{2j}\}_{j=1..5}$ can be chosen small with appropriate R_2 and ϵ_2.

Proof: The proof of this lemma is very similar to the last one and we may leave it out. Notice that k large enough can be chosen so that $\sup(F'(w_0) - k\psi_0^2) \le -2$. Then, $\sup |F'(w_0)| \le dk$. Once more, inequality (3.11) for $\dot{E}_2$ follows for $R \ge R_2$ and $\epsilon \le \epsilon_2$. ■

Lemma 3.2.10 *Under assumptions (H), there exist positive constants c_0, d, R_2, ϵ_2 such that for any $t \in (0,T]$ and any $R \ge R_2$, $\epsilon \le \epsilon_2$,*

$$\dot{E}_4(t) \le -d \frac{E_5}{(R+ct)^2} + c_0 E_4^{\frac{1}{2}} \left(\frac{E_5^{\frac{3}{4}}}{(R+ct)^{\frac{3}{2}}} \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{4}} + A + B + G \right).$$

where

$$G = \frac{E_5^{\frac{1}{4}}}{(R+ct)^{\frac{3}{2}}} \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{4}} \left(E_5^{\frac{1}{2}} + E_1^{\frac{1}{2}} E_5^{\frac{1}{4}} \left(\frac{E_6}{(R+ct)^2} \right)^{\frac{1}{4}} + \sqrt{R+ct}\, E_2^{\frac{1}{2}} \right).$$

Consequently, there exist positive constants $e_{41}, e_{42}, e_{43}, e_{45}$ and e_{46} such that

$$\begin{aligned} \dot{E}_4(t) \le & -d \frac{E_5}{(R+ct)^2} + e_{41} E_1 + e_{42} E_2 + e_{43} E_3 + e_{45} \frac{E_5}{(R+ct)^2} \\ & + e_{46} \frac{E_6}{(R+ct)^2} + \frac{c_0}{(R+ct)^2}, \end{aligned}$$

where $\{e_{4j}\}_{j=1..6}$ can be chosen small with appropriate R_2 and ϵ_2.

Proof: From equation (3.9), we bound $\dot{E}_4$ term by term. The only difficulty which has not been seen yet in the previous lemmas is the term G which can be bounded by

$$\int_\Omega \frac{\rho \rho_\theta}{r^2} \psi w_y \frac{\lambda_\theta(\infty)}{\lambda^2(\infty)} dr d\theta$$

with Cauchy-Schwartz's inequality.

Let us recall that $\lambda_\theta = \int_0^\infty \rho_\theta\left(\psi_{yy}v - \psi_y w_y - \psi w_{yy}\right) - \psi_y v_\theta dr$. Then,

$$\|\lambda_\theta\|_{L^2(0,2\pi)} \leq c_0 \left(\|\rho_\theta\|_{L^\infty} \frac{E_1^{\frac{1}{2}}}{\sqrt{R+ct}} + E_5^{\frac{1}{2}} + \sqrt{R+ct} E_2^{\frac{1}{2}} \right)$$

and the inequality $\|\rho_\theta\|_{L^\infty} \leq (E_5 E_6)^{\frac{1}{4}}$ ends the proof. ■

Lemma 3.2.11 *Under assumptions (H), there exist positive constants c_0, R_2, ϵ_2 such that for any $t \in (0,T]$ and any $R \geq R_2$, $\epsilon \leq \epsilon_2$,*

$$\dot{E}_5(t) \leq -d\frac{E_6}{(R+ct)^2} + c_0\left(E_5^{\frac{1}{2}}\frac{E_6}{(R+ct)^2} + E_6^{\frac{1}{2}}(A+B)\right).$$

Consequently, there exist positive constants e_{51}, e_{52}, e_{53} and e_{56} such that

$$\begin{aligned}\dot{E}_5(t) \leq & - d\frac{E_6}{(R+ct)^2} + e_{51}E_1 + e_{52}E_2 + e_{53}E_3 + (e_{56} + \frac{d}{4})\frac{E_6}{(R+ct)^2} \\ & + \frac{c_0}{(R+ct)^2} + \frac{2}{d}(R+ct)(E_1+E_2)^2\end{aligned}$$

where $\{e_{5j}\}_{j=1..6}$ can be chosen small with appropriate R_2 and ϵ_2.

Proof: Once more, the proof is very similar to the previous ones, using Cauchy-Schwartz' inequality. However, we may detail how we get, from the first result, inequality (3.11) for $\dot{E}_5$. Using inequalities such as $ab \leq \frac{a^2+b^2}{2}$, the only difficulties come from the terms $E_6^{\frac{1}{2}}\left(\frac{E_1+E_2}{\sqrt{R+ct}}\right)$ and $\frac{E_6^{\frac{1}{2}}}{(R+ct)^2}$ which appear in $E_6^{\frac{1}{2}}B$: for any $d > 0$,

$$\begin{aligned}\frac{E_6^{\frac{1}{2}}}{(R+ct)^2} &\leq \frac{2}{d(R+ct)^2} + \frac{d}{8}\frac{E_6}{(R+ct)^2} \\ E_6^{\frac{1}{2}}\left(\frac{E_1+E_2}{\sqrt{R+ct}}\right) &\leq \left(\frac{E_6}{(R+ct)^2}\right)^{\frac{1}{2}}\left(\sqrt{R+ct}(E_1+E_2)\right) \\ &\leq \frac{d}{8}\frac{E_6}{(R+ct)^2} + \frac{2}{d}(R+ct)(E_1+E_2)^2.\end{aligned}$$

This ends the proof of inequalities (3.11). ■

These six lemmas end the proof of proposition 3.2.5 and hence of theorem 3.2.3. Equipped with these energy estimates, we are able to prove the end of theorem 2.

3.3 Example and density of non radial profiles

In this section, the end of theorem 2 is proved thanks to theorem 3.2.3.

Lemma 3.3.1 *Under the assumptions of theorem 3.2.3, there exists a function $\rho_\infty \in L^2(0,2\pi)$ such that $\rho(.,t)$ converges in the $L^2(0,2\pi)$ norm to ρ_∞ as t goes to infinity.*

Proof: By lemma 3.2.6 and theorem 3.2.3, we get

$$\|\rho_t(t)\|_{L^2(0,2\pi)} \leq c_0 \left(\frac{E_1 + E_2}{\sqrt{R+ct}} + \frac{E_3}{R+ct} + \frac{1}{\sqrt{R+ct}} \frac{E_6}{(R+ct)^2} + \frac{1}{(R+ct)^{\frac{3}{2}}} \right).$$

Then, by inequality (3.19),

$$\int_0^t \|\rho_t(s)\|_{L^2(0,2\pi)} ds \leq c_0 \left(\frac{1+\tilde{\epsilon}}{\sqrt{R}} \right).$$

As this bound is independent of t, $\int_0^\infty \|\rho_t(s)\|_{L^2(0,2\pi)} ds$ is convergent and there exists a function $\rho_\infty \in L^2(0,2\pi)$ such that

$$\|\rho_\infty - \rho(.,t)\|_{L^2(0,2\pi)} \leq \int_t^\infty \|\rho_t(s)\|_{L^2(0,2\pi)} ds$$

converges to zero as t goes to infinity. This completes the proof. ■

Lemma 3.3.2 *There exist positive constants R and ϵ such that if $u_0(r,\theta) = w(r - R - \sqrt{\frac{\epsilon}{2\pi}} \sin\theta, r)$, the solution $u(r,\theta,t)$ of equation (1.1) with initial data u_0 converges to a non radial profile.*

Proof: Take R_1 and ϵ_1 as in theorem 3.2.3 and $R \geq R_1, \epsilon \leq \epsilon_1$. Then u_0 satisfies the assumptions of theorem 3.2.3. Indeed, $u_0 = w(r - R - \rho_0, r) + v_0$ where $\rho_0(\theta) = \sqrt{\frac{\epsilon}{2\pi}} \sin\theta$ and $v_0 = 0$. Thus, $R^{\frac{1}{2}} \|v_0\|_W^2 + \|\rho_0\|_Z^2 = \epsilon$. Since $v_0 = 0$, notice that R and ϵ can be chosen independently. Therefore, choose R sufficiently large so that $\sqrt{\frac{\epsilon}{2}} > c_0 \frac{1+\tilde{\epsilon}}{\sqrt{R}}$. Let $u(r,\theta,t) = w(r - s(\theta,t), r) + v(r,\theta,t)$ be the solution of equation (1.1) with initial data u_0 where $s(\theta,t)$ is defined by (1.9) . Then, by lemma 3.3.1, $\int_0^t \|\rho_t(s)\|_{L^2(0,2\pi)} ds \leq c_0 \frac{1+\tilde{\epsilon}}{\sqrt{R}}$. Finally,

$$\|\rho(\theta,t) - \sqrt{\frac{\epsilon}{2\pi}} \sin\theta\|_{L^2(0,2\pi)} \leq c_0 \left(\frac{1+\tilde{\epsilon}}{\sqrt{R}} \right)$$

for any $t \geq 0$. If there exists some $t > 0$ such that $\rho(\theta,t) = \rho$ is independent of θ, then

$$\|\rho - \sqrt{\frac{\epsilon}{2\pi}} \sin\theta\|_{L^2(0,2\pi)} = \sqrt{2\pi\rho^2 + \frac{\epsilon}{2}} > c_0 \left(\frac{1+\tilde{\epsilon}}{\sqrt{R}} \right).$$

This contradicts the latter inequality. Therefore, for any $t \geq 0$, $\rho(\theta,t)$ is not constant.

Moreover, as theorem 3.2.3 is satisfied, $\|v\|_W$ converges to zero as t goes to infinity and $u(r,\theta,t)$ converges to a non radial profile as t goes to infinity. ■

This ends the proof of theorem 2. We give a few more information by introducing two new spaces as follows:

$$\mathcal{S}_1 = \{u_0 \in H^1(\mathbf{R}^2) \mid \text{for some } R \geq \max(R_1, R_1'), \tilde{u}_0(r,\theta) - w(r-R,r) \equiv \\ \xi(r,\theta) \text{ satisfies lemma 3.2.1 and } (v_0,\rho_0) \in W \times Z \text{ satisfy theorem 3.2.3}\}$$

Moreover, there exists, for any function $u_0 \in \mathcal{S}_1$, a unique function $\rho_\infty \in L^2(0,2\pi)$ satisfying lemma 3.3.1. We call $\mathcal{S}_2$ the set of all these functions $\rho_\infty \in L^2(0,2\pi)$ satisfying the above properties for $u_0 \in \mathcal{S}_1$.

Lemma 3.3.3 *$\mathcal{S}_2$ is a subset of $L^2(0,2\pi)$ which contains some non constant functions and $\mathcal{S}_2$ is dense in the ball $B(0,\min(\delta_1', \sqrt{\epsilon_1}))$ of Z.*

Proof: For any $\rho_\infty \in \mathcal{S}_2$, we know that $\rho_\infty \in L^2(0,2\pi)$. Moreover, there exists, by lemma 3.3.2, some $u_0 \in \mathcal{S}_1$ such that $\rho_\infty \in \mathcal{S}_2$ is not constant.

Take now $\rho \in B(0,\min(\delta_1', \sqrt{\epsilon_1}))$ and $R \geq \max(R_1, R_1')$. Define $u_0 \in H^1(\mathbf{R}^2)$ by $\tilde{u}_0(r,\theta) = w(r-R-\rho(\theta), r)$. Then, $\|\tilde{u}_0 - w(r-R,r)\|_W \leq \|\rho\|_Z \leq \delta_1'$, and by lemma 3.2.1, there exists a unique pair $(v_0,\rho_0) \in W \times Z$ satisfying

$$\begin{aligned}
&\tilde{u}_0(r,\theta) = w(r-R-\rho_0(\theta), r) + v_0(r,\theta) \\
&< v_0, \psi >= 0 \\
&\|v_0\|_W + \|\rho_0\|_Z \leq K' \min(\delta_1', \sqrt{\epsilon_1}).
\end{aligned}$$

As a consequence, $\rho_0 \equiv \rho$ and $v_0 \equiv 0$ and

$$R^{\frac{1}{2}} \|v_0\|_W^2 + \|\rho_0\|_Z^2 = \|\rho\|_Z^2 \leq \epsilon_1.$$

Notice that as $v_0 = 0$, this last inequality is still valid for arbitrary large R. Finally by theorem 3.2.3 and lemma 3.3.1, there exist $(v,\rho,\rho_\infty) \in C(\mathbf{R}^+, W \times Z) \times L^2(0,2\pi)$ such that

$$\begin{aligned}
&(R+ct)^{\frac{1}{2}} \|v\|_W^2 + \|\rho\|_Z^2 \leq n\left(\epsilon_1 + \frac{1}{R}\right) \\
&\lim_{t\to+\infty} \|\rho(.,t) - \rho_\infty\|_{L^2(0,2\pi)} = 0 \\
&\|\rho_0 - \rho_\infty\|_{L^2(0,2\pi)} \leq \frac{c_1}{\sqrt{R}}
\end{aligned}$$

As R can be chosen as large as we need it, the last inequality shows that $\mathcal{S}_2$ is dense in $B_Z(0,\min(\delta_1', \sqrt{\epsilon_1}))$. ■

Chapitre 4

Appendix

4.1 Perturbation theorem for evolution operators

Theorem 4.1.1 *Let A be a sectorial operator on a Banach space X such that $Re(\sigma(A)) \geq a > 0$ and $\alpha \in [0,1)$. We set $X^\alpha \equiv D(A^\alpha)$. Let $\beta > 0$, $M > 0$ so that*

$$\|e^{-tA}\|_{\mathcal{L}(X)} \leq Me^{-\beta t} \text{ and } \|e^{-tA}x\|_{X^\alpha} \leq \frac{M}{t^\alpha}e^{-\beta t}\|x\|_X$$

for all $t > 0$ and $x \in X$. Suppose $B : [t_0; +\infty) \to \mathcal{L}(X^\alpha, X)$ is locally Hölder continuous with

$$\|B(t)\|_{\mathcal{L}(X^\alpha, X)} \leq \gamma$$

for all $t \geq t_0 \geq 0$ and some $\gamma > 0$. Let $T(t,\tau)$, $t_0 \leq \tau \leq t$, be the family of evolution operators so that the unique solution of

$$\begin{aligned} &\frac{dx}{dt} + Ax = B(t)x, \quad t \geq \tau, \\ &x(\tau) = x_0, \end{aligned} \tag{4.1}$$

is $x(t;\tau,x_0) = T(t,\tau)x_0$, $t_0 \leq \tau \leq t$. Then, there exists $\gamma_0 > 0$ such that for any $\gamma \in (0,\gamma_0)$, there exists $\delta \in (0,\beta)$ such that for any $t_0 \leq s \leq t$,

$$\|T(t,s)\|_{\mathcal{L}(X)} \leq M_1 e^{-\delta(t-s)}. \tag{4.2}$$

Proof: Given $x_0 \in X$, $t_0 \leq \tau \leq T$ and $\delta \in (0,\beta)$, we shall solve (4.1) in the Banach space

$$V = \{x \in C^0([\tau,T],X) \cap C^0((\tau,T],X^\alpha) \mid \|x\|_V < \infty\}$$
$$\text{where } \|x\|_V = \sup_{\tau \leq t \leq T} e^{\delta(t-\tau)}\|x(t)\|_X + \sup_{\tau < t \leq T} (t-\tau)^\alpha e^{\delta(t-\tau)}\|x(t)\|_{X^\alpha}.$$

First, given $x \in V$, we define the function F from V to V by

$$F(x)(t) = e^{-A(t-\tau)}x_0 + \int_\tau^t e^{-A(t-s)}B(s)x(s)ds.$$

For $r > 0$, let $\gamma_0 > 0$ and $R > 0$ be chosen so that

$$\begin{aligned} c(T) &= \sup_{\tau \le t \le T} \int_\tau^t \frac{ds}{(t-s)^\alpha (s-\tau)^\alpha} \\ R &= 4Mr \\ C_1 &= M\gamma_0 e^{(\delta-\beta)(T-\tau)} (T-\tau)^\alpha c(T) \le \frac{1}{4} \\ C_2 &= M\gamma_0 e^{-\delta(T-\tau)} \frac{(T-\tau)^{1-\alpha}}{1-\alpha} \le \frac{1}{4}. \end{aligned}$$

Then, for any $x_0 \in X$ with $\|x_0\|_X \le r$, F maps the ball $B_V(0,R)$ of V into itself and has a unique fixed point in the ball $B_V(0,R)$. Using Gronwall's lemma, it is then straightforward to show that this fixed point is actually the unique solution of (4.1) in the space V. Finally, since $\|x\|_V \le 2M\|x_0\|_X + (C_1 + C_2)\|x\|_V$, the solution $x(t)$ is defined for all $t > 0$ and the bound (4.2) holds with $M_1 = 4M$. ■

4.2 A few lemmas

4.2.1 Schur's lemma

Lemma 4.2.1 *Let P be an operator of $L^2(\boldsymbol{R}^2)$ defined in polar coordinates by*

$$Pu(r,\theta) = \int_0^\infty u(z,\theta) K(z,r,\theta)dz, \qquad u \in L^2(\boldsymbol{R}^2)$$

so that

$$\begin{aligned} c_1 &= \sup_{z\ge 0, \theta\in[0,2\pi)} \int_0^\infty |K(z,r,\theta)| \sqrt{\frac{r}{z}} dr < \infty \\ c_2 &= \sup_{r\ge 0, \theta\in[0,2\pi)} \int_0^\infty |K(z,r,\theta)| \sqrt{\frac{r}{z}} dz < \infty. \end{aligned}$$

Then, P is continuous on $L^2(\boldsymbol{R}^2)$ and for any $u \in L^2(\boldsymbol{R}^2)$,

$$\|Pu\|_{L^2(\boldsymbol{R}^2)} \le \sqrt{c_1 c_2} \|u\|_{L^2(\boldsymbol{R}^2)}.$$

Proof: We fix $\theta \in [0, 2\pi)$. Then, by Hölder's inequality and Fubini's theorem,

$$\begin{aligned} \int_0^\infty \left[\int_0^\infty K(z,r,\theta)u(z,\theta)dz\right]^2 rdr &\le \int_0^\infty \left(\int_0^\infty K\frac{dz}{\sqrt{z}}\right)\left(\int_0^\infty K u^2 \sqrt{z}dz\right) rdr \\ &\le c_2 \int_0^\infty u^2(z) z \int_0^\infty K \sqrt{\frac{r}{z}} drdz \\ &\le c_1 c_2 \int_0^\infty u^2(z,\theta) z dz. \end{aligned}$$

Integrating in $\theta \in (0, 2\pi)$ the above inequality, we get the continuity of P. ■

Throughout the proof of lemma 3.2.5, we use Schur's lemma in the following way, most of the time without mentioning it. For instance, the following inequality

$$\int_0^{2\pi}\int_0^{R+ct}\left(\int_0^r \psi v_\theta dz\right)^2 rdrd\theta \le c_0(R+ct)^2\int_0^{2\pi}\int_0^{\infty}\frac{v_\theta^2}{r^2}rdrd\theta$$

is proved by Schur's lemma by writing

$$K(z,r,\theta,t)=\mathbb{I}_{z\le r\le(R+ct)}\psi(z-s(\theta,t),z)z \text{ and } u(z,\theta,t)=\frac{v_\theta}{z}.$$

Then, $c_i(t)\le c_0(R+ct)$ for $i=1,2$. This concludes the proof of the above inequality.

4.2.2 Jensen's Inequality

Proposition 4.2.2 *Let ϕ be a convex function and ν a probability measure on a measurable set A. Then, for any $f\in L^1(A,d\nu)$,*

$$\phi\left(\int_A f d\nu\right)\le\int_A\phi(f)d\nu.$$

Corollary 4.2.3

$$\int_0^{2\pi}\left(\int_0^\infty v(r,\theta,t)\psi(r-s(\theta,t),r)dr\right)^2 d\theta\le c_0\int_0^{2\pi}\int_0^\infty v^2\psi(r-s(\theta,t),r)drd\theta$$

Proof: For any $\theta\in(0,2\pi)$ and any $t>0$, let $d\nu=\tilde\alpha\psi(r-s(\theta,t),r)dr$ where $\tilde\alpha$ is chosen so that $\int_{\mathbf{R}}\tilde\alpha\psi(r-s(\theta,t),r)dr=1$. Then, ν is a probability measure for any fixed t and θ, and $\phi(x)=x^2$ is convex in $\mathbf{R}^2$. By Jensen's inequality,

$$\int_0^{2\pi}\left(\int_0^\infty v(r,\theta,t)\psi(r-s(\theta,t),r)dr\right)^2 d\theta\le\int_0^{2\pi}\int_0^\infty\frac{1}{\tilde\alpha}v^2\psi(r-s(\theta,t),r)drd\theta.$$

As $\tilde\alpha^{-1}$ can be bounded independently of θ and t, this ends the proof. ■

4.3 Log-concave functions

Proposition 4.3.1 *Let $F\in C^3(\boldsymbol{R})$ be a function satisfying the following conditions:*

$$F(0)=F(1)=0,\quad F'(0)=\alpha<0,\quad F'(1)=\beta<0,$$
$$\exists\mu\in(0,1) \text{ so that } F(u)>0 \text{ for } u\in(\mu,1),\quad F(u)<0 \text{ for } u\in(0,\mu),$$
$$\int_0^1 F(u)du>0,\quad F^{(3)}(u)\le 0 \text{ for all } u\in[0,1].$$

Let $c>0$ and $w_0\in C^2(\boldsymbol{R})$ be a monotone solution of the ODE

$$w_0''+cw_0'+F(w_0)=0,\qquad x\in\boldsymbol{R}, \tag{4.3}$$

with the boundary conditions at infinity

$$\lim_{x\to-\infty} w_0(x) = 1 \text{ and } \lim_{x\to+\infty} w_0(x) = 0.$$

Define $\phi_0 = w_0' < 0$. *Then,* ϕ_0 *is log-concave:*

$$-\left(\frac{\phi_0'}{\phi_0}\right)' > 0.$$

Proof: As $-\frac{\phi_0'}{\phi_0} = -\frac{w_0''}{w_0'} = c + \frac{F(w_0)}{w_0'} \equiv c + g$, it is sufficient to prove that g is increasing on $\mathbf{R}$, i.e. that $h \equiv g'$ is positive. We first study the behaviour of g and h as $|x|$ goes to infinity. It is a standard result that w_0 (respectively $1 - w_0$) decreases exponentially fast to zero as x goes to $+\infty$ (resp $-\infty$). Let us begin with the behaviour of w_0 at $-\infty$:

$$w_0(x) = 1 - e^{\lambda x} + Ae^{2\lambda x} + o(e^{2\lambda x}),$$

where $\lambda > 0$. Then,

$$\begin{aligned} w_0'(x) &= -\lambda e^{\lambda x} + 2\lambda A e^{2\lambda x} + o(e^{2\lambda x}), \\ w_0''(x) &= -\lambda^2 e^{\lambda x} + 4\lambda^2 A e^{2\lambda x} + o(e^{2\lambda x}), \end{aligned}$$

and by Taylor's theorem,

$$\begin{aligned} F(w_0(x)) &= F'(1)(w_0(x) - 1) + \frac{1}{2}F''(1)(w_0(x) - 1)^2 + o(e^{2\lambda x}) \\ &= \beta(-e^{\lambda x} + Ae^{2\lambda x}) + \frac{1}{2}F''(1)e^{2\lambda x} + o(e^{2\lambda x}). \end{aligned}$$

As w_0 is a solution of (4.3), the first order of the expansion says that λ is the positive root of

$$\lambda^2 + c\lambda + \beta = 0.$$

The second order gives

$$A(4\lambda^2 + 2c\lambda + \beta) + \frac{1}{2}F''(1) = 0,$$

i.e. $A(3\lambda^2 + c\lambda) + \frac{1}{2}F''(1) = 0$. Notice that the above assumptions on F forces $F''(1)$ to be negative. Therefore, $A > 0$. Finally,

$$g = -\frac{w_0''}{w_0'} - c = -(c + \lambda) + 2A\lambda e^{\lambda x} + o(e^{2\lambda x})$$

and $h \sim 2A\lambda^2 e^{\lambda x}$ as x goes to $-\infty$. We can then conclude from this study that h is positive for $x < 0$ sufficiently large.

A similar study in $+\infty$ with $w_0(x) = e^{\mu x} - Be^{2\mu x} + o(e^{2\mu x})$ where μ is the negative root of $\mu^2 + c\mu + \alpha = 0$, gives that $-B(2\mu^2 - \alpha) + \frac{1}{2}F''(0) = 0$ which implies that $B > 0$. Finally, as $g(x) = -(c + \mu) + 2B\mu e^{\mu x} + o(e^{2\mu x})$,

$$h(x) \sim 2B\mu^2 e^{\mu x} \text{ when } x \to +\infty$$

and h is positive for $x > 0$ sufficiently large.

Suppose now that there exists some $x_0 \in \mathbf{R}$ such that $h(x_0) \leq 0$ and define

$$x_1 = \inf\{x \in \mathbf{R} \,|\, h(x) \leq 0\}$$
$$x_2 = \sup\{x \in \mathbf{R} \,|\, h(x) \leq 0\}.$$

Then, $h'(x_1) \leq 0$ and $h'(x_2) \geq 0$. As $h = cg + g^2 + F'(w_0)$, we get

$$h' = c(1 + 2g)h + F''(w_0)w_0'. \tag{4.4}$$

Then, $F''(w_0(x_1)) \geq 0$ and $F''(w_0(x_2)) \leq 0$. As $x_1 \leq x_2$ and $F''(w_0)$ is increasing, we conclude that

$$F''(w_0(x)) = 0 \text{ for all } x \in [x_1, x_2].$$

Then, $F''(w_0(x)) \geq 0$ for all $x \geq x_2$ and by (4.4),

$$\begin{cases} h'(x) \leq c(1 + 2g(x))h(x) & x \in [x_2, +\infty) \\ h(x_2) = 0. \end{cases}$$

Finally, by the maximum principle, $h(x) \leq 0$ for all $x \geq x_2$ which contradicts the definition of x_2. Therefore, h is positive on $\mathbf{R}$ and g is increasing. This concludes the proof. ■

Part II

Long-Time Asymptotics of Navier-Stokes and Vorticity equations in a three-dimensional layer

Le deuxième paragraphe de ce chapitre constitue un article soumis dans "Communications in Partial Differential Equations" en 2003

Résumé: On étudie le comportement pour les grands temps des solutions de l'équation de Navier-Stokes dans la bande $\mathbf{R}^2 \times (0,1)$ avec ou sans force exterieure de Coriolis. Après reformulation du problème à l'aide de la vorticité et de variables auto-similaires, on calcule un développement asymptotique en temps de la vorticité, en supposant que la vorticité initiale est suffisamment petite et décroît polynomialement à l'infini. Dans un deuxième temps, sans cette hypothèse de petitesse sur la donnée initiale, on montre que, de nouveau, le comportement asymptotique des solutions de l'équation de Navier-Stokes est régi par le tourbillon d'Oseen et l'équation de Navier-Stokes bidimensionnelle. Dans un troisième temps, on montre que pour une rotation assez rapide, il existe une unique solution globale pour l'équation de Navier-Stokes avec force de Coriolis dans des espaces de Sobolev homogènes.

Abstract: We study the long-time behavior of solutions of the Navier-Stokes equation in $\mathbf{R}^2 \times (0,1)$ with or without external Coriolis force. After introducing the vorticity and self-similar variables, we compute the long-time asymptotics of the rescaled vorticity, assuming the initial vorticity is sufficiently small and has polynomial decay at infinity. Afterwards, we release this assumption of smallness on initial data and we prove again that the long-time behavior of solutions of the Navier-Stokes equation is governed by Oseen vortices and the two-dimensional Navier-Stokes equation. Finally, we prove that under high rotation, there exists a unique global solution to the Navier-Stokes equation with Coriolis force in homogeneous Sobolev spaces.

Keywords: Navier-Stokes equation, rotating fluids, long-time asymptotics, three dimensional layer, self-similar variables.

AMS classification codes (2000): 35B40, 35Q30, 35G10, 76D05

Chapitre 1

Introduction

We consider the motion of an incompressible viscous fluid filling a three dimensional layer $\mathbf{R}^2 \times (0, L)$ where L is a given length scale (for example, the depth of ocean). We denote by $x = (x_1, x_2) \in \mathbf{R}^2$ the horizontal variable and by $z \in (0, L)$ the vertical coordinate. If no external force is applied, the velocity field $u = (u_1, u_2, u_3)^T$ of the fluid is given by the Navier-Stokes equation

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \frac{1}{\rho} \nabla p\,, \quad \text{div } u = 0\,, \tag{1.1}$$

where ρ is the density of the fluid, ν the kinematic viscosity and p the pressure field. Replacing x, z, t, u, p with the dimensionless quantities

$$\frac{x}{L}, \frac{z}{L}, \frac{\nu t}{L^2}, \frac{Lu}{\nu}, \frac{L^2 p}{\rho \nu^2},$$

equation (1.1) is transformed into

$$\partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p\,, \quad \text{div } u = 0\,, \tag{1.2}$$

where $u = u(x, z, t) \in \mathbf{R}^3$, $p = p(x, z, t) \in \mathbf{R}$, $(x, z, t) \in \mathbf{R}^2 \times (0, 1) \times \mathbf{R}^+$. We will also consider the motion of an incompressible viscous rotating fluid. In this case, an external Coriolis force is applied to the fluid and the velocity field u is given by the Navier-Stokes Coriolis equation

$$\partial_t u + (u \cdot \nabla) u + \Omega e_3 \wedge u = \Delta u - \nabla p\,, \quad \text{div } u = 0\,, \tag{1.3}$$

where $\Omega \in \mathbf{R}$ is the angular speed of the rotation and e_3 the unit vector of the vertical axis. We supplement (1.2) and (1.3) with the initial condition

$$u(x, z, 0) = u_0(x, z)\,, \quad (x, z) \in \mathbf{R}^2 \times (0, 1)\,.$$

Studying the asymptotic behavior of the velocity u in both cases is the aim of the second part of this thesis. In chapter 2, we are interested in the Navier-Stokes equation (1.2). In chapter 3, we deal with rotating fluids and study the Navier-Stokes Coriolis equation (1.3).

As we shall see, the Coriolis term does not intervene much in the long-time asymptotics of (1.3) and the methods as well as the results are very similar in chapter 3 to those of chapter 2. Different studies are carried out when dealing with global convergence (see sections 2.4 and 3.4).

As far as the three-dimensional Navier-Stokes equation is concerned, there have been numerous studies in the recent past years to precise the asymptotic decay in time of global solutions. Let us quote in particular the papers of M.E. Schonbek [82], [83], [84], M. Wiegner [94], A. Carpio [20], [21], and more recently of T. Miyakawa and M.E. Schonbek [65], Th. Gallay and C.E. Wayne [40]. In the present work, we show how the methods developped in [40] can be adapted to the case of a three-dimensional layer.

The previous works on rotating fluids mostly deal with global existence and uniqueness of solutions of (1.3) for large Coriolis parameter Ω. These studies are often divided in three steps. First, they look at the nature of the limit equations when Ω goes to infinity and at the regularity of their solutions. Afterwards, they prove the convergence of solutions of the Navier-Stokes Coriolis equation (1.3) to the solutions of the limit equations when Ω goes to infinity. Finally, they conclude from the first two steps that the solutions of (1.3) exist globally and are regular for Ω large enough but finite. Among those papers, we refer to A. Babin, A. Mahalov and B. Nicolaenko [3], [4] for the periodic case, to J.Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier [25], [26] for the whole space and to T. Colin and P. Fabrie for other boundary conditions such as a free surface [27]. In all cases, the authors prove that the asymptotics of (1.3) (when Ω goes to infinity) are driven by the two-dimensional Navier-Stokes equation. As we shall see, the long-time behavior of (1.2) and (1.3) are also governed by the two-dimensional Navier-Stokes equation studied in [39].

We are therefore interested in the asymptotic behavior of the two dimensional Navier-Stokes equation. Let us quote three important papers which show with different methods that the first order asymptotics are driven by Oseen vortices. Y. Giga and T. Kambe [42] show the stability of the Gauss kernel with estimates on the integral equation. A. Carpio [20] proves the convergence to the fundamental solution of the heat equation with rescaling methods. Finally, Th. Gallay and C.E. Wayne [39] construct finite dimensional invariant manifolds and use the idea that these manifolds control the long-time behavior of solutions to prove the stability of Oseen vortices. This method also allows to compute the asymptotics of the two-dimensional Navier-Stokes equation to any order.

We supplement (1.2) and (1.3) with boundary conditions: for all $(x, z, t) \in \mathbf{R}^3 \times \mathbf{R}^+$,

$$u(x, z+1, t) = u(x, z, t) \,. \tag{1.4}$$

This periodic boundary conditions (1.4) are not physically realistic. Nevertheless, space periodic flows are of interest in the study of homogeneous turbulence and, from the mathematical point of view, periodic boundary conditions enable us to solve functional analysis problems with the use of Fourier transformation (see [88]).

Although Dirichlet boundary conditions would be more realistic, they are of less interest in our case as the solutions converge exponentially fast to zero. The asymptotic behavior observed in the periodic case and the formation of Oseen vortices do not occur with the Dirichlet boundary conditions.

Alternatively, we will also consider stress-free boundary conditions for equation (1.2). In this case, the force applied by the boundary on the fluid is normal to the surface and there is no shearing stress, see [89]. The mathematical translation of this situation reads for all $(x,t) \in \mathbf{R}^2 \times \mathbf{R}^+$,

$$\begin{aligned} \frac{\partial u_1}{\partial z}(x,0,t) &= \frac{\partial u_1}{\partial z}(x,1,t) = 0 \\ \frac{\partial u_2}{\partial z}(x,0,t) &= \frac{\partial u_2}{\partial z}(x,1,t) = 0 \\ u_3(x,0,t) &= u_3(x,1,t) = 0\,. \end{aligned} \tag{1.5}$$

In this work, we use the vorticity formulation to study the long-time behavior of solutions of the Navier-Stokes equation. Setting $\omega = \text{rot } u$, equations (1.2) and (1.3) are transformed respectively into

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \Delta\omega\,, \qquad \text{div } \omega = 0\,, \tag{1.6}$$

$$\partial_t\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = \Delta\omega - \Omega\partial_z u\,, \quad \text{div } \omega = 0\,, \tag{1.7}$$

together with the initial condition

$$\omega(x,z,0) = \omega_0(x,z) = \text{rot } u_0(x,z)\,.$$

The velocity field u can be reconstructed from ω via the Biot-Savart law (see appendix 4.1). Boundary conditions can also be expressed in terms of the vorticity. Periodic conditions read for all $(x,z,t) \in \mathbf{R}^3 \times \mathbf{R}^+$,

$$\omega(x,z+1,t) = \omega(x,z,t)$$

and stress-free conditions can be written for all $(x,t) \in \mathbf{R}^2 \times \mathbf{R}^+$ as

$$\begin{aligned} \omega_1(x,0,t) &= \omega_1(x,1,t) = 0 \\ \omega_2(x,0,t) &= \omega_2(x,1,t) = 0 \\ \frac{\partial \omega_3}{\partial z}(x,0,t) &= \frac{\partial \omega_3}{\partial z}(x,1,t) = 0\,. \end{aligned} \tag{1.8}$$

Although (1.2) and (1.6) or (1.3) and (1.7) are equivalent in some spaces (see [57]), we believe it is more convenient to compute long-time asymptotics in the vorticity formulation. Indeed, it has been shown, for instance by Wiegner in [94], that the decay rate in time of the velocity $u(t)$ is governed by the spacial decay rate of the initial data u_0. However, this spacial decay is not preserved under the evolution defined by (1.2) or (1.3) and $\{u_0 \in L^2(\mathbf{R}^2\times(0,1))^3 \,|\, (1+|x|)u_0 \in L^1(\mathbf{R}^2\times(0,1))^3\}$ for instance is not an invariant set of initial data. On the other hand, the evolution of the vorticity (1.6) or (1.7) is not affected by this disadvantage. If $(1+|x|)^m\omega_0 \in L^2(\mathbf{R}^2\times(0,1))^3$ for some $m \geq 0$, then the solution $\omega(t)$ of (1.6) or (1.7), whenever it exists, satisfies $(1+|x|)^m\omega(t) \in L^2(\mathbf{R}^2\times(0,1))^3$. The spatial decay rate of the vorticity ω is preserved under the evolution defined by (1.6) or (1.7). Thus, we believe it is more convenient to use the vorticity formulation of the Navier-Stokes equations to compute the long-time asymptotics of the solutions.

In the first three sections of chapter 2 and chapter 3, we assume ω is small and decreases sufficiently fast as $|x|$ goes to infinity. The first property allows to deal with global bounded solutions of (1.6) and (1.7) and the second one is very helpful to study long-time asymptotics.

To actually compute the asymptotics, we use methods of infinite dynamical systems and spectral projections to reduce the study of (1.6) or (1.7) to the one of a finite number of ODE's. This idea has been developped by Th. Gallay and C.E. Wayne in [39] when building invariant manifolds to derive the long-time behavior of the vorticity. However, if we linearize equation (1.6) or (1.7) around the zero solution, the linearised equation has continuous spectrum all the way from minus infinity to zero and it is not clear how to build such manifolds. The usual idea for parabolic equations is then to express the vorticity $\omega(x,z,t)$ in terms of self-similar variables (ξ, z, τ) defined by $\xi = x/\sqrt{1+t}$, $\tau = \log(1+t)$, see (2.1) below. As the scaling in time has been blown up, the rescaled linearised operator has remarkable spectral properties in weighted Lebesgue spaces that we use to compute the asymptotics of ω. Indeed, we find as in [39] that the asymptotics are governed by $\bar{R}w$, the projection of the rescaled vorticity w onto z-independent functions. Moreover, $\bar{R}w$ satisfies an evolution equation whose operator has a countable set of real, isolated eigenvalues with finite multiplicities. The essential spectrum can be pushed arbitrarily far away into the left-half plane by choosing appropriate function spaces (i.e. spatial decay rate of the vorticity). Thus, the long-time asymptotics in a neigborhood of the origin are determined, up to second order, by a finite system of ordinary differential equations.

In sections 2.1 and 3.1, we prove the existence and uniqueness of global bounded solutions of the vorticity equations (1.6) and (1.7) with periodic boundary conditions in a neighborhood of the origin. Sections 2.2 and 3.2 are devoted to the first order asymptotics. Under appropriate conditions, we show that

$$\omega(x,z,t) \sim \frac{\alpha}{1+t}\mathbf{G}\left(\frac{x}{\sqrt{1+t}}\right), \quad \mathbf{G}(\xi) = \frac{1}{4\pi}\begin{pmatrix} 0 \\ 0 \\ e^{-|\xi|^2/4} \end{pmatrix},$$

as t goes to infinity, where α is a real coefficient which can be easily computed from the initial data. Notice that $\mathbf{G}$ is independent of z and the corresponding velocity field obtained from the Biot-Savart law is horizontal, i.e. the third coordinate u_3 is zero. This velocity field is called Oseen vortex and also governs the long-time asymptotics of the two-dimensional Navier-Stokes equation (see [20], [42], [39]). In section 2.3, we give a higher order asymptotic expansion of ω, solution of (1.6) in case $\alpha = 0$. This case represents the velocity of finite energy. We prove in this situation that the long-time behavior of the velocity field is two-dimensional (i.e. does not depend on z) but not horizontal (i.e. u_3 is not trivially equal to zero). Actually, we show that under appropriate conditions

$$\omega(x,z,t) \sim \sum_{i=1}^{3} \frac{\beta_i}{(1+t)^{\frac{3}{2}}}\mathbf{F}_i\left(\frac{x}{\sqrt{1+t}}\right),$$

when t goes to infinity, where $(\beta)_{i=1,..,3}$ are real coefficients computed easily from the initial data. The vectors $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3)$ made of derivatives of $\mathbf{G}$ are linearly independent. The three of them are two-dimensional but only F_1 and F_2 correspond to horizontal velocities. The velocity obtained from F_3 has non-trivial coordinate u_3.

The methods developped in these three sections 2.2, 3.2 and 2.3 are very general and could be applied to compute the asymptotics of (1.6) and (1.7) up to any order, as soon as the spectral properties of the rescaled linearised operator mentioned above are well-known.

In section 3.3, we compute higher-order asymptotics for equation (1.7). The idea is to see how the rotation, whose influence is masked in the previous computations, intervenes in the long-time behavior of equation (1.7). Therefore, we consider an initial condition ω_0 of mean-value zero in z and look at its evolution. The previous coefficients α, $(\beta)_{i=1,..,3}$ are, in this case, zero and we could even prove that all asymptotics in $1/(1+t)^{1+\frac{n}{2}}$, at any finite order $n \in \mathbf{N}$, are zero. We show however , at a higher rate, that the solution of the linearised equation of (1.7) around the origin satisfies

$$\omega(t) \sim \frac{e^{-4\pi^2 t}}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) R_{\Omega t}\left(\int_{\mathbf{R}^2\times(0,1)} \omega_0(y,z')\cos(2\pi(z-z'))dydz'\right) ,$$

as t goes to infinity, where $R_{\Omega}t$ is a rotation of angle Ωt and axis $\mathbf{R}e_3$. The behavior of the solution of the full non-linear equation still has to be understood.

So far, our results concern small solutions only. In section 2.4, we show how they can be extended to all global bounded solutions of (1.6). Following [41], we relax the smallness assumption on the vorticity and compute with different methods the asymptotics of the vorticity in the same weighted spaces. Using the ω-limit set of a trajectory, Lyapunov function and LaSalle's principle, we show once more that the asymptotics are governed by the z-independent part of the vorticity. More precisely, we prove that the velocity converges to Oseen vortices.

At the time of writing, it is well-known however that the existence and uniqueness of such global bounded solutions of (1.6) remain an open problem. Therefore, in section 3.4, we highlight a particular case for which we are able to prove such results. Considering the limit when the rotation speed Ω is high, we prove that equation (1.3) has a unique global solution

$$u \in \mathcal{C}^0(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}) .$$

For the purpose of this section, we deal with the velocity formulation in homogeneous Sobolev spaces, following ideas developped by J.Y. Chemin, B. Desjardins, I. Gallagher, E. Grenier in [25] on energy and dispersion estimates.

In section 2.5, we prove analogous results for the Navier-Stokes equation in the case of stress-free boundary conditions. We show that the long-time behavior of the velocity is two-dimensional and horizontal. In particular, $\omega(t)$ behaves, when t goes to infinity, as $(0,0,\omega_{2D})^T$, where ω_{2D} is the solution of the two-dimensional vorticity equation studied in [39].

Finally, appendix 4.1 deals with the Biot-Savart laws in a three-dimensional layer and contains useful estimates of the velocity field in terms of the vorticity in weighted Lebesgue spaces. Appendix 4.2 is a generalisation of the study carried out in [39] on the spectrum of the two-dimensional operator $\mathcal{L}$ which governs the asymptotics of our three-dimensional equation. Next, appendix 4.3 describes the properties of generator $S(\tau,\sigma)$ of the evolution equation satisfied by the rescaled vorticity w for the Navier-Stokes equation (1.6). We compute useful estimates on $\partial^\alpha S(\tau,\sigma)$ in weighted Lebesgue spaces. Appendix 4.4 describes similar results on $\mathcal{S}(\tau,\sigma)$ for equation (1.7). Finally, appendix 4.5 gives some technical bounds on series and integrals used throughout this second part of the thesis.

Notations: Throughout the second part, we denote by $\|.\|_Z$ the norm in the Banach space Z and by $|.|$ the usual euclidean norm in $\mathbf{R}^n$. For any $p \in [1,+\infty]$, if $f \in L^p(\mathbf{R}^2 \times (0,1))^3$, we set $\|f\|_{L^p(\mathbf{R}^2\times(0,1))} = \|\,|f|\,\|_{L^p(\mathbf{R}^2\times(0,1))}$. Weighted norms play an important role in this part. We always denote $b(\xi) = (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbf{R}^2$, the weight function. For any $m \geq 0$, we set $\|f\|_m = \|b^m f\|_{L^2(\mathbf{R}^2\times(0,1))}$. If $f \in \mathcal{C}^0([0,T];L^p(\mathbf{R}^2\times(0,1))^3)$, we often write $f(\tau)$ to denote the map $(\xi,z) \mapsto f(\xi,z,\tau)$. Finally, we denote by C a generic positive constant, which may differ from place to place, even in the same chain of inequalities.

Chapitre 2

Navier-Stokes and Vorticity equations

2.1 The Cauchy problem

In this section, we describe existence and uniqueness results for solutions of the vorticity equation (1.6). As stressed in the introduction, our approach is to study the behavior of solutions of (1.6) and then to derive information about the solutions of the Navier-Stokes equation as a corollary.

In $\mathbf{R}^2 \times (0,1)$, the vorticity equation is

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = \Delta \omega\,, \quad \mathrm{div}\ \omega = 0$$

where $\omega = \omega(x,z,t) \in \mathbf{R}^3$ is 1-periodic in z, $(x,z,t) \in \mathbf{R}^2 \times (0,1) \times \mathbf{R}^+$ and the velocity field u is defined in terms of the vorticity via the Biot-Savart law (see appendix 4.1).

As our analysis of the long-time asymptotics of (1.6) depends on rewriting the equations in terms of scaling variables, we deal with the Cauchy problem in the new variables

$$\xi = \frac{x}{\sqrt{1+t}}\,,\ \tau = \log(1+t)\,.$$

As no scaling of type $z \mapsto \lambda z$ preserves the domain $(0,1)$, the third coordinate z remains unchanged. If $\omega(x,z,t)$ is a solution of (1.6) and u the corresponding velocity field, we introduce new functions $w(\xi,z,\tau)$ and $v(\xi,z,\tau)$ by

$$\begin{aligned} \omega(x,z,t) &= \frac{1}{1+t}\, w\left(\frac{x}{\sqrt{1+t}}, z, \log(1+t)\right)\,, \\ u(x,z,t) &= \frac{1}{\sqrt{1+t}}\, v\left(\frac{x}{\sqrt{1+t}}, z, \log(1+t)\right)\,. \end{aligned} \tag{2.1}$$

As the transformation is time-dependent for the first two coordinates $\xi \in \mathbf{R}^2$, the divergence operator becomes a time-dependent operator. Namely,

$$\mathrm{div}\ \omega(t) = 0 \text{ for any } t \geq 0 \Leftrightarrow \mathrm{div}_\tau w(\tau) = 0 \text{ for any } \tau \geq 0\,, \tag{2.2}$$

where

$$\mathrm{div}_\tau w(\tau) = \nabla_\tau \cdot w = \nabla_\xi \cdot w_\xi + e^{\frac{\tau}{2}} \partial_z w_z \,,$$

and

$$w_\xi = (w_1, w_2, 0)^T \,, \quad \nabla_\xi = (\partial_{\xi_1}, \partial_{\xi_2}, 0)^T \,.$$

Using the same notations, notice that the relation between w and v reads

$$w(\tau) = \mathrm{rot}_\tau v(\tau) = \nabla_\tau \wedge v(\tau) \,, \quad \tau \geq 0 \,.$$

Then, w satisfies the evolution equation

$$\partial_\tau w = \Lambda(\tau) w + N(w)(\tau) \,, \quad \mathrm{div}_\tau w(\tau) = 0 \,, \tag{2.3}$$

where

$$\begin{aligned}
\Lambda(\tau) &= \mathcal{L} + e^\tau \partial_z^2 \\
\mathcal{L} &= \Delta_\xi + \frac{1}{2}\xi \cdot \nabla_\xi + 1 \\
N(w)(\tau) &= (w \cdot \nabla_\tau) v - (v \cdot \nabla_\tau) w \\
&= (w_\xi \cdot \nabla_\xi) v - (v_\xi \cdot \nabla_\xi) w + e^{\frac{\tau}{2}} (w_z \partial_z v - v_z \partial_z w)
\end{aligned}$$

and the velocity field v is given by the Biot-Savart law described in appendix 4.1. Scaling variables have been previously used to study the evolution of the vorticity in [20], [42] and [39]. In those articles, the scaling variables are very convenient as they transformed an autonomous system into another one. Indeed, in $\mathbf{R}^n$, Navier-Stokes equation is invariant under the scaling transformation

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t) \,,\; p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t) \,.$$

In the three-dimensional layer $\mathbf{R}^2 \times (0,1)$, this property is no more satisfied and the new system (2.3) in scaling variables is not autonomous. However, as stressed in the introduction, we shall prove that the asymptotics of (1.6) are governed by the two-dimensional Navier-Stokes equation in $\mathbf{R}^2$ which is autonomous.

As in the two-dimensional case [39], we shall solve the rescaled vorticity equation in weighted L^2-spaces. For any $m \geq 0$, we define the Hilbert space $L^2(m)$ by

$$L^2(m) = \{ f(\xi, z) : \mathbf{R}^3 \to \mathbf{R}^3 \,|\, f \text{ is 1-periodic in } z,\, \|f\|_m < \infty \} \tag{2.4}$$

where

$$\|f\|_m = \left(\int_{\mathbf{R}^2 \times (0,1)} (1 + |\xi|^2)^m |f(\xi, z)|^2 dz d\xi \right)^{\frac{1}{2}} = \|b^m f\|_{L^2(\mathbf{R}^2 \times (0,1))} .$$

On the contrary to what is usually done on Navier-Stokes equation (see R. Temam [89]), we do not include the condition of incompressibility in the definition of function spaces we use. As shown in (2.2), the divergence-free condition on ω becomes time-dependent in scaling variables and therefore cannot be taken into account to define $L^2(m)$. However, as this assumption on incompressibility is crucial, we always mention it in our various theorems.

In appendix 4.3, we show that the time-dependent operator $\Lambda(\tau)$ is the generator of a family of evolution operators (or evolution system) $S(\tau,\sigma)$ in $L^2(m)$ for any $m \geq 0$. Since $\partial_i \Lambda(\tau) = (\Lambda(\tau) + \frac{1}{2})\partial_i$ for $i = 1$ or 2 (where $\partial_i = \partial_{\xi_i}$) and $\partial_z \Lambda(\tau) = \Lambda(\tau)\partial_z$, it is clear that $\partial_i S(\tau,\sigma) = e^{\frac{\tau-\sigma}{2}} S(\tau,\sigma)\partial_i$ for all $0 < \sigma < \tau$ and $i = 1$ or 2. Thus, using the fact that $\mathrm{div}_\tau w(\tau) = \mathrm{div}_\tau v(\tau) = 0$, we can rewrite (2.3) in integral form:

$$w_i(\tau) = S(\tau,0)w_i(0) + \int_0^\tau \sum_{j=1}^{2} e^{-\frac{\tau-\sigma}{2}} \partial_j S(\tau,\sigma) M_{ij}(\sigma) + e^{\frac{\sigma}{2}} \partial_z S(\tau,\sigma) M_{i3}(\sigma) d\sigma \qquad (2.5)$$

where $i = 1,..,3$ and

$$M_{ij} = w_j v_i - v_j w_i.$$

The main result of this section states that, if the initial data are small, (2.5) has global bounded solutions in $L^2(m)$.

Theorem 2.1.1 *Let $m > 1$. There exists $K_0 > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K_0$, equation (2.5) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $\mathrm{div}_\tau w(\tau) = 0$. In addition, there exists $K_1 > 0$ such that*

$$\|w(\tau)\|_m \leq K_1 \|w_0\|_m\,, \quad \tau \geq 0. \qquad (2.6)$$

Proof: Given $w_0 \in L^2(m)$ with div $w_0 = 0$, we shall solve (2.5) in the Banach space

$$X = \{w \in \mathcal{C}^0([0,+\infty); L^2(m)) \,|\, \mathrm{div}_\tau w(\tau) = 0,\ \|w\|_X = \sup_{\tau \geq 0} \|w(\tau)\|_m < \infty\}.$$

We first note that $\tau \mapsto S(\tau,0)w_0 \in X$ as by proposition 4.3.1.(a) with $\alpha = 0, q = 2, m > 1$, there exists $C_1 > 0$ such that for any $\tau \geq 0$,

$$\|S(\tau,0)w_0\|_m \leq C_1 \|w_0\|_m. \qquad (2.7)$$

Next, given $w \in \mathcal{C}^0([0,+\infty); L^2(m))$, we define $F(w) \in \mathcal{C}^0([0,+\infty); L^2(m))$ coordinate by coordinate. For $i = 1,..,3$,

$$F_i(w)(\tau) = \int_0^\tau \sum_{j=1}^{2} e^{-\frac{\tau-\sigma}{2}} \partial_j S(\tau,\sigma) M_{ij}(\sigma) + e^{\frac{\sigma}{2}} \partial_z S(\tau,\sigma) M_{i3}(\sigma) d\sigma,\ \tau \geq 0. \qquad (2.8)$$

We shall prove that F maps X into X and that there exists $C_2 > 0$ such that

$$\|F(w)\|_X \leq C_2 \|w\|_X^2, \quad \|F(w) - F(w')\|_X \leq C_2 \|w - w'\|_X (\|w\|_X + \|w'\|_X), \qquad (2.9)$$

for all $(w,w') \in X^2$. As is easily verified, the bounds (2.7) and (2.9) imply that the map $w \mapsto S(\tau,0)w_0 + F(w)$ has a unique fixed point in the ball $\{w \in X \,|\, \|w\|_X \leq R\}$

if $R < (2C_2)^{-1}$ and $\|w_0\|_m \le (2C_1)^{-1}R$. Using Gronwall's lemma, it is then straightforward to show that this fixed point is actually the unique solution of (2.5) in the space $\mathcal{C}^0([0,+\infty); L^2(m))$. Finally, since $\|w\|_X \le C_1\|w_0\|_m + C_2\|w\|_X^2 \le C_1\|w_0\|_m + \frac{1}{2}\|w\|_X$, the bound (2.6) holds with $K_1 = 2C_1$.

To prove (2.9), we use the bounds on $S(\tau,\sigma)$ proved in appendix 4.3. First,

$$\|F_i(w)(\tau)\|_m \le \int_0^\tau \sum_{j=1}^{2} e^{-\frac{\tau-\sigma}{2}} \|\partial_j S(\tau,\sigma) M_{ij}(\sigma)\|_m d\sigma + \int_0^\tau e^{\frac{\sigma}{2}} \|\partial_z S(\tau,\sigma) M_{i3}(\sigma)\|_m d\sigma\,.$$

The first integral is bounded by proposition 4.3.1(a) with $\alpha = (1,0,0)$ or $(0,1,0)$, $q = \frac{3}{2}$ and $m > 1$. The second one is bounded by proposition 4.3.1(b) with $\alpha = (0,0,1)$, $q = \frac{3}{2}$ and $m > 1$. Then, for $i = 1,..,3$,

$$\begin{aligned}\|F_i(w)(\tau)\|_m \le & C\int_0^\tau \sum_{j=1}^{2} \frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{2}{3}} a(e^\tau - e^\sigma)^{\frac{1}{12}}} \|b^m M_{ij}(\sigma)\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} d\sigma \\ & + C\int_0^\tau \frac{e^{\frac{\sigma}{2}} e^{-4\pi^2(e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{6}} a(e^\tau - e^\sigma)^{\frac{7}{12}}} \|b^m M_{i3}(\sigma)\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} d\sigma\,.\end{aligned}$$

As $a(\tau-\sigma) \le a(e^\tau - e^\sigma)$, it is clear that

$$\int_0^\tau \frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{2}{3}} a(e^\tau - e^\sigma)^{\frac{1}{12}}} d\sigma \le \int_0^\infty \frac{e^{-u/2}}{a(u)^{\frac{3}{4}}} du < +\infty \tag{2.10}$$

and by appendix 4.5.2 with $(\alpha,\beta,\gamma,\delta) = (\frac{1}{2}, 0, \frac{1}{6}, \frac{7}{12})$, we get

$$\int_0^\tau \frac{e^{\frac{\sigma}{2}} e^{-4\pi^2(e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{6}} a(e^\tau - e^\sigma)^{\frac{7}{12}}} d\sigma \le Ce^{-\frac{\tau}{3}} < +\infty\,.$$

Then, we just need to bound $\|b^m M_{ij}\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}$ in terms of $\|w\|_X^2$ to get the first inequality of (2.9). Using Hölder's inequality, we get

$$\|b^m w_j v_i\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} \le \|b^m w_j\|_{L^2(\mathbf{R}^2\times(0,1))} \|v_i\|_{L^6(\mathbf{R}^2\times(0,1))}.$$

Dividing v_i into two parts as in appendix 4.1, $v_i = \bar{v}_i + \tilde{v}_i$, see (4.1) and using the Biot-Savart laws proved in appendix 4.1.4, we get

$$\begin{aligned}\|\bar{v}_i\|_{L^6(\mathbf{R}^2)} &\le C\|\bar{w}\|_{L^{\frac{3}{2}}(\mathbf{R}^2)} \\ \|\tilde{v}_i\|_{L^6(\mathbf{R}^2\times(0,1))} &\le C\|\tilde{w}\|_{L^2(\mathbf{R}^2\times(0,1))}\,.\end{aligned}$$

Finally, by Hölder's inequality, we have $L^2(m) \hookrightarrow L^q(\mathbf{R}^2\times(0,1))$ for all $q \in [1,2]$, $m > 1$, and the following estimates

$$\|\bar{w}\|_{L^{\frac{3}{2}}(\mathbf{R}^2)} \le C\|w\|_m \text{ and } \|\tilde{w}\|_{L^2(\mathbf{R}^2\times(0,1))} \le C\|w\|_m$$

lead to the conclusion $\|b^m w_j v_i\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} \le C\|w\|_m^2$. Then, $\|F(w)\|_X \le C_2\|w\|_X^2$ and the second inequality in (2.9) can be proved along the same lines. The proof of theorem 2.1.1 is now complete. ■

Since the semi-group $e^{\tau\mathcal{L}}$ is not analytic in $L^2(m)$ (see [39]), the evolution system $S(\tau,\sigma)$ is not smooth in (τ,σ) and the solution w given by theorem 2.1.1 is in general not a smooth function of τ. In particular, $\tau \mapsto w(\tau) \notin C^1((0,+\infty), L^2(m))$ so that w is not a classical solution of (2.3) in $L^2(m)$. Nevertheless, following the common use, we shall often refer to w as the (mild) solution of (2.3).

We translate theorem 2.1.1 in terms of the vorticity $\omega(x,z,t)$ in the original variables:

Corollary 2.1.2 *Let $m > 1$. There exists $\epsilon_0 > 0$ such that for all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$ and $\|\omega_0\|_m \le \epsilon_0$, equation (1.6) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and div $\omega = 0$. In addition, for any $p \in [1,2]$, there exists $\epsilon_1 > 0$ such that*

$$\|\omega(t)\|_{L^p(\mathbf{R}^2\times(0,1))} \le \frac{\epsilon_1}{(1+t)^{1-\frac{1}{p}}}\|\omega_0\|_m\,, \quad t \ge 0. \tag{2.11}$$

Proof: First take $\epsilon_0 = K_0$. If $\omega_0 \in L^2(m)$ satisfies div $\omega_0 = 0$ and $\|\omega_0\|_m \le \epsilon_0$, the function w_0 defined by (2.1) is in $L^2(m)$ for $m > 1$ and $\|w_0\|_m \le K_0$. By theorem 2.1.1, there exists a unique solution $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ to (2.5) satisfying $w(0) = w_0$ and for any $\tau \ge 0$, $\mathrm{div}_\tau w(\tau) = 0$. Let ω be the corresponding vorticity defined by (2.1). Then, as $\omega(t) \in L^2(m)$ and $L^2(m) \hookrightarrow L^p(\mathbf{R}^2\times(0,1))$ for $p \in [1,2]$, $m > 1$,

$$\begin{aligned}\|\omega(t)\|_{L^p(\mathbf{R}^2\times(0,1))} &= (1+t)^{-1+\frac{1}{p}}\|w(\tau)\|_{L^p(\mathbf{R}^2\times(0,1))}\\ &\le C(1+t)^{-1+\frac{1}{p}}\|w(\tau)\|_m\\ &\le \frac{CK_1}{(1+t)^{1-\frac{1}{p}}}\|w_0\|_m\,.\end{aligned}$$

Taking $\epsilon_1 = CK_1$ ends the proof of the corollary. ■

Remark: Due to the embedding $L^2(m) \hookrightarrow L^p(\mathbf{R}^2\times(0,1))$ which is true for $p \in [1,2]$, the proof only holds for this range of p. However, due to the regularising effect, (2.11) holds for all $p \in [1,+\infty]$ if $t \ge 1$.

In order to compare these estimates with other known results on Navier-Stokes, it is worth stating the previous corollary in terms of the physical variables which appear in equation (1.1). Define the physical vorticity Ω by

$$\frac{L^2}{\nu}\Omega\left(Lx, Lz, \frac{L^2}{\nu}t\right) = \omega(x,z,t)\,, \quad (x,z,t) \in \mathbf{R}^2\times(0,1)\times\mathbf{R}^+\,.$$

Then, Ω satisfies the vorticity equation associated to (1.1) and corollary 2.1.2 states that if $L^{\frac{1}{2}}\|(1+\frac{|x|}{L})^m\Omega_0\|_{L^2(\mathbf{R}^2\times(0,1))} \le \epsilon_0\nu$, then

$$L^{2-3/p}\|\Omega(t)\|_{L^p(\mathbf{R}^2\times(0,1))} \le \frac{\epsilon_1\nu}{\left(1+\frac{\nu t}{L^2}\right)^{1-\frac{1}{p}}}\,.$$

This last inequality clearly shows the influence of the kinematic viscosity ν. In particular, the smallness assumption of the initial data required in the first three sections is, in fact, a comparison between the physical vorticity and the viscosity.

2.2 First-Order asymptotics

In this section, we consider the behavior of small solutions of the integral equation (2.5) in $L^2(m)$ for $m > 1$. In $\bar{R}(L^2(m))$ where $\bar{R}$ is defined in (4.9), the discrete spectrum of $\mathcal{L}$ contains at least a simple isolated eigenvalue $\lambda_0 = 0$ (see appendix 4.2.1) with eigenfunction $\mathbf{G} = (0,0,G)^T$ where here and in the sequel, G is the gaussian function:

$$G(\xi) = \frac{1}{4\pi}\exp\left(-\frac{|\xi|^2}{4}\right) \; , \; \xi \in \mathbf{R}^2 \, .$$

Let $\mathbf{v}^G$ denote the corresponding velocity field, satisfying rot $\mathbf{v}^G = \mathbf{G}$. Then,

$$\mathbf{v}^G(\xi) = \frac{1}{2\pi}\frac{e^{-|\xi|^2/4} - 1}{|\xi|^2}\begin{pmatrix} \xi_2 \\ -\xi_1 \\ 0 \end{pmatrix} \; , \; \xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \, ,$$

and $(\mathbf{v}^G \cdot \nabla)\mathbf{G} = (\mathbf{G} \cdot \nabla)\mathbf{v}^G = 0$. As a consequence, for any $\alpha \in \mathbf{R}$, $w(\xi, z) = \alpha\mathbf{G}(\xi)$ is a stationary solution of (2.3) whose velocity $\alpha\mathbf{v}^G$ is called *Oseen Vortex*. Using these notations and appendix 4.2.3, any solution w of (2.5) in $L^2(m)$ for $m > 1$ can be decomposed as

$$\begin{aligned} w(\xi, z, \tau) &= P_0 w(\xi, z, \tau) + Q_0 w(\xi, z, \tau) + \tilde{R} w(\xi, z, \tau) \\ &\equiv \alpha(\tau)\mathbf{G}(\xi) + q_0(\xi, \tau) + r(\xi, z, \tau) \end{aligned} \tag{2.12}$$

where the projections $P_0, Q_0, \tilde{R}$ and the coefficient α are defined in appendix 4.2 by (4.9, 4.13, 4.15). Then, q_0 belongs to the subspace $\mathcal{W}_0$ of $\bar{R}(L^2(m))$ defined in (4.14) which is also the spectral subspace associated with the strictly stable part of the spectrum of $\mathcal{L}$ in $\bar{R}(L^2(m))$. In particular, $\int_{\mathbf{R}^2} q_0(\xi, \tau) d\xi = 0$ for all $\tau \geq 0$. Moreover, $\int_0^1 r(\xi, z, \tau) dz = 0$. Notice that the notations r and $\tilde{w}$ are equivalent.

As in the two-dimensional case [39], an important property of (2.5) is the *conservation of mass*:

Lemma 2.2.1 *Assume $m > 1$ and $w \in \mathcal{C}^0([0,T]; L^2(m))$ is a solution of (2.5). Then, the coefficient α defined in (4.13) is constant in time.*

Proof: As $\alpha(\tau) = \int_{\mathbf{R}^2 \times (0,1)} w_3(\xi, z, \tau) d\xi dz$, integrating by parts shows that

$$\begin{aligned} \dot{\alpha}(\tau) &= \int_{\mathbf{R}^2 \times (0,1)} \left(\mathcal{L}w_3 + e^\tau \partial_z^2 w_3 + N(w)_3\right) \, dz \, d\xi \\ &= \int_{\mathbf{R}^2 \times (0,1)} \left(\nabla_\xi \cdot \left(\nabla_\xi w_3 + \frac{1}{2}\xi w_3\right) - v_3 \mathrm{div}_\tau w(\tau) + w_3 \mathrm{div}_\tau v(\tau)\right) \, dz \, d\xi \\ &= 0 \end{aligned}$$

as w and v are 1-periodic in z and decreasing in ξ at infinity. ■

In particular, it follows from lemma 2.2.1 that $\mathcal{W}_0$ is invariant under the evolution defined by (2.5). The remainder terms q_0 and r defined in (2.12) satisfy the equations:

$$\partial_\tau q_0 = \mathcal{L} q_0 + Q_0(N(w)) \,, \qquad \text{div } q_0 = 0 \,, \xi \in \mathbf{R}^2 \,, \tau > 0 \tag{2.13}$$

$$\partial_\tau r = \Lambda(\tau) r + \tilde{R}(N(w)) \,, \qquad \text{div}_\tau r(\tau) = 0 \,, (\xi, z) \in \mathbf{R}^2 \times (0,1) \,, \tau > 0 \tag{2.14}$$

where Q_0 and $\tilde{R}$ are the projections defined in (4.15) and (4.9). The main result of this section states that, if the initial data are small, the solution of (2.5) converges to the vorticity associated to Oseen Vortex:

Theorem 2.2.2 *Let $0 < \mu < \frac{1}{2}$ and $m > 1 + 2\mu$. There exists $K_0' > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\|_m \le K_0'$, equation (2.5) has a unique global solution $w \in \mathcal{C}^0([0, +\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \ge 0$, $div_\tau w(\tau) = 0$. In addition, there exists $K_2 > 0$ such that*

$$\|w(\tau) - \alpha \boldsymbol{G}\|_m \le K_2 \, e^{-\mu\tau} \|w_0\|_m \,, \quad \tau \ge 0 \,,$$

where $\alpha = \int_{\boldsymbol{R}^2 \times (0,1)} (w_0)_3(\xi, z) \, dz \, d\xi$.

Remark: In fact, one can show that theorem 2.2.2 remains true for $\mu = \frac{1}{2}$, but the proof below is limited to $\mu < \frac{1}{2}$ for technical reasons.

Proof: If $w \in \mathcal{C}^0([0, \infty); L^2(m))$ is the solution of (2.5) given by theorem 2.1.1 for $K_0' \le K_0$ and v the corresponding velocity field, we define α, q_0 and r as in (2.12). By lemma 2.2.1, $\alpha(\tau) = \alpha(0) = \alpha$ for all $\tau \ge 0$. To bound the remainder q_0 and r, we use the integral equations

$$q_0(\tau) = e^{\tau \mathcal{L}} q_0(0) + \int_0^\tau e^{(\tau - \sigma)\mathcal{L}} Q_0(N(w)(\sigma)) d\sigma \,, \; \tau \ge 0, \tag{2.15}$$

$$r(\tau) = S(\tau, 0) r(0) + \int_0^\tau S(\tau, \sigma) \tilde{R}(N(w)(\sigma)) d\sigma \,, \; \tau \ge 0. \tag{2.16}$$

We first easily prove that

$$\|e^{\tau \mathcal{L}} q_0(0)\|_m \le C e^{-\mu\tau} \|w_0\|_m \,. \tag{2.17}$$

Indeed, (2.17) follows from proposition 4.2.1 with $n = 0$, $\alpha = 0$, $q = 2$ and $\epsilon \in (0, m - 1 - 2\mu)$.

In the same way, by proposition 4.3.1(b) with $\alpha = 0$ and $q = 2$, there exists $C > 0$ such that

$$\|S(\tau, 0) r(0)\|_m \le C e^{-4\pi^2 e^\tau} \|w_0\|_m \,.$$

To estimate integrals in (2.15, 2.16), we proceed as in the proof of inequalities (2.9) in theorem 2.1.1. However, $\|w(\tau)\|_m$ does not converge (in general) to zero and since we want to prove that $w(\tau)$ converges to $\alpha \mathbf{G}$, the above method is not sufficient to conclude. The right procedure is to bound the integrals in (2.15, 2.16) by $\|q_0\|_m + \|r\|_m$ which will converge to zero. Therefore, we first notice that the non-linearity $N = (w \cdot \nabla_\tau) v - (v \cdot \nabla_\tau) w$

does not contain any terms in α^2. If we decompose w as $\alpha\mathbf{G}+q_0+r$ and v as $\alpha\mathbf{v}^G+\mathbf{v}^q+\mathbf{v}^r$ where v^q and v^r are the velocities associated via the Biot-Savart law to the vorticities q_0 and r respectively, N can be written as:

$$\begin{aligned} N =&\alpha^2\left((\mathbf{G}\cdot\nabla_\tau)\mathbf{v}^G-(\mathbf{v}^G\cdot\nabla_\tau)\mathbf{G}\right)\\ &+\alpha\left((\mathbf{G}\cdot\nabla_\tau)(\mathbf{v}^q+\mathbf{v}^r)-((\mathbf{v}^q+\mathbf{v}^r)\cdot\nabla_\tau)\mathbf{G}\right)\\ &+\alpha\left(((q_0+r)\cdot\nabla_\tau)\mathbf{v}^G-(\mathbf{v}^G\cdot\nabla_\tau)(q_0+r)\right)\\ &+((q_0+r)\cdot\nabla_\tau)(\mathbf{v}^q+\mathbf{v}^r)-((\mathbf{v}^q+\mathbf{v}^r)\cdot\nabla_\tau)(q_0+r)\,. \end{aligned}$$

Since $(\mathbf{G}\cdot\nabla_\tau)\mathbf{v}^G=(\mathbf{v}^G\cdot\nabla_\tau)\mathbf{G}=0$, N depends linearly on α. Then, for $i=1,..,3$,

$$N_i=\sum_{j=1}^{2}\partial_j\hat{M}_{ij}+e^{\frac{\tau}{2}}\partial_z\hat{M}_{i3}$$

where for $(i,j)\in\{1,..,3\}^2$,

$$\begin{aligned} \hat{M}_{ij} =&\alpha(\mathbf{G}_j(\mathbf{v}_i^q+\mathbf{v}_i^r)-\mathbf{G}_i(\mathbf{v}_j^q+\mathbf{v}_j^r)+\mathbf{v}_i^G(q_{0j}+r_j)-\mathbf{v}_j^G(q_{0i}+r_i))\\ &+(q_{0j}+r_j)(\mathbf{v}_i^q+\mathbf{v}_i^r)-(q_{0i}+r_i)(\mathbf{v}_j^q+\mathbf{v}_j^r)\,. \end{aligned}$$

As $\int_{\mathbf{R}^2}q_0(\xi,\tau)d\xi=\int_0^1 r(\xi,z,\tau)dz=0$,

$$\begin{aligned} Q_0(N(w)(\sigma))&=\int_0^1 N(w)dz=\left(\sum_{j=1}^{2}\partial_j\int_0^1\hat{M}_{ij}\,dz\right)_{i=1,..,3}\\ \tilde{R}(N(w)(\sigma))&=\left(\sum_{j=1}^{2}\left(\partial_j\hat{M}_{ij}-\partial_j\int_0^1\hat{M}_{ij}\,dz\right)+e^{\frac{\sigma}{2}}\partial_z\hat{M}_{i3}\right)_{i=1,..,3} \end{aligned}$$

Easily, we find for $i=1,..,3$,

$$\begin{aligned} e^{(\tau-\sigma)\mathcal{L}}Q_0(N(w))_i&=\sum_{j=1}^{2}e^{-\frac{\tau-\sigma}{2}}e^{(\tau-\sigma)\mathcal{L}}\int_0^1\hat{M}_{ij}dz\,,\\ S(\tau,\sigma)\tilde{R}(N(w))_i&=\sum_{j=1}^{2}e^{-\frac{\tau-\sigma}{2}}\partial_jS(\tau,\sigma)\tilde{R}(\hat{M}_{ij})+e^{\frac{\sigma}{2}}\partial_zS(\tau,\sigma)\hat{M}_{i3}\,. \end{aligned}$$

Noticing that by Jensen's inequality,

$$\|b^m\int_0^1\hat{M}_{ij}dz\|_{L^{\frac{3}{2}}(\mathbf{R}^2)}\le\|b^m\hat{M}_{ij}\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}\,,$$

we obtain as in theorem 2.1.1:

$$\|e^{(\tau-\sigma)\mathcal{L}}Q_0(N)d\sigma\|_m \leq C \sum_{(i,j)\in I} \frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{2}{3}}a(e^\tau-e^\sigma)^{\frac{1}{12}}}\|b^m\hat{M}_{ij}\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}$$

$$\|S(\tau,\sigma)\tilde{R}(N)d\sigma\|_m \leq C \sum_{(i,j)\in I} \frac{e^{-\frac{\tau-\sigma}{2}}e^{-4\pi^2(e^\tau-e^\sigma)}}{a(\tau-\sigma)^{\frac{2}{3}}a(e^\tau-e^\sigma)^{\frac{1}{12}}}\|b^m\hat{M}_{ij}\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}$$
$$+ C\sum_{i=1}^{3} \frac{e^{\frac{\sigma}{2}}e^{-4\pi^2(e^\tau-e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{6}}a(e^\tau-e^\sigma)^{\frac{7}{12}}}\|b^m\hat{M}_{i3}\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}$$

where $I=\{1,2,3\}\times\{1,2\}$. Next, proceeding as in theorem 2.1.1, by Hölder's inequality and Biot-Savart law,

$$\begin{aligned}\|b^m\hat{M}_{ij}\|_{L^{\frac{3}{2}}} \leq& \alpha\left(\|b^m\mathbf{G}_j(\mathbf{v}_i^q+\mathbf{v}_i^r)\|_{L^{\frac{3}{2}}} + \|b^m\mathbf{G}_i(\mathbf{v}_j^q+\mathbf{v}_j^r)\|_{L^{\frac{3}{2}}}\right)\\ &+ \alpha\left(\|b^m\mathbf{v}_j^G(q_{0i}+r_i)\|_{L^{\frac{3}{2}}} + \|b^m\mathbf{v}_i^G(q_{0j}+r_j)\|_{L^{\frac{3}{2}}}\right)\\ &+ \|b^m(q_{0j}+r_j)(\mathbf{v}_i^q+\mathbf{v}_i^r)\|_{L^{\frac{3}{2}}} + \|b^m(q_{0i}+r_i)(\mathbf{v}_j^q+\mathbf{v}_j^r)\|_{L^{\frac{3}{2}}}\\ \leq& C(w)(\|q_0\|_m+\|r\|_m)\end{aligned}$$

where $C(w)=2C(2\alpha\|\mathbf{G}\|_m+\|q_0+r\|_m)$. Then, by theorem 2.1.1, $|C(w)|\leq C_0\|w\|_m\leq C_0K_1\|w_0\|_m$ and $|C(w)|$ can be taken as small as we want by choice of appropriate initial data w_0.

Finally, denoting $f(\tau)=e^{\mu\tau}(\|q_0(\tau)\|_m+\|r(\tau)\|_m)$, we get

$$f(\tau)\leq C\|w_0\|_m + C(w)\int_0^\tau\left(\frac{e^{(\mu-\frac{1}{2})(\tau-\sigma)}}{a(\tau-\sigma)^{\frac{2}{3}}a(e^\tau-e^\sigma)^{\frac{1}{12}}} + \frac{e^{\frac{\sigma}{2}}e^{\mu(\tau-\sigma)}e^{-4\pi^2(e^\tau-e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{6}}a(e^\tau-e^\sigma)^{\frac{7}{12}}}\right)f(\sigma)d\sigma.$$

As $0<\mu<1/2$ and $m>1+2\mu$, the first part of the integral is bounded as in (2.10) and the second one by appendix 4.5.2 with $(\alpha,\beta,\gamma,\delta)=(\frac{1}{2},\mu,\frac{1}{6},\frac{7}{12})$ there exist positive constants C_1,C_2 such that for any $T>0$,

$$\|f\|_{L^\infty(0,T)}\leq C_1\|w_0\|_m + C_2C(w)\|f\|_{L^\infty(0,T)}.$$

Taking $K_0'>0$ such that $|C_2C(w)|\leq C_0C_2K_1K_0'\leq 1/2$, we get

$$\|f\|_{L^\infty(0,T)}\leq 2C_1\|w_0\|_m.$$

Then, for any $\tau\geq 0$,

$$\|w(\tau)-\alpha\mathbf{G}\|_m\leq 2C_1\|w_0\|_m e^{-\mu\tau}.$$

Then $K_2=2C_1$ and the proof of theorem 2.2.2 is complete. ∎

Finally, we translate theorem 2.2.2 in terms of the vorticity $\omega(x,z,t)$ in the original variables:

Corollary 2.2.3 *Let $0 < \mu < 1/2$ and $m > 1 + 2\mu$. There exists $\epsilon_0' > 0$ such that for all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$ and $\|\omega_0\|_m \le \epsilon_0'$, equation (1.6) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and div $\omega = 0$. In addition, for any $p \in [1,2]$, there exists $\epsilon_2 > 0$ such that*

$$\|\omega(t) - \omega_{app}(t)\|_{L^p(\boldsymbol{R}^2\times(0,1))} \le \frac{\epsilon_2}{(1+t)^{1-\frac{1}{p}+\mu}}\|\omega_0\|_m\,, \quad t \ge 0\,,$$

where $\omega_{app}(x,z,t) = \frac{\alpha}{1+t}\boldsymbol{G}\left(\frac{x}{\sqrt{1+t}}\right)$ and $\alpha = \int_{\boldsymbol{R}^2\times(0,1)}(\omega_0)_3 dzdx$.

2.3 Second-Order asymptotics

We now turn our attention to solutions of the integral equation (2.5) of finite energy, namely when $v(\tau)$ is in $L^2(\mathbf{R}^2 \times (0,1))$ for any $\tau \ge 0$. This case can be also caracterised by the vorticity since $v(\tau) \in L^2(\mathbf{R}^2 \times (0,1))$ is equivalent to $\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3(\xi,z,\tau)d\xi dz = 0$. We showed in section 2.2 that small solutions of (2.5) converge to $\alpha\mathbf{G}$ when τ goes to infinity and we want to precise this behavior when $\alpha = 0$. We study the asymptotics in $L^2(m)$ for $m > 2$. In $\bar{R}(L^2(m))$, the discrete spectrum of $\mathcal{L}$ contains at least a simple isolated eigenvalue $\lambda_0 = 0$ with eigenfunction $\mathbf{G}$ and another isolated eigenvalue $\lambda_1 = -\frac{1}{2}$ of multiplicity 3 with eigenfunctions defined in appendix 4.2.1:

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 0 \\ F_1 \end{pmatrix} ; \mathbf{F}_2 = \begin{pmatrix} 0 \\ 0 \\ F_2 \end{pmatrix} ; \mathbf{F}_3 = \begin{pmatrix} -F_2 \\ F_1 \\ 0 \end{pmatrix} .$$

Using these notations and appendix 4.2.3, any solution w of (2.5) in $L^2(m)$ can be decomposed as

$$\begin{aligned} w(\xi,z,\tau) &= P_1 w(\xi,z,\tau) + Q_1 w(\xi,z,\tau) + \tilde{R}w(\xi,z,\tau) \\ &\equiv \alpha(\tau)\mathbf{G}(\xi) + \sum_{i=1}^{3}\beta_i(\tau)\mathbf{F}_i(\xi) + q_1(\xi,\tau) + r(\xi,z,\tau) \end{aligned} \tag{2.18}$$

where the projections $P_1, Q_1, \tilde{R}$ and the coefficients α, β_i are defined in 4.2 by (4.9, 4.13, 4.15). Then, q_1 belongs to the subspace $\mathcal{W}_1$ of $\bar{R}(L^2(m))$ defined in (4.14) which is also the spectral subspace associated with the remainder part of the spectrum $\sigma(\mathcal{L})$ in $\bar{R}(L^2(m))$. In particular,

$$\int_{\mathbf{R}^2} q_1(\xi,\tau)d\xi = 0\,, \quad \int_{\mathbf{R}^2} \xi_1 q_1(\xi,\tau)d\xi = 0\,, \quad \int_{\mathbf{R}^2} \xi_2 q_1(\xi,\tau)d\xi = 0\,.$$

Moreover, $\int_0^1 r(\xi,z,\tau)\,dz = 0$ for all $\tau \ge 0$.

The coefficients $(\beta_i)_{i=1,..,3}$ satisfy very simple ODE's:

Lemma 2.3.1 *Assume $m > 2$ and let $w \in \mathcal{C}^0([0,T], L^2(m))$ be a solution of (2.5) such that $\alpha = 0$. Then, the coefficients $(\beta_i)_{i=1,..,3}$ defined by (4.13) satisfy*

$$\dot{\beta}_i(\tau) = -\frac{1}{2}\beta_i(\tau)\,, \quad \tau \in [0,T]\,.$$

Proof: We write the proof for $i = 1$. The two other cases can be proved along the same lines, even for $\beta_3(\tau)$ which is defined in a slightly different way.

$$\dot{\beta}_1(\tau) = \int_{\mathbf{R}^2\times(0,1)} \xi_1 \left(\mathcal{L}w_3 + \sum_{j=1}^{2} \partial_j(w_j v_3 - v_j w_3) \right) dz\, d\xi \tag{2.19}$$

as w and v are periodic in z. Since $\mathrm{div}_\tau v(\tau) = \mathrm{div}_\tau w(\tau) = 0$ and $\mathrm{rot}_\tau v(\tau) = w(\tau)$, we find the identities

$$\xi_1 \mathcal{L}w_3 + \frac{1}{2}\xi_1 w_3 = \partial_1(\xi_1\partial_1 w_3 + \frac{1}{2}\xi_1^2 w_3 - w_3) + \partial_2(\xi_1\partial_2 w_3 + \frac{1}{2}\xi_1\xi_2 w_3)$$

$$\int_{\mathbf{R}^2\times(0,1)} \xi_1 \sum_{j=1}^{2} \partial_j(w_j v_3 - v_j w_3)\, dz\, d\xi = -\int_{\mathbf{R}^2\times(0,1)} \partial_2 \left(\frac{v_1^2 + v_3^2 - v_2^2}{2}\right) dz\, d\xi\,.$$

This last equality requires integrations by parts and the fact that v is in $L^2(\mathbf{R}^2 \times (0,1))$. Then, $\dot{\beta}_1(\tau) = -\frac{1}{2}\beta_1(\tau)$. ■

In particular, it follows from lemma 2.3.1 that the subspace $\mathcal{W}_1$ is invariant under the evolution defined by (2.5). The remainder q_1 and r in (2.18) satisfy the equations:

$$\partial_\tau q_1 = \mathcal{L}q_1 + Q_1(N(w))\,, \qquad \mathrm{div}\ q_1 = 0\,, \xi \in \mathbf{R}^2\,, \tau \geq 0$$

$$\partial_\tau r = \Lambda(\tau) r + \tilde{R}(N(w))\,, \qquad \mathrm{div}_\tau r(\tau) = 0\,, (\xi, z) \in \mathbf{R}^2 \times (0,1), \tau \geq 0$$

where Q_1 and $\tilde{R}$ are defined in (4.15) and (4.9). The following result describes the second order asymptotics of $w(\tau)$ as τ goes to infinity if v is of finite energy.

Theorem 2.3.2 *Let $\frac{1}{2} < \nu < 1$, $m > 1 + 2\nu$. There exists $K_0'' > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$, $\int_{\mathbf{R}^2\times(0,1)} w_0 d\xi dz = 0$ and $\|w_0\|_m \leq K_0''$, equation (2.5) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $\mathrm{div}_\tau w(\tau) = 0$, $\int_{\mathbf{R}^2\times(0,1)} w(\xi, z, \tau) d\xi dz = 0$. In addition, there exists $K_3 > 0$ such that*

$$\|w(\tau) - \sum_{i=1}^{3} \beta_i \boldsymbol{F}_i e^{-\frac{\tau}{2}}\|_m \leq K_3\, e^{-\nu\tau} \|w_0\|_m\,, \quad \tau \geq 0\,,$$

where

$$\beta_1 = \int_{\mathbf{R}^2\times(0,1)} \xi_1 (w_0)_3(\xi, z)\, dz\, d\xi\,,$$

$$\beta_2 = \int_{\mathbf{R}^2\times(0,1)} \xi_2 (w_0)_3(\xi, z)\, dz\, d\xi\,,$$

$$\beta_3 = \int_{\mathbf{R}^2\times(0,1)} \frac{1}{2}(\xi_1 (w_0)_2 - \xi_2 (w_0)_1)(\xi, z)\, dz\, d\xi\,.$$

Proof: If $w \in \mathcal{C}^0([0,\infty); L^2(m))$ is the solution of (2.5) given by theorem 2.1.1 for $K_0'' \leq K_0$ and v the corresponding velocity field, we define $\alpha(\tau)$, $\beta_i(\tau)$, q_1 and r as in (4.13) and

(2.18). By lemma 2.2.1, $\alpha(\tau)=\alpha(0)=0$ and v is of finite energy for any time. By lemma 2.3.1, $\dot{\beta}_i(\tau)=-\frac{1}{2}\beta_i(\tau)$ for $\tau\geq 0$ and $i=1,..,3$. Then, $\beta_i(\tau)=e^{-\tau/2}\beta_i$ where $\beta_i=\beta_i(0)$. To bound the remainder terms q_1 and r, we use the integral equations

$$q_1(\tau)=e^{\tau\mathcal{L}}q_1(0)+\int_0^\tau e^{(\tau-\sigma)\mathcal{L}}Q_1(N(w)(\sigma))d\sigma\,,\quad \tau\geq 0, \tag{2.20}$$
$$r(\tau)=S(\tau,0)r(0)+\int_0^\tau S(\tau,\sigma)\tilde{R}(N(w)(\sigma))d\sigma\,,\quad \tau\geq 0.$$

As far as q_1 is concerned, we bound $\|q_1\|_m$ in three steps.

First step: We easily prove that

$$\|e^{\tau\mathcal{L}}q_1(0)\|_m\leq Ce^{-\nu\tau}\|w_0\|_m\,. \tag{2.21}$$

Indeed, (2.21) follows from proposition 4.2.1 with $n=1$, $q=2$, $\alpha=0$ and $\epsilon\in(0,m-1-2\nu)$.

Second step: To bound the integral term in (2.20), we notice that the moments up to order 1 of N are zero and

$$Q_1(N(w))=Q_0(N(w))=\int_0^1 N(w)dz=\left(\sum_{j=1}^2\int_0^1\partial_j M_{ij}dz\right)_{i=1,..,3}\,.$$

We cut (2.20) into two integrals between 0 and $\tau-1$ and between $\tau-1$ and τ in order to obtain an optimal decay rate in time. The first term is bounded by proposition 4.2.1 with $n=1$, $q=2$, $\alpha=0$, $\epsilon>0$, $m\in(2,3]$ and another time with $n=-1$, $q=\frac{3}{2}$, $\alpha=(1,0,0)$ or $(0,1,0)$, $\epsilon>0$, and by Jensen's inequality. We then get

$$\begin{aligned}\|\int_0^{\tau-1}e^{(\tau-\sigma)\mathcal{L}}Q_1(N(w))d\sigma\|_m&=\|\int_0^{\tau-1}e^{(\tau-\sigma-1)\mathcal{L}}Q_1e^{\mathcal{L}}Q_1(N(w))d\sigma\|_m\\
&\leq C\int_0^{\tau-1}e^{\frac{\tau-\sigma-1}{2}(1-m+\epsilon)}\|e^{\mathcal{L}}Q_1N(w)\|_m d\sigma\\
&\leq C\int_0^{\tau-1}e^{\frac{\tau-\sigma-1}{2}(1-m+\epsilon)}\sum_{(i,j)\in I}\|\partial_j e^{\mathcal{L}-\frac{1}{2}}\int_0^1 M_{ij}dz\|_m d\sigma\\
&\leq C\int_0^{\tau-1}e^{\frac{\tau-\sigma}{2}(1-m+\epsilon)}\|w(\sigma)\|_m^2 d\sigma\,,\end{aligned}$$

where $I=\{1,2,3\}\times\{1,2\}$. According to theorem 2.2.2, for $0<\mu<1/2$ and $m>1+2\mu$, $\|w(\sigma)\|_m\leq K_2e^{-\mu\sigma}\|w_0\|_m$. The previous term is then bounded by

$$CK_2^2\|w_0\|_m^2e^{-2\mu\tau}\int_1^\tau e^{\frac{u}{2}(1-m+\epsilon+4\mu)}du\,.$$

Taking $\nu=2\mu$, $m>1+2\nu$ and $\epsilon\in(0,m-1-2\nu)$, the second step leads to the following estimate

$$\|\int_0^{\tau-1}e^{(\tau-\sigma)\mathcal{L}}Q_1N(w)d\sigma\|_m\leq Ce^{-\nu\tau}\|w_0\|_m^2\,. \tag{2.22}$$

The same arguments are still valid when $m > 3$ and proposition 4.2.1 with $n = 1$ and $\gamma = 1$ leads to estimate (2.22) for $\frac{1}{2} < \nu < 1$.

Thrid step: In a similar way, the third step can be driven as follows

$$\begin{aligned}\|\int_{\tau-1}^{\tau} e^{(\tau-\sigma)\mathcal{L}} Q_1 N(w) d\sigma\|_m &\leq C \int_{\tau-1}^{\tau} e^{-\frac{\tau-\sigma}{2}} \sum_{(i,j)\in I} \|\partial_j e^{(\tau-\sigma)\mathcal{L}} \int_0^1 M_{ij} dz\|_m d\sigma \\ &\leq C \int_{\tau-1}^{\tau} \frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{2}{3}}} \|w(\sigma)\|_m^2 d\sigma \\ &\leq C K_2^2 \|w_0\|_m^2 e^{-2\mu\tau} \int_0^1 \frac{e^{-\frac{u}{2}(1-4\mu)}}{a(u)^{\frac{2}{3}}} du \\ &\leq C e^{-\nu\tau} \|w_0\|_m^2 \,. \end{aligned} \tag{2.23}$$

It is clear that the previous bound would not have been sharp for the interval $(0, \tau)$ as $1 - 4\mu < 0$.

Joining inequalities (2.21, 2.22) and (2.23), we prove that for $\nu \in (\frac{1}{2}, 1)$ and $m > 1+2\nu$, there exists $C_0 > 0$ such that for any $\tau \geq 0$,

$$\|q_1(\tau)\|_m \leq C_0 e^{-\nu\tau} \|w_0\|_m \,.$$

To turn to the bound of $\|r(\tau)\|_m$, we refer to the proof of theorem 2.2.2 and the previous result on the decreasing of $\|q_1(\tau)\|_m$:

$$\begin{aligned}\|r(\tau)\|_m &\leq C e^{-4\pi^2 e^{\tau}} \|w_0\|_m \\ &\quad + C(w) \int_0^{\tau} \frac{e^{-\frac{\tau-\sigma}{2}} e^{-4\pi^2(e^{\tau}-e^{\sigma})}}{a(\tau-\sigma)^{\frac{2}{3}} a(e^{\tau}-e^{\sigma})^{\frac{1}{12}}} \left(e^{-\nu\sigma}\|w_0\|_m + \|r(\sigma)\|_m\right) d\sigma \\ &\quad + C(w) \int_0^{\tau} \frac{e^{\frac{\sigma}{2}} e^{-4\pi^2(e^{\tau}-e^{\sigma})}}{a(\tau-\sigma)^{\frac{1}{6}} a(e^{\tau}-e^{\sigma})^{\frac{7}{12}}} \left(e^{-\nu\sigma}\|w_0\|_m + \|r(\sigma)\|_m\right) d\sigma \\ &\leq C e^{-\nu\tau} \|w_0\|_m + C C(w) \int_0^{\tau} \phi(\tau,\sigma) \|r(\sigma)\|_m d\sigma \,.\end{aligned}$$

The last inequality is obtained by appendix 4.5.2 with $(\alpha, \beta, \gamma, \delta)$ equal to $(0, \nu - \frac{1}{2}, \frac{2}{3}, \frac{1}{12})$ and $(\frac{1}{2}, \nu, \frac{1}{6}, \frac{7}{12})$. Moreover, $\int_0^{\tau} \phi(\tau,\sigma) e^{\nu(\tau-\sigma)} d\sigma$ can be bounded independtly of τ by appendix 4.5.2 with $(\alpha, \beta, \gamma, \delta)$ equal to $(0, \nu - \frac{1}{2}, \frac{2}{3}, \frac{1}{12})$ or $(\frac{1}{2}, \nu, \frac{1}{6}, \frac{7}{12})$. Denote $f(\tau) = e^{\nu\tau}\|r(\tau)\|_m$. There exist positive constants C_1, C_2 such that for any $T > 0$,

$$\|f\|_{L^{\infty}(0,T)} \leq C_1 \|w_0\|_m + C_2 C(w) \|f\|_{L^{\infty}(0,T)} \,.$$

Since $C(w)$ can be taken as small as we want by choice of appropriate initial data, we take K_0'' such that $|C_2 C(w)| < \frac{1}{2}$. Then,

$$\|r(\tau)\|_m \leq 2 C_1 e^{-\nu\tau} \|w_0\|_m$$

and taking $K_3 = C_0 + 2C_1$ ends the proof of theorem 2.3.2. ∎

We now translate this second asymptotics theorem in terms of the original variables:

Corollary 2.3.3 *Let $\frac{1}{2} < \nu < 1$ and $m > 1 + 2\nu$. There exists $\epsilon_0'' > 0$ such that for all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$, $\int_{\boldsymbol{R}^2\times(0,1)} \omega_0 dzdx = 0$ and $\|\omega_0\|_m \leq \epsilon_0''$, equation (1.6) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and for any $t \geq 0$, div $\omega(t) = 0$ and $\int_{\boldsymbol{R}^2\times(0,1)} \omega(t) dzdx = 0$. Moreover, for any $p \in [1,2]$, there exists $\epsilon_3 > 0$ such that*

$$\|\omega(t) - \omega_{app}(t)\|_{L^p(\boldsymbol{R}^2\times(0,1))} \leq \frac{\epsilon_3}{(1+t)^{1-\frac{1}{p}+\nu}} \|\omega_0\|_m , \quad t \geq 0$$

where

$$\begin{aligned} \omega_{app}(t) &= \sum_{i=1}^{3} \frac{\beta_i}{(1+t)^{3/2}} \boldsymbol{F}_i \left(\frac{x}{\sqrt{1+t}} \right) \\ \beta_1 &= \int_{\boldsymbol{R}^2\times(0,1)} x_1 (\omega_0)_3 dzdx \\ \beta_2 &= \int_{\boldsymbol{R}^2\times(0,1)} x_2 (\omega_0)_3 dzdx \\ \beta_3 &= \int_{\boldsymbol{R}^2\times(0,1)} \frac{1}{2} (x_1 (\omega_0)_2 - x_2 (\omega_0)_1) dzdx \,. \end{aligned}$$

2.4 Global Convergence

We adopt in this section a more general point of view. We are still interested in the asymptotics of bounded global solutions of the integral equation (2.5) together with periodic boundary conditions (1.4) but we relax the assumption of smallness of the initial data. Of course, in this more general case, we do not know how to show existence of solutions. However, as $w = \alpha \mathbf{G}$ is such a global bounded solution, we may assume that there exist some of them. Under this assumption, we prove, following [41], that the result of the second section can be generalised. Indeed, the asymptotics of such solutions are still governed by Oseen vortices. The main result of this section can be stated as follows:

Theorem 2.4.1 *Let $m > 1$ and $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be a global solution of (2.5) that is uniformly bounded in time in $L^2(m)$. Then,*

$$\lim_{\tau \to +\infty} \|w(\tau) - \alpha \boldsymbol{G}\|_m = 0 \,,$$

where $\alpha = \int_{\boldsymbol{R}^2\times(0,1)} w_3(\xi, z, 0) dzd\xi$.

Proof: As the proof of this theorem is quite long, we cut it in six lemmas and corollaries. The main idea is to study the ω-limit set of the trajectory $\{w(\tau)\}_{\tau\geq 0}$ and to prove that its elements are two-dimensional (i.e. independent of z). We are then able to use the result of [41] where it is shown that Oseen vortices are global attractors of any solution of the two-dimensional Navier-Stokes equation with initial conditions in $L^2_{2D}(m)$, (see appendix

4.2). Finally, using a Lyapunov function, we prove that the ω-limit set of $\{w(\tau)\}_{\tau\geq 0}$ is actually reduced to one element: $\alpha\mathbf{G}$. Let us begin with the following lemma:

Lemma 2.4.2 *Let $w \in \mathcal{C}^0([0,+\infty);L^2(m))$ be as in theorem 2.4.1. Then, there exist positive constants K_0, K_1 such that*

$$\|w(\tau)\|_m \leq K_0\,, \quad \tau \geq 0 \tag{2.24}$$

$$\|\nabla w(\tau)\|_m \leq K_1\,, \quad \tau \geq 1\,. \tag{2.25}$$

Proof of lemma 2.4.2: Inequality (2.24) holds by assumptions. Regarding inequality (2.25), we proceed as in lemma 2.1 in [41]. Consider for $m>1$ the Banach space

$$X = \{W \in \mathcal{C}^0([0,T];L^2(m)) \cap \mathcal{C}^0((0,T];H^1(m)) \,|\, \mathrm{div}_\tau W(\tau) = 0\,,\, \|W\|_X < +\infty\}$$

where the X norm is defined by

$$\|W\|_X = \sup_{\tau\in[0,T]} \|W(\tau)\|_m + \sup_{\tau\in(0,T]} a(\tau)^{\frac{1}{2}} \|\nabla W(\tau)\|_m$$

and $a(\tau) = 1-e^{-\tau}$. We shall prove that there exist positive constants T and K_1 such that for all initial data $W_0 \in L^2(m)$ with $\|W_0\|_m \leq K_0$, equation (2.5) has a unique solution W in X which satisfies $\|W\|_X \leq K_1$. Applying this result to $W_0 = w(\frac{nT}{2})$ with $n \in \mathbf{N}$, we find that

$$\|\nabla w(\tau + \frac{nT}{2})\|_m \leq K_1\,, \quad \forall n \in \mathbf{N}\,, \quad \forall \tau \in (0,T]$$

which implies (2.25).

To prove existence and uniqueness of solutions of (2.5) in X, we use a fixed point theorem as in the proof of theorem 2.1.1. First note that $\tau \mapsto S(\tau,0)W_0 \in X$ since by proposition 4.3.1(a) with $|\alpha| = 0$ or $\frac{1}{2}$, $q=2$, $m>1$, there exists $C_1>0$ such that for any $\tau \geq 0$,

$$\|S(\tau,0)W_0\|_m \leq C_1 \|W_0\|_m$$

$$\|\nabla S(\tau,0)W_0\|_m \leq \frac{C_1}{a(\tau)^{\frac{1}{2}}} \|W_0\|_m\,.$$

Then, considering F defined in (2.8), we shall prove that F maps X into X and that there exists $C(T)>0$ such that $C(T)$ converges to zero as T goes to zero and

$$\|F(W)\|_X \leq C(T)\|W\|_X^2\,.$$

As we already saw, $\|F(W)\|_m \leq C\|W\|_m^2$ for any $\tau \geq 0$. Moreover,

$$\nabla F(W)(\tau) = \int_0^\tau \nabla S(\tau,\sigma) N(W)(\sigma)d\sigma\,,$$

and $\nabla S(\tau,\sigma)N(W)(\sigma) = \nabla S(\tau,\sigma)(W\cdot\nabla_\sigma V - V\cdot\nabla_\sigma W)(\sigma)$. Applying proposition 4.3.1(a) with $|\alpha|=1$, $q=\frac{3}{2}$ and $m>1$, there exists $\phi(\tau,\sigma)>0$ such that

$$\|\nabla S(\tau,\sigma)N(W)\|_m \leq \phi(\tau,\sigma)\left(\|b^m(V\cdot\nabla_\sigma)W\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} + \|b^m(W\cdot\nabla_\sigma)V\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))}\right)$$

where $\int_0^\tau \phi(\tau,\sigma)\frac{e^\sigma}{a(\sigma)^{\frac{1}{2}}}d\sigma$ is finite and goes to zero when τ goes to zero. Using, as usual, Hölder's inequality and the Biot-Savart law, we get

$$\begin{aligned}\|b^m(V\cdot\nabla_\sigma)W\|_{L^{\frac{3}{2}}} \le& C\|V\|_{L^6}\|b^m\nabla_\sigma W\|_{L^2}\\ \le& Ce^{\frac{\sigma}{2}}\|W\|_m\|\nabla W\|_m\,.\end{aligned}$$

To bound the second term $\|b^m(W\cdot\nabla_\sigma)V\|_{L^{3/2}}$, we need new estimates coming from the Biot-savart law. Therefore, we divide V as $\bar V+\tilde V$ and bound the two terms separately. By lemma 2.1 in [39], we first have for any $p\in(1,+\infty)$

$$\|\nabla\bar V(\tau)\|_{L^p(\mathbf{R}^2)}\le Ce^{\frac{\tau}{2}}\|\bar W(\tau)\|_{L^p(\mathbf{R}^2)}\,.$$

Moreover, using estimates (4.5), we get

$$\|\nabla\tilde V(\tau)\|_{L^6(\mathbf{R}^2\times(0,1))}\le C\|\nabla\tilde V(\tau)\|_{H^2(m)}\le C\|\tilde W(\tau)\|_{H^1(m)}\,.$$

Thus, the second term is bounded as follows

$$\begin{aligned}\|b^m(W\cdot\nabla_\sigma)V\|_{L^{\frac{3}{2}}} \le& C\|b^mW\|_{L^2(\mathbf{R}^2\times(0,1))}\|\nabla_\sigma V\|_{L^6(\mathbf{R}^2\times(0,1))}\\ \le& Ce^{\frac{\sigma}{2}}\|W\|_m\left(e^{\frac{\sigma}{2}}\|\bar W\|_{L^6(\mathbf{R}^2)}+\|\tilde W\|_{H^1(m)}\right)\\ \le& Ce^\sigma\|W\|_m\left(\|\bar W\|^{\frac{1}{3}}_{L^2(\mathbf{R}^2)}\|\nabla\bar W\|^{\frac{2}{3}}_{L^2(\mathbf{R}^2)}+\|\tilde W\|_m+\|\nabla\tilde W\|_m\right)\\ \le& Ce^\sigma\|W\|_m(\|W\|_m+\|\nabla W\|_m)\end{aligned}$$

Finally,

$$\|\nabla S(\tau,\sigma)N(W)(\sigma)\|_m\le Ce^\sigma\phi(\tau,\sigma)\|W(\sigma)\|\left(\|\nabla W(\sigma)\|_m+\|W(\sigma)\|_m\right)\,,$$

and $a(\tau)^{\frac{1}{2}}\nabla F(W)(\tau)\le C(T)\|W\|_X^2$ where $C(T)$ goes to zero as $T\to 0$. Choosing $T>0$ sufficiently small, a fixed point argument completes the proof. Property (2.25) is called the *regularizing effect.* ■

Since $m>1$, we decompose the solution w with the spectral projection P_0 defined in appendix 4.2.3 by (4.13):

$$w(\tau)=\alpha(\tau)\mathbf{G}+f(\xi,z,\tau)$$

where $\alpha(\tau)=\int_{\mathbf{R}^2\times(0,1)}w_3(\xi,z,\tau)dz\,d\xi$ so that $P_0f=0$. Then, as shown in lemma 2.2.1, $\dot\alpha(\tau)=0$ and f satisfies the following equation

$$\partial_\tau f=\Lambda(\tau)f+(Q_0+\tilde R)N(w)(\tau)\,,$$

which can be also written in its integral form

$$f(\tau)=S(\tau,0)f(0)+\int_0^\tau S(\tau,\sigma)(Q_0+\tilde R)N(w)(\sigma)d\sigma\equiv F_1(\tau)+F_2(\tau)\,,\qquad(2.26)$$

where the projection $\tilde{R}$ is defined in (4.9).

Lemma 2.4.3 *Let F_1 and F_2 be defined as in (2.26). There exist positive constants γ, C, K_2 such that*

$$\|F_1(\tau)\|_m \le Ce^{-\gamma\tau}\|w(0)\|_m\,, \quad \tau \ge 0 \tag{2.27}$$

$$\|F_2(\tau)\|_{m+1} \le K_2\,, \quad \tau \ge 0 \tag{2.28}$$

$$\|\nabla F_2(\tau)\|_{m+1} \le K_2\,, \quad \tau \ge 1 \tag{2.29}$$

Proof of lemma 2.4.3: Inequality (2.27) is an easy consequence of propositions 4.3.1 and 4.2.1. Indeed, $F_1(\tau) = e^{\tau\mathcal{L}}Q_0 w(0) + S(\tau,0)\tilde{R}w(0)$. The first term is bounded by proposition 4.2.1 with $n = 0$, $q = 2$ and $\alpha = 0$. The second one is bounded by proposition 4.3.1(b) with $\alpha = 0$ and $q = 2$. To prove (2.28), notice as in section 2.2 that for $i = 1,..,3$,

$$(Q_0 + \tilde{R})N(w)(\sigma)_i = \sum_{j=1}^{2} \partial_j \hat{M}_{ij} + e^{\frac{\sigma}{2}}\partial_z \hat{M}_{i3}$$

where for $(i,j) \in \{1,..,3\}^2$ and v^f the velocity corresponding to f

$$\hat{M}_{ij} = \alpha\left(\mathbf{G}_j v_i^f - \mathbf{G}_i v_j^f + f_j v_i^G - f_i v_j^G\right) + f_j v_i^f - f_i v_j^f\,.$$

Then, using proposition 4.3.1 for $q \in (\frac{6}{5}, 2]$, we get

$$\begin{aligned}\|F_2(\tau)\|_{m+1} \le{}& C\int_0^\tau \sum_{(i,j)\in I} \frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{1}{q}} a(e^\tau - e^\sigma)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{2})}} \|b^{m+1}\hat{M}_{ij}\|_{L^q} d\sigma \\ &+ C\int_0^\tau \sum_{i=1}^{3} \frac{e^{\frac{\sigma}{2}} e^{-4\pi^2(e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{q}-\frac{1}{2}} a(e^\tau - e^\sigma)^{\frac{1}{2}+\frac{1}{2}(\frac{1}{q}-\frac{1}{2})}} \|b^{m+1}\hat{M}_{i3}\|_{L^q} d\sigma\end{aligned}$$

where $I = \{1,2,3\} \times \{1,2\}$. Bounding $\|b^{m+1}\hat{M}_{ij}\|_{L^q}$ is slightly different from what we did before. Indeed, for any $(i,j) \in \{1,..,3\}^2$, using Hölder's inequality and the Biot-Savart law, we get for $q = \frac{3}{2}$

$$\begin{aligned}\|b^{m+1}\alpha\mathbf{G}_j v_i^f\|_{L^{\frac{3}{2}}(\mathbf{R}^2\times(0,1))} &\le C|\alpha|\|G\|_{m+1}\|v^f\|_{L^6(\mathbf{R}^2\times(0,1))} \\ &\le C|\alpha|\|G\|_{m+1}\left(\|\tilde{f}\|_{L^2(\mathbf{R}^2\times(0,1))} + \|\bar{f}\|_{L^{\frac{3}{2}}(\mathbf{R}^2)}\right) \\ &\le C|\alpha|\|G\|_{m+1}\|f\|_m \le C|\alpha|\|G\|_{m+1}K_0\,.\end{aligned}$$

Notice that $\tilde{f}$ and $\bar{f}$ are the usual notations defined in appendix 4.1. More easily, for $q = 2$,

$$\|b^{m+1}\alpha v_i^G f_j\|_{L^2(\mathbf{R}^2\times(0,1))} \le C|\alpha|\|f\|_m\|bv^G\|_{L^\infty} \le C|\alpha|\|bv^G\|_{L^\infty}K_0\,.$$

The last bound uses the idea that, since f has mean value zero, quantities like $v^f f$ decay a little bit faster at infinity than f itself. Using Hölder's inequality and splitting v^f

as in appendix A, we first get for $(i,j) \in \{1,..,3\}^2$

$$\begin{aligned}||b^{m+1} f_i v_j^f||_{L^q(\mathbf{R}^2\times(0,1))} &\le C||b^m f||_{L^2(\mathbf{R}^2\times(0,1))}||bv^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2\times(0,1))} \\ &\le C||f||_m \left(||b\bar{v}^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2)} + ||b\tilde{v}^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2\times(0,1))}\right).\end{aligned}$$

As far as the first part, independent of z, is concerned, we use proposition B.1 in [39]. For $q = \frac{2}{m}$ and $m \in (1,2)$, we get

$$\begin{aligned}||b\bar{v}^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2)} &\le ||b\bar{v}_3^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2)} + \left\|b\begin{pmatrix}\bar{v}_1^f\\ \bar{v}_2^f\end{pmatrix}\right\|_{L^{\frac{2q}{2-q}}(\mathbf{R}^2)} \\ &\le C\left(\left\|b^m\begin{pmatrix}\bar{f}_1\\ \bar{f}_2\end{pmatrix}\right\|_{L^2(\mathbf{R}^2)} + ||b^m\bar{f}_3||_{L^2(\mathbf{R}^2)}\right) \le C||f||_m\,.\end{aligned}$$

For m greater than 2, the embedding $L^2(m) \hookrightarrow L^2(m')$ for $m \le m'$ leads to the same conclusion.

For the second part, the Biot-Savart law 4.1.2 helps to conclude for $q \in (\frac{6}{5};\frac{3}{2}]$ and $m > 1$,

$$\begin{aligned}||b\tilde{v}^f||_{L^{\frac{2q}{2-q}}(\mathbf{R}^2\times(0,1))} &\le ||b\tilde{v}^f||_{H^1(\mathbf{R}^2\times(0,1))} \le C||\tilde{v}^f||_{H^1(1)} \\ &\le C||\tilde{f}||_{L^2(1)} \le C||\tilde{f}||_m \le C||f||_m\,.\end{aligned}$$

Finally, $||b^{m+1} f_i v_j^f||_{L^q(\mathbf{R}^2\times(0,1))} \le C||f||_m^2 \le CK_0^2$.

Collecting information from the last three steps where we used different values of q for the different terms and using appendix 4.5.2, we get inequality (2.28) for $\tau \ge 0$. Once more, (2.29) is due to the regularizing effect and we omit the details. ■

Corollary 2.4.4 *Let $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be as in theorem 2.4.1. Then, the trajectory $\{w(\tau)\}_{\tau\ge 0}$ is relatively compact in $L^2(m)$.*

Proof of corollary 2.4.4: We write, as in lemma 2.4.2, $w = \alpha\mathbf{G} + F_1 + F_2$. Observe that, by Rellich's criterion (see [74]), the embedding of $H^1(m+1)$ into $L^2(m)$ is compact. Since $F_1(\tau)$ converges to zero and $F_2(\tau)$ is bounded in $H^1(m+1)$ for $\tau \ge 1$, we conclude that $\{w(\tau)\}_{\tau\ge 0}$ is relatively compact in $L^2(m)$. ■

Let $m > 1$ and $w \in \mathcal{C}^0([0;+\infty); L^2(m))$ be a global solution of (2.5) that is uniformly bounded in time in $L^2(m)$. Let $\Omega \subset L^2(m)$ denote the ω-limit set of this trajectory. By corollary 2.4.4, we know that Ω is nonempty, compact and connected. By lemma 2.4.3, Ω is also bounded in $H^1(m+1)$. By a bootstrap argument, we see that Ω is bounded in $H^1(m')$ for all $m' > 1$.

Lemma 2.4.5 *Let $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be as in theorem 2.4.1. Let Ω be the ω-limit set of $\{w(\tau)\}_{\tau\ge 0}$. Then, the elements of Ω are independent of z. Namely, $\tilde{R}(\Omega) = \{0\}$.*

Proof of lemma 2.4.5: Let $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be a solution of (2.5) that is uniformly bounded in time in $L^2(m)$. We decompose w as in appendix 4.1:

$$w(\xi, z, \tau) = \bar{w}(\xi, \tau) + r(\xi, z, \tau)$$

where $\tilde{R}r = r$ for any $\tau \geq 0$. Then, r satisfies the evolution equation

$$\partial_\tau r = \Lambda(\tau) r + \tilde{R}(N(w)(\tau)) \,,\ \tau \geq 0$$

which can be written in the integral form:

$$r(\tau) = S(\tau, 0) r(0) + \int_0^\tau S(\tau, \sigma) \tilde{R}(N(w)(\sigma)) d\sigma \,,$$

see (2.14) and (2.16). Since for $i = 1, .., 3$,

$$\tilde{R}(N(w))_i = \sum_{j=1}^{2} \partial_j \tilde{R}(\mathcal{M}_{ij}) + e^{\frac{\sigma}{2}} \partial_z \mathcal{M}_{i3}$$

where $\mathcal{M}_{ij} = \bar{w}_j v_i^r - v_j^r \bar{w}_i + r_j \bar{v}_i - \bar{v}_j r_i + r_j v_i^r - v_j^r r_i$, we get by proposition 4.3.1,

$$\|r(\tau)\|_m \leq C e^{-4\pi^2 e^\tau} \|w_0\|_m + C \int_0^\tau \frac{e^{-\frac{\tau-\sigma}{2}} e^{-4\pi^2 (e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{2}{3}} a(e^\tau - e^\sigma)^{\frac{1}{12}}} + \frac{e^{\frac{\sigma}{2}} e^{-4\pi^2 (e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{6}} a(e^\tau - e^\sigma)^{\frac{7}{12}}} d\sigma$$

as $\|w(\sigma)\|_m$ is uniformly bounded in time. According to appendix 4.5.2 with $(\alpha, \beta, \gamma, \delta)$ equal to $(0, 0, \frac{2}{3}, \frac{1}{12})$ or $(\frac{1}{2}, 0, \frac{1}{6}, \frac{7}{12})$, this integral can be uniformly bounded in time and

$$\|r(\tau)\|_m \leq C e^{-\frac{\tau}{3}} \,, \quad \tau \geq 0 \,. \tag{2.30}$$

Obviously, $\lim_{\tau \to +\infty} \|w(\tau) - \bar{w}(\tau)\|_m = 0$ and the ω-limit set Ω is made of functions independent of z. This ends the proof of lemma 2.4.5. ■

Lemma 2.4.6 *Let $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be as in theorem 2.4.1. Let Ω be the ω-limit set of $\{w(\tau)\}_{\tau \geq 0}$. Then, Ω is totally invariant under the evolution defined by the autonomous system*

$$\partial_\tau \hat{w} = \mathcal{L} \hat{w} + (\hat{w} \cdot \nabla_\xi \hat{v} - \hat{v} \cdot \nabla_\xi \hat{w}) \,, \quad \tau \geq 0 \,, \tag{2.31}$$

where $\nabla_\xi = (\partial_{\xi_1}, \partial_{\xi_2}, 0)^T$, $\hat{w}(\xi, \tau) : \boldsymbol{R}^2 \times \boldsymbol{R}^+ \to \boldsymbol{R}^3$ and $\hat{v}$ is given in terms of $\hat{w}$ via the Biot-Savart law in appendix 4.1.

Remark: If the initial condition $\hat{w}_0$ is independent of z, this property is preserved by (2.31) and (2.31) is nothing but equation (2.3) applied to functions independent of z.

Proof of lemma 2.4.6: Denote $S_{2D}(\tau)$ the dynamical system associated with (2.31). Then, any solution $\hat{w}(\tau)$ of (2.31) with initial data $\hat{w}_0$ is given by $S_{2D}(\tau)\hat{w}_0$. We observe that

$$S_{2D}(\tau)\hat{w}_0 = e^{\tau \mathcal{L}} \hat{w}_0 + \int_0^\tau e^{(\tau-\sigma)\mathcal{L}} \hat{N}(S_{2D}(\sigma)\hat{w}_0) d\sigma \tag{2.32}$$

where $\hat{N}(\hat{w}) = (\hat{w}\cdot\nabla_\xi)\hat{v} - (\hat{v}\cdot\nabla_\xi)\hat{w}$. We devide this proof in two steps. First, we prove that Ω is positively invariant (i.e $S_{2D}(\tau)\Omega \subset \Omega$) and then, that Ω is included in $S_{2D}(\tau)\Omega$. Thus, we prove that Ω is totally invariant under the evolution defined by (2.31).

First step: Let $\hat{w}_0 \in \Omega$ and $T > 0$. We shall prove that $S_{2D}(\tau)\hat{w}_0 \in \Omega$ for $\tau \in [0,T]$. By lemma 2.4.5, $\hat{w}_0$ is independent of z and there exists $(\tau_n)_{n\in\mathbf{N}}$ such that $\lim_{n\to+\infty} \tau_n = +\infty$ and $\lim_{n\to+\infty} \|w(\tau_n) - \hat{w}_0\|_m = 0$. The idea is to show that $w(\tau+\tau_n)$ converges to $S_{2D}(\tau)\hat{w}_0$ as n goes to infinity. Therefore, we decompose $w(\tau+\tau_n)$ as in appendix 4.1,

$$w(\tau+\tau_n) = \bar{w}(\tau+\tau_n) + \tilde{w}(\tau+\tau_n) \equiv \bar{w}_n(\tau) + \tilde{w}_n(\tau)\,,$$

and we shall prove that

$$\|\tilde{w}_n(\tau)\|_m \le Ce^{-\frac{\tau+\tau_n}{3}}\,, \tag{2.33}$$

$$\|\bar{w}_n(\tau) - S_{2D}(\tau)\hat{w}_0\|_m \le C\epsilon(n) + C\int_0^\tau \|\bar{w}_n(\sigma) - S_{2D}(\sigma)\hat{w}_0\|_m d\sigma\,. \tag{2.34}$$

where $\epsilon(n)$ goes to zero when n goes to infinity. As easily verified, (2.33, 2.34) and Gronwall's lemma show that

$$\lim_{n\to+\infty} \|w(\tau+\tau_n) - S_{2D}(\tau)\hat{w}_0\|_m = 0\,.$$

Hence, $S_{2D}(\tau)\hat{w}_0 \in \Omega$ and Ω is positively invariant.

To prove (2.33, 2.34), notice that (2.33) is a consequence of (2.30) while (2.34) is obtained by combining two integral equations. Indeed, (2.32) and

$$\bar{w}_n(\tau) = e^{\tau\mathcal{L}}\bar{w}_n(0) + \int_0^\tau e^{(\tau-\sigma)\mathcal{L}}\bar{R}N(w)(\tau_n+\sigma)d\sigma$$

lead to the following equality

$$\begin{aligned}\bar{w}_n(\tau) - S_{2D}(\tau)\hat{w}_0 =& e^{\tau\mathcal{L}}(\bar{w}_n(0) - \hat{w}_0)\\ &+ \int_0^\tau e^{(\tau-\sigma)\mathcal{L}}\left(\bar{R}N(w)(\tau_n+\sigma) - \hat{N}(S_{2D}(\sigma)\hat{w}_0)\right)d\sigma\end{aligned}$$

where

$$\begin{aligned}\bar{R}N(w)(\tau_n+\sigma) &= N(\bar{w}_n)(\sigma) + \bar{R}N(\tilde{w}_n)(\sigma)\\ &= N(\bar{w}_n)(\sigma) + \left(\sum_{j=1}^2 \partial_j \int_0^1 \tilde{M}_{ij}dz\right)_{i=1,..,3}\end{aligned}$$

and $\tilde{M}_{ij} = \tilde{w}_j\tilde{v}_i - \tilde{v}_j\tilde{w}_i$. Using proposition 4.2.1, Jensen's and Hölder's inequalities and the Biot-Savart law, we get for $q \in (1,2)$

$$\begin{aligned}\|e^{(\tau-\sigma)\mathcal{L}}\bar{R}N(\tilde{w}_n)(\sigma)\|_m &\le C\frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{1}{q}}}\|\tilde{w}_n\|_m\|\tilde{v}_n\|_{L^{\frac{2q}{2-q}}(\mathbf{R}^2\times(0,1))}\\ &\le C\frac{e^{-\frac{\tau-\sigma}{2}}}{a(\tau-\sigma)^{\frac{1}{q}}}\|\tilde{w}_n(\sigma)\|_m^2\,.\end{aligned}$$

Finally,

$$\| \int_0^\tau e^{(\tau-\sigma)\mathcal{L}} \bar{R} N(\tilde{w}_n)(\sigma) d\sigma \|_m \leq C e^{-\frac{2\tau_n}{3}} .$$

Then,

$$\begin{aligned} \bar{w}_n(\tau) - S_{2D}(\tau)\hat{w}_0 = & \epsilon(n) + e^{\tau\mathcal{L}}(\bar{w}_n(0) - \hat{w}_0) \\ & + \int_0^\tau e^{(\tau-\sigma)\mathcal{L}} \left(N(\bar{w}_n)(\sigma) - \hat{N}(S_{2D}(\sigma)\hat{w}_0) \right) d\sigma \end{aligned}$$

where $\epsilon(n)$ converges to zero when n goes to infinity. Using proposition 4.2.1 to bound $e^{\tau\mathcal{L}}(\bar{w}_n(0) - \hat{w}_0)$ and inequality (2.9) to bound the integral term, we obtain (2.34).

Second step: Let $\hat{w}_0 \in \Omega$. There exists $(\tau_n)_{n\in\mathbf{N}}$ such that $\lim_{n\to+\infty} \tau_n = +\infty$ and $\lim_{n\to+\infty} \|w(\tau_n) - \hat{w}_0\|_m = 0$. Let $T > 0$. We should prove that $\hat{w}_0 \in S_{2D}(T)\Omega$. As $\{w(\tau)\}_{\tau\geq 0}$ is relatively compact, $(w(\tau_n - T))_{n\in\mathbf{N}}$ converges towards an element of Ω denoted $\hat{w}_{-T}$. According to the first step, $w(\tau_n)$ converges in $L^2(m)$ to $S_{2D}(T)\hat{w}_{-T}$. By the uniqueness of such a limit, $\hat{w}_0 = S_{2D}(T)\hat{w}_{-T} \in S_{2D}(T)\Omega$. This ends the proof of lemma 2.4.6. ■

Lemma 2.4.7 *Let $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be as in theorem 2.4.1. Let Ω be the ω-limit set of $\{w(\tau)\}_{\tau\geq 0}$. Then, $\Omega = \{\alpha \boldsymbol{G}\}$, $\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3(\xi, z, 0) d\xi dz$.*

Proof of lemma 2.4.7: Let $\hat{w}_0 \in \Omega$. As Ω is totally invariant under the evolution defined by (2.31), there exists a complete trajectory of (2.31) in Ω denoted $\{\hat{w}(\tau)\}_{\tau\in\mathbf{R}}$ satisfying $\hat{w}(0) = \hat{w}_0$. As Ω is compact, $\hat{w}(\tau)$ is uniformly bounded in time in $L^2(m)$ and by lemma 2.4.5, it is independent of z. As in appendix 4.1.1, the trajectory divides itself into two independent systems $(\hat{w}_3, \hat{v}_1, \hat{v}_2)$ and $(\hat{w}_1, \hat{w}_2, \hat{v}_3)$.

As far as the first system is concerned, $\{\hat{w}_3(\tau)\}$ is a complete trajectory bounded in $L^2_{2D}(m)$ for the evolution studied in [39]. Then, by lemma 3.3 in [41],

$$\hat{w}_3(\tau) = \alpha G \text{ and } (\hat{v}_1, \hat{v}_2) = \alpha(v_1^G, v_2^G)$$

where $\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3(\xi, z, 0) dz d\xi$. For the second system, we look at the velocity $\hat{v}_3$ which satisfies

$$\partial_\tau \hat{v}_3(\tau) + \left(v_1^G \partial_1 + v_2^G \partial_2\right) \hat{v}_3(\tau) = (\mathcal{L} - \frac{1}{2})\hat{v}_3(\tau) .$$

Since $\hat{w} \in L^2(m)$, $m > 1$, by lemma 4.1.5, $\hat{v}_3 \in L^2(\mathbf{R}^2)$ and

$$\frac{1}{2}\frac{d}{dt}\|\hat{v}_3\|^2_{L^2(\mathbf{R}^2)} = -\int_{\mathbf{R}^2} |\nabla \hat{v}_3|^2 d\xi \leq 0 .$$

Consider $\Phi(\hat{w}) = \int_{\mathbf{R}^2} |\hat{v}_3(\xi)|^2 d\xi$. Then, Φ is a Lyapunov function for the semiflow defined by (2.31). More precisely, Φ is strictly decreasing along the trajectories of (2.31), except along the subset $\{\hat{w} \in L^2(m) \,|\, \hat{v}_3 = 0\}$ where Φ is constant. By LaSalle's invariance principle, the ω-limit and the α-limit sets of this trajectory are contained in the set $\{\hat{w} \in L^2(m) \,|\, \hat{v}_3 = 0\}$. As Φ must be strictly decreasing or zero along the trajectories,

$$\|\hat{v}_3(\tau)\|_{L^2(\mathbf{R}^2)} = 0 , \quad \tau \in \mathbf{R} .$$

As a consequence, $\hat{v}_3$ and $\hat{w}_1$, $\hat{w}_2$ are zero and

$$\hat{w}(\tau) = \alpha \mathbf{G}\,,\ \tau \in \mathbf{R}\,.$$

This implies that $\hat{w}_0 = \alpha\mathbf{G}$ and concludes the proof of lemma 2.4.7. ■

Proof of theorem 2.4.1: Let $m > 1$ and $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ be a global solution of (2.5), uniformly bounded in time in $L^2(m)$. By corollary 2.4.4 and lemma 2.4.7, the trajectory $\{w(\tau)\}_{\tau\geq 0}$ is relatively compact and its ω-limit set Ω is reduced to $\{\alpha\mathbf{G}\}$ where $\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3(\xi, z, 0)d\xi dz$. This shows that

$$\lim_{\tau\to+\infty} \|w(\tau) - \alpha\mathbf{G}\|_m = 0\,.$$

This ends the proof of theorem 2.4.1. ■

2.5 Stress-free boundary conditions

The aim of this section is to rewrite the previous results in the case of stress-free boundary conditions. We consider the vorticity equation (2.3) together with initial condition $w(\xi, z, 0) = w_0(\xi, z)$ and boundary conditions (1.8). Therefore, we work in the weighted Lebesgue space $L^2_{sf}(m)$ defined by

$$L^2_{sf}(m) = \{f : \mathbf{R}^2 \times (0,1) \to \mathbf{R}^3 \,|\, \|f\|_m < \infty \text{ and } f \text{ satisfies (1.8)}\}\,, \tag{2.35}$$

where the norm $\|.\|_m$ is given in (2.4).

The main difference that occurs in this case is the splitting $u = \bar{u} + \tilde{u}$ defined in appendix 4.1. Indeed, $\bar{u}$ is here two-dimensional and horizontal and $\tilde{u}$ is no more of mean value zero in z, see (4.6).

Appendix 4.1.3 and 4.1.4 deal with Biot-Savart laws in this case and show that the same estimates hold as for periodic boundary conditions.

As far as appendix 4.2 is concerned, the spectrum of the linear operator $\mathcal{L}$ in $\bar{R}(L^2_{sf}(m))$ is studied in 4.2.5. It is shown that for $m > 1$, the discrete spectum of $\mathcal{L}$ in $\bar{R}(L^2_{sf}(m))$ consists of isolated eigenvalues $\lambda_k = -\frac{k}{2}$, $k \in \mathbf{N}$, $k < m-1$ with multiplicity $(k+1)$ and that the essential spectrum lies in the half plane $\{\lambda \in \mathbf{C} \,|\, \text{Re}\,(\lambda) \leq \frac{1-m}{2}\}$. Notice that the projection $\bar{R}$, defined in (4.9), has the same notation as in the periodic case but is slightly different.

Equipped with those results, we are now able to deal with the Cauchy problem (2.3)-(1.8) and the asymptotics of solutions.

1. The Cauchy Problem

Theorem 2.1.1 can be easily rewritten in $L^2_{sf}(m)$. In the Banach space X defined as

$$X = \{w \in \mathcal{C}^0([0,+\infty); L^2_{sf}(m)) \,|\, \text{div}_\tau w(\tau) = 0\,,\ \|w\|_X = \sup_{\tau\geq 0} \|w(\tau)\|_m < +\infty\}$$

estimates (2.9) still hold and the fixed point theorem leads to the following theorem:

Theorem 2.5.1 *Let $m > 1$. There exists $K_{0sf} > 0$ such that, for all initial data $w_0 \in L^2_{sf}(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K_{0sf}$, equation (2.3) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2_{sf}(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $div_\tau w(\tau) = 0$. In addition, there exists $K_{1sf} > 0$ such that*

$$\|w(\tau)\|_m \leq K_{1sf} \|w_0\|_m \, , \, \tau \geq 0 \, .$$

2. First-Order Asymptotics

In this section, we determine the first-order asymptotics of solution w given by theorem 2.5.1. Defining projections $\bar{R}$ and $\tilde{R}$ as in (4.9) and P_0, Q_0 as in (4.13, 4.15), we can easily decompose w as

$$\begin{aligned} w(\xi, z, \tau) &= P_0 w(\xi, z, \tau) + Q_0 w(\xi, z, \tau) + \tilde{R} w(\xi, z, \tau) \\ &= \alpha(\tau)\mathbf{G}(\xi) + q_0(\xi, \tau) + r(\xi, z, \tau) \end{aligned} \tag{2.36}$$

The following lemma shows that the conservation of mass still holds:

Lemma 2.5.2 *Assume $m > 1$ and $w \in \mathcal{C}^0([0,T); L^2_{sf}(m))$ is a solution of (2.3). Then, the coefficient α defined in (4.13) is constant in time.*

Proof: As in lemma 2.2.1, integrating by parts, we get

$$\begin{aligned} \dot{\alpha}(\tau) &= \int_{\mathbf{R}^2 \times (0,1)} \left(\mathcal{L} w_3 + e^\tau \partial_z^2 w_3 + N(w)_3\right) \, dz \, d\xi \\ &= \int \nabla_\xi \cdot \left(\nabla_\xi w_3 + \frac{1}{2}\xi w_3\right) + e^\tau \partial_z^2 w_3 - v_3 \mathrm{div}_\tau w(\tau) + w_3 \mathrm{div}_\tau v(\tau) dz d\xi \\ &= 0 \, . \end{aligned}$$

Indeed, $\partial_z w_3(\xi, 0, \tau) = \partial_z w_3(\xi, 1, \tau) = 0$, $\mathrm{div}_\tau w(\tau) = \mathrm{div}_\tau v(\tau) = 0$ and $\lim_{|\xi| \to +\infty} w_j v_i = 0$. ■

From now on, the proof of theorem 2.2.2 can be rewritten in the case of stress-free boundary conditions as the estimates satisfied by q_0 and r are the same as for periodic conditions. It is then straightforward that the following theorem on the first-order asymptotics of solutions holds.

Theorem 2.5.3 *Let $0 < \mu < \frac{1}{2}$ and $m > 1 + 2\mu$. There exists $K'_{0sf} > 0$ such that, for all initial data $w_0 \in L^2_{sf}(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K'_{0sf}$, equation (2.3) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2_{sf}(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $div_\tau w(\tau) = 0$. In addition, there exists $K_{2sf} > 0$ such that*

$$\|w(\tau) - \alpha \boldsymbol{G}\|_m \leq K_{2sf} \, e^{-\mu\tau} \|w_0\|_m \, , \quad \tau \geq 0 \, ,$$

where $\alpha = \int_{\boldsymbol{R}^2 \times (0,1)} (w_0)_3(\xi, z) \, dz \, d\xi$.

3. Second-Order Asymptotics

For the second-order asymptotics, we can once more rewrite the results of section 2.3. With the definitions (4.13, 4.15, 4.9) of appendix 4.2, we decompose any solution $w \in L^2_{sf}(m)$ given by theorem 2.5.1 as

$$\begin{aligned} w(\xi,z,\tau) &= P_1 w(\xi,z,\tau) + Q_1 w(\xi,z,\tau) + \tilde{R} w(\xi,z,\tau) \\ &= \alpha(\tau)\mathbf{G}(\xi) + \sum_{i=1}^{2} \beta_i(\tau)\mathbf{F}_i(\xi) + q_1(\xi,\tau) + r(\xi,z,\tau) \end{aligned}$$

Looking at solutions of finite energy, $\alpha = 0$ and $\dot{\beta}_i(\tau) = -\frac{1}{2}\beta_i(\tau)$ for $\tau \geq 0$ and $i = 1$ or 2. Therefore, the proof of theorem 2.3.2 can be easily rewritten in the case of stress-free boundary conditions and we get:

Theorem 2.5.4 *Let $\frac{1}{2} < \nu < 1$, $m > 1 + 2\nu$. There exists $K''_{0sf} > 0$ such that, for all initial data $w_0 \in L^2_{sf}(m)$ with div $w_0 = 0$, $\int_{\mathbf{R}^2\times(0,1)} w_0 d\xi dz = 0$ and $\|w_0\|_m \leq K''_{0sf}$, equation (2.3) has a unique global solution $w \in C^0([0,+\infty); L^2_{sf}(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $div_\tau w(\tau) = 0$, $\int_{\mathbf{R}^2\times(0,1)} w(\xi,z,\tau) d\xi dz = 0$. In addition, there exists $K_{3sf} > 0$ such that*

$$\|w(\tau) - \sum_{i=1}^{2} \beta_i \boldsymbol{F}_i e^{-\frac{\tau}{2}}\|_m \leq K_{3sf}\, e^{-\nu\tau} \|w_0\|_m\,, \quad \tau \geq 0\,,$$

where $\beta_i = \int_{\mathbf{R}^2\times(0,1)} \xi_i (w_0)_3(\xi,z)\, dz\, d\xi$ for $i = 1$ or 2.

Notice that in the above theorem, the sum $\sum_{i=1}^{2} \beta_i \mathbf{F}_i e^{-\frac{\tau}{2}}$ only occurs on $i = 1$ and 2. The vector F_3 does not appear in the asymptotics of solutions for stress-free boundary conditions. Therefore, the long-time behavior of the velocity is, in this case, two-dimensional and horizontal.

4. Global convergence

As far as global convergence towards Oseen vortices of uniformly bounded solutions of (2.3) is concerned, we can easily read the proof of theorem 2.4.1 with appropriate projections $\bar{R}$ and $\tilde{R}$ defined in (4.9). The same arguments hold as for periodic conditions. A simplification even occurs in lemma 2.4.7 since the decomposition of appendix 4.1.3 gives directly that $\hat{v}_3$, $\hat{w}_1$ and $\hat{w}_2$ are zero. A Lyapunov function is in this case useless (see also the remark at the end of appendix 4.1.5). We thus have the following theorem

Theorem 2.5.5 *Let $m > 1$ and $w \in C^0([0,+\infty); L^2_{sf}(m))$ be a global solution of (2.3) that is uniformly bounded in time in $L^2_{sf}(m)$. Then,*

$$\lim_{\tau\to+\infty} \|w(\tau) - \alpha\boldsymbol{G}\|_m = 0$$

where $\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3(\xi,z,0) dz d\xi$.

Chapitre 3

Rotating fluids

3.1 The Cauchy problem for rotating fluids

We describe in this section existence and uniqueness results for the vorticity equation for rotating fluids. Our approach is very similar to the one we used for the Navier-Stokes equation in a three-dimensional layer. Indeed, we deal with the vorticity ω and derive results on the velocity u as a corollary. We also use scaling variables (ξ, z, τ) defined in (2.1) to analyse the asymptotics of the Navier-Stokes Coriolis equation.

In $\mathbf{R}^2 \times (0,1)$, the vorticity equation for rotating fluids reads as (1.7)

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u + \Omega \partial_z u = \Delta \omega\,, \quad \text{div}\ \omega = 0$$

where $\omega = \omega(x, z, t) \in \mathbf{R}^3$, $(x, z, t) \in \mathbf{R}^2 \times (0,1) \times \mathbf{R}^+$ and the velocity field can be reconstructed from ω via the Biot-Savart law (see appendix 4.1). We supplement (1.7) with initial and boundary conditions:

$$\begin{aligned} \omega(x, z, 0) &= \omega_0(x, z)\,, \quad (x, z) \in \mathbf{R}^2 \times (0,1) \\ \omega(x, z+1, t) &= \omega(x, z, t)\,, \quad (x, z) \in \mathbf{R}^3\,,\ t \geq 0\,. \end{aligned}$$

Translating the latter equation with scaling variables (2.1), equation (1.7) reads in $\mathbf{R}^2 \times (0,1)$,

$$\begin{aligned} \partial_\tau w = \Lambda(\tau)w + N_1(w)(\tau) + N_2(w)(\tau)\,, &\quad \text{div}_\tau w(\tau) = 0\,, \\ w(\xi, z, 0) = w_0(\xi, z)\,, &\quad (\xi, z) \in \mathbf{R}^2 \times (0,1) \\ w(\xi, z+1, \tau) = w(\xi, z, \tau)\,, &\quad (\xi, z) \in \mathbf{R}^3\,,\ \tau \geq 0 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Lambda(\tau) &= \mathcal{L} + e^\tau \partial_z^2 \\ \mathcal{L} &= \Delta_\xi + \frac{1}{2}\xi \cdot \nabla_\xi + 1 \\ N_1(w)(\tau) &= (w \cdot \nabla_\tau)v - (v \cdot \nabla_\tau)w \\ N_2(w)(\tau) &= -\Omega e^{\frac{3\tau}{2}} \partial_z v \\ \text{div}_\tau w(\tau) &= \nabla_\tau \cdot w = \nabla_\xi \cdot w_\xi + e^{\frac{\tau}{2}} \partial_z w_z\,. \end{aligned}$$

Notice that N_1 is the classical bilinear term for the vorticity equation while N_2 is a linear term specific to rotating fluids.

As far as the Cauchy problem is concerned, we choose once more to work in weighted Lebesgue spaces $L^2(m)$ defined in (2.4) and to apply a fixed point theorem on the integral equation corresponding to (3.1) in those Hilbert spaces. Proceeding as in section 2.1 comes to a deadlock as we cannot bound properly the N_2 term. Indeed, in the proof of theorem 2.1.1, inequalities (2.9) have to be at least quadratic in w to lead to the conclusion. As N_2 is linear in w, we will get linear estimates on F and a smallness assumption on Ω will be required to conclude. As we do not want such a restriction, we deal slightly differently with the Coriolis term, including N_2 in the linear operator. Define for any $\tau \geq 0$ and any $w \in L^2(m)$, $m > 1$, the time-dependent operator

$$\mathcal{M}(\tau)w = \Lambda(\tau)w + N_2(w)(\tau)\,.$$

Then, it is shown in appendix 4.4 that $\mathcal{M}(\tau)$ is the generator of a family of evolution operators $\mathcal{S}(\tau,\sigma)$ in $L^2(m)$. Since $\partial_i\mathcal{S}(\tau,\sigma) = e^{\frac{\tau-\sigma}{2}}\mathcal{S}(\tau,\sigma)\partial_i$ for $i = 1$ or 2 and $\partial_z\mathcal{S}(\tau,\sigma) = \mathcal{S}(\tau,\sigma)\partial_z$, we get as in section 2.1 the integral equation corresponding to (3.1):

$$w_i(\tau) = \mathcal{S}(\tau,0)w_i(0) + \int_0^\tau \sum_{j=1}^{2} e^{-\frac{\tau-\sigma}{2}}\partial_j\mathcal{S}(\tau,\sigma)M_{ij}(\sigma) + e^{\frac{\sigma}{2}}\partial_z\mathcal{S}(\tau,\sigma)M_{i3}(\sigma)d\sigma \qquad (3.2)$$

where $i = 1,..,3$ and $M_{ij} = w_jv_i - v_jw_i$. Moreover, we prove in appendix 4.4 that $\mathcal{S}(\tau,\sigma)$ satisfies similar estimates to those proved on $S(\tau,\sigma)$. Then, applying a fixed point theorem to (3.2) follows the same lines as in theorem 2.1.1. The Cauchy problem for (3.2) is indeed solved by the following assertion:

Theorem 3.1.1 *Let $m > 1$. There exists $K_0 > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\|_m \leq K_0$, equation (3.2) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \geq 0$, $div_\tau w(\tau) = 0$. In addition, there exists $K_1 > 0$ such that*

$$\|w(\tau)\|_m \leq K_1\|w_0\|_m\,, \quad \tau \geq 0\,.$$

Proof: The proof is very similar to theorem 2.1.1's one replacing the evolution operator $S(\tau,\sigma)$ by $\mathcal{S}(\tau,\sigma)$. Given $w_0 \in L^2(m)$ with div $w_0 = 0$, we shall solve (3.2) in the Banach space

$$X = \{w \in \mathcal{C}^0([0,+\infty); L^2(m)) \,|\, \mathrm{div}_\tau w(\tau) = 0,\, \|w\|_X = \sup_{\tau\geq 0}\|w(\tau)\|_m < \infty\}.$$

We first note that $\tau \mapsto \mathcal{S}(\tau,0)w_0 \in X$, since by propositions 4.2.1 and 4.4.1, there exists $C_1 > 0$ such that for any $\tau \geq 0$

$$\begin{aligned}\|\mathcal{S}(\tau,0)w_0\|_m &\leq \|e^{\tau\mathcal{L}}\bar{w}_0\|_m + \|\mathcal{S}(\tau,0)\tilde{w}_0\|_m \\ &\leq C_1\|\bar{w}_0\|_m + C_1e^{-4\pi^2e^\tau}\|\tilde{w}_0\|_m \leq C_1\|w_0\|_m\,,\end{aligned}$$

where $w_0 = \bar{w}_0 + \tilde{w}_0$ is the usual decomposition introduced in appendix 4.1. Next, given $w \in \mathcal{C}^0([0,+\infty); L^2(m))$, we define $F(w) \in \mathcal{C}^0([0,+\infty); L^2(m))$ coordinate by coordinate. For $i = 1,..,3$,

$$F_i(w)(\tau) = \int_0^\tau \sum_{j=1}^{2} e^{-\frac{\tau-\sigma}{2}} \partial_j \mathcal{S}(\tau,\sigma) M_{ij}(\sigma) + e^{\frac{\sigma}{2}} \partial_z \mathcal{S}(\tau,\sigma) M_{i3}(\sigma) d\sigma\,, \quad \tau \geq 0.$$

Dividing M_{ij} into $\bar{M}_{ij} + \tilde{M}_{ij}$, we bound $\|F(w)\|_X$ and $\|F(w) - F(w')\|_X$ as in theorem 2.1.1 using appendix 4.2.4 and 4.4. Then, inequalities such as (2.9) and a fixed point theorem end the proof. ■

We can once more translate this result into original variables:

Corollary 3.1.2 *Let $m > 1$. There exists $\epsilon_0 > 0$ such that for all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$ and $\|\omega_0\|_m \leq \epsilon_0$, equation (1.7) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and div $\omega = 0$. In addition, for any $p \in [1,2]$, there exists $\epsilon_1 > 0$ such that*

$$\|\omega(t)\|_{L^p(\boldsymbol{R}^2\times(0,1))} \leq \frac{\epsilon_1}{(1+t)^{1-\frac{1}{p}}}\,,\ t \geq 0\,.$$

3.2 First-Order Asymptotics

Trying once more to generalise the previous results on the Navier-Stokes equation to rotating fluids, we consider in this section the asymptotic behavior of solutions of (3.2) given by theorem 3.1.1 . As shown in section 2.2, the asymptotics of the Navier-Stokes equation in a three-dimensional layer are governed by the two-dimensional Navier-Stokes equation. Here we prove that the Coriolis term does not play any role for the first-order asymptotics, namely that solutions of equations (2.5) and (3.2) for rotating or not fluids behave in the same way when looking at the first-order asymptotics.

Using projectors P_0, Q_0 and $\tilde{R}$ defined in appendix 4.2 by (4.13), (4.15) and (4.9), we decompose the solution w of (3.2) as

$$\begin{aligned} w(\xi,z,\tau) &= \left(P_0 w + Q_0 w + \tilde{R} w\right)(\xi,z,\tau) \\ &= \alpha(\tau)\mathbf{G}(\xi) + q_0(\xi,\tau) + r(\xi,z,\tau)\,. \end{aligned}$$

Since w and v are 1-periodic in z, these parameters satisfy the following equations for any $\tau > 0$:

$$\begin{aligned} \dot{\alpha}(\tau) &= 0\,, \\ \partial_\tau q_0 &= \mathcal{L} q_0 + Q_0(N_1(w))\,, \qquad \text{div } q_0 = 0\,, \\ \partial_\tau r &= \mathcal{M}(\tau) r + \tilde{R}(N_1(w))\,, \quad \text{div}_\tau r(\tau) = 0\,. \end{aligned} \tag{3.3}$$

According to appendix 4.4, $\mathcal{M}(\tau)$ generates a family of evolution operators $\mathcal{S}(\tau,\sigma)$ in $\tilde{R}(L^2(m))$ which satisfies similar estimates to those proved in proposition 4.3.1(b) for

$S(\tau,\sigma)$ in $\tilde{R}(L^2(m))$. Therefore, the previous arguments developped in section 2.2 still hold for equations (3.3) and we easily get the following theorem.

Theorem 3.2.1 *Let $0 < \mu < 1/2$ and $m > 1 + 2\mu$. There exists $K_0' > 0$ such that, for all initial data $w_0 \in L^2(m)$ with div $w_0 = 0$ and $\|w_0\| \le K_0'$, equation (3.2) has a unique global solution $w \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $w(0) = w_0$ and for any $\tau \ge 0$, $div_\tau w(\tau) = 0$. In addition, there exists $K_2 > 0$ such that*

$$\|w(\tau) - \alpha \boldsymbol{G}\|_m \le K_2 e^{-\mu\tau} \|w_0\|_m\,, \quad \tau \ge 0\,,$$

where $\alpha = \int_{\boldsymbol{R}^2\times(0,1)} (w_0)_3(\xi,z)dzd\xi$.

This theorem can be also translated into original variables and we get a new corollary, similar to corollary 2.2.3.

Corollary 3.2.2 *Let $0 < \mu < 1/2$ and $m > 1 + 2\mu$. There exists $\epsilon_0' > 0$ such that for all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$ and $\|\omega_0\|_m \le \epsilon_0'$, equation (1.7) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and div $\omega = 0$. In addition, for any $p \in [1,2]$, there exists $\epsilon_2 > 0$ such that*

$$\|\omega(t) - \omega_{app}(t)\|_{L^p(\boldsymbol{R}^2\times(0,1))} \le \frac{\epsilon_2}{(1+t)^{1-\frac{1}{p}+\mu}} \|\omega_0\|_m\,, \quad t \ge 0\,,$$

where $\omega_{app}(x,z,t) = \frac{\alpha}{1+t}\boldsymbol{G}\left(\frac{x}{\sqrt{1+t}}\right)$ and $\alpha = \int_{\boldsymbol{R}^2\times(0,1)}(\omega_0)_3 dzdx$.

3.3 Higher-Order Asymptotics

From previous section 3.2 "First-Order Asymptotics", it is clear that all results on the asymptotics of the Navier-Stokes equation can be generalised to the Navier-Stokes equation for rotating fluids. The Coriolis term N_2 does not play any role in the long-time behavior of the vorticity when computing the asymptotics at any finite order. Therefore, section 2.3 could also be generalised and a more precise study of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $L^2(m)$ would give the asymptotics of (3.2) at any finite order $n \in \mathbf{N}$ in $e^{-n\tau/2}$ for w or $(1+t)^{-(1+n/2)}$ for ω.

However, it would be false to believe that rotating fluids behave exactly as non-rotating ones. The Coriolis term can be taken into account at a higher rate and has a stabilisation power. Indeed, we consider in this section an initial condition of mean-value zero in z, namely $w_0 \in \tilde{R}(L^2(m))$. Since this property is preserved under the evolution (3.1), the solution w is at any time of mean-value zero in z and the previous computations on the long-time asymptotics of solutions show that w decays to zero as τ goes to infinity faster than any exponential rate $e^{-n\tau/2}$, $n \in \mathbf{N}$. Put differently, ω decays to zero as t goes to infinity faster than any polynomial rate $(1+t)^{-(1+n/2)}$.

In this section, we compute the asymptotics of the linearised equation of (3.1) around the origin at an infinite order, namely in e^{-e^τ} for w or $\frac{e^{-t}}{1+t}$ for ω. Of course, the right procedure would have been to compute the asymptotics of ω, solution of the full non-linear equation (3.1). However, at the time of typing this manuscript, the contribution of the non-linear terms was not yet completely understood. Therefore, we only consider the linear equation

$$\partial_t \omega = \mathcal{M}\omega\,, \quad t > 0\,, \quad x \in \mathbf{R}^2 \times (0,1)\,, \tag{3.4}$$

where $\mathcal{M}$ is defined in (4.20), together with the initial and boundary conditions

$$\begin{aligned} \omega(x,z,0) &= \omega_0(x,z)\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1) \\ \omega(x,z+1,t) &= \omega(x,z,t)\,, \quad (x,z) \in \mathbf{R}^3\,, \quad t \geq 0 \end{aligned}$$

where $\omega_0 \in \tilde{R}(L^2(m))$. We proved in appendix 4.4 that $\mathcal{M}$ is the generator of a linear semi-group $e^{t\mathcal{M}}$. Consequently, for any $\omega_0 \in \tilde{R}(L^2(m))$, there exists a unique global solution $\omega(t) = e^{t\mathcal{M}}\omega_0$ to (3.4). Then, we show that

$$\omega(x,z,t) \sim \frac{e^{-4\pi^2 t}}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) R_{\Omega t}\left(\int_{\mathbf{R}^2\times(0,1)} \omega_0(y,z')\cos(2\pi(z-z'))dydz'\right)$$

as t goes to infinity, where $R_{\Omega t}$ is a rotation of axis (Oz) and angle Ωt. Namely,

$$R_{\Omega t} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{3.5}$$

Notice that we obtain some similarities with the first order asymptotics. The vortex still appears as $\frac{1}{1+t}G\left(\frac{x}{\sqrt{1+t}}\right)$ but the coefficient $\alpha = \int_{\mathbf{R}^2\times(0,1)}(\omega_0)_3 dxdz$ has been replaced by $R_{\Omega t}(\int_{\mathbf{R}^2\times(0,1)}\omega_0(y,z')\cos(2\pi(z-z'))dz'dy)$ which depends on z. Let us now make our ideas more precise.

Theorem 3.3.1 *Let $m > 1$. For all initial data $\omega_0 \in L^2(m)$ with div $\omega_0 = 0$ and $\int_0^1 \omega_0 dz = 0$, equation (3.4) has a unique global solution $\omega \in \mathcal{C}^0([0,+\infty); L^2(m))$ satisfying $\omega(0) = \omega_0$ and div $\omega(t) = 0$. In addition, $\int_0^1 \omega(t)dz = 0$ and for any $p \in [1,2]$, there exists $C > 0$ such that*

$$\|\omega(t) - \omega_\infty(t)\|_{L^p(\boldsymbol{R}^2\times(0,1))} \leq C\frac{e^{-4\pi^2 t}}{(1+t)^{\frac{3}{2}-\frac{1}{p}}}\,, \quad t \geq 0\,,$$

where

$$\omega_\infty(x,z,t) = \frac{e^{-4\pi^2 t}}{1+t} G\left(\frac{x}{\sqrt{1+t}}\right) R_{\Omega t}\left(\int_{\boldsymbol{R}^2\times(0,1)} \omega_0(y,z')\cos(2\pi(z-z'))dydz'\right)\,.$$

Proof: We work for this proof in original variables. For any initial condition $\omega_0 \in \tilde{R}(L^2(m))$, the global solution $\omega(t)$ of (3.4) is given by

$$\omega(t) = e^{t\mathcal{M}}\omega_0\,, \quad t \geq 0\,,$$

where $\mathcal{M}$ is defined in (4.20). Then,

$$\omega(t) - \omega_\infty(t) = e^{t\mathcal{M}}\omega_0 - \omega_\infty(t) \equiv D(t)\,,.$$

The aim of this proof is thus to bound $D(t)$ in the L^p norm using Fourier transformation. Notice that by a change of variables and the embedding $L^2(m) \hookrightarrow L^p(\mathbf{R}^2 \times (0,1))$ for $p \in [1,2]$, $m > 1$, we have

$$\begin{aligned}\|D(t)\|_{L^p(\mathbf{R}^2\times(0,1))} &= \frac{1}{(1+t)^{-\frac{1}{p}}}\|D(x\sqrt{1+t},z,t)\|_{L^p(\mathbf{R}^2\times(0,1))}\\ &\leq \frac{1}{(1+t)^{-\frac{1}{p}}}\|D(x\sqrt{1+t},z,t)\|_m\,.\end{aligned}$$

Thus, it is sufficient to compute $\|D(x\sqrt{1+t},z,t)\|_m$. Using the equivalent norm with Fourier coordinates, we write

$$\|D(x\sqrt{1+t},z,t)\|_m \leq \frac{1}{1+t}\left(\sum_{n\in\mathbf{Z}^*}\int_{\mathbf{R}^2}\sum_{|\alpha|\leq m}\left|\partial^\alpha D_n\left(\frac{k}{\sqrt{1+t}},t\right)\right|^2 dk\right)^{\frac{1}{2}},$$

and all we have to compute is $D_n(k/\sqrt{1+t},t)$ for any $n \in \mathbf{Z}^*$, $k \in \mathbf{R}^2$ and $t \geq 0$. First of all,

$$D_n(k,t) = e^{-tA_n^\Omega(k)}\omega_{0n}(k) - \omega_{\infty n}(k,t)$$

where $A_n^\Omega(k)$ is defined in appendix 4.4 by (4.22). Moreover,

$$\omega_{\infty n}(k,t) = e^{-4\pi^2 t}e^{-(1+t)|k|^2}R_{\Omega t}\left(\int_0^1 e^{-2i\pi nz}\int_{\mathbf{R}^2\times(0,1)}\omega_0(y,z')\cos(2\pi(z-z'))dydz'dz\right).$$

Then, ω_∞ has a very simple Fourier development. Indeed,

$$\begin{aligned}\omega_{\infty 1}(k,t) &= e^{-4\pi^2 t}e^{-(1+t)|k|^2}R_{\Omega t}\left(\omega_{01}(0)\right)\\ \omega_{\infty -1}(k,t) &= e^{-4\pi^2 t}e^{-(1+t)|k|^2}R_{\Omega t}\left(\omega_{0-1}(0)\right)\\ \omega_{\infty n}(k,t) &= 0 \text{ for } |n| > 1\,.\end{aligned}$$

On the other hand,

$$e^{-tA_n^\Omega(k)}\omega_{0n}(k) = e^{-t|k|^2}e^{-4\pi^2 n^2 t}f_n(k,t)\omega_{0n}(k)$$

where $f_n(k,t) = \exp\left(\frac{2i\pi n\Omega t}{|k|^2+4\pi^2 n^2}A_n(k)\right)$, $k \in \mathbf{R}^2$ and $A_n(k)$ is defined in appendix 4.4 by (4.21). Thus,

$$D_n\left(\frac{k}{\sqrt{1+t}}\right) = e^{-\frac{t}{1+t}|k|^2}e^{-4\pi^2 n^2 t}f_n\left(\frac{k}{\sqrt{1+t}},t\right)\omega_{0n}\left(\frac{k}{\sqrt{1+t}}\right) - \omega_{\infty n}\left(\frac{k}{\sqrt{1+t}}\right).$$

To expand the asymptotics of $D_n\left(\frac{k}{\sqrt{1+t}}\right)$ in $1/\sqrt{1+t}$, we use Taylor's formula and

$$\omega_{0n}\left(\frac{k}{\sqrt{1+t}}\right) = \omega_{0n}(0) + O\left(\frac{|k|}{\sqrt{1+t}}\right)$$
$$f_n\left(\frac{k}{\sqrt{1+t}}, t\right) = f_n(0,t) + O\left(\frac{|k|}{\sqrt{1+t}}\right) = R_{\Omega t} + O\left(\frac{|k|}{\sqrt{1+t}}\right).$$

Thus, we get

$$D_1\left(\frac{k}{\sqrt{1+t}}\right) = O\left(\frac{|k|e^{-|k|^2}e^{-4\pi^2 t}}{\sqrt{1+t}}\right)$$
$$D_{-1}\left(\frac{k}{\sqrt{1+t}}\right) = O\left(\frac{|k|e^{-|k|^2}e^{-4\pi^2 t}}{\sqrt{1+t}}\right)$$
$$D_n\left(\frac{k}{\sqrt{1+t}}\right) = O(e^{-4\pi^2 n^2 t}) \text{ for } |n| > 1\,,$$

and replacing those bounds in the above norm of $D(x\sqrt{1+t}, z, t)$, we get

$$\|D(x\sqrt{1+t}, z, t)\|_m \leq C\frac{e^{-4\pi^2 t}}{(1+t)^{\frac{3}{2}}}\,,$$

and the proof of proposition 3.3.1 is complete. ■

Remark: Define $\gamma_n(t) = \int_{\mathbf{R}^2\times(0,1)} \omega(x,z,t)e^{-2i\pi nz}dxdz$. Then, for any $n \in \mathbf{Z}^*$, $\gamma_n(t) = \omega_n(0,t)$ and it is an easy computation to see, using the Biot-Savart law (4.5) that for any $t \geq 0$

$$\dot{\gamma}_n(t) = -A_n^{\Omega}(0)\gamma_n(t)\,.$$

In particular, $\gamma_i(t) = e^{-4\pi^2 t}R_{\Omega t}(\gamma_i(0))$ for $i = \pm 1$. Thus, $\omega_\infty(t)$ can be seen as

$$\frac{1}{1+t}G\left(\frac{x}{\sqrt{1+t}}\right)\left(\gamma_1(t)e^{2i\pi z} + \gamma_{-1}(t)e^{-2i\pi z}\right)\,.$$

3.4 High rotation limit

We want to give in this section some credit to the previous observations on global convergence. We studied in section 2.4 the convergence towards Oseen vortices of global solutions of equation (2.3) that are uniformly bounded in time. We assumed the existence of such solutions by claiming that $w = \alpha\mathbf{G}$ is one of these. However, we were not able to prove the existence of global solutions of (2.3) that are uniformly bounded in time. The aim of this section is precisely to make up for this lack. We show that, in the particular case of equation (1.3) for rotating fluids, there exist global solutions uniformly bounded in time when the rotation Ω is large enough.

Introduction

The methods used in this section are quite different from what we did so far. Following J.Y Chemin, B. Desjardins, I. Gallagher and E. Grenier in [26], we consider the velocity formulation to study the global well-posedness of the Navier-Stokes equation for rotating fluids in a three-dimensional layer. We recall that the velocity $u = (u_1, u_2, u_3)$ is given by

$$\begin{aligned} &\partial_t u - \nu \Delta u + P((u \cdot \nabla) u) + \Omega P(u \wedge e_3) = 0 \\ &\text{div } u = 0 \\ &u(x, z, 0) = u_0(x, z) \end{aligned} \tag{3.6}$$

where the horizontal variables are denoted by $x = (x_1, x_2) \in \mathbf{R}^2$ and the vertical coordinate by $z \in (0,1)$, ν is the kinematic viscosity, Ω the rotation speed, $e_3 = (0,0,1)^T$ and P the Leray's projector onto divergence free vector fields. We also consider periodic boundary conditions

$$u(x, z+1, t) = u(x, z, t)\,, \quad (x, z) \in \mathbf{R}^3\,,\ t \geq 0\,.$$

We prove here that for Ω large enough, equation (3.6) is globally well-posed in $\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$ where the homogeneous Sobolev spaces in a three-dimensional layer $\dot{H}^s(\mathbf{R}^2 \times (0,1))$ are defined in the following way:

Definition 3.4.1 *For any $s \in \boldsymbol{R}$, let $\dot{H}^s(\boldsymbol{R}^2 \times (0,1))$ be the closure of $\mathcal{D}_1$ for the norm $\|.\|_{\dot{H}^s(\boldsymbol{R}^2 \times (0,1))}$. We define $\mathcal{D}_1$ as the set of all functions $\phi \in \mathcal{C}(\boldsymbol{R}^3)$, 1-periodic in z and whose support in $x \in \boldsymbol{R}^2$ is compact and independent of z. Furthermore, the $\|.\|_{\dot{H}^s(\boldsymbol{R}^2 \times (0,1))}$ norm can be defined with Fourier coordinates by*

$$\|\phi\|_{\dot{H}^s(\boldsymbol{R}^2 \times (0,1))} = \left(\int_{\boldsymbol{R}^2} \sum_{n \in \boldsymbol{Z}} (|k|^2 + 4\pi^2 n^2)^s |\phi_n(k)|^2 dk \right)^{\frac{1}{2}}$$

where our conventions for Fourier transform in $\boldsymbol{R}^2 \times (0,1)$ are given in (4.4).

We can define, in a similar way, non-homogeneous Sobolev spaces $H^s(\mathbf{R}^2 \times (0,1))$. Then, $\dot{H}^s(\mathbf{R}^2 \times (0,1))$ and $H^s(\mathbf{R}^2 \times (0,1))$ are Banach spaces for $s < 3/2$ and $H^s(\mathbf{R}^2 \times (0,1))$ satifies the same Sobolev embeddings as the usual non-homogeneous Sobolev space $H^s(\mathbf{R}^3)$ in the whole space. In particular,

$$H^s(\mathbf{R}^2 \times (0,1)) \hookrightarrow L^p(\mathbf{R}^2 \times (0,1))\,, \text{ where } 2 \leq p \leq \frac{6}{3-2s}\,.$$

Moreover, $\dot{H}^s(\mathbf{R}^2 \times (0,1))$ satisfies the following properties:

$$u \in \dot{H}^s(\mathbf{R}^2 \times (0,1)) \Leftrightarrow \bar{u} \in \dot{H}^s(\mathbf{R}^2) \text{ and } \tilde{u} \in H^s(\mathbf{R}^2 \times (0,1))$$

where $\bar{u}$ and $\tilde{u}$ are defined in appendix 4.1. Since, $\dot{H}^s(\mathbf{R}^2) \hookrightarrow L^p(\mathbf{R}^2)$ for $p = 2/(1-s)$, the homogeneous Sobolev space $\dot{H}^s(\mathbf{R}^2 \times (0,1))$ does not satify any Sobolev embeddings in $L^p(\mathbf{R}^2 \times (0,1))$. For simplicity, we often write $\dot{H}^s$ in place of $\dot{H}^s(\mathbf{R}^2 \times (0,1))$ where no confusion is possible.

Since equation (3.6) has the same energy estimate in $\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$ as the three-dimensional Navier-Stokes equation (1.2), there is no hope to prove any global well-posedness results with such estimates. Indeed, it is well-known that energy estimates in $\dot{H}^{\frac{1}{2}}$ prove well-posedness of (1.2) only for small initial data and that such results for general initial conditions remain, at the time of writing, an open problem. However, we expect, as in the previous sections, the velocity u to converge towards $\bar{u} = (\bar{u}_1(x), \bar{u}_2(x), \bar{u}_3(x))$ solution of the two-dimensional Navier-Stokes equation

$$\begin{aligned} &\partial_t \bar{u} - \nu\Delta_x \bar{u} + P(\bar{u}\cdot\nabla_x \bar{u}) = 0\,, \quad x \in \mathbf{R}^2\,, t > 0 \\ &\operatorname{div}\, \bar{u} = 0 \\ &\bar{u}(x,0) = \bar{u}_0(x)\,, \quad x \in \mathbf{R}^2\,. \end{aligned} \tag{3.7}$$

Therefore, it would be meaningful to compute energy estimates on the difference $u - \bar{u}$ and to prove that it goes to zero as Ω goes to infinity. However, this energy estimate is inadequate since it does not contain the rotation term anymore. Indeed, without any rotation, we are not able to prove any convergence results. We thus have to take benefit from the rotation through another estimate.

The idea developped in [26] is to look at the difference $u - \bar{u} - w_F$ where w_F is the solution of the free linear rotating equation

$$\begin{aligned} &\partial_t w_F - \nu\Delta w_F + \Omega P(w_F \wedge e_3) = 0\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1)\,, t > 0 \\ &\operatorname{div}\, w_F = 0 \\ &w_F(x,z,0) = \tilde{u}_0(x,z)\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1)\,, \end{aligned} \tag{3.8}$$

where $w_F : \mathbf{R}^2 \times (0,1) \to \mathbf{R}^3$ is 1-periodic in z. Then, the equation satisfied by $u - \bar{u} - w_F$ is a three-dimensional Navier-Stokes equation with additional linear and bilinear force terms. Moreover, if we cut high frequencies of w_F, we obtain a regular function w which satisfies Strichartz' estimates and $u - \bar{u} - w$ verifies a three-dimensional Navier-Stokes equation whose initial data and force terms are small. In this configuration, a proof similar to Fujita-Kato's one (see [34]) gives the global well-posedness of $u - \bar{u} - w$. Well-known results on equations (3.7) and (3.8) lead finally to the global well-posedness of (3.6). Precisely, we have the following theorem:

Theorem 3.4.2 *Let $u_0 \in H^{\frac{1}{2}}(\boldsymbol{R}^2 \times (0,1))$. There exists $\Omega_0 > 0$ such that for all $\Omega \in \boldsymbol{R}$ with $|\Omega| \geq \Omega_0$, equation (3.6) has a unique global solution*

$$u \in \mathcal{C}^0(\boldsymbol{R}^+, \dot{H}^{\frac{1}{2}}(\boldsymbol{R}^2 \times (0,1))) \cap L^2(\boldsymbol{R}^+, \dot{H}^{\frac{3}{2}}(\boldsymbol{R}^2 \times (0,1)))$$

with initial data u_0.

Before stating the proof of theorem 3.4.2, we first recall some results on equations (3.7) and (3.8). We also compute dispersion and Strichartz' estimates for (3.8).

1. The 2D Navier-Stokes and the free linear rotating equations

If $\bar{u}_0 \in L^2(\mathbf{R}^2)$, equation (3.7) has a unique global solution $\bar{u} \in \mathcal{C}^0(\mathbf{R}^+, L^2(\mathbf{R}^2)) \cap L^2(\mathbf{R}^+, \dot{H}^1(\mathbf{R}^2))$ which satisfies the energy equality:

$$\|\bar{u}(t)\|^2_{L^2(\mathbf{R}^2)} + 2\nu \int_0^t \|\nabla \bar{u}(s)\|^2_{L^2(\mathbf{R}^2)} ds = \|\bar{u}_0\|^2_{L^2(\mathbf{R}^2)}\,, \quad t \geq 0\,. \tag{3.9}$$

As far as equation (3.8) is concerned, it is easy to compute the solutions in Fourier variables. Indeed, Leray's projector P is given for any function f by

$$(Pf)_n(k) = f_n(k) - \frac{1}{|k|^2 + 4\pi^2 n^2}\left(\begin{pmatrix} k \\ n \end{pmatrix} \cdot f_n(k)\right)\begin{pmatrix} k \\ n \end{pmatrix}$$

where $(k, n) = (k_1, k_2, 2\pi n)$. Then, equation (3.8) reads for any $n \in \mathbf{Z}$, $k \in \mathbf{R}^2$,

$$\begin{aligned} \partial_t w_{Fn}(k,t) &= -A_n^\Omega(k) w_{Fn}(k,t) \\ w_{Fn}(k,0) &= (\tilde{u}_0)_n(k)\,, \begin{pmatrix} k \\ n \end{pmatrix} \cdot w_{Fn} = 0\,. \end{aligned}$$

where

$$A_n^\Omega(k) = \nu(|k|^2 + 4\pi^2 n^2) I_3 + \frac{2i\pi n \Omega}{|k|^2 + 4\pi^2 n^2}\begin{pmatrix} 0 & -2i\pi n & ik_2 \\ 2i\pi n & 0 & -ik_1 \\ -ik_2 & ik_1 & 0 \end{pmatrix}. \tag{3.10}$$

Then, $w_{Fn}(k,t) = e^{-tA_n^\Omega(k)}(\tilde{u}_0)_n(k)$ and we can state that if $\tilde{u}_0 \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$, equation (3.8) has a unique global solution $w_F \in \mathcal{C}^0(\mathbf{R}^+, \dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))) \cap L^2(\mathbf{R}^+, \dot{H}^{\frac{3}{2}}(\mathbf{R}^2 \times (0,1)))$. Moreover, w_F satisfies the energy estimate:

$$\|w_F(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))} + 2\nu \int_0^t \|\nabla w_F(s)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))} ds \leq \|\tilde{u}_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))}\,, \; t \geq 0\,. \tag{3.11}$$

As we explained before, w_F is of importance since it contains the influence of the rotation. Indeed, if we restrict w_F to low frequencies, it satisfies Strichartz' and dispersion estimates we will study now.

2. Strichartz' and dispersion estimates

Proposition 3.4.3 *For any $R > 0$, we denote by $\mathcal{C}_R$ the ball*

$$\mathcal{C}_R = \{(k,n) \in \boldsymbol{R}^2 \times \boldsymbol{Z} \,|\, \sqrt{|k|^2 + 4\pi^2 n^2} \leq R\}\,.$$

For any $R > 0$, there exists a positive constant $C_R > 0$ such that if $w_0 \in L^2(\boldsymbol{R}^2 \times (0,1))$ satisfies $\int_0^1 w_0 dz = 0$ and $supp_{(k,n)}(w_0)_n(k) \subset \mathcal{C}_R$ and if w is the solution of (3.8) with initial data w_0, then for any $p \in [1, +\infty]$,

$$\|w\|_{L^p(\boldsymbol{R}^+, L^\infty(\boldsymbol{R}^2\times(0,1)))} \leq C_R |\Omega|^{-\frac{1}{4p}} \|w_0\|_{L^2(\boldsymbol{R}^2\times(0,1))}\,.$$

The previous proposition states that if we get rid of the high frequencies of w_F, it satisfies dispersion estimates in the $L^p(L^\infty)$ norm since this norm decreases when Ω goes to infinity. Notice that w is of mean value zero in z, which means in Fourier variables that the fundamental state $n=0$ is zero.

Proof of proposition 3.4.3: Fix $R>0$. Let w_0 be a function which satisfies the assumptions of proposition 3.4.3 and w be a solution of (3.8) with initial data w_0. Then, for any $(k,n)\in\mathcal{C}_R$, $n\in\mathbf{Z}^*$ and $t\geq 0$,

$$w_n(k,t)=e^{-tA_n^\Omega(k)}w_{0n}(k)\,,$$

where $A_n^\Omega(k)$ is given by (3.10). We first prove proposition 3.4.3 when $p=1$. By a consequence of the Hahn-Banach' theorem,

$$\|w\|_{L^1(\mathbf{R}^+,L^\infty(\mathbf{R}^2\times(0,1)))}=\sup_{\phi\in\mathcal{B}}<w,\phi>_{L^2(\mathbf{R}^+,L^2(\mathbf{R}^2\times(0,1)))}$$

where

$$\begin{aligned}\mathcal{B}&=\{\phi\in\mathcal{D}_2\,|\,\|\phi\|_{L^\infty(\mathbf{R}^+,L^1(\mathbf{R}^2\times(0,1)))}\leq 1\}\\ \mathcal{D}_2&=\{\phi\in\mathcal{C}^\infty(\mathbf{R}^+\times\mathbf{R}^3)\,,\ \text{1-periodic in } z \text{ and whose support in } x\in\mathbf{R}^2\\ &\qquad\text{is compact and independent of } z\,.\}\end{aligned}$$

Then,

$$\|w\|_{L^1(\mathbf{R}^+,L^\infty)}=\sup_{\phi\in\mathcal{B}}\int_0^{+\infty}\int_{\mathbf{R}^2}\sum_{n\in\mathbf{Z}^*}e^{-tA_n^\Omega(k)}w_{0n}(k)\phi_n(k,t)dkdt\,.$$

Notice that in the previous equality, the integral and the sum are taken over $(k,n)\in\mathcal{C}_R$. Moreover, $A_n^\Omega(k)$ has three eigenvalues: $\nu|k,n|^2$ and $\nu|k,n|^2\pm i\Omega a(k,n)$ where

$$|k,n|=|(k,n)|=\sqrt{|k|^2+4\pi^2n^2}\tag{3.12}$$

$$a(k,n)=\frac{2\pi n}{|k,n|}\,.\tag{3.13}$$

Then, $A_n^\Omega(k)$ is diagonalisable and the product $e^{-tA_n^\Omega(k)}w_{0n}(k)$ can be decomposed along the three eigenspaces of $A_n^\Omega(k)$. However, we shall see that $w_{0n}(k)$ only lies in the eigenspaces corresponding to the eigenvalues $\nu|k,n|^2\pm i\Omega a(k,n)$. Indeed, the eigenspace corresponding to the eigenvalue $\nu|k,n|^2$ is the set of all functions f such that for any $(k,n)\in\mathbf{R}\times\mathbf{Z}$

$$\begin{aligned}A_n^\Omega(k)f_n(k)&=\nu|k,n|^2f_n(k)\\ \begin{pmatrix}k\\ n\end{pmatrix}\cdot f_n(k)&=0\,,\end{aligned}$$

which is also equivalent to

$$\begin{aligned}&P(f\wedge e_3)=0\\ &\operatorname{div} f=0\,.\end{aligned}$$

This means that f is independent of z, namely $\tilde{R}(f) = 0$. This can be also read as $f_n(k) \equiv 0$ for $n \neq 0$. Since $\int_0^1 w_0(x,z)dz = 0$, w_0 does not lie in this eigenspace. We prove, in a similar way, that w_0 lies in the union of eigenspaces corresponding to the eigenvalues $\nu|k,n|^2 \pm i\Omega a(k,n)$. Therefore,

$$\|w\|_{L^1(\mathbf{R}^+,L^\infty)} \leq C_R \sup_{\phi\in\mathcal{B}} \int_0^{+\infty} \int_{\mathbf{R}^2} \sum_{n\in\mathbf{Z}^*} \mathrm{Re}\left(e^{-\nu t|k,n|^2+i\Omega t a(k,n)} w_{0n}^+(k)\right) \phi_n(k,t)dkdt\,,$$

where $w_{0n}^+(k)$ is the part of $w_{0n}(k)$ which lies in the eigenspace associated with $-\nu|k,n|^2 + i\Omega a(k,n)$ and $\mathrm{Re}(\lambda)$ is the real part of any complex λ. Let $\psi \in \mathcal{C}^\infty(\mathbf{R}^3)$ be 1-periodic in z, spherically symmetric in x with $\int_0^1 \psi(x,z)dz = 0$. Assume also that the Fourier transform $\psi_n(k)$ is identically equal to 1 on $\mathcal{C}_R\backslash_{\mathbf{R}^2\times\{0\}}$ and to 0 outside $\mathcal{C}_{2R}$. Let $K(A,B,x,z)$ be the function defined in Fourier coordinates by

$$K(A,B)_n(k) = e^{-\nu A|k,n|^2+iBa(k,n)}\psi_n(k)\,. \tag{3.14}$$

Then, by Hölder's inequality,

$$\|w\|_{L^1(\mathbf{R}^+,L^\infty)} \leq C_R\|w_0\|_{L^2(\mathbf{R}^2\times(0,1))} \times$$
$$\sup_{\phi\in\mathcal{B}}\left(\int_{\mathbf{R}^2}\sum_{n\in\mathbf{Z}^*}\int_0^{+\infty}\int_0^{+\infty} K(t+s,\Omega(t-s))_n(k)\phi_n(k,t)\overline{\phi_n(k,s)}dtdsdk\right)^{\frac{1}{2}}.$$

According to Young's inequality and the fact that $\phi \in \mathcal{B}$,

$$\begin{aligned}\|\,(K*\phi(t))\,\phi(s)\|_{L^1(\mathbf{R}^2\times(0,1))} &\leq \|K*\phi(t)\|_{L^\infty(\mathbf{R}^2\times(0,1))}\|\phi(s)\|_{L^1(\mathbf{R}^2\times(0,1))}\\ &\leq \|K\|_{L^\infty(\mathbf{R}^2\times(0,1))}\|\phi(t)\|_{L^1(\mathbf{R}^2\times(0,1))}\|\phi(s)\|_{L^1(\mathbf{R}^2\times(0,1))}\\ &\leq \|K(t+s,\Omega(t-s))\|_{L^\infty(\mathbf{R}^2\times(0,1))}\,.\end{aligned}$$

In order to bound the L^∞ norm of K, we compute dispersion estimates. Actually, the following lemma holds.

Lemma 3.4.4 *Fix $R>0$. Let K be the function defined in Fourier coordinates by (3.14). Then, there exists $C_R > 0$ such that for any $(A,B) \in (\boldsymbol{R}^+)^2$,*

$$\|K(A,B)\|_{L^\infty(\boldsymbol{R}^2\times(0,1))} \leq C_R\frac{e^{-4\pi^2\nu A}}{\sqrt{B}}\,.$$

We first conclude the proof of proposition 3.4.3 and postpone for a while the proof of lemma 3.4.4. Taking $A = t+s$ and $B = |\Omega||t-s|$, we get

$$\begin{aligned}\|w\|_{L^1(\mathbf{R}^+,L^\infty)} &\leq C_R\|w_0\|_{L^2(\mathbf{R}^2\times(0,1))}|\Omega|^{-\frac{1}{4}}\left(\int_0^\infty\int_0^\infty \frac{e^{-\nu(t+s)}}{\sqrt{|t-s|}}dtds\right)^{\frac{1}{2}}\\ &\leq C_R\|w_0\|_{L^2(\mathbf{R}^2\times(0,1))}|\Omega|^{-\frac{1}{4}}\,. \end{aligned}\tag{3.15}$$

Proposition 3.4.3 is thus proved for $p=1$. Moreover,

$$\|w\|_{L^\infty(\mathbf{R}^+,L^2)} = \sup_{t\geq 0} \|e^{-tA_n^\Omega(k)} w_{0n}(k)\|_{L^2_k l^2_n} \leq \|w_0\|_{L^2(\mathbf{R}^2\times(0,1))} .$$

Since $w_n(k)$ is supported in $\mathcal{C}_R$, we then get

$$\|w\|_{L^\infty(\mathbf{R}^+,L^\infty)} \leq C_R \|w\|_{L^\infty(\mathbf{R}^+,L^2)} \leq C_R \|w_0\|_{L^2(\mathbf{R}^2\times(0,1))} , \tag{3.16}$$

and proposition 3.4.3 is proved for $p=\infty$. Interpolating (3.15) with (3.16) ends the proof of proposition 3.4.3. ■

Proof of lemma 3.4.4: We now turn our attention to the proof of lemma 3.4.4. K is given by

$$K(A,B,x,z) = \int_{\mathbf{R}^2} \sum_{n\in\mathbf{Z}^*} \psi_n(k) e^{-\nu A|k,n|^2 + iBa(k,n)} e^{i(k\cdot x + 2\pi n z)} dk ,$$

where the integral and the sum are taken over $(k,n)\in\mathcal{C}_R$ and $n\neq 0$. As K is invariant under any planar rotation around the vertical axis, we can restrict ourselves to the case $x_2=0$ and we use a stationary phase method to prove lemma 3.4.4. Let α and L be defined by

$$\alpha(k,n) = -\partial_{k_2} a(k,n) , \quad L = \frac{1}{1+B\alpha^2(k,n)} \left(Id + i\alpha(k,n)\partial_{k_2}\right) .$$

Then, $L(e^{iBa(k,n)}) = e^{iBa(k,n)}$ and integration by parts leads to

$$K(A,B,x,z) = \int \sum_{(k,n)\in\mathcal{C}_R,\, n\neq 0} e^{iBa(k,n)} e^{i(k\cdot x+2\pi n z)} ({}^tL) \left(\psi_n(k) e^{-\nu A|k,n|^2}\right) dk ,$$

where ${}^tL$ is the adjoint operator of L. A computation gives

$$\begin{aligned} {}^tL\left(\psi_n(k)e^{-\nu A|k,n|^2}\right) &= \left(\frac{1}{1+B\alpha^2} - i(\partial_{k_2}\alpha)\frac{1-B\alpha^2}{(1+B\alpha^2)^2}\right)\psi_n(k)e^{-\nu A|k,n|^2} \\ &\quad - \frac{i\alpha}{1+B\alpha^2}\partial_{k_2}\left(\psi_n(k)e^{-\nu A|k,n|^2}\right) . \end{aligned}$$

Since $(k,n)\in\mathcal{C}_R$ and $n\neq 0$, we get

$$\begin{aligned} &\frac{|k_2|}{R^3} \leq |\alpha(k,n)| \leq R^2 , \quad |\partial_{k_2}\alpha(k,n)| \leq C_R \\ &\frac{1}{1+B\alpha^2} + \frac{|1-B\alpha^2|}{(1+B\alpha^2)^2} + \frac{|\alpha|}{1+B\alpha^2} \leq \frac{C_R}{1+Bk_2^2} \\ &\partial_{k_2}\left(\psi_n(k)e^{-\nu A|k,n|^2}\right) \leq C_R e^{-4\pi^2\nu A} . \end{aligned}$$

Then, $\left|{}^tL\left(\psi_n(k)e^{-\nu A|k,n|^2}\right)\right| \leq C_R\frac{e^{-4\pi^2\nu A}}{1+Bk_2^2}$ and

$$\|K(A,B,.)\|_{L^\infty(\mathbf{R}^2\times(0,1))} \leq C_R e^{-4\pi^2\nu A}\int_{\mathbf{R}} \frac{dk_2}{1+Bk_2^2} \leq C_R\frac{e^{-4\pi^2\nu A}}{\sqrt{B}} .$$

The proof of lemma 3.4.4 is thus complete. ■

3. Proof of theorem 3.4.2

Let us now prove theorem 3.4.2. As outlined before, our strategy is to compute some energy estimates in $\dot{H}^{\frac{1}{2}}$ on $u - \bar{u} - w$ where w is a regularisation of w_F by getting rid of high frequencies. As initial data and force tems in the Navier-Stokes equation satisfied by $u - \bar{u} - w$ are small by a choice of appropriate parameters, classical arguments following Fujita-Kato [34] prove well-posedness for this equation. Let us now precise these ideas.

Proof of theorem 3.4.2: Let $u_0 \in H^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$. We divide this initial condition as in appendix 4.1:

$$u_0(x,z) = \bar{u}_0(x) + \tilde{u}_0(x,z)\,, \qquad \int_0^1 \tilde{u}_0(x,z)dz = 0\,.$$

Since $u_0 \in H^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$ and $\bar{u}_0(x) = \int_0^1 u_0(x,z)dz$, we easily get that $\bar{u}_0 \in H^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$ and it follows that

$$\bar{u}_0 \in L^2(\mathbf{R}^2)\,, \qquad \tilde{u}_0 \in H^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))\,.$$

Thus, there exist $\bar{u}$ and w_F solutions of equations (3.7) and (3.8) with initial data $\bar{u}_0$ and $\tilde{u}_0$ respectively. Moreover, they satisfy energy estimates (3.9) and (3.11). Let us now make precise the regularisation of w_F to express Strichartz' estimates properly.

Define $\chi \in \mathcal{D}(\mathbf{R}^3)$ such that $\chi(\lambda) = 1$ for $|\lambda| \leq \frac{1}{2}$ and $\chi(\lambda) = 0$ for $|\lambda| \geq 1$. For any $R > 0$, let

$$w = \chi\left(\frac{|k,n|}{R}\right) w_F \tag{3.17}$$

which reads for any $(k,n) \in \mathbf{R}^2 \times \mathbf{Z}$,

$$w_n(k) = \chi\left(\frac{\sqrt{|k|^2 + 4\pi^2 n^2}}{R}\right) w_{Fn}(k)\,.$$

Then, w satisfies equation (3.8) with initial data $w_0 = \chi\left(\frac{|k,n|}{R}\right)\tilde{u}_0$. As $\tilde{u}_0$ is of mean value zero in z, w satisfies the assumptions of proposition 3.4.3 and for any $p \in [1,+\infty]$, the Strichartz' estimates read

$$\|w\|_{L^p(\mathbf{R}^+, L^\infty(\mathbf{R}^2 \times (0,1)))} \leq C_R |\Omega|^{-\frac{1}{4p}} \|\tilde{u}_0\|_{L^2(\mathbf{R}^2 \times (0,1))}\,.$$

Moreover,

$$\lim_{R \to +\infty} \|w - w_F\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))} = 0\,.$$

Thus, for any $\eta > 0$, there exists $R > 0$ such that for any $t \geq 0$,

$$\|w(t) - w_F(t)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))} + 2\nu \int_0^t \|\nabla(w - w_F)(s)\|^2_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))} ds \leq \eta^2\,. \tag{3.18}$$

Equipped with those results, we are now able to prove theorem 3.4.2. Let u be the local unique solution of equation (3.6) given by Fujita-Kato's theorem in [34]. Then, there exists $T_0 > 0$ such that

$$u \in \mathcal{C}^0([0,T_0); \dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))) \cap L^2((0,T_0); \dot{H}^{\frac{3}{2}}(\mathbf{R}^2 \times (0,1)))\,.$$

Define also for $t \in [0, T_0)$ the difference

$$\delta(t) = u(t) - \bar{u}(t) - w(t) \tag{3.19}$$

where $\bar{u}$ and w have been defined above. Then, δ satisfies the following Navier-Stokes equation

$$\begin{aligned}
&\partial_t \delta - \nu\Delta\delta + \Omega P(\delta \wedge e_3) = \sum_{j=1}^{5} P(F_j)\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1)\,,\, t \in (0,T_0) \qquad (3.20)\\
&\mathrm{div}\ \delta = 0\\
&\delta(x,z,0) = \left(\mathbf{1} - \chi\left(\frac{|k,n|}{R}\right)\right)\tilde{u}_0(x,z)\,, \quad (x,z) \in \mathbf{R}^2 \times (0,1)
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= Q(\delta,\delta)\\
F_2 &= 2Q(\delta,\bar{u})\\
F_3 &= 2Q(\delta,w) \qquad (3.21)\\
F_4 &= Q(w,w)\\
F_5 &= 2Q(\bar{u},w)\\
Q(f,g) &= -\frac{1}{2}\left((f\cdot\nabla)g + (g\cdot\nabla)f\right)\,.
\end{aligned}$$

Thus, taking the scalar product of (3.20) with δ in $\dot{H}^{\frac{1}{2}}(\mathbf{R}^2 \times (0,1))$, we get for any $t \in [0,T_0)$,

$$\|\delta(t)\|^2_{\dot{H}^{\frac{1}{2}}} + 2\nu\int_0^t \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds \le \|\delta(0)\|^2_{\dot{H}^{\frac{1}{2}}} + 2\sum_{j=1}^{5}\int_0^t < P(F_j), \delta(s) >_{\dot{H}^{\frac{1}{2}}} ds\,. \tag{3.22}$$

To bound the force term, we refer to the following proposition whose proof will be postponed to the end of the section.

Proposition 3.4.5 *Let $\bar{u}$ and w_F be the solutions of equations (3.7) and (3.8) with initial data $\bar{u}_0$ and $\tilde{u}_0$ respectively. Let w be the truncature of w_F defined as in (3.17). Let u be the solution of (3.6) on $(0,T_0)$ and δ the difference defined in (3.19). Let $(F_j)_{j=1,..,5}$ be as in (3.21). Then, there exists $C_0 > 0$ such that for any $t \in (0,T_0)$, the following estimates* *hold*

$$2\sum_{j=1}^{5}\int_0^t < F_j,\delta >_{\dot H^{\frac12}} ds \le \ \nu\int_0^t \|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds + C_0\int_0^t \|\delta(s)\|_{\dot H^{\frac12}}\|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds$$
$$+ C_0\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot H^{\frac12}} ds + C_0\int_0^t \mathcal{F}_2(s)ds$$

where for $i = 1$ *or* 2, $\mathcal{F}_i \in L^1(\boldsymbol{R}^+)$ *and*

$$\begin{aligned}\|\mathcal{F}_1\|_{L^1(\boldsymbol{R}^+)} &\le C_0\left(\|\bar u_0\|^2_{L^2(\boldsymbol{R}^2)} + \|\bar u_0\|^4_{L^2(\boldsymbol{R}^2)} + \|\tilde u_0\|^2_{\dot H^{\frac12}} + \|\tilde u_0\|^4_{\dot H^{\frac12}}\right)\\ &\quad + C_R|\Omega|^{-\frac14}\|\tilde u_0\|^2_{L^2(\boldsymbol{R}^2\times(0,1))}\\ \|\mathcal{F}_2\|_{L^1(\boldsymbol{R}^+)} &\le C_R|\Omega|^{-\frac14}\|\tilde u_0\|^2_{L^2(\boldsymbol{R}^2\times(0,1))}\|\tilde u_0\|_{\dot H^{\frac12}(\boldsymbol{R}^2\times(0,1))}\\ &\quad + C_R\|\tilde u_0\|_{L^2(\boldsymbol{R}^2\times(0,1))}\left(|\Omega|^{-\frac18}\|\tilde u_0\|_{L^2(\boldsymbol{R}^2\times(0,1))}\|\bar u_0\|_{L^2(\boldsymbol{R}^2)} + |\Omega|^{-\frac14}\|\bar u_0\|^2_{L^2(\boldsymbol{R}^2)}\right)\end{aligned}$$

Proof of theorem 3.4.2 (concluded): Equipped with proposition 3.4.5 and inequality (3.22), we can state that

$$\|\delta(t)\|^2_{\dot H^{\frac12}} + \nu\int_0^t \|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds \le \|\delta(0)\|^2_{\dot H^{\frac12}} + C_0\int_0^t \|\delta(s)\|_{\dot H^{\frac12}}\|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds$$
$$+ C_0\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot H^{\frac12}} ds + C_0\int_0^t \mathcal{F}_2(s)ds\,. \tag{3.23}$$

The aim of the remainder of the proof is to adjust parameters η, R and Ω such that δ exists globally in $\mathcal{C}^0(\mathbf{R}^+,\dot H^{\frac12})\cap L^2(\mathbf{R}^+,\dot H^{\frac32})$. Choose $\eta > 0$ small enough in comparison with ν, then R large enough so that (3.18) holds and finally, $\Omega_0 > 0$ large enough such that

$$\eta \in \left(0,\frac{\nu}{2C_0}\right) \qquad C_R\,\Omega_0^{-\frac14} \le 1 \tag{3.24}$$
$$2\eta^2\exp\left(C_0^2\|\bar u_0\|^2_{L^2} + C_0^2\|\bar u_0\|^4_{L^2} + C_0^2\|\tilde u_0\|^2_{\dot H^{\frac12}} + C_0^2\|\tilde u_0\|^4_{\dot H^{\frac12}} + C_0C_R\Omega_0^{-\frac14}\|\tilde u_0\|^2_{L^2}\right) \le \left(\frac{\nu}{4C_0}\right)^2$$
$$C_0\|\tilde u_0\|_{L^2}\left(\Omega_0^{-\frac14}\|\tilde u_0\|_{L^2}\|\tilde u_0\|_{\dot H^{\frac12}} + \Omega_0^{-\frac18}\|\tilde u_0\|_{L^2}\|\bar u_0\|_{L^2} + \Omega_0^{-\frac14}\|\bar u_0\|^2_{L^2}\right) \le \frac{\eta^2}{C_R}$$

where C_0 and C_R are the positive constants that appear in proposition 3.4.5. Then, for any $\Omega \in \mathbf{R}$ with $|\Omega| \ge \Omega_0$,

$$\|\delta(0)\|_{\dot H^{\frac12}} \le \eta\,.$$

Define T^* the maximal time such that δ remains small:

$$T^* = \sup\left\{T > 0\,|\,\forall t \in [0,T)\,,\,\|\delta(t)\|_{\dot H^{\frac12}} \le \frac{\nu}{2C_0}\right\}\,.$$

Of course $T^* > 0$ and assuming $T^* < T_0$, we have

$$\forall t < T^*\,,\ \|\delta(t)\|_{\dot H^{\frac12}} \le \frac{\nu}{2C_0}\,.$$

Plugging this estimate in (3.23) shows that

$$\|\delta(t)\|^2_{\dot H^{\frac12}} + \frac{\nu}{2}\int_0^t \|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds \le \|\delta(0)\|^2_{\dot H^{\frac12}} + C_0\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot H^{\frac12}} ds + C_0\int_0^t \mathcal{F}_2(s)ds\,.$$

Gronwall's lemma and proposition 3.4.5 finally give for $t < T^*$,

$$\begin{aligned}\|\delta(t)\|^2_{\dot H^{\frac12}} &\le \left(\|\delta(0)\|^2_{\dot H^{\frac12}} + C_0\int_0^t \mathcal{F}_2(s)ds\right)\exp\left(C_0\int_0^t \mathcal{F}_1(s)ds\right)\\ &\le \left[\eta^2 + C_0C_R\|\tilde u_0\|_{L^2}\left(|\Omega|^{-\frac14}\|\tilde u_0\|_{L^2}\|\tilde u_0\|_{\dot H^{\frac12}} + |\Omega|^{-\frac18}\|\tilde u_0\|_{L^2}\|\bar u_0\|_{L^2} + |\Omega|^{-\frac14}\|\bar u_0\|^2_{L^2}\right)\right]\\ &\qquad \times\exp\left(C_0^2\left(\|\bar u_0\|^2_{L^2} + \|\bar u_0\|^4_{L^2} + \|\tilde u_0\|^2_{\dot H^{\frac12}} + \|\tilde u_0\|^4_{\dot H^{\frac12}}\right) + C_0C_R|\Omega|^{-\frac14}\|\tilde u_0\|^2_{L^2}\right)\,.\end{aligned}$$

We have chosen η and Ω_0 in (3.24) such that this last inequality implies for any $t < T^*$,

$$\|\delta(t)\|_{\dot H^{\frac12}} \le \frac{\nu}{4C_0}\,.$$

This contradicts the definition of T^* and $T^* = T_0$. Thus, for any $t \in (0, T_0)$,

$$\|\delta(t)\|^2_{\dot H^{\frac12}} + \frac{\nu}{2}\int_0^t \|\nabla\delta(s)\|^2_{\dot H^{\frac12}} ds \le C_1\eta^2 \tag{3.25}$$

where C_1 is a positive constant independent of t, Ω, R and η. C_1 depends on the initial data $\bar u_0$ and $\tilde u_0$. The first consequence of this uniform bound is that for $|\Omega| \ge \Omega_0$, δ exists globally in time in $\mathcal{C}^0(\mathbf{R}^+, \dot H^{\frac12}) \cap L^2(\mathbf{R}^+, \dot H^{\frac32})$ and since δ is the unique solution of (3.20) on $(0, T_0)$, we conclude that (3.20) has a unique global solution in $\mathcal{C}^0(\mathbf{R}^+, \dot H^{\frac12}) \cap L^2(\mathbf{R}^+, \dot H^{\frac32})$.

Since $\bar u$ and w_F are the unique global solutions of (3.7) and (3.8) with initial data $\bar u_0$ and $\tilde u_0$ respectively, it follows that for $|\Omega| \ge \Omega_0$, (3.6) has a unique global solution

$$u \in \mathcal{C}^0(\mathbf{R}^+, \dot H^{\frac12}) \cap L^2(\mathbf{R}^+, \dot H^{\frac32})\,.$$

and theorem 3.4.2 follows immediately. ■

Let us now come to the proof of proposition 3.4.5. The idea is to bound the force term in equation (3.20) by Sobolev embeddings and Strichartz' estimates.

4. Bounding the force terms

Proof of proposition 3.4.5: We deal separately with the five $(F_j)_{j=1,..,5}$ terms involved in (3.20) even if the methods used are similar. We first recall some inequalities we often use. There exists $C > 0$ such that for any $(a, b) \in \dot H^{\frac12}(\mathbf{R}^2 \times (0,1))$,

$$< a, b >_{\dot H^{\frac12}(\mathbf{R}^2\times(0,1))} \le C < a, \nabla b >_{L^2(\mathbf{R}^2\times(0,1))}\,.$$

By interpolation, we also have

$$\|a\|_{\dot H^1}^2 \le C\|a\|_{\dot H^{\frac12}}\|a\|_{\dot H^{\frac32}} \le C\|a\|_{\dot H^{\frac12}}\|\nabla a\|_{\dot H^{\frac12}}$$

and finally, we recall some Sobolev' embeddings in $\mathbf{R}^2\times(0,1)$ for the functions of mean value zero in z,

$$\tilde R(\dot H^{\frac12}(\mathbf{R}^2\times(0,1)))\hookrightarrow L^3(\mathbf{R}^2\times(0,1))\,,\quad \tilde R(\dot H^1(\mathbf{R}^2\times(0,1)))\hookrightarrow L^6(\mathbf{R}^2\times(0,1))$$

and in $\mathbf{R}^2$, $\dot H^{\frac12}(\mathbf{R}^2)\hookrightarrow L^4(\mathbf{R}^2)$. Let us now deal with the first force term F_1.

Bound of $< F_1,\delta >_{\dot H^{\frac12}}$: Using the above definitions of F_1 and Q (see (3.21)), Hölder's inequality, Sobolev' embeddings and interpolation's results, we prove that there exists $C>0$ such that

$$\begin{aligned}\left|< F_1,\delta >_{\dot H^{\frac12}(\mathbf{R}^2\times(0,1))}\right| &\le C < \delta\cdot\nabla\delta,\nabla\delta >_{L^2(\mathbf{R}^2\times(0,1))}\\ &\le C\|\delta\|_{L^6}\|\nabla\delta\|_{L^3}\|\nabla\delta\|_{L^2}\\ &\le C\|\delta\|_{\dot H^1}^2\|\nabla\delta\|_{\dot H^{\frac12}}\\ &\le C\|\delta\|_{\dot H^{\frac12}}\|\nabla\delta\|_{\dot H^{\frac12}}^2\,.\end{aligned}$$

Thus, $\int_0^t < F_1,\delta >_{\dot H^{\frac12}(\mathbf{R}^2\times(0,1))} ds \le C\int_0^t \|\delta(s)\|_{\dot H^{\frac12}}\|\nabla\delta(s)\|_{\dot H^{\frac12}}^2 ds$.

Bound of $< F_2,\delta >_{\dot H^{\frac12}}$: By definition, we have

$$\left|< F_2,\delta >_{\dot H^{\frac12}(\mathbf{R}^2\times(0,1))}\right| \le C\left|< \bar u\cdot\nabla\delta,\delta >_{\dot H^{\frac12}}\right| + C\left|< \delta\cdot\nabla\bar u,\delta >_{\dot H^{\frac12}}\right|\,.$$

Since $\bar u$ only depends on x, its scaling and therefore Sobolev embeddings are different from those of δ. To counter this difficulty, we use anisotropic estimates as follows:

$$\begin{aligned}\left|< \bar u\cdot\nabla\delta,\delta >_{\dot H^{\frac12}}\right| &\le C\left|< \bar u\cdot\nabla\delta,\nabla\delta >_{L^2(\mathbf{R}^2\times(0,1))}\right|\\ &\le C\int_0^1 \|\bar u\|_{L^4(\mathbf{R}^2)}\|\nabla\delta(.,z)\|_{L^4(\mathbf{R}^2)}\|\nabla\delta(.,z)\|_{L^2(\mathbf{R}^2)}dz\\ &\le C\|\bar u\|_{\dot H^{\frac12}(\mathbf{R}^2)}\int_0^1 \|\nabla\delta(z)\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\delta(z)\|_{L^2(\mathbf{R}^2)}\\ &\le C\|\bar u\|_{\dot H^{\frac12}(\mathbf{R}^2)}\|\nabla\delta\|_{L^2((0,1),\dot H^{\frac12}(\mathbf{R}^2))}\|\nabla\delta\|_{L^2(\mathbf{R}^2\times(0,1))}\,.\end{aligned}$$

Since for any $a\in\dot H^{\frac12}(\mathbf{R}^2\times(0,1))$, $\|a\|_{L^2((0,1),\dot H^{\frac12}(\mathbf{R}^2))}\le\|a\|_{\dot H^{\frac12}(\mathbf{R}^2\times(0,1))}$, we get with estimates such as $ab\le(a^2+b^2)/2$ and interpolation's results

$$\begin{aligned}\left|< \bar u\cdot\nabla\delta,\delta >_{\dot H^{\frac12}}\right| &\le \frac{\nu}{24}\|\nabla\delta\|_{\dot H^{\frac12}}^2 + C\|\bar u\|_{\dot H^{\frac12}}^2\|\delta\|_{\dot H^1}^2\\ &\le \frac{\nu}{12}\|\nabla\delta\|_{\dot H^{\frac12}}^2 + C\|\bar u\|_{\dot H^{\frac12}}^4\|\delta\|_{\dot H^{\frac12}}^2\,.\end{aligned}$$

Finally, interpolating (3.9), we know that

$$\|\bar{u}\|_{L^4(\mathbf{R}^+,\dot{H}^{\frac{1}{2}}(\mathbf{R}^2))} \leq \frac{C}{\nu^{\frac{1}{2}}}\|\bar{u}_0\|_{L^2(\mathbf{R}^2)}\,.$$

Thus, $\int_{\mathbf{R}^+} \|\bar{u}(s)\|^4_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2)} ds \leq C\|\bar{u}_0\|^4_{L^2(\mathbf{R}^2)}$ and

$$\int_0^t \left|< \bar{u}\cdot\nabla\delta, \delta >_{\dot{H}^{\frac{1}{2}}}\right| ds \leq \frac{\nu}{12}\int_0^t \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds + C\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds$$

where $\mathcal{F}_1$ satisfies proposition (3.4.5). The second term can be bounded along the same lines:

$$\begin{aligned}
\left|< \delta\cdot\nabla\bar{u}, \delta >_{\dot{H}^{\frac{1}{2}}}\right| &\leq C\int_0^1 \|\nabla\bar{u}\|_{L^2(\mathbf{R}^2)}\|\delta(.,z)\|_{L^4(\mathbf{R}^2)}\|\nabla\delta(.,z)\|_{L^4(\mathbf{R}^2)} dz \\
&\leq C\|\nabla\bar{u}\|_{L^2(\mathbf{R}^2)}\|\delta\|_{L^2((0,1),\dot{H}^{\frac{1}{2}}(\mathbf{R}^2))}\|\nabla\delta\|_{L^2((0,1),\dot{H}^{\frac{1}{2}}(\mathbf{R}^2))} \\
&\leq C\|\nabla\bar{u}\|_{L^2(\mathbf{R}^2)}\|\delta\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))}\|\nabla\delta\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))} \\
&\leq \frac{\nu}{12}\|\nabla\delta\|^2_{\dot{H}^{\frac{1}{2}}} + C\|\nabla\bar{u}\|^2_{L^2(\mathbf{R}^2)}\|\delta\|^2_{\dot{H}^{\frac{1}{2}}}\,.
\end{aligned}$$

By (3.9), $\int_0^t \|\nabla\bar{u}(s)\|^2_{L^2(\mathbf{R}^2)} ds \leq \frac{C}{\nu}\|\bar{u}_0\|^2_{L^2(\mathbf{R}^2)}$ and

$$\int_0^t \left|< \delta\cdot\nabla\bar{u}, \delta >_{\dot{H}^{\frac{1}{2}}}\right| ds \leq \frac{\nu}{12}\int_0^t \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds + C\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds\,.$$

Bound of $< F_3, \delta >_{\dot{H}^{\frac{1}{2}}}$***:*** By definition, we have

$$\left|< F_3, \delta >_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^2\times(0,1))}\right| \leq C\left|< \delta\cdot\nabla w, \delta >_{\dot{H}^{\frac{1}{2}}}\right| + C\left|< w\cdot\nabla\delta, \delta >_{\dot{H}^{\frac{1}{2}}}\right|\,.$$

This term is very similar to the previous one and it is bounded along the same lines. Notice that anisotropic estimates are in this case useless. Then,

$$\begin{aligned}
\left|< \delta\cdot\nabla w, \delta >_{\dot{H}^{\frac{1}{2}}}\right| &\leq C\|\delta\|_{L^6}\|\nabla w\|_{L^3}\|\nabla\delta\|_{L^2} \\
&\leq C\|\nabla\delta\|^2_{L^2}\|\nabla w\|_{\dot{H}^{\frac{1}{2}}} \\
&\leq C\|\delta\|_{\dot{H}^{\frac{1}{2}}}\|\nabla\delta\|_{\dot{H}^{\frac{1}{2}}}\|\nabla w\|_{\dot{H}^{\frac{1}{2}}} \\
&\leq \frac{\nu}{12}\|\nabla\delta\|^2_{\dot{H}^{\frac{1}{2}}} + C\|\delta\|^2_{\dot{H}^{\frac{1}{2}}}\|\nabla w\|^2_{\dot{H}^{\frac{1}{2}}}\,.
\end{aligned}$$

By (3.11), $\int_0^t \|\nabla w(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds \leq \frac{C}{\nu}\|\tilde{u}_0\|^2_{\dot{H}^{\frac{1}{2}}}$, then

$$\int_0^t \left|< \delta\cdot\nabla w, \delta >_{\dot{H}^{\frac{1}{2}}}\right| ds \leq \frac{\nu}{12}\int_0^t \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds + C\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds\,.$$

A similar proof shows that

$$\int_0^t \left|< w\cdot\nabla\delta, \delta >_{\dot{H}^{\frac{1}{2}}}\right| ds \leq \frac{\nu}{12}\int_0^t \|\nabla\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds + C\int_0^t \mathcal{F}_1(s)\|\delta(s)\|^2_{\dot{H}^{\frac{1}{2}}} ds\,.$$

Bound of $< F_4, \delta >_{\dot H^{\frac12}}$: The term $F_4 = Q(w,w)$ is bilinear in w and its bound requires the Strichartz' estimates proved in proposition 3.4.3. The other ideas of the proof are similar to the previous cases.

$$\begin{aligned}
\left|< F_4, \delta >_{\dot H^{\frac12}}\right| &\le C\|w\|_{L^\infty}\|\nabla w\|_{L^2}\|\nabla\delta\|_{L^2}\\
&\le C\|w\|_{L^\infty}\|w\|^{\frac12}_{\dot H^{\frac12}}\|\nabla w\|^{\frac12}_{\dot H^{\frac12}}\|\nabla\delta\|_{L^2}\\
&\le C\|w\|^2_{L^\infty}\|w\|_{\dot H^{\frac12}} + C\|\nabla w\|_{\dot H^{\frac12}}\|\delta\|_{\dot H^{\frac12}}\|\nabla\delta\|_{\dot H^{\frac12}}\\
&\le C\|w\|^2_{L^\infty}\|w\|_{\dot H^{\frac12}} + C\|\nabla w\|^2_{\dot H^{\frac12}}\|\delta\|^2_{\dot H^{\frac12}} + \frac{\nu}{12}\|\nabla\delta\|^2_{\dot H^{\frac12}}\,.
\end{aligned}$$

Once more by the energy estimate (3.11) and by proposition 3.4.3, we get

$$\int_0^{+\infty}\|w(s)\|^2_{L^\infty}\|w(s)\|_{\dot H^{\frac12}}ds \le C\|\tilde u_0\|_{\dot H^{\frac12}}\|w\|^2_{L^2(\mathbf{R}^+,L^\infty)} \le C\|\tilde u_0\|_{\dot H^{\frac12}}C_R|\Omega|^{-\frac14}\|\tilde u_0\|^2_{L^2}\,.$$

Thus, F_4 satisfies the bound

$$\int_0^t\left|< F_4, \delta >_{\dot H^{\frac12}}\right|ds \le \frac{\nu}{12}\int_0^t\|\nabla\delta\|^2_{\dot H^{\frac12}}ds + C\int_0^t\mathcal{F}_1(s)\|\delta(s)\|^2_{\dot H^{\frac12}}ds + \int_0^t\mathcal{F}_2(s)ds$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy the estimates of proposition 3.4.5.

Bound of $< F_5, \delta >_{\dot H^{\frac12}}$: This last term mixes up all the functions $\bar u$, w and δ and is bounded again with Hölder's inequality, interpolation, inequalities such as $ab \le (a^2+b^2)/2$ and Strichartz' estimates. In this case, we do not need anisotropic estimates since $\|\bar u\|_{L^2(\mathbf{R}^2\times(0,1))} = \|\bar u\|_{L^2(\mathbf{R}^2)}$. Notice that this is a fundamental difference with the case in the whole space $\mathbf{R}^3$ studied in [26]. In particular, we do not need anisotropic Strichartz' estimates (see [26]) nor anisotropic spaces [49]. A new idea is the bound $\|\nabla w\|_{L^\infty} \le R\|w\|_{L^\infty}$ since w is a regular function obtained from w_F by getting rid of high frequencies. Then,

$$\begin{aligned}
\left|< F_5, \delta >_{\dot H^{\frac12}}\right| &\le C\|\nabla\delta\|_{L^2}\left(\|w\|_{L^\infty}\|\nabla\bar u\|_{L^2(\mathbf{R}^2)} + \|\bar u\|_{L^2(\mathbf{R}^2)}\|\nabla w\|_{L^\infty}\right)\\
&\le C\|\nabla\delta\|_{L^2}\|w\|_{L^\infty}\left(\|\nabla\bar u\|_{L^2(\mathbf{R}^2)} + R\|\bar u\|_{L^2(\mathbf{R}^2)}\right)\\
&\le C\|\nabla\bar u\|_{L^2}\left(\|\nabla\delta\|^2_{L^2} + \|w\|^2_{L^\infty}\right) + C\|w\|_{L^\infty}\left(\|\nabla\delta\|^2_{L^2} + R^2\|\bar u\|^2_{L^2(\mathbf{R}^2)}\right)\\
&\le C\|\nabla\bar u\|_{L^2(\mathbf{R}^2)}\left(\|\delta\|_{\dot H^{\frac12}}\|\nabla\delta\|_{\dot H^{\frac12}} + \|w\|^2_{L^\infty}\right)\\
&\quad + C\|w\|_{L^\infty}\left(\|\delta\|_{\dot H^{\frac12}}\|\nabla\delta\|_{\dot H^{\frac12}} + R^2\|\bar u\|^2_{L^2(\mathbf{R}^2)}\right)\\
&\le \frac{\nu}{12}\|\nabla\delta\|^2_{\dot H^{\frac12}} + C\|\delta\|^2_{\dot H^{\frac12}}\left(\|\nabla\bar u\|^2_{L^2(\mathbf{R}^2)} + \|w\|^2_{L^\infty}\right)\\
&\quad + C\|\nabla\bar u\|_{L^2(\mathbf{R}^2)}\|w\|^2_{L^\infty} + CR^2\|w\|_{L^\infty}\|\bar u\|^2_{L^2(\mathbf{R}^2)}\\
&\le \frac{\nu}{12}\|\nabla\delta\|^2_{\dot H^{\frac12}} + Cf_1(t)\|\delta\|^2_{\dot H^{\frac12}} + f_2(t)\,,
\end{aligned}$$

where

$$\int_{\mathbf{R}^+} f_1(s)ds \le C\int_0^\infty \|\nabla \bar{u}(s)\|^2_{L^2(\mathbf{R}^2)}ds + \int_0^\infty \|w(s)\|^2_{L^\infty}ds$$
$$\le C\|\bar{u}_0\|^2_{L^2(\mathbf{R}^2)} + C_R|\Omega|^{-\frac{1}{4}}\|\tilde{u}_0\|^2_{L^2}$$
$$\int_{\mathbf{R}^+} f_2(s)ds \le CC_R\|\bar{u}_0\|_{L^2(\mathbf{R}^2)}|\Omega|^{-\frac{1}{8}}\|\tilde{u}_0\|^2_{L^2(\mathbf{R}^2\times(0,1))}$$
$$+ C_R\|\bar{u}_0\|^2_{L^2(\mathbf{R}^2)}|\Omega|^{-\frac{1}{4}}\|\tilde{u}_0\|_{L^2(\mathbf{R}^2\times(0,1))}\,.$$

This completes the proof of proposition 3.4.5. ■

Chapitre 4

Appendix

4.1 The Biot-Savart Law

Let ω be the vorticity given by $\omega(x,z):\mathbf{R}^3\to\mathbf{R}^3$, 1-periodic in z and div $\omega=0$. Define the associated velocity field by $u(x,z):\mathbf{R}^3\to\mathbf{R}^3$, 1-periodic in z such that

$$\begin{cases} \text{div } u = 0, \\ \text{rot } u = \omega. \end{cases}$$

The aim of this section is to express the velocity field u in terms of the vorticity ω via the Biot-Savart law and to collect useful estimates for the velocity u in terms of ω. We first decompose the functions $(\omega,\ u)$ into two parts which still satisfy periodic boundary conditions. The first one $(\bar{\omega},\ \bar{u})$ is independent of z and the other one $(\tilde{\omega},\ \tilde{u})$ is of mean-value zero in z: for any $(x,z)\in\mathbf{R}^3$, we thus set

$$\begin{gathered} \omega(x,z)=\bar{\omega}(x)+\tilde{\omega}(x,z)\,, \quad u(x,z)=\bar{u}(x)+\tilde{u}(x,z)\,, \\ \int_0^1 \tilde{\omega}(x,z)dz=0\,, \quad \int_0^1 \tilde{u}(x,z)dz=0\,. \end{gathered} \tag{4.1}$$

Then, $\bar{\omega}=\text{rot }\bar{u}$ and $\tilde{\omega}=\text{rot }\tilde{u}$. Moreover, as div $\omega=0$ and as ω is 1-periodic in z, notice that $\partial_1\bar{\omega}_1+\partial_2\bar{\omega}_2=0$, hence also div $\tilde{\omega}=0$. This means that our decomposition leads to two independent systems with their own Biot-Savart laws:

$$A.1\begin{cases} \bar{\omega},\bar{u}:\mathbf{R}^2\to\mathbf{R}^3 \\ \text{div } \bar{\omega}=0 \\ \text{div } \bar{u}=0 \\ \bar{\omega}=\text{rot } \bar{u} \end{cases} \qquad A.2\begin{cases} \tilde{\omega},\tilde{u}:\mathbf{R}^3\to\mathbf{R}^3\,,\ \text{1-periodic in } z \\ \int_0^1\tilde{\omega}(x,z)dz=\int_0^1\tilde{u}(x,z)dz=0 \\ \text{div } \tilde{u}=\text{div } \tilde{\omega}=0 \\ \tilde{\omega}=\text{rot } \tilde{u} \end{cases}$$

4.1.1 The Biot-Savart law for $(\bar{\omega},\bar{u})$.

Let $(\bar{\omega},\bar{u}):\mathbf{R}^2\to\mathbf{R}^3$ such that

$$\begin{cases} \partial_1\bar{u}_1+\partial_2\bar{u}_2=0 \\ \partial_1\bar{\omega}_1+\partial_2\bar{\omega}_2=0 \\ \bar{\omega}=\text{rot } \bar{u}\,. \end{cases}$$

This system divides itself into two independent systems:

$$(a)\left\{\begin{array}{l}\bar\omega_1=\partial_2\bar u_3\\ \bar\omega_2=-\partial_1\bar u_3\\ \partial_1\bar\omega_1+\partial_2\bar\omega_2=0\end{array}\right.\qquad (b)\left\{\begin{array}{l}\bar\omega_3=\partial_1\bar u_2-\partial_2\bar u_1\\ \partial_1\bar u_1+\partial_2\bar u_2=0\,.\end{array}\right.$$

The second system (b) is equivalent to the Biot-Savart law in $\mathbf{R}^2$ (see [39]). Then,

$$\begin{pmatrix}\bar u_1\\ \bar u_2\end{pmatrix}(x)=\frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\bar\omega_3(y)dy\,. \tag{4.2}$$

If $x=(x_1,x_2)\in\mathbf{R}^2$, we denote $x^\perp=(-x_2,x_1)^T$. To solve the first system (a), notice that

$$-\Delta_x\bar u_3=\partial_1\bar\omega_2-\partial_2\bar\omega_1\,.$$

By the fundamental solution of the Laplacian Δ_x in $\mathbf{R}^2$,

$$\bar u_3(x)=-\frac{1}{2\pi}\int_{\mathbf{R}^2}\log(|x-y|)(\partial_1\bar\omega_2(y)-\partial_2\bar\omega_1(y))dy\,,$$

and integrating by parts, we get

$$\bar u_3(x)=-\frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{(x-y)}{|x-y|^2}\wedge\begin{pmatrix}\bar\omega_1\\ \bar\omega_2\end{pmatrix}(y)dy. \tag{4.3}$$

Equalities (4.2) and (4.3) are very similar to the Biot-Savart law in $\mathbf{R}^2$ and we refer to [86] and [39] for detailed proofs of the following estimates:

Proposition 4.1.1 *Let $\bar u$ be the velocity field obtained from $\bar\omega$ via the Biot-Savart laws (4.2-4.3). Assume $1<p<2<q<\infty$ and $\frac{1}{q}=\frac{1}{p}-\frac{1}{2}$. If $\bar\omega\in L^p(\boldsymbol{R}^2)$, then $\bar u\in L^q(\boldsymbol{R}^2)$ and there exists $C>0$ such that*

$$\|\bar u\|_{L^q(\boldsymbol{R}^2)}\le C\|\bar\omega\|_{L^p(\boldsymbol{R}^2)}.$$

4.1.2 The Biot-Savart law for $(\tilde\omega,\tilde u)$

Let $(\tilde\omega,\tilde u):\mathbf{R}^3\to\mathbf{R}^3$, 1-periodic in z such that

$$\left\{\begin{array}{l}\int_0^1\tilde\omega(x,z)dz=\int_0^1\tilde u(x,z)dz=0\\ \mathrm{div}\ \tilde\omega=\mathrm{div}\ \tilde u=0\\ \tilde\omega=\mathrm{rot}\ \tilde u\,.\end{array}\right.$$

We use a decomposition of $\tilde\omega$ and $\tilde u$ in Fourier variables (k,n) where our conventions for Fourier transformation are

$$f(x,z)=\int_{\mathbf{R}^2}\sum_{n\in\mathbf{Z}}f_n(k)e^{i(k\cdot x+2\pi nz)}dk$$

$$f_n(k)=\frac{1}{2\pi}\int_{\mathbf{R}^2}\int_0^1 f(x,z)e^{-i(k\cdot x+2\pi nz)}dzdx. \tag{4.4}$$

Using the relations between the derivatives of $\tilde{u}$ and $\tilde{\omega}$, we easily get for any $k \in \mathbf{R}^2$ and any $n \in \mathbf{Z}^*$,

$$\tilde{u}_n(k) = \frac{1}{|k|^2 + 4\pi^2 n^2} \begin{pmatrix} 0 & -i2\pi n & ik_2 \\ i2\pi n & 0 & -ik_1 \\ -ik_2 & ik_1 & 0 \end{pmatrix} \tilde{\omega}_n(k)\,. \tag{4.5}$$

As in section 2.1, we shall work in weighted spaces $L^2(m)$ defined by (2.4). The $L^2(m)$-norm is also equivalent to another one defined by Fourier coefficients:

$$\left(\sum_{n \in \mathbf{Z}} \int_{\mathbf{R}^2} \sum_{|\alpha| \leq m} |\partial^\alpha f_n(k)|^2 dk \right)^{\frac{1}{2}}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{N}^2$, $\partial^\alpha = \partial_{k_1}^{\alpha_1} \partial_{k_2}^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$. Weighted Sobolev spaces can be defined in a similar way. For instance,

$$H^1(m) = \{ f \in L^2(m) \,|\, \partial_i f \in L^2(m), i = 1,..,3 \}$$

and the norm in $H^1(m)$ is given by one of the following equivalent expressions:

$$\left(\int_{\mathbf{R}^2} \int_0^1 (1+|x|^2)^m \left(|f(x,z)|^2 + |\nabla f(x,z)|^2 \right) dz dx \right)^{\frac{1}{2}},$$

$$\left(\sum_{n \in \mathbf{Z}} \int_{\mathbf{R}^2} (1+|k|^2+n^2) \sum_{|\alpha| \leq m} |\partial^\alpha f_n(k)|^2 dk \right)^{\frac{1}{2}}.$$

Using these norms and relation (4.5) where n is different from 0, we get the following proposition:

Proposition 4.1.2 *Let $\tilde{u}$ be the velocity field obtained from $\tilde{\omega}$ via the Biot-Savart law (4.5). For any $m \in \mathbf{N}$, if $\tilde{\omega} \in L^2(m)$, then $\tilde{u} \in H^1(m)$ and there exists $C > 0$ such that*

$$\|\tilde{u}\|_{H^1(m)} \leq C \|\tilde{\omega}\|_{L^2(m)}.$$

As a consequence, using Sobolev embedding $H^1(\mathbf{R}^2 \times (0,1)) \hookrightarrow L^q(\mathbf{R}^2 \times (0,1))$ for all $q \in [2,6]$ and proposition 4.1.2 for $m = 0$, we get

Corollary 4.1.3 *Let $\tilde{u}$ be the velocity field obtained from $\tilde{\omega}$ via the Biot-Savart law (4.5). If $\tilde{\omega} \in L^2(\mathbf{R}^2 \times (0,1))$, then $\tilde{u}$ is in $L^q(\mathbf{R}^2 \times (0,1))$ for any $q \in [2,6]$ and there exists $C > 0$ such that*

$$\|\tilde{u}\|_{L^q(\mathbf{R}^2 \times (0,1))} \leq C \|\tilde{\omega}\|_{L^2(\mathbf{R}^2 \times (0,1))}\,.$$

4.1.3 The Biot-Savart law in stress-free

Let $\omega(x,z) : \mathbf{R}^2 \times (0,1) \to \mathbf{R}^3$ be the vorticity field which satisfies div $\omega = 0$ and stress-free boundary conditions (1.8). Define $u(x,z) : \mathbf{R}^2 \times (0,1) \to \mathbf{R}^3$ the corresponding velocity field such that u satisfies stress-free conditions (1.5) and

$$\begin{cases} \text{div } u = 0, \\ \text{rot } u = \omega. \end{cases}$$

We prove in this subsection that the previous Biot-Savart laws stated in appendix 4.1.1 and 4.1.2 still hold with stress-free boundary conditions. To obtain a decomposition of ω and u which fits with the boundary conditions, we split ω and u as follows:

$$\omega(x,z) = \begin{pmatrix} 0 \\ 0 \\ \bar{\omega}_3(x) \end{pmatrix} + \tilde{\omega}(x,z) \, , \quad u(x,z) = \begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \\ 0 \end{pmatrix} + \tilde{u}(x,z) \, , \tag{4.6}$$

where $\tilde{u}_1$, $\tilde{u}_2$ and $\tilde{\omega}_3$ satisfy Neuman boundary conditions in $z = 0$ and $z = 1$ and are of mean-value zero in z, while $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{u}_3$ satisfy Dirichlet boundary conditions in $z = 0$ and $z = 1$, see (1.5) and (1.8). Notice that the last three functions, $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{u}_3$, are no more of mean-value zero in z. We can then divide our problem into two independent systems with their own Biot-Savart laws:

$$\begin{cases} \bar{\omega}, \bar{u} : \mathbf{R}^2 \to \mathbf{R}^3 \\ \bar{\omega}_1 = \bar{\omega}_2 = \bar{u}_3 = 0 \\ \partial_1 \bar{u}_1 + \partial_2 \bar{u}_2 = 0 \\ \bar{\omega}_3 = \partial_1 \bar{u}_2 - \partial_2 \bar{u}_1 \end{cases} \qquad \begin{cases} \tilde{\omega}, \tilde{u} : \mathbf{R}^2 \times (0,1) \to \mathbf{R}^3 \\ \text{div } \tilde{u} = \text{div } \tilde{\omega} = 0 \\ \tilde{\omega} = \text{rot } \tilde{u} \\ \tilde{u}_1, \, \tilde{u}_2, \, \tilde{\omega}_3 \text{ Neumann and mean value zero in } z \\ \tilde{\omega}_1, \, \tilde{\omega}_2, \, \tilde{u}_3 \text{ Dirichlet}. \end{cases}$$

The first system is exactly system 4.1.1(b) which is solved in (4.2). Therefore, proposition 4.1.1 still holds.

As far as the second system is concerned, we use a Fourier decomposition which takes into account boundary conditions (1.5) and (1.8). Namely, for $i = 1$ or 2,

$$\tilde{\omega}_i(x,z) = \int_{\mathbf{R}^2} \sum_{n=1}^{\infty} \omega_{in}(k) e^{ik\cdot x} \sin(n\pi z) dk$$

$$\tilde{\omega}_3(x,z) = \int_{\mathbf{R}^2} \sum_{n=1}^{\infty} \omega_{3n}(k) e^{ik\cdot x} \cos(n\pi z) dk$$

$$\tilde{u}_i(x,z) = \int_{\mathbf{R}^2} \sum_{n=1}^{\infty} u_{in}(k) e^{ik\cdot x} \cos(n\pi z) dk$$

$$\tilde{u}_3(x,z) = \int_{\mathbf{R}^2} \sum_{n=1}^{\infty} u_{3n}(k) e^{ik\cdot x} \sin(n\pi z) dk$$

and the relation $\tilde{\omega} = \text{rot } \tilde{u}$ reads for any $k \in \mathbf{R}^2$ and any $n \geq 1$,

$$\tilde{u}_n(k) = \frac{1}{|k|^2 + \pi^2 n^2} \begin{pmatrix} 0 & n\pi & ik_2 \\ -n\pi & 0 & -ik_1 \\ -ik_2 & ik_1 & 0 \end{pmatrix} \tilde{\omega}_n(k) \,.$$

As this relation is very similar to (4.5), proposition 4.1.2 and corollary 4.1.3 still hold in $L^2_{sf}(m)$ and $H^1_{sf}(m)$.

4.1.4 The Biot-Savart law in scaling variables

As we work in this article with scaling variables, we translate the above results in terms of (ξ, z, τ) and (w, v) defined in section 1 by (2.1). As for any $\tau \geq 0$ and any $q \geq 1$,

$$\|u(t)\|_{L^q(\mathbf{R}^2\times(0,1))} = e^{\tau(\frac{1}{q}-\frac{1}{2})}\|v(\tau)\|_{L^q(\mathbf{R}^2\times(0,1))}$$

and

$$\|\omega(t)\|_{L^q(\mathbf{R}^2\times(0,1))} = e^{\tau(\frac{1}{q}-1)}\|w(\tau)\|_{L^q(\mathbf{R}^2\times(0,1))} \,,$$

the above sections 4.1.1, 4.1.2 and 4.1.3 easily imply the following estimates:

Proposition 4.1.4 *Let v be the velocity field obtained from w via the Biot-Savart law. Decompose v and w as in (4.1) or (4.6). If $\tilde{w} \in L^2(\boldsymbol{R}^2 \times (0,1))$, then $\tilde{v} \in L^q(\boldsymbol{R}^2 \times (0,1))$ for any $q \in [2,6]$. If $\bar{w} \in L^p(\boldsymbol{R}^2)$, then $\bar{v} \in L^{\frac{2p}{2-p}}(\boldsymbol{R}^2)$ for any $p \in (1,2)$ and*

$$\begin{aligned} \|\tilde{v}\|_{L^q(\boldsymbol{R}^2\times(0,1))} &\leq Ce^{-\frac{\tau}{q}}\|\tilde{w}\|_{L^2(\boldsymbol{R}^2\times(0,1))} \\ \|\bar{v}\|_{L^{\frac{2p}{2-p}}(\boldsymbol{R}^2)} &\leq C\|\bar{w}\|_{L^p(\boldsymbol{R}^2)} . \end{aligned}$$

4.1.5 Another bound for the velocity

For the purpose of section 2.4, we need some further estimates on the velocity field in scaling variables in case it is independent of the third coordinate z.

Lemma 4.1.5 *Let $m > 1$. If $w \in L^2(m)$ satisfies div $w = 0$ and $\partial_z w = 0$, then the corresponding velocity field v given by the Biot-Savart law in appendix 4.1.1 satisfies*

$$v_3 \in L^2(\boldsymbol{R}^2) \,.$$

Proof: Using appendix 4.1.1, we get

$$\begin{aligned} v_3(\xi) &= -\frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(\xi-\eta)}{|\xi-\eta|^2} \wedge \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(\eta)d\eta \\ -\Delta_\xi v_3 &= \partial_1 w_2 - \partial_2 w_1 \,. \end{aligned} \tag{4.7}$$

By the Biot-Savart law, we obtain that if $w \in L^q(\mathbf{R}^2)$ for $q \in (1,2)$, then $v \in L^{\frac{2q}{2-q}}(\mathbf{R}^2)$. But this result is not sufficient to prove this lemma. The idea is to take benefit from another property satisfied by the vorticity. Indeed, the first moments of w_i are zero. This result, together with lemma B.2 in [39], enable us to conclude. We divide this proof in three steps.

First step: We show that the moments of the vorticity are zero. Indeed, $w_i = \text{div}\,(\xi_i w)$ and $\xi_i w \in L^2(m-1)$. As $m > 1$, $L^2(m) \hookrightarrow L^1(\mathbf{R}^2 \times (0,1))$ and for all $p \in (1,2)$, $p > \frac{2}{m+1}$, we get easily the continuous embedding $L^2(m) \hookrightarrow L^p(\mathbf{R}^2 \times (0,1))$. Then, $\xi_i w \in L^p(\mathbf{R}^2 \times (0,1))$ for $p \in (\frac{2}{m}, 2)$ and $w_i \in L^1(\mathbf{R}^2 \times (0,1))$. As w is independent of z, this implies for $i = 1, 2$ that

$$\int_{\mathbf{R}^2} w_i(\xi) d\xi = 0\,. \tag{4.8}$$

Second step: According to (4.7) and (4.8),

$$v_3(\xi) = -\frac{1}{2\pi} \int_{\mathbf{R}^2} \left(\frac{\xi - \eta}{|\xi - \eta|^2} - \frac{\xi}{|\xi|^2} \right) \wedge \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} (\eta) d\eta\,.$$

For all $(\xi, \eta) \in \mathbf{R}^2$ with $\xi \neq 0$ and $\xi \neq \eta$, we have the identity

$$\frac{\xi_1 - \eta_1}{|\xi - \eta|^2} - \frac{\xi_1}{|\xi|^2} = \frac{1}{|\xi|^2 |\xi - \eta|^2} \left((\xi_1 - \eta_1)\xi \cdot \eta + (\xi_2 - \eta_2)\xi \wedge \eta \right)$$

where $\xi \cdot \eta = \xi_1\eta_1 + \xi_2\eta_2$ and $\xi \wedge \eta = \xi_1\eta_2 - \xi_2\eta_1$. A similar estimate holds for $\frac{\xi_2 - \eta_2}{|\xi - \eta|^2} - \frac{\xi_2}{|\xi|^2}$. Therefore,

$$|v_3(\xi)| \leq C \int_{\mathbf{R}^2} \frac{1}{|\xi||\xi - \eta|} |\eta| (|w_1(\eta)| + |w_2(\eta)|) d\eta\,.$$

Combining this estimate with (4.7), we obtain

$$|b(\xi) v_3(\xi)| \leq C \int_{\mathbf{R}^2} \frac{1}{|\xi - \eta|} |b(\eta) w(\eta)| d\eta\,.$$

Third step: Let $1 < m < 2$ and $w \in L^2(m)$. From lemma B.2 in [39] with $u = |bv_3|$ and $\omega = |bw|$, we get for all $q \in (2, +\infty)$,

$$\| b^{m-1-\frac{2}{q}} b v_3 \|_{L^q(\mathbf{R}^2)} \leq C \|w\|_m\,.$$

Finally, by Hölder's inequality and $q = \frac{2}{m-1} > 2$, we get

$$\|v_3\|_{L^2(\mathbf{R}^2)} \leq C \|b^{-1}\|_{L^{\frac{2q}{q-2}}(\mathbf{R}^2)} \|b v_3\|_{L^q(\mathbf{R}^2)} \leq C \|w\|_m\,,$$

hence $v_3 \in L^2(\mathbf{R}^2)$.

For m greater than 2, $L^2(m) \hookrightarrow L^2(m')$ for some $m' \in (1,2)$ and the previous result ends the proof of lemma 4.1.5. ∎

Remark: This result has no interest in case of stress-free conditions since $w \in L^2_{sf}(m)$ and $\partial_z w = 0$ imply $v_3 = 0$.

4.2 Spectrum of the linear operator $\mathcal{L}$

In this appendix, we are interested in the spectrum of the linear operator $\mathcal{L}$,

$$\mathcal{L} = \Delta_\xi + \frac{1}{2}\xi \cdot \nabla_\xi + 1\,,\; \xi \in \mathbf{R}^2\,.$$

A complete study has already been carried out in [39] when $\mathcal{L}$ is applied to scalar functions, namely in weighted L^2-spaces defined for $m \geq 0$ by

$$\begin{aligned} L^2_{2D}(m) &= \{w : \mathbf{R}^2 \to \mathbf{R}\,,\, \|w\|_{2D(m)} < \infty\} \\ \|w\|_{2D(m)} &= \left(\int_{\mathbf{R}^2} (1+|\xi|^2)^m |w(\xi)|^2\, d\xi\right)^{\frac{1}{2}} = \|b^m w\|_{L^2(\mathbf{R}^2)}\,. \end{aligned}$$

We use the same notation $\mathcal{L}$ for the operator applied to scalar or vectorial functions, as for any vectorial function w,

$$\mathcal{L}w = \mathcal{L}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \mathcal{L}w_1 \\ \mathcal{L}w_2 \\ \mathcal{L}w_3 \end{pmatrix}\,.$$

The spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $L^2_{2D}(m)$, $m \geq 0$, is

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbf{C} \,\Big|\, \mathrm{Re}(\lambda) \leq \frac{1-m}{2}\right\} \cup \left\{-\frac{k}{2} \,\Big|\, k \in \mathbf{N}\right\}\,.$$

Moreover, under the assumption $m > 1$, the discrete spectrum of $\mathcal{L}$ in $L^2_{2D}(m)$ consists of isolated eigenvalues $\lambda_k = -\frac{k}{2}$, $k \in \mathbf{N}$, $k < m-1$, with multiplicity $(k+1)$ and the essential spectrum lies in the half plane $\{\lambda \in \mathbf{C} \,|\, Re(\lambda) \leq \frac{1-m}{2}\}$.

We want to generalize this property for vectorial functions, that is to say, we study the spectrum of $\mathcal{L}$ in the space $L^2(m)$ or $L^2_{sf}(m)$ of vectorial functions defined respectively by (2.4) or (2.35) together with the incompressibility condition (2.2). As $\mathcal{L}$ only acts on the first two components $\xi \in \mathbf{R}^2$, we consider its action on functions independent of z. Hence, the first idea is to split the vorticity w into $\bar{w}$ and $\tilde{w}$ as we did in appendix 4.1. Let us define some useful projections:

$$\begin{array}{llll} \bar{R}: & L^2(m) \text{ or } L^2_{sf}(m) & \longrightarrow & L^2(m) \text{ or } L^2_{sf}(m) \\ & w & \longmapsto & \bar{w} \\ \tilde{R}: & L^2(m) \text{ or } L^2_{sf}(m) & \longrightarrow & L^2(m) \text{ or } L^2_{sf}(m) \\ & w & \longmapsto & \tilde{w}\,. \end{array} \tag{4.9}$$

Then, $\mathbf{1} = \bar{R} + \tilde{R}$ and the projectors $\bar{R}$ and $\tilde{R}$ are well defined by (4.1) in $L^2(m)$ or (4.6) in $L^2_{sf}(m)$ depending on the boundary conditions we consider. Notice that for periodic boundary conditions,

$$\bar{R}(L^2(m)) = \{w \in L^2(m) \,|\, \partial_z w = 0\}$$

while for stress-free conditions,

$$\bar{R}(L^2_{sf}(m)) = \{w \in L^2_{sf}(m) \,|\, w_1 = w_2 = 0\,,\, \partial_z w_3 = 0\}\,.$$

The incompressibility condition (2.2) states in those two-dimensional spaces that

$$\nabla_\xi \cdot w_\xi = \partial_1 w_1 + \partial_2 w_2 = 0\,. \tag{4.10}$$

We now want to study the spectrum of $\mathcal{L}$ in $\bar{R}(L^2(m))$ or $\bar{R}(L^2_{sf}(m))$. Notice that if λ is an eigenvalue of $\mathcal{L}$ with eigenfunction $w = (w_1, w_2, w_3)^T$ in $\bar{R}(L^2(m))$ or $\bar{R}(L^2_{sf}(m))$, then for $i \in \{1, 2, 3\}$, w_i is an eigenfunction of $\mathcal{L}$ in $L^2_{2D}(m)$ with eigenvalue λ. In the next four subsections, we deal with periodic boundary conditions and we postpone the study of stress-free conditions to appendix 4.2.5.

4.2.1 The discrete spectrum of $\mathcal{L}$.

For the purpose of this article, we only turn our attention to the first two eigenvalues. In $L^2_{2D}(m)$, $\lambda_0 = 0$ is a simple eigenvalue of $\mathcal{L}$ with eigenfunction $G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}$ and $\lambda_1 = -\frac{1}{2}$ is an eigenvalue of multiplicity 2 with eigenfunctions $F_1(\xi) = \frac{\xi_1}{2} G(\xi)$ and $F_2(\xi) = \frac{\xi_2}{2} G(\xi)$. As a consequence, 0 and $-\frac{1}{2}$ are eigenvalues of $\mathcal{L}$ in $\bar{R}(L^2(m))$ with multiplicity less than 3 and 6 respectively. Among the possible eigenfunctions, we must check which ones are in $\bar{R}(L^2(m))$ and satisfy the incompressibility condition (4.10).

As far as the first eigenvalue $\lambda_0 = 0$ is concerned, the three possible eigenfunctions are

$$\begin{pmatrix} G \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ G \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ G \end{pmatrix}$$

As $\partial_z G = 0$, the only suitable eigenfunction is $\mathbf{G} = (0, 0, G)^T$. Then, $\lambda_0 = 0$ is a simple eigenvalue of $\mathcal{L}$ in $\bar{R}(L^2(m))$ with eigenfunction $\mathbf{G}$.

The same arguments are valid for the second eigenvalue $\lambda_1 = -\frac{1}{2}$. Six vectorial eigenfunctions can be built from F_1 and F_2 and we must check which ones are suitable. If $w \in \bar{R}(L^2(m))$ satisfies (4.10), it follows for $i = 1$ or 2 that

$$\begin{aligned} \int_{\mathbf{R}^2 \times (0,1)} w_i(\xi, z) d\xi dz &= \int_{\mathbf{R}^2 \times (0,1)} \text{div } (\xi_i w) d\xi dz \\ &= \int_{\mathbf{R}^2} \xi_i w(\xi, 1) d\xi - \int_{\mathbf{R}^2} \xi_i w(\xi, 0) d\xi = 0\,. \end{aligned}$$

Moreover, for $(i, j) \in \{1, 2\}^2$, div $(\xi_i \xi_j w) = \xi_i w_j + \xi_j w_i$ and as $\partial_z w = 0$,

$$\begin{aligned} \int_{\mathbf{R}^2 \times (0,1)} \xi_1 w_1(\xi, z)\, d\xi dz &= \int_{\mathbf{R}^2 \times (0,1)} \xi_2 w_2(\xi, z)\, d\xi dz = 0\,, \\ \int_{\mathbf{R}^2 \times (0,1)} \xi_1 w_2(\xi, z)\, d\xi dz &= - \int_{\mathbf{R}^2 \times (0,1)} \xi_2 w_1(\xi, z)\, d\xi dz\,. \end{aligned}$$

As $\int_{\mathbf{R}^2\times(0,1)} \xi_1 F_1 \, d\xi dz = \int_{\mathbf{R}^2\times(0,1)} \xi_2 F_2 \, d\xi dz = 1$ and $\int_{\mathbf{R}^2\times(0,1)} \xi_j F_i \, d\xi dz = 0$ for $i \neq j$, the only vectorial eigenfunctions which satisfy the above conditions are

$$\mathbf{F}_1 = \begin{pmatrix} 0 \\ 0 \\ F_1 \end{pmatrix}, \ \mathbf{F}_2 = \begin{pmatrix} 0 \\ 0 \\ F_2 \end{pmatrix}, \ \mathbf{F}_3 = \begin{pmatrix} -F_2 \\ F_1 \\ 0 \end{pmatrix}.$$

Since these three vectors are independent vectors, $\lambda_1 = -\frac{1}{2}$ is an eigenvalue of $\mathcal{L}$ in $L^2(m)$ of multiplicity 3 with eigenfunctions $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$.

Remark: Even if we do not need for the purpose of this article more information on the discrete spectrum of $\mathcal{L}$ in $\bar{R}(L^2(m))$, we can state that for any $k \in \mathbf{N}$, $-\frac{k}{2}$ is an eigenvalue of $\mathcal{L}$ in $\bar{R}(L^2(m))$ with multiplicity $(2k+1)$. Indeed, $-\frac{k}{2}$ is an eigenvalue of $\mathcal{L}$ in $L^2_{2D}(m)$ with multiplicity $(k+1)$. Hence, there could be a maximum of $3(k+1)$ suitable vectorial eigenfunctions. However, to be in $\bar{R}(L^2(m))$ and verify the incompressibility condition (4.10), the vectorial eigenfunctions must satisfy $(k+2)$ relations on moments of order k. Indeed,

$$\int_{\mathbf{R}^2\times(0,1)} \xi_1^k w_1(\xi, z) \, d\xi dz = \int_{\mathbf{R}^2\times(0,1)} \xi_2^k w_2(\xi, z) \, d\xi dz = 0$$

and the k other moments of order k of w_1 can be expressed by the k other moments of order k of w_2. Therefore, only $3(k+1) - (k+2) = 2k+1$ vectorial eigenfunctions are suitable.

As a consequence, the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $\bar{R}(L^2(m))$ satisfies

$$\sigma(\mathcal{L}) \supset \left\{ -\frac{k}{2} \,\middle|\, k \in \mathbf{N} \right\}. \tag{4.11}$$

4.2.2 The essential spectrum of $\mathcal{L}$.

In [39], it is proved that the essential spectrum of $\mathcal{L}$ in $L^2_{2D}(m)$ lies in the half plane $\{\lambda \in \mathbf{C} \,|\, Re(\lambda) \leq \frac{1-m}{2}\}$. For any $\lambda \in \mathbf{C}$ with $Re(\lambda) < \frac{1-m}{2}$, there exists $\psi_\lambda \in C^\infty(\mathbf{R}^2, \mathbf{R})$ such that $\mathcal{L}\psi_\lambda = \lambda\psi_\lambda$. Then, for any $\lambda \in \mathbf{C}$ with $Re(\lambda) < \frac{1-m}{2}$, $(0, 0, \psi_\lambda)^T$ is a vectorial eigenfunction of $\mathcal{L}$ in $\bar{R}(L^2(m))$ which satisfies the incompressibility condition (4.10) and since the spectrum of $\mathcal{L}$ is closed,

$$\sigma(\mathcal{L}) \supset \left\{ \lambda \in \mathbf{C} \,\middle|\, Re(\lambda) \leq \frac{1-m}{2} \right\}. \tag{4.12}$$

4.2.3 The spectral projections.

Assume $m \geq 0$. For $n \in \{-1, 0, 1\}$ and $n+1 < m$, we define P_n the spectral projection onto the $\sum_{k=0}^{n}(2k+1)$-dimensional subspace of $\bar{R}(L^2(m))$ spanned by the eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalues $\{-\frac{k}{2}\,|\,k=0,..,n\}$. For any $w \in \bar{R}(L^2(m))$,

$$P_{-1}w = 0$$
$$P_0 w = \alpha \mathbf{G}$$
$$P_1 w = \alpha \mathbf{G} + \sum_{i=1}^{3} \beta_i \mathbf{F}_i$$

where

$$\alpha = \int_{\mathbf{R}^2\times(0,1)} w_3 \, d\xi dz \tag{4.13}$$
$$\beta_1 = \int_{\mathbf{R}^2\times(0,1)} \xi_1 w_3 \, d\xi dz$$
$$\beta_2 = \int_{\mathbf{R}^2\times(0,1)} \xi_2 w_3 \, d\xi dz$$
$$\beta_3 = \int_{\mathbf{R}^2\times(0,1)} \frac{1}{2}(\xi_1 w_2 - \xi_2 w_1) \, d\xi dz\,.$$

We also denote $\mathcal{W}_n$ the complement of the corresponding spectral subspace

$$\mathcal{W}_n = \{w \in \bar{R}(L^2(m))\,|\,P_n w = 0\}\,. \tag{4.14}$$

Finally, we also define the complementary spectral projection Q_n by

$$Q_n = \bar{R} - P_n\,. \tag{4.15}$$

Then, $\mathbf{1} = P_n + Q_n + \tilde{R}$ in case of periodic boundary conditions with $\tilde{R}$ defined in (4.9).

4.2.4 The semigroup $e^{\tau\mathcal{L}}$.

The operator $\mathcal{L}$ is the generator of a linear semigroup $e^{\tau\mathcal{L}}$ in $\bar{R}(L^2(m))$ which satisfies the following estimates:

Proposition 4.2.1 *Let $n \in \{-1,0,1\}$, $m > n+1$ and $q \in [1,2]$. For all $\alpha = (\alpha_1,\alpha_2,0) \in \boldsymbol{N}^2 \times \{0\}$ and all $\epsilon > 0$, there exists $C > 0$ such that for all $w \in \bar{R}(L^2(m))$ and all $\tau > 0$,*

$$\|b^m \partial^\alpha e^{\tau\mathcal{L}} Q_n w\|_{L^2(\boldsymbol{R}^2)} \leq \frac{Ce^{-\gamma\tau}}{a(\tau)^{\frac{1}{q}-\frac{1}{2}+\frac{|\alpha|}{2}}} \|b^m w\|_{L^q(\boldsymbol{R}^2)}$$

where

$$a(\tau) = 1 - e^{-\tau}$$
$$\gamma = \frac{m-1-\epsilon}{2} \quad \text{if } n+1 < m \leq n+2$$
$$\gamma = \frac{n+1}{2} \quad \text{if } m > n+2\,.$$

Proof: We deal separately with the different values of $n \in \{-1, 0, 1\}$. The idea of this proof is to come back to scalar functions to take benefit of Th. Gallay and C.E. Wayne's work in two dimensions, see [39] and proposition 4.2.2 below. Therefore, we introduce other spectral projections for the scalar and two-dimensional case (see Appendix A in [39]). For any $n \in \mathbf{N}$, let $\bar{P}_n$ be the spectral projection onto the $\sum_{k=0}^{n}(k+1)$-dimensional subspace of $L^2_{2D}(m)$ spanned by the eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalues $\{-\frac{k}{2} \,|\, k = 0, .., n\}$. Notice that the condition $\bar{P}_n f = 0$ is also equivalent to

$$\int_{\mathbf{R}^2} \xi^\alpha f(\xi) d\xi = 0 \text{ for all } \alpha \in \mathbf{N}^2 \text{ with } |\alpha| \leq n\,.$$

For any $n < 0$, define $\bar{P}_n = 0$. Moreover, we denote for any $n \in \mathbf{Z}$

$$\bar{Q}_n = \mathbf{1} - \bar{P}_n\,. \tag{4.16}$$

Let $m \geq 0$, $q \in [1, 2]$, $\alpha = (\alpha_1, \alpha_2, 0)$ and $\epsilon > 0$. Assume $w \in \bar{R}(L^2(m))$.

Case 1: $n = -1$. Then, $Q_{-1}w = (\bar{R} - P_{-1})w = w$ and by proposition 4.2.2, we get

$$\begin{aligned} \|b^m \partial^\alpha e^{\tau\mathcal{L}} Q_{-1} w\|_{L^2(m)} &\leq \sum_{i=1}^{3} \|b^m \partial^\alpha e^{\tau\mathcal{L}} \bar{Q}_{-1} w_i\|_{L^2(\mathbf{R}^2)} \\ &\leq \frac{Ce^{-\gamma\tau}}{a(\tau)^{(\frac{1}{q}-\frac{1}{2}+\frac{|\alpha|}{2})}} \sum_{i=1}^{3} \|b^m w_i\|_{L^q(\mathbf{R}^2)} \\ &\leq \frac{Ce^{-\gamma\tau}}{a(\tau)^{(\frac{1}{q}-\frac{1}{2}+\frac{|\alpha|}{2})}} \|b^m w\|_{L^q(\mathbf{R}^2)} \end{aligned}$$

where γ is defined in the statement of proposition 4.2.1.

Case 2: $n = 0$. Then,

$$Q_0 w = (\bar{R} - P_0)w = w - \alpha\mathbf{G} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 - \alpha G \end{pmatrix}.$$

As stressed before in this appendix, $\int_{\mathbf{R}^2\times(0,1)} w_1 d\xi dz = 0$, $\int_{\mathbf{R}^2\times(0,1)} w_2 d\xi dz = 0$, $\int_{\mathbf{R}^2\times(0,1)} (w_3 - \alpha G) d\xi dz = 0$. As w is independent of z, these equalities precisely state that

$$\bar{Q}_0 w_1 = w_1\,,\ \bar{Q}_0 w_2 = w_2\,, \bar{Q}_0 w_3 = w_3 - \alpha G\,.$$

Then,

$$Q_0 w = \begin{pmatrix} \bar{Q}_0 w_1 \\ \bar{Q}_0 w_2 \\ \bar{Q}_0 w_3 \end{pmatrix}$$

and by proposition 4.2.2, we get

$$\begin{aligned} \|b^m \partial^\alpha e^{\tau\mathcal{L}} Q_0 w\|_{L^2(\mathbf{R}^2)} &\leq \sum_{i=1}^{3} \|b^m \partial^\alpha e^{\tau\mathcal{L}} \bar{Q}_0 w_i\|_{L^2(\mathbf{R}^2)} \\ &\leq \frac{Ce^{-\gamma\tau}}{a(\tau)^{(\frac{1}{q}-\frac{1}{2}+\frac{|\alpha|}{2})}} \|b^m w\|_{L^q(\mathbf{R}^2)} \end{aligned}$$

where γ is defined as in proposition 4.2.1.

Case 3: $n = 1$. Then,

$$Q_1 w = (\bar{R} - P_1)w = w - (\alpha \mathbf{G} + \sum_{i=1}^{3} \beta_i \mathbf{F}_i) = \begin{pmatrix} w_1 + \beta_3 F_2 \\ w_2 - \beta_3 F_1 \\ w_3 - \alpha G - \beta_1 F_1 - \beta_2 F_2 \end{pmatrix}.$$

As α, β_1, β_2 and β_3 have been chosen as in (4.13) so that the moments up to order one of $Q_1 w$ are zero,

$$Q_1 w = \begin{pmatrix} \bar{Q}_1 w_1 \\ \bar{Q}_1 w_2 \\ \bar{Q}_1 w_3 \end{pmatrix}$$

and proposition 4.2.2 applies coordinate by coordinate,

$$\|b^m \partial^\alpha e^{\tau \mathcal{L}} Q_1 w\|_{L^2(\mathbf{R}^2)} \leq \frac{C e^{-\gamma \tau}}{a(\tau)^{(\frac{1}{q} - \frac{1}{2} + \frac{|\alpha|}{2})}} \|b^m w\|_{L^q(\mathbf{R}^2)}.$$

Then, proposition 4.2.1 holds for all values of $n \in \{-1, 0, 1\}$. ■

For easy reference, we reproduce here the main estimates of the study of $e^{\tau \mathcal{L}}$ in [39].

Proposition 4.2.2 ***(Th. Gallay and C.E. Wayne)*** *Let $m \geq 0$, $n \in \boldsymbol{Z}$ and $q \in [1, 2]$ such that $n + 1 < m$. For all $\alpha = (\alpha_1, \alpha_2) \in \boldsymbol{N}^2$ and all $\epsilon > 0$, there exists $C > 0$ such that for all $w \in L^2_{2D}(m)$ and all $\tau > 0$,*

$$\|b^m \partial^\alpha e^{\tau \mathcal{L}} \bar{Q}_n w\|_{L^2(\boldsymbol{R}^2)} \leq \frac{C e^{-\gamma \tau}}{a(\tau)^{(\frac{1}{q} - \frac{1}{2} + \frac{|\alpha|}{2})}} \|b^m w\|_{L^q(\boldsymbol{R}^2)}$$

where

$$\gamma = \frac{m - 1 - \epsilon}{2} \quad \text{if } n + 1 < m \leq n + 2$$
$$\gamma = \frac{n + 1}{2} \quad \text{if } m > n + 2,$$

and where $\bar{Q}_n$ is defined in (4.16).

Proof: If $q = 2$, proposition 4.2.2 follows from proposition A.2 in [39]. If $q < 2$, and $\tau \in (0, 2)$, proposition 4.2.2 follows from proposition A.5 in [39]. If $q < 2$ and $\tau \geq 2$, using the above results, we get

$$\begin{aligned} \|b^m \partial^\alpha e^{\tau \mathcal{L}} \bar{Q}_n w\|_{L^2(\mathbf{R}^2)} &= \|b^m \partial^\alpha e^{(\tau - 1)\mathcal{L}} \bar{Q}_n e^{\mathcal{L}} \bar{Q}_n w\|_{L^2(\mathbf{R}^2)} \\ &\leq \frac{C e^{-\gamma(\tau - 1)}}{a(\tau - 1)^{\frac{|\alpha|}{2}}} \|b^m e^{\mathcal{L}} \bar{Q}_n w\|_{L^2(\mathbf{R}^2)} \\ &\leq \frac{C e^{-\gamma \tau}}{a(\tau - 1)^{\frac{|\alpha|}{2}} a(1)^{(\frac{1}{q} - \frac{1}{2})}} \|b^m w\|_{L^q(\mathbf{R}^2)} \\ &\leq \frac{C e^{-\gamma \tau}}{a(\tau)^{(\frac{|\alpha|}{2} + \frac{1}{q} - \frac{1}{2})}} \|b^m w\|_{L^q(\mathbf{R}^2)}. \end{aligned}$$

This ends the proof of proposition 4.2.2. ■

Remark: Using proposition 4.2.1, it is easy to complete the study of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $\bar{R}(L^2(m))$. Let $n \in \mathbf{Z}$ and $m \geq 0$ such that $n+1 < m \leq n+2$. Then, $\sigma(\mathcal{L}) = \sigma(\mathcal{L}P_n) \cup \sigma(\mathcal{L}Q_n)$. By construction, $\sigma(\mathcal{L}P_n) = \emptyset$ if $n < 0$ and $\sigma(\mathcal{L}P_n) = \{0, -\frac{1}{2}, .., -\frac{n}{2}\}$ if $n \in \mathbf{N}$. On the other hand, by the Hille-Yosida theorem (see [67]) and proposition 4.2.1, $\sigma(\mathcal{L}Q_n) \subset \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}(\lambda) \leq \frac{1-m}{2}\}$. Thus, using (4.11) and (4.12),

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbf{C} \,\middle|\, \mathrm{Re}(\lambda) \leq \frac{1-m}{2}\right\} \cup \left\{-\frac{k}{2} \,\middle|\, k \in \mathbf{N}\right\} .$$

4.2.5 Stress-free boundary conditions

In an analogous way, we can study the spectrum of $\mathcal{L}$ in $\bar{R}(L^2_{sf}(m))$ where the projector $\bar{R}$ for stress-free boundary conditions has been defined in (4.9) and (4.6). We recall that in this case

$$\bar{R}(L^2_{sf}(m)) = \{w \in L^2_{sf}(m) \,|\, w_1 = w_2 = 0 \,,\, \partial_z w_3 = 0\} .$$

Then, the study of $\sigma(\mathcal{L})$ in $\bar{R}(L^2_{sf}(m))$ with stress-free boundary conditions can be brought back to the study of [39] for the two-dimensional Navier-Stokes equation. The discrete spectrum of $\mathcal{L}$ in $\bar{R}(L^2_{sf}(m))$, $m > 1$, consists of isolated eigenvalues $\lambda_k = -\frac{k}{2}$, $k \in \mathbf{N}$, $k < m-1$, with multiplicity $(k+1)$ and eigenvalues $(0,0,\phi_\alpha)^T$ where for any $\alpha \in \mathbf{N}^2$, $\phi_\alpha \in \mathcal{S}(\mathbf{R}^n)$ is the Hermite function defined by $\phi_\alpha = \partial_\xi^\alpha G$ and $|\alpha| = k$. Namely, $\phi_{(0,0)} = G$, $\phi_{(1,0)} = F_1$, $\phi_{(0,1)} = F_2$. Moreover, the essential spectrum lies in the half plane $\{\lambda \in \mathbf{C} \,|\, Re(\lambda) \leq \frac{1-m}{2}\}$. We can also define spectral projections as in appendix 4.2.3. If for any $n < 0$, $P_n = 0$ and for any $n \in \mathbf{N}$, P_n is the spectral projection onto the $\sum_{k=0}^{n}(k+1)$-dimensional subspace of $\bar{R}(L^2_{sf}(m))$ spanned by the eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalues $\{-\frac{k}{2} \,|\, k = 0, .., n\}$, we have for any $w \in \bar{R}(L^2_{sf}(m))$,

$$\begin{aligned} P_{-1}w &= 0 \\ P_0 w &= \alpha \mathbf{G} \\ P_1 w &= \alpha \mathbf{G} + \sum_{i=1}^{2} \beta_i \mathbf{F}_i \,, \end{aligned}$$

where α, β_1 and β_2 are defined in (4.13). Then, proposition 4.2.1 still holds since it is an easy consequence of proposition 4.2.2.

4.3 Bounds on the evolution operator $S(\tau, \sigma)$

In this section, we consider the operator $\Lambda(\tau)$ given for any $\tau \geq 0$ by

$$\Lambda(\tau) = \mathcal{L} + e^\tau \partial_z^2 = (\Delta_\xi + \frac{1}{2}\xi \cdot \nabla_\xi + 1) + e^\tau \partial_z^2 .$$

Since its coefficients depends linearly on the space variables (ξ, z), $\Lambda(\tau)$ becomes a first order differential operator when expressed in the Fourier variables (k, n) defined in (4.4).

Indeed, for any $n \in \mathbf{Z}$, $k \in \mathbf{R}^2$ and $\tau > 0$,

$$(\Lambda(\tau)f)_n(k) = -(|k|^2 + \frac{1}{2}k \cdot \nabla_k + 4\pi^2 e^\tau n^2) f_n(k) .$$

Then, $\Lambda(\tau)$ is the generator of a family of evolution operators (or evolution system) $S(\tau,\sigma)$ given for any $0 \le \sigma \le \tau$ by

$$S(\tau,\sigma) = e^{(\tau-\sigma)\mathcal{L}} \circ e^{(e^\tau - e^\sigma)\partial_z^2} = e^{(e^\tau - e^\sigma)\partial_z^2} \circ e^{(\tau-\sigma)\mathcal{L}} ,$$

or in Fourier variables $(k,n) \in \mathbf{R}^2 \times \mathbf{Z}$ by

$$(S(\tau,\sigma)f)_n(k) = e^{-a(\tau-\sigma)|k|^2} e^{-4\pi^2(e^\tau - e^\sigma)n^2} f_n\left(k e^{-\frac{\tau-\sigma}{2}}\right) ,$$

where $a(\tau) = 1 - e^{-\tau}$. We refer to Henry [48] chapter 7.1 and Pazy [67] chapter 5 for more information on evolution operators or evolution systems. The aim of this section is to prove the following estimates on the evolution system $S(\tau,\sigma)$ for any $0 < \sigma < \tau$:

Proposition 4.3.1 ***(a)*** *Fix $m > 1$. For all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \boldsymbol{N}^3$ and $q \in [1,2]$, there exists $C > 0$ such that for all $w \in L^2(m)$ or $L^2_{sf}(m)$ and all $0 < \sigma < \tau$,*

$$\|\partial^\alpha S(\tau,\sigma) w\|_m \le \frac{C}{a(\tau-\sigma)^{\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2}} a(e^\tau - e^\sigma)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{2})+\frac{\alpha_3}{2}}} \|b^m w\|_{L^q(\boldsymbol{R}^2 \times (0,1))}$$

where $a(\tau) = 1 - e^{-\tau}$.
(b) *Fix $m > 1$. For all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \boldsymbol{N}^3$ and $q \in [1,2]$, there exists $C > 0$ such that for all $w \in L^2(m)$ or $L^2_{sf}(m)$ and $0 < \sigma < \tau$, assuming $\alpha_3 \neq 0$ or $\tilde{R}w = w$,*

$$\|\partial^\alpha S(\tau,\sigma) w\|_m \le \frac{C e^{-4\pi^2(e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2}} a(e^\tau - e^\sigma)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{2})+\frac{\alpha_3}{2}}} \|b^m w\|_{L^q(\boldsymbol{R}^2 \times (0,1))} .$$

Remark: $L^2(m)$ is defined in (2.4), $L^2_{sf}(m)$ in (2.35) and $\tilde{R}$ in (4.9).
Proof: To prove (a), we expand $\partial^\alpha S(\tau,\sigma)w$ in Fourier series. In the case of periodic conditions,

$$\partial^\alpha S(\tau,\sigma) w(\xi,z) = \sum_{n\in\mathbf{Z}} (2i\pi n)^{\alpha_3} e^{-4\pi^2(e^\tau - e^\sigma)n^2} \partial^{(\alpha_1,\alpha_2)} e^{(\tau-\sigma)\mathcal{L}} w_n(\xi) e^{2i\pi n z} \tag{4.17}$$

where for any $n \in \mathbf{Z}$,

$$w_n(\xi) = \int_0^1 w(\xi,z) e^{-2i\pi n z} dz.$$

Then, using Parseval's equality, we get

$$\|\partial^\alpha S(\tau,\sigma) w\|_m^2 = \sum_{n\in\mathbf{Z}} (2\pi n)^{2\alpha_3} e^{-8\pi^2(e^\tau - e^\sigma)n^2} \|b^m \partial^{(\alpha_1,\alpha_2)} e^{(\tau-\sigma)\mathcal{L}} w_n\|^2_{L^2(\mathbf{R}^2)} .$$

In case of stress-free conditions, Fourier series in sinus and cosinus lead to a similar Parseval's equality. By proposition 4.2.1 with $n = -1$, $m > 1$, for any $\epsilon > 0$, any $q \in [1, 2]$, there exists $C > 0$ independent of n such that

$$\|\partial^\alpha S(\tau,\sigma)w\|_m \leq \frac{C}{a(\tau-\sigma)^{(\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2})}} \left(\sum_{n\in\mathbf{Z}} g_n^2 \|b^m w_n\|^2_{L^q(\mathbf{R}^2)}\right)^{\frac{1}{2}}$$

where $g_n = (2\pi n)^{\alpha_3} e^{-4\pi^2(e^\tau - e^\sigma)n^2}$. Finally, using Hölder's inequality, we get

$$\|\partial^\alpha S(\tau,\sigma)w\|_m \leq \frac{C}{a(\tau-\sigma)^{(\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2})}} \|g_n\|_{l^p} \left(\|\left(\|b^m w_n\|_{L^q(\mathbf{R}^2)}\right)\|_{l^{q'}} \right)$$

where p and q' satisfy the relation $\frac{1}{p} + \frac{1}{q'} = \frac{1}{2}$. By appendix 4.5.1 with $\gamma = \alpha_3 p$ and $A = 4\pi^2 p(e^\tau - e^\sigma)$, there exists $C > 0$ such that for any $0 < \sigma < \tau$,

$$\|g_n\|_{l^p} \leq \frac{C}{a(e^\tau - e^\sigma)^{\frac{1}{2p}+\frac{\alpha_3}{2}}}. \tag{4.18}$$

Moreover, Riesz-Thorin's interpolation's theory [73] asserts that if $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\| \|b^m w_n\|_{L^q(\mathbf{R}^2)}\|_{l^{q'}} \leq C \|b^m w\|_{L^q(\mathbf{R}^2\times(0,1))}.$$

Indeed, $\|\left(\|b^m w_n\|_{L^1(\mathbf{R}^2)}\right)\|_{l^\infty} \leq \|b^m w\|_{L^1(\mathbf{R}^2\times(0,1))}$ and by Parseval's equality, $\|\|\|b^m w_n\|_{L^2(\mathbf{R}^2)}\|_{l^2} \leq \|b^m w\|_{L^2(\mathbf{R}^2\times(0,1))}$. This concludes the proof of (a).

As far as the property (b) is concerned, the only difference appears in the bound of $\|g_n\|_{l^p}$ in (4.18). As $\tilde{R}w = w$ or $\alpha_3 \neq 0$, the sum (4.17) over $n \in \mathbf{Z}$ only appears in the proof (b) over $n \neq 0$. Indeed, in case of periodic boundary conditions,

$$\partial^\alpha S(\tau,\sigma)w(\xi,z) = \sum_{n\neq 0} i^{\alpha_3} g_n \partial^{(\alpha_1,\alpha_2)} e^{(\tau-\sigma)\mathcal{L}} w_n(\xi) e^{2i\pi n z}$$

and the same phenomenon occurs in case of stress-free boundary conditions. Then, by appendix 4.5.1,

$$\left(\sum_{n\neq 0} |g_n|^p\right)^{1/p} \leq C \frac{e^{-4\pi^2(e^\tau - e^\sigma)}}{a(e^\tau - e^\sigma)^{\frac{1}{2p}+\frac{\alpha_3}{2}}}.$$

This concludes the proof of (b). ■

4.4 Bounds on the evolution operator $\mathcal{S}(\tau,\sigma)$

In this section, we consider the operator $\mathcal{M}(\tau)$ given for any $\tau \geq 0$ and any $w \in L^2(m)$ by

$$\mathcal{M}(\tau)w = \Lambda(\tau)w + N_2(w) = \mathcal{L}w + e^\tau \partial_z^2 w - \Omega e^{\frac{3\tau}{2}} \partial_z v\,,$$

where v is given in terms of w via the Biot-Savart law (see appendix 4.1). We recall that the time-dependent operator $\Lambda(\tau)$ has been studied in appendix 4.3 and the operator $\mathcal{L}$ in appendix 4.2.

If we consider periodic boundary conditions in $\mathbf{R}^2 \times (0,1)$, any function $w \in L^2(m)$ can be decomposed as in appendix 4.1 into $w(x,z) = \bar{w}(x) + \tilde{w}(x,z)$ where $\int_0^1 \tilde{w}(x,z)dz = 0$. Plugging this decomposition into the above definition of $\mathcal{M}(\tau)$ shows that

$$\begin{aligned}\mathcal{M}(\tau)w &= \Lambda(\tau)(\bar{w}+\tilde{w}) + N_2(\bar{w}+\tilde{w}) \\ &= \mathcal{L}\bar{w} + \mathcal{M}(\tau)\tilde{w} = \mathcal{L}\bar{R}(w) + \mathcal{M}(\tau)\tilde{R}(w)\,.\end{aligned}$$

Therefore, we are only interested in the action of $\mathcal{M}(\tau)$ in $\tilde{R}(L^2(m))$. In this space, the Biot-Savart law is given in appendix 4.1.2 by the matricial equality (4.5). Since this equality is given in terms of Fourier coordinates, it is natural to express $\mathcal{M}(\tau)w$ in Fourier variables. Indeed, for any $n \in \mathbf{Z}^*$, $k \in \mathbf{R}^2$, $\tau > 0$ and $w \in \tilde{R}(L^2(m))$,

$$(\mathcal{M}(\tau)w)_n(k) = -\left(|k|^2 + \frac{1}{2}k\cdot\nabla_k + 4\pi^2 e^\tau n^2 - \Omega e^\tau 2i\pi n \frac{\tilde{A}_n(k,\tau)}{|k|^2+4\pi^2n^2}\right) w_n(k)\,, \qquad (4.19)$$

where $\tilde{A}_n(k,\tau)$ is the matrix corresponding to (4.5) in scaling variables. $\mathcal{M}(\tau)$ is then the generator of a family of evolution operators $\mathcal{S}(\tau,\sigma)$ which satisfies the following estimates:

Proposition 4.4.1 *Fix $m > 1$. For all $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \boldsymbol{N}^3$ and $q \in [1,2]$, there exists $C > 0$ such that for all $w \in \tilde{R}(L^2(m))$ and all $0 < \sigma < \tau$,*

$$\|\partial^\alpha \mathcal{S}(\tau,\sigma)w\|_m \le \frac{Ce^{-4\pi^2(e^\tau - e^\sigma)}}{a(\tau-\sigma)^{\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2}}\, a(e^\tau - e^\sigma)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{2})+\frac{\alpha_3}{2}}}\|b^m w\|_{L^q(\boldsymbol{R}^2\times(0,1))}\,.$$

Remark: Proposition 4.3.1(b) is a particular case of proposition 4.4.1 when $\Omega = 0$.

Proof: The first narural idea is to work in Fourier coordinates as in appendix 4.3. However, as the operator $k\cdot\nabla_k$ in (4.19) does not commute with $\tilde{A}_n(k)$, it is not possible to get an easy exponential formula for $\mathcal{S}(\tau,\sigma)$ as we did in appendix 4.3 for $S(\tau,\sigma)$. However, this disadvantage is due to the operator $k\cdot\nabla_k$ which comes from self-similar variables. Therefore, we choose to return to initial variables (x,z) and initial functions (u,ω) to study $\mathcal{M}(\tau)$. ■

Denote by $\mathcal{M}$ the corresponding operator to $\mathcal{M}(\tau)$ in original variables. Then, for any $\omega \in \tilde{R}(L^2(m))$,

$$\mathcal{M}\omega = \Delta\omega - \Omega\partial_z u\,, \qquad (4.20)$$

where u is given in terms of ω via the Biot-Savart law (see appendix 4.1.2). Expressed in Fourier coordinates, this equality gives for any $n \in \mathbf{Z}^*$ and $k \in \mathbf{R}^2$,

$$(\mathcal{M}\omega)_n(k) = -\left(|k|^2 + 4\pi^2n^2 + \frac{2i\pi n\Omega}{|k|^2+4\pi^2n^2}A_n(k)\right)\omega_n(k)$$

where

$$A_n(k) = \begin{pmatrix} 0 & -i2\pi n & ik_2 \\ i2\pi n & 0 & -ik_1 \\ -ik_2 & ik_1 & 0 \end{pmatrix} . \qquad (4.21)$$

Then, $\mathcal{M}$ generates a linear semigroup $e^{t\mathcal{M}}$ given by the following expression

$$\left(e^{t\mathcal{M}}\omega\right)_n(k) = e^{-tA_n^{\Omega}(k)}\omega_n(k)$$

where

$$A_n^{\Omega}(k) = (|k|^2 + 4\pi^2 n^2)I_3 + \frac{2i\pi n\Omega}{|k|^2 + 4\pi^2 n^2} A_n(k) . \qquad (4.22)$$

Notice that this matrix $A_n^{\Omega}(k)$ appears in section 3.4 in the study of the free linear rotating equation (3.8). There, we prove that $A_n^{\Omega}(k)$ is diagonalisable and that its three eigenvalues are $|k|^2 + 4\pi^2 n^2$, $|k|^2 + 4\pi^2 n^2 \pm \frac{2i\pi n\Omega}{\sqrt{|k|^2+4\pi^2 n^2}}$. Then, the linear semigroup $e^{t\mathcal{M}}$ satisfies the following estimates:

Proposition 4.4.2 *Fix $m > 1$. For all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \boldsymbol{N}^3$ and $q \in [1, 2]$, there exists $C > 0$ such that for all $\omega \in \tilde{R}(L^2(m))$ and all $0 < s < t$,*

$$\|\partial^{\alpha} e^{(t-s)\mathcal{M}}\omega\|_m \le \frac{Ce^{-4\pi^2(t-s)}}{(t-s)^{\frac{1}{q}-\frac{1}{2}+\frac{\alpha_1+\alpha_2}{2}} a(t-s)^{\frac{1}{2}(\frac{1}{q}-\frac{1}{2})+\frac{\alpha_3}{2}}} \|b^m\omega\|_{L^q(\boldsymbol{R}^2\times(0,1))} .$$

Proof: We use for this proof the expression of $e^{t\mathcal{M}}$ in Fourier coordinates and proceed as in the previous appendix 4.3 on $S(\tau, \sigma)$. Thus, we get

$$\begin{aligned} \|\partial^{\alpha} e^{(t-s)\mathcal{M}}\omega\|_m^2 &\le C \sum_{n\in\mathbf{Z}^*} \int_{\mathbf{R}^2} \sum_{|\beta|\le m} \left|\partial_k^{\beta}\left(\partial^{\alpha} e^{(t-s)\mathcal{M}}\omega\right)_n(k)\right|^2 dk \\ &\le C \sum_{n\in\mathbf{Z}^*} \int_{\mathbf{R}^2} (2\pi n)^{2\alpha_3} \sum_{|\beta|\le m} \left|\partial_k^{\beta}\left(k_1^{\alpha_1}k_2^{\alpha_2} e^{-(t-s)(|k|^2+4\pi^2n^2)} P_n(k)\omega_n(k)\right)\right|^2 dk \end{aligned}$$

where $P_n(k)$ is a matrix due to diagonalisation. Since $\|P_n(k)\|_{L^{\infty}(\mathcal{M}_3(\mathbf{C}))} \le 1$,

$$\|\partial^{\alpha} e^{(t-s)\mathcal{M}}\omega\|_m^2 \le C \sum_{n\in\mathbf{Z}^*} \int_{\mathbf{R}^2} (2\pi n)^{2\alpha_3} |k_1^{\alpha_1}k_2^{\alpha_2}|^2 e^{-2(t-s)(|k|^2+4\pi^2n^2)} \sum_{|\beta|\le m} \left|\partial_k^{\beta}\omega_n(k)\right|^2 dk .$$

By Hölder's inequality for any $(p, p') \in \mathbf{N}^2$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\|\partial^{\alpha} e^{(t-s)\mathcal{M}}\omega\|_m^2 \le C\int_{\mathbf{R}^2} |k_1^{\alpha_1}k_2^{\alpha_2}|^2 \left(\sum_{n\in\mathbf{Z}^*} \left|n^{\alpha_3} e^{-(t-s)(|k|^2+4\pi^2n^2)}\right|^{2p}\right)^{\frac{1}{p}} \left\|\sum_{|\beta|\le m} \left|\partial_k^{\beta}\omega_n(k)\right|\right\|_{l^{2p'}}^2 dk$$

Applying appendix 4.5.1 with $\gamma = 2\alpha_3 p$ and $A = 8\pi^2 p(t-s)$ and once more Hölder's inequality, we obtain

$$\|\partial^\alpha e^{(t-s)\mathcal{M}}\omega\|_m^2 \leq \frac{Ce^{-8\pi^2(t-s)}}{a(t-s)^{\alpha_3+\frac{1}{2p}}} \left(\int_{\mathbf{R}^2} |k_1^{\alpha_1} k_2^{\alpha_2}|^{2p} e^{-2p(t-s)|k|^2} dk\right)^{\frac{1}{p}} \Big\| \sum_{|\beta|\leq m} \left|\partial_k^\beta \omega_n(k)\right| \Big\|_{L^{2p'} l^{2p'}}^2$$

Using interpolation theory to obtain the L^q norm of $b^m\omega$ with $q = (2p')' \in [1,2]$ and a change of variables $u = \sqrt{t-s}\,k$ to bound $\int_{\mathbf{R}^2} |k_1^{\alpha_1} k_2^{\alpha_2}|^{2p} e^{-2p(t-s)|k|^2} dk$ conclude the proof. ■

4.5 Bounds on integrals and series

The aim of this technical appendix is to precise the bound of a sum which appears in appendix 4.3 and to give some details in the bound of an integral used quite often throughout this paper.

4.5.1 How to bound $\sum_{n\in\mathbf{Z}} |n|^\gamma e^{-An^2}$?

Proposition 4.5.1 *Let γ be a positive constant. There exists $C > 0$ such that for any $A > 0$,*

$$S(A) \equiv \sum_{n\in \boldsymbol{Z}} |n|^\gamma e^{-An^2} \leq \frac{C}{a(A)^{\frac{\gamma+1}{2}}},$$

$$T(A) \equiv \sum_{n\in \boldsymbol{Z}^*} |n|^\gamma e^{-An^2} \leq \frac{Ce^{-A}}{a(A)^{\frac{\gamma+1}{2}}},$$

where $a(A) = 1 - e^{-A}$.

Proof: Since the function S is continous on $(0, +\infty)$ and uniformly bounded on $[\epsilon, +\infty)$ for any $\epsilon > 0$, the bound on $S(A)$ follows from the computation

$$\lim_{A\to 0} A^{\frac{\gamma+1}{2}} S(A) = \int_{\mathbf{R}} |x|^\gamma e^{-x^2} dx = \Gamma\left(\frac{\gamma+1}{2}\right),$$

where Γ is the Euler function. The bound on $T(A)$ is then an easy consequence of the previous result by a change of index. ■

4.5.2 Bound on integrals

Proposition 4.5.2 *Let $(\alpha, \beta, \gamma, \delta) \in (\boldsymbol{R}^+)^4$ such that $\gamma + \delta < 1$. Then, there exists a positive constant $C > 0$ such that for any $t \geq 0$,*

$$I(t) \equiv \int_0^t \frac{e^{\alpha s} e^{\beta(t-s)} e^{-4\pi^2(e^t - e^s)}}{a(t-s)^\gamma a(e^t - e^s)^\delta} ds \leq Ce^{(\alpha+\gamma-1)t},$$

where $a(t) = 1 - e^{-t}$.

Proof: First note some easy estimates: $e^t - e^s = e^t a(t-s)$ and by the mean-value theorem,

$$e^s(t-s) \le e^t - e^s \le e^t(t-s)\,,\; 0 \le s \le t\,.$$

According to the properties of function a, mentionned in appendix 4.5.1, we divide our study in two steps depending if t is greater or smaller than 1.

First case: If $t \in [0,1]$, it is sufficient to prove that $I(t)$ is uniformly bounded in time. Since

$$I(t) \le C \int_0^t \frac{ds}{(t-s)^{\gamma+\delta}} \le C \int_0^1 \frac{du}{u^{\gamma+\delta}}$$

and $\gamma + \delta < 1$, the first step is finished.

Second case: If $t > 1$, we divide the integral $I(t)$ at a critical point $s_0 \in (0,t)$ such that $e^t - e^{s_0} = 1$. Then, $s_0 = t + \ln a(t)$. We denote I_1 and I_2 the two parts of $I(t)$ obtained by this cut:

$$I_1(t) = \int_0^{s_0} \frac{e^{\alpha s} e^{\beta(t-s)} e^{-4\pi^2(e^t - e^s)}}{a(t-s)^\gamma a(e^t - e^s)^\delta} ds\,,$$

$$I_2(t) = \int_{s_0}^{t} \frac{e^{\alpha s} e^{\beta(t-s)} e^{-4\pi^2(e^t - e^s)}}{a(t-s)^\gamma a(e^t - e^s)^\delta} ds\,.$$

We first bound I_1. With the first easy estimate recalled above, we get

$$I_1 = e^{(\beta+\gamma)t} e^{-4\pi^2 e^t} \int_0^{s_0} \frac{e^{(\alpha-\beta)s} e^{4\pi^2 e^s}}{(e^t - e^s)^\gamma a(e^t - e^s)^\delta} ds\,.$$

Taking into account that $0 \le s \le s_0$ implies $e^t - e^s \ge 1$, we have

$$I_1 \le C e^{(\beta+\gamma)t} e^{-4\pi^2 e^t} \int_0^t e^{(\alpha-\beta)s} e^{4\pi^2 e^s} ds\,.$$

By a change of variables $r = e^s$ and some integrations by parts, we bound the last integral as follows

$$\int_0^t e^{(\alpha-\beta)s} e^{4\pi^2 e^s} ds = \int_1^{e^t} r^{\alpha-\beta-1} e^{4\pi^2 r} dr \le C e^{(\alpha-\beta-1)t} e^{4\pi^2 e^t}\,.$$

Then, $I_1(t) \le C e^{(\alpha+\gamma-1)t}$ for any $t > 1$.

As far as I_2 is concerned, s is greater than s_0 and $e^t - e^s$ and $(t-s)$ are in $[0,1]$. Hence, we get

$$I_2 \le C e^{\alpha t} \int_{s_0}^t \frac{ds}{a(t-s)^\gamma a(e^t - e^s)^\delta}\,.$$

Using once more the first easy estimate and a change of variables, we obtain

$$I_2 \le C e^{(\alpha-\delta)t} \int_0^{t-s_0} \frac{du}{u^{\gamma+\delta}} \le C e^{(\alpha-\delta)t} (t-s_0)^{1-\gamma-\delta}\,.$$

As $s_0 = t + \ln a(t)$, we finally get

$$I_2(t) \le C e^{(\alpha-\delta)t} e^{-(1-\gamma-\delta)t}\,.$$

This completes the proof. ■

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