Intégrales matricielles et Probabilités Non-Commutatives

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Thèse de Doctorat de l'Université Paris 6

Spécialité:
Mathématiques

présentée par
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pour obtenir le grade de Docteur de l'Université Paris 6

Sujet de la thèse :

Intégrales matricielles et Probabilités Non-Commutatives

Soutenue le 20 Janvier 2003 devant le Jury composé de :

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A mes parents.

Je remercie sincèrement Philippe Biane d'avoir accepté d'encadrer cette thèse. Cette dernière doit beaucoup à sa grande disponibilité, à la richesse des idées qu'il a su me communiquer, et je garderai longtemps en exemple l'art de la recherche qu'il a su m'enseigner.

Masaki Izumi et Jean-Bernard Zuber ont accepté le rôle de rapporteurs de ma thèse, et l'ont effectué avec beaucoup de soin. Qu'ils en soient ici sincèrement remerciés.

Je remercie Dan Voiculescu, Gilles Pisier, Philippe Bougerol et Jean-Bernard Zuber de me faire l'honneur de participer à mon jury.

Je remercie les différentes équipes de recherche aux portes desquelles j'ai frappé pendant cette thèse, à commencer par l'excellent accueil de Tadahisa Funaki, Yasuyuki Kawahigashi et Masaki Izumi au Japon. Cette thèse doit beaucoup aux discussions avec eux-mêmes et leurs élèves.

Je remercie les thésards du laboratoire de probabilités de Paris 6, dont les séminaires informels m'ont beaucoup appris, ainsi que les membres du groupe de travail "matrices aléatoires". Merci aussi à Alice Guionnet et Ofer Zeitouni pour plusieurs échanges de points de vue sur certaines questions, ainsi qu'aux probabilistes de l'université de Tokyo.

Un grand merci à tous les membres du GdR d'algèbres d'opérateurs, pour leur accueil toujours chaleureux et de nombreux échanges stimulants, en particulier avec Roland Vergnioux, Stéphane Vassout, Frédéric Cadet, Teodor Banica, Franck Lesieur, Kroum Tzanev, Yi-Jun Yao...

Merci aussi à Gilles Pisier et Eric Ricard. Cette thèse doit beaucoup au cours de ce premier et aux explications du second. Merci à Paul Zinn-Justin et Jean-Bernard Zuber pour de nombreuses discussions sur l'interface avec la physique théorique.

Les divers cours et rencontres avec Dietmar Bisch, Marek Bozejko, Andu Nica, Andrei Okounkov, Roland Speicher et Dan Voiculescu ont beaucoup influencé mon travail de thèse, et je les en remercie.

J'ai eu la chance d'effectuer cette thèse au sein du DMA, et remercie ses membres pour leur accueil, ainsi que l'équipe administrative pour son dynamisme.

Je n'oublie pas mes camarades de labeur, mais aussi de soirées du bureau V6. Merci à Jean-Philippe, Thomas et Benoît, ainsi qu'à Jef pour ses explications patientes sur la théorie des groupes. Merci aussi à Sylvain et Florent pour la relecture attentive de mes œuvres et bonne route à ces derniers !

Merci à mes amis qui ont su me distraire de ma thèse ; les voileux et beaucoup d'autres se reconnaîtront ici. Merci aux universitaires de Limoges et à mes enseignants qui ont orienté mes choix vers la recherche en mathématiques (ils se reconnaîtront aussi). Merci à mes parents, frère et sœurs pour leurs encouragements dans les moments difficiles, et surtout, merci à Sonoko.

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Chapitre 1

Introduction

Ce travail de thèse s'articule en trois parties bien distinctes. Elles ont pour point commun d'utiliser de la combinatoire et la théorie des représentations de certains groupes classiques, et de l'appliquer au calcul des probabilités. La partie 2 (article accepté ; IMRN/20917) traite du problème de l'intégration sur les groupes unitaires, et utilise les résultats théoriques sur l'intégration unitaire pour résoudre divers problèmes de probabilités libres et d'intégrales matricielles. La partie 3 (article soumis aux annales scientifiques de l'IHP) utilise des méthodes combinatoires et de théorie des groupes pour définir une théorie de Martin des fonctions harmoniques positives non nécessairement bornées du dual d'un groupe compact. Le calcul de frontière est effectué de manière complètement explicite dans le cas de la restriction à certaines algèbres de fusion. Un cas particulier de marche aléatoire pour le dual du groupe quantique $\mathrm{SU}_q(2)$ est aussi traité.

Dans la partie 4 (article en cours de rédaction), nous calculons la mesure image de la mesure de Haar sur le groupe unitaire par certaines contractions. Ceci nous permet d'établir de nouveaux liens entre certains modèles matriciels et les probabilités libres. Cette partie fournit une approche alternative à la partie 2 pour la description des fonctions Wg, qui sont à la base de l'intégration de tout polynôme sur le groupe unitaire.

1.1 Matrices aléatoires

1.1.1 Introduction

Une première partie sintéresse à l'intégration de variables coordonnées de matrices aléatoires unitaires ayant pour loi la mesure de Haar invariante.

Cette étude s'inscrit dans le cadre beaucoup plus général de la théorie des matrices aléatoires. On peut considérer que ce domaine est né avec le célèbre article de Wigner [Wig58].

Il démontre que la mesure d'intensité de la loi empirique des valeurs propres d'une matrice $d \times d$ symétrique réelle aléatoire dont la loi admet une densité multiple de $\exp -d\mathrm{Tr}X^2$ par rapport à la mesure de Lebesgue, converge lorsque d tend vers l'infini, vers la loi du demi-cercle :

$$\frac{1}{2\pi} 1_{|x|\leq 2}\sqrt{4-x^2}dx$$

Une suite de matrices aléatoires vérifiant les propriétés ci-dessus s'appelle un GOE (Gaussian Orthogonal Ensemble). Le GUE (resp. GSE) s'obtient en remplaçant des matrices réelles par des matrices Hermitiennes (resp. symplectiques).

Wigner a d'abord établi une convergence en moments, puis plusieurs mathématiciens (Mehta, Pastur), ont renforcé de différentes manières les conclusions sur la nature de cette convergence, et ont établi d'autres théorèmes limites en $d \to \infty$ pour d'autres types de convergences.

De manière générale, la répartition exacte des valeurs propres en dimension finie, ainsi que les théorèmes limites qui leur sont associés lorsque la dimension tend vers l'infini, sont les centres principaux d'étude pour les matrices aléatoires. Les modèles d'études sont foison ; citons entre autres les matrices de loi "proche" des GOE (resp. GUE, GSE) invariantes par multiplication à gauche et à droite par une matrice orthogonale (resp. unitaire, symplectique). Citons aussi les matrices réparties suivant une mesure de Haar sur un sous-groupe compact de $GL(n)$, les matrices à coefficients indépendants (de Wigner), les matrices de Wishart.

La référence en matrices aléatoires est [Meh91]. Dans les dix dernières années, la discipline a connu un renouveau sans précédent. Le développement des méthodes "ponctuelles déterminantales" (voir [Sos00] pour une introduction) a permis d'établir des liens avec d'autres domaines inattendus des mathématiques, tels que les modèles de croissance (Johansson, Prähöfer-Spohn), ou bien la théorie des groupes symétriques. En particulier, les travaux de [TW99] et [BDJ00] puis ceux de Johannson, [BOO00], puis [Oko00], ont permis de démontrer une analogie de comportement asymptotique entre le groupe symétrique et les matrices aléatoires.

L'article [Oko00], ainsi que [Sos99], dans lequel est démontré un résultat important d'universalité asymptotique du bord du spectre pour des matrices de Wigner, marque le retour en force des méthodes combinatoires originelles, qui avaient permis les premières preuves de la théorie des matrices aléatoires.

Par ailleurs, les matrices aléatoires sont depuis longtemps appliquées à la physique théorique et nucléaire pour modéliser des Hamiltoniens dont on ne connaît que peu d'information. Elles ont aussi été utilisées plus récemment en théorie des noeuds et en théorie des cordes [ZJZ02a], ainsi que pour résoudre des problèmes de combinatoire (voir [HT98], [HZ86]).

L'article de Voiculescu ([Voi95]) est sans doute celui qui a établi le lien le plus frappant entre matrices aléatoires et un autre domaine des mathématiques : les probabilités libres et les algèbres d'opérateurs. Pour une très bonne introduction, on peut consulter [VDN92] ou [Voi00], ainsi que [Bia].

Rappelons, que dans une algèbre de von Neumann A munie d'un état ϕ, des sous-algèbres de von Neumann $A_1, \ldots, A_k$ sont libres si et seulement si, pour tout mot $a = a_1 \ldots a_l$ avec $a_i \in A_{i_j}$ et $i_1 \neq i_2, i_2 \neq i_3 \ldots i_{l-1} \neq i_l$, et tels que $\phi(a_i) = 0$, alors on a $\phi(a) = 0$.

La preuve que des copies indépendantes d'ensembles gaussiens sont asymptotiquement libres a permis de résoudre beaucoup de conjectures sur l'algèbre de von Neumann du groupe symétrique, notamment par l'introduction des microétats et de l'entropie libre ([Voi94], [Voi93], [Voi98], [Ge98]). Ces travaux ont motivé de nombreux résultats sur les grandes déviations des mesures empiriques des matrices aléatoires ([BAG97]). Ils ont aussi renouvelé l'intérêt pour l'étude des mouvements Browniens Hermitiens ([BS98], [CD]).

1.1.2 Intégrales sur le groupe unitaire

Les matrices unitaires aléatoires distribuées selon la mesure de Haar sur le groupe unitaire sont des exemples intéressants de matrices aléatoires. La plupart des résultats sur les matrices unitaires passent par la décomposition polaire d'une matrice aléatoire d'un ensemble gaussien.

Beaucoup de questions d'apparence élémentaire restent ouvertes. Par exemple, comprendre la norme asymptotique d'une combinaison linéaire de variables unitaires indépendantes réparties suivant la mesure de Haar reste un problème ouvert, dont la résolution aurait des conséquences importantes pour les algèbres d'opérateurs (voir [Pis00]). Une question très semblable a été résolue récemment de manière combinatoire par Haagerup-Thorbjornsen ([HT99]) dans le cas de matrices à entrées gaussiennes indépendantes. Leur approche souligne l'importance d'avoir une compréhension directe des matrices unitaires aléatoires.

Par ailleurs, une approche combinatoire a déja été entamée par Rains ([Rai98]), donnant des interprétations combinatoires très élégantes de certaines fonctionnelles d'une matrice aléatoire unitaire. Il utilise la correspondance RSK (voir [Ful97]) ainsi que différents résultats de représentation de groupes classiques. Malheureusement, les méthodes de Rains ne se généralisent pas à une fonctionnelle arbitraire du groupe unitaire. Signalons toutefois son article [Rai97] dans lequel un algorithme de calcul d'espérances de produits de traces de polynômes de matrices aléatoires quelconques est proposé. Cet algorithme n'est toutefois pas implémentable dans la pratique.

Nous proposons une nouvelle approche, inspirée par celle de Weingarten ([Wei78]) et reprise par F. Xu ([Xu97]). Nous résolvons le problème général de calculer des espérances de fonctions polynômiales de degré quelconque sur les groupes unitaires de matrices de taille supérieure au degré, à l'aide de moyens combinatoires.

Plus précisément, soit dU la mesure de Haar normalisée du groupe unitaire $\mathbb{U}_d$, et q, q' des entiers et $\mathbf{i} = (i_1, \ldots, i_q)$, $\mathbf{i}' = (i'_1, \ldots, i'_{q'})$, $\mathbf{j} = (j_1, \ldots, j_q)$, $\mathbf{j}' = (j'_1, \ldots, j'_{q'})$ deux q-uples et deux q'-uples d'indices dans $[1, d]$. Posons

$$I_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'} = \int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U^*_{j'_1 i'_1} \cdots U^*_{j'_{q'} i'_{q'}} dU$$

L'invariance de la mesure de Haar par la multiplication par des matrices scalaires de norme 1 entraîne que cette intégrale vaut zéro si $q \neq q'$. Ainsi, nous ne considérons que le cas $q = q'$.

Soit $(E_{ij})_{i,j \in [1,d]}$ la base canonique de $\mathbb{M}_d(\mathbb{C})$, et

$$E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'} := E_{i_1 i'_1} \otimes \cdots \otimes E_{i_q i'_q} \otimes E_{j_1 j'_1} \otimes \cdots \otimes E_{j_q j'_q} \in \mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$$

Soit la forme linéaire $I_{d,q}$ sur $\mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$ donnée par $I_{d,q}(E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}) = I_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}$. Nous appelons $\mathcal{S}_q$ le groupe des permutations d'un ensemble à q éléments. Pour $\sigma, \tau \in \mathcal{S}_q$, appelons $\delta_{(\sigma,\tau)}$ la forme linéaire sur $\mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$ vérifiant

$$\delta_{(\sigma,\tau)}(E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}) := \delta_{i_1 i'_{\sigma(1)}} \cdots \delta_{i_q i'_{\sigma(q)}} \delta_{j_1 j'_{\tau(1)}} \cdots \delta_{j_q j'_{\tau(q)}}$$

Rappelons qu'une suite décroissante d'entiers positifs $\lambda = (\lambda_i)_{i \geq 0}$ est une *partition* du nombre q (on note $\lambda \vdash q$) si la somme de ses termes est q. Nous appelons $s_\lambda(x_1, \ldots, x_d)$ la fonction de Schur (voir [Ful97] ou [Mac95]). Nous utilisons l'abréviation $s_{\lambda,d}(x) = s_{\lambda,d}(x, \ldots, x)$.

Théorème 1.1.1 (C. 2.2.1). *Soient q, d deux entiers tels que $d \geq q$. Alors,*

$$I_{d,q} = \sum_{\sigma,\tau \in \mathcal{S}_q} \delta_{(\sigma,\tau)} \mathrm{Wg}(d, q, \sigma\tau^{-1}) \tag{1.1.1.1}$$

où

$$\mathrm{Wg}(d, q, \sigma) = \frac{1}{q!^2} \sum_{\lambda \vdash q} \frac{\chi_\lambda(1)^2 \chi^\lambda(\sigma)}{s_{\lambda,d}(1)}$$

Comme $s_{\lambda,d}(1)$, est un polynôme en d de degré q, Wg est une fraction rationnelle de degré au plus $-q$. La dépendance de Wg en q est implicitement donnée par σ, et nous utiliserons la notation $\mathrm{Wg}(d,\sigma)$ quand cela ne prête pas à confusion.

Wg est une fraction rationnelle en d de degré au plus $-q$, donc elle admet un développement de Laurent

$$\mathrm{Wg}(d,\sigma) = \sum_{l=0}^{\infty} A[\sigma,l]d^{-q-l} \tag{1.1.1.2}$$

Le théorème suivant précise la détermination des coefficients de ce développement. Rappelons que pour permutation $\sigma \in \mathcal{S}_q$, $|\sigma|$ est le nombre minimal de transpositions nécessaire pour réaliser σ.

Théorème 1.1.2 (C, 2.2.2). *Soit* $\sigma \in \mathcal{S}_q$.

(i) $A[\sigma,0] = 0$ *pour* $\sigma \neq e$, *et* $A[e,0] = 1$.

(ii) Pour $l \neq 0$, *soit* $A[\sigma,k,l]$ *le nombre de* k*-uples de permutations* $(\sigma_1, \dots, \sigma_k)$, *toutes différentes de l'identité, telles que* $\sum_{i=1}^k |\sigma_i| = l$ *et* $\sigma\sigma_1 \dots \sigma_k = e$. *Alors,*

$$A[\sigma,l] = \sum_{k=1}^{l} (-1)^k A[\sigma,k,l]$$

(iii) En particulier, on montre que $\mathrm{Wg}(d,\sigma)$ *est une fraction rationnelle de degré* $-q-|\sigma|$ *en* d.

Cette description admet une prolongation pour tout cumulant classique en les fonctions Wg, que nous expliquons à la partie 2.2.4. Pour tous nos calculs explicites, nous utilisons un résultat combinatoire récent de Bousquet-Mélou et Schaeffer [BMS00].

1.1.3 Intégrales de Itzykson-Zuber

Une des premières applications de notre théorie de l'intégration sur le groupe unitaire provient de la physique théorique. Nous appellerons fonction de Itzykson-Zuber la fonction analytique :

$$IZ_{d,X,Y} : z \to E(\exp(dz\mathrm{Tr}(A_d))) \tag{1.1.1.3}$$

où $A_d = X_d U Y_d U^*$, et X_d, Y_d sont deux matrices $d \times d$ hermitiennes.

La fonction IZ a fait son apparition en physique théorique à la fin des années 1970, et une première formule a été proposée dans [IZ80]. Il s'agit de la transformée de Fourier d'une orbite pour l'action par conjugaison du groupe unitaire sur la partie hermitienne de $\mathbb{M}_d(\mathbb{C})$. Elle a été calculée explicitement par Harisch-Chandra à condition que $x_i = x_j$ ssi $i = j$:

$$E(\exp(d\mathrm{Tr}(A))) = \frac{\det(e^{d \cdot x_i y_j})_{1 \leq i,j \leq d}}{\Delta(X)\Delta(Y)} \quad (1.1.1.4)$$

où $(x_i)_{i=1}^d$ et $(y_j)_{j=1}^d$ sont les valeurs propres de X_d et Y_d, et $\Delta(X) = \prod_{1 \leq i < j \leq d}(x_j - x_i)$ (resp. $\Delta(Y) = \prod_{1 \leq i < j \leq d}(y_j - y_i)$), leur déterminant de Vandermonde.

Nous nous intéressons à la fonction analytique en un voisinage de 0 :

$$F_{d,X,Y} : z \to d^{-2} \log E(\exp(d^2 z \mathrm{tr}(A_d))) \quad (1.1.1.5)$$

Dans [Mat94], Matytsin exprime comme solution d'une EDP la limite de $F_{d,X,Y}(1)$ lorsque d tend vers l'infini. Il suppose que les mesures empiriques de X et Y admettent une limite absolument continue par rapport à la mesure de Lebesgue, et sont Hermitiennes. La preuve mathématique de l'existence d'une limite pour $d \to \infty$ demeure toutefois un problème mathématique ardu, qui n'a été résolu que très récemment par A. Guionnet et O. Zeitouni dans [GZ02], sous certaines restrictions pour X et Y.

Nous démontrons les résultats de convergence suivants :

Théorème 1.1.3 (C, 2.4.1). *Soit W une famille de matrices déterministes admettant une loi jointe asymptotique. Soient $U_1, \dots, U_k$ des matrices aléatoires de Haar unitaires indépendantes, et $(P_{i,j})_{1 \leq i \leq k, 1 \leq j \leq k}$, $(Q_{i,j})_{1 \leq i \leq k, 1 \leq j \leq k}$ deux familles de polynômes non-commutatifs en $U_1, U_1^*, \dots, U_k, U_k^*$ et W. Soit A_d la variable*

$$\sum_{i=1}^{k} \prod_{j=1}^{k} \mathrm{tr} P_{i,j}(U, U^*, W)$$

et $B_d = \sum_{i=1}^k \prod_{j=1}^k \mathrm{tr} Q_{i,j}(U, U^*, W)$. Alors,

- *(i) Pour tout d, la fonction analytique*

$$z \to d^{-2} \log E \exp(z d^2 A_d) = \sum_{q \geq 1} a_q^d z^q$$

est telle, que pour tout q, $\lim_d a_q^d$ *existe dans* $\mathbb{C}$. *Cette limite ne dépend que de la loi limite de* W *et des polynômes* $P_{i,j}$.

- *(ii) Pour tout d, la fonction analytique*

$$z \to \frac{E\exp(zB_d + zd^2A_d)}{E\exp(zd^2A_d)} = 1 + \sum_{n\geq 1} b_n^d z^n$$

est telle, que pour tout q, $\lim_d b_q^d$ *existe dans* $\mathbb{C}$. *Cette limite ne dépend que de la loi limite de* W *et des polynômes* $P_{i,j}$ *et* $Q_{i,j}$.

Dans le cas du logarithme de l'intégrale de Harisch-Chandra, il est possible de faire des calculs explicites et de les interpréter en termes de graphes planaires.

Soit G_q l'ensemble des graphes planaires finis à q arêtes dont chacune des composantes connexes est dessinée sur une sphère. On suppose de plus dans la définition de G_q, que

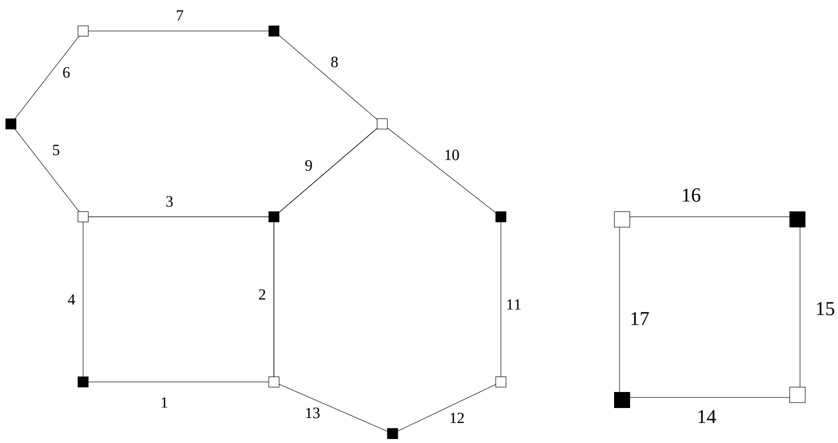
- (i) toute face a un nombre pair d'arêtes
- (ii) les arêtes sont numérotées de 1 à q.
- (iii) il y a une bicoloration en blanc et noir de chacun des sommets de telle sorte que chaque sommet noir n'est relié par une arête qu'à des sommets blancs, et vice versa.

À chaque tel graphe nous associons les permutations $\sigma(g)$ (resp. $\tau(g)$) de $\mathcal{S}_q$ obtenues en tournant dans le sens trigonométrique (resp. dans le sens des aiguilles d'une montre) autour des sommets blancs (resp. noirs), ainsi qu'un nombre Moeb(g) qui sera défini à la partie 2.4.1. Précisons toutefois ici, que Moeb est un nombre entier non-nul.

Par exemple, sur le dessin,

$$\sigma = (1\,13\,2)(3\,5\,4)(6\,7)(8\,9\,10)(11\,12)(16\,17)(14\,15)$$
$$\tau = (5\,6)(7\,8)(10\,11)(2\,3\,9)(12\,13)(1\,4)(14\,17)(15\,16)$$
$$\tau\sigma^{-1} = (1\,3)(5\,9\,7)(6\,8\,11\,13\,4)(2\,12\,10)(17\,15)(14\,16)$$

Deux graphes de G_q sont dits équivalents s'il existe un difféomorphisme préservant les orientations des unions disjointes de chacune des sphères sur lesquelles sont inscrites les composantes connexes des graphes, transformant



un graphe en l'autre tout en respectant le coloriage ainsi que la numérotation des arêtes. Soit $\sim$ cette relation d'équivalence. Si $\sigma \in \mathcal{S}_q$ a d_i cycles avec i éléments et $C(\sigma)$ cycles, soit

$$\langle X \rangle_\sigma = \prod_i \mathrm{Tr}(X^{d_i}) d^{-C(\sigma)}$$

On peut montrer le résultat suivant :

Théorème 1.1.4 (C, 2.4.4).

$$\lim_d \frac{\partial^q}{\partial z^q} F_{X,Y,d}(z)|_{z=0} = \sum_{g \in G_q/\sim} \langle X \rangle_{\tau(g)} \langle Y \rangle_{\sigma(g)} \mathrm{Moeb}(g) \qquad (1.1.1.6)$$

1.1.4 Probabilités libres

Un des premiers résultats de Voiculescu en probabilités libres a consisté à introduire une série formelle dépendant de la mesure spectrale d'un élément autoadjoint, permettant de linéariser la convolution libre. Il s'agit de la R-transformée, dont nous rappelons la définition.

Soit a un élément d'une algèbre de von Neumann A munie d'une trace normalisée ϕ. La fonction holomorphe $z \to E((z-a)^{-1})$ est inversible, d'inverse G dans un voisinage de ∞. La fonction $R_A(z) = G(z) - z^{-1} = \sum_{q \geq 0} k_{q+1}(a) z^q$ est telle que si a et a' sont libres dans A, alors

$$R_{a+a'} = R_a + R_{a'}$$

Les méthodes introduites dans cette thèse permettent de donner une preuve mathématique d'un résultat combinatoire établissant un rapport entre la fonction $I - Z$ et les probabilités libres :

Théorème 1.1.5 (C, 2.4.5). *Pour tout $d \in \mathbb{N}^*$, soit $X_{1,d}$ un projecteur de rang 1 et $X_{2,d} \in \mathbb{M}_d(\mathbb{C})$ tel que*

$$\lim_d d^{-1}\mathrm{Tr}(X_{2,d}^q) = x_{2,q}$$

Soit A une variable aléatoire de moments $x_{2,q}$, et F la fonction d'Itzykson-Zuber complexe :

$$F_{X_1,X_2,d} \,:\, z \to d^{-2} \log E(\exp zd\mathrm{Tr}(X_{1,d}UX_{2,d}U^*))$$

où U est un variable aléatoire unitaire uniforme de dimension d. On a alors

$$\lim_d \frac{\partial^q}{\partial z^q} F_{X,Y,d}(0) = (q-1)!k_q(A) \tag{1.1.1.7}$$

où k_q est le cumulant non-croisé de Speicher, défini à la partie A.3.

La preuve de ce résultat ne fait pas intervenir la définition de Voiculescu ni sa preuve originale, mais la caractérisation combinatoire de Speicher de la liberté [Wei78].

Par ailleurs, notre analyse nous permet aussi de donner des preuves purement combinatoires de résultats de liberté asymptotique de Voiculescu, sans faire usage de calcul fonctionnel, et en affaiblissant les hypothèses :

Théorème 1.1.6 (C, 2.3.1). *Soient U_i^d des variables matricielles indépendantes suivant la loi de Haar sur le groupe unitaire, et D_i^d des matrices $d \times d$ admettant une loi jointe. Alors les familles $\{U_i^d, D_i^d\}$ sont asymptotiquement libres.*

1.2 Analyse harmonique

La partie suivante de cette thèse traite de certains aspects probabilistes de problèmes d'analyse harmonique. Il s'agit de classifier dans certains cas les solutions de l'équation

$$Pf = f$$

où P est un opérateur Markovien ou sous-Markovien sur un espace quantique.

Les résultats présentés dans cette partie font pour la plupart écho aux recherches de Philippe Biane ([Bia90], [Bia91a], [Bia91b], [Bia92a], [Bia92b] [Bia94]) dans le début des années 1990. Nous rappelons brièvement les fondements probabilistes de la théorie de Martin, puis les définitions des probabilités quantiques. Ceci nous permet de citer les résultats de Biane, puis les résultats de cette thèse dans ce domaine.

Ensuite, nous rappelons quelques idées sur les sous-facteurs, et décrivons les applications de la théorie de Martin quantique aux probabilités quantiques, développées par Izumi à partir de la théorie des sous-facteurs.

1.2.1 Théorie de Martin

La théorie de Martin s'intéresse au problème de la représentation probabiliste et topologique des fonctions harmoniques positives par rapport à un opérateur Markovien (resp. sous-markovien).

Soit $\mathcal{E}$ un espace d'états discret et P l'opérateur de Markov : $Pf(x) = \sum_{y\in\mathcal{E}} P(x,y)f(y)$, où $P(x,y)$ est un noyau réel positif, tel que $P1 = 1$ (resp. $P1 \leq 1$).

Quand cela ne prête pas à confusion, nous utilisons la même notation pour l'opérateur P et pour l'opérateur $P(x,y)$. Définissons par récurrence $P^0(x,y) = \delta_{x,y}$, $P^{n+1}(x,y) = \sum_{z\in\mathcal{E}} P^n(x,z)P(z,y)$ et $U(x,y) = \sum_{n\in\mathbb{N}} P^n(x,y)$. De plus, nous supposons que pour tous $x, y \in \mathcal{E}$, on a $0 < U(x,y) < \infty$.

Le *noyau de Martin* par rapport à un élément e, choisi au préalable, est le noyau k tel que pour tous $x, y \in \mathcal{E}$,

$$k(x,y) = U(x,y)/U(e,y)$$

On remarque que pour tout $x \in \mathcal{E}$, la fonction $k(x,.)$ est bornée. La *compactification de Martin* MS de $\mathcal{E}$ est le plus petit espace compact dans lequel $\mathcal{E}$ est plongé densément, tel que toute fonction $k(x,.)$ se prolonge par continuité à MS. Soit MB le complémentaire dans MB du plongement de $\mathcal{E}$.

Théorème 1.2.1. *Il existe un sous-ensemble MB^{min} de MB tel que pour toute fonction positive harmonique f vérifiant $f(e) = 1$, on ait une unique mesure de probabilité μ_f sur MB^{min} telle que pour tous $x \in \mathcal{E}$,*

$$f(x) = \int_{\xi\in MB^{min}} k(x,\xi)d\mu_f(\xi)$$

La preuve de ce théorème se trouve dans [Rev84], ou bien [KSK76]. Un exemple remarquable de calcul explicite de k et MS se trouve dans [NS66]. De manière générale, dans cette thèse, nous disons que nous obtenons un théorème *de type Ney-Spitzer*, quand nous parvenons 1^o à calculer toutes les fonctions harmoniques minimales, 2^o à calculer la frontière de Martin, et 3^o à calculer la correspondance entre les deux, donnée par le noyau de Martin dans le théorème ci-dessus.

1.2.2 Probabilités quantiques

Rappelons qu'une algèbre de von Neumann A est une algèbre de Banach munie d'une involution $*$, isomorphe à une sous-algèbre de l'ensemble des opérateurs continus $B(H)$ d'un espace de Hilbert H, contenant l'identité, stable par involution, et égale à son bicommutant.

Un espace de probabilités non-commutatif (A, ϕ) est la donnée d'une algèbre de von Neumann A munie d'un état ϕ normal. Ses éléments sont appelés *variables aléatoires non-commutatives*. Dans le cas où l'algèbre est Abélienne, les théorèmes de classification des algèbres de von Neumann (voir [Dix69] ou bien [Tak79]) nous permettent de conclure que (A, ϕ) est isomorphe à un certain $(L^\infty(\Omega, \mathcal{F}, P), E)$, où E est l'espérance associée à P, de telle sorte que cet isomorphisme entrelace ϕ et E. De plus il est unique à isomorphisme d'espace de probabilités près. Ceci justifie la terminologie d'espace de probabilités non-commutatif.

En probabilités classiques, on ne peut pas systématiquement se contenter de variables aléatoires bornées. De même, on a interêt à s'autoriser d'autres algèbres d'opérateurs non nécessairement bornés.

Il est possible de définir différentes notions d'indépendance sur ces espaces de probabilités non-commutatifs, notamment l'indépendance tensorielle, et l'indépendance libre. Pour des définitions, et de nombreuses analogies avec les probabilités classiques, on pourra se reporter à [VDN92], [Bia90], [Mey86], [Mey87], [Mey88a], [Mey88b], [Mey89a], [Mey89b], [SS98]. Biane s'est intéressé à la théorie de Martin en probabilités quantiques dans le contexte des algèbres de Hopf.

Une algèbre de Hopf-von Neumann est une algèbre A de von Neumann munie de deux morphismes normaux ε et $\widehat{\Delta}$ de A à valeurs respectivement dans $\mathbb{C}$ et $A \otimes A$. Ces applications doivent vérifier les deux axiomes suivants :

- $(\widehat{\Delta} \otimes id)\widehat{\Delta} = (id \otimes \widehat{\Delta})\widehat{\Delta}$ (propriété de *coassociativité*)

- $(\varepsilon \otimes id)\widehat{\Delta} = id$

On appelle $\widehat{\Delta}$ le coproduit et ε la counité.

Cette structure permet d'étudier des marches au hasard en probabilités non-commutatives dans d'autres contextes que celui des groupes. Dans le cas de la marche au hasard sur un groupe G, l'algèbre de von Neumann à considérer pour être en adéquation avec l'axiomatique ci-dessus est $A = l^\infty(G)$, avec $\varepsilon(a) = a(e)$ pour $a \in A$, et e l'élément neutre du groupe. Le coproduit $\widehat{\Delta}$ est défini pour tout $a \in A$, par

$$\widehat{\Delta}(1_a) = \sum_{x,y \in G, xy=a} 1_x \otimes 1_y$$

comme un élément de $A \otimes A \cong l^\infty(G \times G)$. Dans l'exemple, le triplet $(l^\infty(G), \widehat{\Delta}, \varepsilon)$ code toute l'information du monoïde G. Ainsi, si μ est une mesure de probabilités sur G et X_t une marche au hasard à gauche dont la loi des sauts est μ, son opérateur d'évolution sur $l^\infty(G)$ est $P = (id \otimes \mu)\widehat{\Delta}$.

La théorie de Martin traite des fonctions harmoniques positives. Ainsi, on a besoin de la notion de positivité qui est déja contenue dans A, et on doit aussi pouvoir considérer la convolution par des opérateurs non-bornés. Pour ce faire, il est nécessaire d'étendre la notion d'algèbre de Hopf à des algèbres plus grosses. Ceci est expliqué en détail dans la partie 3.2.

Les résultats de la partie 3 concernent pour leur majorité la résolution et la classification des solutions de l'équation $Pf = f$ pour des algèbres de Hopf non-commutatives. Les exemples sur lesquels nous travaillons en majorité de telles algèbres, proviennent des groupes compacts.

Soit G un groupe compact, $L^2(G)$ l'espace de Hilbert associé à sa mesure de Haar normalisée, et $M(G)$ l'algèbre de von Neumann engendrée dans $B(L^2(G))$ par les fonctions

$$\lambda_g : f \to (x \to f(g^{-1}x))$$

pour tout $g \in G$. On la munit d'une structure d'algèbre de Hopf en étendant par linéarité et par continuité la fonction $\varepsilon(\lambda_g) = 1$ et $\widehat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$. Une telle algèbre s'appelle *algèbre de Hopf-von Neumann* du groupe G. Il est intéressant de considérer des groupes compacts, car toutes leurs représentations sont de dimension finie d'après le théorème de Peter-Weyl. Ceci implique que l'algèbre de von Neumann $M(G)$ est finie et de type I (i.e. c'est une somme directe d'algèbres de matrices).

1.2.3 Théorie de Martin quantique

Soit G un groupe de Lie compact semi-simple simplement connexe contenu dans $\mathbb{U}_n$, V sa représentation fondamentale, et μ un poids sur l'algèbre de von Neumann $M(G)$ du groupe G. Soit $\widehat{\Delta}$ le coproduit sur $M(G)$, que l'on peut étendre par continuité à $\widehat{M(G)}$, l'algèbre des opérateurs affiliés à $M(G)$. Soit $P_\mu = (id \otimes \mu)\widehat{\Delta}$. L'ensemble des éléments positifs harmoniques

$$\mathcal{H}^+_{P_\mu} = \{h \in \widehat{M(G)}, P_\mu(h) = h, h \geq 0, \varepsilon(h) = 1\}$$

est un convexe compact. Alors,

Théorème 1.2.2 ([Bia92a]). *Les points extrémaux de $\mathcal{H}^+_{P_\mu}$ sont les éléments tels que $\widehat{\Delta}h = h \otimes h$ et $\mu(h) = 1$. De plus, l'ensemble*

$$Ex(G) = \{h \in \widehat{M(G)}, \widehat{\Delta}h = h \otimes h\}$$

est un sous-groupe du groupe des éléments inversibles de $\widehat{M(G)}$ isomorphe au complexifié de G. La projection de ce groupe sur $End(V)$ est un isomorphisme.

L'analogue du théorème de Choquet-Deny obtenu dans [Bia92a] ne caractérise pas la structure topologique de la frontière de Martin. Le théorème ci-dessous propose une caractérisation :

Théorème 1.2.3 (C, 3.3.4). *Soit μ une fonction polynômiale de type positif sur G, telle que $\mu(1) < 1$, et P_μ l'opérateur de convolution par μ sur $\widehat{M(G)}$. L'ensemble*

$$\mathcal{H}^+_{P_\mu} = \{f \in M(\widehat{SU(n)}), f \geq 0, Pf = f, \varepsilon(f) = 1\}$$

est un convexe compact et le sous-ensemble de ses points extremaux est fermé et homéomorphe à la sphère S^{n^2-2}.

En probabilités classiques, le noyau de Martin a été introduit pour donner une représentation de toutes les fonctions harmoniques positives par rapport à une chaîne de Markov. Nous donnons une définition naturelle de noyau de Martin en probabilités non-commutatives, et montrons que tout élément positif harmonique peut être représenté comme un état sur la frontière de Martin.

Nous supposons que le poids μ est tel que $\mu_d(1) < 1$, et que $\varepsilon U = \sum_{n\geq 0} \mu^{*n}$ est fidèle.

Le théorème ci-dessous prouve que la définition naturelle du noyau de Martin en géométrie non-commutative permet bien de représenter toute fonction harmonique positive.

Théorème 1.2.4 (C, 3.3.3). *Supposons que $\mu(1) < 1$ et qu'il existe un poids tracial $\widetilde{\mu}$ par rapport auquel μ admet une densité h telle que $\widehat{\Delta} h = h \otimes h$. Alors tout élément harmonique positif admet une représentation intégrale par rapport à la convolution par μ.*

Dans le cas $n = 2$, [Bia94] raffine ce résultat et fournit un théorème de type Ney-Spitzer. La C^* -algèbre engendrée par les K_ν est celle des opérateurs pseudo-différentiels invariants sur $SU(2)$. De plus, le quotient de cette C^*-algèbre par celle des opérateurs compacts de $M(SU(2))$ est la compactification de Martin, au sens où pour toute fonction harmonique positive f pour P_ν telle $\varepsilon(f) = 1$, il existe un unique état χ_f positif du quotient tel que pour tout poids ν de $M(SU(2))$ on ait $\nu(f) = \chi_f(K_\nu)$.

Dans le chapitre 3, partie 3.4, nous nous intéressons à la restriction de l'opérateur P au centre de $M\widehat{(SU(n))}$ dans le cas où la convolution est donnée par l'état $\mu = n^{-1}\mathrm{Tr}$. Soit T_{n-1} un tore maximal de $SU(n)$, et L le dual de T^{n-1}, engendré par les formes linéaires $e_i = E_{i,i}$. Il vérifie, en notation additive, $\sum_i e_i = 0$. Etant donné $x \in L$, il existe un unique moyen de l'écrire sous la forme $x_1 e_1 + \ldots + x_{n-1} e_{n-1}$ avec tous les $x_i \in \mathbb{N}$, et l'un au moins qui vaut 0. Nous notons alors $||x|| = \sum x_i$. $d(x, y) = ||x - y||$ définit alors une distance sur L. Soit $\overset{o}{W}$ le sous-ensemble de L défini par

$$\overset{o}{W} = \{x, x_1 > x_2 > \ldots > x_n\}$$

La restriction de P correspond à un opérateur de Markov sur l'espace d'états $\overset{o}{W}$, dont nous définissons la compactification par le simplexe

$$\Sigma = \{y' = (y'_1, \ldots, y'_n) \,:\, y'_1 \geq \ldots \geq y'_n = 0, \sum_i y'_i = 1\} \subset \mathbb{R}^n \qquad (1.2.1.1)$$

de la manière suivante : soit $y^d = (y_1^d > y_2^d > \ldots > y_n^d = 0)_{d\geq 0}$ une suite d'éléments de W. On dit que y^d converge vers $(y'_1, \ldots, y'_n)$ si y^d tend vers l'infini et $y_i^d / \sum_i y_i^d$ converge vers y'_i pour tout i. On a alors

Théorème 1.2.5 (C, 3.4.1). *La compactification de W par Σ est la compactification de Martin. De plus, la frontière de Martin Σ est exactement la frontière de Martin minimale.*

1.2.4 Probabilités non-commutatives et sous-facteurs

La théorie des sous-facteurs est née avec l'article de Jones [Jon83], dans lequel il est démontré que les indices possibles d'une inclusion de facteurs de type II_1 varie dans l'ensemble $\{4\cos^2(\pi/q), q \geq 3\} \cup [4, \infty[$. Depuis, la classification et les exemples de sous-facteurs ont fait l'objet de beaucoup de recherches. Citons notamment les contributions de Jones, Popa, Bisch, Haagerup, A. Wassermann.

Les sous-facteurs revêtent des aspects combinatoires et algébriques, mais aussi analytiques et probabilistes. Par exemple, la formule de Pimsner-Popa donne l'indice d'un sous-facteur à l'aide d'un minimum sur une espérance conditionnelle.

Récemment, Izumi ([Izu00]) a considéré l'action diagonale "ITP" du groupe quantique $\mathrm{SU}_q(2)$ sur la C^*-algèbre AFD $\bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$. Cette algèbre admet un unique état de Powers invariant par l'action de $\mathrm{SU}_q(2)$. On peut la compléter dans la représentation GNS associée à cet état en un facteur d'Araki-Woods R de type III_{q^2}. L'algèbre des points de R sous l'action de $\mathrm{SU}_q(2)$ est un facteur F de type II_1. Le commutant de F dans R est une algèbre de von Neumann que l'on peut identifier naturellement à la frontière de Poisson d'une marche au hasard quantique P très simple sur le dual de $\mathrm{SU}_q(2)$. Si $\mathrm{SU}_q(2) = SU(2)$, l'état invariant de $\bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ est tracial, et R est le facteur hyperfini de type II_1. Il est alors connu ([Was88]) que le commutant de F dans R est trivial. Ce résultat est une conséquence des résultats de Biane au vu du travail de Izumi.

La frontière de Poisson que Izumi obtient fait intervenir un analogue quantique de la sphère S^2. Cette dernière apparaît par ailleurs dans les travaux de Biane. Ceci suggère que la théorie de Martin développée dans la partie 3 de cette thèse s'applique au groupe quantique $\mathrm{SU}_q(2)$ et qu'elle redonne une interprétation du résultat d'Izumi. Nous montrons en particulier dans cette thèse que l'on peut définir un noyau de Martin K associé à P. Nous en donnons la définition dans l'équation 3.6.3.9 La clôture normique dans l'algèbre $M(\mathrm{SU}_q(2))$ des K_ν engendre une algèbre Ψ^0, et on a :

Théorème 1.2.6 (C, 3.6.11). *Le noyau K donne un théorème de type Ney-*

Spitzer pour Q. L'algèbre Ψ^0 contient les compacts $\mathcal{K}$ de $M(\mathrm{SU}_q(2))$, et son quotient par $\mathcal{K}$ est isomorphe à une sphère de Podles. Toute fonction positive harmonique correspond de manière unique à un état de $\Psi^0/\mathcal{K}$.

Remarquons que le récent préprint [NT02] traite de manière différente et plus générale de questions liées au théorème ci-dessus. Nos résultats ont été obtenus indépendamment de [NT02].

1.2.5 Restriction à des algèbres Abéliennes

Nous donnons une description explicite de certaines fonctions harmoniques positives, et en particulier, des fonctions harmoniques positives dont la diagonale est bornée dans une base correctement choisie.

L'opérateur P introduit par Izumi a beaucoup de propriétés très intéressantes. Il laisse stable la sous-algèbre de von Neumann d'un tore maximal $M(T^{n-1})$, ainsi que son centre $Z(\mathrm{SU}_q(2))$. En ce sens, il définit deux chaînes de Markov classiques. Biane a déja étudié les rapports entre ces deux chaînes dans le cas où $q = 1$ dans [Bia91a].

Théorème 1.2.7 ([Bia91a]).

- *La marche au hasard déterminée par P sur L est celle dont les sauts ont pour loi $n^{-1}\sum_{i=1}^{n}\delta_{e_i}$.*
- *La marche au hasard déterminée par P sur W s'obtient à partir de la précédente en la conditionnant au sens de Doob à ne pas quitter $\overset{o}{W}$. Il n'existe qu'un seul moyen d'effectuer ce conditionnement.*

Ce théorème admet une contrepartie intéressante dans le cas de $SU_q(n)$, $q \neq 1$. Le simplexe Σ peut naturellement être considéré comme une face de la sphère unité S^{n^2-2}. Soit alors le point

$$y_q = \sum_{i=1}^{n} q^{n+1-2i} e_i / || \sum_{i=1}^{n} q^{n+1-2i} e_i || \in S^{n-2}$$

Nous obtenons la caractérisation suivante :

Théorème 1.2.8 (C, 3.5.4). *L'opérateur P restreint à $\overset{o}{W}$ s'obtient à partir de la marche au hasard sur L d'incrément $\alpha \sum \delta_{e_i}/n$, avec $\alpha = n/[n]_q$, conditionnée au sens de Doob à tendre vers le point y_q avant de sortir et/ou de mourir.*

Remarquons que très récemment, plusieurs groupes de mathématiciens ([KOR02], [OY02], [Joh02], [Joh00], [BJ02]) ont publié des résultats dans l'esprit du théorème ci-dessus.

1.3 Images de mesures de Haar

Dans cette partie, nous revenons sur l'étude des matrices aléatoires unitaires et proposons un point de vue très différent de celui développé dans la partie 2.

Notre point de départ est le théorème suivant :

Théorème 1.3.1 (C, 4.2.1). *Soient d, q_1, q_2 des entiers tels que $d \geq 2q_1 \geq 2q_2$. Soit μ_d la mesure de Haar normalisée sur le groupe unitaire $\mathbb{U}_d$, et π_{d,q_1,q_2} la projection de $\mathbb{M}_d(\mathbb{C})$ sur son coin supérieur gauche $\mathbb{M}_{q_1,q_2}(\mathbb{C})$. La mesure image par π_{d,q_1,q_2} est la loi*

$$c_{d,q_1,q_2} \det(1 - A^*A)^{d-q_1-q_2}$$

où c_{d,q_1,q_2} est une constante de normalisation.

En particulier, si $A_{d,q}$ est une matrice aléatoire de $\mathbb{M}_q(\mathbb{C})$ extraite d'une matrice unitaire d'un ensemble unitaire $\mathbb{U}_d$, alors on peut montrer que

$$2A_{d,q}^* A_{d,q} - 1$$

est un ensemble de Jacobi de paramètre $(0, q-2q)$ pour $2q \leq d$. Cet ensemble a la propriété que le processus des valeurs propres qui lui est associé est déterminantal (voir [Sos00] pour une introduction). Par ailleurs, la distribution limite des valeurs propres peut être décrite au sens de la convergence faible, lorsque $q \sim \alpha d$, $d \to \infty$, $\alpha \in]0,1[$. Il est donc naturel de s'interroger sur les propriétés d'universalité de cet ensemble. Les techniques de polynômes orthogonaux ([CI91]) nous ont permis d'établir les résultats suivants :

Théorème 1.3.2 (C, 4.3.3). *Soit a un nombre réel dans $]0, 1/2[$, et pour tout d, soit q tel que $2q \leq d$ et $q/d \sim a$. Soit K_q le processus determinantal associé aux valeurs propres $\lambda_1 \geq \ldots \geq \lambda_q$.*

- *Soit K' un compact de $]-1, 8a - 8a^2 - 1[$. Alors*

$$\lim_q \frac{1}{q} K_q(x,x) = \frac{1}{2a} \frac{\sqrt{8a - 8a^2 - 1 - x}}{2\pi\sqrt{1+x}(1-x)} = f(x)$$

  et cette limite est uniforme sur K.

- *Pour $u, v \in]0, \infty[$, on a, pour $q \to \infty$,*

$$\frac{1}{qf(x)} K_q^{d-2q,0}(x + \frac{u}{nf(x)}, x + \frac{v}{nf(x)}) = \frac{\sin \pi(u-v)}{\pi(u-v)}$$

- *Soit K' un compact de $]8a - 8a^2 - 1, 1[$. Presque sûrement, pour q assez grand, il ne contient aucune valeur propre du processus.*

Remaquons que Michel Ledoux, [Led02] dans un preprint récent, démontre que la plus grande valeur propre d'un ensemble de Jacobi se comporte comme celle d'un ensemble gaussien unitaires. Notons aussi que M. Capitaine et M. Casalis [CC02] font apparaitre des propriétés de liberté asymptotique des ensembles de Jacobi de maniere différente.

Par ailleurs, le théorème 4.2.1 nous permet d'améliorer un résultat de Diaconis, Eaton et Lauritzen [DEL92] :

Théorème 1.3.3 (C, 4.4.2). *Soit ν_q la mesure de probabilités $c_{d,q} e^{-q \mathrm{Tr} MM^*} dM$ sur $\mathbb{M}_q(\mathbb{C})$, et q_d une suite d'entiers tendant vers l'infini tels qu'il existe une constante $C > 0$ telle que $q_d^3 \leq Cd$. Alors,*

$$|\sqrt{d/q_d} \pi^*_{d,q_d}(\mu_d) - \nu_{q_d}| = o(1)$$

où $| \cdot |$ est la mesure variation totale.

La connaissance de la mesure explicite de l'image de μ_d par la projection π_{d,q_1,q_2} nous permet de donner une interprétation combinatoire simple des résultats 2.2.2 et 2.2.6 de la partie 2, au théorème 4.5.3.

Cette partie se termine par une majoration de Wg uniforme en q au théorème 4.5.4.

1.4 Organisation, remarques

Je remercie J.B. Zuber de me signaler que le théorème 2.2.1 avait déjà, et de manière complètement indépendante, fait l'objet de littérature physique dans le début des années 1980, avec notamment [Sam80] et [DZ83].

Dans le chapitre 2 se trouvent les résultats combinatoires pour l'intégration de matrices unitaires, et leurs applications aux modèles matriciels et aux probabilités libres.

Le chapitre 3 contient les résultats de probabilités quantiques.

Le chapitre 4 contient des résultats analytiques pour l'intégration de matrices unitaires. Il contient plusieurs nouvelles preuves et améliorations de résultats du chapitre 2.

En annexe, nous mettons

- quelques rappels sur la représentation des groupes, utiles tout au long de cette thèse. Ces résultats sont très classiques, mais contiennent le nécessaire pour comprendre l'algèbre contenue dans cette thèse.

- un court listing maple/ACE permettant d'effectuer certains calculs d'intégrales sur les groupes unitaires.

Chapitre 2

Intégration unitaire

Moments and Cumulants of Polynomial random variables on unitary groups, the Itzykson-Zuber integral and free Probability

HTTP://FR.ARXIV.ORG/ABS/MATH-PH/0205010
IMRN/20917

We consider integrals on unitary groups $\mathbb{U}_d$ of the form

$$\int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U^*_{j'_1 i'_1} \cdots U^*_{j'_{q'} i'_{q'}} dU$$

We give an explicit formula in terms of characters of symmetric groups and Schur functions, which allows us to rederive an asymptotic expansion as $d \to \infty$. Using this we rederive and strengthen a result of asymptotic freeness due to Voiculescu.

We then study large d asymptotics of matrix model integrals and of the logarithm of Itzykson-Zuber integrals and show that they converge towards a limit when considered as power series. In particular we give an explicit formula for

$$\lim_{d \to \infty} \frac{\partial^n}{\partial z^n} d^{-2} \log \int_{\mathbb{U}_d} e^{zd\mathrm{Tr}(XUYU^*)} dU|_{z=0}$$

assuming that the normalized traces $d^{-1}\mathrm{Tr}(X^k)$ and $d^{-1}\mathrm{Tr}(Y^k)$ converge in the large d limit. We consider as well a different scaling and relate its asymptotics to Voiculescu's R-transform.

Sur le groupe unitaire $\mathbb{U}_d$, nous considérons des intégrales du type

$$\int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U^*_{j'_1 i'_1} \cdots U^*_{j'_{q'} i'_{q'}} dU$$

Nous donnons une formule explicite à l'aide des caractères du groupe symétrique et des fonctions de Schur, qui nous permettent de retrouver un développement asymptotique lorsque d tend vers l'infini. A l'aide de ce résultat, nous retrouvons et renforçons un résultat de liberté asymptotique de Voiculescu.

Ensuite, nous étudions des intégrales de modèles matriciels et leurs asymptotiques, et en particulier le logarithme de l'intégrale d'Itzykson-Zuber, dans

la limite des grandes dimensions. Nous montrons que ces intégrales convergent vers une limite en tant que séries formelles. En particulier, nous donnons une expression explicite pour

$$\lim_{d\to\infty} \frac{\partial^n}{\partial z^n} d^{-2} \log \int_{\mathbb{U}_d} e^{zd\mathrm{Tr}(XUYU^*)} dU|_{z=0}$$

en supposant que les traces normalisées $d^{-1}\mathrm{Tr}(X^k)$ et $d^{-1}\mathrm{Tr}(Y^k)$ convergent lorsque d tend vers l'infini. Nous considérons aussi la limite pour un autre changement d'échelle, et relions son asymptotique à la R-transformée de Voiculescu.

2.1 Introduction

The starting point of this paper is a recent result by Guionnet and Zeitouni (see [GZ02]). For all integers d, let X_d and Y_d be Hermitian matrices in $\mathbb{M}_d(\mathbb{C})$ such that for all $k \geq 0$, $\lim_d d^{-1}\mathrm{Tr}(X_d^k)$ (resp. $\lim_d d^{-1}\mathrm{Tr}(Y_d^k)$) exists in $\mathbb{R}$ and equals x_k (resp. y_k). Assume furthermore that there exists a majorant A independent on d for all $||X_d||$ and $||Y_d||$. Consider the following integral on the unitary group $\mathbb{U}_d$ with its normalized Haar measure dU:

$$F_d = d^{-2} \log \int_{\mathbb{U}_d} e^{d\mathrm{Tr}(X_d U Y_d U^*)} dU$$

It is proved in [GZ02], that $\lim_d F_d$ exists and depends only on the sequences (x_k) and (y_k).

This result, whose proof implies subtle computations and results about large deviation bounds for Brownian motion, raises a number of questions. For example, it is of interest to understand what would happen if the matrices fail to be Hermitian or to have a norm bounded independently on d.

Our approach to this problem is based on an explicit algebraic computation of the integral of polynomial functions on unitary groups. This approach yields a different convergence concept than that of [GZ02] and the results of this paper don't allow to recover the results of Guionnet and Zeitouni. On the bright side, we show that a very large class of integrals similar to F_d converge in our sense and we obtain an explicit expression for their limit.

Furthermore, we provide for the first time a combinatorial unified method allowing to deal with large unitary matrices in various domains such as free probability theory and matrix integrals and establish new connections between them.

The paper [Wei78] was among the first try to understand the integral of polynomial functions on unitary groups for large d. We supply a new proof of Weingarten's result and generalize it by giving an explicit way of computing an integral of a polynomial random variable on the unitary group $\mathbb{U}_d$ with characters and Schur functions (Theorem 2.2.1 of Section 2.2). In Theorems 2.2.2 and 2.2.6, we give a way of computing these integrals and cumulants associated to them by summing over sequences of permutations satisfying specific properties.

Section 2.3 is an application of Theorem 2.2.1 to a new proof (Theorem 2.3.1) and an improvement (Theorem 2.3.5) of an asymptotic freeness result of Voiculescu and Feng Xu. For a set W and for each d, let $(w_{i,d})_{i\in W}$ be a family of matrices in $\mathbb{M}_d(\mathbb{C})$ such that for any non-commutative polynomial P in W, $\lim_d(d^{-1}\mathrm{Tr}(P(w_{i,d})))$ admits a finite limit in $\mathbb{C}$.

Let $U_1, U_2, \dots$ be a sequence of Haar distributed unitary independent matrices. Take the convention that $\mathrm{tr} = d^{-1}\mathrm{Tr}$.

Theorem.
- *(i) For any non-commutative polynomial Q in the variables $W, U_1, U_1^*, U_2, U_2^* \dots$, $d^{-1}\mathrm{Tr}(Q(w_{i,d}, U_1^*, U_2, U_2^* \dots))$ admits a finite limit in $\mathbb{C}$.*
- *(ii) The family of sets of variables $W, \{U_1, U_1^*\}, \dots$ is asymptotically free.*
- *(iii) Furthermore, we have*

$$P(|\mathrm{tr}(Q(w_{i,d}, U_1, U_1^* \dots)) - E(\mathrm{tr}(Q(w_{i,d}, U_1, U_1^* \dots)))| \geq \varepsilon) = O(d^{-2})$$

The two first points are stated in Theorem 2.3.1 and are due to Feng Xu [Xu97]. We supply a new proof of these results. The third point is the object of Theorem 2.3.5. The methods gathered in this section are used for section 2.4, in which we first show the convergence Theorem 2.4.1:

Theorem. *Let $(P_{l,j})_{1\leq l\leq k, 1\leq j\leq k}$ and $(Q_{l,j})_{1\leq l\leq k, 1\leq j\leq k}$ be two families of non commutative polynomials in $U_1, U_1^*, \dots, U_k, U_k^*$ and W. Let A_d be the variable $\sum_{l=1}^k \prod_{j=1}^k \mathrm{tr}P_{l,j}(U, U^*, w_{i,d})$ and B_d the variable $\sum_{l=1}^k \prod_{l=1}^k \mathrm{tr}Q_{i,j}(U, U^*, w_{i,d})$.*

- *The function*

$$z \to d^{-2}\log E(\exp(zd^2A_d)) = \sum_{q\geq 1} a_q^d z^q$$

is such that for all q, $\lim_d a_q^d$ exists and depends only on $P_{l,j}$ and the limit distribution of W.

- *The function*

$$z \to \frac{E\exp(zB_d + zd^2A_d)}{E\exp zd^2A_d} = 1 + \sum_{q\geq 1} b_q^d z^q$$

is such that for all q, $\lim_d b_q^d$ exists and depends only on $P_{l,j}, Q_{l,j}$ and the limit distribution of W.

The explicit link with the work of Guionnet and Zeitouni is the following: let F be the function defined by

$$F_{d,X,Y} = d^{-2} \log E(\exp z d^2 \mathrm{tr}(XUYU^*))$$

where X and Y are arbitrary $d \times d$ matrices. Then $\frac{\partial^q}{\partial z^q} F_{d,X,Y}(0)$ is a polynomial function of the variables $\mathrm{tr}X^i$ and $\mathrm{tr}Y^i$, whose coefficients are rational fractions of d. We prove that these coefficients converge as $d \to \infty$ to some limit, for which we give an explicit formula (Theorem 2.4.2) and a diagrammatic interpretation (Theorem 2.4.4).

We finish this paper by establishing in Theorem 2.4.5 the following link between free probability and the Itzykson-Zuber integral

Theorem. *Assume that X is a one dimensional projection and that Y admits a limit distribution. Then for all q, the number $d \cdot \partial^q / \partial z^q F_{d,X,Y}(0)$ converges towards $(q-1)! k_q(Y)$, i.e. the coefficient of the primitive of Voiculescu's R -transform R_Y of the limit distribution of Y.*

This article is a part of the author's Ph.D. work. The author would like to thank his advisor Philippe Biane for numerous discussions and for introducing him to the subjects to which this paper is related. He also thanks Paul Zinn-Justin for useful conversations.

2.2 Integrals of polynomial functions on $\mathbb{U}_d$

2.2.1 An exact formula for Weingarten's function

Let $\mathbb{U}_d$ be the group of unitary $d \times d$ matrices and dU be its normalized Haar measure. Let q, q' be a positive integers and $\mathbf{i} = (i_1, \ldots, i_q)$, $\mathbf{i'} = (i'_1, \ldots, i'_{q'})$, $\mathbf{j} = (j_1, \ldots, j_q)$, $\mathbf{j'} = (j'_1, \ldots, j'_{q'})$ be two q -tuples and two q'-tuples of indices in $[1, d]$, we define

$$I_{d,\mathbf{i},\mathbf{i'},\mathbf{j},\mathbf{j'}} = \int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U^*_{j'_1 i'_1} \cdots U^*_{j'_{q'} i'_{q'}} dU$$

We shall give an explicit formula for this integral in terms of Schur functions and characters of symmetric groups. Using the invariance of Haar measure under multiplication by scalar unitary matrices one checks that this integral is zero if $q \neq q'$ so we will only consider the case $q = q'$.

Remark 2.1. Let us remark that some results related to a generating function of $I_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}$, the so called external-source integral

$$\int \exp \operatorname{Tr}(A^*U + U^*A)dU$$

have already been obtained by theoretical physicists ([BG80] and [BRT81]) but they is no direct relation to our results.

In order to state our result we need to introduce some notations. Let $(E_{ij})_{i,j\in[1,d]}$ be the canonical basis of $\mathbb{M}_d(\mathbb{C})$, we define

$$E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'} := E_{i_1i'_1} \otimes \cdots \otimes E_{i_qi'_q} \otimes E_{j_1j'_1} \otimes \cdots \otimes E_{j_qj'_q} \in \mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$$

and the linear form $I_{d,q}$ on $\mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$ such that $I_{d,q}(E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}) = I_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}$. We denote $\mathcal{S}_q$ the symmetric group on $\{1,\dots,q\}$. For $\sigma,\tau\in\mathcal{S}_q$ we define $\delta_{(\sigma,\tau)}$ as the linear form on $\mathbb{M}_d(\mathbb{C})^{\otimes q} \otimes \mathbb{M}_d(\mathbb{C})^{\otimes q}$ such that

$$\delta_{(\sigma,\tau)}(E_{d,\mathbf{i},\mathbf{i}',\mathbf{j},\mathbf{j}'}) := \delta_{i_1i'_{\sigma(1)}} \cdots \delta_{i_qi'_{\sigma(q)}} \delta_{j_1j'_{\tau(1)}} \cdots \delta_{j_qj'_{\tau(q)}}$$

We shall follow the standard notations concerning partitions, Schur functions, and characters of symmetric groups, namely if $\lambda \vdash q$, i.e. λ is a partition of q, we denote by χ^λ the corresponding character of $\mathcal{S}_q$ and by $s_{\lambda,d}(x_1,\dots,x_d)$ the Schur function, see [Ful97] or [Mac95]. Whenever convenient we shall let $s_{\lambda,d}(x) = s_{\lambda,d}(x,\dots,x)$. If $\mu \vdash q$ we denote C_μ the corresponding conjugacy class of $\mathcal{S}_q$, and Z_μ the number of elements of C_μ. Finally, if $\sigma\in\mathcal{S}_q$ we denote by $|\sigma|$ the minimal number k such that σ can be written as a product of k transpositions. Recall that $|\sigma| = q - c(\sigma)$, where $c(\sigma)$ is the number of cycles of σ.

We can now state the main result of this section.

Theorem 2.2.1. *Let q, d be integers satisfying $d \geq q$; one has*

$$I_{d,q} = \sum_{\sigma,\tau\in\mathcal{S}_q} \delta_{(\sigma,\tau)} \frac{1}{q!^2} \sum_{\lambda\vdash q} \frac{\chi^\lambda(e)^2\chi^\lambda(\sigma\tau^{-1})}{s_{\lambda,d}(1)} \tag{2.2.2.1}$$

We shall denote Wg the function of $d, q, \sigma\tau^{-1}$ which occurs in the above result. For integers d, q and $\sigma\in\mathcal{S}_q$ one has

$$\operatorname{Wg}(d,q,\sigma) = \frac{1}{q!^2} \sum_{\lambda\vdash q} \frac{\chi^\lambda(1)^2\chi^\lambda(\sigma)}{s_{\lambda,d}(1)} \tag{2.2.2.2}$$

Observe that $s_{\lambda,d}(1)$, which is the dimension of the irreducible representation of $\mathbb{U}_d$ associated with λ, is a polynomial function of d, of degree q, therefore, Wg is a rational function of d, of degree at most $-q$. Since the dependence of Wg in q is implicitly given by σ, we shall drop it from notations and use $\mathrm{Wg}(d,\sigma)$ when it causes no confusion.

Proof.

We consider the left action π of $\mathbb{U}_d$ on $\mathbb{M}_d(\mathbb{C})$ by conjugation, i.e.

$$\pi(U)M := UMU^*$$

and $\pi^{\otimes q}$ the corresponding q -fold tensor product action of $\mathbb{U}_d$ on $\mathbb{M}_d(\mathbb{C})^{\otimes q}$. Then we consider the action $\pi^{\otimes q}\otimes\pi^{\otimes q}$ of $U_d\times U_d$ on $W=\mathbb{M}_d(\mathbb{C})^{\otimes q}\otimes\mathbb{M}_d(\mathbb{C})^{\otimes q}$.

The linear form $\delta_{(\sigma,\tau)}$ is invariant under the action of $\mathbb{U}_d\times\mathbb{U}_d$. By Schur-Weyl duality, the collection $\{\delta_{(\sigma,\tau)}\}$ is a generating family of the vector space of $\mathbb{U}_d\times\mathbb{U}_d$-linear maps, furthermore it is a basis for $d\geq q$, so that one can write

$$I_{d,q}=\sum_{\sigma,\tau\in S_q}\delta_{(\sigma,\tau)}\cdot\Gamma(d,\tau,\sigma)$$

and the coefficients $\Gamma(d,\tau,\sigma)$ are uniquely defined for $d\geq q$. Let Φ_σ be the linear endomorphism of $\mathbb{M}_d(\mathbb{C})^{\otimes q}\otimes\mathbb{M}_d(\mathbb{C})^{\otimes q}$ defined by

$$\begin{aligned}&\Phi_\sigma(E_{i_1i'_1}\otimes\cdots\otimes E_{i_qi'_q}\otimes E_{j_1j'_1}\otimes\cdots\otimes E_{j_qj'_q}):=\\&E_{i_1i'_{\sigma(1)}}\otimes\cdots\otimes E_{i_qi'_{\sigma(q)}}\otimes E_{j_1j'_{\sigma(1)}}\otimes\cdots\otimes E_{j_qj'_{\sigma(q)}}\end{aligned}\tag{2.2.2.3}$$

For all $\sigma\in\mathcal{S}_q$ one has $I_q\circ\Phi_\sigma=I_q$. On the other hand one has $\delta_{(\sigma,\sigma')}\circ\Phi_\tau=\delta_{(\sigma\tau,\sigma'\tau)}$. This proves that $\Gamma(d,\sigma,\sigma')=\Gamma(d,\sigma\tau,\sigma'\tau)$. In the same way, defining

$$\begin{aligned}&\Phi'_\sigma(E_{i_1i'_1}\otimes\cdots\otimes E_{i_qi'_q}\otimes E_{j_1j'_1}\otimes\cdots\otimes E_{j_qj'_q}):=\\&E_{i_{\sigma(1)}i'_1}\otimes\cdots\otimes E_{i_{\sigma(q)}i'_q}\otimes E_{j_{\sigma(1)}j'_1}\otimes\cdots\otimes E_{j_{\sigma(q)}j'_q}\end{aligned}\tag{2.2.2.4}$$

gives that $\Gamma(d,\sigma,\sigma')=\Gamma(d,\tau\sigma,\tau\sigma')$. This proves that Γ only depends on the conjugacy class of $\sigma\sigma'^{-1}$. We call $\mathrm{Wg}(d,\tau\sigma^{-1})$ the function such that $\Gamma(d,\sigma,\tau)=\mathrm{Wg}(d,\tau\sigma^{-1})$.

Note that choosing $\mathbf{i}=(1,\ldots,q)$, $\mathbf{i'}=(1,\ldots,q)$, $\mathbf{j}=(1,\ldots,q)$ and $\mathbf{j'}=(\sigma(1),\ldots,\sigma(q))$ one finds that

$$\mathrm{Wg}(d,\sigma)=\int_{\mathbb{U}_d}U_{11}\ldots U_{qq}\overline{U_{1\sigma(1)}}\ldots\overline{U_{q\sigma(q)}}dU\tag{2.2.2.5}$$

Consider now

$$A = \bigotimes_{i=1}^{q} E_{ii} \in \mathbb{M}_d(\mathbb{C})^{\otimes q} \quad \text{and} \quad B = \bigotimes_{i=1}^{q} E_{i\sigma(i)} \in \mathbb{M}_d(\mathbb{C})^{\otimes q}$$

Denoting $\rho(U)$ the matrix $U \otimes \ldots \otimes U \in \mathbb{M}_d(\mathbb{C})^{\otimes q}$ one has

$$\mathrm{Wg}(d,\sigma) = \int_{\mathbb{U}_d} (\mathrm{Tr}(A\rho(U)B\rho(U)^*))dU$$

For $\tau \in \mathcal{S}_q$ let $A_\tau = \bigotimes_{i=1}^q E_{\tau(i)\tau(i)}$ and $B_\tau = \bigotimes_{i=1}^q E_{\tau(i)\tau(\sigma(i))}$. The unitary permutation matrix $U_\tau = \sum_{i,j} E_{ij}\delta_{i\tau(j)} \in \mathbb{M}_d(\mathbb{C})$ satisfies $A_\tau = \rho(U_\tau)A\rho(U_\tau^*)$ and $B_\tau = \rho(U_\tau)B\rho(U_\tau^*)$ so that, since dU is a Haar measure, we have, for $\tau, \tau' \in \mathcal{S}_q$,

$$\int_{\mathbb{U}_d} (\mathrm{Tr}(A_\tau\rho(U)B_{\tau'}\rho(U)^*)dU = \int_{\mathbb{U}_d} (\mathrm{Tr}(A\rho(U)B\rho(U)^*)dU$$

Let $\tilde{A} = \sum_{\tau\in\mathcal{S}_q} \bigotimes_{i=1}^q E_{\tau(i)\tau(i)}$ and $\tilde{B} = \sum_{\tau\in\mathcal{S}_q} \bigotimes_{i=1}^q E_{\tau(i)\tau(\sigma(i))}$, we have thus

$$q!^2 \int_{\mathbb{U}_d} (\mathrm{Tr}(A\rho(U)B\rho(U)^*))dU = \int_{\mathbb{U}_d} (\mathrm{Tr}(\tilde{A}\rho(U)\tilde{B}\rho(U)^*)dU \qquad (2.2.2.6)$$

We compute the r.h.s. of 2.2.2.6. Let Ψ be the action of $\mathcal{S}_q$ on $\mathbb{M}_d(\mathbb{C})^{\otimes q}$ defined by

$$\Psi(\sigma)E_{i_1j_1} \otimes \ldots \otimes E_{i_qj_q} = E_{i_{\sigma(1)}j_{\sigma(1)}} \otimes \ldots \otimes E_{i_{\sigma(q)}j_{\sigma(q)}}$$

Consider the algebra of elements of $\mathbb{M}_d(\mathbb{C})^{\otimes q}$ fixed under the action $\pi^{\otimes q} \circ \Psi$ of $\mathbb{U}_d \times \mathcal{S}_q$. It is known from representation theory that this algebra is Abelian and that its minimal projections are indexed by $\lambda \vdash q$. We denote by $\{I_\lambda\}_{\lambda\vdash q}$ these minimal projections. The dimension of the range of I_λ is $\chi^\lambda(e)s_{\lambda,d}(1)$. One has $\mathrm{Tr}(\tilde{B}I_\lambda) = \chi^\lambda(e)\chi^\lambda(\sigma)$, therefore $\int \rho(U)\tilde{B}I_\lambda\rho(U^*)dU$, which is is a multiple of I_λ, is fully determined by its trace. thus,

$$\int_{\mathbb{U}_d} \rho(U)\tilde{B}I_\lambda\rho(U^*)dU = \frac{\chi^\lambda(e)\chi^\lambda(\sigma)}{s_{\lambda,d}(1)\chi^\lambda(e)} I_\lambda \qquad (2.2.2.7)$$

Multiplying by $\tilde{A}$ and taking the trace yields

$$\int_{\mathbb{U}_d} (\mathrm{Tr}(\tilde{A}\rho(U)\tilde{B}I_\lambda\rho(U)^*)) = \frac{\chi^\lambda(e)^2\chi^\lambda(\sigma)}{s_{\lambda,d}(1)} \qquad (2.2.2.8)$$

Summing over $\lambda \vdash q$ gives the result. □

2.2.2 Asymptotic expansion for Wg

Remember from the remark following the statement of Theorem 2.2.1 that Wg is a rational function of d, of degree at most $-q$, therefore it has a Laurent expansion

$$\mathrm{Wg}(d,\sigma)=\sum_{l=0}^{\infty}A[\sigma,l]d^{-q-l} \tag{2.2.2.9}$$

Since the roots of the polynomials $s_{\lambda,d}(1)$ belong to the set $\{0,1,\ldots,q-1\}$ we know that the radius of convergence in $1/d$ of the series is at least $1/(q-1)$, hence 2.2.2.9 converges for all $d\geq q$. We now determine the coefficients of this expansion.

Theorem 2.2.2. *Let $\sigma\in\mathcal{S}_q$.*

(i) One has $A[\sigma,0]=1$ if $\sigma=e$ and $A[\sigma,0]=0$ for $\sigma\neq e$.

(ii) For $l\neq 0$, let $A[\sigma,k,l]$ be the number of k-tuples of permutations $(\sigma_1,\ldots,\sigma_k)$, each different from the identity, satisfying both $\sum_{i=1}^{k}|\sigma_i|=l$ and $\sigma\sigma_1\ldots\sigma_k=e$, then one has

$$A[\sigma,l]=\sum_{k=1}^{l}(-1)^kA[\sigma,k,l].$$

Before we prove this result, we deduce a useful corollary.

Corollary 2.2.3. *For any $\sigma\in\mathcal{S}_q$, the rational function $d^{q+|\sigma|}\mathrm{Wg}(d,\sigma)$ is even, and its degree is less or equal than 0.*

Proof. If $\sum_{i=1}^{k}|\sigma_i|=l<|\sigma|$, then by the triangle inequality $|\sigma\sigma_1\ldots\sigma_k|>0$, in other words the product $\sigma_1\ldots\sigma_k$ can not be e. So $A[\sigma,l]=0$. This proves the assertion regarding the degree. Let $\varepsilon(\sigma)\in\{-1,1\}$ denote the sign of a permutation σ. If $\sigma\sigma_1\ldots\sigma_k=e$ then $\varepsilon(\sigma\sigma_1\ldots\sigma_k)=1=\varepsilon(\sigma)\varepsilon(\sigma_1)\cdots\varepsilon(\sigma_k)$ Since for every τ one has $\varepsilon(\tau)=(-1)^{|\tau|}$ we see that $A[\sigma,l]=0$ if l and $|\sigma|$ do not have the same parity. □

In order to prove Theorem 2.2.2 we shall need some preliminary results. First we introduce some notation. Let k be an integer and $\mu,\mu_1,\ldots,\mu_k$ be partitions of q. Let $\sigma\in\mathcal{S}_q$ be such that $\sigma\in C_\mu$. We call $A[\mu;\mu_1,\ldots,\mu_k]$ the number of k-tuples $(\sigma_1,\ldots,\sigma_k)\in\mathcal{S}_q$ such that $\sigma\cdot\sigma_1\cdots\sigma_k=e$ and $\sigma_i\in C_{\mu_i}$ for all $i\in[1,k]$. Let Z_σ be the cardinal of the conjugacy class of σ.

Lemma 2.2.4. *Let $\mu, \mu_1, \dots, \mu_k$ be partitions of q and $\sigma \in C_\mu$ then*

$$\frac{1}{q!}\sum_{\lambda} \chi^\lambda(\sigma)\chi^\lambda_{\mu_1}\dots\chi^\lambda_{\mu_k}(\chi^\lambda(e))^{1-k}\cdot Z_{\sigma_1}\dots Z_{\sigma_k} = A[\mu;\mu_1,\dots,\mu_k].$$

Proof. A standard computation in the group algebra of $\mathcal{S}_q$, using Fourier analysis. □

Next we recall another well known formula, see e.g. [Ful97] or [Mac95].

Lemma 2.2.5. *For any $\lambda \vdash q$ one has $s_{\lambda,d}(1) = \frac{1}{q!}\sum_{\tau\vdash q} d^{q-|\tau|}\chi^\lambda(\tau)Z_\tau$.*

We now proceed to the proof of Theorem 2.2.2.

Proof. From Lemma 2.2.5 one has

$$s_{\lambda,d}(1) = \frac{d^q}{q!}(1 + \sum_{\tau\vdash q,\tau\neq 1^q} d^{-|\tau|}\chi^\lambda(\tau)Z_\tau)$$

Expand $1/s_{\lambda,d}(1)$ in the expression

$$\mathrm{Wg}(d,\sigma) = \frac{1}{d^q q!}\sum_{\lambda\vdash q}\frac{\chi^\lambda(\sigma)\chi^\lambda(e)}{\sum_{\tau\vdash q} d^{-|\tau|}\chi^\lambda(\tau)Z_\tau/\chi^\lambda(e)}$$

to find that the coefficient of d^{-q-l} is

$$\frac{1}{q!}\sum_{\lambda\vdash q}\chi^\lambda(\sigma)\chi^\lambda(e)\sum_{k=1}^{l}(-1)^k \sum_{\mu_1,\dots,\mu_k\vdash q,\sum_{i=1}^k|\mu_i|=l}\prod_{i=1}^{k}\frac{\chi^\lambda_{\mu_i}}{\chi^\lambda(e)}Z_{\sigma_i}$$

$$= \frac{1}{q!}\sum_{k=1}^{l}(-1)^k\sum_{\mu_1,\dots,\mu_k\vdash q,\sum_{i=1}^k|\mu_i|=l}\sum_{\lambda\vdash q}\frac{\chi^\lambda(\sigma)\chi^\lambda(\sigma_1)\dots\chi^\lambda(\sigma_k)}{(\chi^\lambda(e))^{k-1}}\cdot Z_{\sigma_1}\dots Z_{\sigma_k} \tag{2.2.2.10}$$

Using Lemma 2.2.4, this expression is

$$\sum_{k=1}^{l}(-1)^k\sum_{\mu_1,\dots,\mu_k\vdash q,\sum_{i=1}^k|\mu_i|=l} A[\sigma;\mu_1,\dots,\mu_k]$$

From the definition of $A[\sigma,k,l]$ we have

$$\sum_{\mu_1,\dots,\mu_k\vdash q,\sum_{i=1}^k|\mu_i|=l} A[\sigma;\mu_1,\dots,\mu_k] = A[\sigma,k,l]$$

which concludes the proof. □

2.2.3 Cumulants of random variables

For reference and proofs about the beginning of this section, we refer to [Zvo97] or [Rot64]. We consider the lattice $\mathcal{P}$ of partitions of $[1,q]=\{1,\dots,q\}$, and say that $\Pi' \leq \Pi$ if, for any block V of Π' there exists a block of Π that contains V. For any partitions Π and Π' define $\Pi \vee \Pi'$ (resp. $\Pi \wedge \Pi'$) to be the smallest majorant (resp. the greatest minorant), and $1_q = \{\{1,\dots,q\}\}$ (resp. $0_q=\{\{1\},\dots,\{q\}\}$) the greatest (resp. smallest) element.

The permutation group $\mathcal{S}_q$ admits a natural left action on the partitions of $[1,q]$. Call $\mathcal{O}_\Pi$ the orbit to which Π belongs. These orbits are in natural one to one correspondence with the partitions of the integer q. For $\sigma \in \mathcal{S}_q$, let Π_σ be the partition of $[1,q]$ whose blocks are the orbits of σ. It is obvious but useful to remark that $C_\sigma = \mathcal{O}_{\Pi_\sigma}$.

For each $\Pi_1, \Pi_2 \in \mathcal{P}$ such that $\Pi_1 \leq \Pi_2$, there exists a partition Π such that the interval $[\Pi_1,\Pi_2]$ is isomorphic as a lattice to the lattice $[0_q,\Pi]$. If $\mathcal{O}_\Pi = (p_1,\dots,p_k)$ then define

$$\mathrm{Moeb}(\Pi_1,\Pi_2) = \prod_i ((-1)^{i-1}(i-1)!)^{p_i}$$

For a set $A_1,\dots,A_q$ of random variables and $\Pi=\{V_1,\dots,V_k\}$ a partition of $[1,q]$, let

$$E_\Pi(A_1,\dots,A_q) = \prod_{i=1}^k E(\prod_{j\in V_i} A_j)$$

With the same notations we define the *classical cumulant* by

$$C_\Pi(A_1,\dots,A_q) = \sum_{\Pi' \leq \Pi} E_{\Pi'}(A_1,\dots,A_q)\mathrm{Moeb}(\Pi',\Pi)$$

and

$$C_q(A_1,\dots,A_q) = C_{1_q}(A_1,\dots,A_q)\mathrm{Moeb}(\Pi',\Pi)$$

We also introduce the notion of *relative cumulant*

$$C_{\Pi_1,\Pi_2}(A_1,\dots,A_q) = \sum_{\Pi_1\leq\Pi\leq\Pi_2} E_\Pi(A_1,\dots,A_q)\mathrm{Moeb}(\Pi,\Pi_2) \qquad (2.2.2.11)$$

This cumulant is the cumulant of the function E_Π associated to the lattice $[\Pi_1, 1_q]$. It is known that $\sum_{q\geq 1} C_q(A,\dots,A)z^q/q!$ is the power series expansion in a neighborhood of 0 of the analytic function $z \to \log E(\exp(zA))$. Let $\Pi_1 \leq \Pi_2$ be partitions of $[1,q]$ and $A_1,\dots A_q$ be random variables. We have

$$E_{\Pi_2}(A_1,\dots,A_q) = \sum_{\Pi_1\leq\Pi\leq\Pi_2} C_{\Pi_1,\Pi}(A_1,\dots,A_q) \qquad (2.2.2.12)$$

Let q be an integer and σ a permutation of $\mathcal{S}_q$. For any $i \in [1,q]$ call B_i the random variable $B_i^d = U_{ii}\overline{U_{i\sigma(i)}}$, where U_{ij} is the complex random variable on the unitary group U_d that was introduced in section 2.2.1. For any subset $V \subset [1,q]$ we define the random variable

$$B_V = \prod_{i\in V} B_i \qquad (2.2.2.13)$$

In particular, for $\Pi = \{V_1,\dots,V_k\}$ a partition of $[1,q]$ satisfying $\Pi \geq \Pi_\sigma$, let

$$E_\Pi(\sigma,d) = \prod_{i=1}^{k} E(B_{V_i}) \qquad (2.2.2.14)$$

and

$$C_\Pi(\sigma,d) = \sum_{\Pi\leq\Pi_2} E_\Pi(\sigma,d)\mathrm{Moeb}(\Pi,\Pi_2)$$
$$C_{\Pi_1,\Pi_2}(\sigma,d) = \sum_{\Pi_1\leq\Pi\leq\Pi_2} E_\Pi(\sigma,d)\mathrm{Moeb}(\Pi,\Pi_2)$$

the classical (resp. relative) cumulants related to $E_\Pi(\sigma,d)$.

Theorem 2.2.6. *Let σ be an element of $\mathcal{S}_q$. Let Π be a partition of $[1,q]$, such that $\Pi_\sigma \leq \Pi$. Let τ be a permutation such that $\Pi_\tau = \Pi$. For integers k,l, let $\gamma_{\sigma,\Pi,k,l}$ the number of solutions of $\sigma\sigma_1\dots\sigma_k = e$ such that for all i, one has $\sigma_i \neq e$, $\sum_{i=1}^k |\sigma_i| = l$ and the group generated by $\tau,\sigma_1,\dots,\sigma_k$ acts transitively on $[1,q]$. Let*

$$\gamma_{\sigma,\Pi,l} = \sum_{k=1}^{l} (-1)^k \gamma_{\sigma,\Pi,k,l} \qquad (2.2.2.15)$$

Let d be an integer such that $d \geq q$, then

$$C_{\Pi,1_q}(\sigma,d) = \sum_{l\geq 1} \gamma_{\sigma,\Pi,l} d^{-q-l} \tag{2.2.2.16}$$

In order to prove Theorem 2.2.6 we need some elementary technical lemmas. We first derive an important corollary. For a partition Π of $[1,q]$, call $C(\Pi)$ the number of its blocks.

Corollary 2.2.7. *Let d, q be integers, $\sigma \in \mathcal{S}_q$ be a permutation and Π be a partition of $[1,q]$ such that $d \geq q$. Then $C_{\Pi_1,\Pi_2}(\sigma,d)$ is a rational fraction of order at most $-q-|\sigma|+2(C(\Pi_2)-C(\Pi_1))$.*

We first prove Corollary 2.2.7.

Proof. From Theorem 2.2.6 it is enough to prove that if $\tau \in \mathcal{S}_q$ is such that $\Pi_\tau = \Pi$ and $\sigma_1, \ldots, \sigma_k$ are permutations of $\mathcal{S}_q$ such that $\sigma\sigma_1\ldots\sigma_k = e$, and the group generated by $\tau, \sigma_1, \ldots, \sigma_k$ acts transitively on $[1,q]$, then $|\sigma_1| + \ldots + |\sigma_k| \geq |\sigma| + 2(C(\Pi)-1)$.

It is known (see e.g. [BMS00]), that if $\sigma_1, \ldots, \sigma_k$ act transitively on $[1,q]$ and satisfy $\sigma_1\ldots\sigma_k = \sigma$, then $|\sigma|+|\sigma_1|+\ldots+|\sigma_k| \geq 2q-2$. In order to prove this, the difference is interpreted as the genus of a given graph drawn on a two dimensional compact manifold. This implies that if τ' is a permutation such that $\Pi_{\tau'} = \vee_{i=1}^k \Pi_{\sigma_i}$, then $|\sigma_1| + \ldots + |\sigma_k| \geq 2|\tau'|$. Indeed, restricting the equation $\sigma_1\ldots\sigma_k = \sigma$ to an orbit of τ' turns the action into a transitive one, so the inequality of [BMS00] holds. Conclusion follows by summing on the orbits.

Thus it is enough to show that if the group generated by τ and τ' acts transitively on $[1,q]$ then $|\tau'|+1 \geq C(\Pi_\tau)$. Let $V_1, \ldots, V_k$ be the blocks of Π, and $\tau_1\ldots\tau_l = \tau'$ be a transposition decomposition of τ'. This decomposition yields a graph on $V_1, \ldots, V_k$ in the following way: if $\tau_i = (ab)$ with $a \in V_a$ and $b \in V_b$ then put an edge between V_a and V_b. This diagram has to be connected, so that $l \geq k-1$. □

Lemma 2.2.8. *Let k_1, k_2 be two integers satisfying $1 \leq k_1 \leq k_2$. We have*

$$\sum_{\substack{k\geq 1, V_1,V_2\subset[1,k],\, V_1\cup V_2=[1,k] \\ \mathrm{Card}\,V_1=k_1,\, \mathrm{Card}\,V_2=k_2}} (-1)^k = (-1)^{k_1+k_2} \tag{2.2.2.17}$$

Proof. Consider the formal power series

$$A = (1+X)^{-1}(1+Y)^{-1} = \sum_{k_1,k_2\geq 0} A_{k_1,k_2} X^{k_1} Y^{k_2} \in \mathbb{C}[[X,Y]]$$

It is clear that $A_{k_1,k_2} = (-1)^{k_1+k_2}$. And if we rewrite A as $(1+(X+Y+XY))^{-1} = \sum_k (-1)^k (X+Y+XY)^k$, then

$$A_{k_1,k_2} = \sum_k \sum_{a_1,\dots,a_k\in\{X,Y,XY\},\prod_i a_i = X^{k_1}Y^{k_2}} (-1)^k$$

To a sequence $(a_1,\dots,a_k)$ we can associate the pair (V_1,V_2) of subsets of $[1,k]$ defined by $i\in V_1$ iff $a_i = X$ or XY and $i \in V_2$ iff $a_i = Y$ or XY. We have $\mathrm{Card}V_1 = k_1$, $\mathrm{Card}V_2 = k_2$ and $V_1\cup V_2 = [1,k]$. This correspondence is clearly one to one. □

Lemma 2.2.9. *Let f and g be functions $\mathbb{N}^* \to \mathbb{C}$, almost everywhere zero. Define*

$$S(f) = \sum_{k\geq 1} (-1)^k f(k) \tag{2.2.2.18}$$

And define the operation

$$h = f * g : \mathbb{N}^* \to \mathbb{C} \tag{2.2.2.19}$$

$$h(k) = \sum_{\Pi_1,\Pi_2\subset[1,k],\Pi_1\cup\Pi_2=[1,k]} f(\mathrm{Card}\Pi_1)g(\mathrm{Card}\Pi_2) \tag{2.2.2.20}$$

This operation is associative. Besides, we have for all f,g, $S(f*g) = S(f)S(g)$.

Proof. Associativity is elementary. For the multiplicativity, one just needs to remark that $f(k_1)g(k_2)$ contributes to $(-1)^{k_1+k_2}$ in $S(f)S(g)$, and that it gives a contribution of

$$\sum_{k\geq 1,\,\Pi_1,\Pi_2\subset[1,k],|\Pi_1|=k_1,|\Pi_2|=k_2,\Pi_1\cup\Pi_2=[1,k]} (-1)^k$$

in $S(h)$, so this is a consequence of Lemma 2.2.8. □

Let $\Pi = \{V_1, \ldots, V_j\}$ be a partition of $[1, q]$; k and l be integers and consider a k -tuple $(\sigma_1, \ldots, \sigma_k)$ such that $\sigma\sigma_1 \ldots \sigma_k = e$, $|\sigma_1| + \ldots + |\sigma_k| = e$, for all i, $\Pi_{\sigma_i} \leq \Pi$ and $\sigma_i \neq e$. We call $A[\sigma, k, l](\Pi)$ the number of such k -tuples. We also need the notation $A[\sigma, k, l](V_i) = A[\sigma_{|V_i}, k, l]$. We denote $A[\sigma, l](\Pi) = S(A[\sigma, \cdot, l](\Pi))$ and $A[\sigma, l](V_i) = S(A[\sigma, \cdot, l](V_i))$.

Lemma 2.2.10. *Under the preceding notations, we have*

$$A[\sigma, l](\Pi) = \sum_{l_1, \ldots, l_j, \sum_{r=1}^{j} l_r = l} \prod_{r=1}^{j} A[\sigma, l_r](V_r) \tag{2.2.2.21}$$

Proof. If $\Pi = \{V_1, \ldots, V_j\}$, and $l_1, \ldots, l_j$ are integers we call $A[\sigma, k, [l_1, \ldots, l_j]](\Pi)$ the number of solutions of the equation $\sigma\sigma_1 \ldots \sigma_k = e$ in $\mathcal{S}_q - \{e\}$ satisfying for all $r \in [1, j]$ $\sum_{i=1}^{k} |\sigma_{i|V_r}| = l_r$ and for all $i \in [1, k]$, $\Pi_{\sigma_i} \leq \Pi$. We have

$$A[\sigma, l](\Pi) = \sum_{l_1, \ldots, l_j, \sum_{r=1}^{j} l_r = l} A[\sigma, [l_1, \ldots, l_j]](\Pi)$$

so it is enough to prove that

$$A[\sigma, k, [l_1, \ldots, l_j]](\Pi) = \prod_{r=1}^{j} A[\sigma, l_r](V_r)$$

For this, introduce the functions f_r for $1 \leq r \leq j$ defined as $k \geq 1$ by $f_r(k) = A[\sigma, k, l_r](V_r)$. Define f by $f(k) = A[\sigma, k, [l_1, \ldots, l_j]](\Pi)$.

Considering the convolution that we introduced in equation 2.2.2.19 we have $f = f_1 * \ldots * f_k$ so by the lemma 2.2.9 we get $S(f) = S(f_1) \ldots S(f_k)$. □

Lemma 2.2.11. *Let* Π *be a partition of* $[1, k]$ *then* $E_\Pi(B)[l] = A[\sigma, l](\Pi)$

Proof. By Theorem 2.2.2, we have $E_{V_i}(B)[l] = A[\sigma, l](V_i)$ so $E_\Pi(d, \sigma)[l] = \sum_{l_1, \ldots, l_j, \sum_{r=1}^{j} l_r = l} \prod_{r=1}^{j} A[\sigma, l_r](V_r)$, which equals $A[\sigma, l](\Pi)$ by Lemma 2.2.10.
□

Now we can go through the proof of Theorem 2.2.6.

Proof. We have by definition $C_{\Pi, 1_q}(\sigma, d)[l] = \sum_{\Pi' \geq \Pi} E_B(\Pi)[l] \mathrm{Moeb}(\Pi', 1_q)$. By lemma 2.2.11 this is $\sum_{\Pi' \geq \Pi} A[\sigma, l](\Pi') \mathrm{Moeb}(\Pi', 1_q)$.

Let Π be a partition such that $\Pi_\sigma \leq \Pi$, and let $A[\sigma,k,l][\Pi]$ be the number of k -tuples of permutations $(\sigma_1,\dots,\sigma_k)$ such that $\sigma_i \neq e$ for all i ; $\sigma\sigma_1\dots\sigma_k = e$ and $\sum_i |\sigma_i| = l$ and $\vee_i[\sigma_i] = \Pi$. We have the obvious relation between $A[\sigma,l][\Pi]$ and $A[\sigma,l](\Pi)$

$$A[\sigma,l](\Pi) = \sum_{\Pi'} 1_{\Pi\geq\Pi'} A[\sigma,l][\Pi'] \tag{2.2.2.22}$$

By proposition 2.2.2.12 this implies that $C_{\Pi,1_q}(\sigma,d)[l] = A[\sigma,l][\Pi]$, which is the expected result. □

2.2.4 A first order expansion for $C_{\Pi,1_q}(d,\sigma)$

So far we have obtained upper bounds for the degrees of cumulants of unitary polynomial integrals. However, except in some very elementary cases, it is far from obvious that they are optimal. In this section we show that our bounds are optimal and perform some explicit computations.

Let Moeb : $\mathcal{S}_q \to \mathbb{C}$ be the central function defined by

$$\mathrm{Moeb}(\sigma) := \prod_{i=1}^{k} c_{|C_i|} \cdot (-1)^{|C_i|} \tag{2.2.2.23}$$

where $\sigma \in \mathcal{S}_q$ is a permutation whose cycle decomposition is $\sigma = C_1 \cdots C_k$ and $c_n = (2n)!/(n!(n+1)!)$. The integer c_n is the n^{th} *Catalan number* and satisfies the inductive relation $c_n = \sum_{i=0}^{n-1} c_i \cdot c_{n-i-1}$. Its first values are

$$c_o = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, c_5 = 42, c_6 = 132, c_7 = 429, c_8 = 1430$$

Remark 2.2. This function is related to Speicher's function Moeb defined in [Spe94] in the sense that a non crossing partition Π (that will be defined at section 2.4.2) of $[1,q]$ with blocks of length $|C_1|,\cdots,|C_k|$ satisfies $\mathrm{Moeb}(\Pi) = \mathrm{Moeb}(\sigma)$.

Theorem 2.2.12. • *(i) Let σ be a permutation of $[1,q]$ such that in its cycle product decomposition it has d_i cycles of length $i-1$. Then*

$$\gamma_{\sigma,\Pi_\sigma,3q-2-|\sigma|} = (-1)^{|\sigma|} \frac{2^{q-|\sigma|}(3q-3-|\sigma|)!}{(2q)!} \prod_{i=1}^{q} \left(\frac{(2i-1)!}{(i-1)!^2}\right)^{d_i} \tag{2.2.2.24}$$

where γ was defined in Theorem 2.2.6, equation 2.2.2.15.

- *(ii) In particular,*

$$\lim_{d\to\infty} d^{q+|\sigma|}\mathrm{Wg}(d,\sigma) = \mathrm{Moeb}(\sigma) \tag{2.2.2.25}$$

- *(iii) Let Π' be a partition having blocks $V_1,\ldots,V_k$ of length $q_1,\ldots,q_k$ such that $\Pi_\sigma \leq \Pi'$ and define $\sigma_i = \sigma_{|V_i}$. Call*

$$g_{\sigma,\Pi} = \sum_{\substack{\Pi'\in\mathcal{P}_q,\Pi'\geq\Pi_\sigma,\Pi'\vee\Pi=1_q\\ C(\Pi_\sigma)-C(\Pi')=C(\Pi)-1}} \prod_{i=1}^{k} \frac{(3q_i-3-|\sigma_i|)!}{(2q_i)!} \tag{2.2.2.26}$$

we have

$$\gamma_{\sigma,\Pi,q+|\sigma|+2(C(\Pi)-1)} = g_{\sigma,\Pi}\prod_{i=1}^{q}\left(\frac{(2i-1)!}{(i-1)!^2}\right)^{d_i}(-1)^{|\sigma|}2^{q-|\sigma|} \tag{2.2.2.27}$$

Note that this it is not zero.

Proof. Formula 2.2.2.27 is a consequence of 2.2.2.24 and 2.2.2.25 together with lemma 2.2.10. The fact that it is non zero comes from the fact that all summands of the right hand side of 2.2.2.26 have the same sign and are non zero.

So we focus on proving 2.2.2.24 and 2.2.2.25. In [BMS00], the following result is proved: the number of k -tuples $(\sigma_1,\ldots,\sigma_k)$ of permutations of $\mathcal{S}_q$ such that $\sigma_1\ldots\sigma_k\sigma = e$, the group generated by $\sigma_1,\ldots,\sigma_k$ acts transitively on $[1,q]$ and $|\sigma|+|\sigma_1|+\ldots+|\sigma_k| = 2q-2$, is

$$\tilde{A}[\sigma,k] = k\frac{(qk-q-1)!}{(qk-2q+|\sigma|+2)!}\prod_{i\geq 1}\left[i\binom{ki-1}{i}\right]^{d_i}$$

where d_i denotes the number of cycles with i elements of σ.

This set allows the σ_i's to be identity. An application of the exclusion-inclusion principle shows that: the number of k -tuples $(\sigma_1,\ldots,\sigma_k)$ of permutations of $\mathcal{S}_q$ *different from identity* such that $\sigma_1\ldots\sigma_k\sigma = e$, the group generated by $\sigma_1,\ldots,\sigma_k$ acts transitively on $[1,q]$ and $|\sigma|+|\sigma_1|+\ldots+|\sigma_k| = 2q-2$, is

$$\gamma_{\sigma,\Pi_\sigma,k,2q-2-|\sigma|} = \sum_{l=1}^{2q-2-|\sigma|}\binom{2q-2-|\sigma|}{l}\tilde{A}[\sigma,k,l](-1)^l \tag{2.2.2.28}$$

We need to estimate $\gamma_{\sigma,\Pi_\sigma,2q-2-|\sigma|} = S(\gamma_{\sigma,\Pi_\sigma,k,2q-2-|\sigma|})$. The fact that for $k' \geq k$ one has $\sum_{n=k}^{k'} \binom{n}{k} = \binom{k'+1}{k+1}$ implies that

- (i) If $|\sigma| < q-1$ then

$$\gamma_{\sigma,\Pi_\sigma,2q-2-|\sigma|} = \sum_{k=2}^{2q-2-|\sigma|} (-1)^k \binom{2q-1-|\sigma|}{k+1} k \frac{(qk-q-1)!}{(qk-2q+|\sigma|+2)!} \prod_{i\geq 1} \left[i \binom{ki-1}{i} \right]^{d_i} \tag{2.2.2.29}$$

- (ii) if $|\sigma| = q-1$ then

$$\gamma_{\sigma,\Pi_\sigma,2q-2-|\sigma|} = \sum_{k=2}^{q} (-1)^{q-k} \frac{(qk-q)!}{k!(q-k)!(qk-2q+1)!} \tag{2.2.2.30}$$

Now we need to prove that for $\sigma \in \mathcal{S}_q$ having a_i cycles with i elements, one has

$$\gamma_{\sigma,\Pi_\sigma,3q-2-|\sigma|} = \frac{2^q(3q-3-|\sigma|)!}{(2q)!} \prod_{i=1}^{q} \left(\frac{(2i-1)!}{(i-1)!^2(-2)^{i-1}} \right)^{d_i} \tag{2.2.2.31}$$

We recall that for $0 \leq r < n$ we have

$$\sum_{k=0}^{n} \binom{n}{k} k^r (-1)^k = 0 \tag{2.2.2.32}$$

Indeed, the holomorphic function $z \to (1-\exp z)^n$ has a zero of order n at 0 and the left hand side of equation 2.2.2.32 is its r-th derivative at zero. We first handle equation 2.2.2.29. The expression

$$P(k) = k \frac{(qk-q-1)!}{(qk-2q+|\sigma|+2)!} \prod_{i\geq 1} \left[i \binom{ki-1}{i} \right]^{d_i}$$

is a polynomial in k, of degree at most $2q-2-|\sigma|$. It is easily checked that $P(1) = P(0) = 1$. So according to equation 2.2.2.32.

$$\sum_{k=2}^{2q-2-|\sigma|} (-1)^k \binom{2q-1-|\sigma|}{k+1} k \frac{(qk-q-1)!}{(qk-2q+|\sigma|+2)!} \prod_{i\geq 1} \left[i \binom{ki-1}{i} \right]^{d_i} = -P(-1)$$

But

$$-P(-1) = \frac{2^q(3q-3-|\sigma|)!}{(2q)!}\prod_{i=1}^{q}\left(\frac{(2i-1)!}{(i-1)!^2(-2)^{i-1}}\right)^{d_i} \qquad (2.2.2.33)$$

This proves 2.2.2.24 in the case of 2.2.2.29. The case of 2.2.2.30 is done in the same way. □

2.3 An application to free probability theory

In this section we first recall basic definitions about asymptotic freeness. The main results of this section are Theorems 2.3.1 and 2.3.5. In order to prove 2.3.5 we introduce material that will be used as well in Section 2.4.

2.3.1 Asymptotic freeness

Our definition of a *non commutative probability space* is the following: it is an algebra with unit endowed with a faithful tracial state ϕ. In particular we do not make any assumption about the linear form ϕ, except that $\phi(ab)=\phi(ba)$ and $\phi(1)=1$. We do not need the $*$ -structure of the matrix algebras (and a fortiori faithfulness or positivity assumptions). We denote such a space by (A,ϕ). An element of this space will be called a (non-commutative) random variable.

Let $A_1,\cdots,A_k$ be subalgebras of A having the same unit as A. They are said to be *free* iff for all $a_i\in A_{j_i}$ ($j\in[1,l]$) such that $\phi(a_i)=0$, one has $\phi(a_1\cdots a_l)=0$ as soon as $j_1\neq j_2$, $j_2\neq j_3,\cdots,j_{l-1}\neq j_l$. Random variables are said to be free iff the unital subalgebras that they generate are free.

Let $(a_1,\cdots,a_k)$ be a k -tuple of random variables and let $\mathbb{C}\langle X_1,\cdots,X_k\rangle$ be the free algebra of non commutative polynomials on $\mathbb{C}$ generated by the k indeterminates $X_1,\cdots,X_k$. The *joint distribution* of the family a_i is the linear form $\mu_{(a_1,\cdots,a_k)}:\mathbb{C}\langle X_1,\cdots,X_k\rangle\to\mathbb{C}$ defined in the obvious sense.

Given a k -tuple $(a_1,\cdots,a_k)$ of free random variables and given each law μ_{a_i}, the joint law $\mu_{(a_1,\cdots,a_k)}$ is uniquely determined by induction with the μ_{a_i}'s. We shall say that a family $(a_1^d,\cdots,a_k^d)_d$ of k -uples of random variables *converges in law* towards $(a_1,\cdots,a_k)$ iff for all $P\in\mathbb{C}\langle X_1,\cdots,X_k\rangle$, $\mu_{(a_1^d,\cdots,a_k^d)}(P)$ converges towards $\mu_{(a_1,\cdots,a_k)}(P)$ as $d\to\infty$.

This gives the following obvious sense to *asymptotic freeness*: a sequence of families $(a_1^d, \cdots, a_k^d)_d$ is asymptotically free as $d \to \infty$ iff it converges in law towards a free random variable.

By now we are able to state the main result of this section. Note that it was originally contained in [VDN92] under stronger hypotheses. Feng Xu [Xu97] obtains an analogous result using geometric methods.

Theorem 2.3.1. *Let $U_1, \cdots, U_k, \cdots$ be a collection of independent Haar distributed random matrices of $\mathbb{M}_d(\mathbb{C})$ and $(W_i^d)_{i\in I}$ be a set of constant matrices of $\mathbb{M}_d(\mathbb{C})$ admitting a joint limit distribution for large d. Then the family $((U_1, U_1^*), \cdots, (U_k, U_k^*), \cdots, (W_i))$ admits a limit distribution, and is asymptotically free.*

Note that this statement holds under the very weak hypothesis that the joint law of the family W admits a weak limit, i.e. it does not make any assumption of boundedness of the elements of W as $d \to \infty$. In other words we do not have to consider asymptotic $*$ -freeness, our method works in the framework of asymptotic algebraic freeness. This allows us to escape from the machinery of functional calculus and weaken the hypotheses of [Voi98]. Besides, we do not have to restrict to diagonal elements or self adjoint elements for the $(W_i)_i$ as in the previous proofs. In particular they might have no $*$ -asymptotic limit.

We split the proof of this theorem into two parts. The first one consists in showing that there exists an asymptotic joint law for the families of random variables. The second step is to show that these families are asymptotically free.

Let $W_1, \cdots, W_n$ be matrices of $\mathbb{M}_d(\mathbb{C})$ and α a permutation of $[1, n]$. If the cycle decomposition of α is $(a_{11} \cdots a_{1i_1}) \cdots (a_{k1} \cdots a_{ki_k})$, then we define

$$\langle W_1 \cdots W_n \rangle_\alpha := \mathrm{tr}(W_{a_{11}} \cdots W_{a_{1i_i}}) \cdots \mathrm{tr}(W_{a_{k1}} \cdots W_{a_{ki_k}}) \qquad (2.3.2.1)$$

These notations are well defined because we consider the state tr is tracial.

For an integer n, let ξ be a permutation of $\mathcal{S}_n$ such that there exists disjoint subsets V_1 and V_2 with q elements of $[1, n]$ such that $\xi(V_1) = V_2$, $\xi(V_2) = V_1$ and ξ stabilizes pointwise $(V_1 \cup V_2)^c$. We can check promptly that ξ^2 stabilizes both V_1 and V_2, and that the restrictions of ξ^2 to V_1 and to V_2 are conjugate. This allows to define

$$\widetilde{\mathrm{Wg}}(\xi) = \mathrm{Wg}(d, \xi^2_{|V_1})$$

Besides it is routine to check that $\widetilde{\mathrm{Wg}}(\xi) = 0(d^{-|\xi|})$. From this we can state

Proposition 2.3.2. *Let $U_1, \ldots, U_k$ be k independent Haar distributed random matrices. Let $n = 2q$ be an even integer and for $i \in [1, n]$ let $\widetilde{W}_i$ be a random matrix of the form $W_i U_{j_i}^{\varepsilon_i}$ with $j_i \in [1, k]$ and $\varepsilon_i \in \{-1, 1\}$. For $j \in [1, k]$, let $\mathcal{T}_j$ be the subset of $\mathcal{S}_n$ of permutations ξ such that if $j_i = k$ then $j_{\xi(i)} = k$ and $\varepsilon_{\xi(i)} = -\varepsilon_i$ and if $j_i \neq k$ then $\xi(i) = i$. Let α be a permutation of $[1, n]$. Then*

$$E(\langle \widetilde{W}_1 \ldots \widetilde{W}_n \rangle_\alpha) = \sum_{\xi_1 \in \mathcal{T}_1, \ldots, \xi_k \in \mathcal{T}_k} \langle W_1 \ldots W_n \rangle_{\alpha \xi_1 \ldots \xi_k} d^{|\alpha| - |\alpha \xi_1 \ldots \xi_k|} \prod_{i=1}^{k} \widetilde{\mathrm{Wg}}(\xi_i) \tag{2.3.2.2}$$

Proof. This is done by induction on k. For $k = 1$ this is a standard combinatorial computation in view of Theorem 2.2.1. For the general k case, it is enough to apply Fubini's Theorem and remark that the elements of $\mathcal{T}_i$ and $\mathcal{T}_j$ commute. □

Each summand of equation 2.3.2.2 is of order $0(1)$ because by the triangle inequality $|\alpha| - |\alpha\xi_1 \ldots \xi_k| - |\xi_1| - \ldots - |\xi_k| \leq 0$. This proves the existence of a joint distribution.

Proposition 2.3.3.

- *(i) In equation 2.3.2.2, if for a given $\xi_1 \in \mathcal{T}_1, \ldots, \xi_k \in \mathcal{T}_k$, one has $|\alpha| - |\alpha\xi_1 \ldots \xi_k| - |\xi_1| - \ldots - |\xi_k| = 0$ then $\alpha\xi_1 \ldots \xi_k$ admits at least two fixed points. Besides if j is a fixed point, it can not be such that $\varepsilon_{j-1} = \varepsilon_j$ and $i_{j-1} = i_j$.*

- *(ii) In particular the family $((W), (U_1, U_1^*), \cdots, (U_k, U_k^*))$ is asymptotically free.*

Proof.

The fact that the first point implies the second one is as for the proof of 2.3.2.2, a straightforward consequence of the definition of freeness. Let us consider the situation $|\alpha| - |\alpha\xi_1 \ldots \xi_k| - |\xi_1| - \ldots - |\xi_k| = 0$. We have $|\alpha\xi_1 \ldots \xi_k| \leq |\alpha| - q \leq q - 1$. Consequently ϕ moves at most $2(q-1) = n - 2$ elements, hence has at least two fixed points. Besides, according to the definition of ϕ and of $\mathcal{T}_i$, the permutation $\alpha\xi_1 \ldots \xi_k$ has to move the point i if $\varepsilon_{j-1} = \varepsilon_j$ and $i_{j-1} = i_j$. □

Since the proposition holds for all k, it holds for an arbitrary family of independent random variables, therefore this proposition proves the Theorem 2.3.1 in full generality.

The following corollary can be found in [Voi98] and [Xu97] and it is a consequence from Theorem 2.3.1.

Corollary 2.3.4. *Let W be a family of constant matrices with a limiting distribution and $U_1, \cdots, U_n$ be random unitary matrices. Then the variables $W, U_1 W U_1^*, \cdots, U_k W U_k^*$ are asymptotically free.*

Proof. This is a straightforward consequence of the Theorem 2.3.1 and of the definition of asymptotic freeness. □

Remark 3.1. The results we obtained are the same if one replaces $\mathbb{U}_d$ by SU_d. Indeed, a careful reading of the proof of Theorem 2.2.1 shows that it also holds for SU_d as soon as $d > q$. Another way to see it is to note that the multiplication map $SU_d \times \mathbb{U}_1 \to \mathbb{U}_d$ is both a group morphism and a probability space morphism (for these groups endowed with their respective normalized Haar measure).

Remark 3.2. A slight modification of the argument of the proof of Theorem 2.3.1 shows that the result holds as well provided that for all $q \in \mathbb{N}$ and for all $\sigma \in \mathcal{S}_q$,

$$E(\langle W \rangle_\sigma) = \prod_{c \text{ cycle of } \sigma} E(\langle W \rangle_c) + o(1)$$

as $d \to \infty$. In particular, this yields a new proof of asymptotic freedom of independent GUE's (albeit much more complicated).

2.3.2 Refinements and corollaries of Theorem 2.3.1

Our methods allow to derive an interesting consequence of Theorem 2.3.1:

Theorem 2.3.5. *Let W be a family of matrices admitting a limit law and $U_1, \cdots, U_k$ be unitary independent random variables. Let $w, u_1, \cdots, u_k$ be non-commutative random variables whose law is the limit joint law of $W, U_1, \cdots, U_k$. If $\varepsilon > 0$, then one has*

$$P(|\langle W_1 U_{i_1}^{\varepsilon_1} \cdots W_n U_{i_n}^{\varepsilon_n} \rangle - \langle w_1 u_{i_1}^{\varepsilon_1} \cdots w_n u_{i_n}^{\varepsilon_n} \rangle| \geq \varepsilon) = O(d^{-2}) \qquad (2.3.2.3)$$

In particular, the random variable $\langle W_1 U_{i_1}^{\varepsilon_1} \cdots W_n U_{i_n}^{\varepsilon_n} \rangle$ *converges in probability towards* $\langle w_1 u_{i_1}^{\varepsilon_1} \cdots w_n u_{i_n}^{\varepsilon_n} \rangle$

In order to prove this we need to estimate the expectation of the product of the traces of random variables in terms of the product of the expectations of the traces of the random variables. Corollary 2.3.7 of the following proposition does the job, but we shall also use proposition 2.3.6 in section 2.4. We define the relative cumulants $C_{\Pi_1,\Pi_2}(\langle \widetilde{W}_1 \ldots \widetilde{W}_n \rangle_\alpha)$ by modifying the definition of equation 2.2.2.11 in the obvious way.

Proposition 2.3.6. *Take the same notations as in Lemma 2.3.2. Then*

$$C_{\Pi_1,\Pi_2}(\langle \widetilde{W}_1 \ldots \widetilde{W}_n \rangle_\alpha) = 0(d^{2(C(\Pi_2)-C(\Pi_1))}) \tag{2.3.2.4}$$

Proof. We have, according to lemma 2.3.2,

$$C_{\Pi_1,\Pi_2}(\langle \widetilde{W}_1 \ldots \widetilde{W}_n \rangle_\alpha) = \sum_{\xi_1 \in \mathcal{T}_1, \ldots, \xi_k \in \mathcal{T}_k} \langle W_1 \ldots W_n \rangle_{\alpha \xi_1 \ldots \xi_k} C_{\Pi,\Pi_2}(\prod_{i=1}^{k} \widetilde{\mathrm{Wg}}(\xi_i))$$

with $\Pi = \Pi_1 \vee [\xi_1] \vee \ldots \vee [\xi_k]$. But it is straightforward to check that

$$C_{\Pi,\Pi_2}(\prod_{i=1}^{k} \widetilde{\mathrm{Wg}}(\xi_i)) = O(d^{-|\xi_1 \ldots \xi_k| + 2(C(\Pi_2)-C(\Pi))})$$

and for $\xi_1, \ldots, \xi_k$ such that $\Pi = \Pi_1 \vee \Pi_{\xi_1} \vee \ldots \vee \Pi_{\xi_k}$ one has

$$|\alpha| - |\alpha \xi_1 \ldots \xi_k| \leq 2(C(\Pi) - C(\Pi_1)) + |\xi_1| + \ldots + |\xi_k|$$

□

This proposition implies the following corollary:

Corollary 2.3.7. *Let* W_i *and* U *be as in the equation 2.3.2.1. The following holds*

$$E(\langle W_1 U^{\varepsilon_1} \cdots W_n U^{\varepsilon_n} \rangle_{\alpha_1} \langle W_{n+1} U_{i_{n+1}}^{\varepsilon_{n+1}} \cdots W_m U_{i_n}^{\varepsilon_m} \rangle_{\alpha_2}) =$$
$$E(\langle W_1 U_{i_1}^{\varepsilon_1} \cdots W_n U_{i_n}^{\varepsilon_n} \rangle_{\alpha_1}) E(\langle W_{n+1} U_{i_{n+1}}^{\varepsilon_{n+1}} \cdots W_m U_{i_m}^{\varepsilon_m} \rangle_{\alpha_2}) + O(d^{-2})$$

Now we can proceed to the proof of Theorem 2.3.5

Proof. We prove a slightly better result, namely

$$E(|\langle W_1U_{i_1}^{\varepsilon_1}\cdots W_nU_{i_n}^{\varepsilon_n}\rangle - \langle w_1u_{i_1}^{\varepsilon_1}\cdots w_nu_{i_n}^{\varepsilon_n}\rangle|^2) = O(d^{-2})$$

Our result will follow by Tshebyshev inequality. Developing the above square yields that it is enough to show that

$$E(|\langle W_1U_{i_1}^{\varepsilon_1}\cdots W_nU_{i_n}^{\varepsilon_n}\rangle|^2) - |E(\langle W_1U_{i_1}^{\varepsilon_1}\cdots W_nU_{i_n}^{\varepsilon_n}\rangle)|^2 = O(d^{-2})$$

This is an immediate consequence of the Corollary 2.3.7 on the factorization of the expectation. □

Remark 3.3.

- In corollary 2.3.5, for the convergence in probability to hold in equation 2.3.2.3, it is enough for the family $\{W_i\}$ to have joint $2n^{th}$ moments that are bounded uniformly in d.
- If we assume furthermore that the random variables $(U_i^d \in \mathbb{M}_d(\mathbb{C}))_{d\geq 1}$ are defined on the same large probability space Ω, then the convergence holds almost surely. This is just Borel-Cantelli's lemma and the fact that $\sum d^{-2}$ converges.

2.4 Large d behavior of the I-Z integral

We start with two theorems which were one of our initial motivations for performing explicit computations on functions of the kind $d \to C_{\Pi_1,\Pi_2}(d,\sigma)$.

Theorem 2.4.1. *Let W be a family of matrices admitting a limit joint distribution. Let $U_1,\ldots,U_k$ be independent Haar distributed unitary matrices. Let $(P_{i,j})_{1\leq i\leq k, 1\leq j\leq k}$ and $(Q_{i,j})_{1\leq i\leq k,1\leq j\leq k}$ be two families of non commutative polynomials in $U_1,U_1^*,\ldots,U_k,U_k^*$ and W. Let A_d be the variable $\sum_{i=1}^k\prod_{j=1}^k \mathrm{tr}P_{i,j}(U,U^*,W)$ and B_d the variable $\sum_{i=1}^k\prod_{j=1}^k \mathrm{tr}Q_{i,j}(U,U^*,W)$*

- *(i) For each d, the analytic function*

$$z \to d^{-2}\log E\exp(zd^2A_d) = \sum_{n\geq 1} a_n^d z^n$$

is such that for all q, $\lim_d a_q^d$ exists and is finite. It depends only on the limit distribution of W and on the polynomials $P_{i,j}$.

- *(ii) For each d, the analytic function*

$$z \to \frac{E\exp(zB_d + zd^2A_d)}{E\exp(zd^2A_d)} = 1 + \sum_{n\geq 1} b_n^d z^n$$

is such that for all q, $\lim_d b_q^d$ exists. It depends only on the limit distribution of W and on the polynomials $P_{i,j}$ and $Q_{i,j}$.

Proof. For the first point, remark that if q is greater than 1 then $q!a_q^d = C_q(d^2A)d^{-2}$. By Proposition 2.3.6 this is asymptotically bounded. For the second point we show that equivalently, defining

$$\log(1 + \sum_{q\geq 1} b_q^d z^q) = \sum_{q\geq 1} b_q'^d z^q$$

the coefficients $b_q'^d$ are such that for all q greater than 1, $\lim_d b_q'^d$ exists, is bounded. But

$$q!b_q'^d = C_q(B + d^2A) - C_q(d^2A)$$

Developing this by multilinearity and applying Proposition 2.3.6 shows that $q!b_q'^d = qC_q(B, d^2A, \ldots, d^2A) + o(1)$ and $\lim_d C_q(B, d^2A, \ldots, d^2A)$ itself exists, so this concludes our proof. □

Remark 4.1. Heuristically, point (ii) of Theorem 2.4.1 is a way to give a meaning to $E(\exp(B + d^2A))/E(\exp d^2A)$ in the large d limit. However, it is important to note that conclusion of (ii) does not hold any more if one considers, for example $z \to E(B\exp zd^2A)/E(\exp zd^2A)$.

In the remainder, we set $(X_d)_{d\geq 1}$, $(Y_d)_{d\geq 1}$ to be two families of matrices of $\mathbb{M}_d(\mathbb{C})$ and consider them as non-commutative variables of the non commutative probability space $(\mathbb{M}_d(\mathbb{C}), \mathrm{tr})$. Let U be a matricial random variable with values in the unitary group $\mathbb{U}_d$ and with law the left and right invariant Haar measure. Consider the commutative random variable $A_d = \mathrm{tr}(X_dUY_dU^*)$. We denote by A the family $A = (A_d)_{d\geq 1}$. The *Itzykson-Zuber integral* (or IZ integral) is the following analytic function:

$$IZ_{d,X,Y} : z \to E(\exp(d^2zA_d)) \tag{2.4.2.1}$$

Let (x_i) and (y_j) be the eigenvalues of X and Y and $\Delta(X) = \prod_{1\leq i<j\leq d}(x_j - x_i)$ (resp. $\Delta(Y) = \prod_{1\leq i<j\leq d}(y_j - y_i)$) be the Vandermonde determinant associated to the roots (x_i) (resp. (y_j)). The IZ integral first appeared in relation to matrix models and two-dimensional quantum gravity ([IZ80]). It can be computed explicitely by the Harisch-Chandra formula provided that $x_i = x_j$ iff $i = j$:

$$E(\exp(d^2 \mathrm{tr}(A))) = \frac{\det(e^{d\cdot x_i y_j})_{1\leq i,j\leq d}}{\Delta(X)\Delta(Y)} \tag{2.4.2.2}$$

In this section we investigate the function

$$F_{d,X,Y} : z \to d^{-2}\log E(\exp(d^2 z \mathrm{tr}(A_d))) \tag{2.4.2.3}$$

In [Mat94], Matytsin identifies the limit of $F_{d,X,Y}(1)$ with the solution of some PDE with boundary conditions, under the assumption that the distributions of X and Y admit a smooth limit. The existence of a large d limit remains a non-obvious mathematical problem. It has been solved very recently by A. Guionnet and O. Zeitouni in [GZ02] under the assumption that X and Y admits a limit bounded distribution and are Hermitian. However, here we investigate the limit of F by focusing on the study of the coefficients of its series in 0:

$$d^{-2}C_q(d^2 A_d) = \frac{\partial^q}{\partial z^q} F_{d,X,Y}(0)$$

In the remainder we define $\langle X\rangle_\sigma$ to be $\langle X^q\rangle_\sigma$, which was defined in Equation 2.3.2.1.

Theorem 2.4.2. *We have* $\lim_d d^{-2}C_q(d^2 A_d) =$

$$\sum_{\substack{\sigma,\tau\in\mathcal{S}_q \\ |\tau|+|\sigma|+|\tau\sigma^{-1}|=2q-2C(\Pi_\tau\vee\Pi_\sigma)}} \langle X\rangle_\sigma \langle Y\rangle_\tau \gamma_{\tau\sigma^{-1},\Pi_\tau\vee\Pi_\sigma,q+|\tau\sigma^{-1}|+2(C(\Pi_\tau\vee\Pi_\sigma)-1)} \tag{2.4.2.4}$$

where the number $\gamma_{\tau\sigma^{-1},\Pi_\tau\vee\Pi_\sigma,q+|\tau\sigma^{-1}|+2(C(\Pi_\tau\vee\Pi_\sigma)-1)}$ *was defined in 2.2.2.15 and computed in Theorem 2.2.12.*

This theorem is a consequence of the following lemma:

Lemma 2.4.3. • *(i) We have, for $d \geq q$,*

$$E(\mathrm{tr}((XUYU^*)^q)) = \sum_{\sigma,\tau\in\mathcal{S}_q} d^{q-|\tau|-|\sigma|}\langle X\rangle_\tau\langle Y\rangle_\sigma \mathrm{Wg}(d,\tau\sigma^{-1}) \quad (2.4.2.5)$$

- *(ii)*

$$d^{-2}C_q(d^2A) = \sum_{\sigma,\tau\in\mathcal{S}_q} d^{2q-2-|\tau|-|\sigma|}\langle X\rangle_\sigma\langle Y\rangle_\tau C_{\Pi_\tau\vee\Pi_\sigma,1_q}(\tau\sigma^{-1},d) \quad (2.4.2.6)$$

Proof. The second point is a straightforward consequence of the first one together with the definition of $C_{\Pi,1_q}(\tau\sigma^{-1},d)$.

For the first point, take the notations of 2.3.2: the set V is the set of odd numbers of $[1,2q]$ and the permutation α is $(12)(34)\cdots(2q-1\,2q)$. The rest follows by inspection, together with the fact that $|\tau|+|\sigma|+|\tau\sigma^{-1}|$ is always an even number. □

Now we can prove Theorem 2.4.2.

Proof. Observe by Theorem 2.2.7 that $C_{\Pi,1_q}(\sigma,d) = O(d^{-q-|\sigma|-2(C(\Pi)-1)})$. This implies that for any $\sigma,\tau\in\mathcal{S}_q$, one has

$$d^{2q-2-|\tau|-|\sigma|}C_{\Pi_\sigma\vee\Pi_\tau,1_q}(\tau\sigma^{-1},d) = O(d^{2q-|\tau|-|\sigma|-|\tau\sigma^{-1}|-2C(\Pi_\sigma\vee\Pi_\tau)})$$

which is known to be asymptotically bounded (see 2.2.7). Theorem 2.4.2 follows. □

2.4.1 A geometric interpretation of the formula for the limit

In this section we discuss briefly an interpretation of Theorem 2.4.2 in terms of sum over equivalence class of planar graphs. Let G_q be the set of (not-necesssarily connected) planar graphs (such that any connected component is drawn on a distinct sphere) with q edges together with the following data and conditions:

- (i) each face has an even number of edges.
- (ii) the edges are labeled from 1 to q.

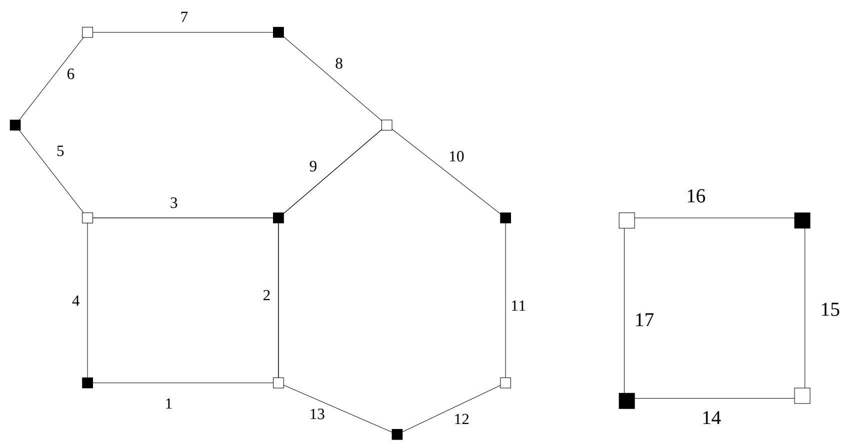
- (iii) there is a bicolouration in white and black of the vertices such that each black vertex has only white neighbors and vice versa.

To each such graph $g \in G_q$ we associate the permutations $\sigma(g)$ (resp. $\tau(g)$) of $\mathcal{S}_q$ defined by turning clockwise (resp. counterclockwise) around the white (resp. black) vertices and the function

$$\mathrm{Moeb}(g) = \gamma_{\tau\sigma^{-1},\Pi_\tau \vee \Pi_\sigma, q+|\tau\sigma^{-1}|+2(C(\Pi_\tau\vee\Pi_\sigma)-1)}$$

For example in the picture,

$$\sigma = (1\,13\,2)(3\,5\,4)(6\,7)(8\,9\,10)(11\,12)(16\,17)(14\,15)$$
$$\tau = (5\,6)(7\,8)(10\,11)(2\,3\,9)(12\,13)(1\,4)(14\,17)(15\,16)$$
$$\tau\sigma^{-1} = (1\,3)(5\,9\,7)(6\,8\,11\,13\,4)(2\,12\,10)(17\,15)(14\,16)$$



Two graphs are said to be equivalent if there is a positive oriented diffeomorphism of the plane transforming one to the other and respecting the coloring of the vertices and the labeling of the edges. We call $\sim$ this equivalence relation.

Theorem 2.4.4. *We have*

$$\lim_d d^{-2} C_q(d^2 A) = \sum_{g\in G_q/\sim} \langle X\rangle_{\tau(g)} \langle Y\rangle_{\sigma(g)} \mathrm{Moeb}(g) \qquad (2.4.2.7)$$

Proof. It is enough to check that our combinatorial objects of $G_q/\sim$ yield a one-to-one encoding of the situation $|\sigma|+|\tau|+|\tau\sigma^{-1}| = 2(q - C(\mathcal{O}_{\Pi_\tau} \vee \mathcal{O}_{\Pi_\sigma}))$. This is a consequence of the results of [BMS00]. □

2.4.2 A new scaling for $F_{d,X,Y}$

We start by recalling some definitions and facts contained in [Spe94]. Let NC_q be the set of non-crossing partitions of the ordered set $[1,q]$. A partition $V=\{V_1,\ldots,V_k\}$ of $[1,q]$ is said to be *non-crossing* iff it is not possible to find four elements $a_1<a_2<a_3<a_4$ of $[1,q]$ such that a_1 and a_3 are in one block V_i and a_2 and a_4 are in one other block V_j, $j\neq i$. This is a lattice for the refinement order, admitting 0_q as a minimal element (the discrete partition) and 1_q as a maximal element (the one block partition). More generally, if $\Pi_1\leq\Pi_2$, Π_1 and $\Pi_2\in NC_q$, then the interval $[\Pi_1,\Pi_2]$ is a lattice, and there exists a $\Pi\in NC_q$ such that $[\Pi_1,\Pi_2]$ is lattice-isomorphic to the interval $[O_q,\Pi]$.

Let $\sigma\in\mathcal{S}_q$ be a permutation such that $\Pi_\sigma=\Pi$. The Möbius function is known (cf [Spe94]) to satisfy

$$\mathrm{Moeb}(\Pi_1,\Pi_2)=\mathrm{Moeb}(\sigma)$$

where $\mathrm{Moeb}(\sigma)$ was defined in equation 4.5.4.2. For $A=(X_1,\cdots,X_q)$ a q -tuple of random variables in a non-commutative probability space $(\mathcal{A},\phi)$, consider a partition $V=\{V_1,\cdots,V_k\}$, with $V_i=\{m_{i1}<\cdots<m_{il_i}\}$. Associate to this partition the scalar

$$A_V=\phi(a_{m_{11}}\cdots a_{m_{1l_1}})\cdots\phi(a_{m_{k1}}\cdots a_{m_{kl_k}})$$

For any integer q and $\Pi\in NC(q)$, the *free cumulant* associated to Π is the

$$k_\Pi(A)=\sum_{\Pi'\leq\Pi\,,\,\Pi'\in NC_q}A_{\Pi'}\mathrm{Moeb}(\Pi',\Pi) \qquad (2.4.2.8)$$

For convenience we define $k_q(A)=k_{1_q}(A)$ and $k_q(X)=k_q(A)$ for $A=(X,\cdots,X)$. It has been known since Speicher (see [Spe94]) that if two random variables X and Y of a non-commutative space are free, then for all $q\geq 1$,

$$k_q(X+Y)=k_q(X)+k_q(Y)$$

Remark 4.2. The elements $X\to k_q(X)$ were already known to Voiculescu under the form

$$R_X:z\to\sum_{q\geq 0}z^qk_{q+1}(X)$$

as the R -transform, the analytic transformation linearizing the free additive convolution.

The following theorem is a stronger version of a result of P. Zinn-Justin contained in [ZJ99] (also see [ZJ98]).

Theorem 2.4.5. *Let X_d be a rank one projection and assume that Y_d has a limit distribution.*

$$\lim_{d} d^{-1} \cdot C_q(d^2 A_d) = (q-1)! k_q(Y) \tag{2.4.2.9}$$

In order to prove this theorem we need some notation and some preliminary results. For a partition $V = \{V_1, \cdots, V_k\}$ of $[1, q]$, enumerate the elements of V_i as an increasing sequence $v_{i1} \leq \cdots \leq v_{il_i}$. Then we define the right converse r of the surjection by its cycle product decomposition:

$$r(V) = \sigma = (v_{11} \cdots v_{1l_1}) \cdots (v_{k1} \cdots v_{kl_k})$$

This injection satisfies $\Pi_{r(V)} = V$.

Lemma 2.4.6. *Let Z be the permutation $(1 \cdots q)$ of $\mathcal{S}_q$. The set of τ 's satisfying*

$$|\tau| + |Z\tau^{-1}| = q - 1$$

is in one to one bijection with the set $NC(q)$. This bijection is $\tau \to \Pi_\tau$ and its converse is r.

Proof. See paragraph 2.7 of [Bia98], or [Bia97]. □

The *Kreweras complementation* $K(\pi)$ is defined, for π a non-crossing partition of NC_q, as the partition $\Pi_{Z^{-1}r(\Pi)}$, with $Z = (1 \cdots q)$. We first establish a formula for the free cumulant, slightly different from the usual ones.

Lemma 2.4.7. *One has*

$$k_q(Y) = \sum_{p \in NC(q)} \phi_p(Y) \cdot \mathrm{Moeb}(K(p)) \tag{2.4.2.10}$$

Proof. The interval $[p, 1_q]$ is isomorphic as a lattice to the interval $[0, K(p)]$ because the Kreweras complementation is an antiautomorphism of the lattice of non-crossing partitions. Consequently we get that $\mathrm{Moeb}(p, 1_q) = \mathrm{Moeb}(0_q, K(p)) = \mathrm{Moeb}(K(p))$ and thus:

$$k_q(Y) = \sum_{p \in NC_q} \phi_p(Y) \cdot \mathrm{Moeb}(p, 1_q) = \sum_{p \in NC_q} \phi_p(Y) \cdot \mathrm{Moeb}(K(p))$$

□

Now we can proceed to the proof of Theorem 2.4.5.

Proof. Recall from 2.4.2.6 that

$$d^{-1} C_q(d^2 A) = \sum_{\sigma, \tau \in \mathcal{S}_q} d^{1+2q-|\tau|-|\sigma|-|\tau\sigma^{-1}|-2} \phi_\sigma(X) \phi_\tau(Y) C_{\Pi_\sigma \vee \Pi_\tau, 1_q}(\tau\sigma^{-1}, d)$$

In particular here $\phi_\sigma(X) = d^{-q+|\sigma|}$, so every summand of Equation 2.4.2.6 is $O(d^x)$ with $x = 2q - |\tau| - |\sigma| - |\tau\sigma^{-1}| - 2C(\Pi_\sigma \vee \Pi_\tau) - q + |\sigma| + 1$. We already know that for all permutations σ and τ we have $2q - |\tau| - |\sigma| - |\tau\sigma^{-1}| - 2C(\Pi_\sigma \vee \Pi_\tau) \leq 0$; and it is also clear that $-q + |\sigma| + 1 \leq 0$. If the exponent x is zero then $|\sigma|$ has to be $q - 1$. In such a case $\Pi_\sigma \vee \Pi_\tau = 1_q$ and $x = q - 1 - |\tau| - |\tau\sigma^{-1}|$. The permutation σ has to be conjugated to the permutation $Z = (1 \ldots q)$ and its conjugacy class has $(q-1)!$ elements. Therefore

$$d^{-1} C_q(d^2 A) = (q-1)! \sum_{\tau \in \mathcal{S}_q, |\tau| + |\tau Z^{-1}| = q-1} \phi_\tau(Y) \mathrm{Wg}(\tau\sigma^{-1}, d) + o(1) \tag{2.4.2.11}$$

Lemmas 2.4.7 and 2.4.6 imply that Equation 2.4.2.11 is the same as Equation 2.4.2.9, which is the expected result. □

2.5 Conclusion and numerical values

2.5.1 Conclusion

The results of this paper raise a number of questions: Is there any chance to obtain a reasonably short explicit formula for the limit in Theorem 2.4.2, for example in terms of some generating function? In this direction, P. Zinn-Justin recently communicated me a paper ([ZJ02]) in which he computes the coefficients of $d^{-2}C_q(d^2\mathrm{tr}A)$ of the kind $P(y)x_{i_1}x_{i_2}$ and $P(y)x_{i_1}x_{i_2}x_{i_3}$ (see 2.5.3 for notation).

In another direction, does our result hold in the topology of uniform convergence on compact subsets of $\mathbb{C}$? This question was the one we hoped to answer in view of the results of [GZ02], but it still remains open to us.

Finally, we believe that our approach to integration of polynomial functions on unitary groups could lead to a better understanding of statistical properties of eigenvalues for the product of independent unitary random variables. For example, if U and V are independent unitary random variables of $\mathbb{M}_d(\mathbb{C})$, we obtain thanks to lemma 2.3.2:

$$\begin{array}{ll} E(\mathrm{tr}(UVU^*V^*)) = d^{-2} & E(\mathrm{tr}(U^2VU^{*2}V^*)) = 2d^{-2} \\ E(\mathrm{tr}(UVUVU^*V^*U^*V^*)) = 2d^{-2} & E(\mathrm{tr}(U^2V^2U^{*2}V^{*2})) = \frac{3d^2-4}{d^2(d^2-1)} \\ E(\mathrm{tr}(U^3VU^{*3}V^*)) = 3d^{-2} & E(\mathrm{tr}((UVU^*V^*)^2) = \frac{-4}{d^2(d^2-1)} \end{array}$$

An elementary use of symmetric properties of the algebra generated by U and V shows that the above list allows one to compute the expectation of the trace of all reduced words in U, V, U^*, V^* of length eight or less. The method can be generalized to arbitrary length words.

2.5.2 Values of Wg for small q.

We give a table of the first Wg functions. Note that the asymptotics of these rational fractions fit with Theorem 2.2.12, (iii):

$$\begin{array}{ll} \mathrm{Wg}((1),d) = \frac{1}{d} & \mathrm{Wg}((2),d) = \frac{-1}{d(d^2-1)} \\ \mathrm{Wg}((1,1),d) = \frac{1}{(d^2-1)} & \mathrm{Wg}((3),d) = \frac{2}{(d^2-1)(d^2-4)d} \\ \mathrm{Wg}((2,1),d) = \frac{-1}{(d^2-1)(d^2-4)} & \mathrm{Wg}((1,1,1),d) = \frac{d^2-2}{(d^2-1)(d^2-4)d} \end{array}$$

We also observe that

$$\mathrm{Wg}((q),d) = (-1)^{q+1}c_{q-1} \prod_{-q+1\leq j\leq q-1} (d-j)^{-1} \qquad (2.5.2.1)$$

Indeed, a classical combinatorial result about the Schur polynomials shows that if one writes $\mathrm{Wg}((q),d)$ as an irreducible rational fraction, its denominator has to be $\prod_{-q+1\leq j\leq q-1}(d-j)$. An argument of degree of $\mathrm{Wg}((q),d)$ together with the knowledge of the asymptotics concludes the proof.

2.5.3 Asymptotic values of cumulants for small q.

We give an asymptotic value of the first cumulants. The four first values are due to Biane and we performed others with a computer. Since they are semi-invariant, it is enough to assume that $\mathrm{tr}X=0$ and $\mathrm{tr}Y=0$. This avoids (even more) cumbersome results. We use the following notation: $A=XUYU^*$. $\lim_d \mathrm{tr}X^i = x_i$ and $\lim_d \mathrm{tr}Y^i = y_i$.

$$\begin{aligned}
d^{-2}C_1(d^2\mathrm{tr}A)/0! &= 0\\
d^{-2}C_2(d^2\mathrm{tr}A)/1! &= x_2y_2\\
d^{-2}C_3(d^2\mathrm{tr}A)/2! &= x_3y_3\\
d^{-2}C_4(d^2\mathrm{tr}A)/3! &= x_4y_4+3\,{x_2}^2{y_2}^2-2\,y_4{x_2}^2-2\,x_4{y_2}^2\\
d^{-2}C_5(d^2\mathrm{tr}A)/4! &= x_5y_5+20\,x_2x_3y_2y_3-5\,x_2x_3y_5-5\,x_5y_2y_3\\
d^{-2}C_6(d^2\mathrm{tr}A)/5! &= -6\,y_6x_2x_4+30\,x_2x_4y_2y_4+27\,{x_2}^3{y_2}^3-30\,{x_2}^3y_2y_4\\
&\quad -30\,{y_2}^3x_2x_4-16\,{y_2}^3{x_3}^2+15\,x_2x_4{y_3}^2+15\,{x_3}^2y_2y_4\\
&\quad -6\,x_6y_2y_4+x_6y_6-16\,{x_2}^3{y_3}^2+7\,x_6{y_2}^3+7\,{x_2}^3y_6\\
&\quad +6\,{x_3}^2{y_3}^2-3\,y_6{x_3}^2-3\,x_6{y_3}^2\\
d^{-2}C_7(d^2\mathrm{tr}A)/6! &= -7\,y_7x_2x_5+42\,x_2x_5y_3y_4+42\,x_3x_4y_2y_5-7\,x_7y_2y_5\\
&\quad +462\,x_3y_3{x_2}^2{y_2}^2-7\,x_7y_3y_4+x_7y_7+28\,x_7{y_2}^2y_3\\
&\quad -7\,x_3x_4y_7+35\,x_3x_4y_3y_4-140\,x_3x_4{y_2}^2y_3+28\,{x_2}^2x_3y_7\\
&\quad -140\,{x_2}^2x_3y_3y_4+42\,x_2x_5y_2y_5-147\,x_2x_5{y_2}^2y_3-147\,x_2^2x_3y_2y_5
\end{aligned}$$

These computations have been performed with Maple and its ACE package on a Free BSD Dell personal computer at the École Normale Supérieure.

2.6 Addendum

2.6.1 A comparison to previous physical results

In this short addendum, we show why our Theorem 2.2.6 is equivalent to a older result of Zuber and O'Brien [OZ84], see also [ZJZ02b]. I thank P. Zinn-Justin and J-B Zuber for communicating the above references.

In [OZ84], it is proved that

$$\lim_d d^{-2} \frac{C_{2q}(d\mathrm{Tr}(AU + A^*U^*))}{(2q)!} = \sum_{\lambda \vdash q} \prod \frac{\mathrm{tr}(AA^*)^{\lambda_i}}{\lambda_i!} W_\lambda \qquad (2.6.2.1)$$

where $(\lambda_1, \ldots, \lambda_q)$ is a partition of q, and

$$W_\lambda = (-1)^q \frac{(2q + \sum_i \lambda_i - 3)!}{(2q)!} \prod_{i=1}^{q} \left(\frac{-(2i)!}{i!(i-1)!} \right)^{\lambda_i}$$

By multilinearity and elementary symmetry properties of the Haar measure,

$$C_{2q}(d\mathrm{Tr}(AU + A^*U^*)) = \binom{2q}{q} C_{2q}(d\mathrm{Tr}(AU), \ldots, d\mathrm{Tr}(A^*U^*), \ldots) \quad (2.6.2.2)$$

where $\mathrm{Tr}(AU)$ and $\mathrm{Tr}(A^*U^*)$ appear q times. It is then a consequence of Proposition 2.3.2, that

$$\lim_d d^{-2} C_{2q}(d\mathrm{Tr}(AU), \ldots, d\mathrm{Tr}(A^*U^*), \ldots) = q! \sum_{\sigma \in \mathcal{S}_q} \langle AA^* \rangle_\sigma \gamma_{\sigma, \Pi_\sigma, 3q-2-|\sigma|} \qquad (2.6.2.3)$$

Putting together equations 2.6.2.1, 2.6.2.2 and 2.6.2.3 together with the fact that there are $\prod(\lambda_i - 1)!$ permutations of type λ, one recovers the formulae (i) and (ii) of Theorem 2.2.12.

Therefore we have proved that our result is equivalent to the result of [OZ84]. It is interesting to remark that the approach of [OZ84] is very different from ours, since we enumerate graphs and they have an inductive proof with a recursion formula.

2.6.2 Higher order graphs

In this part we suggest a way to modify Theorem 2.4.4 to obtain a matrix model converging towards a sum over fixed genus nonconnected planar diagrams.

Define inductively $f_0(d,q) = d^{-2}C_q(d^2A)$ where $F_{d,X,Y}$ was defined at equation 2.4.2.3, and $f_{n+1}(d,q) = d^2(f_n(2d,q) - f_n(d,q))$. One can rename the set G_q defined in section 2.4.1 by $G_{q,0}$ and define by the obvious modification the sets $G_{q,n}$ of graphs of genus n. One can also associate permutations $\tau(g)$, $\sigma(g)$ and

$$\mathrm{Moeb}(g) = \gamma_{\tau\sigma^{-1},\Pi_\tau\vee\Pi_\sigma,q+|\tau\sigma^{-1}|+2(C(\Pi_\tau\vee\Pi_\sigma)-1-n)}$$

One can then prove:

Theorem 2.6.1. *There exists a function $i : \mathbb{N} \to \mathbb{R}$ such that*

$$\lim_d f_n(d,q) = \sum_{g\in G_{q,n}/\sim} \langle X\rangle_{\tau(g)}\langle Y\rangle_{\sigma(g)}\mathrm{Moeb}(g)i(n) \tag{2.6.2.4}$$

Obtaining a closed expression for $\mathrm{Moeb}(g)$ in full generality seems to be a hard task and we are not able to complete it yet.

For example, for $n = 1$, this theorem admits the following equivalent statement: let $A, B \in \mathbb{M}_d(\mathbb{C})$ admitting a limit distribution. Then

$$z \to \frac{E_{\mathbb{U}_{2d}}(e^{z2d\mathrm{Tr}(diag(A,A)Udiag(B,B)U^*)})}{E_{\mathbb{U}_d}(e^{zd\mathrm{Tr}(AUBU^*)})^4}$$

admits a power series development around zero whose coefficients converge pointwise as $d \to \infty$.

Chapitre 3

Frontière de Martin quantique

HTTP://WWW.ARXIV.ORG/ABS/MATH.PR/0211356

In this paper we define a general setting for Martin boundary theory associated to quantum random walks, and prove a general representation theorem. We show that under natural assumptions, the extremal Martin boundary is homeomorphic to a sphere. Then, we investigate restriction of quantum random walks to Abelian subalgebras of group algebras, and establish a Ney-Spitzer theorem for an elementary random walk on the fusion algebra of $SU(n)$, generalizing a previous result of Biane. We also consider the restriction of a quantum random walk on $SU_q(n)$ introduced by Izumi to specific Abelian subalgebras, and generalize a result of [Bia91a].

Dans cet article, nous définissons un cadre général pour la théorie de Martin associée à une large classe de marches au hasard sur le dual de groupes compacts, et établissons un théorème de représentation intégral. Ensuite, nous montrons sous des hypothèses naturelles, que la frontière de Martin extremale est homéomorphe à une sphère. Nous nous concentrons alors sur la restriction de marches au hasard quantiques à certaines sous-algèbres Abéliennes d'algèbres de groupes, et établissons un théor'eme de Ney-Spitzer pour une marche au hasard "de Bernoulli" sur l'algèbre de fusion de $SU(n)$. Nous considérons aussi la restriction d'une marche au hasard quantique introduite par Izumi à deux sous-algèbres abéliennes distinctes, et relions les chaînes de Markov sous-jacentes par des procédés probabilistes classiques. Ces deux derniers résultats généralisent des résultats de [Bia91a].

3.1 Introduction: classical Martin boundary

The classical Martin boundary theory gives a geometric and probabilistic solution to the problem of describing positive harmonic functions with respect to a transient (sub)Markov operator.

Let $\mathcal{E}$ be a discrete countable state space and P be a Markov (resp. submarkov) operator defined by $Pf(x) = \sum_{y \in \mathcal{E}} P(x, y) f(y)$, where $P(x, y)$ is an array of positive real numbers assumed to satisfy $P1 = 1$ (resp. $P1 \leq 1$). In order to avoid technical difficulties, we assume that for each x, every $P(x, y)$ is zero except finitely many of them. Throughout the whole paper, we will identify, whenever relevant, the operator P and the kernel $P(x, y)$. One defines inductively $P^0(x, y) = \delta_{x,y}$, $P^{n+1}(x, y) = \sum_{z \in \mathcal{E}} P^n(x, z) P(z, y)$ and the Green kernel $U(x, y) = \sum_{n \in \mathbb{N}} P^n(x, y)$. We make the usual assumptions of irreduciblity and transience:

Assumption 1.1. For all $x, y \in \mathcal{E}$, one has $0 < U(x, y) < \infty$.

A function f is said to be *harmonic* with respect to P if $Pf = f$. The *Martin kernel* with a base point x_0 is defined as

$$k(x, y) = U(x, y)/U(x_0, y)$$

Harnack inequalities imply that for all $x \in \mathcal{E}$, the function $k(x, .)$ is bounded. The *Martin compactification* MS of $\mathcal{E}$ is defined as the smallest compact subspace in which $\mathcal{E}$ can be continuously and densely embedded and such that every function $k(x, .)$ can be uniquely extended by continuity to MS. Let MB be the boundary of $\mathcal{E}$ in MS. A positive harmonic function f is said to be *minimal* if any harmonic function g satisfying $0 \leq g \leq f$ is a multiple of f. There exists a measurable subset MB^{min} of MB such that $x \in MB^{min}$ if and only if $k(., x)$ is a positive minimal harmonic function.

Theorem 3.1.1. *For every positive harmonic function f satisfying $f(e) = 1$, there exists a unique probability measure μ_f on MB such that $\mu_f(MB^{min}) = 1$ and for all $x \in \mathcal{E}$, $f(x) = \int_{\xi \in MB^{min}} k(x, \xi) d\mu_f(\xi)$.*

For the proof of the above theorem, see [Rev84] or [KSK76]. In view of this, it is natural to try to compute explicit examples. This turns out to be a difficult task, and one remarkable example of such a computation is done in [NS66].

Let $\mathcal{E}$ be the state space $\mathbb{Z}^d$, for some $d \geq 2$, and μ be a finitely supported measure (the hypothesis of finite support can be considerably weakened, but we do not enter into such technical considerations) whose mean on $\mathbb{R}^d$ is different from zero. We identify canonically $\mathbb{Z}^d$ with a lattice of the Euclidean space $\mathbb{R}^d$ with its scalar product $\langle .,. \rangle$. Then, the set

$$E = \{x, \int_{\mathbb{Z}^d} \exp\langle x, X\rangle d\mu(X) = 1\}$$

is a C^∞ submanifold of $\mathbb{R}^d$. It is diffeomorphic to the sphere S^{d-1}. It is a consequence on a theorem of Choquet and Deny (see [CD60]) that any positive function on $\mathbb{Z}^d$ harmonic with respect to the operator of convolution by μ (we call its operator P_μ) admits an unique integral representation as a linear combination of functions $y \to e^{\langle x,y\rangle}$, $x \in E$.

Let β be a continuous increasing bijection from $[0,\infty]$ to $[0,1]$ Let Π be the map from $\mathbb{Z}^d$ to the canonical unit ball $B(0,1)$ of $\mathbb{R}^d$ given by $\Pi(x) = \beta(||x||)x/||x||$. This map is a topological injection, and the metrics it inherits gives rise to a compactification of $\mathbb{R}^d$ by S^{d-1}. As such, the map Π extends continuously to $\mathbb{Z}^d \cup S^{d-1}$ with value in $B(0,1)$. The map u sending an element of E to its normed outer normal vector is a C^∞ diffeomorphism from E to S^{d-1}.

Theorem 3.1.2 (NS, [NS66]). *The Martin compactification of $\mathbb{Z}^d$ is $\mathbb{Z}^d \cup S^{d-1}$. The correspondence between S^{d-1} and E arising from the Martin kernel is given by the map u. The extremal Martin boundary MB^{min} thus coincides with MB.*

Throughout the paper, we define

$$\mathcal{H}^+_{\mathcal{E},P} = \{f \in \mathbb{C}^{\mathcal{E}}, f \geq 0, Pf = f, f(e) = 1\} \quad (3.1.3.1)$$

$$\mathcal{H}_{\mathcal{E},P} = \{f \in L^\infty(\mathcal{E}), Pf = f\} \quad (3.1.3.2)$$

The set $\mathcal{H}^+_{\mathcal{E},P}$ is convex compact for the topology of pointwise convergence. Therefore it admits extremal points. By definition, let $\mathcal{H}^{+,ex}_{\mathcal{E},P}$ be this subset. By the Krein-Milman theorem, the closure of the convex hull of these extremal points is exactly $\mathcal{H}^+_{\mathcal{E},P}$.

Thus, Theorem 3.1.1 identifies the extremal points of the above set with MB^{min}. We shall say in this paper that we obtain a *Ney-Spitzer like* theorem when we give a compactification $MS = \mathcal{E} \cup MB$ of a state space, describe the extremal positive harmonic functions, describe a subset MB^{min} of MB and a bijection between $\mathcal{H}_{\mathcal{E},P}^{+,ex}$ and MB^{min}.

This paper is organized as follows. In part 3.2, we define the framework of Hopf algebras in which we will study noncommutative probability theory. Part 3.3 is devoted to defining a quantum Martin boundary theory and showing that any positive harmonic element can be represented with respect to an adapted Martin kernel. By passing, we show that the minimal Martin boundary is isomorphic to a sphere under weak assumptions. In Part 3.4, we consider the restriction of a "Bernoulli" quantum walk to the center of the Hopf algebra and establish a Ney-Spitzer like theorem. Part 3.5 is an application of Part 3.4 to a quantum random walk on the dual of $SU_q(n)$ introduced by Izumi in [Izu02]: it provides a generalization of a result of Biane.

3.2 Quantum spaces, random walks and harmonic elements

3.2.1 Hopf algebras

We fix some classical notations of operator algebra theory, and remind some elementary definitions and results of Hopf algebra theory.

Let G be a topological compact group, and $d\mu$ its left and right invariant probability Haar measure. Let $L^2(G)$ be the L^2 space associated to this Haar measure. For $g \in G$, the unitary operator $\lambda_g \in B(L^2(G))$ is defined by $\lambda_g : f \to (x \to f(g^{-1}x))$. The mapping $g \to \lambda_g$ is continuous for the strong operator topology in $B(L^2(G))$. The vector space $Vect(\lambda_g, g \in G)$ is a $*$ -subalgebra of $B(L^2(G))$. Let $M(G)$ be the von Neumann algebra of G, i.e. the bicommutant of $Vect(\lambda_g, g \in G)$ in $B(L^2(G))$.

The set of equivalence classes of irreducible finite dimensional unitary representations of G is denoted by Γ. For $x \in \Gamma$, d_x is the corresponding dimension. By Peter-Weyl's theorem, we have the following isomorphism of von Neumann algebras:

$$M(G) \cong \oplus_{x\in\Gamma}\mathbb{M}_{d_x}(\mathbb{C})$$

The counit is the map $M(G) \to \mathbb{C}$, defined as the continous linear expansion of the map $\varepsilon(\lambda_g) = 1$.

In the same way, the antipode is the continous map $M(G) \to M(G)$ such that $S(\lambda_g) = \lambda_{g^{-1}}$, and the coproduct is the map $M(G) \to M(G \times G) \cong M(G) \otimes M(G)$ such that $\widehat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$.

The quadruple $(M(G), \varepsilon, \widehat{\Delta}, S)$ is called the Hopf-von Neumann algebra of the group G.

For a von Neumann algebra A, we shall call $\widehat{A}$ the set of elements affiliated to A. By the Peter-Weyl theorem, $\widehat{M(G)}$ is a $*$ -algebra endowed with a natural pointwise convergence topology. As a topological $*$ -algebra, it is isomorphic to $\prod_{x\in\Gamma} \mathbb{M}_{d_x}(\mathbb{C})$.

One can see (see [Bia92a]), that $\widehat{\Delta}$, ε and S are also continuous for the topology of pointwise convergence, so that there is a unique way of extending them from $\widehat{M(G)}$ to $\widehat{M(G) \otimes M(G)}$ (resp., $\widehat{M(G)}$, $\mathbb{C}$). According to Effros and Ruan (see [ER94]; also see [Bia92a]) we call this structure a *topological* $*$ *-Hopf algebra* .

Let $\overset{o}{M}(G)$ be the set of finite rank operators in $M(G)$. A linear form $\nu : \widehat{M(G)} \to \mathbb{C}$ is said to be *finitely supported* iff it is continuous with respect to the pointwise convergence. Equivalently, there exists a faithful weight τ on $M(G)$ and an element $A \in \overset{o}{M}(G)$, such that for all $B \in \widehat{M(G)}$, one has $\mu(B) = \tau(AB)$. We call $(\widehat{M(G)})_*$ the vector space of finitely supported linear forms.

To summarize, we will be dealing with the following inclusions of algebras :

$$\overset{o}{M}(G) \subset M(G) \subset \widehat{M(G)}$$

The first one is not a Hopf algebra, but the latter two are.

3.2.2 Random walks and harmonic analysis

We use the framework of Hopf algebras in order to define quantum random walks. Several groups of mathematicians have already inspected axiomatics (see [AFL82]) and their properties (see for example [Izu00], [SS98], [BD95]).

For $l \in \Gamma$, let 1_l be the minimal central idempotent in $M(G)$ associated to the irreducible representation $l \in \Gamma$ in $M(G)$. For ν and μ two states on

$M(G)$, we define their convolution $\mu*\nu$ by the equation $\mu*\nu(f) = (\mu\otimes\nu)\widehat{\Delta}(f)$ for each $f \in M(G)$. $\mu * \nu$ is a state and we can define inductively ν^{*n} to be ε if $n = 0$ and $\nu * \nu^{*n-1}$ else. We define the operator P_μ on $M(G)$ by

$$P_\mu(f) = (id \otimes \mu)\widehat{\Delta}(f) \tag{3.2.3.1}$$

and its iterates inductively by $P_\mu^n = id$ if $n = 0$ and $P_\mu^n = P_\mu \circ P_\mu^{n-1}$ else. One has also $P_\mu^n = (id \otimes \mu^{*n})\widehat{\Delta}$.

The operator P_μ is the evolution operator associated to a quantum random walk on the dual of G in the sense of [AFL82]. It is a completely positive operator on $M(G)$. If $\mu(1_l) = 0$ for any but finitely many l's, P_μ extends to a positive continuous operator on $\widehat{M(G)}$.

An element f in $\widehat{M(G)}$ is said to be *harmonic* iff

$$Pf = f \tag{3.2.3.2}$$

Biane showed in [Bia91b], that $\mathcal{H}_{P_\mu}^{+,ex}$ is the set

$$E = \{f \in \widehat{M(G)}, f \geq 0, \widehat{\Delta} f = f \otimes f, \mu(f) = 1\} \tag{3.2.3.3}$$

We shall say that an element $f \in \widehat{M(G)}$ such that $\widehat{\Delta} f = f \otimes f$ is an *exponential*, and call $Ex(G)$ be the set of exponentials.

3.3 Quantum Martin boundary theory

3.3.1 Representation of positive harmonic elements

In this section, we define a Martin compactification and a Martin kernel, and show that every positive harmonic element can be represented by a state on the Martin boundary. For a completely positive continuous operator Q from $M(G)$ into itself and ν a weight, νQ is again a weight defined by $\nu Q f = \nu(Q(f))$. We write $\nu \leq \mu$ iff for any positive f, $\nu(f) \leq \mu(f)$. We need the following assumption:

Assumption 3.1. The weight μ is such that $\mu(1) = q \in]0,1[$ and that $\varepsilon U = \sum_{n\geq 0} \mu^{*n}$ is faithful. There exists a positive $A \in Ex(G)$ and a tracial operator $\widetilde{\mu}$ such that for any $f \in \widehat{M(G)}$, $\mu(f) = \widetilde{\mu}(Af)$.

Let $U = \sum_{n\geq 0} P^n$. This operator has operator norm less than $(1-q)^{-1}$ and is the quantum analogue of the *Green kernel*. Let $\overset{o}{M}(G)$ be the subalgebra of $M(G)$, consisting of finite rank operators. Note that this is not a Hopf subalgebra.

We define the *Martin Kernel* to be the linear map

$$K : (\widehat{M(G)})_* \to \widehat{M(G)} \tag{3.3.3.1}$$

such that for any $\nu \in (\widehat{M(G)})_*$, K_ν satisfies for all $f \in \overset{o}{M}(G)$,

$$\nu U(f) = \varepsilon U(A^{-1/2} K_\nu A^{1/2} f) \tag{3.3.3.2}$$

K_ν is well defined because the weight εU is faithful. A definition equivalent to this one has already appeared in P. Biane's papers (see [Bia94]) in the context of $SU(2)$ for a tracial weight.

Lemma 3.3.1. *Let $g \in M(G)$. Then one has $g = U(g - P(g))$*

Proof. It is enough to remark that Uf is defined for any $f \in M(G)$, that under the Assumption 3.3.1 it is the norm limit of $\sum_{k=0}^n P^k f$, and that $\sum_{k=0}^n P^k(f - Pf) = f - P^{n+1} f$. But $P^{n+1} f$ has norm tending towards zero as n tends to infinity. □

Proposition 3.3.2.

- *K is positive and its image is contained in $M(G)$.*
- *The norm closure of* span$\{K_\nu, \nu \in (\widehat{M(G)})_*\}$ *contains the C^* -algebra $\mathcal{K}$ of compact operators.*

Proof. The weight εU is faithful. It is known that $\phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ is a weight iff there exists a positive matrix B such that for all C, $\phi(C) = \mathrm{Tr}(CB)$. Therefore, if ν is a weight then $A^{1/2} K_\nu A^{1/2}$ is positive. Since A is invertible, K_ν is also positive.

For ν a finitely supported weight, there exists by Assumption 3.3.1 an integer n and a constant α such that $\nu \leq \alpha \sum_{i=0}^n \varepsilon P^i$. By positivity of P this implies that $\nu U \leq (n+1)\alpha\varepsilon U$. This implies that K_ν is bounded.

If $K_\nu = 0$, then by faithfulness of εU, one has $\nu U = 0$, and by Lemma 3.3.1, $\nu = 0$, which proves the "into".

For the second statement, it is enough to show that for every positive finite dimensional operator $f \in \widehat{M(G)}$, there exists ν such that $K_\nu = f$.

Let ν be the linear form such that for any $g \in M(G)$, $\nu(g) = \varepsilon U(f(g - Pg))$. Then ν is finitely supported, so that K_ν is well defined; and one can check using Lemma 3.3.1 that it satisfies $K_\nu = f$. $\square$

We define the *Martin space* MS to be the C^*-algebra

$$MS = C^*(K_\nu, \nu \in (\widehat{M(G)})_*)$$

By Proposition 3.3.2, $\mathcal{K}$ is an ideal of MS. Let the *Martin boundary* be the C^* -algebra $MB = MS/\mathcal{K}$. The Martin compactification of the dual $\widehat{G}$ of G is then defined to be the following exact sequence:

$$0 \to \mathcal{K} \to MS \to MB \to 0 \tag{3.3.3.3}$$

We are now able to prove the following representation theorem:

Theorem 3.3.3.
- *For each positive harmonic element h in $\widehat{M(G)}$ there exists a state ϕ_h on MB such that for every finitely supported linear form ν on $M(G)$, one has $\nu(h) = \phi_h(K_\nu)$*
- *This representation is unique if* span$\{K_\nu, \nu \in (\widehat{M(G)})_*\}$ *is dense in* MS.

Proof. If ϕ is a state on MB then the element $h_\phi \in \widehat{M(G)}$ defined by $\nu(h_\phi) = \phi(K_\nu)$ for all $\nu \in (\widehat{M(G)})_*$, is positive. Let us first show that it is harmonic.

We need to show that $Ph_\phi = h_\phi$, or equivalently, that for all $\nu \in (\widehat{M(G)})_*$, $\nu(Ph_\phi) = \nu(h_\phi)$. But $\nu Ph_\phi = (\mu * \nu)h_\phi = \phi(K_{\mu*\nu})$ and $\nu(h_\phi) = \phi(K_\nu)$, therefore it is enough to show that $\phi(K_{\mu*\nu}) = \phi(K_\nu)$.

We have, for all $f \in \widehat{M(G)}$

$$\begin{aligned} &\varepsilon U((A^{-1/2}K_{\mu*\nu} - K_\nu)A^{1/2}f) = \\ &\qquad (\mu * \nu - \nu)Uf = \nu f \end{aligned}$$

(second equality arises from Lemma 3.3.1). But since $\nu \in (\widehat{M(G)})_*$, this implies that $K_{\mu*\nu} - K_\nu$ has finite rank, thus is compact. Therefore $\phi(K_{\mu*\nu} - K_\nu) = 0$

Furthermore, h_ϕ satisfies $\varepsilon h_\phi = 1$ and one sees that the linear map

$$\Xi : (MS)^* \to \widehat{M(G)}$$

that maps ϕ to h_ϕ is continuous for the pointwise convergence topology.

We will now show that for every extremal harmonic element h, there exists a weight ϕ on MS such that for any $\nu \in (\widehat{M(G)})_*$, one has

$$\nu(h) = \phi(K_\nu)$$

Since Ξ is linear, any convex combination of extremal harmonic elements can be represented. Furthermore, Ξ is continuous, therefore, any harmonic element that can be approximated in the pointwise convergence topology by a convex combination of extremal harmonic elements can be represented, therefore, any element can be represented.

Let h be such a minimal harmonic element. By Biane's theorem ([Bia92a]), it satisfies $\widehat{\Delta} h = h \otimes h$ and $\mu(h) = 1$. Let $\widehat{M(h)}$ be the closure in $\widehat{M(G)}$ of the algebra generated by h. This is obviously a topological Hopf $*$-subalgebra of $\widehat{M(G)}$. This algebra is commutative, cocommutative and closed, therefore one can show directly that the operations inf and sup are well defined inside this algebra.

In $\widehat{M(h)}$, consider a sequence h_k of positive elements such that Uh_k tends and increases towards h as k goes towards infinity. The existence of such a sequence is a consequence of standard probabilistic considerations, but we justify it nonetheless.

Let $f_k = inf(kId, h)$, where the infimum is taken on the commutative affiliated algebra $\widehat{M(h)}$. f_k is bounded and satisfies $P(f_k) \leq f_k$. Let $h_k = f_k - Pf_k$. This element is positive and it can not be zero because h is extremal and non bounded, thus f_k would have to be a multiple of h, which would result in $h = 0$. By Lemma 3.3.1 this implies that $f_k = Uh_k$. Last, it is obvious that f_k tends towards h in the pointwise convergence topology, as $k \to \infty$.

Consider $\phi_k = \varepsilon U(A^{-1/2} h_k A^{1/2} \cdot)$. It is a state on the norm closed operator system generated by K_μ, and it satisfies $\phi_k(1) = \varepsilon U h_k \leq 1$. By a classical result (see [Dix64], p. 50, lemme 2.10.1), it extends to a state on MS.

Furthermore, $\phi_k(K_\nu) = \nu(Uh_k)$ tends towards $\nu(h)$ as k tends towards infinity. This proves that ϕ_k converges weakly towards a state ϕ on MS. This state vanishes on compact operators, so is actually a state of MB. $\square$

3.3.2 Topological structure of the boundary

In this section, we assume that G is a compact simply connected Lie subgroup of $\mathbb{U}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexified Lie algebra, $G_{\mathbb{C}}$ be the complexified Lie group, and (ρ, V) be the fundamental representation of G.

The left regular representation yields an identification of $\mathfrak{g}_{\mathbb{C}}$ with a Lie subalgebra of $\widehat{M(G)}$. If $f \in \mathfrak{g}_{\mathbb{C}}$, then $\widehat{\Delta} f = f \otimes 1 + 1 \otimes f$. The map $EXP : \widehat{M(G)} \to \widehat{M(G)}$ defined by the usual series is such that for any $f \in \mathfrak{g}_{\mathbb{C}}$, $\widehat{\Delta} EXP f = EXP f \otimes EXP f$.

By a slight modification of a result of Biane ([Bia91b], Proposition 11 and Lemme 12), if any irreducible representation of G is contained in some tensor power of V, then the set of non-zero exponentials in $\widehat{M(G)}$ is the group generated by $EXP\mathfrak{g}_{\mathbb{C}}$. It is exactly $G_{\mathbb{C}}$, and the restriction of $Ex(G)$ to $End(V)$ is a group isomorphism between $Ex(G)$ and $G_{\mathbb{C}}$. An explicit isomorphism is obtained by restricting $Ex(G)$ to the fundamental representation. We call

$$i : G_{\mathbb{C}} \to Ex(G) \tag{3.3.3.4}$$

the converse of this isomorphism.

The following theorem answers a question raised by Biane about the topology of the boundary.

Theorem 3.3.4. *Let μ be a weight on $\widehat{M(G)}$ satisfying $\mu(1) = q \in]0, 1[$ and assumption 3.3.1.*

Then, the set $H_{P_\nu}^{+,ex}$ of extremal harmonic elements is diffeomorphic to the sphere S^{k-1}, where k is the dimension of the Lie algebra $\mathfrak{g}$.

Proof. Let $\mathfrak{g}_{sa}$ be the real vector subspace of $\widehat{M(G)}$ of self adjoint elements of $\mathfrak{g}_{\mathbb{C}}$ in $\widehat{M(G)}$. Let $x \in \mathfrak{g}_{sa}$ be non-zero, and f_x the map $\mathbb{R} \to \mathbb{R}$ given by $f_x(t) = \nu(EXPtx)$. This map is always positive. Since $\mathrm{tr} x = 0$ and x is Hermitian, it has one negative eigenvalue and one positive eigenvalue. Therefore $\lim_{\pm\infty} f_x = \infty$. Besides by definition, $f_x(0) = q < 1$. The function f_x admits the second derivate $\nu(x^2 EXPtx)$ at t, therefore it is positive. Thus, the function f_x is convex; therefore there exists only two real numbers t_x^+ (resp. t_x^-) satisfying $t_x^+ > 0$ and $f_x(t_x^+) = 1$ (resp. $t_x^- < 0$ and $f_x(t_x^- = 1)$). But the map EXP is a diffeomorphism from $\mathfrak{g}_{sa}$ onto $E = \{x \in \widehat{M(G)}, \widehat{\Delta} x = x \otimes x\}$. Therefore the inverse image of $\mathcal{H}_P^{+,ex}$ under exp is a closed star-like subset around 0, therefore it is homeomorphic to the sphere S^{k-1}. □

3.4 A Ney-Spitzer theorem for a RW on a fusion algebra

It would be interesting and seems challenging to obtain nice generalizations of the result of [Bia94] in the framework developed above. We are not able to perform fully such computations. Yet, it is possible to obtain a Ney-Spitzer like theorem if one restrict a tracial quantum random walk on $SU(n)$ to the center of its Hopf algebra. In this section, we establish a Ney-Spitzer like theorem for the most elementary quantum random walk, improving previous results of [Bia91a].

3.4.1 Main result

In the Euclidean space $\mathbb{R}^n$, $n \geq 3$ with canonical basis $(\tilde{e_i})_{i=1}^n$, we consider the lattice L spanned by $e_i = \tilde{e_i} - (\tilde{e_1} + \ldots + \tilde{e_n})/n$. There is a unique way to write $x \in L$ under the form $x = \sum_{i=1}^n x_i e_i$ such that every $x_i \in \mathbb{N}$ and one at least is zero. We call $(x_1, \ldots, x_n)$ the *coordinates* of x and $\sum x_i = |x|$ the *length* of x. Let

$$\begin{aligned} W &= \{x \in L, x_1 \geq x_2 \geq \ldots \geq x_n = 0\} \\ \overset{o}{W} &= \{x \in L, x_1 > x_2 > \ldots > x_n = 0\} \end{aligned} \tag{3.4.3.1}$$

The lattice L is compactified by the sphere S^{n-2} in the following sense: a sequence x^d of L tends towards $y \in S^{n-2}$ iff its Euclidean norm $||x_n||$ tends to infinity and $x^d/||x^d|| \to y$.

Consider the simplex

$$\Sigma = \{y' = (y'_1, \ldots, y'_n) \, : \, y'_1 \geq \ldots \geq y'_n = 0, \sum_i y'_i = 1\} \subset \mathbb{R}^n \tag{3.4.3.2}$$

We embed it into S^{n-2} by the map $y' \to y'/||y'||$. By doing so, the above compactification induces a compactification of $\overset{o}{W}$ by Σ.

Namely, let $y^d = (y_1^d > y_2^d > \ldots > y_n^d = 0)_{d \geq 0}$ be a series of elements of $\overset{o}{W}$. Then it converges iff y_d is constant for d large enough or if $|y_d|$ tends towards infinity and for all i, $y_i^d/|y^d|$ admits a limit y'_i.

Let $0 < q < 1$ be a real number. Consider the measure

$$\mu = \sum_{i=1}^n \frac{q}{n} \delta_{e_i} \tag{3.4.3.3}$$

where δ_x is the Dirac mass at x.

The Martin theory with respect to P_μ (the convolution operator by the measure μ on the lattice L) is completely understood.

Let us now define a new random walk on the state space $\overset{o}{W}_n$. The vector $\rho = (n-1, n-2, \dots, 0)$ is such that $W_n = \rho + \overset{o}{W}_n$. In the sequel, we abbreviate P_μ by P. Our random walk in $\overset{o}{W}_n$ is obtained from P and conditioned not to hit $\partial W_n = W_n - \overset{o}{W}_n$. Call $\overset{o}{P}$ its transition kernel. For $x \in \overset{o}{W}_n$, its transition kernel satisfies $\overset{o}{P}(x,y) = q/n$ iff $y - x = e_i$ and $y \in \overset{o}{W}_n$ and 0 otherwise. The main result of this section is:

Theorem 3.4.1. *The Martin boundary associated to $\overset{o}{P}$, the state space $\overset{o}{W}$ and the vector e, is homeomorphic to Σ. Furthermore, $MB = MB^{min}$.*

The remainder of this section is devoted to proving this theorem. In part 3.4.2, we compute MB. In part 3.4.3, we consider $\overset{o}{W}$ as a canonical basis for the fusion algebra of $SU(n)$, and consider the Markov operator $\widetilde{P}$ obtained by considering the convolution by the normalized fundamental representation. Thanks to a result of Biane, its abstract Martin boundary can be computed and identified with that of $\overset{o}{P}$. We use that to show that $MB = MB^{min}$.

3.4.2 Asymptotics of the Martin kernel

The translation invariance of P implies that we can define for any l the one parameter functions

$$P^l(y-x) = P^l(x,y)$$

From now on, we take the convention that $x!^{-1} = 0$ if $x < 0$.

Lemma 3.4.2. *With the notations of section 3.4.1, we have*

- $$P^l(y) = \begin{cases} \frac{(|y|+kn)!}{\prod_{i=1}^n (y_i+k)!}(q/n)^l \text{ if } l = kn + |y| \\ 0 \text{ otherwise} \end{cases}$$

- $$\overset{o}{P}{}^l(x,y) = (|y-x|+kn)!(q/n)^{|y-x|+kn}\det(y_i - x_j + k)!^{-1}$$

Proof. The first point is elementary combinatorics. For the second one, if $x = (x_1, \dots, x_n)$, then for σ a permutation of $[1, n]$, let $x_\sigma = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We have, by the reflexion principle,

$$\overset{o}{P}{}^{l}(x, y) = \sum_{\sigma \in \mathcal{S}_n} P^l(x_\sigma, y)\varepsilon(\sigma)$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ, and the result follows. $\square$

Let $Y = (Y_1, \dots, Y_n)$ be an n-tuple of formal variables. Recall that the Vandermonde determinant is the polynomial $V(Y) = \prod_{1 \le i < j \le n}(Y_j - Y_i)$. Let $x \in W$. The function

$$s_x(Y_i) = \frac{\det(Y_i^{x_j+n-j})}{V(Y_i)} \tag{3.4.3.4}$$

is a symmetric polynomial in Y homogeneous of degree $|x| - |e|$. It is known as the *Schur polynomial* (see [Ful97]). It is classical (see [FH91]) that W is in one to one correspondance with the set of classes of irreducible representations of $SU(n)$ up to isomorphism. By Weyl character formula, s_x is known to be the character of the irreducible representation associated to x evaluated on $\mathrm{diag}(Y_1, \dots, Y_n)$.

Lemma 3.4.3. *Let*

$$f(x, y, l) = \frac{(|y - x| + kn)!}{(kn + |x| + |y - x|)!}$$

and

$$c_{x,y} = |x| + |y - x| - |y|$$

Note that by the triangle inequality, $0 \le c_{x,y} \le |x|$. *One has*

- *For each* $x, y \in \overset{o}{W}_n$ *and* $l \in \mathbb{N}$ *such that* $l = |y - x| + kn$ *for some positive integer* k,

  $$\overset{o}{P}{}^{l}(x, y) = (q/n)^{-|x|}V(y_i)f(x, y, l)p^{l+c_{x,y}}(y)s_{x-\rho}(y_i + k)(1 + o(1))$$

  where the symbol $o(1)$ *has to be understood pointwise in* x, *as* $|y| \to \infty$, *uniformly in* $k \ge 0$.

- *f is equivalent to $(|y|+kn)^{-|x|}$ as $|y|\to\infty$, uniformly in $k\geq 0$. Furthermore $f(x,y,l)s_{x-\rho}(y_i+k)(q/n)^{-|x|}$ is bounded independently on $|y|$ and k.*

Proof. Since

$$\overset{o\,l}{P}(x,y)=(|y-x|+kn)!(q/n)^{|y-x|+kn}\det(y_i-x_j+k)!^{-1}$$

we have

$$\overset{o\,l}{P}(x,y)/p^{l+c_{x,y}}(y)=(q/n)^{-|x|}f(x,y,l)\det(\frac{(y_i+k)!}{(y_i-x_j+k)!})$$

The expression $(y_i+k)!/(y_i-x_j+k)!$ is a polynomial in the variable y_i+k whose leading term is $(y_i+k)^{x_j}$. By multilinearity of the determinant and the definition of Schur polynomials this implies that

$$\det(\frac{(y_i+k)!}{(y_i-x_j+k)!})=V(y)s_{x-\rho}(y_i+k)(1+o(1))$$

The element $s_{x-\rho}(y_i+k)$ is positive. The fact that the function

$$(y,k)\to f(x,y,l)s_{x-\rho}(y_i+k)$$

is bounded is elementary. □

Let

$$A_q=\{y''=(y''_1\geq\ldots\geq y''_n\geq 0)),\ \prod_{i=1}^{n}y''_i=1,\ \sum_{i=1}^{n}y''_i=nq^{-1}\}\qquad(3.4.3.5)$$

To $y'\in\Sigma$ we associate an element $y''=\phi(y')\in A_q$ defined by $y''_i=q^{-1}n(y'_i+\alpha)/(1+n\alpha)$ where α is the only real number such that $\prod_{i=1}^{n}(y'_i+\alpha)/(1+n\alpha)=1$. The fact that this map is well defined (i.e. the fact that the real number α is unique) is a consequence of the proof of 3.4.5, in which it is showed that $\alpha\to\prod_{i=1}^{n}(y'_i+\alpha)/(1+n\alpha)$ is non increasing. The fact that it is continuous is a consequence of the continuity of the roots of a polynomial with respect to its coefficients. The map ϕ admits a left and right inverse, that, to an element of y'' of A_q associates y' defined by $y'_i=(y''_i-y''_n)/(\sum_j y''_j-y''_n)$. It is also continuous. As a summary, we have

Lemma 3.4.4. *The map $\phi : \Sigma \to A_q$ is a homeomorphism.*

The key to the proof of the main result of this section is a precise understanding of the asymptotics of the summands of the kernel u.

Lemma 3.4.5. *For all d it is possible to choose two integers $a_d < b_d$ such that*

$$U^{|y^d|}(y^d) \sim \sum_{k=a_d}^{b_d} P^{|y_d|+kn}(y^d)$$

$$(q/n)^n(|y^d| + kn) \prod_{i=1}^{n} (y_i^d + k)^{-1} \sim 1$$

uniformly in $k \in [a_d, b_d]$ as $d \to \infty$.

Proof. Let

$$f_k(y) = p^{|y|+(k+1)n}(y)/p^{|y|+kn}(y) = (q/n)^n \frac{(|y| + (k+1)n)!}{(|y| + kn)! \prod_{i=1}^{n}(y_i + k + 1)}$$

This function is defined a priori only for $k \in \mathbb{N}^*$, but it admits a natural extension on the index set $k \in \mathbb{R}_+^*$.

As $y^d \to y'$, the function $g_d : t \to f_{t|y^d|}(y^d)$ converges pointwise on $]0, \infty[$ towards

$$g_\infty : t \to \frac{q^n(1/n + t)^n}{\prod_{i=1}^{n}(y_i' + t)}$$

This function is strictly decreasing. Indeed, its logarithmic derivative is

$$\frac{g_\infty'}{g_\infty} : t \to \frac{n}{1/n + t} - \sum_{i=1}^{n} \frac{1}{y_i' + t}$$

and the inequality between harmonic mean and natural mean implies that this logarithmic derivative is always < 0.

Let $[a, b]$ be a closed subinterval of $]0, \infty[$. For d large enough, $t \to g_d(t)$ admits a logarithmic derivative that is non positive everywhere on $[a, b]$. Indeed, the function

$$(t, d) \to \frac{\partial}{\partial t} \log g_d(t)$$

is easily seen to be a continuous function on the set $[a, b] \times (\mathbb{N} \cup \{+\infty\})$. Therefore for d large enough, g_d is non-increasing. By Dini's theorem this implies that the convergence of g_d towards g_∞ holds uniformly on compact subsets of $]0, \infty[$.

Let $\varepsilon_d = |y_d|^{-1/3}$ and let $[a_d, b_d]$ be the greatest interval such that $|f_k(y_d) - 1| \leq \varepsilon_d$ for all $k \in [a_d, b_d]$. By the property of uniform convergence on compact subsets and by the non increasing property of the limit, this interval is well defined and there exist non-negative constants C_1 and C_2 such that $C_1|y^d|^{2/3} \leq |b_d - a_d| \leq C_2|y^d|^{2/3}$. For $k \in [0, a_d - 1]$, we have

$$P^{|y_d|+kn+n}(y^d)/P^{|y_d|+kn}(y^d) \geq 1 + |y_d|^{-1/3}$$

and for $k \geq b_d$,

$$P^{|y_d|+kn+n}(y^d)/P^{|y_d|+kn}(y^d) \leq 1 - |y_d|^{-1/3}$$

An immediate recursion together with a geometric series summation argument shows that

$$\sum_{k=0}^{a_d-1} P^{|y_d|+kn}(y^d) \leq P^{|y_d|+a_d n}(y^d)|y_d|^{1/3}$$

and

$$\sum_{k \geq b_d} P^{|y_d|+kn}(y^d) \leq P^{|y_d|+b_d n}(y^d)|y_d|^{1/3}$$

Let $[a_d^1, b_d^1]$ (resp.$[a_d^2, b_d^2]$)be the greatest interval such that $f_k(y_d) \in [1, 1+\varepsilon_d]$ (resp.$f_k(y_d) \in [1-\varepsilon_d, 1]$). There is a non-negative constant C_3 such that $[a_d^1, b_d^1]$ and $[a_d^2, b_d^2]$ are of length more than $C_3|y^d|^{2/3}$. Furthermore, by the definition of f_k there exists an index $i \in \{1, 2\}$ such that for any $k \in [a_d^i, b_d^i]$,

$$P^{|y_d|+kn}(y^d) \geq \max\{P^{|y_d|+a_d n}(y^d), P^{|y_d|+b_d n}(y^d)\}$$

This shows that $U^{|y^d|}(y^d) \sim \sum_{k=a_d}^{b_d} P^{|y_d|+kn}(y^d)$. $\square$

Proposition 3.4.6. *Let $(y^d)_{d\in\mathbb{N}}$ be a sequence of $\overset{o}{W}$ converging towards an element $y' \in \Sigma$ as above. Then*

$$\lim_d \frac{\overset{o}{U}(x, y^d)}{\overset{o}{U}(\rho, y^d)} = s_{x-\rho}(y''_i)$$

where $y'' = \phi(y')$, as defined in Lemma 3.4.4.

Proof. Lemma 3.4.3 and 3.4.5 imply that

$$\overset{o}{U} \sim (q/n)^{-|\rho|} \sum_{k\in[a_d,b_d]} P^{|y^d-x|+kn+c_{x,y^d}}(y^d)V(y_i^d)s_{x-\rho}(y_i'')(|y^d|+kn)^{-|\rho|} \tag{3.4.3.6}$$

as d tends to infinity. Equivalently, we have

$$\overset{o}{U}(\rho,y^d) \sim (q/n)^{-|\rho|} \sum_{k\in[a_d,b_d]} P^{|y^d-\rho|+kn+c_{\rho,y^d}}(y^d)V(y_i^d)(|y^d|+kn)^{-|\rho|} \tag{3.4.3.7}$$

The asymptotics of 3.4.3.6 (resp. 3.4.3.7) do not change if one replaces the range $[a_d,b_d]$ of the sum by $[a_d-c_{x,y^d}/n, b_d-c_{x,y^d}/n]$ (resp. $[a_d-c_{e,y^d}/n, b_d-c_{e,y^d}/n]$) because c_{x,y^d} takes only a finite number of values (contained in $[0,|x|]$). Furthermore $k\to(|y^d|+kn)^{-|e|}$ is a rational fraction, so Equation 3.4.3.6 (resp. 3.4.3.7) still holds if one replaces c_{x,y^d} by 0 in the r.h.s. (resp. c_{e,y^d} by 0). This implies that

$$\frac{\overset{o}{U}(x,y^d)}{\overset{o}{U}(e,y^d)} \sim s_{x-\rho}(y_i'')$$

□

This proves the main part of Theorem 3.4.1, namely that the Martin compactification of W is the compactification by Σ that we defined in equation 3.4.3.2. In particular, the minimal harmonic functions are of the kind $x\to s_{x-\rho}(y_i'')$, where $y_i''\in A_q$.

3.4.3 Extremal Martin Boundary

This part is devoted to computing MB^{min}, and showing that $MB^{min}=MB$. Let $\mathfrak{sl}_n$ be the complex Lie algebra of $SL(n)$.

$SU(n)$ admits a natural left action by conjugation on $M(\widehat{SU(n)})$, which we denote by Ad. The normalized trace tr of the fundamental representation. extends by linearity and continuity to a positive linear functional on $M(\widehat{SU(n)})$. This allows to define the positive convolution operator

$$\widetilde{P} = (\mathrm{tr}\otimes id)\widehat{\Delta} \tag{3.4.3.8}$$

Let $Z(\widehat{SU(n)})$ be the center of $M(\widehat{SU(n)})$. Since tr commutes with the action Ad, so does $\widetilde{P}$. The algebra $Z(\widehat{SU(n)})$ is the fixed point algebra of Ad, therefore $\widetilde{P}$ leaves $Z(\widehat{SU(n)})$ invariant and defines a new submarkovian operator on $\overset{o}{W}_n$ (upon the obvious identification of $\mathbb{C}^{\overset{o}{W}_n}$ with $Z(\widehat{SU(n)})$).

Proposition 3.4.7. *The minimal harmonic functions with respect to the operator P (resp. $\widetilde{P}$) on $\overset{o}{W}$ are the functions*

$$x \to s_{x-\rho}(y_1'', \dots, y_n'')$$

(resp. $x \to s_{x-\rho}(y_1'', \dots, y_n'')/s_{x-\rho}(1)$), where $(y_1'', \dots, y_n'')$ run in A_q.

Proof. It is equivalent to have $f : \overset{o}{W} \to \mathbb{R}_+^*$ harmonic with respect to $\overset{o}{P}$, and $\tilde{f} : x \in \overset{o}{W} \to f(x)/s_{x-\rho}(1)$ harmonic with respect to $\widetilde{P}$.

By [Bia92a], for any positive harmonic element f, there exists a finite positive measure μ_f on the set of positive elements of $SL(n)$ (we call it $SL(n)_+$), such that $f = \int_{SL(n)} i(A) d\mu(A)$. Since f is invariant under Ad, μ_f is also invariant under Ad. Let $\widetilde{\mu_f}$ be the image of μ_f under the canonical projection of $SL(n)_+$ onto $SL(n)_+/\mathrm{Ad}$. The quotient space $SL(n)_+/\mathrm{Ad}$ contains naturally A_q, and the support of $\widetilde{\mu_f}$ is a subset of A_q upon this inclusion.

Conversely, a finite positive measure $d\mu$ in A_q represents an element of $Z(\widehat{SU(n)})$. Indeed, let Y be the matrix $diag(y_1, \dots, y_n)$. Recall that i was defined at equation 3.3.3.4. Then,

$$\int_{SU(n)} \int_{A_q} i(UYU^*) dU d\mu$$

defines an harmonic element in $Z(\widehat{SU(n)})$. This defines a one to one correspondence between positive finite measures on A_q and harmonic elements of $Z(\widehat{SU(n)})$.

Let $y = (y_1 \geq \dots \geq y_n) \in A_q$. Then for any $U \in SU(n)$ and for any $x \in \overset{o}{W}$, we have $s_{x-\rho}(UYU^*) = s_{x-\rho}(y_i)$. Therefore, denoting by p_x the minimal central idempotent of $\mathfrak{U}(\mathfrak{sl}_n)$, the element $\int_{SU(n)} i(UMU^*) dU$ is harmonic, central and by equality of traces,

$$p_x \int_{SU(n)} i(UMU^*) dU = p_x s_{x-\rho}(y)/s_{x-\rho}(1)$$

Therefore, to each element y of A_q corresponds the harmonic function

$$x \to s_{x-\rho}(y)/s_{x-\rho}(1)$$

This proves f has to be $x \to s_{x-\rho}(y)/s_{x-\rho}(1)$ for some $y \in A_q$. □

3.5 QRW on $SU_q(n)$ and Abelian subalgebras

In this part, we apply the results of section 3.4 to a quantum random walk on the dual of $SU_q(n)$ that was first introduced in [Izu00].

3.5.1 Quantum compact groups

We start with the definition of matrix pseudogroup due to Woronowicz (see [Wor88] and [Wor87a]). Let A be a C^* -algebra with unit. The set $\mathbb{M}_N(A)$ of matrices with entries belonging to A is identified with the C^*-algebra $B(\mathbb{C}^N) \otimes A$. A pair $(u = (u_{ij}) \in \mathbb{M}_N(A), A)$ is said to be a *compact matrix pseudogroup* iff

- the $*$-subalgebra $\mathcal{A}$ generated by matrix elements of u is dense in A.
- there exists a C^*-homomorphism

  $$\Delta : A \to A \otimes A$$

  such that

  $$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

- u is invertible and there exists a linear antimultiplicative mapping

  $$\kappa : \mathcal{A} \to \mathcal{A}$$

  such that $\kappa(\kappa(a^*)^*) = a$ and

  $$(\mathrm{id} \otimes \kappa)u = u^{-1}$$

An element $w = (w_{ij}) \in \mathbb{M}_n(\mathbb{C}) \otimes A$ is called a *unitary corepresentation* if the following holds:

$$\Delta w_{ij} = \sum_k w_{ik} \otimes w_{kj}$$

A vector space V with basis v_i and with a map $\Phi : V \to V \otimes A$ is called a comodule if there exists a corepresentation of A such that

$$\Phi v_j = \sum_k v_k \otimes w_{kj}$$

For example, $vect\{w_{i1}, \dots, w_{ik}\}$ is a comodule if $w = (w_{ij})$ is a unitary corepresentation. It is possible to define in an obvious way a notion of sub-comodule, irreducible comodule, and equivalent comodules.

In this paper we shall focus on the specific example of $A(SU_q(n))$. It is the universal C^* -algebra generated by n^2 elements u_{kl} $(k, l = 1, 2, \dots, n)$ such that

- $$\sum_k u_{kl}^* u_{km} = \delta_{lm} I \ , \ \sum_k u_{mk} u_{lk}^* = \delta_{lm} I \tag{3.5.3.1}$$

- $$\sum_{k_1, \dots, k_n} u_{l_1 k_1} \dots u_{l_n k_n} E_{k_1, k_2, \dots, k_n} = E_{l_1, l_2, \dots, l_n} I \tag{3.5.3.2}$$

where, for $q \in]0, 1]$,

$$E_{i_1, i_2, \dots, i_n} = \begin{cases} 0 \text{ if } \mathrm{i_k} = \mathrm{i_l} \text{ for some } \mathrm{k} \neq \mathrm{l} \\ (-q)^{I(i_1, i_2, \dots, i_n)} \text{ otherwise} \end{cases} \tag{3.5.3.3}$$

with $I(i_1, i_2, \dots, i_n)$ denoting the number of inversed pairs in the sequence $(i_1, i_2, \dots, i_n)$. Then $SU_q(n) = (A(SU_q(n)), u)$ is a compact matrix pseudogroup. Furthermore, for $q = 1$, it coincides with the algebra of continuous functions on the classical $SU(n)$ group.

For any matrix pseudogroup, there exists a unique state h called *Haar measure*, satisfying

$$h(x) \cdot 1 = (h \otimes id) \cdot \Delta(x) = (id \otimes h) \cdot \Delta(x), x \in A$$

The state h is faithful in the case of $SU_q(n)$. Let (π_h, H_h, Ω_h) be the GNS triple of h, and Λ_h the natural map from $A(SU_q(n))$ to $B(H_h)$. The *multiplicative unitary* is defined as the bounded extension of the following operator:

$$V(\Lambda_h(x) \otimes \xi) = \Delta(x)(\Omega_h \otimes \xi), x \in A, \xi \in H_h$$

V is unitary and satisfies the following pentagon equation: (see [BS93])

$$V_{12}V_{13}V_{23} = V_{23}V_{12}$$

The dual von Neumann algebra $M(SU_q(n))$ is the bicommutant in $B(H_h)$ of the set $\{(id \otimes \mu)V\}$ where μ runs over $B(H_h)_*$. The dual coproduct is defined by

$$\widehat{\Delta}(x) = V^*(1 \otimes x)V \tag{3.5.3.4}$$

therefore $M(SU_q(n))$ is endowed with a Hopf-von Neumann algebra structure. This von Neuman algebra is well understood: since the representation theory of $SU_q(n)$ is the same as that of $SU(n)$, it has the same von Neumann algebra structure as the von Neumann algebra of $SU(n)$, therefore is isomorphic to

$$\bigoplus_{x \in W} \mathbb{M}_{d_x}(\mathbb{C})$$

For any representation $s \in W$, let $\{f_z\}_{z \in \mathbb{C}}$ be the family of Woronowicz characters. For its definition and basic properties, we refer to [Wor87a]. We only need to know that there exists a unique positive $\rho \in \mathbb{M}_n(\mathbb{C})$ with normalized trace, such that $f_z = \rho^z$, and ρ intertwines the fundamental representation with its double contragredient. Let s be the fundamental representation, $\mu = \mathrm{tr}_s(\rho \cdot)$, and

$$\tilde{P} := (id \otimes \mu)\widehat{\Delta} \tag{3.5.3.5}$$

The operator $\tilde{P}$ is completely positive. It has already been considered by Izumi (see [Izu99], [Izu98] and his preprint [Izu00]). It leaves invariant the center $Z(SU_q(n))$ of $M(SU_q(n))$ (this is a consequence of [Izu00], lemma 3.2, (3)). In order to prove this, one needs to define the morphism Φ:

$$M(SU_q(n)) \to M(SU_q(n)) \otimes L^\infty(SU_q(n))$$
$$\Phi(x) = U(x \otimes 1)U^*$$

This is a von Neumann action of $SU_q(n)$ on $M(SU_q(n))$. The center $Z(SU_q(n))$ is invariant under this action, therefore it makes sense to restrict it to Peter-Weyl blocks and to extend it to $M(\widehat{SU_q(n)})$.

$\tilde{P}$ intertwines Φ, therefore it leaves invariant the center $Z(SU_q(n))$ of $M(SU_q(n))$. We define the twisted integer

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

With this, one can show

Proposition 3.5.1. *In the canonical basis of* $\mathbb{M}_n(\mathbb{C})$, *we have*

$$\rho = \frac{1}{[n]_q} \begin{pmatrix} q^{-n+1} & 0 & \dots & 0 \\ 0 & q^{-n+3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & q^{n-1} \end{pmatrix} \tag{3.5.3.6}$$

It is a standard computation. See for example [Wor87a].

3.5.2 Restrictions and h-processes

Let $A(T^{n-1})$ be the C^*-algebra of continuous function on the torus T^{n-1}. It is the universal Abelian C^*-algebra generated by the n unitaries $u_1, \dots, u_n$ satisfying $u_1 \dots u_n = 1$. The coproduct $\widehat{\Delta} u_i = u_i \otimes u_i$ defines a C^* -Hopf algebra structure.

Let ϕ be the algebra morphism $A(SU_q(n)) \to A(T^{n-1})$ such that $\phi(u_{ij}) = \delta_{ij} u_i$ with the induced relations. Obviously ϕ is a morphism of C^* -Hopf algebras. This allows to define a subalgebra $M(T^{n-1})$ of $M(SU_q(n))$. It is the von Neumann algebra generated by $\{(id \otimes \mu)V\}$, where μ runs over the characters of $A(T^{n-1})$ seen as elements of $B(H_h)_*$. One can show that it is isomorphic to the group von Neumann algebra of T^{n-1} and has a Hopf-von Neumann algebra structure. Therefore it makes sense to restrict P defined at Equation 3.5.3.5 to this von Neumann subalgebra.

Furthermore, $\tilde{P}$ is invariant under the adjoint action Φ, thus it leaves invariant the center $Z(SU_q(n))$.

Therefore, $\tilde{P}$ restricts to two natural Abelian subalgebras of $M(SU_q(n))$. $M(T^{n-1})$ is in natural correspondance with $L^\infty(L)$ defined in section 3.4 and

$Z(SU_q(n))$ with $L^\infty(\overset{o}{W}_n)$. Thus P induces a classical Markov chain on L and an other one on $\overset{o}{W}_n$. It is of natural interest to investigate a (probabilistic) link between these two Markov chains.

This has already been done by Biane in [Bia91a] in the case $q = 1$. He shows that under the embedding of $\overset{o}{W}_n$ into L described in the preceeding section 3.4, the Markov chain on $\overset{o}{W}_n$ is obtained from that on L by conditioning in Doob's sense the Chain on L not to leave $\overset{o}{W}_n$. The chain on L itself is a nearest neighbor centered random walk.

Consider, for $q < 1$, the point of $\Sigma \subset S^{n-2}$

$$y_q = \sum_{i=1}^{n} q^{n+1-2i} e_i / || \sum_{i=1}^{n} q^{n+1-2i} e_i || \in S^{n-2} \tag{3.5.3.7}$$

Proposition 3.5.2. *The random walk induced by the restriction of $\tilde{P}$ to $M(T^{n-1})$ corresponds on the lattice L, to the convolution by the probability measure*

$$\mu(e_i) = \frac{q^{-n+2i-1}}{[n]_q} \tag{3.5.3.8}$$

It is obtained by conditioning, in Doob's sense, the submarkov random walk

$$\tilde{\mu}(e_i) = \frac{1}{[n]_q} \tag{3.5.3.9}$$

to hit the point y_q before dying.

Proof. It is enough to remark that the function

$$h : \begin{cases} L \to \mathbb{R} \\ \sum x_i e_i \to q^{\sum_i x_i(-n+2i-1)} \end{cases}$$

is well defined and harmonic with respect to $\tilde{\mu}$. Furthermore the Doob conditioning of the convolution operator by $\tilde{\mu}$ with respect to h is the convolution operator by μ and by the law of large numbers, this random walk almost surely tends to y_q. □

Now we focus on the restriction to the center.

Proposition 3.5.3. *The transition probability of the random walk restricted to the center is*

$$p_{l,l'} = \begin{cases} \frac{s_{l'}(q^{-n+1},q^{-n+3},\dots,q^{n-1})}{s_l(q^{-n+1},q^{-n+3},\dots,q^{n-1})[q]_n} \text{ if } \exists \mathrm{i},\ \mathrm{l}' = \mathrm{l} + \mathrm{e_i} \text{ and } \mathrm{l}' \in \overset{o}{\mathrm{W}} \\ 0 \text{ otherwise} \end{cases} \tag{3.5.3.10}$$

Proof. This is a consequence of [Izu00] and [HI98] together with the fact that the quantum dimension of the representation l is

$$s_l(q^{-n+1}, q^{-n+3}, \dots, q^{n-1})$$

This last fact results from the fact that the representation theory of $SU_q(n)$ and $SU(n)$ are the same, that the maximal torus remains non-deformed, and that we therefore have a Weyl character formula. □

Proposition 3.5.4. *The restriction of $\widetilde{P}$ to the center corresponds to the random walk on L with increment $\alpha \sum \delta_{e_i}/n$ with $\alpha = n/[n]_q$, conditioned to tending towards the point y_q before dying or hitting ∂W.*

Proof. According to the previous section, Proposition 3.4.7 and Theorem 3.4.1, $s \to s_x(q^{-n+1}, q^{-n+3}, \dots, q^{n-1})$ is harmonic with respect to $\overset{o}{P}$ (when replacing q by α in the defining equation 3.4.3.3) and in the Martin theory, it corresponds to the point y_q. This implies that the Doob conditioning of $\overset{o}{P}$ with respect to this harmonic function corresponds to conditioning to tend towards y_q and remain inside $\overset{o}{P}$. But this operator is also $\tilde{P}$, as it was computed in Proposition 3.5.3. □

L'exemple de $SU_q(2)$

Dans ce qui suit, nous établissons une théorie de Martin pour le groupe quantique $\mathrm{SU}_q(2)$ et donnons une description géométrique d'une large classe de fonctions harmoniques.

Notons que très récemment et indépendamment, des résultats généralisant ceux de cette partie ont été obtenus dans [NT02].

3.6 The example of $\mathrm{SU}_q(2)$

In this section we give an example of an explicit Ney-Spitzer like theorem in the framework of discrete quantum groups. The results of this section go beyond the results of section 3.3.1, in the sense that we compute explicitly the Martin boundary, and match it with a known object.

In [Bia94], the Martin kernel K is computed for $SU(2)$, with μ defined by $\mu(\lambda_g) = q\mathrm{Tr}(g)/2$, with $q \in [0,1]$. It is shown that the closure of the range of K is the C^*-algebra of invariant pseudo-differential operators, and one obtains a Ney-Spitzer like theorem.

The quotient of invariant pseudo-differential operators of order zero by compact operators of $M(SU(2))$ is isomorphic to the algebra of continuous functions of the sphere S^2, and it turns out that our description also involves quantum analogues of spheres.

In particular, for $q = 1$, the set of bounded harmonic elements is a one dimensional space. This is in sharp contrast with the quantum groups case, as it was established in [Izu00], and has a nice interpretation in the theory of subfactors.

Therefore, it is of interest to establish Martin boundary theory in our setting of section 3.3.1 for quantum groups. In this section, we use Izumi's results and do so for $\mathrm{SU}_q(2)$.

3.6.1 Quantum $\mathrm{SU}_q(2)$ group

In the case $n = 2$, we take an equivalent definition, and enumerate some useful properties, proved in [Wor87b].

$A(\mathrm{SU}_q(2))$, which we shall abbreviate by A, is the universal C^*-algebra generated by four elements x, u, v and y satisfying the following relations:

$$\begin{aligned} ux = qxu, vx = qxv, yu = quy, yv = qvy \\ uv = vu, xy - q^{-1}uv = yx - quv = 1, x^* = y, u^* = -q^{-1}v \end{aligned} \tag{3.6.3.1}$$

This C^*-algebra is nuclear. Let $w = (w_{ij})$ be the element of $\mathbb{M}_2(\mathbb{C}) \otimes A$ defined by

$$\begin{pmatrix} x & u \\ v & y \end{pmatrix}$$

The representation theory of $\mathrm{SU}_q(2)$ is the same as that of $SU(2)$. Namely, for each integer n there exists a unitary corepresentation $w = (w_{ij}) \in$

$\mathbb{M}_n(\mathbb{C}) \otimes A$ of $\mathrm{SU}_q(2)$, up to unitary equivalence. We adopt the spin notation, i.e. for $l \in \frac{1}{2}\mathbb{N}$, $i, j \in \{-l, -l+1, \ldots, l-1, l\}$, we call $w(l)$ the unitary corepresentation of dimension $2l+1$ such that (see for example [Izu00])

$$h(w(l)^*_{i,j} w(l')_{i',j'}) = \delta_{l,l'}\delta_{i,i'}\delta_{j,j'} q^{2(s+i)} \frac{1-q^2}{1-q^{2(2s+1)}}$$
$$h(w(l)^*_{i,j} w(l')_{i',j'}) = \delta_{l,l'}\delta_{i,i'}\delta_{j,j'} q^{2(s-j)} \frac{1-q^2}{1-q^{2(2s+1)}}$$

The vector space $vect_{i,j,l} w^l_{i,j}$ generates the $*$-algebra $\mathcal{A}$ which is dense in A. For each $l \in \mathbb{N}/2$, there exists a minimal central projection p_l such that $p_l p_{l'} = \delta_{l,l'}$ and $\sum_l p_l = 1$. Furthermore, $p_l M(\widehat{\mathrm{SU}_q(2)}) \cong \mathbb{M}_{2l+1}(\mathbb{C})$.

For i, j, l, we define the matrix element $e^l_{i,j}$ of $M(\widehat{\mathrm{SU}_q(2)})$ by

$$e^l_{i,j} \Lambda_h(w^{l'}_{i',j'}) = \delta_{l,l'}\delta_{j,j'} \Lambda_h(w^l_{i',i}) \tag{3.6.3.2}$$

In the multiplier algebra of $M(\widehat{\mathrm{SU}_q(2)}) \otimes A$, we have

$$V = \sum_{i,j,l} e^l_{i,j} \otimes w^l_{i,j}$$

Note that $M(\widehat{\mathrm{SU}_q(2)})$ is also the vector space of linear forms on $\mathcal{A}$ via the pairing $e^l_{i,j}(w^{l'}_{i',j'}) = \delta_{i,i'}\delta_{j,j'}\delta_{l,l'}$.

Let π be the adjoint action of $\mathrm{SU}_q(2)$ on $p_{1/2} M(\widehat{\mathrm{SU}_q(2)}) \cong \mathbb{M}_2(\mathbb{C})$, i.e.

$$\pi(e^{1/2}_{ij}) = \sum_{a,b} e^{1/2}_{ab} \otimes w^{1/2}_{ai} w^{1/2*}_{bj}$$

The minimal central projection of $M(\widehat{\mathrm{SU}_q(2)})$ onto $\mathbb{M}_2(\mathbb{C})$ intertwines Φ and π. It is also a C^*-algebra morphism. On the AFD C^*-algebra $B = \bigotimes_{k=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ we call π_n the action of π on the n-th leg and $\pi^{\otimes n} = \pi_1 \otimes \ldots \otimes \pi_n$. It is shown in [Izu00] (also see [Kon92]) that $\pi^{\otimes\infty} = \lim_n \pi^{\otimes n}$ is an action of $\mathrm{SU}_q(2)$ on B.

Furthermore, $\mu^{\otimes\infty} = \bigotimes_{k=1}^{\infty} \mu$ is an invariant state, in the sense that

$$(\mu^{\otimes\infty} \otimes id)\pi^{\otimes\infty}(x) = 1 \cdot \mu^{\otimes\infty}(x)$$

Therefore, this action extends to von Neumann algebra actions by taking the GNS representations associated to $\mu^{\otimes\infty}$ for B and h for $\mathrm{SU}_q(2)$. Let $\widetilde{B}$ be the bicommutant of B in its GNS representation associated to $\mu^{\otimes\infty}$, F the fixed point algebra and $R = \widetilde{B} \cap F'$ be the relative commutant.

Let I be the ideal of $A(\mathrm{SU}_q(2))$ generated by $u = 0$ and $v = 0$. The canonical projection of $A(\mathrm{SU}_q(2))$ onto $A(\mathrm{SU}_q(2))/I =: A(T)$ gives rise to a C^*-hopf algebra morphism, thus inducing an inclusion of the maximal torus T into $\mathrm{SU}_q(2)$. The fixed point algebra of the action of T by left multiplication on $\mathrm{SU}_q(2)$ is called the quantum sphere $A(\mathrm{SU}_q(2)/T)$. One shows that this subalgebra is still a $\mathrm{SU}_q(2)$-space for the action by conjugation (see [Pod87]). Taking the GNS construction shows that the action by conjugation holds at a von Neumann algebra level. Let $L^\infty(\mathrm{SU}_q(2)/T)$ be this algebra.

Theorem 3.6.1 ([Izu00]). • *Let*

$$\mathcal{H}_P = \{f \in M(\mathrm{SU}_q(2)), Pf = f\} \tag{3.6.3.3}$$

This set is an operator system, i.e. a closed subspace of $M(\mathrm{SU}_q(2))$ *stable under conjugation. There exists a completely positive normal map from* $\mathcal{H}$ *to* R *intertwining the actions of* $\mathrm{SU}_q(2)$

- R *and* $L^\infty(\mathrm{SU}_q(2))$ *are isomorphic von Neumann algebras. Furthermore, this isomorphism can be chosen so that it intertwines the respective actions of* $\mathrm{SU}_q(2)$.

The next section is devoted to defining a quantum analogue of Martin boundary theory and describing in these terms all harmonic elements.

3.6.2 An enumeration of harmonic elements on $\mathrm{SU}_q(2)$

The following exposition is standard and can be found in [VK92]. $\mathcal{U}_q(su(2))$ is the Hopf algebra generated by the four elements k^+, k^-, e, f satisfying the following relations:

$$\begin{gathered} k^+k^- = k^-k^+ = 1, k^+ek^- = qe, k^+fk^- = q^{-1}ef - fe = \frac{k^{+2} - k^{-2}}{q - q^{-1}} \\ \widehat{\Delta}(k^\pm) = k^\pm \otimes k^\pm, \widehat{\Delta}(e) = e \otimes k^+ + k^- \otimes f, \widehat{\Delta}(f) = f \otimes k^+ + k^- \otimes e \\ e^* = f, k = k^* \end{gathered} \tag{3.6.3.4}$$

The generators k, e, f act on the canonical basis $(\xi_i^l)_{i\in\{-l,-l+1,\dots,l\}}$ of the spin l $SU_q(2)$ left comodule V_l the following way:

$$\begin{aligned} k(\xi_i^l) &= q^i \xi_i^l \\ e(\xi_i^l) &= \sqrt{[l-i][l+i+1]}\xi_{i+1}^l \\ f(\xi_i^l) &= \sqrt{[l+i][l-i+1]}\xi_{i-1}^l \end{aligned} \tag{3.6.3.5}$$

Equivalently, an element of $\mathcal{U}_q(su(2))$ is also a linear form on $M(\widehat{\mathrm{SU}_q(2)})$:

$$\begin{aligned} k(w_{i,j}^l) &= \delta_{i,j} q^i \\ e(w_{i,j}^l) &= \delta_{j,i+1}\sqrt{[l+j][l-j+1]} \\ f(w_{i,j}^l) &= \delta_{j,i+1}\sqrt{[l-j][l+j+1]} \end{aligned}$$

We use Sweedler notation for $\mathcal{U}_q(su(2))$ and write $\widehat{\Delta}x = x^1 \otimes x^2$. With this notation we define the *adjoint* representation of $\mathcal{U}_q(su(2))$ by

$$\mathrm{Ad}(X)Y = X^1 Y \kappa(X^2)$$

For the generators e, f, k, this action is given by

$$\begin{aligned} \mathrm{Ad}(k)Y &= kYk^{-1} \\ \mathrm{Ad}(e)Y = eYk^{-1} - qk^{-1}Ye, \mathrm{Ad}(f)Y &= fYk{-1} - q^{-1}Yf \end{aligned}$$

This action extends by continuity to an action of $\mathcal{U}_q(su(2))$ on the algebra of elements affiliated to $M(\widehat{\mathrm{SU}_q(2)})$. This action is the same as that given by Φ, in the sense that for any $X, Y \in \mathcal{U}_q(su(2))$,

$$\mathrm{Ad}(X)Y = (id \otimes X)\Phi(Y)$$

This can be found in [Izu00], Lemma 4.2.

By quantum Clebsch-Gordan rules, $p_l M(\mathrm{SU}_q(2))$ is isomorphic as a $\mathcal{U}_q(su(2))$-modules to $\bigoplus_{l'=0}^{2l} V_{l'}$.

We say that a vector Y has weight l iff $\mathrm{Ad}(k)(Y) = q^l Y$. Additionnaly, it has *maximal weight* if $\mathrm{Ad}(e)Y = 0$. One can check with the relations that the vector $e^n k^n$ has maximal weight for any $n \in \mathbb{N}$. Furthermore, if p_l is the minimal central projection onto the spin l representation, the action of $\mathcal{U}_q(su(2))$ restricts obviously to an action of $M(\widehat{\mathrm{SU}_q(2)})_l$, and $p_l, p_l ek, p_l e^2k^2, \ldots, p_l e^{2l}k^{2l}$ corresponds to a maximal weight module for each irreducible submodule. For $n \leq 2l$, $-n \leq k \leq n$, let

$$v_{l,k,n} = p_l \mathrm{Ad}(f^{n-k})(e^n k^n)$$

This is a linearly free family of vectors in $M(\widehat{\mathrm{SU}_q(2)})$ and it generates a weakly dense space in $M(\widehat{\mathrm{SU}_q(2)})$. The *Casimir operator* is the operator

$$C = (\frac{q^{1/2}k - q^{-1/2}k^{-1}}{q - q^{-1}})^2 + fe$$

It is known that $p_l C = [l+1/2]^2 p_l$. In [Izu00], Lemma 5.2, this property is used to show

$$\begin{aligned} P(v_{n/2,k,n}) &= \frac{1}{(q+q^{-1})[n+2]_q} v_{n/2+1/2,k,n} \\ P(v_{l,k,n}) &= \frac{[2l+1+n]_q}{(q+q^{-1})[2l]_q} v_{l-1/2,k,n} + \frac{[2l+1-n]_q}{(q+q^{-1})[2l+2]_q} v_{l+1/2,k,n}, \\ & \qquad 2l \geq n+1 \end{aligned} \tag{3.6.3.6}$$

It implies that for any k, n, there exists a sequence of integers $(\alpha_{l,k,n})_{2l \geq n}$ and an harmonic element $h_{k,n} \in M(\widehat{\mathrm{SU}_q(2)})$ such that for any l'

$$p_{l'} h_{k,n} = \alpha_l v_{l,k,n}$$

if one requires $\alpha_{2l} = 1$. Under this assumption, none of the α_l is zero.

Theorem 3.6.2 (Izumi, [Izu00]). *The element $h_{k,n}$ is bounded. For every P-harmonic element h, there exists a family $\beta_{k,n}$ of complex numbers such that for any l',*

$$p_{l'} h = \sum p_{l'} \beta_{k,n} h_{k,n}$$

Note that in the l.h.s, there is only a finite number of non-zero elements, so that this sum is algebraically well-defined.

3.6.3 Martin Boundary of $\mathrm{SU}_q(2)$

Let us define inductively $\mu^0 = \varepsilon$, and $\mu^{*n+1} = (\mu^{*n} \otimes \mu)\widehat{\Delta}$. For $l \in \mathbb{N}/2$, let μ_l be the element of $M(\mathrm{SU}_q(2))$ defined by

$$\mu_l(f) = \frac{1}{[2l+1]_q} \mathrm{Tr}_l \begin{pmatrix} q^{-2l} & 0 & \dots & 0 \\ 0 & q^{-2l+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & q^{2l} \end{pmatrix} f \tag{3.6.3.7}$$

Lemma 3.6.3. *One has*

- $$\mu_l * \mu = q^{-1}[2l+2]/([2][2l+1])\mu_{l+1/2} + q[2l]/([2][2l+1])\mu_{l-1/2} \tag{3.6.3.8}$$

  By induction, this implies that for any n, μ^{*n} is a linear combination of the μ_l's.

- *For any l, $\sum_{n\geq 0} \mu^{*n}(p_l)$ is finite.*

Proof. The first point is an elementary computation. It is done in[Izu00] with the help of equations 3.6.2. We also refer to Lemma 3.6.13 for a more general computation. For the second point, consider the Markov chain on $\mathbb{N}$ given by $p(n, n \pm 1) = 1/[2]$ killed at a rate $1 - 2/[2]$ for $n \geq 1$ and $p(0,1) = 1/[2]$ killed at a rate $1 - 1/[2]$. Every point of $\mathbb{N}$ is transient. We remark that the function $n \to [n]$ is harmonic for the this process, and that we obtain Equation 3.6.3.8 by conditioning in Doob's sense by this function. Therefore it remains transient. □

This connection with Doob conditioning is investigated for general $SU_q(n)$ in section 3.5.

For any finitely supported measure ν, we call *Martin kernel* K_ν the only element of $M(\widehat{\mathrm{SU}_q(2)})$ such that for any f,

$$\nu * \varepsilon U f = \varepsilon U(k^{-1} K_\nu k f) \tag{3.6.3.9}$$

Lemma 3.6.4. *The Martin element K_ν is actually bounded (i.e., an element of $M(\mathrm{SU}_q(2))$). Furthermore, if ν is a weight, K_ν is positive.*

Proof. In order to prove that K_ν is bounded, it is enough to prove it if ν is a finitely supported weight. By Lemma 3.6.3, there exists an integer N and a constant β such that $\nu \leq \beta \sum_{k=0}^{N} \mu^{*k}$. This implies that

$$\nu P^n \leq \beta\varepsilon \sum_{l=n}^{N+n} P^l$$

therefore

$$\nu U \leq \beta(N+1)\varepsilon U$$

By Lemma 3.6.3 again, $kK_\nu k$ is positive. The above equation implies that it is smaller than $\beta(N+1)k^2$; therefore K_ν is positive and bounded. □

Remark 6.1. This Martin kernel makes sense at a probabilistic level: indeed, if one replaces $\mathrm{SU}_q(2)$ by any compact Abelian group G, one recovers the Martin kernel of a random walk started from ε on the dual group of G. Furthermore, setting $q = 1$ gives Biane's definition in [Bia94].

Let $C_0 \in M(\mathrm{SU}_q(2))$ be the compact central operator defined by

$$C_0 = \frac{\sqrt{[2]}}{(1-q^2)(C + 2(q-q^{-1})^{-2})} \tag{3.6.3.10}$$

We define Ψ_0, the algebra of *quantum pseudo-differential operators* as the unital C^* algebra generated by

$$x_1 = ekC_0, \;\; x_0 = \mathrm{Ad}(f)(ek)C_0, \;\; x_{-1} = fkC_0$$

One checks by hand with relations 3.6.2, that the operators x_1, x_0, x_{-1} are bounded, and satisfy

$$x_0^2 - qx_{-1}x_1 - q^{-1}x_1x_{-1} = 1 + C_1 \tag{3.6.3.11}$$

where C_1 is the compact operator

$$C_1 = \sum_l p_l \frac{-(q+q^{-1})}{(q^{2l+1} + q^{-2l-1})^2} \tag{3.6.3.12}$$

We also refer to Lemma 5.8 of [Izu00].

We remark that Ψ_0 contains all compact operators $\mathcal{K}$. Indeed, it contains C_1 therefore by functional calculus it contains all projectors p_l, and one checks that $p_l x_0, p_l x_1, p_l x_{-1}$ generates $p_l M(\mathrm{SU}_q(2))$.

We can thus consider the exact sequence

$$0 \to \mathcal{K} \to \Psi_0 \to \Psi_0/\mathcal{K} \to 0 \tag{3.6.3.13}$$

the above arrows are both C^*-algebra morphisms and $\mathrm{SU}_q(2)$-linear.

One also checks that x_0, x_1, x_{-1} are independent modulo $\mathcal{K}$. It is therefore a consequence from Podles' theorem of classification of quantum spheres ([Pod87]), that

Proposition 3.6.5. *There exists a* $\mathrm{SU}_q(2)$*-linear* C^**-algebra isomorphism between* $\Psi_0/\mathcal{K}$ *and* $C(\mathrm{SU}_q(2)/T)$.

Remark 6.2. The algebra $\Psi_0/\mathcal{K}$ is also the space of principal symbols of the pseudo differential operators. In the classical case, left invariant pseudo-differential operators on compact Lie groups have already been considered, see for example [Was88].

Lemma 3.6.6. *There exists constants* $c_{l,k,n}$ *such that for any* l, l', k, k', n, n', *one has* $\mu(v_{l,k,n} v^*_{l',k',n'}) = \delta_{l,l'}\delta_{k,k'}\delta_{n,n'}c_{l,k,n}$

Proof. The state μ_l is an invariant under Φ. It means that for all X

$$(\mu_l \otimes id)\Phi X = 1\mu_l(X)$$

and implies that $\mu(v_{l,k,n}v^*_{l',k',n'}) = (\mu \otimes \mu_h)\Phi(v_{l,k,n}v^*_{l',k',n'})$. But $\Phi(v_{l,k,n}) = \sum_{k'} v_{l,k',n} \otimes w^n_{k',k}$. Since Φ is also a $*$-morphism of algebras, and since $\mu_h(w_{k,k'}w^*_{j,j'}) = 0$ unless $k = j$ and $k' = j'$, we get the result. □

The following two Lemmas are crucial for the main theorem of this part.

Lemma 3.6.7. *Let* l *be a half integer,* $f \in M(\widehat{\mathrm{SU}_q(2)})$ *such that* $p_l f = f$. *Then* $U(f)$ *is well defined, and if* $p_l U(f) = \sum_{k,n} c_{n,k} v_{l,k,n}$, *then for any* $l' \leq l$, $p_{l'} U f = p_{l'} \sum h_{k,n} c_{n,k}/\alpha_{l,k,n}$.

Proof. The condition $p_l f = f$ implies that f is bounded, so by transience of P, $U(f)$ is well defined. One checks that $Uf - PUf = f$, therefore for any $l' < l$,

$$p_{l'}(P(p_{l'-1/2}Uf) + P(p_{l'+1/2}Uf)) = p_{l'}Uf$$

By definition of the elements $h_{k,n}$, replacing Uf in the above equation by $\sum h_{k,n}c_{n,k}/\alpha_l$ gives the same result. This concludes the proof. □

Lemma 3.6.8. *For any l, n, k such that $2l \geq n$ and $-n \leq k \leq n$ we, there exists a constant $C_{l,n}$ such that*

$$p_l U(v_{l,n,k}) = C_{l,n} v_{l,n,k}$$

Furthermore, for any n,

$$\lim_l C_{l,n} = \frac{1+q^2}{1-q^2}$$

Proof. According to Equations 3.6.3.6, one can show that for any $m \in \mathbb{N}$, $p_l P^{2m}(v_{l,k,n}) = \gamma v_{l,k,n}$ with

$$0 \leq \gamma \leq (q+q^{-1})^{-2m}\binom{2m}{m}$$

If one replaces $2m$ by an odd number, γ is zero. For any n, $\gamma \sim (q + q^{-1})^{-2m}\binom{2m}{m}$ as $l \to \infty$. Since $\sum (q+q^{-1})^{-2m}\binom{2m}{m}$ is a convergent series whose sum is $(1+q^2)/(1-q^2)$, the Lemma follows by dominated convergence. □

Proposition 3.6.9. *Let ν be a finitely supported linear form on $M(\widehat{\mathrm{SU}_q(2)})$. Then $K_\nu \in \Psi_0$.*

Proof. It is enough to prove the result if ν is such that for any $f \in M(\widehat{\mathrm{SU}_q(2)})$,

$$\nu(f) = \varepsilon U(f v^*_{l',k',n'})$$

Indeed, any finitely supported linear form on $M(\widehat{\mathrm{SU}_q(2)})$ can be realized as a finite combination of such ν's. Assume that $f = v_{l,k,n}$. There exists constants c_l such that $U(f) = \sum_l c_l v_{l,k,n}$, and by definition,

$$\nu U(f) = \varepsilon U(\sum_l c_l v_{l,k,n} v^*_{l',k',n'})$$

By an application of Lemma 3.6.6 and since, by Lemma 3.6.3, any μ_l admits a central density with respect to εU, we get

$$\nu U(f) = \varepsilon U(c_{l'} v_{l',k,n} v^*_{l',k',n'})$$

By definition of K_ν, (Equation 3.6.3.9) we therefore have

$$\varepsilon U(c_{l'} v_{l',k,n} v^*_{l',k',n'}) = \varepsilon U(k^{-1} K_\nu k v_{l,k,n}) = \varepsilon U(k^{-1} v_{l,k,n} k \cdot K_\nu)$$

Since this holds for all (l,k,n), and since $k^{-1} v_{l,k,n} k$ is a multiple of $v_{l,k,n}$, we get by Lemma 3.6.6, that there exists complex numbers α_l such that

$$K^*_\nu = \sum_l \alpha_l v_{l,k',n'} C_0^{n'} \tag{3.6.3.14}$$

We know that $\sum_l v_{l,k',n'} C_0^{n'}$ is a pseudo-differential operator. Besides, we know from Stone-Weierstrass Theorem and the existence of the operator C_1 in Ψ_0, that any central operator that can be written under the form

$$\sum_l \beta_l p_l$$

where β_l admits a finite limit in $\mathbb{C}$, is in Ψ_0.

By Lemma 3.6.8 and thanks to Lemma 5.4 of [Izu00], $U(v_{l,k,n} C_0^n)$ admits a nontrivial limit in the topology of pointwise convergence as $l \to \infty$. Therefore $\lim_l \nu U(v_{l,k,n} C_0^n)$ exists as $l \to \infty$. But by definition,

$$\nu U(v_{l,k,n} C_0^n) = \varepsilon U(k^{-1} K_\nu k v_{l,k,n} C_0^n)$$

Therefore, by equation 3.6.3.14

$$\varepsilon U(k^{-1} v_{l,k,n} k (\sum_j \bar{\alpha}_j v^*_{j,k',n'}) C_0^{n+n'})$$

But one checks with relations 3.6.3.5 and Lemma 3.6.3, that

$$\lim_l \varepsilon U(v_{l,k,n} v^*_{l,k',n'} C_0^{n+n'})$$

always exist in $\mathbb{C}$ and is non-zero iff $k = k'$ and $l = l'$. Therefore $\lim_l \alpha_l$ exists in $\mathbb{C}$. Combining these results and the fact that Ψ_0 is stable under multiplication shows that Equation 3.6.3.14 defines a pseudo-differential operator. □

We show that the C^* -algebra Ψ_0 allows to represent any harmonic elements. Let

$$\mathcal{H}^+_P = \{f \in M(\widehat{\mathrm{SU}_q(2)}), f \geq 0, Pf = f, \varepsilon f = 1\} \tag{3.6.3.15}$$

Theorem 3.6.10. *Let h be a P-harmonic element in $M(\widehat{\mathrm{SU}_q(2)})$. The element $U(p_l h)(1+q^2)/(1-q^2)$ tends towards h as $l \to \infty$.*

Proof. We consider $f_l = p_l h/C$, where C was introduced in Lemma 3.6.8, and we wish to show that Uf_l converges pointwise towards h. Let l be an half integer, and $\varepsilon \geq 0$. It is enough to show that there exists l' such that for each $l'' \geq l'$, $||p_l(U(f_{l''}) - h)|| \leq \varepsilon$.

Let $\varepsilon' > 0$, and consider the only constants $\beta_{k,n}$ such that $h = \sum_{k,n} \beta_{k,n} h_{k,n}$. Then $p_l h = \sum_{n \leq 2l, k \in \{-n,n\}} \beta_{k,n} v_{k,n}$. Fix l' according to ε' so that Lemma 3.6.8 is satisfied. Thus, we have $p_l(Uf_{l''} - h) = p_l \beta'_{k,n} h_{k,n}$, where $|\beta'_{k,n}| \leq 2\varepsilon'/C$. Since $p_l M(\mathrm{SU}_q(2))$ is a finite dimensional Banach space, the norms $\sup_{k,n} |\beta_{k,n}|$ and the operator norm are equivalent. Therefore it is possible to choose ε' such that $|\beta'_{k,n}| \leq 2\varepsilon/C$ implies $||p_l(U(f_{l''}) - h)|| \leq \varepsilon$. The corresponding l' such that for each $l'' \geq l'$, $||p_l(U(f_{l''}) - h)|| \leq \varepsilon$, and concluding this proof. □

Corollary 3.6.11. *For any element $h \in \mathcal{H}_P^+$, there exists a state ϕ on Ψ_0 vanishing on $\mathcal{K}$ such that for any weight ν,*

$$\nu(h) = \phi(K_\nu)$$

Proof. Let ϕ_l be the state defined by $\phi_l(K_\nu) = \varepsilon U(k^{-1} K_\nu k p_l h/C)$. It is obviously a state, and by Theorem 3.6.10, its limit exists and defines a state on Ψ_0 satisfying

$$\nu(h) = \phi(K_\nu)$$

In order to check that ϕ vanishes on compact operators, it is enough to check that it vanishes on finite rank operators. But by transience, any finite rank f can be written as $g - Pg$ with $g = Uf$. Thus there exists a finitely supported ν such that $f = K_\nu - K_{\nu * \mu}$. But by definition, $\nu(h) = \nu * \mu(h)$, so this concludes the proof. □

We also investigate a converse of this representation theorem:

Proposition 3.6.12. *Let ϕ be a state on Ψ_0 vanishing on $\mathcal{K}$. Then, one can define a element h_ϕ by the equation*

$$\nu(h_\phi) = \phi K_\nu$$

Furthermore, the map $\phi \to h_\phi$ is injective. Endowing the cone of states with the pointwise convergence topology and $\mathcal{H}_P^+$ with the topology induced by $M(\widehat{\mathrm{SU}_q(2)})$, this map is a continuous bijection.

Proof. For any ν, the equation $\nu(h) = \phi K_\nu$ obviously defines a positive element $h \in M(\widehat{\mathrm{SU}_q(2)})$. Let ν an element of $M(\widehat{\mathrm{SU}_q(2)})_*$. We have seen in the preceding proof that $K_\nu - K_{\nu * \tau}$ is a finite rank operator in $M(\mathrm{SU}_q(2))$. Therefore $\phi(K_\nu) = \phi(K_{\nu*\tau})$, hence $\nu * \tau(h) = \nu P h = \nu h$. Since this holds for any ν, $Ph = h$. □

Remark 6.3. The above proposition shows that in our case, the situation is analogue to the classical case when the Martin boundary is analogue to the case when $MB = MB^{min}$.

3.6.4 A discussion on minimal harmonic elements

In the preceding section, we gave a quantum Ney-Spitzer like theorem without making use of minimal harmonic element. This is the first example of a Ney-Spitzer like theorem obtained by circumventing any knowledge on minimal harmonic function. Yet it is natural to wonder about the minimal harmonic elements. By the last proposition of the previous part, the minimal harmonic elements are in one to one correspondence with the pure states of Podles' quantum sphere. The minimal states of Podles' quantum sphere are all known; yet, it is not easy to describe the related h_ϕ in full generality.

In this section we give a "numerical" description of a large class of minimal harmonic element. This description is another highlight of the sharp contrast existing between the case $q = 1$ and $q \neq 1$. It also provides a counterpart of Izumi's description.

For all $k \in \mathbb{Z}$, let

$$\mathcal{E}_k = Vect\{e^l_{i,i-k}\ l \in \mathbb{N}/2,\ \} = Vect\{v_{l,k,n},\} \tag{3.6.3.16}$$

It is a consequence of equations 3.6.3.6 that these vector spaces are invariant under the action of P. Let $\widetilde{\mathcal{E}_k}$ be the closure of $\mathcal{E}_k$ for the pointwise convergence in $M(\widehat{\mathrm{SU}_q(2)})$. For any $g \in M(\widehat{\mathrm{SU}_q(2)})$, for any $k \in \mathbb{Z}$, there exists unique $g_k \in \widetilde{\mathcal{E}_k}$ such that for any l, $p_l g = \sum_{k\in\{-2l,-2l+1,\dots,2l\}} p_l g_k$.

Let $\mathcal{E}$ be the discrete space $\mathcal{E} = \cup_{i,j,l} e^l_{i,j}$. We partition it into a family indexed over $\mathbb{Z}$ of subsets $E_k = \cup_{i,l} e^l_{i,i-k}$ and identify naturally $\mathcal{E}_k$ with the vector space of functions almost everywhere zero on E_k. Equivalently, $M(\widehat{\mathrm{SU}_q(2)})$ is naturally identified as a vector space as the space of complex valued functions on $\mathcal{E}$.

In this setting, we prove in Lemma 3.6.13 below, that P can be considered as a concrete operator on E_k, whose transition probabilities are:

$$\begin{aligned} p(e^l_{i,j}, e^{l+1/2}_{i-1/2,j-1/2}) &= \frac{q^{-1}}{[2]} q^{-(l+i/2+j/2)} \frac{\sqrt{[l-i+1][l-j+1]}}{[2l+1]} \\ p(e^l_{i,j}, e^{l-1/2}_{i-1/2,j-1/2}) &= \frac{q^{-1}}{[2]} q^{l-i/2-j/2+1} \frac{\sqrt{[l+i][l+j]}}{[2l+1]} \\ p(e^l_{i,j}, e^{l+1/2}_{i+1/2,j+1/2}) &= \frac{q}{[2]} q^{l-i/2-i/2} \frac{\sqrt{[l+i+1][l+j+1]}}{[2l+1]} \\ p(e^l_{i,j}, e^{l-1/2}_{i+1/2,j+1/2}) &= \frac{q}{[2]} q^{-(l+i/2+i/2+1)} \frac{\sqrt{[l-i][l-j]}}{[2l+1]} \end{aligned} \qquad (3.6.3.17)$$

By the Cauchy-Schwarz inequality, this is a sub Markov random walk. We can recover a random walk X_t by adding a cemetery point c to E. It is easy to see that for any k, the sets $E_k \cup \{c\}$ are classes for X_t. Additionally, it is a consequence of the quantum transience that $\mathcal{E}$ is a transient set for X_t.

Lemma 3.6.13. *Let e^l_{ij} be the matrix element of $M(\mathrm{SU}_q(2))$ defined in Equation 3.6.3.2. We have*

$$P(e^l_{ij}) = a^+_+ e^{l-1/2}_{i-1/2,i-1/2} + a^+_- e^{l-1/2}_{i+1/2,i+1/2} + a^-_+ e^{l+1/2}_{i-1/2,i-1/2} + a^-_- e^{l+1/2}_{i+1/2,i-1/2} \qquad (3.6.3.18)$$

where

$$\begin{aligned} a^+_+ &= \frac{q}{[2]} q^{l-i/2-j/2} \frac{\sqrt{[l+i][l+j]}}{[2l]} \\ a^+_- &= \frac{q^{-1}}{[2]} q^{-(l+i/2+j/2)} \frac{\sqrt{[l-i][l-j]}}{[2l]} \\ a^-_+ &= \frac{q}{[2]} q^{-(l+i/2+j/2)} \frac{\sqrt{[l-i+2][l-j+2]}}{[2l+2]} \\ a^-_- &= \frac{q^{-1}}{[2]} q^{l-i/2-j/2+2} \frac{\sqrt{[l+i][l+j]}}{[2l+2]} \end{aligned} \qquad (3.6.3.19)$$

Proof. This is a consequence of Vilenkin's Clebsch-Gordan formulae, p. 40 (4) and (4'), [VK92] □

The space $\mathcal{E}_0$ is a maximal Abelian von Neumann subalgebra of $M(\mathrm{SU}_q(2))$. According to Lemma 3.6.13, it is stable under P and it induces a random walk whose transition probabilities are

$$\begin{aligned} p(e^l_{i,i}, e^{l+1/2}_{i-1/2,i-1/2}) &= \frac{q^{-1}}{[2]} q^{-(l+i)} \frac{[l-i+1]}{[2l+1]} \\ p(e^l_{i,i}, e^{l-1/2}_{i-1/2,i-1/2}) &= \frac{q^{-1}}{[2]} q^{l-i+1} \frac{[l+i]}{[2l+1]} \\ p(e^l_{i,i}, e^{l+1/2}_{i+1/2,i+1/2}) &= \frac{q}{[2]} q^{l-i} \frac{[l+i+1]}{[2l+1]} \\ p(e^l_{i,i}, e^{l-1/2}_{i+1/2,i+1/2}) &= \frac{q}{[2]} q^{-(l+i+1)} \frac{[l-i]}{[2l+1]} \end{aligned} \tag{3.6.3.20}$$

In the remainder of the paper, we take the convention that if $x = e^c_{ab}$, then $i(x) = a$, $j(x) = b$ and $l(x) = c$. We define a compactification of $\mathcal{E}_0$ in the following sense. A sequence $(e^l_{ii})_n$ admits a limit iff l tends to infinity and $l(X_t) + i(X_t)$ converges in $\mathbb{N} \cup \{\infty\}$. This gives a compactification of $\mathcal{E}_0$ by the space $MB_0 \cong \mathbb{N} \cup \{\infty\}$, and the compactification is $MS_0 \cong \mathcal{E}_0 \cup MB_0$.

Proposition 3.6.14. *Let $x = (e^l_{ii})$ be any starting point on $\mathcal{E}_0$ and X_n the Markov chain started from x whose evolution operator is given by lemma 3.6.13. This process is transient and irreducible. Moreover its Green kernel is finite. The process X_t is such that almost surely, $l(X_t) + i(X_t)$ admits a finite limit in $\mathbb{N}$.*

Proof. Irreducibility of the process is obvious. By Lemma 3.6.13, the random process $i(X_t)$ has probability $q^{-1}/[2]$ to decrease by $1/2$ and $q/[2]$ to increase by $1/2$. This implies that the process is transient.

This also implies by the law of large numbers, that for all $\varepsilon > 0$, almost surely, for t large enough $l(X_t) \geq t(q^{-1} - q - \varepsilon)/[2]$. The green kernel associated to the process $i(X_t)$ on $\mathbb{Z}$ is finite (it is the kernel of a non-centered random walk), therefore so is the kernel of X_t.

It is straightforward from Equations 3.6.3.20 to check that the probability that $l(X_t) + i(X_t)$ jumps is smaller than q^{-i}. An application of Borel-

Cantelli's Lemma then implies that almost surely, $l(X_t) + i(X_t)$ only jumps a finite number of times, therefore it converges towards a finite limit. □

We extend the definition of MB_0 to arbitrary k by saying that $e^{l_d}_{i_d, i_d+k} \in E_k$ tends towards $a \in \mathbb{N} \cup \{\infty\} \cong MB_k$ iff $l_d \to \infty$ and $i_d + l_d \to a$. This defines a compactification of E_k that we call $MS_k = MB_k \cup E_k$.

The following probabilistic lemma will be needed throughout the remainder of the paper. We use the standard Markov theoretic notations, in which the subscript x in front of P or E means "starting from x".

Lemma 3.6.15. *Let P be a Markov chain on the state space $\mathbb{Z} \cup \{c\}$ such that*

$$\begin{array}{cc} P(n, n+1) = a - c_n & P(n, n-1) = b - c'_n \\ P(n, c) = c_n + c'_n & P(c, c) = 1 \end{array}$$

with $0 < b < a < 1$, $a + b = 1$, $0 < c_n < a$, $0 < c'_n < b$. We also assume that there exists two constants $C > 0$, $\alpha < 1$ such that $c_n + c'_n \leq C\alpha^{|n|}$. For any $x \in \mathbb{Z}$, $P_x(X_\infty = \infty) > 0$. Furthermore, $\lim_{x \to \infty}(X_\infty = \infty) = 1$.

Proof. Let L_n be the random variable counting the time spent in on the site n. We first evaluate $P_x(X_\infty = c)$, i.e. the probability to die. It satisfies

$$1 - P_x(X_\infty = c) = E_x(\prod_{n \in \mathbb{Z}} (1 - c_n - c'_n)^{L_n})$$

Let $\widetilde{P}$ be the Markov operator of the random walk on $\mathbb{Z}$, obtained by $P(n, n+1) = a$ and $P(n, n-1) = b$. It is known that the associated process $\widetilde{X_t}$ is transient, and the expectation of the occupation time of the site n satisfies $E_x(\widetilde{L_n}) = f(n - x)$, where

$$f(n) = \exp(k(|n| - n))2/(a - b)$$

with k the only negative number satisfying $e^{2k}a + e^{-2k}b = 1$. It is obvious that $E_x(L_n) \leq f(n - x)$.

Let β be a positive real constant that we shall fix later on. The key estimate for this lemma is

$$E_x(\prod_{n\in\mathbb{Z}}(1-c_n-c'_n)^{L_n}) \geq$$
$$\prod_{n\in\mathbb{Z}}(1-c_n-c'_n)^{f(n-x)\beta(1+(n-x)^2)}P_x(\forall n, L_n \leq f(n-x)\beta(1+(n-x)^2))$$

An application of Chebyshev inequality shows that

$$P_x(L_n > f(n-x)\beta(1+(n-x)^2)) \leq \frac{1}{\beta(1+(n-x)^2))}$$

Therefore, with the rough estimate $\sum_{n\in\mathbb{Z}}(n^2+1)^{-1} \leq 5$,

$$P_x(\forall n, L_n \leq \beta(1+(n-x)^2)) \geq 1-5/\beta$$

For any $\beta > 0$, we have

$$\prod_{n\in\mathbb{Z}}(1-c_n-c'_n)^{f(n-x)\beta(1+(n-x)^2)} > 0$$

because $\sum_{n\in\mathbb{Z}}\alpha^{|n|}f(n-x)(1+(n-x)^2)$ converges. Therefore, taking $\beta > 5$ shows that $P_x(X_\infty = c) < 1$. Furthermore, when $x \to \infty$,

$$\lim_{x\to\infty}\prod_{n\in\mathbb{Z}}(1-c_n-c'_n)^{f(n-x)(1+(n-x)^2)} = 1$$

So, according to the definition, there exists a β_x tending towards ∞ such that $\prod_{n\in\mathbb{Z}}(1-c_n-c'_n)^{f(n-x)\beta_x(1+(n-x)^2)}$ tends to 1 as $x \to \infty$. For example, $\beta_x = \left(\sum_{n\in\mathbb{Z}}\alpha^{|n|}f(n-x)(1+(n-x)^2)\right)^{-1/2}$ is convenient for x large enough.

We need to show that $P_x(X_\infty = \infty)+P_x(X_\infty = c) = 1$. By an elementary combinatorial argument, for any $\varepsilon > 0$,

$$P_x(X_t/t < (a-b)/2-\varepsilon) < P_x(\widetilde{X_t}/t < (a-b)/2-\varepsilon)$$

But the last one is well known to tend towards 0 as $t \to \infty$. This concludes the proof. □

Remark 6.4. This lemma is sufficient for our purposes, but with some additional work, one could prove that its conclusion still holds if $\sum_{n\geq 0} c_n + c'_n < \infty$.

Proposition 3.6.16. *Let x be any starting point on E_k. The corresponding process X_t has a probability < 1 to die (i.e., $P_x(X_\infty = c) < 1$). The process conditioned not to die converges almost surely in MB_k.*

Proof. By Lemma 3.6.13, we have

$$p(i(e^l_{i,j}), i(e^l_{i,j}) - 1)$$
$$= \frac{q^{-1}}{[2]}(q^{-(l+i/2+j/2)}\frac{\sqrt{[l-i+1][l-j+1]}}{[2l+1]} + q^{l-i/2-j/2+1}\frac{\sqrt{[l+i][l+j]}}{[2l+1]})$$

An elementary identity shows that

$$(p(i(e^l_{i,j}), i(e^l_{i,j}) - 1)[2]/q^{-1})^2$$
$$= 1 - q^{-2i-2j-1}\left(\frac{\sqrt{[l-i+1][l+j]} - \sqrt{[l-j+1][l+i]}}{[2l+1]}\right)^2 \tag{3.6.3.21}$$

Since $q^{-1}((\sqrt{[l-i+1][l+j]} - \sqrt{[l-j+1][l+i]})/[2l+1])^2$ is bounded uniformly on E_k by a constant C, after considering the counterpart of 3.6.3.21 for $p(i(e^l_{i,j}), i(e^l_{i,j}) - 1)$, we can apply Lemma 3.6.15 with $\alpha = q^4$ and the above constant C.

If we condition X_t by $i(X_\infty) = -\infty$, which has probability > 0 by the above, we the proof of Lemma 3.6.15 shows that for any $\varepsilon > 0$,

$$P_x(i(X_t) \sim t(q - q^{-1})/[2]) = 1$$

An application of Borel-Cantelli Lemma as in the end of the proof of Proposition 3.6.14 shows that almost surely, $i(X_t) + l(X_t)$ admits a finite limit. □

Theorem 3.6.17. *Let x be an operator in $B(l^2(\mathbb{N}))$, and $(x_{i,j})_{i,j\geq 0}$ be its matrix in the canonical basis. Let*

$$g_x = \sum_{l\in\mathbb{N}/2, 0\leq i,j\leq 2l} x_{i,j} e^l_{-l+i,-l+j} \tag{3.6.3.22}$$

Then $\lim_n P^n g_x$ admits a limit f_x. This limit is harmonic. The map

$$\Xi : B(l^2(\mathbb{N})) \to \mathcal{H}$$
$$x \to f_x$$

is a completely positive isometry.

Note that this is an alternative and more geometric description of Izumi's Poisson boundary.

Proof. We write in an unique way $g_x = \sum_{k\in\mathbb{Z}} g_k$ with $g_k \in \mathcal{E}_k$. By definition, g_k, seen as a function on E_k, is continuous on MB_k.

Let $k \in \mathbb{Z}$ and $x \in E_k$. It is enough to show that the complex numbers $P^t g_k(x)$ converge as $t \to \infty$. For any t, the random variable $g_k(X_t)$ is bounded by $||x||$, and converges almost surely by Lemma 3.6.16. Therefore, by the dominated convergence theorem, $\lim_t P^t(g_k)(x) = E(g_k(X_\infty))$.

This proves the existence of $\lim_n P^n g_x$. It is clear that this limit is harmonic. It is also straightforward that the map Ξ is completely positive.

We need to prove that Ξ is an isometry. By complete positivity of Ξ, it is enough to show that if x is positive, then $||f_x|| = ||x||$. It satisfies $||f_x|| \leq ||x||$ because P is a contraction. Let x_d be a sequence of points in E_k and suppose that $x_d \to x \in MB_k$. By a slight modification of Lemma 3.6.15 and application of Lemma 3.6.13, $P_{x_d}(X_\infty = c)$ tends towards 1 as $d \to \infty$. For p_n the projection in $B(l^2(\mathbb{N}))$ the projection on the vector space generated by the first n elements of the canonical basis, we thus have for any $\varepsilon > 0$, for any $n \in \mathbb{N}$, $||f_x|| \geq ||p_n x p_n|| - \varepsilon$. Therefore, $||f_x|| = ||x||$.

We need to prove that the range of Ξ is $\mathcal{H}$. Let f be a bounded harmonic element and write it as usual as $\sum_{k\in\mathbb{Z}} f_k$. Let y be a starting point in E_k, and g be a bounded harmonic element. Since g_k is an harmonic and bounded function on E_k, the stochastic process $g_k(X_t)$ is a l^∞ martingale and it converges almost surely. Since this holds for any starting point x, it implies by the same slight modification of Lemma 3.6.15 that g_k admits a continuous extension to MS_k. Let $x \in B(l^2(\mathbb{N}))$ be defined by $x_{i,j} = \lim_l g^l_{-l+i,-l+j}$.

It is straightforward to check that this definition makes sense and defines an operator in $B(l^2(\mathbb{N})).f_x - f$ is thus a bounded harmonic function, tending towards 0 at any point of MB_k, for any k. An application of Lemma 3.6.16 and of the martingale convergence theorem shows that $f_x = g$. Thus we have proved surjectivity. □

From this Theorem we derive the obvious corollary:

Corollary 3.6.18. *The bounded minimal harmonic elements are the image of rank one projectors in* $B(l^2(\mathbb{N}))$

Remark 6.5. We have an analogous description of a large class of non bounded minimal harmonic elements. For instance, we can obtain the same description for minimal harmonic functions f satisfying that there exists a constant C such that $f^l_{i,i} \leq C$ for any i, l.

Chapitre 4

Images de mesures de Haar

We study the law of a corner of a unitary Haar distributed random matrix, and exhibit its density with respect to the Lebesgue. For two randomly rotated projectors π_1 and π_2 of rank ad over $\mathbb{M}_d(\mathbb{C})$, the eigenvalue distribution of the operator $\pi_1\pi_2\pi_1$ is expected to converge towards the multiplicative free convolution of a Bernoulli distribution of parameter a by itself. We strengthen this convergence and establish universality of the level spacing inside the bulk of the spectrum, give a rough estimate of the behavior of the largest eigenvalue.

Then, we improve a result of Diaconis-Lauritzen-Eaton, and outline a new approach to the combinatorial problem of expanding a polynomial random variable in the entries of classical compact groups.

Nous étudions la loi d'un coin d'une matrice unitaire distribuée suivant la mesure de Haar normalisée et donnons la densité de sa loi par rapport à la mesure de Lebesgue.

Pour deux projecteurs aléatoires π_1 et π_2 de rang ad dans $\mathbb{M}_d(\mathbb{C})$, π_2 étant une rotation aléatoire de π_1, nous montrons que la distribution des valeurs propres de $\pi_1\pi_2\pi_1$ est un ensemble de Jacobi unitaire. De plus, un théorème général de probabilités libres affirme que la distribution des valeurs propres converge vers la convolution muliplicative libre d'une loi de Bernoulli de parametre a par elle-même. Nous renforçons cette convergence et établissons l'universalité de l'écartement des valeurs propres à l'interieur du spectre, et donnons une estimée approximative du comportement de la plus grande valeur propre.

En particulier, nous étudions la convergence vers un ensemble circulaire gaussien, et améliorons un résultat de Diaconis-Lauritzen-Eaton. Par ailleurs, nous proposons une approche plus directe pour comprendre le développement en puissances de $1/d$ de la fonction Wg introduite dans [Col02].

4.1 Introduction

It is known that for a Haar distributed Unitary (resp. Real, Symplectic) random matrix, an upper left corner of fixed size converges in law, up to a suitable renormalization constant, to a standard complex (resp. real, quaternionic) Gaussian matrix, as the dimension of the compact group tends to infinity. On the other hand, in chapter 2, an explicit formula is provided for moments of the law of such a corner, of order less than the dimension of the group. However, this formula excludes the real and symplectic Gaussian cases. This paper provides new ideas to overcome these important limitations.

The solution proposed is based on an explicit formula for the density of the law of a corner with respect to the Lebesgue measure, and our first result is Theorem 4.2.1. A result very close to this one has already been obtained by Olshanski in [Ol'90], and a good review can be found in [For02].

The general problem of the convergence of the eigenvalue counting measure for matrix models, and in particular the asymptotics of the local spacing of eigenvalues, is known as the universality problem for random matrices. See [Meh91], conjectures 1.2.1 and 1.2.2 for a statement of this universality problem. Many important breakthroughs towards these conjectures have been achieved for various matrix models during the last years (see works of Pastur, Johansson, etc...).

In free probability theory, properties of contractions of noncommutative probability spaces by free projections have been widely studied. A matrix model for this is $\pi_d A \pi_d$, where $A \in \mathbb{M}_d(\mathbb{C})$ is a constant hermitian matrix with limit distribution, and the selfadjoint random projector π_d has unitarily invariant distribution and rank q_d with $q_d/d \sim \alpha \in]0,1[$. It is natural to expect universality of local spacing for such distributions, and it turns out that we are able to establish such a property in the particular case when $A = \pi'_d$ is a constant selfadjoint projector of rank q_d.

Namely, a corollary of Theorem 4.2.1 is that $\pi_d \pi'_d \pi_d$ is, up to an affine transformation, a Jacobi unitary ensemble. Moreover, random matrix models for free probability (for references and results, see chapter 2) and older results about the S-transform (see [VDN92] example 3.6.7) show that the eigenvalue counting measure converges towards an explicitely computable measure if $q_d/d \sim \alpha < 1/2$. Therefore, our particular model $\pi_d \pi'_d \pi_d$ can be worked out with the machinery of orthogonal polynomials, and that results of [CI91] about Jacobi polynomials allow us not only to sharpen convergence predictions given by free probability theory, but also to establish some universality of the level spacing. This is the contents of Theorem 4.3.3.

Theorem 4.2.1 has also other applications. In particular, it allows us to estimate how quickly the push forward measure converges towards a Gaussian measure (Theorem 4.4.2).

In chapter 2, Catalan numbers appeared via some polynomial identities, as the leading term of the expansion in d^{-1} of a class of polynomials with variables in an upper left corner. But a clear combinatorial interpretation of these numbers was missing. The approach proposed here gives a geometric *raison d'être* for the Catalan numbers, with Theorem 4.5.3. We finish this article by giving an uniform estimate for the functions Wg.

4.2 Push forward of the Haar measure of the unitary group

Let $\mathbb{U}_d$ be the group of $d\times d$ complex unitary matrices, and μ_d its normalized Haar measure. For $d\geq q_1$, $d\geq q_2$, let π_{d,q_1,q_2} be the canonical projection of $\mathbb{M}_d(\mathbb{C})$ onto its upper left corner $\mathbb{M}_{q_1,q_2}(\mathbb{C})$ with q_1 lines and q_2 columns. Let dA be the standard Lebesgue measure on $\mathbb{M}_{q_1,q_2}(\mathbb{C})$

Theorem 4.2.1. *For $d\geq 2q_1\geq 2q_2$,*

$$\pi^*_{d,q_1,q_2}(\mu_d) = c_{q_1,q_2,d}\det(1-AA^*)^{d-q_1-q_2}1_{||A||\leq 1}dA$$

where $c_{q_1,q_2,d}$ is a normalization constant.

Note that a computation similar to this one has already been performed in [Ol'90].

Proof. In this proof we shall split π_{d,q_1,q_2} into $\pi_1\circ\pi_2$, where $\pi_2:\mathbb{M}_d(\mathbb{C})\to\mathbb{M}_{d,q_2}(\mathbb{C})$, and $\pi_1:\mathbb{M}_{d,q_2}(\mathbb{C})\to\mathbb{M}_{q_1,q_2}(\mathbb{C})$ are the canonical projections. Conversely, we introduce the canonical injection $i:\mathbb{M}_{q_1,q_2}(\mathbb{C})\to\mathbb{M}_d(\mathbb{C})$ which we split into the canonical injections $i_1:\mathbb{M}_{q_1,q_2}(\mathbb{C})\to\mathbb{M}_{d,q_2}(\mathbb{C})$ and $i_2:\mathbb{M}_{d,q_2}(\mathbb{C})\to\mathbb{M}_d(\mathbb{C})$. We have $i=i_2\circ i_1$, $\pi_1\circ i_1 = id_{\mathbb{M}_q(\mathbb{C})}$, $\pi_2\circ i_2 = id_{\mathbb{M}_{d,q}(\mathbb{C})}$.

The set $\pi_2(\mathbb{U}_d)$ is a real sub-manifold of $\mathbb{M}_{d,q_2}(\mathbb{C})$ of real dimension

$$2dq_2-q_2-2q_2(q_2-1)/2 = q_2(2d-q_2)$$

The push forward $\pi_2^*(\mu_d)$ is a probability measure invariant under the natural left $\mathbb{U}_d$ action and the natural right $\mathbb{U}_{q_2}$ action. Since the action of $\mathbb{U}_d \times \mathbb{U}_{q_2}$ on $\pi_2(\mathbb{U})$ is transitive, this measure is the only normalized invariant one.

Since $d - q_2 \geq q_2$, the map $\pi_1 : \pi_2(\mathbb{U}_d) \to \mathbb{M}_{q_1,q_2}(\mathbb{C})$ is surjective onto the unit ball of $\mathbb{M}_{q_1,q_2}(\mathbb{C})$ endowed with the operator norm. Indeed, let $A \in \mathbb{M}_{q_1,q_2}(\mathbb{C})$, such that $A^*A \leq 1$, and $V \in \mathbb{M}_{q_2}(\mathbb{C})$ be a square root of the positive matrix $1 - A^*A$. The matrix

$$X = \begin{pmatrix} A \\ 0 \\ V \end{pmatrix}$$

where $A \in \mathbb{M}_{q_1,q_2}(\mathbb{C}), 0 \in \mathbb{M}_{d-q_1-q_2,q_2}(\mathbb{C}), V \in \mathbb{M}_{q_2}(\mathbb{C})$, satisfies $\pi_1(X) = A$ and is in $\pi_2(\mathbb{U}_d)$.

Let Ξ be the open subset of $\pi_2(\mathbb{U})$:

$$\{X \in \pi_2(\mathbb{U}), \det(1 - \pi_1(X)^*\pi_1(X)) \neq 0\}$$

It has measure 1 with respect to $\pi_2^*(\mu_d)$. Indeed, its complement in $\pi_2(\mathbb{U})$ is a sub-manifold of dimension strictly less than $q_2(2d - q_2)$.

On Ξ, π_1 is a submersion. Indeed, for an element

$$\begin{pmatrix} A \\ V \end{pmatrix}$$

of Ξ, the tangent space $T_X\Xi$ is the space of matrices

$$\begin{pmatrix} dA \\ dV \end{pmatrix}$$

satisfying

$$A^*dA + dA^*A + V^*dV + dV^*V = 0$$

Let $dA \in \mathbb{M}_{q_1,q_2}(\mathbb{C})$, then $A^*dA + dA^*A = 2ReA^*dA$ is hermitian in $\mathbb{M}_{q_2}(\mathbb{C})$. But V is invertible, because $V^*V + A^*A = 1$ and $1 - A^*A$ is invertible. Therefore there exists a vector $dV \in \mathbb{M}_{d-q_1,q_2}(\mathbb{C})$ such that $V^*dV = ReA^*dA$. Thus we have proved that the tangent map is onto.

Therefore $\pi_{d,q_1,q_2}^*(\mu_d)$ admits a density with respect to the Lebesgue measure on $\mathbb{M}_q(\mathbb{C})$.

Let Ψ be the sub-manifold $\{V \in \mathbb{M}_{d-q_1,q_2}(\mathbb{C}), V^*V = 1_{q_2}\}$, and $B = \{A \in \mathbb{M}_{q_1,q_2}(\mathbb{C}), A^*A < 1\}$. Let

$$\alpha : \Xi \to B \times \Psi$$

be defined by

$$X = \begin{pmatrix} A \\ V \end{pmatrix} \to (A, V(1 - A^*A)^{-1/2})$$

The map α is a diffeomorphism. For $X \in \Xi$, let $S = ker(d\pi_1(X))$, where $d\pi_1$ is a map from $T_X\Xi$ to $\mathbb{M}_{q_1,q_2}(\mathbb{C})$, and $S^\perp$ its orthogonal complement in $T_X\Xi$. The map $\pi_1 : S^\perp \to \mathbb{M}_{q_1,q_2}(\mathbb{C})$ is a bijection. Besides, one can show that the image under π_1 of the natural Euclidean measure on $S^\perp$ is

$$\det(1 - A^*A)^{-q_2/2} \tag{4.2.4.1}$$

times the Euclidean measure on $\mathbb{M}_{q_1,q_2}(\mathbb{C})$.

Let S' be the tangent subspace of $V(1-A^*A)^{-1/2}$ in Ψ. Then the tangent map $T_X\alpha : T_X\Xi \to T_{\alpha(X)}B_q(0,1) \times \Psi$ admits the following matrix in the decomposition $S^\perp \oplus S \to S^\perp \oplus S'$:

$$\begin{pmatrix} I_q & * \\ 0 & K \end{pmatrix}$$

where K is the restriction to S of the map $\mathbb{M}_{d-q_1,q_2}(\mathbb{C}) \to \mathbb{M}_{d-q_2,q_2}(\mathbb{C})$ given by

$$V \to V(1 - A^*A)^{-1/2}$$

Let $\lambda_1, \ldots, \lambda_{q_2}$ be the eigenvalues of $V(1 - A^*A)^{-1/2}$. The (real) endomorphism K has $2d - 2q_1 - q_2$ eigenvalues λ_1, $2d - 2q_1 - q_2$ eigenvalues λ_2, etc... Therefore K has determinant

$$\det(1 - A^*A)^{d-q_1-q_2/2} \tag{4.2.4.2}$$

and our result follows by putting together equations 4.2.4.1 and 4.2.4.2 and an application of the change of variables formula and Fubini's Theorem. □

Remark 2.1. Replacing the unitary group by the orthogonal group gives that the push forward of the Haar measure has a density with respect to the Lebesgue measure proportional to $\det(1 - AA^*)^{d/2-1/2-q_1/2-q_2/2}$ provided that $d \geq 2q_1 \geq 2q_2$.

4.3 Jacobi ensembles

Let $(\alpha, \beta) \in \mathbb{R}^+$, and $J_{\alpha,\beta}$ be the probability distribution on the Hermitian matrices $\mathbb{M}_d(\mathbb{C})_{sa}$ given by

$$Z_d^{-1} \det(1-M)^\alpha \det(1+M)^\beta 1_{-1\leq M\leq 1} dM$$

where Z_d is some normalization constant. The probability $J_{\alpha,\beta}$ is called *Jacobi unitary ensemble.*

Theorem 4.2.1 admits the following corollary:

Corollary 4.3.1. *Let A be a random matrix in $\mathbb{M}_q(\mathbb{C})$ having law $\pi^*_{d,q,q}(\mu_d)$ for some d large enough. Then the random matrix $W = 2A^*A - 1$ has distribution $J_{d-2q,0}$.*

Proof. It is enough to remark, by change of variable, that A^*A has law

$$c1_{0\leq M\leq 1} \det(1-M)^{d-2q} dM$$

where c is a normalization constant and dM is the Lebesgue measure on the Hermitian matrices. □

In the remainder of this section, let us fix a real number $a \in]0, 1/2[$, and let for all d, $\pi_{1,d}$ be a self adjoint projector of $\mathbb{M}_d(\mathbb{C})$ of rank q where asymptotically $q \sim ad$. Let $\pi_{2,d} = U^*\pi_{1,d}U$ where U is a Haar distributed unitary random matrix. Then it is known by results of Voiculescu (see for example [Voi98]), that $\pi_{1,d}$ and $\pi_{2,d}$ are asymptotically free. Therefore $\pi_{1,d}\pi_{2,d}\pi_{1,d}$ has an empirical eigenvalues distribution converging towards $\mu \boxtimes \mu$, where μ is the probability

$$(1-a)\delta_0 + a\delta_1$$

By a standard S-transform argument (see [VDN92], example 3.6.7),

$$\mu \boxtimes \mu = (1-\alpha)\delta_0 + 1_{x\in[0,4a(1-a)]} \frac{\sqrt{4a(1-a)-x}}{2\pi\sqrt{x}(1-x)} dx$$

An immediate consequence of this is that the empirical eigenvalues distribution on $\mathbb{M}_q(\mathbb{C})$ of W converges towards

$$1_{-1\leq x\leq 8a-8a^2-1} \frac{1}{2a} \frac{\sqrt{8a-8a^2-1-x}}{2\pi\sqrt{1+x}(1-x)} dx$$

We shall see that the convergence of measure predicted by free probability theory can be strengthened to a uniform convergence for density functions, that some universality of the level spacing holds, and that the largest eigenvalue converges almost surely towards the upper bound of the spectrum.

The purpose of the following results is to give answers to these questions.

We make use of the Jacobi polynomials $P_n^{\alpha,\beta}$. We recall that for $\alpha, \beta \geq 0$, $(P_n^{\alpha,\beta})_{n\geq 0}$ is a series of orthogonal polynomials with respect to the measure $1_{[-1,1]}w^{\alpha,\beta}(x)dx$ where

$$w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$$

For the normalization constant, we refer to [Sze75], equation (4.3.4). Let $p_n^{\alpha,\beta}$ be the corresponding orthonormal polynomials, and γ_n their (nonnegative) leading term.

The function

$$K_q^{\alpha,\beta}(x,y) = \sqrt{w^{\alpha,\beta}(x)}\sqrt{w^{\alpha,\beta}(y)}\sum_{j=0}^{q-1} p_j^{\alpha,\beta}(x)p_j^{\alpha,\beta}(y)$$

satisfies for $x \neq y$ by the Christoffel-Darboux formula

$$K_q^{\alpha,\beta}(x,y) = \sqrt{w^{\alpha,\beta}(x)}\sqrt{w^{\alpha,\beta}(y)}\frac{\gamma_{q-1}}{\gamma_q}\frac{p_q^{\alpha,\beta}(x)p_{q-1}^{\alpha,\beta}(y) - p_{q-1}^{\alpha,\beta}(x)p_q^{\alpha,\beta}(y)}{x-y}$$

It is known that $q^{-1}K_q^{\alpha,\beta}(x)d_x$ is the distribution of eigenvalues of a unitary Jacobi process of parameter (α,β) on $\mathbb{M}_q(\mathbb{C})$; see for example [Meh91], A.10.

We first state an important lemma.

Lemma 4.3.2. *For $a \geq 0$ and $x \in]-1,1[$, let*

$$\Delta = (a(x+1))^2 - 4(a+1)(1-x^2)$$

Let C be a nonnegative real number, and α, β two real numbers such that $|\alpha| \leq C$ and $|\beta| \leq C$.

There exist a real parameter α_1 depending continuously on $(a,x) \in \mathbb{R}^+\times]-1,1[$, a parameter α_2 depending continuously on α, β, a, x, such that $\alpha_1 = \alpha_2 = 0$ as soon as $\Delta \geq 0$, and satisfying the following conditions:

- *Let K be a compact subset of* $\mathbb{R}^+ \times]-1,1[$ *such that for any* $(a,x) \in K$, $\Delta < 0$. *For any* $x \in K$, *for any* α, β *(bounded by* C*),*

$$\sqrt{w(x)}P_n^{\alpha+an,\beta}(x) = \left(\frac{4\sqrt{-\Delta}}{\pi n}\right)^{1/2} \Phi(\alpha,\beta,a,n)\cos(\alpha_1 n + \alpha_2) + o_{n\to\infty}(\Phi(\alpha,\beta,a,n)) \tag{4.3.4.1}$$

where

$$\Phi(\alpha,\beta,a,n) = 2^{(an+\alpha+\beta)/2}(a+1)^{-\beta/2-1/4}$$

Furthermore, this estimate is uniform in α, β *bounded, and* $(a,x) \in K$.

- *Let*

$$f(x) = \frac{\sqrt{8a - 8a^2 - 1 - x}}{2\pi\sqrt{1+x}(1-x)}$$

For any (a,x) *in the above* K,

$$\alpha_1(x + u/(n(f(x))), a) = \alpha_1(x,a) + \pi u/n + o(1/n)$$

- *Let K be a compact subset of* $\mathbb{R}^+ \times]-1,1[$ *such that for any* $(a,x) \in K$, $\Delta < 0$. *There exists constants* $c_1 > 0$ *and* $c_2 \in]0,1[$ *such that*

$$\sqrt{w(x)}P_n^{\alpha+an,\beta}(x) \leq c_1 c_2^n \Phi(\alpha,\beta,a,n) \tag{4.3.4.2}$$

This estimate is uniform in α, β *bounded and* $(a,x) \in K$.

Proof. The first and third point of this Lemma are contained under a weaker form in the paper [CI91]. The feature that remains to be proved for points one and three is the uniformity. This can be done by refining the comparison function close to the singularities by a comparison function such that the derivative of the difference stays bounded on the closed ball of radius the first singularity. Then, one applies the Bessel inequality to the derivative of the difference between the new comparison function and the generating series. For the third point, one should remark that equalities (2.8) and (2.9) of [CI91] become strict inequalities if $\sqrt{\Delta}$ is replaced by $|\sqrt{\Delta}|$.

The second point is a tedious computation of $\partial\alpha_1/\partial x$ using formulae (2.11), (2.12), (2.13) and (2.16) of [CI91]. □

Theorem 4.3.3. *Let a be a real number in $]0,1/2[$, and to each d, associate a number q_d such that $2q_d \leq d$ and $q_d/d \sim a$.*

- *Let K' be a compact of $]-1, 8a-8a^2-1[$. We have*

$$\lim_d \frac{1}{q} K_q^{d-2q,0}(x,x) = \frac{1}{2a} \frac{\sqrt{8a-8a^2-1-x}}{2\pi\sqrt{1+x}(1-x)} = f(x)$$

  and this limit holds uniformly on K.

- *For $u, v \in]0,\infty[$, we have as $d \to \infty$,*

$$\frac{1}{qf(x)} K_q^{d-2q,0}\left(x+\frac{u}{qf(x)}, x+\frac{v}{qf(x)}\right) = \frac{\sin\pi(u-v)}{\pi(u-v)}$$

  This limit is uniform for $x \in K$ and for u, v in compact subsets of $\mathbb{R}$.

- *For K' a compact of $]8a-8a^2-1, 1[$, almost surely for q large enough, there is no eigenvalue of W in K'.*

Proof. According to [Sze75], formula (4.5.2), one has

$$K_q^{\alpha,\beta}(x,y) = \sqrt{w^{\alpha,\beta}(x)}\sqrt{w^{\alpha,\beta}(y)} \frac{2^{-\alpha-\beta}}{2q+\alpha+\beta} \frac{\Gamma(q+1)\Gamma(q+\alpha+\beta+1)}{\Gamma(q+\alpha)\Gamma(q+\beta)} \frac{P_q^{\alpha,\beta}(x)P_{q-1}^{\alpha,\beta}(y) - P_{q-1}^{\alpha,\beta}(x)P_q^{\alpha,\beta}(y)}{x-y} \tag{4.3.4.3}$$

therefore

$$K_q^{\alpha,\beta}(x,x) = w^{\alpha,\beta}(x) \frac{2^{-\alpha-\beta}}{2q+\alpha+\beta} \frac{\Gamma(q+1)\Gamma(q+\alpha+\beta+1)}{\Gamma(q+\alpha)\Gamma(q+\beta)} (-P_q(x)P'_{q-1}(x) + P_{q-1}(x)P'_q(x))$$

Furthermore, one has ([Sze75], formula (4.21.7))

$$(P_n^{\alpha,\beta})'(x) = 1/2(n+\alpha+\beta+1)P_{n-1}^{\alpha+1,\beta+1}(x) \tag{4.3.4.4}$$

Putting equations 4.3.4.3, 4.3.4.4 and 4.3.4.2 together show that whenever $q = ad + \alpha_q$ where α_q ranges in a bounded interval,

$$\frac{1}{q} K_q^{d-2q,0}(x,x) = \frac{1}{2a} \frac{\sqrt{8a-8a^2-1-x}}{2\pi\sqrt{1+x}(1-x)} + o(1)$$

where $o(1)$ is uniform in $x \in K'$

The second point is proved in the same way with the help of 4.3.4.3, 4.3.4.4 and 4.3.4.2. Since, by the previous Lemma,

$$\alpha_1(x + u/(n(f(x))), a) = \alpha_1(x, a) + \pi u/n + o(1/n)$$

our result follows by standard trigonometric considerations (see for example [Meh91], A.10).

For the third point, it is a consequence of [CI91], that there exists constants $\zeta_1 > 0$ and $\zeta_2 \in]0, 1[$ such that for any $x \in K'$,

$$\frac{1}{q} K_q(x, x) \leq \zeta_1 \zeta_2^n$$

□

4.4 Asymptotic independence

We first recover a well-known result.

Proposition 4.4.1. *For any q, the random variable $\sqrt{d/q}\pi_{q,d}(U_d)$ converges towards a standard complex non-hermitian Gaussian ensemble as $d \to \infty$*

Proof. It is just a matter of remarking that

$$\det(1 - qAA^*/d)^{d-2q} \to e^{-q\mathrm{Tr}(AA^*)}$$

as $d \to \infty$. □

We can also improve a result by Diaconis-Eaton-Lauritzen [DEL92].

Theorem 4.4.2. *Let ν_q be the probability measure $c_{d,q} e^{-q\mathrm{Tr}MM^*} dM$ on $\mathbb{M}_q(\mathbb{C})$, and q_d be a sequence of integers tending towards infinity such that there exists a $C > 0$ such that $q_d^3 \leq Cd$. Then*

$$|\sqrt{d/q_d}\pi^*_{d,q_d}(\mu_d) - \nu_{q_d}| = o(1)$$

where $|\cdot|$ denotes the total variation measure.

This result was already known to [DEL92] under the assumption that $q_d^3 = o(d)$.

Proof.

Let n be any integer and A a random matrix on $\mathbb{M}_q(\mathbb{C})$ following the law ν_q. By a classical combinatorial result (see chapter 2), there exists a constant C_n such that

$$\lim_q q^2(E(q^{-1}\mathrm{Tr}(A^*A)^{2n}) - (E(q^{-1}\mathrm{Tr}(A^*A)^n))^2) = C_n$$

Therefore, by Jensen's inequality, there exists a constant C'_n such that for any K

$$P_{\nu_{q_d}}(|q_d^{-1}\mathrm{Tr}(A^*A)^n - E(q_d^{-1}\mathrm{Tr}(A^*A)^n)| \geq K) \leq \frac{C'_n}{K^2 q_d^2}$$

In particular, there exists a constant C''_2 such that for any K,

$$P_{\nu_{q_d}}(|\frac{q_d^2}{d-2q_d}\mathrm{Tr}(A^*A)^2 - E(\frac{q_d^2}{d-2q_d}\mathrm{Tr}(A^*A)^n))| \geq K) \leq \frac{C''_2}{K^2 q_d^4}$$

By Theorem 4.2.1,

$$\sqrt{(d-2q_d)/q_d}\pi^*_{d,q_d}(\mu_d) = \widetilde{c_{d,q_d}}\det(1 - qA^*A/(d-2q))^{d-2q}dA$$

where

$$\widetilde{c_{d,q_d}}^{-1} = \int_{\mathbb{M}_q(\mathbb{C})} \det(1 - q_dA^*A/(d-2q_d))^{d-2q_d}dA$$

Let $K = q_d^{-1}$. In equation 4.4, one has $K\frac{q_d^3}{(d-2q_d)} \to 0$ and $\leq \frac{C''_n}{K^2q_d^2} \to 0$.

Introduce on $\mathbb{M}_{q_d}(\mathbb{C})$ the sets

$$A_1 = \{A, |\frac{q_d^n}{n(d-2q_d)^3}\mathrm{Tr}(A^*A)^2 - E(\frac{q_d^2}{n(d-2q_d)}\mathrm{Tr}(A^*A)^n)| \leq K \ , \ ||A|| \leq 3\}$$

$$A_2 = B(0,3) \cap A_1^c \ ; \ A_3 = B(0,3)^c$$

A Taylor expansion shows that uniformly on A_1,

$$\det(1 - q_dA^*A/(d-2q_d))^{d-2q_d} \sim e^{-q_d\mathrm{Tr}A^*A - \frac{q_d^2}{2(d-2q_d)}E(\mathrm{Tr}(A^*A)^2)}$$

Therefore,

$$\int_{A_1} \det(1 - q_d A^*A/(d-2q_d))^{d-2q_d} \sim \int_{A_1} e^{-q_d \mathrm{Tr} A^*A - \frac{q_d^2}{2(d-2q_d)} E(\mathrm{Tr}(A^*A)^2)}$$

By a classical result, $P_{\nu_{q_d}}(||M|| \geq 3\sqrt{q_d/d}) = o(1)$ as $d \to \infty$. Therefore

$$\int_{A_2 \cup A_3} e^{-q\mathrm{Tr}(A^*A)} dA \ll \int_{A_1} e^{-q\mathrm{Tr}(A^*A)}$$

Furthermore, since

$$\det(1 - q_d A^*A/(d-2q_d))^{d-2q_d} \leq e^{-q_d \mathrm{Tr} A^*A}$$

we have

$$\int_{A_2 \cup A_3} \det(1 - q_d A^*A/(d-2q_d))^{d-2q_d} \leq \int_{A_2 \cup A_3} e^{-q_d \mathrm{Tr} A^*A} dA$$

This and the fact that

$$\frac{q_d^2}{2(d-2q_d)} E(\mathrm{Tr}(A^*A)^2)$$

is bounded as $d \to \infty$ end the proof. □

4.5 Integration of polynomial random variables on $\mathbb{U}_d$

4.5.1 A new approach to the Wg function

In chapter 2, we introduced, for any permutation $\sigma \in \mathcal{S}_q$ and any integer $d \geq q$, a function $\mathrm{Wg}(d, \sigma)$ allowing to perform computations of polynomial integrals on the unitary group. It is defined by

$$\mathrm{Wg}(d, \sigma) = \frac{1}{q!^2} \sum_{\lambda \vdash q} \frac{\chi^\lambda(e)^2 \chi^\lambda(\sigma)}{s_{\lambda,d}(1)} \tag{4.5.4.1}$$

And it allows to compute

$$\int_{\mathbb{U}_d} U_{i_1 j_1} \cdots U_{i_q j_q} U^*_{j'_1 i'_1} \cdots U^*_{j'_{q'} i'_{q'}} dU$$

as a sum over all possible 4-uples of q-uples of indices in $[1, d]$, $\mathbf{i} = (i_1, \ldots, i_q)$, $\mathbf{i'} = (i'_1, \ldots, i'_{q'})$, $\mathbf{j} = (j_1, \ldots, j_q)$, $\mathbf{j'} = (j'_1, \ldots, j'_{q'})$, of

$$\delta_{i_1 i'_{\sigma(1)}} \cdots \delta_{i_q i'_{\sigma(q)}} \delta_{j_1 j'_{\tau(1)}} \cdots \delta_{j_q j'_{\tau(q)}} \mathrm{Wg}(d, \tau\sigma^{-1})$$

One of its main property was

$$\mathrm{Wg}(d, \sigma) = O(d^{-q-|\sigma|})$$

as $d \to \infty$. We explain here how one can recover this result with our first part.

Let $\mathrm{Moeb} : \mathcal{S}_q \to \mathbb{C}$ be the central function defined by

$$\mathrm{Moeb}(\sigma) := \prod_{i=1}^{k} c_{|C_i|} \cdot (-1)^{|C_i|} \tag{4.5.4.2}$$

where $\sigma \in \mathcal{S}_q$ is a permutation whose cycle decomposition is $\sigma = C_1 \cdots C_k$ and $c_n = (2n)!/(n!(n+1)!)$. The integer c_n is the n^{th} *Catalan number* . It is the number of pairing edges of a $2n$-gon and gluing them in order to obtain a sphere.

Lemma 4.5.1. *Let $W = AA^*$. If $p_1 + \ldots + p_k - k < |\sigma|$, then*

$$\int_{A \in \mathbb{M}_q(\mathbb{C})} A_{11} \ldots A_{qq} \overline{A_{1\sigma(1)}} \ldots \overline{A_{q\sigma(q)}} \mathrm{Tr} W^{p_1} \ldots \mathrm{Tr} W^{p_k} e^{-\mathrm{Tr} AA^*} dA = 0$$

If $p_1 + \ldots + p_k - k = |\sigma|$, then

$$\int_{A \in \mathbb{M}_q(\mathbb{C})} A_{11} \ldots A_{qq} \overline{A_{1\sigma(1)}} \ldots \overline{A_{q\sigma(q)}} \mathrm{Tr} W^{p_1} \ldots \mathrm{Tr} W^{p_k} e^{-\mathrm{Tr} AA^*} dA$$

is non-zero iff σ has k cycle (including the trivial ones) with $p_1, \ldots, p_k$ elements. In such a case, this integral is

$$p_1 \ldots p_k \mathrm{Moeb}(\sigma)$$

Proof. A product of elements A_{ij} and $\overline{A_{ij}}$ has integral non-zero with respect to $e^{-\mathrm{Tr}AA^*}dA$ if and only if for any pair of indices (i,j), the degree in A_{ij} is the same as that in $\overline{A_{ij}}$.

Therefore, if

$$\int_{A\in\mathbb{M}_q(\mathbb{C})} A_{11}\dots A_{qq}\overline{A_{1\sigma(1)}}\dots\overline{A_{q\sigma(q)}}\mathrm{Tr}W^{p_1}\dots\mathrm{Tr}W^{p_k}e^{-\mathrm{Tr}AA^*}dA$$

is non-zero, σ has to be conjugate to a permutation σ' of $[1,r]$ for some $r\in[1,p_1+\dots+p_k]$, such that σ' leaves globally invariant the blocks $[1,p_1],[p_1+1,p_2],\dots,[p_1+\dots+p_{k-1},p_1+\dots p_k]$.

This is possible only if $p_1+\dots+p_k-k\geq|\sigma|$. If $p_1+\dots+p_k-k=|\sigma|$, one needs to find out all sequences $\overline{A_{11}}A_{1\sigma(1)}\dots\overline{A_{qq}}A_{q\sigma(q)}$ appearing in $\mathrm{Tr}W^{p_1}\dots\mathrm{Tr}W^{p_k}$. This is exactly the number

$$p_1\dots p_k\mathrm{Moeb}(\sigma)$$

□

We need a second technical Lemma:

Lemma 4.5.2. *For any* $i_1,\dots,i_{2q}\in[1,q]$, *for any* $k\in\mathbb{N}$ *we have*

$$\int_{||A||\geq\sqrt{d}} A_{i_1i_2}\dots A_{i_{2q-1}i_{2q}}A_{i'_1i'_2}\dots A_{i'_{2q-1}i'_{2q}}e^{-\mathrm{Tr}AA^*}dA=O(d^{-k})$$

Proof. Standard estimate of the tail of a Gaussian random variable together with the equivalence of Banach norms on a finite dimensional space. □

Theorem 4.5.3. *For any* $\sigma\in\mathcal{S}_q$,

$$d^{q+|\sigma|}\mathrm{Wg}(d,\sigma)=\mathrm{Moeb}(\sigma)+o(1)$$

Proof. By chapter 2, Theorem 2.1 we have

$$\mathrm{Wg}(d,\sigma)=\int_{\mathbb{M}_d(\mathbb{C})} A_{11}\dots A_{qq}\overline{A_{1\sigma(1)}}\dots\overline{A_{q\sigma(q)}}d\mu_{q,d}$$

Equivalently,

$$(d+2q)^q \mathrm{Wg}(d,\sigma) = \int_{||A||\leq\sqrt{d-2q}} A_{11}\ldots A_{qq}\overline{A_{1\sigma(1)}}\ldots\overline{A_{q\sigma(q)}}\det(1-A^*A/(d-2q))^{d-2q}dA$$

On the set $x < ||A||^{-1/2}$, the following holds:

$$\det(1-xA^*A)^{x^{-1}} = e^{-\mathrm{Tr}A^*A + x\mathrm{Tr}A^*AA^*A/2 + x^2\mathrm{Tr}A^*AA^*AA^*A/3+\ldots)}$$

therefore the function

$$f_A : x \to \det(1-xAA^*)^{x^{-1}}$$

is analytic in a neighborhood of zero, and its derivatives at zero are polynomials with rational coefficients in the variables $x^{i-1}\mathrm{Tr}(A^*A)^i$. Furthermore, one shows by induction that the n-th derivative of f_A is

$$(f_A)^{(n)} = f_A f_n$$

where f_n is a continuous function with polynomial growth.

An application of Taylor formula with integral remainder to the order $|\sigma|+1$ shows that:

$$f_A((d-2q)^{-1}) = \sum_{k=0}^{|\sigma|}\frac{f_A^{(k)}(0)}{k!(d-2q)^k} + \frac{1}{|\sigma|!(d-2q)^{|\sigma|+1}}\int_{t\in[0,1]}(1-t)^{|\sigma|}f_A(t(d-2q)^{-1})f_{|\sigma|+1}(t(d-2q)^{-1})dt$$

Therefore

$$\int_{||A||\leq\sqrt{d-2q}} A_{11}\ldots A_{qq}\overline{A_{1\sigma(1)}}\ldots\overline{A_{q\sigma(q)}}\det(1-A^*A/(d-2q))^{d-2q}dA = \sum_{k=0}^{|\sigma|}\int_{||A||\leq\sqrt{d-2q}}\frac{f_A^{(k)}(0)}{k!(d-2q)^k}dA + \int_{||A||\leq\sqrt{d-2q}}\int_{t\in[0,1]}(1-t)^{|\sigma|}f_A(t(d-2q)^{-1})f_{|\sigma|+1}(t(d-2q)^{-1})dtdA$$

By the polynomial growth property of f_n,

$$\int_{t\in[0,1]}(1-t)^{|\sigma|}f_A(t(d-2q)^{-1})f_{|\sigma|+1}(t(d-2q)^{-1})dtdA = O(d^{-|\sigma|+1})$$

By Lemmas 4.5.1 and 4.5.2, for any $k\in[0,|\sigma|-1]$, $l\in\mathbb{N}$,

$$\int_{||A||\leq\sqrt{d-2q}} A_{11}\dots A_{qq}\overline{A_{1\sigma(1)}}\dots\overline{A_{q\sigma(q)}}\frac{f_A^{(k)}(0)}{k!(d-2q)^k}dA = O(d^{-l})$$

and

$$A_{11}\dots A_{qq}\overline{A_{1\sigma(1)}}\dots\overline{A_{q\sigma(q)}}\int_{||A||\leq\sqrt{d-2q}}\frac{f_A^{(|\sigma|)}(0)}{|\sigma|!}dA = \mathrm{Moeb}(\sigma)+O(d^{-l})$$

□

Remark 5.1.

- This theorem furnishes an algorithm to compute asymptotics of functions on orthogonal groups, where no Schur-Weyl duality holds.
- We are not able so far to compute cumulants of polynomial functions on the unitary group. Yet, we believe that the method presented here could give a nicer combinatorial interpretation than that of Theorem 2.2.12 of chapter 2.

4.5.2 A uniform bound for Wg

This section is independent from the previous ones. Theorem 4.5.3 implies that for each σ, there exists a constant C such that for d large enough,

$$|\mathrm{Wg}(d,\sigma)|\leq d^{-q-|\sigma|}C \tag{4.5.4.3}$$

We are not able to give an explicit value for C and tell from which d on the above inequality holds, except in very specific cases.

In the following theorem we provide an answer to this question by weakening the asymptotics.

Theorem 4.5.4. *Let k, q and d be nonnegative integers such that $q^k \leq d$. Then there exists a constant C depending only on k such that for any σ,*

$$|\mathrm{Wg}(d,\sigma)| \leq Cd^{-q-|\sigma|(1-2/k)} \tag{4.5.4.4}$$

Proof. We recall, see [Ful97], that for any partition $\lambda \vdash q$ of the integer q,

$$s_{\lambda,d}(1) = \frac{\chi^\lambda(e)}{q!}\prod_{i=1}^{q}(d-\lambda_i)$$

where λ_i is an integer in $\{0,\dots,q-1\}$. Equivalently, Equation 4.5.4.1 becomes

$$\mathrm{Wg}(d,q,\sigma) = \frac{1}{d^q q!}\sum_{\lambda\vdash q}\frac{\chi^\lambda(e)\chi^\lambda(\sigma)}{\prod_{i=1}^{q}(1-\lambda_i/d)} \tag{4.5.4.5}$$

Consider the function

$$f_\lambda : z \to \prod_{i=1}^{q}(1-z\lambda_i)$$

This function is holomorphic in a neighborhood of zero. Moreover, since $q^2 \leq d$, we have, for any $z \leq q^{-2}$,

$$|f_\lambda(z)| \leq e \tag{4.5.4.6}$$

As a consequence, writing $f_\lambda(z) = 1 + \sum_{i\geq 1} a_{i,\lambda} z^i$, we obtain the Cauchy estimate

$$a_{i,\lambda} \leq q^{2i}$$

But Equation 4.5.4.1 implies that

$$\mathrm{Wg}(d,q,\sigma) = \frac{1}{d^q q!}\sum_{\lambda\vdash q}\chi^\lambda(e)\chi^\lambda(\sigma)(1+\sum_{i\geq 1} a_{i,\lambda}d^{-i})$$

Therefore the coefficient in d^{-q-i} has norm smaller than eq^{2i}. But we know that all coefficients are zero until $i = |\sigma|$, therefore

$$\mathrm{Wg}(\sigma,d) \leq q^{-d-|\sigma|}e(1+\frac{q^2}{d}+(\frac{q^2}{d})^2+\dots)q^{2|\sigma|}$$

For $d > 1$, and since $q^k \leq d$, this implies $q^{2|\sigma|} \leq d^{2|\sigma|/k}$. Furthermore $(1 + \frac{q^2}{d} + (\frac{q^2}{d})^2 + \dots)$ can bounded by an independent constant (5, for example). The result follows. □

Remark 5.2. In the case of σ being a one cycle permutation $(1 \ldots q)$, the explicit value of Wg is known, and one can show a much better estimate of the kind $2^q d^{-2q+1}$. Therefore, and according to computer computations it is tempting to conjecture that Wg is close to its first order non-zero value, i.e. that we have an uniform bound of the kind $2^{|\sigma|} d^{-q-|\sigma|}$ for $q = o(d)$.

Appendice A

Représentation de groupes classiques

Cette thèse fait appel à certains résultats élémentaires de théorie des groupes, ou bien à de la combinatoire (notamment de la combinatoire des probabilités libres) qui ne sont pas fréquemment utilisés en calcul des probabilités, matrices aléatoires ou algèbres d'opérateurs. L'objectif de cet appendice est de rassembler les éléments nécessaires à la lecture de cette thèse, quand ils ne sont pas déjà inclus dans les chapitres concernés. Les preuves ne sont généralement pas incluses, sauf quand elles sont inédites, ou bien qu'elles ont été jugées particulièrement instructives.

A.1 Représentation de groupes classiques

A.1.1 Algèbre semi-simple

Soit A une algèbre de dimension finie sur $\mathbb{C}$. On dit qu'un $\mathbb{C}$-espace vectoriel V est un *A-module à gauche* (resp. à droite) s'il est muni d'un morphisme (resp. morphisme antimultiplicatif) d'algèbres de A dans $\mathrm{End}(V)$.

En particulier A possède une structure de A-module à droite (resp. à gauche) sur lui-même donnée par $a \cdot b = ab$ (resp. $a \cdot b = ba$). Par défaut, on considère des modules à gauche. Pour un A-module V, un sous-espace vectoriel W est appelé un A-sous module si $\forall w \in W, \forall a \in A, a \cdot w \in W$. Un module V est dit *simple* si ses seuls sous-modules sont 0 et lui-même. Il est dit *semi-simple* si l'une des conditions équivalentes suivantes est vérifiée :

- (i) V est la somme d'une famille de modules simples.
- (ii) V est la somme directe d'une famille de modules simples.
- (iii) Tout sous-module W admet un sous-module supplémentaire W'.

Si V_1 et V_2 sont deux A-modules, une application linéaire T entre les deux est dite A-linéaire si elle commute à l'action de A. On note $\mathrm{End}_A(V_1, V_2)$ l'ensemble de ces applications. Si V est un A-module à gauche et un B-module à droite, on dit que c'est un *$A-B$-bimodule* si les actions de A et de B commutent. En particulier, A est un $A-A$-bimodule sur lui-même.

L'algèbre A est dite *semi-simple* ssi elle est semi-simple comme module à gauche sur elle-même. Dans ce cas, tout A-module simple V est équivalent à un sous-module de A. Tout A-module V de dimension finie est semi-simple

On a alors les deux théorèmes fondamentaux suivants, dont une preuve se trouve dans [Lan02].

Théorème A.1.1. *Toute algèbre semi-simple de dimension finie est isomorphe (en tant qu'algèbre et que bimodule) à une somme directe d'algèbres de matrices.*

Réciproquement, toute somme directe d'algèbres de matrices est semi-simple.

Théorème A.1.2 (Burnside). *Soient $A \subset B$ une inclusion d'algèbres semi-simples de dimension finie, A' le commutant de A dans B et A'' le commutant de A'. Alors $A = A''$.*

Soit G un groupe fini. L'algèbre du groupe $\mathbb{C}[G]$ est l'espace vectoriel engendré par $\{\lambda_g\}_{g\in G}$ et muni de la multiplication $\lambda_g\lambda_{g'} = \lambda_{gg'}$.

Proposition A.1.1. *Pour G un groupe fini, l'algèbre $\mathbb{C}[G]$ est semi-simple.*

Proof. En effet, soit V un $\mathbb{C}[G]$-module de dimension finie et $\langle\cdot,\cdot\rangle$ un produit scalaire sur V. Définissons le produit scalaire

$$\langle\langle u, v\rangle\rangle = \sum_{g\in G}\langle\lambda_g u, \lambda_g v\rangle$$

Cette forme bilinéaire est à nouveau un produit scalaire. De plus elle est invariante par G. Si W est un sous-module de V, alors son orthogonal l'est aussi, Q.E.D. □

On en déduit que $\mathbb{C}[G]$ n'a qu'un nombre fini de classes d'équivalences de modules de dimension finie.

Une *représentation* de G dans un espace vectoriel V est un morphisme de groupe de G dans $GL(V)$. Une représentation V de G munit V d'une structure de $\mathbb{C}[G]$-module à gauche. Réciproquement, une structure de $\mathbb{C}[G]$-module à gauche de V fournit une représentation de G. Deux représentations sont dites *équivalentes* (resp. *irréductibles*) si les modules qui leur sont associés sont isomorphes (resp. irréductibles).

On a alors comme conséquence du théorème A.1.1,

Proposition A.1.2.

- *Le nombre de classes de représentations irréductibles de G est égal à la dimension du centre de $\mathbb{C}[G]$*
- *La dimension de ce centre est égale au nombre de classes de conjugaison dans G.*

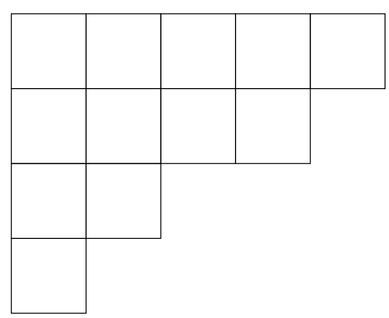
Soit $\rho : g \to GL(V)$ une représentation de G. On appelle *caractère* χ_ρ d'une représentation la fonction à valeurs complexes $\chi_\rho = \mathrm{Tr} \circ \rho$. Cette fonction est centrale, i.e. $\chi_\rho(gg') = \chi_\rho(g'g)$ pour tous g et $g' \in G$. A l'aide du théorème A.1.1, on peut montrer que deux représentations sont équivalentes ssi elles ont même caractère et que les caractères forment une base de l'espace vectoriel des fonctions centrales. En particulier, soit Irr l'ensemble des classes d'équivalences de représentations, et e l'élément neutre de G. On a

$$\#G\delta_{e,g} = \sum_{\lambda \in Irr} \chi^\lambda(e)\chi^\lambda(g)$$

Cette relation est connue sour le nom de *première formule d'orthogonalité pour les caractères.*

A.1.2 Représentations de $\mathcal{S}_q$

Rappelons qu'un *tableau de Young* de taille q est une suite décroissante d'entiers $l = (l_1, l_2, \dots)$ telle que $\sum_i l_i = q$. Par exemple, la partition $l = (4, 3, 2, 2, 1)$ est représentée par le tableau



L'ensemble des tableaux de Young de taille q est appelé $\mathcal{Y}_q$ et la réunion pour $q \geq 1$ des $\mathcal{Y}_q$ est appelée $\mathcal{Y}$.

Les classes de conjugaison de $\mathcal{S}_q$ admettent la description classique suivante en termes de diagrammes de Young : une permutation $\sigma \in \mathcal{S}_q$ est dans la classe $l \in \mathbb{Y}_q$ si et seulement si elle a un cycle contenant l_1 element(s), un deuxième contenant l_2 éléments, etc.

Par conséquent, il y a autant de représentations irréductibles de $\mathcal{S}_q$ que de tableaux de Young dans $\mathcal{Y}_q$. Une question naturelle est donc de chercher à associer de manière explicite à chaque tableau de Young une représentation irréductible de telle sorte que cette construction atteigne un élément de chaque classe de représentation irréductible.

La construction suivante répond à cette question, et nous renvoyons à [Ful97] ou [Mac95] pour les preuves. Soit l un tableau de Young. On choisit une identification de $\mathcal{S}_q$ et du groupe des permutations des cases de l. Soit L (resp. C) le sous-groupe de $\mathcal{S}_q$ laissant invariantes globalement les lignes (resp. les colonnes) du diagramme. Soit e l'élément de $\mathbb{C}[\mathcal{S}_q]$ défini par $e = \sum_{\sigma \in L} \lambda_\sigma$ et f défini par $f = \sum_{\sigma \in C} \varepsilon(f)\lambda_\sigma$, où ε est la signature. Alors $\mathbb{C}[\mathcal{S}_q]ef$ est un idéal à gauche minimal. De plus, deux tableaux de forme différente donnent lieu à deux représentations non-équivalentes de $\mathcal{S}_q$, et deux numérotations différentes d'un même diagramme donnent des représentations équivalentes.

Signalons que récemment, Okounkov et Vershik ([OV96]) développé une nouvelle approche de la représentation des groupes symétriques donnant en plus une preuve élémentaire de la règle d'induction.

A.1.3 Représentation de SU(d) et U(d)

Nous commençons par rappeler la description des représentations de $GL(d)$. Une représentation de $GL(d)$ dans un espace vectoriel V est dite *polynômiale* (resp. rationnelle, holomorphe) si, dans une base de $\mathrm{End}(V)$, chacune des coordonnées est polynômiale (resp. rationnelle, holomorphe). Il est aisé de voir que ces notions sont indépendantes de la base choisie. Nous ne nous intéresserons qu'à ce type de représentations.

Soit $H = \mathrm{diag}(x_1, \ldots, x_d)$ le sous-groupe diagonal et B le sous groupe de Borel des matrices triangulaires supérieures. Un vecteur v est dit *de poids* $\alpha = (\alpha_1, \ldots, \alpha_d)$ si $x \cdot v = x_1^{\alpha_1} \ldots x_d^{\alpha_d} v$ pour tous $x \in H$. On admet que toute représentation V de $GL(d)$ est somme directe d'espaces de poids

$$V = \bigoplus V_\alpha, V_\alpha = \{v \in V : x \cdot v = (\prod x_i^{\alpha_i}), x \in H\}$$

Un vecteur v dans une représentation V est dit *de plus haut poids* si $B \cdot v = \mathbb{C}^* \cdot v$. On montre qu'une représentation est irréductible ssi elle a un unique vecteur de plus haut poids à un scalaire près. Deux représentations irréductibles sont équivalentes si et seulement si leur vecteur de plus haut poids a le même poids. Soit $\{e_1, \ldots, e_n\}$ la base canonique du $GL(d)$-module $\mathbb{C}^d$. On définit de manière naturelle le produit tensoriel $\otimes$ et le produit alterné

$\wedge$ de $GL(d)$-modules. Pour $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$, soit E^λ la sous-représentation de poids λ de $(\mathbb{C}^d)^{\wedge\lambda_1} \otimes \ldots \otimes (\mathbb{C}^d)^{\wedge\lambda_d}$ de vecteur de plus haut poids $e_1^{\wedge\lambda_1} \otimes \ldots \otimes e_d^{\wedge\lambda_d}$. Soit D la représentation de dimension 1 donnée par le déterminant.

Théorème A.1.3. *Pour tout $\alpha = (\alpha_1, \ldots, \alpha_d)$ avec $\alpha_1 \geq \ldots \geq \alpha_n$, posons $k = \alpha_d$ et $\lambda_i = \alpha_i - d$. Il existe, à équivalence près, une unique représentation de $GL(n)$ de plus haut poids α. De plus, elle peut être réalisée comme $E^\lambda \otimes D^{\otimes k}$.*

En particulier, toute représentation de dimension finie de $GL(d)$ est rationnelle.

Théorème A.1.4. *Pour tout $\alpha = (\alpha_1, \ldots, \alpha_d)$ avec $\alpha_d = 0$, il existe, à équivalence près, une unique représentation de $SL(d)$ de plus haut poids α. De plus, elle peut être réalisée comme restriction à $SL(d)$ du $GL(d)$-module $E^\lambda \otimes D^{\otimes k}$.*

Soit E^λ une représentation irréductible de $GL(d)$. Son caractère, évalué sur une matrice $x = \mathrm{diag}(x_1, \ldots, x_d)$ de H, vaut

$$s_\lambda(x_1, \ldots, x_d) = \frac{\det(x_i^{\lambda_j + d - j})_{i,j}}{\det(x_i^{d-j})} \qquad \text{(A.1.A.1)}$$

Le fait que la représentation associée à λ soit polynômiale implique que $s_{d,\lambda}$ est un polynôme symétrique en les x_i.

La décomposition en espaces poids $V_\lambda = \oplus V_\alpha$ entraîne que chaque coefficient en les monômes élémentaires est positif. La formule A.1.A.1 est un cas particulier de la formule des caractères de Weyl (voir [BtD95]).

Les représentations continues (resp. polynômiales, rationnelles) irréductibles de $SU(d)$ (resp. $U(d)$) sont les mêmes que celles de $SL(d)$ (resp $GL(d)$), et s'obtiennent par restriction des représentations holomorphes (resp. polynômiales, rationnelles). Pour des preuves, on peut consulter [BtD95].

A.1.4 Dualité de Schur-Weyl

L'espace $(\mathbb{C}^d)^{\otimes q}$ est muni d'une structure de $\mathcal{S}_q$-module par prolongation par linéarité de l'action par permutation de $\mathcal{S}_q$ sur un tenseur élémentaire. De même, on le munit d'une structure de $GL(d)$-module par l'action diagonale.

Ces deux actions commutent, si bien que $(\mathbb{C}^d)^{\otimes q}$ devient un $\mathcal{S}_q \times GL(d)$-module. On dit d'un A-module V qu'il est sans multiplicité si sa décomposition en somme directe de modules irréductibles fait apparaître des modules deux-à-deux non-équivalents. De manière équivalente, $\mathrm{End}_A(V)$ est abélien.

Les représentations de $\mathcal{S}_q$ et de $GL(n)$ sont étroitement reliées par le théorème de dualité de Schur-Weyl.

Théorème A.1.5. *Le $\mathcal{S}_q \times GL(d)$-module $(\mathbb{C}^d)^{\otimes q}$ est sans multiplicité.*

Généralement les "utilisateurs" de cette formule renvoient à [Wey97], mais nous en donnons une démonstration plus concise et moderne inspirée de Howe ([How95]).

Preuve. Posons $V = \mathbb{C}^n$. Le groupe $GL(V)$ agit sur l'algèbre $\mathrm{End}_{\mathcal{S}_q}(V^{\otimes q})$ des endomorphismes de $V^{\otimes q}$ en tant que $\mathcal{S}_q$ module. Cette algèbre est $\mathrm{End}(V^{\otimes q})^{\mathcal{S}_q}$ des endomorphismes invariants par toute permutation de $\mathcal{S}_q$. On vérifie aisément qu'elle est isomorphe en tant que $GL(d)$-module à $(\mathrm{End}(V)^{\otimes q})^{\mathcal{S}_q}$ qui, par définition, est l'algèbre symétrique $S^q(\mathrm{End}(V))$.

Par ailleurs, $GL(V)$ engendre $\mathrm{End}(V)$ comme $\mathbb{C}$-espace vectoriel, et pour tout espace vectoriel W, et $S^q(W)$ est engendré par les $w \otimes \ldots \otimes w$, $w \in W$ en tant qu'espace vectoriel.

Cela prouve donc que $\mathrm{End}_{\mathcal{S}_q}$ est engendré en tant que $\mathbb{C}$-espace vectoriel par l'image de $GL(d)$ dans $\mathrm{End}_{\mathbb{C}}(V^{\otimes q})$. Donc, par le théorème du bicommutant, le morphisme $\mathbb{C}[\mathcal{S}_q] \to \mathrm{End}_{GL(V)}(V^{\otimes q})$ est surjectif. □

Ce théorème implique que à chaque $s_{d,\lambda}$ on peut associer un caractère $\chi^{\lambda'}$ de $\mathcal{S}_q$ tel que le caractère du $\mathcal{S}_q \times GL(d)$-module $(\mathbb{C}^d)^{\otimes q}$ soit $\sum s_{d,\lambda}\chi^{\lambda'}$. En fait, on peut montrer que compte tenu des définitions des caractères, $\lambda = \lambda'$. Nous illustrons ce résultat par un calcul élémentaire mais fondamental dans cette thèse :

Proposition A.1.3. *Nous avons*

$$E((\mathrm{Tr}(XUYU^*))^n) = \sum_{\lambda \vdash q} \frac{s_{d,\lambda}(X) s_{d,\lambda}(Y) \chi^\lambda(e)}{s_{d,\lambda}(1)} \qquad \text{(A.1.A.2)}$$

A.2 Méthodes combinatoires en matrices aléatoires

Le premier grand résultat en la matière remonte à Wigner [Wig58] et nous donnons une idée de sa méthode.

Théorème A.2.1. *Pour tout d, soit $X_d = (x_{ij})$ une matrice aléatoire hermitienne telle que les $\{x_{ij}, i \geq j\}$ soient iid, centrées, et aient un moment d'ordre 2 égal à $1/d$. Soit μ_d la mesure de comptage des valeurs propres. Pour tout $q \in \mathbb{N}$,*

$$E(\mu_d(x^q)) \to \frac{2}{\pi} \int_{[-1,1]} \sqrt{1-x^2} x^q dx$$

La loi $\frac{2}{\pi} 1_{[-1,1]} \sqrt{1-x^2} x^q dx$ est connue sous le nom de *loi du demi-cercle.* Dans certains cas particuliers (en particulier dans le cas des ensembles gaussiens), des versions beaucoup plus fortes de cette convergence ont été démontrées : la mesure μ_d converge presque sûrement étroitement vers la loi du demi-cercle. Nous renvoyons à [Meh91] pour des énoncés et des preuves, et ne donnons ici qu'une preuve dans l'esprit de l'article de Wigner.

Preuve. Écrivons

$$E(\mu_d(x^{2q})) = E(\mathrm{Tr}(X^{2q})) = \sum_{i_1, i_2, \ldots, i_q} E(x_{i_1 i_2} x_{i_2 i_3} \ldots x_{i_{2q} i_1})$$

Compte tenu de l'indépendance et de la symétrie des variables, pour qu'un élément de la somme soit non-nul, il faut que à chaque (i_j, i_{j+1}) corresponde un couple (i_k, i_{k+1}), $k \neq j$ tel que $i_{j+1} = i_k$ et $i_{k+1} = i_j$. On dit qu'il doit être possible d'*apparier* les couples consécutifs d'indices. En particulier, une puissance impaire est d'espérance nulle. Lorsque $d \to \infty$, on peut se restreindre, à un $o(1)$ près, à une somme sur les familles d'indices $(i_1, i_2, \ldots, i_q)$ appariables telles que les i_j prennent q valeurs distinctes. Un argument combinatoire prouve que le nombre de tels couples est

$$c_q d^q$$

où c_q est le nombre de Catalan

$$c_q = \frac{(2q)!}{q!(q+1)!}$$

□

Il est intéressant de remarquer que cette preuve n'utilise que le fait que les matrices sont hermitiennes à entrées i.i.d. symétriques et admettent des moments de tous ordres, le résultat final ne dépendant que de la variance de la loi des entrées. On appelle de telles matrices *Matrices de Wigner*. La technique de preuve utilisée ci-dessus s'appelle la méthode des moments.

L'exemple fondamental de matrice invariante en loi par rotation aléatoire, le GUE, admet un rapport étroit avec la combinatoire des surfaces.

Théorème A.2.2 (Wick). *Soit H un GUE, s k_i-gones orientés $k_1, \ldots, k_s$ avec un sommet fixé, et S une surface compacte orientable de dimension 2 et de genre $\chi(S)$ On appelle $|Map_S(k_1, \ldots, k_s)|$ le nombre de manières de coller ces polygones de manière consistante avec leur orientation en une surface isomorphe à S. On a alors*

$$E(\prod_{i=1}^{s} \mathrm{Tr}(H^{k_i})) = \sum_S d^{\sum_i k_i/2-s-\chi(S)}|Map_S(k_1, \ldots, k_s)| \qquad \text{(A.2.A.1)}$$

où la somme est sur les classes d'homéomorphismes de S.

Pour la preuve de ce théorème, on peut consulter [Zvo97]. Ce théorème a permis de résoudre des problèmes de comptage de cartes en genre quelconque, voir [HZ86].

A.3 Matrices aléatoires et Probabilités libres

Rappelons qu'un *espace de probabilités non commutatif* est une C^* -algèbre A sur $\mathbb{C}$ avec unité, munie d'une forme linéaire continue ϕ telle que $\phi(Id) = 1$ et que ϕ soit une trace ($\phi(ab) = \phi(ba)$) positive ($\phi(aa^*) \geq 0$). Les éléments de A sont appelés *variables aléatoires non-commutatives* .

Soient $(A_i, i \in I)$ des sous-algèbres unitaires de A. On dit que les A_i sont *libres* si pour tout $n \in \mathbb{N}$, pour tous $a_1, ..., a_n$ tels que $a_i \in A_{i_j}$ et $\phi(a_i) = 0$, on a $\phi(a_1...a_n) = 0$ dès que $i_1 \neq i_2$, $i_2 \neq i_3$,..., $i_{n-1} \neq i_n$.

On dit que des variables $a_1, ..., a_n$ sont libres si les $*$-algèbres qu'elles engendrent sont libres. L'algèbre $\mathbb{C}.Id$ est libre avec toute sous-algèbre de A.

Soit $(a_i, i \in I)$ une famille de variables aléatoires non commutatives (ie d'éléments de A). Leur *loi jointe* est l'application linéaire μ de l'espace $\mathbb{C}\langle X_i, i \in I\rangle$ des polynômes non commutatifs dans $\mathbb{C}$ qui, en P vaut $\phi(P((a_i)_{i \in I}))$.

Dans le cas d'une seule variable autoadjointe, la distribution est uniquement déterminée par les moments $\phi(a^k)$, et il existe alors une unique mesure de probabilité μ_a sur $\mathbb{R}$ à support compact telle que $\phi(a^k) = \int_{\mathbb{R}} x^k d\mu_a$. C'est la mesure spectrale associée à a.

Étant données $\mu, \mu' : \mathbb{C}[X] \to \mathbb{C}$, il existe un espace de probabilités non commutatif $(\mathcal{A}, \phi)$ et a, $a' \in \mathcal{A}$ tels que a, a' soient libres dans $\mathcal{A}$ et aient pour lois respectives μ et μ'. On définit alors la convolée libre additive $\mu \boxplus \mu'$ (resp. multiplicative $\mu \boxtimes \mu'$) comme la loi de la variable $a + a'$ (resp. de aa'). Ces deux opérations sont bien définies. De plus, elles sont associatives et commutatives.

Une famille de variables aléatoires $(a_{(i,d)}, (i,d) \in I \times \mathbb{N})$, où $a_{(i,d)}$ est dans un espace de probabilités (A_d, ϕ_d) est dite avoir une *distribution limite* si la famille des distributions de $(a_{(i,d)}, (i,d) \in I \times \mathbb{N})$ dans (A_d, ϕ_d) converge simplement pour $d \to \infty$.

Le théorème suivant est le premier à avoir justifié l'introduction des matrices aléatoires en probabilités libres.

Théorème A.3.1 (Voiculescu). *Soient X_i^d des variables matricielles indépendantes suivant la loi du GUE et D_i^d des matrices diagonales $d \times d$ uniformément bornées en d admettant une loi jointe. Alors les familles $\{X_i^d, D_i^d\}$ sont asymptotiquement libres.*

Pour une preuve et d'autres références, on pourra consulter [VDN92]. Le résultat ci-dessous permet de linéariser la convolution libre additive et multiplicative :

Théorème A.3.2 (Voiculescu).

- *Si A est une variable L^∞, la fonction holomorphe $z \to E((z-A)^{-1})$ est inversible, d'inverse G dans un voisinage de ∞. La fonction $R_A(z) = G(z) - z^{-1} = \sum_{q \geq 0} k_{q+1}(A) z^q$ est telle que si A et A' sont libres, alors*

$$R_{A+A'} = R_A + R_{A'}$$

- *De même, soit $\psi_A(z) = \sum_{q=1}^{\infty} E(A^q) z^q$ et χ_A l'inverse formel de ψ_a, tel que $\psi_A(\chi_A(z)) = z$. Alors, $S_A(z) = \chi_A(z) z^{-1}(1+z)$ est telle que si A et A' sont libres, alors*

$$S_{AA'} = S_A S_{A'}$$

Pour démontrer le théorème 2.4.5 nous avons besoin de la théorie des cumulants non-croisés de Speicher, dont nous rappelons les grandes lignes. Nous renvoyons à [Spe98] pour plus de détails. Soit $V = \{V_1, \dots, V_k\}$ une partition de $\{1, 2, \dots, n\}$. Elle est dite *non croisée* si il n'existe pas (a, b, i, j, k, l) tels que $V_b \neq V_a$, $i < j < k < l$, $i, k \in V_a$ et $j, l \in V_b$. Soit $NC(n)$ l'ensemble des partitions non croisées de n. Cet ensemble est muni de l'ordre partiel $V \leqslant V'$ ssi pour tout i, il existe i' tel que $V_i \subset V'_{i'}$.

On injecte les partitions non croisées de $NC(n)$ dans $\mathcal{S}_n$ en associant à V la permutation de cycles V_i qui à chaque élément de V_i associe le plus petit élément de V_i suivant *mod n*.

Pour $\pi \in NC(n)$, on note

$$\phi[\pi](a_1, ..., a_n) = \prod_{i \in [1,k]} \phi(a_{i_1} ... a_{i_{j_i}})$$

où les éléments de V_i sont $i_1, ..., i_{j_i}$ pris dans l'ordre croissant, et $\{V_1, .., V_k\}$ est la partition associée à π.

Les *cumulants non-croisés* sont les formes n-linéaires

$$R_n[\pi](a_1, ..., a_n) = \sum_{\pi' \leqslant \pi} \phi[\pi'](a_1, ..., a_n) Moeb(\pi')$$

où $Moeb(\pi') = \prod_{i \in [1,k]} c_{Card(V_i)-1} (-1)^{Card(V_i)-1}$, avec $c_n = \frac{(2n)!}{n!(n+1)!}$ le n-ème nombre de Catalan.

Théorème A.3.3 (Speicher).
- *Si $(\mathcal{B}_i)_{i \in I}$ sont des sous-algèbres libres de $\mathcal{A}$, et $a_1, .., a_n \in \mathcal{A}$ tels que a_j appartient à un $\mathcal{B}_{i_j}$ pour tout $j \in [1, n]$. Alors $R(a_1, ..., a_n) = 0$ dès qu'il existe j et k tels que $i_j \neq i_k$.*
- *On a la formule d'inversion*

$$\phi^n[\pi](a_{i_1} ... a_{i_n}) = \Sigma_{\pi' \leq \pi} R_n[\pi'](a_{i_1}, ..., a_{i_n})$$

Appendice B

Calculs effectifs de traces

Dans cet appendice, nous reproduisons le code Maple (muni de son Package ACE) que nous avons utilisé pour faire des calculs de explicites de trace sur le groupe unitaire dans le chapitre 2, et pour formuler ou vérifier expérimentalement nos résultats/conjectures. Le code n'a aucune prétention d'être programmé de manière optimale.

B.1 Code

Les partitions de n sont contenues dans la liste $ListPart(n)$, et sont numérotées dans l'ordre lexicographique par $ListPart(n)[k]$. La procédure ci-dessous permet de calculer $Z(k,n)$ le nombre de partitions dans la classe de la partition $ListPart(n)[k]$. Notons que nous passons par de l'analyse de Fourier pour des raisons de confort de programmation dans la suite (nous voulons disposer d'une numérotation des partitions), car la formule explicite est très simple.

```
Z := proc (k,n) local i,a; option remember; a:=0;
  for i from 1 to ListPart(n,'nb') do
    a:=a+SgCharTable(n,ListPart(n)[i],ListPart(n)[k])^2 od;
RETURN( (n!)/a ) end:
```

La procédure ci-dessous permet de calculer $sl(k,n) = s_{\lambda,d}(1)$, où $\lambda = ListPart(n)[k]$. La variable formelle de dimension est x.

```
sl := proc (k,n) local i,a; option remember; a:=0;
  for i from 1 to ListPart(n,'nb') do
    a:=a+Z(i,n)*SgCharTable(n,ListPart(n)[k],ListPart(n)[i])
              *x^(Part2Conjugate(ListPart(n)[i])[1]) od;
RETURN( a/(n!) ) end:
```

La procédure ci-dessous calcule $modb(k,n)$, qui est la fonction $\mathrm{Wg}(d,\sigma)$ avec $\sigma \in \mathcal{S}_n$ dans la classe de la partition $\lambda = ListPart(n)[k]$ de n.

```
moeb :=proc (k,n) local i,a; option remember; a:=0;
 for i from 1 to ListPart(n,'nb') do
  a:=a+(SgCharTable
      (n,ListPart(n)[i],ListPart(n)[ListPart(n,'nb')])^2)
    *SgCharTable(n,ListPart(n)[i],ListPart(n)[k])/sl(i,n) od;
RETURN( simplify(a)/(n!)^2 ) end:
```

Notons que le temps de calcul et la mémoire requis augmentent très vite avec la taille de n. Avec cet algorithme, on ne peut pas dépasser $n = 9$ ou 10 pour *moeb* dans des temps raisonnables.

B.2 Résultats numériques

$$\mathrm{Wg}(d,[1]) = d^{-1} \quad \mathrm{Wg}(d,[2]) = -\frac{1}{d\,(d+1)\,(d-1)}$$

$$\mathrm{Wg}(d,[1,1]) = \frac{1}{(d+1)\,(d-1)} \quad \mathrm{Wg}(d,[3]) = 2\,\frac{1}{(d-1)\,(d-2)\,(d+1)\,(d+2)\,d}$$

$$\mathrm{Wg}(d,[2,1]) = -\frac{1}{(d-1)\,(d-2)\,(d+1)\,(d+2)}$$

$$\mathrm{Wg}(d,[1,1,1]) = \frac{d^2-2}{(d-1)\,(d-2)\,(d+1)\,(d+2)\,d}$$

$$\mathrm{Wg}(d,[4]) = -5\,\frac{1}{(d-1)\,(d-2)\,(d-3)\,(d+1)\,(d+2)\,(d+3)\,d}$$

$$\mathrm{Wg}(d,[3,1]) = \frac{2\,d^2-3}{(d-1)\,(d-2)\,(d-3)\,(d+1)\,(d+3)\,(d+2)\,d^2}$$

$$\mathrm{Wg}(d,[2,2]) = \frac{d^2+6}{(d-1)\,(d-2)\,(d-3)\,(d+1)\,(d+3)\,(d+2)\,d^2}$$

$$\mathrm{Wg}(d,[2,1,1]) = -\frac{1}{(d-1)\,(d-3)\,(d+1)\,(d+3)\,d}$$

$$\mathrm{Wg}(d,[1,1,1,1]) = \frac{d^4-8\,d^2+6}{(d-1)\,(d-2)\,(d-3)\,(d+1)\,(d+2)\,(d+3)\,d^2}$$

$$\mathrm{Wg}(d,[5]) = 14\,\frac{1}{(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+4)\,d}$$

$$\mathrm{Wg}(d,[4,1]) = -\frac{5\,d^2-24}{(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+4)\,(d+3)\,d^2}$$

$$\mathrm{Wg}(d,[3,2]) = -2\,\frac{d^2+12}{(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+4)\,d^2}$$

$$\mathrm{Wg}(d,[3,1,1]) = 2\,\frac{1}{d\,(d-1)\,(d-2)\,(d-4)\,(d+1)\,(d+2)\,(d+4)}$$

$$\mathrm{Wg}(d,[2,2,1]) = \frac{d^2-2}{d\,(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+4)\,(d+3)}$$

$$\mathrm{Wg}(d,[2,1,1,1]) = -\frac{\left(d^2-12\right)\left(d^2-2\right)}{(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+4)\,d^2}$$

$$\mathrm{Wg}(d,[1,1,1,1,1]) = \frac{d^4-20\,d^2+78}{d\,(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+4)}$$

$$\mathrm{Wg}(d,[6]) = -42\,\frac{1}{(d-5)\,(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+4)\,(d+5)\,d}$$

$$\mathrm{Wg}(d,[5,1]) = 14\,\frac{d^2-10}{(d-5)\,(d-1)\,(d-2)\,(d-3)\,(d-4)\,(d+1)\,(d+2)\,(d+3)\,(d+5)\,(d+4)\,d^2}$$

$$\mathrm{Wg}(d,[4,2]) = 5\,\frac{d^4+15\,d^2+8}{(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d-4)\,(d+1)^2\,(d+2)\,(d+3)\,(d+4)\,(d+5)\,d^2}$$

$$\mathrm{Wg}(d,[4,1,1]) = -\frac{5\,d^2-13}{d\,(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d+1)^2\,(d+2)\,(d+3)\,(d+5)}$$

$$\mathrm{Wg}(d,[3,3]) = 4\,\frac{d^4+29\,d^2-90}{(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d-4)\,(d+1)^2\,(d+2)\,(d+3)\,(d+4)\,(d+5)\,d^2}$$

$$\mathrm{Wg}(d,[3,2,1]) = -\frac{2\,d^2+13}{d\,(d-5)\,(d-1)^2\,(d-2)\,(d-4)\,(d+1)^2\,(d+2)\,(d+5)\,(d+4)}$$

$$\mathrm{Wg}(d,[3,1,1,1]) = \frac{2\,d^6-51\,d^4+229\,d^2-60}{(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d-4)\,(d+1)^2\,(d+2)\,(d+3)\,(d+4)\,(d+5)\,d^2}$$

$$\mathrm{Wg}(d,[2,2,2]) = -\frac{d^4+d^2+358}{d\,(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d-4)\,(d+1)^2\,(d+2)\,(d+3)\,(d+4)\,(d+5)}$$

$$\mathrm{Wg}(d,[2,2,1,1]) = \frac{d^4-3\,d^2+10}{(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d+1)^2\,(d+2)\,(d+3)\,(d+5)\,d^2}$$

$$\mathrm{Wg}(d,[2,1,1,1,1]) = -\frac{d^4-24\,d^2+38}{d\,(d-5)\,(d-1)^2\,(d-2)\,(d-4)\,(d+1)^2\,(d+2)\,(d+4)\,(d+5)}$$

$$\mathrm{Wg}(d,[1,1,1,1,1,1]) = \frac{d^8-41\,d^6+458\,d^4-1258\,d^2+240}{(d-5)\,(d-1)^2\,(d-2)\,(d-3)\,(d-4)\,(d+1)^2\,(d+2)\,(d+3)\,(d+4)\,(d+5)\,d^2}$$

Bibliographie

[AFL82] Luigi Accardi, Alberto Frigerio, and John T. Lewis. Quantum stochastic processes. *Publ. Res. Inst. Math. Sci.*, 18(1):97–133, 1982.

[BAG97] G. Ben Arous and A. Guionnet. Large deviations for Wigner's law and Voiculescu's non-commutative entropy. *Probab. Theory Related Fields*, 108(4):517–542, 1997.

[BD95] P. Biane and R. Durrett. *Lectures on probability theory.* Springer-Verlag, Berlin, 1995. Lectures from the Twenty-third Saint-Flour Summer School held August 18–September 4, 1993, Edited by P. Bernard.

[BDJ00] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geom. Funct. Anal.*, 10(4):702–731, 2000.

[BG80] E. Brézin and David J. Gross. The external field problem in the large N limit of QCD. *Phys. Lett. B*, 97(1):120–124, 1980.

[Bia] Philippe Biane. Free probability for probabilists. MSRI Preprint 1998-040.

[Bia90] Philippe Biane. Marches de Bernoulli quantiques. In *Séminaire de Probabilités, XXIV, 1988/89*, pages 329–344. Springer, Berlin, 1990.

[Bia91a] Philippe Biane. Quantum random walk on the dual of su(n). *Probab. Theory Related Fields*, 89(1):117–129, 1991.

[Bia91b] Philippe Biane. Some properties of quantum Bernoulli random walks. In *Quantum probability & related topics*, pages 193–203. World Sci. Publishing, River Edge, NJ, 1991.

[Bia92a] Ph. Biane. Équation de Choquet-Deny sur le dual d'un groupe compact. *Probab. Theory Related Fields*, 94(1):39–51, 1992.

[Bia92b] Philippe Biane. Minuscule weights and random walks on lattices. In *Quantum probability & related topics*, pages 51–65. World Sci. Publishing, River Edge, NJ, 1992.

[Bia94] Philippe Biane. Théorème de Ney-Spitzer sur le dual de su(2). *Trans. Amer. Math. Soc.*, 345(1):179–194, 1994.

[Bia97] Philippe Biane. Some properties of crossings and partitions. *Discrete Math.*, 175(1-3):41–53, 1997.

[Bia98] Philippe Biane. Representations of symmetric groups and free probability. *Adv. Math.*, 138(1):126–181, 1998.

[BJ02] Philippe Bougerol and Thierry Jeulin. Paths in weyl chambers and random matrices. 2002.

[BMS00] Mireille Bousquet-Mélou and Gilles Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[BOO00] Alexei Borodin, Andrei Okounkov, and Grigori Olshanski. Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13(3):481–515 (electronic), 2000.

[BRT81] Richard Brower, Paolo Rossi, and Chung I Tan. The external field problem for QCD. *Nuclear Phys. B*, 190(4, FS3):699–718, 1981.

[BS93] Saad Baaj and Georges Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.

[BS98] Philippe Biane and Roland Speicher. Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields*, 112(3):373–409, 1998.

[BtD95] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[CC02] M. Capitaine and M. Casalis. Asymptotic freeness by generalized moments for gaussian and wishart matrices. applications to beta random matrices. November 2002.

[CD] Thierry Cabanal-Duvillard. Probabilités libres et calcul stochastique. application aux grandes matrices aléatoires.

[CD60] Gustave Choquet and Jacques Deny. Sur l'équation de convolution $\mu = \mu * \sigma$. *C. R. Acad. Sci. Paris*, 250:799–801, 1960.

[CI91] Li-Chen Chen and Mourad E. H. Ismail. On asymptotics of Jacobi polynomials. *SIAM J. Math. Anal.*, 22(5):1442–1449, 1991.

[Col02] Benoît Collins. Moments and cumulants of polynomial random variables on unitary groups, the itzykson-zuber integral and free probability, 6 Jun 2002.

[DEL92] Persi W. Diaconis, Morris L. Eaton, and Steffen L. Lauritzen. Finite de Finetti theorems in linear models and multivariate analysis. *Scand. J. Statist.*, 19(4):289–315, 1992.

[Dix64] Jacques Dixmier. *Les C^*-algèbres et leurs représentations*. Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.

[Dix69] Jacques Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*. Gauthier-Villars Éditeur, Paris, 1969. Deuxième édition, revue et augmentée, Cahiers Scientifiques, Fasc. XXV.

[DZ83] Jean-Michel Drouffe and Jean-Bernard Zuber. Strong coupling and mean field methods in lattice gauge theories. *Phys. Rep.*, 102(1-2):1–119, 1983.

[ER94] Edward G. Effros and Zhong-Jin Ruan. Discrete quantum groups. I. The Haar measure. *Internat. J. Math.*, 5(5):681–723, 1994.

[FH91] William Fulton and Joe Harris. *Representation theory*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[For02] Peter Forrester. *Log-gases and Random matrices, Chapter 2.* http://www.ms.unimelb.edu.au/ matpjf/matpjf.html, 2002.

[Ful97] William Fulton. *Young tableaux.* Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[Ge98] Liming Ge. Applications of free entropy to finite von Neumann algebras. II. *Ann. of Math. (2)*, 147(1):143–157, 1998.

[GZ02] Alice Guionnet and Ofer Zeitouni. Large deviations asymptotics for spherical integrals. *J. Funct. Anal.*, 188(2):461–515, 2002.

[HI98] Fumio Hiai and Masaki Izumi. Amenability and strong amenability for fusion algebras with applications to subfactor theory. *Internat. J. Math.*, 9(6):669–722, 1998.

[How95] Roger Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.

[HT98] U. Haagerup and S. Thorbjørnsen. Random matrices with complex gaussian entries, 1998.

[HT99] U. Haagerup and S. Thorbjørnsen. Random matrices and K-theory for exact C^*-algebras. *Doc. Math.*, 4:341–450 (electronic), 1999.

[HZ86] J. Harer and D. Zagier. The Euler characteristic of the moduli space of curves. *Invent. Math.*, 85(3):457–485, 1986.

[IZ80] C. Itzykson and J. B. Zuber. The planar approximation. II. *J. Math. Phys.*, 21(3):411–421, 1980.

[Izu98] Masaki Izumi. Actions of compact quantum groups on operator algebras. *Sūrikaisekikenkyūsho Kōkyūroku*, (1024):55–60, 1998. Profound development of operator algebras (Japanese) (Kyoto, 1997).

[Izu99] Masaki Izumi. Actions of compact quantum groups on operator algebras. In *XIIth International Congress of Mathematical Physics (ICMP '97) (Brisbane)*, pages 249–253. Internat. Press, Cambridge, MA, 1999.

[Izu00] Masaki Izumi. Non commutative poisson boundaries and compact quantum group actions, November 29, 2000.

[Izu02] Masaki Izumi. Non-commutative Poisson boundaries and compact quantum group actions. *Adv. Math.*, 169(1):1–57, 2002.

[Joh00] Kurt Johansson. Transversal fluctuations for increasing subsequences on the plane. *Probab. Theory Related Fields*, 116(4):445–456, 2000.

[Joh02] Kurt Johansson. Non-intersecting paths, random tilings and random matrices. *Probab. Theory Related Fields*, 123(2):225–280, 2002.

[Jon83] V. F. R. Jones. Index for subfactors. *Invent. Math.*, 72(1):1–25, 1983.

[Kon92] Yuji Konishi. A note on actions of compact matrix quantum groups on von Neumann algebras. *Nihonkai Math. J.*, 3(1):23–29, 1992.

[KOR02] Wolfgang König, Neil O'Connell, and Sébastien Roch. Noncolliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electron. J. Probab.*, 7:no. 5, 24 pp. (electronic), 2002.

[KSK76] John G. Kemeny, J. Laurie Snell, and Anthony W. Knapp. *Denumerable Markov chains.* Springer-Verlag, New York, second edition, 1976. With a chapter on Markov random fields, by David Griffeath, Graduate Texts in Mathematics, No. 40.

[Lan02] Serge Lang. *Algebra.* Springer-Verlag, New York, third edition, 2002.

[Led02] Michel Ledoux. Differential operators and spectral distributions of invariant ensembles from the classical orthogonal polynomials part i: the continuous case. November 2002.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials.* The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Mat94] A. Matytsin. On the large-N limit of the Itzykson-Zuber integral. *Nuclear Phys. B*, 411(2-3):805–820, 1994.

[Meh91] Madan Lal Mehta. *Random matrices.* Academic Press Inc., Boston, MA, second edition, 1991.

[Mey86] P.-A. Meyer. Éléments de probabilités quantiques. I–V. In *Séminaire de Probabilités, XX, 1984/85*, pages 186–312. Springer, Berlin, 1986.

[Mey87] P.-A. Meyer. Éléments de probabilités quantiques. VI–VIII. In *Séminaire de Probabilités, XXI*, pages 33–78. Springer, Berlin, 1987.

[Mey88a] P.-A. Meyer. Eléments de probabilités quantiques. IX. Calculs antisymétriques et "supersymétriques" en probabilités. In *Séminaire de Probabilités, XXII*, pages 101–123. Springer, Berlin, 1988.

[Mey88b] P.-A. Meyer. Éléments de probabilités quantiques. X. Calculs avec des noyaux discrets. In *Séminaire de Probabilités, XXII*, pages 124–128. Springer, Berlin, 1988.

[Mey89a] P.-A. Meyer. Éléments de probabilités quantiques. X [bis]. Approximation de l'oscillateur harmonique (d'après L. Accardi et A. Bach). In *Séminaire de Probabilités, XXIII*, pages 175–182. Springer, Berlin, 1989.

[Mey89b] P.-A. Meyer. Éléments de probabilités quantiques. XI. Caractérisation des lois de Bernoulli quantiques d'après K. R. Parthasarathy. In *Séminaire de Probabilités, XXIII*, pages 183–185. Springer, Berlin, 1989.

[NS66] P. Ney and F. Spitzer. The Martin boundary for random walk. *Trans. Amer. Math. Soc.*, 121:116–132, 1966.

[NT02] S. Neshveyev and L. Tuset. The martin boundary of a discrete quantum group, 2002.

[Oko00] Andrei Okounkov. Random matrices and random permutations. *Internat. Math. Res. Notices*, (20):1043–1095, 2000.

[Ol'90] G.I. Ol'shanskij. Unitary representations of infinite dimensional pairs (g,k) and the formalism of r. howe. *Representation of Lie groups and related topics, Adv. Stud. Contemp. Math. 7*, pages 269–463, 1990.

[OV96] Andrei Okounkov and Anatoly Vershik. A new approach to representation theory of symmetric groups. *Selecta Math. (N.S.)*, 2(4):581–605, 1996.

[OY02] Neil O'Connell and Marc Yor. A representation for non-colliding random walks. *Electron. Comm. Probab.*, 7:1–12 (electronic), 2002.

[OZ84] K. H. O'Brien and J.-B. Zuber. A note on U(N) integrals in the large N limit. *Phys. Lett. B*, 144(5-6):407–408, 1984.

[Pis00] Gilles Pisier. *Espaces d'Opérateurs.* Notes de Cours du Centre Émile Borel, 13 septembre 1999 - 11 février 2000.

[Pod87] P. Podleś. Quantum spheres. *Lett. Math. Phys.*, 14(3):193–202, 1987.

[Rai97] E. M. Rains. Combinatorial properties of Brownian motion on the compact classical groups. *J. Theoret. Probab.*, 10(3):659–679, 1997.

[Rai98] E. M. Rains. Increasing subsequences and the classical groups. *Electron. J. Combin.*, 5(1):Research Paper 12, 9 pp. (electronic), 1998.

[Rev84] D. Revuz. *Markov chains.* North-Holland Publishing Co., Amsterdam, second edition, 1984.

[Rot64] Gian-Carlo Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:340–368 (1964), 1964.

[Sam80] Stuart Samuel. U(N) integrals, $1/N$, and the De Wit-'t Hooft anomalies. *J. Math. Phys.*, 21(12):2695–2703, 1980.

[Sos99] A. Soshnikov. Universality at the edge of the spectrum in wigner random matrices, 1999.

[Sos00] A. Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55(5(335)):107–160, 2000.

[Spe94] Roland Speicher. Multiplicative functions on the lattice of non-crossing partitions and free convolution. *Math. Ann.*, 298(4):611–628, 1994.

[Spe98] Roland Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. *Mem. Amer. Math. Soc.*, 132(627):x+88, 1998.

[SS98] Michael Schürmann and Michael Skeide. Infinitesimal generators on the quantum group $\mathrm{su}_q(2)$. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 1(4):573–598, 1998.

[Sze75] Gábor Szegő. *Orthogonal polynomials.* American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[Tak79] Masamichi Takesaki. *Theory of operator algebras. I.* Springer-Verlag, New York, 1979.

[TW99] Craig A. Tracy and Harold Widom. Random unitary matrices, permutations and Painlevé. *Comm. Math. Phys.*, 207(3):665–685, 1999.

[VDN92] D. V. Voiculescu, K. J. Dykema, and A. Nica. *Free random variables.* American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

[VK92] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 3.* Kluwer Academic Publishers Group, Dordrecht, 1992. Classical and quantum groups and special functions, Translated from the Russian by V. A. Groza and A. A. Groza.

[Voi93] Dan Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. I. *Comm. Math. Phys.*, 155(1):71–92, 1993.

[Voi94] Dan Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. II. *Invent. Math.*, 118(3):411–440, 1994.

[Voi95] Dan Voiculescu. Free probability theory: random matrices and von Neumann algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 227–241, Basel, 1995. Birkhäuser.

[Voi98] Dan Voiculescu. A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Internat. Math. Res. Notices*, (1):41–63, 1998.

[Voi00] Dan Voiculescu. Lectures on free probability theory. In *Lectures on probability theory and statistics (Saint-Flour, 1998)*, pages 279–349. Springer, Berlin, 2000.

[Was88] Antony Wassermann. Coactions and Yang-Baxter equations for ergodic actions and subfactors. In *Operator algebras and applications, Vol. 2*, pages 203–236. Cambridge Univ. Press, Cambridge, 1988.

[Wei78] Don Weingarten. Asymptotic behavior of group integrals in the limit of infinite rank. *J. Mathematical Phys.*, 19(5):999–1001, 1978.

[Wey97] Hermann Weyl. *The classical groups.* Princeton University Press, Princeton, NJ, 1997. Their invariants and representations, Fifteenth printing, Princeton Paperbacks.

[Wig58] Eugene P. Wigner. On the distribution of the roots of certain symmetric matrices. *Ann. of Math. (2)*, 67:325–327, 1958.

[Wor87a] S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111(4):613–665, 1987.

[Wor87b] S. L. Woronowicz. Twisted su(2) group. An example of a noncommutative differential calculus. *Publ. Res. Inst. Math. Sci.*, 23(1):117–181, 1987.

[Wor88] S. L. Woronowicz. Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted su(N) groups. *Invent. Math.*, 93(1):35–76, 1988.

[Xu97] Feng Xu. A random matrix model from two-dimensional Yang-Mills theory. *Comm. Math. Phys.*, 190(2):287–307, 1997.

[ZJ98] P. Zinn-Justin. Universality of correlation functions of Hermitian random matrices in an external field. *Comm. Math. Phys.*, 194(3):631–650, 1998.

[ZJ99] P. Zinn-Justin. Adding and multiplying random matrices: a generalization of Voiculescu's formulas. *Phys. Rev. E (3)*, 59(5, part A):4884–4888, 1999.

[ZJ02] P. Zinn-Justin. Hciz integral and 2d toda lattice hierarchy, 2002.

[ZJZ02a] P. Zinn-Justin and J.-B. Zuber. Matrix integrals and the counting of tangles and links. *Discrete Math.*, 246(1-3):343–360, 2002. Formal power series and algebraic combinatorics (Barcelona, 1999).

[ZJZ02b] P. Zinn-Justin and J.-B. Zuber. On some integrals over the U(N) unitary group and their large N limit, 2002.

[Zvo97] A. Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Math. Comput. Modelling*, 26(8-10):281–304, 1997. Combinatorics and physics (Marseilles, 1995).

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