Equations elliptiques semilineaires avec potentiel singulier

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LINEAR AND SEMILINEAR ELLIPTIC EQUATIONS WITH A SINGULAR POTENTIAL

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ABSTRACT OF THE DISSERTATION

Linear and semilinear elliptic equations with a singular potential

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Dissertation Director: H. Brezis

This dissertation is concerned with simple elliptic partial differential equations of the form

$$\begin{cases} -\Delta u = F(x,u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ and F can depend nonsmoothly on the variable x. In this setting, uniqueness, existence and regularity results of the standard theory may fail, even in the linear case. An educating example is the so-called inverse-square potential with a power nonlinearity, i.e., when

$$F(x,u) = \frac{c}{|x|^2}u + u^p + \lambda,$$

where $c, \lambda > 0$ and $p > 1$. We show that existence of solutions depends highly on the values of the parameters. Optimal regularity, uniqueness and stability results are also considered. For the general case, we first look at linear right-hand sides

$$F(x,u) = a(x)u + b(x)$$

and obtain important comparison principles, which enable us in the general case to obtain a sharp criterion of existence for a wide class of nonlinear F.

Preface

This dissertation is a compilation of the research papers written by the author during the course of the Ph. D. As a result, each chapter contains one paper and its own references. Part of the material has been written in collaboration (with J. Davila for chapters 3 and 5 and with G. Nedev for chapter 4). Some of the papers have already appeared in specialized journals, while others have been submitted for publication. Minor changes were made from the original papers.

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Chapter 1

Introduction

1.1 Overview

This work studies the interaction between singular coefficients and nonlinear terms in some simple partial differential equations of the form

$$\begin{cases} -\Delta u &= F(x,u) \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, Δ the usual Laplacian and F a function which is *a priori* nonsmooth in the variable x.

Brézis and Vazquez have observed that this type of problem seemed to contradict the inverse function theorem. Numerous other apparent contradictions can thus appear : loss of maximum principle for coercive elliptic operators, lack of Green's function, blow-up in finite time for linear evolution equations, etc.

From the viewpoint of application, particularily the Gelfand problem or standard combustion models, some complete blow-up phenomena (in the stationnary case) and instantaneous complete blow-up (for parabolic equations) can be obtained.

Starting from a simple example (including a one-point singularity, via the inverse-square potential $1/|x|^2$, coupled with a power nonlinearity u^p), we show the existence of a critical exponent beyond which such phenomena appear.

We then obtain a more general result, allowing one to consider "fatter" singular sets and more complicated nonlinearities. The question of blow-up in the *nonlinear* case can then be reduced to the study of a *linear* problem.

1.2 Linear theory

Even when the right-hand side of (1.1) is linear, i.e.

$$F(x,u) = a(x)u + b(x), \tag{1.2}$$

new phenomena appear as soon as $a(x)$ is singular. A striking example is given by the inverse-square potential :

$$a(x) = \frac{c}{|x|^2} \qquad \text{,where} \qquad 0 < c \leq \frac{(n-2)^2}{4} \qquad \text{and} \qquad n \geq 3. \tag{1.3}$$

By Hardy's inequality, the operator $-\Delta - c/|x|^2$ is (formally) coercive. But (assuming of course that $0 \in \Omega$), one loses the maximum principle. Existence and uniqueness of weak solutions (belonging to $L^1(\Omega)$) can also fail.

At the same time, the usual elliptic regularity theorems no longer hold. More precisely, the L^p theory remains true only in an interval $1 < p_1 < p < p_2 < \infty$. Finally, it is known that all nontrivial solutions of (1.1)-(1.2)-(1.3) are singular at the origin. For all these results, see chapter 2.

One can better understand these anomalies by going back to the (more) general case where $a(x) \in L^1_{loc}(\Omega)$, $a(x) \geq 0$. In an appropriate functional setting ($H^1_0(\Omega)$ suffices in the generic case) and under the natural coercivity assumption, one easily recovers the maximum principle for the operator $L = -\Delta - a(x)$. Under a slightly stronger condition, we obtain the following comparison principle : if ζ_0 solves

$$\begin{cases} -\Delta\zeta_0 - a(x)\zeta_0 &= 1 \quad \text{in } \Omega \\ \zeta_0 &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and $\phi_1 > 0$ is an eigenfunction associated to the first eigenvalue of L, then there exists a constant $C > 0$ such that

$$C^{-1}\zeta_0 \leq \phi_1 \leq C\zeta_0. \tag{1.5}$$

Thus ζ_0 and ϕ_1 have the same singularities and we can take ζ_0 as a reference function. Inequality (1.5) is obtained by combining a technique of reduction to the case of a bounded potential and the Moser iteration method. This procedure has the advantage of being general and flexible : one can thus obtain fine maximum principles and treat evolution problems as well.

It is nevertheless useful to work in the general setting of $L^1(\Omega)$-weak solutions. One can then characterize $b(x)$ for which problem (1.1)-(1.2) has at least one solution, and either recover the maximum principle, provided *a priori* regularity of the considered solution is assumed, or use a concept of minimal solution of (1.1) (by means of the maximum principle for the Laplacian) and thus construct a monotone inverse of L:

$$G = L^{-1} = (-\Delta - a(x))^{-1}. \tag{1.6}$$

1.3 Semilinear equations

First consider equation (1.1) with

$$F(x,u) = \frac{c}{|x|^2}u + u^p + \lambda, \tag{1.7}$$

where $p > 1$, $\lambda > 0$ and c is chosen as in (1.3).

There exists a critical exponent $p_0 = p_0(c,n)$ such that problem (1.1)-(1.7) has no solution given any pair (p,λ) when $p \geq p_0$, whereas solutions exist for $p < p_0$ (and λ small).

In the first case (supercritical), we also have a result of complete blow-up for solutions of the regularized problem.

In the second case, we recover a certain number of phenomena that are well-known when $a(x) \equiv 0$: for each $p < p_0$, there exists an extremal parameter $\lambda^* > 0$ such that (1.1)-(1.7) has a solution u if and only if $\lambda \leq \lambda^*$. When $\lambda < \lambda^*$, this solution has on the one hand optimal regularity, i.e.

$$C^{-1}\zeta_0 \leq u \leq C\zeta_0, \tag{1.8}$$

where $C > 0$ and ζ_0 solves (1.4)-(1.3). On the other hand, u is stable (in a linearized sense) and is the unique stable solution of (1.1)-(1.7) belonging tho $H_0^1(\Omega)$. In the extremal case $\lambda = \lambda^*$, the solution is unique and can in certain cases satisfy (1.8) and be unstable, and in other cases show stronger singular behaviour at the origin while remaining stable.

Nonexistence in the supercritical case can be demonstrated in two different ways. In the first approach, one first obtains an a priori regularity result, allowing one to use the maximum principle. Arguing by contradiction, one then constructs more and more singular subsolutions, eventually reaching a contradiction (see chapter 2). In the second approach, one starts from a solution of the *nonlinear* problem to construct a supersolution of a *linear* problem. Comparing the latter with the minimal solution of the linear problem, we then obtain a restriction on p.

Using this method, we can treet the more general case where

$$F(x,u) = a(x)u + c(x)f(u) + \lambda b(x), \tag{1.9}$$

where $c(x) \geq 0$, $c(x) \in L^1_{loc}(\Omega)$ and where $f \geq 0$ is a convex superlinear function. One can then classify nonlinearities in two categories : existence (for small $\lambda > 0$) or nonexistence. More precisely, problem (1.1)-(1.9) has solutions if and only if there exist $\epsilon, C > 0$ such that

$$G(c(x)f(\epsilon\zeta_0)) \leq C\zeta_0, \tag{1.10}$$

where ζ_0 solves (1.4), G is the inverse evoked in (1.6) and where we supposed $b(x) \in L^\infty(\Omega)$ to simplify the exposition. Observe that (1.10) states that the solutions of two *linear* problems are comparable.

This criterium can be conveniently applied. For example, if

$$F(x,u) = \frac{c}{d(x,\Sigma)^2}u + u^p + \lambda \tag{1.11}$$

where Σ is a (compact imbedded) submanifold of codimension $k \neq 2$ (and $c > 0$ small enough), we obtain a new critical exponent $p_0 = p_0(c,n,k)$, which somewhat surprisingly decreases with k. In particular, if $\Sigma = \partial\Omega$, one can take $p_0 = \infty$.

An important step towards this result consists in obtaining the coercivity of the corresponding linear operator L, which we refer to as a generalized Hardy inequality and demonstrate in full generality in chapter 4.

1.4 Perspectives

Singular potentials lead to a number of interesting questions. Let us cite three of them.

The parabolic analogue of problem (1.1) is a natural extension of the above work.

One major open question is to determine whether condition (1.10) is sufficient to guarantee the well-posedness of the Cauchy problem. We have obtained partial results on this matter and believe that full generality can be reached provided sharp pointwise upper bounds on the corresponding heat kernel are known.

Another direction of investigation consists in considering potentials $a(x)$ that change sign and more importantly in stating problem (1.1) without a sign condition on u : can nonexistence results still be obtained ?

The study of the Yamabe problem in the supercritical case can be attacked from the viewpoint of singular potentials. Consider for example the equation $-\Delta u = u^2$ in dimension 7 or higher. It is easy to derive a solution of this equation (in $\mathbb{R}^n \setminus \{0\}$) of the form $u_0 = c_n|x|^{-2}$ and we can then look for solutions

$$u = u_0 + v$$

to obtain the equation

$$-\Delta v = \frac{2c_n}{|x|^2} v + v^2.$$

The main problem here is that $2c_n > (n-2)^2/4$ and th operator $-\Delta - 2c_n/|x|^2$ is no longer coercive. Can nontrivial solutions still be obtained ?

Chapter 2

A nonlinear elliptic PDE with the inverse square potential

2.1 Introduction

2.1.1 Statement of the problem

This section is concerned with the following equation :

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = u^p + tf & \text{in } \Omega \\ \qquad\quad u > 0 & \text{in } \Omega \\ \qquad\quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{$P_{t,p}$}$$

Here, Ω is a smooth bounded open set of $\mathbb{R}^n$ $(n \geq 3)$ containing the origin, $c > 0$, $p > 1$, $t > 0$ are constants and $f \not\equiv 0$ is a smooth, bounded, nonnegative function.

We assume from now on that

$$0 < c \leq c_0 := \frac{(n-2)^2}{4} \tag{0.1}$$

The relevance of the constant c_0 will appear after we clarify the notion of a solution of $(P_{t,p})$.

Three types of solution are defined thereafter : weak solutions, which provide a good setting for non-existence proofs (see Theorem 1 and Proposition 2.1), $H_0^1(\Omega)$ solutions, for which uniqueness results can be established (see Theorem 2) and strong solutions, which set the optimal regularity one can hope for (see Theorem 1 and Lemma 1.5.)

We shall say that $u \in L^1(\Omega)$ is a **weak solution** of $(P_{t,p})$ if $u \geq 0$ a.e. and if it satisfies the two following conditions :

$$\begin{cases} \int_\Omega \left(\dfrac{u}{|x|^2} + u^p \right) dist(x, \partial\Omega)\, dx < \infty \\ \int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi \right) = \int_\Omega (u^p + tf)\phi \qquad \text{for } \phi \in C^2(\bar{\Omega}) \text{ , } \phi|_{\partial\Omega} = 0 \end{cases}$$

Observe that the first condition merely ensures that the integrals in the second equation make sense.

An $\mathbf{H_0^1(\Omega)}$ **solution** is a function $u \in H_0^1(\Omega)$ such that $u \geq 0$ a.e., $u^p \in L^{\frac{2n}{n+2}}(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi - \int_\Omega \frac{c}{|x|^2} u\phi = \int_\Omega (u^p + tf)\,\phi \qquad \text{for all } \phi \in H_0^1(\Omega)$$

All integrals are well defined because of Sobolev's and Hardy's inequalities (see (0.3) for the latter.)

Finally, a **strong solution** u is a $C^2(\bar{\Omega} \setminus \{0\})$ function satisfying the system of equations $(P_{t,p})$ everywhere except possibly at the origin, such that for some $C > 0$,

$$0 \leq u \leq C\,|x|^{-a}$$

where

$$a := \frac{n - 2 - \sqrt{(n-2)^2 - 4c}}{2} > 0 \tag{0.2}$$

Observe that $-a$ is the larger root of $P(X) = X(X-1) + (n-1)X + c = 0$. Also define a' by

$$-a' \text{ is the smaller root of P(X)} \tag{0.2'}$$

• **Why are definitions (0.1), (0.2) important ?**

The constant c_0 defined in (0.1) is the best constant in Hardy's inequality :

$$\int_\Omega |\nabla u|^2 \geq c_0 \int_\Omega \frac{u^2}{|x|^2} \qquad \text{for all } u \in H_0^1(\Omega) \tag{0.3}$$

Consequently, when $c < c_0$, the operator $-\Delta - \frac{c}{|x|^2}$ is coercive in $H_0^1(\Omega)$. This turns out to be crucial since Theorem 2.2 in [BG] implies that if $c > c_0$, there is no nonnegative u, $u \not\equiv 0$ such that $-\Delta u - \frac{c}{|x|^2}u \geq 0$ and hence no solution of $(P_{t,p})$, even in the weak sense. We arrive at the same conclusion if $c > 0$ is arbitrary and the space dimension n is 1 or 2, as can be deduced from the first lines of the proof of Theorem 1.2 in [BC]. We therefore restrict to $n \geq 3$.

The constant a defined in (0.2) plays a central role, even in the linear theory. Indeed, if $f \not\equiv 0$ is say, a smooth nonnegative bounded function on Ω and $u \in H_0^1(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f \text{ in } \Omega \\ \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{0.4}$$

then $u(x) \geq C|x|^{-a}$ near the origin, for some $C > 0$ (see Lemma 1.5 .) In particular, strong solutions are the nicest one can hope for. In addition, $\psi := |x|^{-a}$ solves $-\Delta\psi - \frac{c}{|x|^2}\psi = 0$ in $\mathbb{R}^n \setminus \{0\}$.

We introduce a third constant, the exponent

$$p_0 := 1 + \frac{n-2+\sqrt{(n-2)^2-4c}}{c} \tag{0.5}$$

which satisfies

$$a + 2 = p_0\, a$$

Roughly speaking, if u behaves like $|x|^{-a}$, then $-\Delta u - \frac{c}{|x|^2}u \sim |x|^{-(a+2)}$ and $u^p \sim |x|^{-ap}$. Hence, p_0 sets the threshold beyond which the nonlinear term produces a stronger singularity at the origin than the differential operator. In fact, we will show that for $p \geq p_0$, $(P_{t,p})$ has no solution, no matter how small $t > 0$ is. See Theorem 1 for details.

This fact is somewhat surprising : one would expect that working with the map $F(u) := -\Delta u - \frac{c}{|x|^2}u - u^p$, which is such that $F'(0) = -\Delta - \frac{c}{|x|^2}$ is formally bijective and $F(0) = 0$, the inverse function theorem would yield solutions for $t > 0$ sufficiently small. Such an argument fails because there is no functional setting in which it may be applied. See section 7 of [BV] or the introduction of [BC] for a similar situation.

Another interesting property of p_0 is its variation as c decreases from $c = c_0$ to $c = 0$: when $c = c_0$, $p_0 = \dfrac{n+2}{n-2}$ is the Sobolev exponent whereas when $c \to 0$, $p_0 \to \infty$. This is natural in view of the case $c = 0$, for which $p > 1$ can be chosen arbitrarily (see e.g. [D],[BCMR],[CR].)

- **How do strong, $\mathbf{H_0^1(\Omega)}$ and weak solutions relate ?**

Proposition 0.1. *Suppose (0.1) holds and recall (0.2), (0.5). Suppose also that* $1 < p < p_0$.

- *If u is a strong solution of $(P_{t,p})$, then u is an $H_0^1(\Omega)$ solution of $(P_{t,p})$.*

- *If u is an $H_0^1(\Omega)$ solution of $(P_{t,p})$, then u is a weak solution of $(P_{t,p})$.*

- *If u is a weak solution of $(P_{t,p})$ and $0 \leq u \leq C\,|x|^{-a}$ then u is an $H_0^1(\Omega)$ solution of $(P_{t,p})$.*

- *If u is an $H_0^1(\Omega)$ solution of $(P_{t,p})$ and $0 \leq u \leq C\,|x|^{-a}$ then u is a strong solution of $(P_{t,p})$.*

This will be proved in Section 1.

Remark 0.1. *In section 5, we provide examples of both strong and $H_0^1(\Omega)$ solutions. We do not know however if there exist weak solutions that are not $H_0^1(\Omega)$.*

With these definitions in mind, we investigate the existence, uniqueness and regularity of solutions of $(P_{t,p})$:

2.1.2 Main results

Theorem 1. *Suppose (0.1) holds and recall (0.5).*

- *If $1 < p < p_0$, there exists $t_0 > 0$ depending on n, c, p, f such that*
  - *if $t < t_0$ then $(P_{t,p})$ has a minimal strong solution,*
  - *if $t = t_0$ then $(P_{t,p})$ has a minimal weak solution,*
  - *if $t > t_0$ then $(P_{t,p})$ has no solution, even in the weak sense and there is complete blow-up.*

- *If $p \geq p_0$ then, for any $t > 0$,*
  - *$(P_{t,p})$ has no solution, even in the weak sense, and there is complete blow-up.*

This result requires the following definition :

Definition 0.1. *Let* $\{a_n(x)\}$ *and* $\{g_n(u)\}$ *be increasing sequences of bounded smooth functions converging pointwise respectively to* $\dfrac{c}{|x|^2}$ *and* $u \to u^p$ *and let* $\underline{u_n}$ *be the minimal nonnegative solution of*

$$\begin{cases} -\Delta \underline{u_n} - a_n \underline{u_n} = g_n(\underline{u_n}) + tf & \text{in } \Omega \\ \underline{u_n} = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_n)$$

We say that there is **complete blow-up** *in* $(P_{t,p})$ *if, given any such* $\{a_n(x)\}$, $\{g_n(u)\}$ *and* $\{\underline{u_n}\}$,

$$\frac{\underline{u_n}(x)}{\delta(x)} \to +\infty \text{ uniformly on } \Omega,$$

where $\delta(x) := dist(x, \partial\Omega)$.

Theorem 2. *Suppose (0.1) holds and* $1 < p < p_0$, $0 < t < t_0$. *Then if* u_t *denotes the minimal strong solution of* $(P_{t,p})$,

- u_t *is stable*
- u_t *is the only stable* $H_0^1(\Omega)$ *solution of* $(P_{t,p})$

If u_{t_0} *denotes the minimal weak solution of* $(P_{t_0,p})$ *and* $0 < c < c_0$ *and*

if u_{t_0} *solves the problem in the strong sense then* $\lambda_1(u_{t_0}) = 0$

Stability is defined as follows :

Definition 0.2. *We say that* u *is* **stable** *if the generalized first eigenvalue* $\lambda_1(u)$ *of the linearized operator of equation* $(P_{t,p})$ *is positive, i.e., if*

$$\lambda_1(u) := \inf\{J(\phi) : \phi \in C_c^\infty(\Omega) \setminus \{0\}\} > 0$$

where

$$J(\phi) = \frac{\int_\Omega |\nabla\phi|^2 - \int_\Omega \frac{c}{|x|^2}\phi^2 - \int_\Omega p u^{p-1}\phi^2}{\int_\Omega \phi^2}$$

The proof of Theorem 1 is presented in sections 2 and 3, whereas Theorem 2 is proved in section 4.

In section 5, we study the extremal case $t = t_0$ and provide examples of two distinct behaviors of the extremal solution of $(P_{t_0,p})$.

Finally, in section 6, proofs of all previously announced results pertaining to the case $c = c_0$ are given.

2.1.3 Notation and further definitions

Dealing with linear equations of the form (0.4) with $f \in L^1(\Omega, dist(x, \partial\Omega)\, dx)$, a weak solution u is one that satisfies the equation $\int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) = \int_\Omega f\phi$ with the integrability condition $\int_\Omega \frac{|u|}{|x|^2} < \infty$. Strong solutions are defined as in the nonlinear case.

Of course, Proposition 0.1 need not be true in this setting.

Sometimes we shall refer to inequalities holding in the weak sense or talk about (weak) supersolutions. This means that we integrate the equation with nonnegative test functions.

For example, $-\Delta u - \frac{c}{|x|^2}u \geq f$ holds **in the weak sense**,
given $f \in L^1(\Omega, dist(x, \partial\Omega)\, dx)$, if $\dfrac{u}{|x|^2} \in L^1(\Omega)$ and if

$$\int_\Omega u \left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) \geq \int_\Omega f\phi \qquad \text{for all } \phi \in C^2(\bar{\Omega}) \text{ with } \phi \geq 0 \text{ and } \phi|_{\partial\Omega} = 0$$

The following L^q weighted spaces will be used in the sequel :

$L^q_\delta = L^q(\Omega, \delta(x)\, dx)$,

$L^q_m = L^q(\Omega, |x|^m\, dx)$,

$L^q_{m,\delta} = L^q(\Omega, |x|^m\, \delta(x)\, dx)$ and

$L^\infty_m = \{u : u \cdot |x|^{-m} \in L^\infty(\Omega)\}$

where $1 \leq q < \infty$, $\delta(x) = dist(x, \partial\Omega)$ and $m \in \mathbb{R}$.

Also, for $\rho > 0$, B_ρ denotes the open ball of radius ρ centered at the origin. The letter C denotes a generic positive constant.

2.2 Preliminary : linear theory

We construct here a few basic tools to be used later on and start out with the L^2 theory.

Lemma 1.1. *Suppose* $0 < c < c_0$ *and let* $f \in H^{-1}(\Omega)$. *There exists a unique* $u \in H_0^1(\Omega)$, *weak solution of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f \ \ in\ \Omega \\ \\ \qquad\qquad u = 0 \ \ on\ \partial\Omega \end{cases} \tag{1.1}$$

Furthermore,

$$\|u\|_{H_0^1(\Omega)} \leq C\|f\|_{H^{-1}} \tag{1.2}$$

$$f \geq 0 \ \ \textit{in the sense of distributions} \ \Rightarrow u \geq 0 \ \ a.e. \tag{1.3}$$

Proof. Hardy's inequality (0.3) implies that $-\Delta - \frac{c}{|x|^2}$ is coercive in $H_0^1(\Omega)$. (1.2) follows from Lax-Milgram's lemma. Observe that, using approximation in $H_0^1(\Omega)$ by smooth functions and integration by parts in $\bar{\Omega} \setminus B_\epsilon$ with $\epsilon \to 0$, our definition of a weak solution and that of Lax-Milgram's lemma coincide in this setting.

For $u \in H_0^1(\Omega)$, it is well known that $u^- \in H_0^1(\Omega)$. Testing the variational formulation of (1.1) against u^- yields (1.3). □

Next, we consider the L^q theory and restrict ourselves to the radial case.

Lemma 1.2. *Suppose* $0 < c < c_0$ *(with* c_0 *defined in (0.1)) and recall (0.2). Let* $q \in \left(\dfrac{n}{n-a}, \dfrac{n}{2+a}\right)$, $E = W^{2,q}(B_1) \cap W_0^{1,q}(B_1) \cap \{u : \frac{u}{|x|^2} \in L^q(B_1)\}$. *For any radial* $f \in L^q(B_1)$, *there exists a unique radial weak solution* $u \in E$ *of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f \ \ in\ B_1 \\ \\ \qquad\qquad u = 0 \ \ on\ \partial B_1 \end{cases} \tag{1.4}$$

Furthermore,

$$\|u\|_E \leq C\|f\|_{L^q} \tag{1.5}$$

$$f \geq 0 \ \ a.e. \ \Rightarrow u \geq 0 \ \ a.e. \tag{1.5'}$$

Remark 1.2. *Observe that*

- *It can be shown that* $u \in W^{2,q} \cap W_0^{1,q} \Rightarrow \frac{u}{|x|^2} \in L^q$ *for* $1 < q < n/2$*, so that the definition of* E *can be slightly simplified.*

- *The interval* $\left(\dfrac{n}{n-a}, \dfrac{n}{2+a}\right)$ *is nonempty if and only if* $c < c_0$*.*

- *The restrictions on the range of* q *are optimal. If* $q \leq \frac{n}{n-a}$*, uniqueness is lost (see Remark 1.4), whereas if the lemma were to hold for some* $q \geq \frac{n}{2+a}$*, one could construct solutions of* $(P_{t,p})$ *for some* p*,* $p \geq p_0$ *by means of the inverse function theorem, contradicting Theorem 1 (see the methods of Proposition 4.1 .)*

- *It would be natural to extend Lemma 1.2 to the nonradial case. The problem remains open.*

Proof. Uniqueness will follow from the maximum principle (Lemma 1.4) proved in this section, provided we can show that $E \subset L^1_{-a-2}$.

If $u \in E$, $\frac{u}{|x|^2} \in L^q$ and using Hölder's inequality, $u \in L^1_{-a-2}$ if $|x|^{-a} \in L^{\frac{q}{q-1}}$, which is equivalent to asking $q > \dfrac{n}{n-a}$.

For existence, we suppose (without loss of generality in view of estimate 1.5) that $f \in C_c^\infty(0,1)$, $f \geq 0$ and define

$$u(r) := \Phi(f)(r) = \frac{r^{-a}}{\alpha} \int_0^1 f(s) \cdot s^{\frac{n+\alpha}{2}} [\max(s,r)^{-\alpha} - 1] \, ds$$

where $\alpha = \sqrt{(n-2)^2 - 4c}$, $r \in (0,1)$.

(1.5') follows from the definition of u.

Since f is supported away from the origin, it is quite clear that $\dfrac{u}{r^{-a}}$ is smooth everywhere on $[0,1]$ so that $|u| \leq Cr^{-a}$ and $|u'| \leq Cr^{-a-1}$. Also, $u(1) = 0$. Differentiating u, we get

$$-u'' - \frac{n-1}{r} u' - \frac{c}{r^2} u = f \tag{1.6}$$

This equality holds for every $r \neq 0$ and also in the weak sense, using integration by parts in $B_1 \setminus B_\epsilon$ with $\epsilon \to 0$ and the above estimate on u and u'.

So, we just have to prove (1.5), which we shall do using Hardy-inequality-type arguments.

Using the definition of u, we see that

$$0 \le C\frac{u}{r^2} \le r^{-(1+\frac{n+\alpha}{2})}\int_0^r f(s)\cdot s^{\frac{n+\alpha}{2}}\,ds + r^{-(1+n/2)+\alpha/2}\int_r^1 f(s)s^{\frac{n-\alpha}{2}}\,ds$$
$$\equiv A \qquad\qquad\qquad\qquad +B$$

Letting $g(s) = f(s)s^{\frac{n+\alpha}{2}}$ for $0 \le s \le 1$ and $G(r) = \int_0^r g(s)\,ds$ for $0 \le r \le 1$, integration by parts yields

$$I := \int_0^1 r^{-(1+\frac{n+\alpha}{2})q}G^q(r)r^{n-1}\,dr =$$
$$\frac{1}{n-(1+\frac{n+\alpha}{2})q}G^q(1) - \frac{q}{n-(1+\frac{n+\alpha}{2})q}\int_0^1 r^{n-(1+\frac{n+\alpha}{2})q}G^{q-1}(r)g(r)\,dr$$
$$\le C\int_0^1 r^{n-(1+\frac{n+\alpha}{2})q}G^{q-1}(r)g(r)\,dr$$

The last inequality results from the fact that when $q > \frac{n}{n-a}$, $\frac{1}{n-(1+\frac{n+\alpha}{2})q} < 0$.

Applying Hölder,

$$I \le I^{\frac{q-1}{q}}\left(\int_0^1 r^\gamma g^q(r)\,dr\right)^{1/q}$$

where $\gamma = q(n-(1+\frac{n+\alpha}{2}))$. But $r^\gamma g^q(r) = r^{q(n-1)}f^q(r) \le r^{(n-1)}f^q(r)$ so

$$\left(\int_{B_1} A^q\right)^{1/q} = C\cdot I^{1/q} \le C\|f\|_{L^q} \tag{1.7}$$

To bound B, we introduce similarily $h(s) = s^{\frac{n-\alpha}{2}}f(s)$ and $H(r) = \int_r^1 h(s)\,ds$. Then, since $H(1) = 0$ and $(-a-2)q+n > 0$, integration by parts yields

$$\int_0^1 r^{-(a+2)q}H^q(r)r^{n-1}\,dr \le C\int_0^1 r^{-(a+2)q+n}H^{q-1}(r)h(r)\,dr$$
$$\le C\left(\int_0^1 r^{-(a+2)q+n-1}H^q(r)\,dr\right)^{\frac{q-1}{q}}\left(\int_0^1 r^\gamma h^q(r)\,dr\right)^{\frac{1}{q}}$$

where $\gamma = n-1-q(n+\alpha)/2$. Now, $r^\gamma h^q(r) = r^{n-1}f^q(r)$ and it follows that

$$\left(\int_{B_1} B^q\right)^{1/q} \le C\|f\|_{L^q} \tag{1.8}$$

Combining (1.7) and (1.8) gives $\|u/r^2\|_{L^q} \le C\|f\|_{L^q}$.

To get (1.5), using equation (1.6), it suffices to show that $u'/r \in L^q$. From the definition of $u = \Phi(f)$, we see that

$$u'/r = -a \cdot u/r^2 - \alpha A$$

and the estimate follows from our previous analysis. $\square$

Existence or uniqueness hold in other functional spaces, as the following two lemmas show :

Lemma 1.3. *Recall (0.1), (0.2), (0.2'). Let f be such that $\int_\Omega |f| \cdot |x|^{-a} dist(x, \partial\Omega)\, dx < \infty$. There exists at least one weak solution u with $u \cdot |x|^{-2} \in L^1(\Omega)$, of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.9}$$

Furthermore,

$$\|u\|_{L^1_{-2}} \leq C\|f\|_{L^1_{-a,\delta}} \tag{1.10}$$

$$\|u\|_{L^\infty_{-a}} \leq C\|f\|_{L^\infty} \tag{1.11}$$

$$\|u\|_{L^\infty_{-b}} \leq C\|f\|_{L^\infty_{-b-2}} \qquad \text{for } a < b < a' \tag{1.11'}$$

Proof. (Case $0 < c < c_0$)

We assume, without loss of generality, that $f \geq 0$ (for the general case, apply the result to the positive and negative parts of f).

Let $f_k = \min(f, k)$ for $k \in \mathbb{N}$. Then, $f_k \nearrow f$ in $L^1_{-a,\delta}$.

By Lemma 1.1, there exists u_k, unique solution in $H^1_0(\Omega)$ of (1.9) with f_k in place of f. Clearly, $\{u_k\}$ is monotone increasing.

Let ζ_0 be the $H^1_0(\Omega)$ solution of

$$\begin{cases} -\Delta \zeta_0 - \dfrac{c}{|x|^2} \zeta_0 = 1 \text{ in } \Omega \\ \zeta_0 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.12}$$

When $\Omega = B_1$, $\zeta_0 = \zeta_0^1 := C(|x|^{-a} - |x|^2)$, for some $C > 0$. Otherwise, $\Omega \subset B_R$ for some $R > 0$ and $C \cdot \zeta_0^1(x/R)$ is a supersolution of problem (1.12), for some $C > 0$. So,

$$0 \leq \zeta_0 \leq C|x|^{-a}\, \delta(x) \qquad \text{in } \Omega \tag{1.13}$$

Since u_k and $\zeta_0 \in H_0^1(\Omega)$, they are valid test functions in their respective equations and

$$\int_\Omega \nabla u_k \nabla \zeta_0 - \int_\Omega \frac{c}{|x|^2} u_k \zeta_0 = \int_\Omega u_k = \int_\Omega f_k \zeta_0$$

Since $f \geq 0$, so are f_k and u_k and

$$\|u_k\|_{L^1} = \int_\Omega f_k \zeta_0 \leq C \|f_k\|_{L^1_{-a,\delta}} \tag{1.14}$$

Let ζ_1 be the smooth solution of

$$\begin{cases} -\Delta \zeta_1 = 1 \text{ in } \Omega \\ \zeta_1 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.15}$$

and integrate in the equation satisfied by u_k :

$$\int_\Omega u_k - \int_\Omega \frac{c}{|x|^2} u_k \zeta_1 = \int_\Omega f_k \zeta_1 \tag{1.16}$$

Using (1.14) and (1.16) and the inequality $m\delta(x) \leq \zeta_1 \leq M\delta(x)$, where m, M are some positive constants, we get

$$\|u_k\|_{L^1_{-2}} \leq C \|f_k\|_{L^1_{-a,\delta}}$$

It is then easy to construct by monotonicity a solution of (1.9) satisfying (1.10). For estimate (1.11), one should just check that if $f \in L^\infty$, $\|f\|_{L^\infty} \zeta_0$ is a supersolution of (1.9) and apply the maximum principle (see e.g. Lemma 1.4). Hence,

$$u \leq \|f\|_{L^\infty} \zeta_0$$

Applying this estimate to $-u$ yields (1.11).

For estimate (1.11'), $\|f\|_{L^\infty_{-b-2}} \zeta_2$ provides a supersolution of (1.9) where

$$\begin{cases} -\Delta \zeta_2 - \dfrac{c}{|x|^2} \zeta_2 = |x|^{-b-2} \text{ in } \Omega \\ \zeta_2 = 0 \text{ on } \partial\Omega \end{cases} \tag{1.17}$$

Observe that in the radial case $\zeta_2 = C(|x|^{-b} - |x|^{-a})$ so that in general $0 \leq \zeta_2 \leq C|x|^{-b}$ and that Lemma 1.4 may be applied because $a < b < a'$. □

Remark 1.3. .

- *In view of Lemma 1.5, for equation (1.9) to have a solution with $f \in L^1_\delta$, it may be necessary that $f \in L^1_{-a,\delta}$.*
- *In the case $0 < c < c_0$, if $\int_\Omega |f| \cdot |x|^{-a} \cdot |\ln(x)| \cdot \delta(x)\,dx < \infty$, that is, if we ask a little more regularity on f, then $u \in L^1_{-a-2}$ and is therefore unique (using Lemma 1.4 .) For a proof, use the methods of the lemma with ζ_2 solving*

$$\begin{cases} -\Delta\zeta_3 - \dfrac{c}{|x|^2}\zeta_3 = |x|^{-a-2} & \text{in } \Omega \\ \zeta_3 = 0 & \text{on } \partial\Omega \end{cases}$$

When $\Omega = B_1$, $\zeta_3 = C|x|^{-a}\ln(1/|x|)$.

Proof of Proposition 0.1 (case $0 < c < c_0$).

Suppose first that u is a strong solution of $(P_{t,p})$. Let $\zeta_n \in C_c^\infty(\Omega \setminus \{0\})$ be such that $0 \le \zeta_n \le 1$, $|\nabla\zeta_n| \le Cn$, $|\Delta\zeta_n| \le Cn^2$ and

$$\zeta_n = \begin{cases} 0 & \text{if} \quad |x| \le 1/n \text{ and } \delta(x) < 1/n \\ 1 & \text{if} \quad |x| \ge 2/n \text{ and } \delta(x) > 2/n \end{cases}$$

Multiplying $(P_{t,p})$ by $u\zeta_n$ and integrating by parts, it follows that

$$\int_\Omega \left(\frac{c}{|x|^2}u + u^p + tf\right) u\zeta_n = -\int_\Omega \Delta u\, u\zeta_n = \int_\Omega |\nabla u|^2\,\zeta_n + \int_\Omega u\nabla u\nabla\zeta_n$$

Since $u \le C|x|^{-a}$ and $p < p_0$, $u^p \le C|x|^{-a-2}$. Hence, on the one hand,$u^p \in L^{\frac{2n}{n+2}}(\Omega)$ and on the other hand, the left-hand-side integral in the above equation is bounded by $C\int_\Omega |x|^{-2a-2} \le C$, whereas $\left|\int_\Omega u\nabla u\nabla\zeta_n\right| = \left|\dfrac{1}{2}\int_\Omega u^2\Delta\zeta_n\right| \le Cn^2\int_{1/n<|x|<2/n} |x|^{-2a} \to 0$ as $n \to \infty$. Hence $\int_\Omega |\nabla u|^2\,\zeta_n \le C$ and $u \in H^1_0(\Omega)$. Multiplying $(P_{t,p})$ by $\phi\zeta_n$ for $\phi \in C_c^\infty(\Omega)$ yields

$$\int_\Omega \left(\frac{c}{|x|^2}u + u^p + tf\right) \phi\zeta_n = -\int_\Omega \Delta u\, \phi\zeta_n = \int_\Omega \zeta_n\nabla u\nabla\phi + \int_\Omega \phi\nabla u\nabla\zeta_n$$

The last term in the right-hand-side can be rewritten as

$$\int_\Omega \phi\nabla u\nabla\zeta_n = \int_\Omega \nabla(u\phi)\nabla\zeta_n - \int_\Omega u\nabla\phi\nabla\zeta_n = -\int_\Omega u\phi\Delta\zeta_n - \int_\Omega u\nabla\phi\nabla\zeta_n$$

and converges to zero as in the previous case when $n \to \infty$. It follows that u is an $H^1_0(\Omega)$ solution of $(P_{t,p})$. Approximating $u \in H^1_0(\Omega)$ by smooth functions and integrating by parts implies that $H^1_0(\Omega)$ solutions are weak solutions.

Suppose now that u is a weak solution satisfying the estimate $u \leq C|x|^{-a}$. Then as before, $u^p \leq C|x|^{-a-2} \in L^{\frac{2n}{n+2}}(\Omega) \subset H^{-1}(\Omega)$.

Letting $g = u^p + tf$, it follows from Lemma 1.1 that there exists a weak solution $v \in H_0^1(\Omega)$ of (1.9) with g in place of f. u is also a weak solution of (1.9) and by Remark 1.3, we must have $u = v \in H_0^1(\Omega)$. Hence, u is an $H_0^1(\Omega)$ solution.

Finally if u is an $H_0^1(\Omega)$ solution satisfying the estimate $u \leq C|x|^{-a}$, using local elliptic regularity theorems in $\Omega \setminus B_\epsilon$ for an arbitrary $\epsilon > 0$, we may conclude that $u \in C^\infty(\bar{\Omega} \setminus \{0\})$ and satisfies $(P_{t,p})$ in the strong sense.

Lemma 1.4 (Maximum Principle). *If* $\int_\Omega |u| \cdot |x|^{-a-2} < \infty$ *and if*

$$-\Delta u - \frac{c}{|x|^2} u \geq 0 \qquad \text{in the weak sense.} \tag{1.16}$$

then

$$u \geq 0 \ \text{a.e.} \tag{2.1}$$

Proof (case $0 < c < c_0$*).* It is enough to show that $\int_\Omega u\phi \geq 0$ for $\phi \in C_c^\infty(\Omega \setminus \{0\})$, $\phi \geq 0$.

For such a ϕ and $\epsilon > 0$, construct $v_\epsilon \in C^2(\bar{\Omega})$, $v_\epsilon \geq 0$, solving

$$\begin{cases} -\Delta v_\epsilon - \dfrac{c}{|x|^2 + \epsilon} v_\epsilon = \phi \text{ in } \Omega \\ v_\epsilon = 0 \text{ on } \partial\Omega \end{cases}$$

Also let $v \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta v - \dfrac{c}{|x|^2} v = \phi \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega \end{cases}$$

Using Lemma 1.1, since $-\Delta(v_\epsilon - v) - \frac{c}{|x|^2}(v_\epsilon - v) \leq 0$,

$$0 \leq v_\epsilon \leq v \qquad \text{a.e. in } \Omega \tag{1.17}$$

Applying (1.11) in Lemma 1.3 to v,

$$0 \leq v \leq C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{1.18}$$

Combining (1.17) and (1.18),

$$0 \le v_\epsilon \le C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{1.19}$$

Applying (1.16) with $\phi = v_\epsilon$,

$$\int_\Omega u \left(-\Delta v_\epsilon - \frac{c}{|x|^2} v_\epsilon \right) \ge 0$$

Since $-\Delta v_\epsilon - \frac{c}{|x|^2} v_\epsilon = \phi - c \left\{ \frac{1}{|x|^2} - \frac{1}{|x|^2+\epsilon} \right\}$,

$$\int_\Omega u\phi \ge \int_\Omega c \left\{ \frac{1}{|x|^2} - \frac{1}{|x|^2+\epsilon} \right\} u\, v_\epsilon.$$

Clearly, $\{v_\epsilon\}$ is monotone increasing and converges pointwise to a finite value a.e. in Ω by (1.19).

So the integrand in the right hand side of the previous equation converges a.e. to 0. Using (1.19) and $u \in L^1_{-a-2}$, this integrand is dominated by an L^1 function.

By Lebesgue's theorem, we conclude that

$$\int_\Omega u\phi \ge 0.$$

□

Remark 1.4. *This maximum principle is sharp in the following sense :*

if $q > -a$ then there exists $u \in L^1_{q-2}$ such that $-\Delta u - \frac{c}{|x|^2} u = 0$ yet $u \not\equiv 0$.

Just take $\Omega = B_1$ and $u := |x|^{-a'} - |x|^{-a}$, with $-a'$ and $-a$ defined in (0.2), (0.2').

We conclude this section with a lemma giving necessary conditions for the existence of a solution to the linear problem.

Lemma 1.5. *Suppose $f \ge 0$ a.e. , $f \not\equiv 0$, $\int_\Omega f(x)\, dist(x, \partial\Omega)\, dx < \infty$. If u is a nonnegative weak solution of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f \ \ in\ \Omega \\ \qquad\qquad u = 0 \ \ on\ \partial\Omega \end{cases} \tag{1.20}$$

Then there exists a constant $C > 0$ depending only on Ω such that

$$u \ge C \left(\int_\Omega f\zeta_0 \right) \zeta_0 \qquad a.e.\ in\ \Omega$$

with ζ_0 defined by (1.12). In particular, for some $m > 0$

$$u \geq m|x|^{-a} \text{ a.e. near the origin}$$

Furthermore, for any $\epsilon > 0$, if u denotes the minimal solution of (1.20) then

$$\int_\Omega u \cdot |x|^{-a-2+\epsilon} dx \leq C_\epsilon \int_\Omega f \cdot |x|^{-a} dist(x, \partial\Omega)\, dx < \infty$$

Most of the results of this lemma are a direct consequence of a more general theorem on the associated evolution equation, established by Baras and J. Goldstein (see [BG] Th 2.2 page 124.) We give here a simpler proof for convenience of the reader.

Proof (case $0 < c < c_0$).

Step 1. $\mathbf{u \geq m|x|^{-a}}$ near the origin.

Let $f_1 = \min(f, k)$ with $k > 0$ such that $f_1 \not\equiv 0$ and $u_1 \geq 0$ be the minimal solution of

$$\begin{cases} -\Delta u_1 - \dfrac{c}{|x|^2} u_1 = f_1 \text{ in } \Omega \\ \qquad u_1 = 0 \text{ on } \partial\Omega \end{cases}$$

Since u is a supersolution of the above problem, u_1 is well defined and $0 \leq u_1 \leq u$ so it suffices to prove the result for u_1.

Since $f_1 \in L^\infty(\Omega)$, on the one hand $0 \leq u_1 \leq C|x|^{-a}$ by (1.11) and on the other hand the equation has a solution $v \in H_0^1(\Omega)$. By Lemma 1.4, we must have $u_1 = v$.

Now, since $u_1 \not\equiv 0$, $u_1 \geq 0$ and $-\Delta u_1 \geq 0$ in the connected set Ω, we have for some $\epsilon > 0$ and $\eta > 0$,

$$u_1 \geq \epsilon \qquad \text{a.e. in } B_{2\eta}$$

Choose $C > 0$ so that $\epsilon \geq C r^{-a}$ for $r \geq \eta$ and let $z = (u_1 - C|x|^{-a})^-$. Observe that $z \in H_0^1(B_\eta)$.

Next, we multiply $u_1 - C|x|^{-a}$ by z and integrate by parts :

$$\begin{aligned} 0 \geq -\int_\Omega |\nabla z|^2 + \int_\Omega \frac{c}{|x|^2} z^2 &= \int_\Omega \nabla(u_1 - C|x|^{-a}) \nabla z - \int_\Omega \frac{c}{|x|^2}(u_1 - C|x|^{-a}) z \\ &= \int_\Omega f z - C\left(\int_{B_\eta} \nabla |x|^{-a} \nabla z - \int_{B_\eta} \frac{c}{|x|^2} |x|^{-a} z \right) \\ &\geq -C \int_{\partial B_\eta} z \partial_\nu |x|^{-a} \geq 0 \end{aligned}$$

And hence $z \equiv 0$ in B_η.

Step 2. $\mathbf{u \geq C(K,\Omega)\int_\Omega f\zeta_0}$ **in** $\mathbf{K \subset\subset \Omega}$ **when** $\mathbf{f \in L^\infty(\Omega)}$

The proof is an adaptation of Lemma 3.2 in [BC]. Observe that up to replacing u by the minimal nonnegative solution of the problem, we may assume u to be an $H_0^1(\Omega)$ solution satisfting $0 \leq u \leq C|x|^{-a}$.

Let $\rho = dist(K, \partial\Omega)/2$ and take m balls of radius ρ such that

$$K \subset B_\rho(x_1) \cup \cdots \cup B_\rho(x_m) \subset \Omega$$

Let $\zeta_1, \ldots, \zeta_m$ be the solutions (given, say, by Lemma 1.1) of

$$\begin{cases} -\Delta\zeta_i - \dfrac{c}{|x|^2}\zeta_i = \chi_{B_\rho(x_i)} \text{ in } \Omega \\ \zeta_i = 0 \text{ on } \partial\Omega \end{cases}$$

where χ_A denotes the characteristic function of A. There is a constant $C > 0$ such that

$$\zeta_i(x) \geq C\zeta_0(x) \qquad \text{in } \Omega \qquad \text{for } 1 \leq i \leq m$$

Indeed, by Step 1, this inequality must hold near the origin and by Hopf's boundary lemma, we also have $\zeta_i \geq c\delta \geq C\zeta_0$ away from the origin.

Let now $x \in K$, and take a ball $B_\rho(x_i)$ containing x. Then $B_\rho(x_i) \subset B_{2\rho}(x) \subset \Omega$ and, since $-\Delta u \geq 0$ in Ω, we conclude

$$\begin{aligned} u(x) &\geq \fint_{B_{2\rho}(x)} u = C\int_{B_{2\rho}(x)} u \geq C\int_{B_\rho(x_i)} u \\ &= C\int_\Omega u\left(-\Delta\zeta_i - \frac{c}{|x|^2}\zeta_i\right) = C\int_\Omega f\zeta_i \\ &\geq C\int_\Omega f\zeta_0 \end{aligned}$$

Step 3. $\mathbf{u \geq C(\Omega)\left(\int_\Omega f\zeta_0\right)\zeta_0}$ **in** $\mathbf{\Omega}$ **when** $\mathbf{f \in L^\infty(\Omega)}$

Suppose without loss of generality that $B_1 \subset \Omega$ and let $K = \bar{B}_1 \setminus B_{1/2}$. By Step 2, it suffices to prove the inequality in $\Omega \setminus K$. Let w be the solution of

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2}w = 0 \text{ in } \Omega \setminus B_1 \\ w = 0 \text{ on } \partial\Omega \\ w = 1 \text{ on } \partial B_1 \end{cases}$$

and extend w by $w := (2|x|)^{-a}$ in $B_{1/2}$, so that the above equation still holds in $\Omega \setminus K$ with $w|_{\partial K} \equiv 1$. By Hopf's boundary Lemma applied in $\Omega \setminus B_1$, we conclude that

$$w \geq C\zeta_0 \qquad \text{in } \Omega \setminus K$$

u is assumed to be dominated by $C|x|^{-a}$ so we can apply the maximum principle (Lemma 1.4) in $\Omega \setminus K$ to conclude that

$$u \geq C\left(\int_\Omega f\zeta_0\right) w \geq C\left(\int_\Omega f\zeta_0\right)\zeta_0 \qquad \text{in } \Omega \setminus K$$

Step 4. $\int_\Omega |\mathbf{x}|^{-\mathbf{a}}\mathbf{f}\delta(\mathbf{x}) < \infty$**.**

We assume for now that $f \in L^\infty(\Omega)$ and that $u \geq 0$ is the minimal solution of (1.20).

We let $\{\phi_n\}$ be a sequence of smooth, nonnegative and bounded functions converging pointwise and monotonically to $c|x|^{-a-2}$ and construct v_n as the (smooth) solution of

$$\begin{cases} -\Delta v_n = \phi_n \text{ in } \Omega \\ \quad v_n = 0 \text{ on } \partial\Omega \end{cases}$$

Testing v_n in (1.20) yields

$$\int_\Omega f v_n = \int_\Omega u\left(-\Delta v_n - \frac{c}{|x|^2}v_n\right) = \int_\Omega u\left(\phi_n - \frac{c}{|x|^2}v_n\right) \tag{1.21}$$

Now $\phi_n \nearrow c|x|^{-a-2}$ pointwise and in L^1, so, by Lemma 2.1, $v_n \nearrow |x|^{-a} - w$ pointwise and in L^1, where w solves

$$\begin{cases} -\Delta w = 0 \text{ in } \Omega \\ \quad w = |x|^{-a} \text{ on } \partial\Omega \end{cases}$$

Since u is minimal, $0 \leq u \leq C|x|^{-a}$ by (1.11) and we can safely pass to the limit in (1.21) to obtain

$$\int_\Omega (|x|^{-a} - w)f = \int_\Omega u\left(c|x|^{-a-2} - \frac{c}{|x|^2}(|x|^{-a} - w)\right) = c\int_\Omega \frac{u}{|x|^2}w$$

Observe that w is bounded and that $|x|^{-a} - w \geq C|x|^{-a}\delta(x)$, hence

$$\int_\Omega |x|^{-a} f\delta(x) \leq C\int_\Omega |x|^{-2}u$$

This estimate holds when $f \in L^\infty$ and u is minimal but also in the general case, as approximation of f by $f_n = \min(n, f)$ shows.

Step 5. $\mathbf{u \geq C(\Omega)\left(\int_\Omega f\zeta_0\right)\zeta_0}$ **in** $\mathbf{\Omega}$ **when** $\mathbf{f \in L^1_{-a,\delta}}$

Let $k > 0$ be so large that $f_k = \min(f, k) \not\equiv 0$. Then u is a supersolution of (1.20) with f_k in place of f and by Step 3, we have

$$u \geq C(\Omega)\left(\int_\Omega f_k\zeta_0\right)\zeta_0$$

Letting $k \to \infty$, Lebesgue's theorem yields the desired result.

Step 6. $\mathbf{\int_\Omega |x|^{-a-2+\epsilon}u < \infty}$.

We proceed as in Step 4, only this time we let $\phi_n \nearrow -P(-a+\epsilon)|x|^{-a-2+\epsilon}$, where $P(X) = X(X-1) + (n-1)X + c$ and construct v_n solving

$$\begin{cases} -\Delta v_n - \dfrac{c}{|x|^2 + 1/n} v_n = \phi_n \text{ in } \Omega \\ \qquad\qquad v_n = 0 \text{ on } \partial\Omega \end{cases}$$

Hence,

$$\int_\Omega f v_n = \int_\Omega u\phi_n + \int_\Omega \left(\frac{c}{|x|^2 + 1/n} - \frac{c}{|x|^2}\right) v_n u \tag{1.22}$$

If ζ solves

$$\begin{cases} -\Delta \zeta - \dfrac{c}{|x|^2}\zeta = -P(-a+\epsilon)|x|^{-a-2+\epsilon} \text{ in } \Omega \\ \qquad \zeta = 0 \text{ on } \partial\Omega \end{cases}$$

then we have $0 \leq v_n \leq \zeta \leq C|x|^{-a}$. Indeed, if $\Omega = B_1$, then $\zeta = \zeta^1 := C(|x|^{-a} - |x|^{-a+\epsilon})$. Otherwise, $\Omega \subset B_R$ for some $R > 0$ and $C\,\zeta^1(x/R)$ is a supersolution of the problem, for some $C > 0$.

By Step 4, $\int_\Omega f v_n \leq \int_\Omega f\zeta < \infty$. Assuming first that f is bounded (whence $u \leq C|x|^{-a}$) and then working by approximation, it follows from Lebesgue's theorem and from (1.22) that

$$\int_\Omega |x|^{-a-2+\epsilon} u \leq C_\epsilon \int_\Omega f\zeta < \infty$$

□

Remark. *More results about the linear theory of our operator, with $c \in \mathbb{R}$ arbitrary have been detailed by F. Pacard in unpublished work (see [P].)*

2.3 Existence vs. complete blow-up

In this section, we will prove existence or nonexistence of weak solutions of $(P_{t,p})$, using the tools we have just constructed and monotonicity arguments.

2.3.1 Case $\mathbf{p < p_0}$, $\mathbf{c < c_0}$: existence for small $\mathbf{t > 0}$

p_0 has been defined so that $p_0 a = a + 2$. So, for $p < p_0$, $ap < a + 2$ and for some $b \in (a, a')$, the inequality $bp < b + 2$ still holds. We fix such a b and prove that for an appropriate choice of $A > 0$ and for $t > 0$ small,

$$w := A|x|^{-b} \in H^1(\Omega) \text{ is a supersolution of } (P_{t,p}).$$

Observe that $w \in H^1(\Omega)$ as long as b is close enough to a, which may be assumed. We have

$$-\Delta w - \frac{c}{|x|^2} w = -AP(-b)|x|^{-b-2} \qquad \text{where } P(X) = X(X-1) + (n-1)X + c$$

Observe that $P(-b) < 0$ since $b \in (a, a')$ and a' and a are the roots of $P(X)$.

We would like to have $-AP(-b)|x|^{-b-2} \geq A^p|x|^{-pb} + tf$ in Ω. This will be true as soon as

$$\begin{cases} -\dfrac{1}{2}AP(-b)|x|^{-b-2} \geq A^p|x|^{-pb} & \text{and} \\ -\dfrac{1}{2}AP(-b)|x|^{-b-2} \geq tf \end{cases}$$

The first inequality amounts to

$$A \leq \left[-\frac{1}{2}P(-b)|x|^{pb-b-2}\right]^{\frac{1}{p-1}}$$

which will be satisfied, taking $R > 0$ such that $\Omega \subset B_R$, if

$$A \leq \left[-\frac{1}{2}P(-b)R^{pb-b-2}\right]^{\frac{1}{p-1}}$$

since $pb - b - 2 < 0$.

With such a choice of A, pick any $t > 0$ such that

$$-\frac{1}{2}AP(-b)R^{-b-2} \geq t\|f\|_{L^\infty}$$

We have just constructed $w \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2} w \geq w^p + tf \text{ in } \Omega \\ \qquad\qquad w \geq 0 \qquad\qquad \text{on } \partial\Omega \end{cases}$$

Finally we construct an $H_0^1(\Omega)$ supersolution of $(P_{t,p})$. We let w_1 be a smooth extension inside Ω of $w|_{\partial\Omega}$ which is also supported away from the origin. Then $g = \Delta w_1 + \dfrac{c}{|x|^2} w_1$ is smooth and bounded and using Lemma 1.1, there is a unique strong solution z of

$$\begin{cases} -\Delta z - \dfrac{c}{|x|^2} z = g \text{ in } \Omega \\ \qquad\quad z = 0 \text{ on } \partial\Omega \end{cases} \tag{2.1}$$

Letting $w_2 = z + w_1$, it follows that

$$\begin{cases} -\Delta w_2 - \dfrac{c}{|x|^2} w_2 = 0 \text{ in } \Omega \\ \qquad\quad w_2 = w \text{ on } \partial\Omega \end{cases} \tag{2.2}$$

Multiplying by w_2^-, it follows that $w_2 \geq 0$ a.e. in Ω. It is now clear that $\tilde{w} = w - w_2$ is an $H_0^1(\Omega)$ supersolution of $(P_{t,p})$. For convenience, we drop the superscript ˜ thereafter.

Construction of a minimal solution u of $(P_{t,p})$ is now just a matter of monotone iteration. For this purpose we recall the following lemma, proved in [BCMR] :

Lemma 2.1. *Suppose* $\int_\Omega |f(x)|\, dist(x, \partial\Omega) < \infty$. *Then there exists a unique* $v \in L^1(\Omega)$ *which is a weak solution of*

$$\begin{cases} -\Delta v = f \ \textit{in } \Omega \\ \quad v = 0 \ \textit{on } \partial\Omega \end{cases}$$

Moreover,

$$\|v\|_{L^1} \leq C \|f\|_{L^1_\delta}$$

Moreover if $v \in L^1(\Omega)$ *and* $-\Delta v \geq 0$ *weakly, i.e. if*

$$\int_\Omega (-\Delta\phi)\, v \geq 0 \qquad \textit{for all } \phi \in C^2(\bar{\Omega}),\ \phi|_{\partial\Omega} \equiv 0,\ \phi \geq 0 \textit{ in } \Omega$$

then

$$v \geq 0 \qquad \textit{a.e. in } \Omega$$

Define $\{u_k\}$ by induction to be the L^1 weak solutions of

$$\begin{cases} -\Delta u_0 = tf & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \quad \text{for } k = 0$$

$$\begin{cases} -\Delta u_k = \dfrac{c}{|x|^2} u_{k-1} + u_{k-1}^p + tf & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases} \quad \text{for } k \geq 1$$

We now check that this definition makes sense and that (u_k) is monotone and satisfies $0 \leq u_k \leq w$ a.e. in Ω.

For u_0 there is nothing to prove. Suppose the result true up to order $k-1$. Then

$$\begin{aligned} 0 \leq \frac{c}{|x|^2} u_{k-1} + u_{k-1}^p + tf &\leq \frac{c}{|x|^2} w + w^p + tf \\ &\leq C|x|^{-a-2} \in L^1(\Omega) \end{aligned}$$

So u_k is well defined using the previous lemma, $u_k \geq 0$ a.e. and since

$$-\Delta(u_k - u_{k-1}) = \frac{c}{|x|^2}(u_{k-1} - u_{k-2}) + u_{k-1}^p - u_{k-2}^p \geq 0 \text{ by induction hypothesis}$$

and similarly $-\Delta(w - u_k) \geq 0$, we conclude using Lemma 2.1 that

$$0 \leq u_{k-1} \leq u_k \leq w \qquad \text{a.e. in } \Omega$$

By a standard monotone convergence argument, $\{u_k\}$ converges to a weak solution of $(P_{t,p})$.

2.3.2 Pushing t to $\mathbf{t_0}$

We let $t_0 = \sup\{t : (P_{t,p}) \text{ has a weak solution.}\}$ and adapt the methods of [BCMR].

If ϕ_1 is a positive eigenvector of $-\Delta$ (with zero Dirichlet condition) associated to its first eigenvalue λ_1, in other words if $\phi_1 > 0$ in Ω and, for some $\lambda_1 > 0$,

$$\begin{cases} -\Delta \phi_1 = \lambda_1 \phi_1 \text{ in } \Omega \\ \phi_1 = 0 \text{ on } \partial\Omega \end{cases}$$

and if u is a weak solution of $(P_{t,p})$, testing against ϕ_1 yields

$$\int_\Omega \frac{c}{|x|^2} u\,\phi_1 + \int_\Omega u^p \phi_1 + t\int_\Omega f\,\phi_1 = \lambda_1 \int_\Omega u\,\phi_1$$

and, by Young's inequality,

$$\lambda_1 \int_\Omega u\,\phi_1 \le \frac{1}{2}\int_\Omega u^p\phi_1 + C\int_\Omega \phi_1.$$

Thus,

$$t\int_\Omega f\,\phi_1 + \int_\Omega \frac{c}{|x|^2} u\,\phi_1 + \int_\Omega u^p\phi_1 \le C \tag{2.3}$$

which implies $t_0 < \infty$. In particular, there are no weak solutions of $(P_{t,p})$ for $t > t_0$. This implies complete blow-up (see Definition 0.1), as the following proposition shows.

Proposition 2.1. *Suppose (0.1) holds, $p > 1$ and $t > 0$. If $(P_{t,p})$ has no weak solution then there is complete blow-up.*

Proof. The proof is an easy adaptation of Theorem 3.1 in [BC].

Suppose indeed that $(P_{t,p})$ has no weak solution and by contradiction that $\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n \underline{u_n}\,\delta \le C$, where $\{a_n\}, \{g_n\}, \{u_n\}$, are given in Definition 0.1 .

Then, multiplying (P_n) by ζ_1, solution of (1.15) we get

$\int_\Omega \underline{u_n}(-\Delta\zeta_1) - \int_\Omega a_n\underline{u_n}\zeta_1 = \int_\Omega g_n(\underline{u_n})\zeta_1 + \int_\Omega tf\,\zeta_1$.

Hence, $\int_\Omega \underline{u_n} \le C$ and there exists a u such that $\underline{u_n} \nearrow u$ in $L^1(\Omega)$, by monotone convergence.

Since $\{a_n\}$ and $\{g_n\}$ converge monotonically, we can pass to the limit in (P_n), using monotone convergence again and obtain a solution u of $(P_{t,p})$, which is a contradiction.

We have just proved that $\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n\underline{u_n}\,\delta \to \infty$. Now, using (P_n) and Lemma 3.2 in [BC], it follows that

$$\frac{\underline{u_n}(x)}{\delta(x)} \ge C(\Omega)\left(\int_\Omega g_n(\underline{u_n})\,\delta + \int_\Omega a_n\underline{u_n}\,\delta\right) \to \infty \quad \square$$

$\square$

Next, we want to prove that if $(P_{\tau,p})$ has a solution then so does $(P_{t,p})$ for $0 < t \le \tau$. This is true because u_τ is a supersolution of $(P_{t,p})$ in the sense that, weakly,

$$-\Delta u_\tau \geq \frac{c}{|x|^2} u_\tau + u_\tau^p + tf$$

and with the help of Lemma 2.1, we may construct a solution of $(P_{t,p})$ by monotone iteration.

Finally, we prove that $(P_{t_0,p})$ has a weak solution. Choose a nondecreasing sequence $\{t_n\}$ converging to t_0 and for each $n \in \mathbb{N}$, let u_n be a (weak) solution of $(P_{t_n,p})$. Since $\phi_1 \geq m\delta(x)$ for some $m > 0$, equation (2.3) implies that

$$\int_\Omega \frac{c}{|x|^2} u_n\, \delta(x) + \int_\Omega u_n^p \delta(x) \leq C$$

Multiplying by ζ_1, solution of (1.15) then implies boundedness of $\{u_n\}$ in L^1 and hence monotone convergence to a solution of $(P_{t_0,p})$ as $t_n \to t_0$.

2.3.3 Case $\mathbf{0 < c < c_0,\ p \geq p_0}$: blow-up for all $\mathbf{t > 0}$

By Proposition 2.1, we just need to prove that there are no weak solutions of $(P_{t,p})$ for $p \geq p_0$. Assume by contradiction there exists one and call it u. If we apply Lemma 1.5 with $u^p + tf$ in place of f, it follows that

$$\int_\Omega u^p |x|^{-a}\, \delta(x) < \infty \qquad \text{and} \qquad u \geq m|x|^{-a} \qquad \text{a.e. near the origin.}$$

Using Hölder's inequality,

$\int_\Omega u|x|^{-a-2}\, \delta(x) \leq \left(\int_\Omega u^p |x|^{-a}\, \delta(x)\right)^{1/p} \cdot \left(\int_\Omega |x|^{-a-2\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$.

If $p \geq p_0$ and $c < c_0$ then $-a - 2\dfrac{p}{p-1} > -n$, hence, since $u \in L^1(\Omega)$,

$$\int_\Omega u\, |x|^{-a-2} < \infty \tag{2.4}$$

Suppose without loss of generality, that $\Omega \subset B_1$ and define $w = A|x|^{-a} \ln(\frac{1}{|x|})$ for some $A > 0$.

Then $-\Delta w - \frac{c}{|x|^2} w = A\sqrt{(n-2)^2 - 4c}\, |x|^{-a-2}$. Also,

$$-\Delta u - \frac{c}{|x|^2} u \geq u^p \geq m|x|^{-ap} \geq m|x|^{-a-2} \qquad \text{in } B_\eta, \text{ for a fixed small } \eta > 0.$$

Let $A = m(\sqrt{(n-2)^2 - 4c} + c \ln \frac{1}{\eta})^{-1}$ and $C = A\eta^{-a} \ln \frac{1}{\eta}$.

Finally define $z = u + C - w$. Using (2.4), $z \in L^1(B_\eta, |x|^{-a-2}dx)$. Furthermore,

$$\begin{aligned}
-\Delta z - \frac{c}{|x|^2} z &\geq u^p - \frac{cC}{|x|^2} - A\sqrt{(n-2)^2 - 4c}\,|x|^{-a-2} \\
&\geq m|x|^{-ap} - cC|x|^{-2} - A\sqrt{(n-2)^2 - 4c}\,|x|^{-a-2} \\
&\geq |x|^{-2} \left[m|x|^{-a} - cC - A\sqrt{(n-2)^2 - 4c}\,|x|^{-a} \right] \\
&\geq |x|^{-2} \left[(m - A\sqrt{(n-2)^2 - 4c})\, \eta^{-a} - cC \right] \\
&\geq 0
\end{aligned}$$

All these inequalities hold in the weak sense in B_η (since our choice of constants implies $z|_{\partial B_\eta} \geq C - w|_{\partial B_\eta} \geq 0$.)

Applying Lemma 1.4, we conclude

$$u \geq A|x|^{-a} \ln \frac{1}{|x|} - C \qquad \text{a.e. in } B_\eta$$

Choosing A and η smaller, we may assume that

$$u \geq A|x|^{-a} \ln \frac{1}{|x|} \geq 1 \qquad \text{a.e. in } B_\eta$$

The next step is to consider the function $\Phi \in C^1(\mathbb{R})$ defined by

$$\Phi(x) = \begin{cases} \ln x & \text{if } x \geq 1 \\ x - 1 & \text{otherwise.} \end{cases}$$

and apply Lemma 1.7 in [BC] to conclude that in B_η

$$\begin{aligned}
-\Delta(\ln u) &\geq \frac{-\Delta u}{u} \geq u^{p-1} \geq A^{p-1}|x|^{-a(p-1)} \left(\ln \frac{1}{|x|} \right)^{p-1} \\
&\geq A^{p-1}|x|^{-2} \left(\ln \frac{1}{|x|} \right)^{p-1}
\end{aligned}$$

Now if $v = \left(\ln \frac{1}{|x|} \right)^p$, a computation yields

$$-\Delta v \le C|x|^{-2}\left(\ln \frac{1}{|x|}\right)^{p-1}$$

And by the L^1 maximum principle (Lemma 2.1),

$$\ln u \ge d\left(\ln \frac{1}{|x|}\right)^{p} - C \qquad \text{for some } d>0 \text{ and } C>0$$

This clearly violates $u \in L^1_{loc}(\Omega)$.

2.4 Regularity

We start out with a result in the spirit of Lemma 5.3 in [BC] :

Lemma 3.1. *Let $f \in L^1_{-a,\delta}$ and $v = |x|^{-a}$. Then if $u \in L^1_{-2}$ is the solution given by Lemma 1.3 of*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2}u = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and if $\Phi \in C^1(\mathbb{R})$ is concave, $\Phi' \in L^\infty$ and $\Phi(1) = 0$, then $v\Phi\left(\frac{u}{v}\right) \in L^1_{-2}$ and

$$-\Delta\left(v\Phi\left(\frac{u}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{u}{v}\right)\right) \ge \Phi'\left(\frac{u}{v}\right) f \qquad \text{in the weak sense.}$$

Proof (case $0 < c < c_0$). Suppose first $u, v \in C^2(\bar{\Omega})$, $v > 0$ in Ω and $\Phi \in C^2(\mathbb{R})$ and write $L = -\Delta - a(x)$ where $a(x)$ is a smooth bounded function. Applying Lemma 5.3 in [BC], it follows that a.e. in Ω,

$$\begin{aligned} Lw \ge& \Phi'(u/v)(-\Delta u) + \left[\Phi(u/v) - \Phi'(u/v)u/v\right](-\Delta v) - a(x)\Phi(u/v)v \\ \ge& \Phi'(u/v)Lu + \left[\Phi(u/v) - \Phi'(u/v)u/v\right]Lv \\ \ge& \Phi'(u/v)(Lu - Lv) + \left[\Phi(u/v) - \Phi'(u/v)u/v + \Phi'(u/v)\right](Lv) \end{aligned}$$

Since Φ is concave,

$$\Phi(s) + (1-s)\Phi'(s) \ge \Phi(1) \qquad \text{for all } s \in \mathbb{R}$$

Hence, if $w = v\Phi(u/v)$,

$$Lw \ge \Phi'(u/v)\,(Lu - Lv) \qquad \text{a.e. in } \Omega \tag{3.1}$$

Since Φ' is bounded, we see, as in [BC], that

$$|v\Phi(u/v)| = |v\,(\Phi(u/v) - \Phi(0)) + \Phi(0)v| \leq C(u+v) \tag{3.2}$$

Hence, w vanishes on $\partial\Omega$ and integrating by parts, (3.1) holds in the weak sense. By approximation of Φ, we can also say that (3.1) holds even when Φ is only C^1.

In the general case, let $a_n = c/(|x| + 1/n)^2$ and f_n be a smooth bounded function increasing pointwise and respectively to $c/|x|^2$,f. Let u_n solve the equation $L_n u_n = f$ (with zero boundary condition), where $L_n = -\Delta - a_n(x)$. Also write $w_n = v_n\Phi(u_n/v_n)$ where $v_n = (|x| + 1/n)^{-a}$. We can then apply (3.1) to obtain

$$-\Delta w_n - a_n(x)w_n \geq \Phi'(u_n/v_n)f_n \qquad \text{weakly}$$

Clearly, $v\Phi(u/v)$ is well defined a.e. Moreover, it is clear that $u_n \nearrow u$ in L^1 and that $a_n(x)u_n(x) \nearrow \dfrac{c}{|x|^2}u(x)$ in L^1_δ and similarly for v. So that, using the above equation and Lebesgue's theorem

$$w_n \to w \qquad \text{in } L^1 \text{ and} \qquad a_n(x)w_n \to \frac{c}{|x|^2}w \qquad \text{in } L^1_\delta$$

Since Φ' is bounded, we can also easily pass to the limit in the right-hand side and obtain the desired result. □

Lemma 3.2. *Let u be the minimal weak solution of $(P_{t,p})$ for $t < t_0$ (and $p < p_0$). Then u is a strong solution of $(P_{t,p})$*

Remark 3.2. .

- *By Proposition 0.1, we only need to show that* $0 < u \leq C|x|^{-a}$
- *By Lemma 1.5, we also have the lower bound* $u \geq m|x|^{-a}\,dist(x, \partial\Omega)$.

Proof. Recall that ζ_0 solving, for f as in the definition of $(P_{t,p})$,

$$\begin{cases} -\Delta\zeta_0 - \dfrac{c}{|x|^2}\zeta_0 = f & \text{in } \Omega \\ \zeta_0 = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $0 < \zeta_0 \leq C|x|^{-a}$. For $u \in \mathbb{R}^+$, let

$$g(u) = (u + t_0\|\zeta_0/v\|_{L^\infty})^p \qquad \text{and} \qquad \tilde{g}(u) = (u + t\|\zeta_0/v\|_{L^\infty})^p$$

and construct $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$ and

$$\Phi'(u) = \frac{\tilde{g}(\Phi(u))}{g(u)} \tag{3.3}$$

as in Lemma 4 of [BCMR].

Next, if u_0 is the minimal solution of $(P_{t_0,p})$ then $z := u_0 - t_0\zeta_0$ is the minimal solution of

$$\begin{cases} -\Delta z - \dfrac{c}{|x|^2} z = (z + t_0\zeta_0)^p & \text{in } \Omega \\ \qquad z = 0 & \text{on } \partial\Omega \end{cases}$$

Applying Lemma 3.1 to z with the above function Φ and $v = |x|^{-a}$,

$$-\Delta\left(v\Phi\left(\frac{z}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{z}{v}\right)\right) \geq \Phi'\left(\frac{z}{v}\right)(z + t_0\zeta_0)^p \geq$$
$$\left(\frac{\Phi\left(\frac{z}{v}\right) + t\|\zeta_0/v\|_{L^\infty}}{\frac{z}{v} + t_0\|\zeta_0/v\|_{L^\infty}}\right)^p (z + t_0\zeta_0)^p$$

We need the following easy lemma :

Lemma 3.3. *Let $A, B > 0$ such that $A \leq \frac{t}{t_0}B$. Then*

$$F(C) := \frac{A + tC}{B + t_0 C} \text{ is increasing with } C.$$

Observe that, since Φ is concave and Φ' is defined by (3.3), $\Phi'(u) \leq \Phi'(0) = \left(\frac{t}{t_0}\right)^p < \frac{t}{t_0}$ for $u \in \mathbb{R}^+$. Hence, since $\Phi(0) = 0$, $\Phi(u) \leq \frac{t}{t_0}u$ for $u \in \mathbb{R}^+$. Applying Lemma 3.3 with $A = \Phi(\frac{z}{v})$ and $B = \frac{z}{v}$, we get

$$\frac{\Phi\left(\frac{z}{v}\right) + t\frac{\zeta_0}{v}}{\frac{z}{v} + t_0\frac{\zeta_0}{v}} \leq \frac{\Phi\left(\frac{z}{v}\right) + t\|\frac{\zeta_0}{v}\|_{L^\infty}}{\frac{z}{v} + t_0\|\frac{\zeta_0}{v}\|_{L^\infty}}$$

and

$$-\Delta\left(v\Phi\left(\frac{z}{v}\right)\right) - \frac{c}{|x|^2}\left(v\Phi\left(\frac{z}{v}\right)\right) \geq \left(\frac{\Phi\left(\frac{z}{v}\right) + t\zeta_0/v}{\frac{z}{v} + t_0\zeta_0/v}\right)^p (z + t_0\zeta_0)^p$$
$$\geq \left(v\Phi\left(\frac{z}{v}\right) + t\zeta_0\right)^p$$

We finally define $w = v\Phi\left(\frac{z}{v}\right) + t\zeta_0$, which satisfies

$$\begin{cases} -\Delta w - \dfrac{c}{|x|^2} w \geq w^p + tf & \text{in } \Omega \\ \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

We have just constructed a supersolution of problem $(P_{t,p})$ satisfying $0 < w \leq C|x|^{-a}$ (since $\Phi(\infty) < \infty$ by Lemma 4 in [BCMR]) and, of course, the same estimate holds for u, the minimal solution of $(P_{t,p})$.

This completes the proof of Theorem 1 (in the case $0 < c < c_0$.) □

2.5 Stability

We show first that $\lambda_1(u_t) > -\infty$ (recall Definition 0.2) and study the corresponding eigenfunction ϕ_1.

Indeed, if u_t is the minimal solution of $(P_{t,p})$ with $t < t_0$, then $0 \leq u_t \leq C|x|^{-a}$ and

$$\int_\Omega u_t^{p-1}\phi^2 \leq C\int_\Omega |x|^{-a(p-1)}\phi^2 \leq C\left(\int_\Omega |x|^{-2}\phi^2\right)^{\frac{a(p-1)}{2}} \cdot \left(\int_\Omega \phi^2\right)^{1-\frac{a(p-1)}{2}}$$
$$\leq C\|\phi\|_{H_0^1(\Omega)}^{a(p-1)} \|\phi\|_{L^2}^{2-a(p-1)}$$

So $\lambda_1 > -\infty$ and if $\{\phi_n\}$ is a minimizing sequence of J (see Definition 0.2), $\{\phi_n\}$ is bounded in $H_0^1(\Omega)$ and converges (weakly and up to a subsequence) to $\phi_1 \in H_0^1(\Omega)$ solving

$$\begin{cases} -\Delta\phi_1 - \dfrac{c}{|x|^2}\phi_1 = pu_t^{p-1}\phi_1 + \lambda_1\phi_1 & \text{in } \Omega \\ \\ \phi_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

Claim. $0 \leq \phi_1 \leq C|x|^{-a}$

Testing equation (4.1) against ϕ_1^+, it follows that

$$\int_\Omega |\nabla\phi_1^+|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^+}^2 - \int_\Omega pu^{p-1}{\phi_1^+}^2 = \lambda_1\int_\Omega {\phi_1^+}^2$$

Hence ϕ_1^+ is also a minimizer of J and up to replacing ϕ_1 by ϕ_1^+, we may assume that $\phi_1 \geq 0$.

Next, using local elliptic regularity, $\phi_1 \in C^\infty(\bar\Omega \setminus \{0\})$. Also, pick $\tilde c \in (c, c_0)$ and $\eta > 0$ so small that

$$\frac{\tilde c - c}{|x|^2} \geq pu_t^{p-1} + \lambda_1 \qquad \text{a.e. in } B_\eta.$$

Let $z = \phi_1 - M|x|^{-\tilde{a}}$ and $M = \|\phi_1\|_{L^\infty(\partial B_\eta)}\, \eta^{\tilde{a}}$ ($-\tilde{a}$ being the greater root of $P(X) = X(X-1) + (n-1)$ 0). Then,

$$\begin{cases} -\Delta z - \dfrac{\tilde{c}}{|x|^2} z \leq 0 & \text{in } B_\eta \\ \qquad\qquad z \leq 0 & \text{on } \partial B_\eta \end{cases} \tag{4.3}$$

Testing (4.3) against z^+ (which is permitted since $z^+ \in H_0^1(B_\eta)$),

$$\phi_1 \leq M \cdot |x|^{-\tilde{a}} \qquad \text{a.e. in } B_\eta.$$

With $\tilde{c}$ close enough to c, it follows that $pu_t^{p-1}\phi_1 + \lambda_1\phi_1 \leq C|x|^{-a-2+\epsilon}$, for some $\epsilon > 0$. Let $\zeta \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta \zeta - \dfrac{c}{|x|^2}\zeta = |x|^{-a-2+\epsilon} & \text{in } \Omega \\ \qquad\qquad \zeta = 0 & \text{on } \partial\Omega \end{cases} \tag{4.4}$$

As in the proof of Lemma 1.5,

$$0 \leq \phi_1 \leq C\zeta \leq C|x|^{-a} \qquad \text{a.e. in } \Omega \tag{4.5}$$

Next, we prove that there exists $0 < t_1 \leq t_0$ such that u_t is stable for $t < t_1$.

Fix $b \in (a, a')$ such that $pb < b+2$ and $b + a(p-1) < a+2$, and define $F : X \times \mathbb{R} \to Y$, by

- X is the space of functions $v \in C(\bar{\Omega} \setminus \{0\})$ such that there exist a constant $C > 0$ and a function $g \in C(\bar{\Omega} \setminus \{0\})$ satisfying $|v| \leq C|x|^{-b}$, $|g| \leq C|x|^{-b-2}$ and

$$\begin{cases} -\Delta v - \dfrac{c}{|x|^2} v = g & \text{in } \Omega \\ \qquad\qquad v = 0 & \text{on } \partial\Omega \end{cases}$$

  in the weak sense. X is a Banach space for the norm $\|v\|_X = \|\,|x|^b v\|_{L^\infty} + \|\,|x|^{b+2} g\|_{L^\infty}$

- $Y = \{f \in C(\bar{\Omega} \setminus \{0\}) \,:\, |x|^{b+2} f \in L^\infty(\Omega)\}$, $\|f\|_Y = \|\,|x|^{b+2} f\|_{L^\infty}$

- $F(v,t) = -\Delta v - \frac{c}{|x|^2} v - |v|^p - tf$

Observe that F is well defined with our choice of b, that $F \in C^1$ and that $F(u_t, t) = 0$. Also $L := F_u(0,0)$ is an isomorphism between X and Y. Indeed L is injective by Lemma 1.4 and surjective with continuous inverse by Lemma 1.3. These facts and a global form of the implicit function theorem (see e.g. Cor. 3 in [BN]) imply the existence of a maximal $t_1 > 0$ such that $t \to u_t$ is a C^1 map from $(0, t_1)$ to X and $F_u(u_t, t) \in Iso(X, Y)$.

In particular, since $\phi_1 \in X$, $\lambda_1(u_t) \neq 0$ for $t < t_1$. It can also be shown that $t \to \lambda_1(u_t)$ is continuous : if $\tau_n \to \tau < t_1$ and λ_1^n and ϕ_1^n are the corresponding eigenvalues and eigenfunctions with $\|\phi_1^n\|_{L^2} = 1$, looking carefully at the previous claim, we obtain that ϕ_1^n is bounded in $H_0^1(\Omega)$ and that

$$0 \leq \phi_1^n \leq C|x|^{-a}$$

Passing to a subsequence, it is then easy to show that $\lambda_1^n \to \lambda_1(u_\tau)$ and therefore that λ_1 is continuous.

Hence, since $\lambda_1(0) > 0$ and λ_1 cannot vanish, we have $\lambda_1 > 0$ for $t < t_1$.

We now prove that $t_1 = t_0$. If not, we would have for $t_1 < t < t_0$,

$$\begin{aligned} -\Delta\left(u_t - u_{t_1}\right) - \frac{c}{|x|^2}\left(u_t - u_{t_1}\right) - p u_{t_1}^{p-1}\left(u_t - u_{t_1}\right) = \\ u_t^p - u_{t_1}^p - p u_{t_1}^{p-1}\left(u_t - u_{t_1}\right) + (t - t_1) f \geq (t - t_1) f \end{aligned}$$

And testing against ϕ_1, solution of (4.1) with t_1 in place of t, we would obtain

$$0 \geq (t - t_1) \int_\Omega f \phi_1$$

which is impossible. Hence, $t_1 = t_0$.

Next, we prove that if v is another stable $H_0^1(\Omega)$ solution then it must coincide with u_t.

Suppose indeed v is another $H_0^1(\Omega)$ solution such that $\lambda_1(v) \geq 0$.Then $v \geq u_t$ and

$$\begin{aligned} \int_\Omega p v^{p-1}(v - u_t)^2 &\leq \int_\Omega |\nabla(v - u_t)|^2 - \int_\Omega \frac{c}{|x|^2}(v - u_t)^2 \\ &\leq \int_\Omega (v^p + tf - u_t^p - tf)(v - u_t) \end{aligned}$$

So that,

$$\int_\Omega (v-u_t)(v^p-u_t^p-pv^{p-1}(v-u_t))\geq 0$$

Since $u\to u^p$ is strictly convex and $v\geq u_t$, we must have $v=u_t$.

Finally, stability of strong extremal solutions is determined through the following proposition :

Proposition 4.1. *Suppose that $0<c<c_0$ and $1<p<p_0$. If u, the minimal solution of $(P_{t_0,p})$, solves the problem in the strong sense then*

$$\lambda_1(u)=0$$

Proof. Arguing by contradiction, our general strategy is to use the implicit function theorem to extend the curve $t\to u_t$ of minimal solutions of $(P_{t,p})$ beyond t_0 if $\lambda_1(u)>0$.

More precisely assume that $\lambda_1(u)>0$ and define $F:X\times\mathbb{R}\to Y$ as before. If we can prove that $F_u(u,t_0)\in Iso(X,Y)$, the implicit function theorem will yield the desired contradiction.

We first claim that $F_u(u,t_0)$ is injective. If not, there would be a weak solution $\phi_1\in X$ of

$$\begin{cases} -\Delta\phi_1-\dfrac{c}{|x|^2}\phi_1=pu^{p-1}\phi_1 & \text{in } \Omega\\ \phi_1=0 & \text{on } \partial\Omega\end{cases}$$

Since $b+a(p-1)<a+2$, $u^{p-1}\phi_1\in L^{\frac{2n}{n+2}}(\Omega)$ and, using the methods of Proposition 0.1, ϕ_1 is an $H_0^1(\Omega)$ solution. Testing the above equation against ϕ_1 would then imply $J(\phi_1)=0$, which contradicts $\lambda_1(u)>0$. Thus $F_u(u,t_0)$ is injective.

Next we prove that $F_u(u,t_0)$ is surjective.

First observe that $L:=F_u(0,0)$ is an isomorphism between X and Y. Indeed L is injective by Lemma 1.4 and surjective with continuous inverse by Lemma 1.3.

Let $Z:=\{f\,:\,|x|^{-a(p-1)}f\in Y\}$ and define $K\in\mathcal{L}(Z)$ by

$$K:\begin{cases} Z\to \quad Y\to Z\\ \phi\mapsto pu^{p-1}\phi\mapsto \quad L^{-1}(pu^{p-1}\phi)\end{cases}$$

K is compact in Z. Indeed if $\{\phi_n\}$ is a bounded sequence in Z then $u_n:=K\phi_n$ is bounded in X, by continuity of L^{-1}. It follows from standard elliptic theory that up to a subsequence, $u_n \to u$ uniformly on compacts of $\bar{\Omega} \setminus \{0\}$ for some $u \in X$. Also, letting $\gamma = 2 - a(p-1) > 0$, we have for $\epsilon > 0$ small

$$\|u_n - u\|_Z \le C\|u_n - u\|_{L^\infty(\Omega\setminus B_\epsilon)} + \epsilon^\gamma \|u_n - u\|_{L^\infty_{-b}} \le C(\|u_n - u\|_{L^\infty(\Omega\setminus B_\epsilon)} + \epsilon^\gamma)$$

so that

$$\limsup_{n\to\infty} \|u_n - u\|_Z \le C\epsilon^\gamma$$

Letting $\epsilon \to 0$, we obtain that K is compact in Z.

With these notations, our problem reduces to showing that $Id - K$ is surjective. By Fredholm's alternative, we just need to prove that $Id - K$ is injective. Now if for some $\phi \in Z, \phi = K\phi$ then $\phi \in X$ by definition of K, and $F_u(u, t_0)\phi = 0$. But we just showed that $F_u(u, t_0)$ is injective so $\phi \equiv 0$. □

2.6 What happens in the extremal case $t = t_0$?

In this section, we look at two specific sets of conditions on c, p, f and Ω.

In one case, the minimal solution u of $(P_{t_0,p})$ solves the problem in the strong sense. It then follows from Proposition 5.1 that $\lambda_1(u) = 0$.

In the other case, the minimal solution u is not a strong one and its singularity at the origin is worse than $|x|^{-a}$. Moreover, u is stable, i.e., $\lambda_1(u) > 0$.

Situation 1. *Suppose $\Omega = B_1$, $c < c_0$ close to c_0, f radial and $p > 1$ close to 1. Then u, the minimal solution of $(P_{t_0,p})$, solves the problem in the strong sense and $\lambda_1(u) = 0$. Furthemore, $u = u(r)$ is radial and*

$$\begin{aligned} &u = r^{-a}\, w && \textit{where } w \in C[0,1] \cap C^\infty(0,1] \\ &w' \sim m\, r^{-a(p-1)+1} && \textit{for some } m < 0 \textit{ as } r \to 0 \\ &w' < 0 && \textit{in } (0,1) \end{aligned}$$

Proof. We suppose for simplicity that $f \equiv 1$.

First, we note that for any rotation of the space $A \in SO(n, \mathbb{R})$, $u \circ A$ is a solution of $(P_{t_0,p})$ and since u is minimal, we must have $u \le u \circ A$. This inequality holds almost everywhere in B_1, hence for almost all $y = A^{-1}x$ with $x \in \mathbb{R}^n$ so that u must be radial.

Next, define $\alpha := \sqrt{(n-2)^2 - 4c}$ and for $r \in (0,1)$,

$$\Phi(u)(r) := \frac{r^{-a}}{\alpha} \int_0^1 s^{1+\alpha+a} u^p(s)[\max(s,r)^{-\alpha} - 1]\, ds + \frac{t_0}{2n+c}[r^{-a} - r^2] \tag{5.1}$$

In view of Lemma 1.5, $\Phi(u)(r)$ is well defined for $r \neq 0$ and it follows from Lebesgue's theorem that $w := r^a\, \Phi(u) \in C(0,1]$.

Using Lebesgue's theorem again, it is also true that $w \in C^1(0,1]$ and that for $r \in (0,1]$,

$$w'(r) = -r^{-1-\alpha} \left(\int_0^r s^{1+\alpha+a} u^p(s)\, ds \right) - (2+a)\frac{t_0}{2n+c} r^{1+a}$$

Using the fundamental theorem of calculus, w is twice differentiable a.e. in $(0,1)$ and

$$w''(r) = -r^a u^p(r) - (1+\alpha)\frac{1}{r}\left[w'(r) + (2+a)\frac{t_0}{2n+c} r^{1+a} \right] - (2+a)(1+a)\frac{t_0}{2n+c} r^a$$

So that

$$-(w'' + (1+\alpha)\frac{1}{r} w') = r^a u^p(r) + t_0\, r^a \qquad \text{a.e. in } (0,1) \tag{5.2}$$

Using the fundamental theorem of calculus again, this equation also holds in the sense of distributions in $(0,1)$. Furthermore, since u is a weak solution of $(P_{t_0,p})$, it is not hard to see that $\tilde{w} := r^a\, u$ solves (5.2) in $\mathcal{D}'(0,1)$.

So if $z = \tilde{w}' - w'$, it follows from (5.2) and this last remark that

$$z' + (1+\alpha)\frac{1}{r} z = 0 \qquad \text{in } \mathcal{D}'(0,1).$$

And by a straightforward computation, we see that

$$\left[r^{1+\alpha}\, z \right]' = 0 \qquad \text{in } \mathcal{D}'(0,1).$$

Hence $z = Ar^{-(1+\alpha)}$ for some $A \in \mathbb{R}$ and, for some $B \in \mathbb{R}$,

$$\tilde{w} = w + \frac{A}{\alpha} r^{-\alpha} + B \tag{5.3}$$

Since w is C^1 away from $r = 0$ (and hence, so must be $\tilde{w}$), we must have, on the one hand, using the boundary condition of $(P_{t_0,p})$ and equation (5.1), that $w(1) = \tilde{w}(1) = 0$ and $B = 0$ and on the other hand that u is C^1 away from the origin. Bootstrapping this result with the help of (5.1) and (5.3), it follows that $w, \tilde{w} \in C^\infty(0,1]$.

Let us now prove that $A = 0$. Suppose by contradiction that $A > 0$ and let $u_1(x) := |x|^{-a} w(|x|)$ for $x \in B_1$. Then,

$$-\Delta u_1 - \frac{c}{|x|^2} u_1 = u^p + t_0 = \left[u_1 + \frac{A}{\alpha} (|x|^{-a'} - |x|^{-a}) \right]^p + t_0$$
$$\geq u_1^p + t_0$$

This equation holds at every $x \neq 0$ and also in the weak sense, as integration by parts on $B_1 \setminus B_\epsilon$ with $\epsilon \to 0$ shows. But then u_1 would be a nonnegative supersolution of problem $(P_{t_0,p})$, contradicting minimality of u.

We have just shown that $A \leq 0$. We now prove that $A = 0$. Recall that

$$\tilde{w}'(r) = -r^{-1-\alpha} \int_0^r s^{1+\alpha-a} u^p(s)\, ds - (2-a) \frac{t_0}{2n+c} r^{1+a} - A r^{-1-\alpha} \tag{5.4}$$

By Hopf's boundary lemma, $u'(1) = \tilde{w}'(1) < 0$. We claim that

$$\tilde{w}'(r) < 0 \qquad \text{for all } r \in (0,1]$$

Suppose not and let $r_0 = \sup\{r \in (0,1) : \tilde{w}'(r) = 0\}$. Then $\tilde{w}' < 0$ on $(r_0, 1]$ and (5.2) implies that

$$\tilde{w}''(r_0) = -(1+\alpha) \frac{1}{r_0} \tilde{w}'(r_0) - (r_0^a u^p(r_0) + t_0\, r_0^a) < 0$$

So $\tilde{w}$ has a local maximum at r_0. Suppose by contradiction that $\tilde{w}$ has another critical point and let $r_1 < r_0$ so that

$$\tilde{w}'(r_1) = 0 \qquad \text{and} \qquad \tilde{w}'(r) > 0 \qquad \text{for } r \in (r_1, r_0)$$

From (5.2), it follows as before that $\tilde{w}''(r_1) < 0$ and r_1 would be a local maximum of $\tilde{w}$, contradicting $\tilde{w}' > 0$ on (r_1, r_0).

Hence $\tilde{w}$ has an absolute maximum at r_0 and must therefore be bounded, which forces $A = 0$.

But then, using (5.4), $\tilde{w}'(r) < 0$ in $(0, 1]$, contradicting $\tilde{w}'(r_0) = 0$.

So, we have proved that $\tilde{w}' < 0$ in $(0, 1]$.

From (5.4), it follows that if $A < 0$, $\tilde{w}'(r) = -A r^{-1-\alpha}(1 + o(1))$ as $r \to 0$ and we cannot have at the same time $\tilde{w}' < 0$ and $A < 0$. Hence $A = 0$.

So far we know that :

$$\tilde{w} = w \tag{2.2}$$

$$w' < 0 \qquad \text{in } (0,1] \tag{2.3}$$

We now prove that $u \leq Cr^{-a}$. From equation (5.1), we already know that $u = \Phi(u) \leq Cr^{-a-\alpha}$. Plugging this result into (5.1) again, we only need to show that the right-hand-side integral is bounded as $r \to 0$, which holds as soon as

$$1 - a(p-1) - \alpha p > -1$$

This last condition is satisfied for αp small and in particular when c is close to c_0 and p close to 1. This result, combined with (5.4) yields the asymptotic behaviour of w' at the origin.

Finally, by Proposition 4.1, we have that $\lambda_1(u) = 0$. □When p is chosen close to the critical exponent p_0, the minimal solution u of $(P_{t_0,p})$ may become more singular than when $t < t_0$, in such a way that u^{p-1} has a singularity at the origin of same order as $\frac{1}{|x|^2}$:

Situation 2. *Suppose* $0 < c < c_0$ *and* p *close to* p_0*. Then there exists a smooth nonnegative nonzero data* f *such that* u*, the minimal solution of* $(P_{t_0,p})$*, is stable and such that, near the origin,*

$$u = m\,|x|^{-\gamma} \qquad ,\textit{where } m > 0 \qquad \textit{and} \qquad \gamma = \frac{2}{p-1} > a > 0$$

Proof. We adapt a proof given in [D].

Let $v = |P(-\gamma)|^{\frac{1}{p-1}}\,|x|^{-\gamma}$, where $P(X) = X(X-1) + (n-1)X + c$.

Then, $-\Delta v - \frac{c}{|x|^2}v = v^p$ in $\mathbb{R}^n$ and, since when $p \to p_0$, $\gamma \to a$, we may assume that $v \in H^1$.

Lemma 5 in [D] constructs a function $\psi \in C^\infty(\bar{\Omega})$ with the following properties:

- $\psi \geq 0$ in $\bar{\Omega}$
- $\Delta\psi + \dfrac{c}{|x|^2}\psi \geq 0$ in Ω

- $\psi \equiv 0$ in a neighbourhood of 0, and
- $\psi = v$ on $\partial\Omega$

We then let $u = v - \psi$ and see that

$$\begin{aligned}
-\Delta u - \frac{c}{|x|^2}u &= -\Delta v - \frac{c}{|x|^2}v + \Delta\psi + \frac{c}{|x|^2}\psi \\
&= v^p + \Delta\psi + \frac{c}{|x|^2}\psi \\
&\geq 0
\end{aligned}$$

and $u = 0$ on $\partial\Omega$, so, by Lemma 1.1 say, $u \geq 0$.

Taking $f = \Delta\psi + \frac{c}{|x|^2}\psi + v^p - u^p$, we then have

$$-\Delta u - \frac{c}{|x|^2}u = u^p + f.$$

Observe that $f \geq 0$ and is smooth since $u \leq v$ and $u \equiv v$ near the origin.

Next, we prove that $\lambda_1(u) > 0$. Given $\phi \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega pu^{p-1}\phi^2 &\leq \int_\Omega pv^{p-1}\phi^2 \\
&= p|P(-\gamma)|\int_\Omega \frac{\phi^2}{|x|^2} \\
&\leq \int_\Omega |\nabla\phi|^2 - \int_\Omega \frac{c}{|x|^2}\phi^2 - \epsilon\int_\Omega \phi^2
\end{aligned}$$

The last inequality holds, using Hardy's inequality (0.3), provided $c + p|P(-\gamma)| < c_0$ and $\epsilon > 0$ small. This condition is readily satisfied since as $p \to p_0$, $\gamma \to a$ and $P(-\gamma) \to P(-a) = 0$. Hence, we get that $\lambda_1(u) \geq \epsilon > 0$.

We still need to prove that, for our choice of f, $t_0 = 1$ and u is the minimal solution of $(P_{t_0,p})$.

If u_1 denotes the minimal solution of $(P_{1,p})$, it is clear that $0 \leq u_1 \leq u$, hence $u_1^p \leq \frac{C}{|x|^2}$ and using this inequality and $(P_{1,p})$, $u_1 \in H_0^1(\Omega)$.

Since $\lambda_1(u) \geq 0$, it follows that

$$\begin{aligned}
\int_\Omega pu^{p-1}(u-u_1)^2 &\leq \int_\Omega |\nabla(u-u_1)|^2 - \int_\Omega \frac{c}{|x|^2}(u-u_1)^2 \\
&\leq \int_\Omega (u^p + f - u_1^p - f)(u-u_1)
\end{aligned}$$

So that,

$$\int_\Omega (u-u_1)(u^p - u_1^p - pu^{p-1}(u-u_1)) \geq 0$$

Since $u \to u^p$ is strictly convex and $u \geq u_1$, we must have $u = u_1$. And since u is not a strong solution of $(P_{1,p})$, we must have $1 = t_0$. $\square$

2.7 The case $c = c_0$

When $c = c_0$, the operator $-\Delta - \frac{c}{|x|^2}$ is no longer coercive in $H_0^1(\Omega)$. However, one can still make use of the improved Hardy inequality (see [BV] or [VZ])

$$\int_\Omega |\nabla u|^2 - c_0 \int_\Omega \frac{u^2}{|x|^2} \geq C(\Omega) \int_\Omega u^2 \qquad \text{for all } u \in C_c^\infty(\Omega) \tag{6.1}$$

to define a new Hilbert space H in which the operator is coercive, even when $c = c_0$.

Definition. *H is the space obtained by completing $C_c^\infty(\Omega)$ with respect to the norm*

$$\|u\|_H^2 := \int_\Omega |\nabla u|^2 - c_0 \int_\Omega \frac{u^2}{|x|^2}$$

By analogy with the case $c < c_0$, an H solution u will be one such that $u^p \in H^*$ and such that the equation holds in the sense of Lax-Milgrams lemma in H.

We now list the modifications needed to prove Theorem 1 when $c = c_0$. When no proof is given, just replace $H_0^1(\Omega)$ by H in the original demonstration.

Lemma 1.1'. *Lemma 1.1 still holds if $c = c_0$, $H_0^1(\Omega)$ is replaced by H and H^{-1} by H^*, the dual of H.*

Proof. Only the proof of (1.3) needs to be clarified in this setting.

Let $f \in H^*$, $f \geq 0$ and $u \in H$ be the corresponding solution of (1.1).

By definition of H, there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ converging to u in H. Letting $f_n = -\Delta u_n - \frac{c}{|x|^2} u_n$, it follows that $f_n \in H^{-1}(\Omega)$ and $f_n \to f$ in H^*.

Now, $u_n \in H_0^1(\Omega) \Rightarrow u_n^- \in H_0^1(\Omega)$ and integrating the equation satisfied by u_n against u_n^- yields

$$-\|u_n^-\|_H^2 = \langle f_n, u_n^- \rangle_{H*,H}$$

To pass to the limit in this last equation, we just need to prove that $\{u_n^-\}$ remains bounded in H. But

$$\begin{aligned}
\|u_n^-\|_H^2 &= \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{(u_n^-)^2}{|x|^2} \\
&= \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} + \int_\Omega \frac{c_0}{|x|^2}(u_n^+)^2 \\
&\leq \int_\Omega |\nabla u_n^-|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} + \int_\Omega |\nabla u_n^+|^2 = \int_\Omega |\nabla u_n|^2 - c_0 \int_\Omega \frac{u_n^2}{|x|^2} \\
&= \|u_n\|_H^2
\end{aligned} \tag{6.2}$$

where we've used (0.3) in the inequality. □

Proposition 0.1'. *Proposition 0.1 still holds when $c = c_0$ and $H_0^1(\Omega)$ solutions are replaced by H solutions.*

Proof. Suppose first that u is a strong solution of $(P_{t,p})$.

Let $\zeta_n \in C_c^\infty(\Omega \setminus \{0\})$ be such that $0 \leq \zeta_n \leq 1$, $|\Delta\zeta_n| \leq Cn^2$ and

$$\zeta_n = \begin{cases} 0 & \text{if} \quad |x| \leq 1/n \text{ and } \delta(x) \leq 1/n \\ 1 & \text{if} \quad |x| \geq 2/n \text{ and } \delta(x) \geq 2/n \end{cases}$$

Multiplying $(P_{t,p})$ by $u\zeta_n$ and integrating by parts, it follows that

$$\begin{aligned}
\int_\Omega (u^p + tf)\, u\zeta_n &= -\int_\Omega \Delta u\, u\zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n \\
&= \int_\Omega |\nabla u|^2 \zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n + \int_\Omega u \nabla u \nabla \zeta_n
\end{aligned}$$

Since $u \leq C|x|^{-a}$ and $p < p_0$, $u^p \leq C|x|^{-a-2+\epsilon}$, for some $\epsilon > 0$, so that the first integral in the above equation is bounded by $C\int_\Omega |x|^{-2a-2+\epsilon} \leq C$ whereas $\left|\int_\Omega u\nabla u \nabla\zeta_n\right| = \left|\frac{1}{2}\int_\Omega u^2 \Delta\zeta_n\right| \leq Cn^2 \int_{1/n<|x|<2/n} |x|^{-2a} \leq C$ as $n \to \infty$.

Hence $\int_\Omega |\nabla u|^2 \zeta_n - \int_\Omega \frac{c}{|x|^2} u^2 \zeta_n \leq C$ and $u \in H$. Approximating $u \in H$ by smooth functions and integrating by parts in $\Omega \setminus B_\epsilon$ with $\epsilon \to 0$, it follows that u is a weak solution of $(P_{t,p})$. For u to be an H solution, we only need to prove the following :

Claim. Suppose u is a weak solution satisfying the estimate $u \leq C|x|^{-a}$. Then

$$u^p \in H^*$$

For $\phi \in C_c^\infty(\Omega)$, $1 < q < 2$, it follows from Hölder's inequality that

$$\left|\int_\Omega |x|^{-ap}\phi\right| \le \left(\int_\Omega |x|^{-(ap-1)\frac{q}{q-1}}\right)^{\frac{q-1}{q}} \left\|\frac{\phi}{|x|}\right\|_{L^q}$$

On the one hand, since $p < p_0$, the integral in the right hand side will be finite if q is chosen close enough to 2.

On the other hand, using Hardy's inequality in L^q and the inclusion $H \hookrightarrow W_0^{1,q}$ (see section 4 of [VZ]),

$$\left\|\frac{\phi}{|x|}\right\|_{L^q} \le C\|\phi\|_{W_0^{1,q}} \le C\|\phi\|_H$$

and $u^p \in H^*$.

Hence, strong solutions are also H solutions.

Showing that H solutions are weak solutions is similar to the case $c < c_0$, whereas, starting from a weak solution u, we observe as above that $u^p \in H^*$ and define $u_n \ge 0$ to be the minimal weak solution of

$$\begin{cases} -\Delta u_n - \dfrac{c - 1/n}{|x|^2} u_n = u_n^p + tf & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

u is a supersolution of this equation so u_n is well defined and $0 \le u_n \le u \le C|x|^{-a}$. By Proposition 0.1 (case $c < c_0$), it follows that $u_n \in H_0^1(\Omega)$ and testing in the above equation against u_n,

$$\|u_n\|_H^2 \le (\|u_n^p\|_{H^*} + C)\|u_n\|_H$$

Letting $n \to \infty$, we get $u \in H$. Since $H \hookrightarrow W_0^{1,q}$ for $1 \le q < 2$, elliptic regularity can be applied to complete the proof. $\square$

Lemma 1.3'. *Lemma 1.3 still holds when $c = c_0$*

Lemma 1.4'. *Lemma 1.4 still holds when $c = c_0$*

Lemma 1.5'. *Lemma 1.5 still holds when $c = c_0$*

Proof. We assume first that u is the minimal (weak) solution of (1.20) and can therefore be written as the pointwise limit of an increasing sequence $\{u_\epsilon\}$, where u_ϵ solves (1.20) with $c-\epsilon$ in place of c. Since Lemma 1.5 can be applied to u_ϵ, an argument of monotone convergence yields the result.

If u isn't minimal, the above discussion yields all the results up to the conclusion of Step 6 in the original proof. That step can be applied -as is- in our context, which finishes the proof. $\square$

2.1' : Existence of a solution of $(\mathbf{P_{t,p}})$ for $\mathbf{1 < p < p_0}$, small $\mathbf{t > 0}$.

For simplicity, we assume without loss of generality that $\Omega \subset B_{\frac{1}{2}}$. Consider $w = A|x|^{-a}\left(\ln\frac{1}{|x|}\right)^{\frac{1}{4p}}$, with $A > 0$ to be fixed later. Then $-\Delta w - \frac{c}{|x|^2}w = \frac{4p-1}{16p^2}A\,|x|^{-a-2}\left(\ln\frac{1}{|x|}\right)^{\frac{1}{4p}-2}$ and w will be a supersolution of $(P_{t,p})$ as soon as

$$\begin{cases} \dfrac{4p-1}{32p^2}A|x|^{-a-2}\left(\ln\dfrac{1}{|x|}\right)^{\frac{1}{4p}-2} \geq A^p|x|^{-ap}\left(\ln\dfrac{1}{|x|}\right)^{p/4} \\ \dfrac{4p-1}{32p^2}A|x|^{-a-2}\left(\ln\dfrac{1}{|x|}\right)^{\frac{1}{4p}-2} \geq tf \end{cases}$$

The first inequality amounts to

$$A \leq C \min_{r\in(0,1/2]}\left\{r^{-a-2+pa}\left(\ln\frac{1}{r}\right)^{\frac{1}{4p}-2-p/4}\right\}^{\frac{1}{p-1}}$$

and the second to

$$t \leq C\cdot A \min_{r\in(0,1/2]}\left\{r^{-a-2}\left(\ln\frac{1}{r}\right)^{\frac{1}{4p}-2}\right\}$$

Under these conditions, w is a supersolution of $(P_{t,p})$. We now just have to construct a supersolution in H. Let w_1 be a smooth extension inside Ω of $w|_{\partial\Omega}$ such that $w_1 = w$ in $\Omega\setminus B_{1/4}$ and $w_1 = 0$ in $B_{1/8}$. Next, we let $g = \Delta w_1 + \dfrac{c}{|x|^2}w_1$ and construct $z \in H$ solving (2.1) and $w_2 = z + w_1$ solving (2.2).

We would like to show that $w_2 \geq 0$ and remark that $w_2^- \in H$. Indeed, let $\phi_k \in C_c^\infty(\Omega) \to z$ in H. Then $(\phi_k + w_1)^- \in H_0^1(\Omega) \subset H$ and

$$\begin{aligned}\|(\phi_k + w_1)^-\|_H^2 &= \int_{\{\phi_k+w_1<0\}}\left(|\nabla(\phi_k+w_1)|^2 - \frac{c}{|x|^2}(\phi_k+w_1)^2\right) \\ &\leq \|\phi_k\|_H^2 + C + 2\int_{\{\phi_k+w_1<0\}}\left(\nabla\phi_k\cdot\nabla w_1 - \frac{c}{|x|^2}\phi_k w_1\right) \\ &\leq \|\phi_k\|_H^2 + C + \frac{1}{2}\|(\phi_k+w_1)^-\|_H^2\end{aligned}$$

Hence $\|(\phi_k + w_1)^-\|_H \leq C$ and passing to the limit (in the weak topology and for a subsequence), it follows that $w_2^- \in H$.

Letting $\psi_k \in C_c^\infty(\Omega) \to w_2^-$ in H, integration by parts then yields

$$(w_2|\psi_k)_H = \int_\Omega \left(\nabla z \nabla \psi_k - \frac{c}{|x|^2} z\psi_k\right) + \int_\Omega \left(\nabla w_1 \nabla \psi_k - \frac{c}{|x|^2} w_1 \psi_k\right) = \int_{\partial\Omega} \psi_k \partial_\nu w$$

and letting $k \to \infty$ in H,

$$\|w_2^-\|_H^2 = (w_2|w_2^-)_H = \int_{\partial\Omega} \partial_\nu w\, w^- = 0$$

Hence $w_2 \geq 0$.

Finally, letting $\tilde{w} = w - w_2 = w - z - w_1$, we only need to prove that $\tilde{w} \in H$ and the rest of the proof will remain unchanged. Since $z \in H$, it is enough to show that $w - w_1 \in H$.

If $H(\omega)$ denotes the space H relative to the open set ω of $\mathbb{R}^n$, it has been shown in [VZ] (see 5.2) that f defined for $0 < r < r_0 < 1$ by

$$f(r) = r^{-a}(\ln(1/r))^\alpha$$

and continued smoothly up to the boundary of the ball B_1, where $f = 0$, belongs to $H(B_1)$ as long as $\alpha < 1/2$.

$(w - w_1)|_{B_{1/4}}$ precisely satisfies these conditions, hence belongs to $H(B_{1/4})$. Since $w - w_1 \equiv 0$ in $\Omega \setminus B_{1/4}$, it follows that $w - w_1 \in H(\Omega)$.

2.3' : Case $\mathbf{p} \geq \mathbf{p_0}$: blow-up.

By Proposition 2.1, we just need to prove that $(P_{t,p})$ has no weak solution if $p \geq p_0$. Assume by contradiction there exists one and call it u. If we apply Lemma 1.5 with $u^p + tf$ in place of f, it follows that

$$\int_\Omega u^p |x|^{-a}\, \delta(x) < \infty \qquad \text{and} \qquad u \geq m|x|^{-a} \qquad \text{a.e. near the origin.}$$

This is impossible since near the origin,

$$|x|^{-a} u^p \geq m|x|^{-a(p+1)} \geq m|x|^{-n}$$

Lemma 3.1'. *Lemma 3.1 still holds when* $c = c_0$

Lemma 3.2'. *Lemma 3.2 still holds when* $c = c_0$ *and* $t < t_0$

Theorem 2'. *Theorem 2 still holds when* $c = c_0$ *with* $H_0^1(\Omega)$ *solutions replaced by H solutions.*

Proof. For $t < t_0$, u_t the strong minimal solution of $(P_{t,p})$ can be written as the monotone limit of u_n, where u_n is the strong minimal solution of the same problem with c replaced by $c_n = c - 1/n$.By our analysis in the case $c < c_0$, we know that $\lambda_1(u_n) > 0$. Passing to the limit, we easily get that $\lambda_1(u_t) \geq 0$.

To obtain a strict inequality, it is enough to show that $t \to \lambda_1(u_t)$ is a strictly decreasing function. It should be clear from its definition that $t \to \lambda_1(u_t)$ is nonincreasing.

Suppose that $\lambda_1(u_t) = \lambda_1(u_s)$ for some $s \leq t$. Call ϕ_1^t and ϕ_1^s the corresponding eigenfunctions, which can be constructed as in the case $c < c_0$. Then,

$$\begin{aligned}
\lambda_1(u_t) &= \int_\Omega |\nabla\phi_1^t|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^t}^2 - \int_\Omega p u_t^{p-1}{\phi_1^t}^2 \\
&\leq \int_\Omega |\nabla\phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^s}^2 - \int_\Omega p u_t^{p-1}{\phi_1^s}^2 \\
&\leq \int_\Omega |\nabla\phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^s}^2 - \int_\Omega p u_s^{p-1}{\phi_1^s}^2 \\
&= \lambda_1(u_s)
\end{aligned}$$

Hence,

$$\int_\Omega |\nabla\phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^s}^2 - \int_\Omega p u_t^{p-1}{\phi_1^s}^2 = \int_\Omega |\nabla\phi_1^s|^2 - \int_\Omega \frac{c}{|x|^2}{\phi_1^s}^2 - \int_\Omega p u_s^{p-1}{\phi_1^s}^2$$

and

$$u_t = u_s \qquad \text{,which implies} \qquad t = s\ .$$

Hence u_t is a stable solution of $(P_{t,p})$.

To prove that u_t is the only stable H solution, we can argue exactly as in the case $c < c_0$. □

The results of section 5 extend in the following way (we skip the proof) :

Situation 1'. *Suppose $c = c_0$, $\Omega = B_1$, f radial and $1 < p < p_0$. Then u, the minimal solution of $(P_{t_0,p})$, solves the problem in the strong sense.*

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Chapter 3

Comparison results for PDE's with a singular potential

3.1 Introduction

Here we consider comparison results for linear elliptic and parabolic equations with singular potentials. Let $\Omega \subset \mathbb{R}^n$ be a smooth and bounded domain, and let $a \in L^1_{\text{loc}}(\Omega)$, $a \geq 0$. To motivate the discussion assume initially that a is smooth and bounded, and suppose that

$$\lambda_1 = \inf_{\varphi \in C^1_c(\Omega)} \frac{\int_\Omega \left(|\nabla \varphi|^2 - a(x)\varphi^2 \right)}{\int_\Omega \varphi^2} > 0 \tag{3.1}$$

i.e., the first eigenvalue for the problem

$$\begin{cases} -\Delta\varphi_1 - a(x)\varphi_1 &= \lambda_1 \varphi_1 \quad \text{in } \Omega \\ \varphi_1 &= 0 \qquad \text{on } \partial\Omega \end{cases} \tag{3.2}$$

is positive. We call φ_1 the first eigenfunction and we take it to be positive. Since a is smooth, it is well known that

$$C^{-1}\zeta_0 \leq \varphi_1 \leq C\zeta_0 \tag{3.3}$$

for some positive constant C, where ζ_0 is the solution of

$$\begin{cases} -\Delta\zeta_0 - a(x)\zeta_0 &= 1 \quad \text{in } \Omega \\ \zeta_0 &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Note that this problem is well posed and that $\zeta_0 > 0$, since $\lambda_1 > 0$.

We can formulate condition (3.1) without any assumption on the smoothness of a. An interesting example is the so-called inverse-square potential

$$a(x) = \frac{c}{|x|^2}, \tag{3.5}$$

where $n \geq 3$ and $0 < c \leq \frac{(n-2)^2}{4}$. An improved version of Hardy's inequality (see Brezis and Vázquez [BV] or Vázquez and Zuazua [3]) shows that it satisfies (3.1). On the other hand, it just fails to belong to $L^{n/2}(\Omega)$ if $0 \in \Omega$, and therefore the standard elliptic regularity theory is not sufficient to conclude an estimate like (3.3). In fact, for this potential, there exists a constant $\alpha > 0$ (more precisely $\alpha = (n-2)/2 - \sqrt{(n-2)^2/4 - c}$) such that ζ_0 and φ_1 behave like $|x|^{-\alpha}$ near the origin (see Dupaigne [D]), so that (3.3) can be interpreted as : "φ_1 cannot have worse singularities than ζ_0". In this note we prove (3.3) under a slightly stronger condition than (3.1).

We also want to extend the following version of the strong maximum principle for the heat equation (see e.g. Brezis and Cazenave [4] or Martel [14]) : let $T > 0$ and $u = u(x,t) \geq 0$ be a solution of

$$\begin{cases} u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{cases}$$

Then either $u \equiv 0$ or

$$u(x,t) \geq c(t)\delta(x), \tag{3.6}$$

where c is a positive function of $t \in (0,T)$ and $\delta(x) = \text{dist}(x, \partial\Omega)$.

Using Hopf's boundary lemma on one hand, and elliptic regularity on the other, observe that for some $C > 0$,

$$C^{-1}\delta \leq \tilde{\zeta}_0 \leq C\delta$$

where $\tilde{\zeta}_0$ is the solution of

$$\begin{cases} -\Delta\tilde{\zeta}_0 &= 1 \quad \text{in } \Omega \\ \tilde{\zeta}_0 &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Thus (3.6) is equivalent to

$$u(x,t) \geq c(t)\tilde{\zeta}_0(x). \tag{3.7}$$

We would like to extend (3.7) to the case where ζ_0 solves (3.4) and $u > 0$ solves

$$\begin{cases} u_t - \Delta u - a(x)u &= 0 \quad \text{in } \Omega \times (0,T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.8}$$

Inequality (3.7) was already proven for the inverse-square potential in Baras and Goldstein [BG] and the authors mentioned (see Remark 7.1 in [BG]) that their methods apply to potentials of the form $a(x) = -\Delta\phi/\phi$ where ϕ satisfies a certain weighted

Sobolev inequality. In our proof, we derive such an inequality (see (3.34)) under an almost optimal assumption on the potential a: see (3.9). As in [BG], we also make use of Moser iteration type arguments, but our approach is, we believe, simpler.

The comparison results obtained in this note are motivated by and apply to some semilinear parabolic equations studied in Dupaigne and Nedev [12]. As we shall see, they also generalize to problems involving other boundary conditions and complement the results obtained in Dupaigne [D].

3.2 Main results

The assumption on the potential a is the following: there exists $r > 2$ such that

$$\gamma(a) := \inf_{\varphi \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2}{\left(\int_\Omega |\varphi|^r\right)^{2/r}} > 0. \tag{3.9}$$

Remark. Observe that if a satisfies (3.1) then for any small $\varepsilon > 0$, $a_\varepsilon := (1-\varepsilon)a$ satisfies (3.9) with $r = 2^* = 2n/(n-2)$ (when $n = 2$, pick any $r \in (2, \infty)$), by Sobolev's embedding. In particular, (3.1) can be seen as a limiting case of (3.9).

We also observe that if $n \geq 3$ the inverse square potential (3.5) satisfies (3.9), with $r = 2^*$ if $0 \leq c < \frac{(n-2)^2}{4}$ and with any $2 < r < 2^*$ for $c = \frac{(n-2)^2}{4}$ (see [BV, 3]).

Before stating our results we clarify in what sense we consider the solutions to (3.2) and (3.4). This is necessary because in the context of weak solutions, or solutions in the sense of distributions, uniqueness may not hold in general, and (3.3) can fail. For example, in the case of the inverse square potential (3.5), when Ω is the unit ball $B_1(0)$ and $0 < c < \frac{(n-2)^2}{4}$, $n \geq 3$ there is a positive solution u to

$$\begin{cases} -\Delta u - \frac{c}{|x|^2}u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.10}$$

which is smooth except at the origin and belongs to $W^{1,1}(\Omega)$. This shows that uniqueness in general doesn't hold.

Furthermore, there exists a solution ζ_0 of (3.4), smooth in $\Omega \setminus \{0\}$, behaving like $|x|^{-\alpha'}$ near the origin, where $\alpha' = (n-2)/2 + \sqrt{(n-2)^2/4 - c}$, and a solution φ_1 of

(3.2) which behaves like $|x|^{-\alpha}$ where $\alpha = (n-2)/2 - \sqrt{(n-2)^2/4 - c} < \alpha'$. But then (3.3) would fail. For details, see L. Dupaigne [D].

Hence we only consider solutions that belong to the Hilbert space H, defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_H^2 = \int_\Omega |\nabla u|^2 - \int_\Omega a(x)u^2. \tag{3.11}$$

This norm comes from an inner product $(\cdot|\cdot)_H$ in H, and with some abuse of notation we can write

$$(u|v)_H = \int_\Omega \nabla u \cdot \nabla v - \int_\Omega a(x)uv.$$

We denote by H^* the dual of H. Observe that $H_0^1(\Omega) \subset H \subset L^2(\Omega)$ and therefore $L^2(\Omega) \subset H^* \subset H^{-1}(\Omega)$.

Definition 3.1. *If $f \in H^*$ we say that $u \in H$ is an H-solution of*

$$\begin{cases} -\Delta u - a(x)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.12}$$

if

$$(u|v)_H = \langle f, v\rangle_{H^*,H}$$

for all $v \in H$. With the obvious abuse of notation, this is equivalent to

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega a(x)uv = \int_\Omega fv \quad \text{for all } v \in H.$$

From now on, we only deal with solutions in this sense, i.e. H-solutions.

Lemma 3.2. *Suppose (3.1) holds and let $f \in H^*$. Then there exists a unique H solution u of (3.10). Furthermore,*

$$\|u\|_H = \|f\|_{H^*}$$

and if $f \geq 0$ in the sense of distributions then $u \geq 0$ a.e.

See a proof in [12].

We also have to precise how to obtain a first eigenfunction for the operator $-\Delta - a(x)$ with zero Dirichlet boundary data.

Lemma 3.3. *Suppose $a(x) \geq 0$ satisfies (3.9). Then H embeds compactly in $L^2(\Omega)$. In particular the operator $L := -\Delta - a(x) : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ where $D(L) = \{u \in H(\Omega) \mid -\Delta u - a(x)u \in L^2(\Omega)\}$ has a positive first eigenvalue*

$$\lambda_1 = \inf_{\varphi \in H \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi|^2 - \int_\Omega a(x)\varphi^2}{\int_\Omega \varphi^2}.$$

This infimum is attained at a positive $\varphi_1 \in H$ that satisfies (3.2). Moreover λ_1 is a simple eigenvalue for $-\Delta - a(x)$, and, if φ is a non-negative, non-trivial H-solution of

$$\begin{cases} -\Delta\varphi - a(x)\varphi &= \lambda\varphi \quad \text{in } \Omega \\ \varphi &= 0 \quad \text{on } \partial\Omega \end{cases}$$

for some $\lambda \in \mathbb{R}$, then $\lambda = \lambda_1$.

Similarly, we can define H-solutions of the evolution equation (3.8) with initial condition $u(0) = u_0 \in L^2(\Omega)$:

Definition 3.4. *The operator L defined in Lemma 3.3 is a bounded below self-adjoint operator with dense domain and generates an analytic semigroup $(S(t))_{t\geq 0}$ in L^2.*

Hence for $u_0 \in L^2(\Omega)$, there exists a unique

$$u := S(t)u_0 \in C([0,\infty), L^2) \cap C^1((0,\infty), L^2) \cap C((0,\infty), H)$$

solving

$$\begin{cases} u_t + Lu &= 0 \quad \text{for } t > 0 \\ u(0) &= u_0 \end{cases}$$

which we call the H-solution (or simply the solution) of (3.8) with initial condition $u(0) = u_0 \in L^2(\Omega)$.

The main results in this paper are the following.

Theorem 3.5. *Assume $a : \Omega \to [0,\infty)$ satisfies (3.9). Let $\varphi_1 > 0$ denote the first eigenfunction for the operator $-\Delta - a(x)$ with zero Dirichlet boundary condition, normalized by $\|\varphi_1\|_{L^2(\Omega)} = 1$ and ζ_0 denote the solution of (3.4). Then there exists $C = C(\Omega, \gamma(a), r) > 0$ such that*

$$C^{-1}\zeta_0 \leq \varphi_1 \leq C\zeta_0.$$

Theorem 3.6. *Assume that $a : \Omega \to [0,\infty)$ satisfies (3.9). Let $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and let u denote the solution of (3.8) with initial condition u_0. Let ζ_0 denote again the solution of (3.4). Then*

$$u(t) \geq c(t)\zeta_0$$

for some $c(t) > 0$ depending on u_0, Ω, $\gamma(a)$, r and t.

Corollary 3.7. *Under the same assumptions as in Theorem 3.6, we have more precisely*

$$u(t) \geq c(t) \left(\int_\Omega u_0\zeta_0 \right) \zeta_0,$$

where one can choose $c(t) = e^{-K(t+1/t)}$ for some $K = K(\Omega, \gamma(a), r) > 0$.

Corollary 3.8. *Assume $a : \Omega \to [0,\infty)$ satisfies (3.9) and let u solve (3.12) for some $f \geq 0$, then*

$$u \geq c \left(\int_\Omega f\zeta_0 \right) \zeta_0,$$

where $c = c(\Omega, \gamma(a), r)$.

Remarks. 1) All the results apply also for a potential $a(x)$ that changes sign in Ω, under the following additional hypothesis:

$$\begin{cases} a(x) = a^+(x) - a^-(x) \quad a^+, a^- \geq 0 \\ a^+ \in L^1_{\text{loc}}(\Omega) \text{ and } a^-(x) \in L^\infty(\Omega). \end{cases} \tag{3.13}$$

In this case the constants also depend on $\|a^-\|_{L^\infty(\Omega)}$.

2) Theorem 3.6 and Corollary 3.7 also hold under the following less restrictive hypothesis: suppose that

$$\gamma(a)\Big(\int_\Omega |\varphi|^r \Big)^{2/r} \leq \int_\Omega \Big(|\nabla\varphi|^2 - a(x)\varphi^2 + M\varphi^2 \Big) \tag{3.14}$$

for all $\varphi \in C_c^\infty(\Omega)$, for some $M(a) > 0$, $\gamma(a) > 0$ and $r > 2$. In this case we define H as the completion of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_H^2 = \int_\Omega \Big(|\nabla u|^2 - a(x)u^2 + Mu^2 \Big).$$

Theorem 3.9. *Suppose that $a(x)$ satisfies (3.13) and (3.14). Let $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$, and let u denote the H-solution of (3.8) with initial condition u_0. Then*

$$u(t) \geq c(t) \left(\int_\Omega u_0 \varphi_1 \right) \varphi_1,$$

where one can choose $c(t) = e^{-K(t+1/t)}$ for some K depending on Ω, $\gamma(a)$, r and M, and where $0 < \varphi_1 \in H$ is the first eigenfunction for $-\Delta - a(x)$ normalized by $\|\varphi_1\|_{L^2} = 1$.

In Section 3.7 we mention some examples of potentials satisfying (3.14) for which the stronger condition (3.9) may fail.

Observe that condition (3.14) implies the more standard inequality

$$\inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 - a(x)\varphi^2}{\int_\Omega \varphi^2} > -\infty,$$

which is a necessary condition for the existence of global nonnegative solutions with exponential growth to the linear parabolic equation (3.8) (see Cabré and Martel [CM]).

3) The method presented here for the parabolic problem also applies to equations with mixed boundary condition, extending a result of [10] to the parabolic case. Let Γ_1, Γ_2 be a partition of $\partial\Omega$, with $\Gamma_1 \neq \emptyset$. For simplicity we can assume that Γ_1, Γ_2 are smooth, but this is not important.

In this context, let $\bar{\zeta}_0$ denote the solution of

$$\begin{cases} -\Delta \bar{\zeta}_0 &= 1 \quad \text{in } \Omega \\ \bar{\zeta}_0 &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial \bar{\zeta}_0}{\partial \nu} &= 0 \quad \text{on } \Gamma_2, \end{cases}$$

where ν denotes the unit outward normal vector to $\partial\Omega$.

Theorem 3.10. *Let $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ and let u denote the solution to*

$$\begin{cases} u_t - \Delta u &= 0 \quad \text{in } \Omega \times (0,\infty) \\ u &= 0 \quad \text{on } \Gamma_1 \times (0,\infty) \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_2 \times (0,\infty) \\ u(0) &= u_0 \quad \text{in } \Omega \end{cases}$$

Then

$$u(t) \geq c(t) \Big(\int_\Omega u_0 \zeta_0 \Big) \zeta_0$$

where $c(t) = e^{-K(t+1/t)}$ for some $K = K(\Omega, \Gamma_1, \Gamma_2)$.

We omit its proof, which is a slight modification of the one given for Theorem 3.6.

3.3 Some preliminaries

We start this section with some preliminary results on the linear equation

$$\begin{cases} -\Delta u - a(x)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

when the potential $a(x)$ satisfies (3.9). As mentioned before all solutions to (3.15) are assumed to be in H.

The last two lemmas in this section allow to reduce the proofs of the main results of this paper to the case of a bounded potential.

Lemma 3.11. *Assume $a(x)$ satisfies (3.9), and that $f \in L^2(\Omega)$. Then the solution u to (3.15) satisfies*

$$\int_\Omega u(-\Delta\zeta) = \int_\Omega a(x)u\zeta + f\zeta \tag{3.16}$$

for all $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$, and all in the integrals in (3.16) exist and are finite. In particular, by taking ζ to be the solution of

$$\begin{cases} -\Delta\tilde{\zeta_0} &= 1 \quad \text{in } \Omega \\ \tilde{\zeta_0} &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.17}$$

we conclude that $a(x)u + f \in L^1_{\text{loc}}(\Omega)$.

Proof. By working with f^+, f^- we can assume that $f \geq 0$. Let

$$a_k(x) = \min(a(x), k), \quad k > 0$$

and u_k be the solution to (3.15) with the potential $a(x)$ replaced by the potential $a_k(x)$. Then it is easy to check that u_k is nondecreasing in k, and converges to u in $L^2(\Omega)$. Now take $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$. Then

$$\int_\Omega u_k(-\Delta\zeta) = \int_\Omega a_k(x)u_k\zeta + f\zeta, \tag{3.18}$$

and note that here all the integrals are finite. By taking in particular $\zeta = \tilde{\zeta_0}$ (where $\tilde{\zeta_0}$ is the solution of (3.17)), and using Fatou's lemma, we see that $\int_\Omega a(x)u\tilde{\zeta_0}$ exists and is finite. Given any $\zeta \in C^2(\overline{\Omega})$, $\zeta = 0$ on $\partial\Omega$, we can find $C > 0$ so that $|\zeta| \leq C\tilde{\zeta_0}$. It follows that we can pass now to the limit in (3.18) and conclude that (3.16) holds. ☐

Lemma 3.12. *Assume that $a(x)$ satisfies (3.9). Let $T : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by $Tf = u$, where u is the solution to (3.15) (i.e. $L = T^{-1}$ where L was defined in Lemma 3.3). Then T is compact.*

Proof. Let (f_j) be a a bounded sequence in $L^(\Omega)$, and $u_j = Tf_j$. Then u_j is bounded in $L^r(\Omega)$ by (3.9). Let $\tilde{\zeta_0}$ be the solution to (3.17). Then, by (3.16) we have

$$\int_\Omega a(x)u_j\tilde{\zeta_0} \leq \|\tilde{\zeta_0}\|_{L^\infty}\int_\Omega |f_j| + \|\tilde{\zeta_0}\|_{C^2}\int_\Omega |u_j|.$$

Therefore $-\Delta u_j = a(x)u_j + f_j$ is bounded in $L^1_{\text{loc}}(\Omega)$ and by the Gagliardo-Nirenberg inequality u_j is bounded in $W^{1,1}_{\text{loc}}(\Omega)$. We conclude that for a subsequence (denoted the same), $u_j \to u$ in $L^q_{\text{loc}}(\Omega)$ for some fixed $1 \leq q < \frac{n}{n-1}$, and a.e. To conclude that u_j converges strongly in $L^2(\Omega)$, let $\varepsilon > 0$ be given. Then by Egorov's theorem there exists $E \subset \Omega$ measurable with $|E| \leq \varepsilon$ so that $u_j \to u$ uniformly in $\Omega \setminus E$. Hence

$$\begin{aligned}\limsup \int_\Omega |u_j - u|^2 &\leq \limsup \int_{\Omega\setminus E} |u_j - u|^2 + \limsup \int_E |u_j - u|^2 \\ &\leq \|u_j - u\|^2_{L^r}|E|^{1-2/r} \\ &\leq C\varepsilon^{1-2/r}\end{aligned}$$

by the uniform bound of u_j in $L^r(\Omega)$. ☐

To prove that the embedding $H \subset L^2(\Omega)$ is compact we use the following result combined with the previous lemma.

Lemma 3.13. *Let H, V be real Hilbert spaces and $J : H \to V$ a bounded, linear map. Then J is compact if and only if JJ^* is compact.*

Proof. Clearly if J is compact then JJ^* is compact.

Let $\varepsilon > 0$. Then the map $S_\varepsilon := JJ^* + \varepsilon I : V \to V$ is selfadjoint and coercive, in the sense that $\|S_\varepsilon y\|_V \geq \varepsilon\|y\|_V$. It follows that S_ε is invertible. Therefore, given $x \in H$ there is $y \in V$ so that

$$JJ^*y + \varepsilon y = Jx. \tag{3.19}$$

But

$$(J^*y,x)_H \le \tfrac{1}{2}\|x\|_H^2 + \tfrac{1}{2}\|J^*y\|_H^2$$

and so

$$(J^*y, x - J^*y)_H \le \tfrac{1}{2}\|x\|_H^2 - \tfrac{1}{2}\|J^*y\|_H^2 \le \tfrac{1}{2}\|x\|_H^2.$$

In combination with (3.19) this yields

$$\|JJ^*y - Jx\|_V^2 \le \frac{\varepsilon}{2}\|x\|_H^2. \tag{3.20}$$

Now assume that JJ^* is compact and let x_j be a bounded sequence in H. Let $M = \sup_j \|x_j\|_H$, and set $\varepsilon_k = 2^{-2k}$ for $k = 1, 2, \dots$. We start by taking $k = 1$ and letting $y_j = S_{\varepsilon_1}^{-1}(Jx_j)$. Then y_j is a bounded sequence and since JJ^* is compact there is a subsequence (denoted the same) and some $z_1 \in V$, so that $JJ^*y_j \to z_1$. Therefore, using (3.20) we see that there is some j_1 so that

$$\begin{aligned} \|Jx_{j_1} - z_1\|_V &\le \|Jx_{j_1} - JJ^*y_{j_1}\|_V + \|JJ^*y_{j_1} - z_1\|_V \\ &\le \sqrt{\frac{\varepsilon_1}{2}}M + +\|JJ^*y_{j_1} - z_1\|_V = 2M\sqrt{\varepsilon_1}. \end{aligned}$$

Using a diagonal argument one can find a subsequence j_k and $z_k \in V$ so that $\|Jx_{j_l} - z_k\|_V \le 2^{-k+1}M$ for all $l \ge k$. This implies that $\|z_{k+1} - z_k\|_V \le 2^{-k+2}$ and therefore z_k is a Cauchy sequence in V. Thus z_k converges, and so Jx_{j_k} is also convergent. □

We are now in a position to prove Lemma 3.3. *Proof.* of Lemma 3.3 Taking $V = L^2(\Omega)$, "$H = H$", and denoting by $J : H \to L^2(\Omega)$ the usual injection, we see that $T = JJ^*$, where $Tf = u$, and u is the H-solution to (3.15). By Lemma 3.12 $T = JJ^*$ is compact and hence by Lemma 3.13 J is compact.

Since T is selfadjoint and compact, $L = T^{-1}$ has a smallest eigenvalue

$$\lambda_1 = \inf_{\varphi \in H\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2}{\int_\Omega \varphi^2} > 0, \tag{3.21}$$

and the infimum is attained by a positive eigenfunction associated to λ_1, which we denote by φ_1. λ_1 is simple, and this can be proved in the same way as for smooth elliptic operators. In fact, let φ denote another eigenfunction for λ_1. Then for any $\mu \in \mathbb{R}$ we have that $\psi = \varphi_1 - \mu\varphi$ satisfies the equation

$$\begin{cases} -\Delta\psi - a(x)\psi &= \lambda_1\psi \quad \text{in } \Omega \\ \psi &= 0 \quad\quad \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Now, because ψ satisfies (3.22), if $\psi \not\equiv 0$ then it minimizes (3.21). Then also $|\psi|$ minimizes (3.27) and therefore satisfies the equation (3.22). Since

$$\begin{cases} -\Delta|\psi| &= a(x)|\psi| + \lambda_1|\psi| \geq 0 \qquad \text{in } \Omega \\ \quad |\psi| &= 0 \qquad \text{on } \partial\Omega \end{cases}$$

by the strong maximum principle we conclude that if $\psi \not\equiv 0$ then $|\psi| > c\delta$ a.e. in Ω, where $c > 0$ and $\delta(x) = \text{dist}(x, \partial\Omega)$ (see e.g. Brezis and Cabré [BC]). This combined with the fact that $\psi \in W^{1,1}_{\text{loc}}(\Omega)$ (by Lemma 3.1) shows that either $\psi > 0$ or $\psi < 0$ in Ω (assuming Ω connected, see for example Chabi and Haraux [9]). That is, for any $\mu \in \mathbb{R}$ either $\varphi \geq \mu\varphi_1$ or $\varphi \leq \mu\varphi_1$. Setting $\mu_0 = \sup\{\mu : \varphi \geq \mu\varphi_1\}$ we see that $\varphi = \mu_0\varphi_1$. $\square$

Define

$$a_k = \min(a, k), \qquad k > 0. \tag{3.23}$$

We denote by λ_1^k, φ_1^k, ζ_0^k the first eigenvalue, first eigenfunction and solution of (3.4) associated with the potential a_k, which are all defined in the usual sense, since a_k is bounded. Let ζ_0 be the solution to (3.4) in the sense of Lemma 2.1. Since $a(x)$ satisfies (3.9) (hence (3.1)), it is easy to check that $\zeta_0^k \to \zeta_0$ in $L^2(\Omega)$.

Lemma 3.14. *Normalize φ_1^k by $\|\varphi_1^k\|_{L^2(\Omega)} = 1$. Then*

$$\lambda_1^k \to \lambda_1 \qquad \textit{and} \qquad \varphi_1^k \to \varphi_1 \quad \textit{in } H$$

as $k \to \infty$, where λ_1 is given by (3.1) and φ_1 is given by Lemma 3.3, normalized so that $\|\varphi_1\|_{L^2(\Omega)} = 1$.

Proof. Observe that

$$\lambda_1^k = \inf_{\varphi \in C_c^\infty(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a_k(x)\varphi^2}{\int_\Omega \varphi^2} \tag{3.24}$$

is non-increasing as k increases. Therefore the limit $\lim_{k\to\infty} \lambda_1^k$ exists. We claim that

$$\lim_{k\to\infty} \lambda_1^k = \lambda_1.$$

Indeed, note that $\lambda_1 \leq \lambda_1^k$ for all k, and also that for any $\varphi \in C_c^\infty(\Omega)$

$$\int_\Omega a_k(x)\varphi^2 \to \int_\Omega a(x)\varphi^2 \tag{3.25}$$

by monotone convergence. Take now $\varphi \in C_c^\infty(\Omega)$ with $\|\varphi\|_{L^2} = 1$. Then

$$\lambda_1^k \leq \int_\Omega |\nabla\varphi|^2 - a_k(x)\varphi^2,$$

and using (3.25) we see that

$$\limsup \lambda_1^k \leq \int_\Omega |\nabla\varphi|^2 - a(x)\varphi^2.$$

Taking the infimum over φ we obtain

$$\limsup \lambda_1^k \leq \lambda_1.$$

Recall that we normalize φ_1^k by $\|\varphi_1^k\|_{L^2} = 1$ and so

$$\|\varphi_1^k\|_H^2 \leq \int_\Omega |\nabla\varphi_1^k|^2 - \int_\Omega a_k(x)|\varphi_1^k|^2 = \lambda_1^k \to \lambda_1 \qquad \text{as } k \to \infty. \tag{3.26}$$

In particular φ_1^k is bounded in H and by Lemma 3.3 we can find a subsequence such that $\varphi_1^k \to \varphi_1$ in $L^2(\Omega)$. We observe that $\varphi_1 \geq 0$ and $\|\varphi_1\|_{L^2} = 1$.

Claim. φ_1 minimizes

$$\lambda_1 = \inf_{\varphi \in H \setminus \{0\}} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2}{\int_\Omega \varphi^2}. \tag{3.27}$$

Indeed, testing the equation of φ_1^k with $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ we find

$$\int_\Omega \nabla\varphi_1^k \cdot \nabla\varphi - \int_\Omega a_k(x)\varphi_1^k\varphi = \lambda_1^k \int_\Omega \varphi_1^k\varphi$$

and therefore

$$\int_\Omega \nabla\varphi_1^k \cdot \nabla\varphi - \int_\Omega a(x)\varphi_1^k\varphi \leq \lambda_1^k \int_\Omega \varphi_1^k\varphi.$$

Taking limits on both sides, we find

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi - \int_\Omega a(x)\varphi_1\varphi \leq \lambda_1 \int_\Omega \varphi_1\varphi.$$

By density this is true for all $\varphi \in H$, $\varphi \geq 0$ and taking $\varphi = \varphi_1$ we find that

$$\frac{\int_\Omega |\nabla\varphi_1|^2 - \int_\Omega a(x)\varphi_1^2}{\int_\Omega \varphi_1^2} \leq \lambda_1$$

and the claim is proved.

Then the standard arguments of the calculus of variations show that φ_1 satisfies (3.2), and hence φ_1 is indeed the first eigenfunction of $-\Delta - a(x)$. The strong convergence $\varphi_1^k \to \varphi_1$ in H, is a consequence of

$$\|\varphi\|_H = \lambda_1 \leq \|\varphi_1^k\|_H \leq \lambda_1^k,$$

which implies that $\|\varphi_1^k\|_H \to \|\varphi\|_H$. □

Lemma 3.15. *It suffices to prove Theorems 3.5 and 3.6 and Corollaries 3.7 and 3.8 in the case where the potential $a(x)$ is bounded.*

Proof. We only give the argument for Theorem 3.5, which can be easily carried out for the other results. Let $a(x) \geq 0$ be any potential satisfying (3.9) and a_k its truncation defined by (3.23). Observe that

$$\inf_{\varphi \in C_c^1(\Omega)} \frac{\int_\Omega |\nabla\varphi|^2 - \int_\Omega a_k(x)\varphi^2}{\left(\int_\Omega |\varphi|^r\right)^{2/r}} \geq \gamma(a).$$

So if Theorem 3.5 holds for bounded potentials, we must have

$$C^{-1}\zeta_0^k \leq \varphi_1^k \leq C\zeta_0^k, \tag{3.28}$$

where ζ_0^k, φ_1^k were defined at the beginning of this section and $C = C(\Omega, \gamma(a), r) > 0$ is independent of k. Since $\zeta_0^k \to \zeta_0$ in L^2 and Lemma 3.14 holds, we can pass to the limit in (3.28). □

3.4 Proof of Theorem 3.5

By Lemma 3.15 in the previous section it is enough to establish the result in the case that $a(x)$ is bounded.

The main idea is to consider the function

$$w = \frac{\varphi_1}{\zeta_0}$$

and notice that it satisfies (formally) an elliptic equation

$$\begin{cases} -\nabla \cdot (\zeta_0^2 \nabla w) &= \lambda_1 \varphi_1 \zeta_0 - \varphi_1 \quad \text{in } \Omega \\ \zeta_0^2 \nabla w \cdot \nu &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.29}$$

where ν denotes the outer unit normal to the boundary $\partial\Omega$. Then we will use Moser's iteration argument, combined with a Sobolev inequality to prove that w is bounded.

Step 1. Formal derivation of an iteration formula: there exists $q > 2$ and $C > 0$ such that for all $j \geq 1$

$$\Big(\int_\Omega \zeta_0^2 w^{qj}\Big)^{2/q} \leq Cj \int_\Omega \zeta_0^2 w^{2j}. \tag{3.30}$$

Multiplying (3.29) by w^{2j-1} where $j \geq 1$, and integrating by parts we obtain:

$$\frac{2j-1}{j^2}\int_\Omega \zeta_0^2 |\nabla w^j|^2 = \int_\Omega (\lambda_1\varphi_1\zeta_0 - \varphi_1)w^{2j-1} \leq \lambda_1 \int_\Omega \zeta_0^2 w^{2j}. \tag{3.31}$$

Now we use the next lemma, which is a kind of Sobolev inequality.

Lemma 3.16. *Assume u satisfies*

$$\begin{cases} -\Delta u - a(x)u &= c(x)u + f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.32}$$

where c, $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$. Assume also that a satisfies (3.9). Then for any $2 \leq q \leq r$ there is a constant C depending only Ω, r, $\gamma(a)$, $\|c\|_{L^\infty}$, $\|f\|_{L^\infty}$ and $\big(\int_\Omega f\delta\big)^{-1}$ such that

$$\Big(\int_\Omega u^s|\varphi|^q\Big)^{2/q} \leq C\int_\Omega u^2\big(|\nabla\varphi|^2 + \varphi^2\big)$$

for all $\varphi \in C^1(\overline{\Omega})$, where s is given by the relation

$$\frac{s}{r} = \frac{q-2}{r-2}. \tag{3.33}$$

(A proof of this lemma is given in Step 4.)

Proof. of Step 1 continued Taking $u = \zeta_0$, $f \equiv 1$, $c \equiv 0$ and $s = 2$, by Lemma 3.16 there is $q = 4\frac{r-1}{r} > 2$ and $C > 0$ such that

$$\Big(\int_\Omega \zeta_0^2|\varphi|^q\Big)^{2/q} \leq C\int_\Omega \zeta_0^2(|\nabla\varphi|^2 + \varphi^2) \tag{3.34}$$

for all $\varphi \in C^1(\overline{\Omega})$. This applied to $\varphi = w^j$ and combined with (3.31) yields (3.30).

Step 2. We derive now the estimate

$$\varphi_1 \leq C\zeta_0. \tag{3.35}$$

Proof. We iterate (3.30): define $\mu = q/2 > 1$ and $j_k = 2\mu^k$, for $k = 0, 1, \ldots$. Let

$$\theta_k = \Big(\int_\Omega \zeta_0^2 w^{j_k} \Big)^{1/j_k}.$$

Then (3.30) can be rewritten as

$$\theta_{k+1} \le \Big(C\mu^k \Big)^{1/\mu^k} \theta_k.$$

Using this recursively yields

$$\theta_k \le C\theta_0 = C\Big(\int_\Omega \zeta_0^2 \Big)^{1/2} < \infty$$

for all $k = 0, 1, 2, \ldots$ with C independent of k. But

$$\lim_{k\to\infty} \theta_k = \sup_\Omega w$$

(because $\zeta_0 > 0$ in Ω) and this shows that $w \le C$.

Step 3. Justification of Step 1. To be rigorous, we need to justify the derivation of (3.30), which has been formal only. One possible approach is the following. *Proof.* of (3.30) Consider the family of smooth domains

$$\Omega_\varepsilon = \Big\{ x \in \mathbb{R}^n \mid \mathrm{dist}(x, \Omega) < \varepsilon \Big\},$$

where $\varepsilon > 0$ is small. Let ζ_0^ε be the solution to

$$\begin{cases} -\Delta\zeta_0^\varepsilon - a(x)\zeta_0^\varepsilon &= 1 \quad \text{in } \Omega_\varepsilon \\ \zeta_0^\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3.36}$$

where a is extended by 0 outside Ω. Then $\zeta_0^\varepsilon \searrow \zeta_0$ as $\varepsilon \to$ uniformly in $\overline{\Omega}$ (because we are in the case $a(x) \in L^\infty$ and therefore we have a uniform bound for ζ_0^ε in $C^{1,\alpha}(\overline{\Omega})$.) Furthermore, $\zeta_0^\varepsilon \ge c_\varepsilon > 0$ in Ω, by the strong maximum principle. Letting

$$w_\varepsilon = \frac{\varphi_1}{\zeta_0^\varepsilon},$$

it follows that $w_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$, $w_\varepsilon = 0$ on $\partial\Omega$, and all the formal computations done with w apply rigorously to w_ε so that (3.30) holds for w_ε in place of w and ζ_0^ε in place of ζ_0. It is then easy to pass to the limit as $\varepsilon \to 0$, using e.g. monotone convergence.

Step 4. *Proof.* of Lemma 3.16 First observe that $u \geq c\delta$ for some $c > 0$ (see Brezis and Cabré [BC] for example), and recall Hardy's inequality

$$\int_\Omega \frac{\psi^2}{\delta^2} \leq C \int_\Omega |\nabla\psi|^2 \qquad \text{for all } \psi \in C_c^1(\Omega)$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. Using this with $\psi = \delta\varphi$ Using this with $\psi = \delta\varphi$ as in [10] it is easy to check that

$$\int_\Omega \varphi^2 \leq C \int_\Omega \delta^2(|\nabla\varphi|^2 + \varphi^2) \tag{3.37}$$

for all $\varphi \in C^1(\overline{\Omega})$. This shows that

$$\int_\Omega \varphi^2 \leq C \int_\Omega u^2\big(|\nabla\varphi|^2 + \varphi^2\big). \tag{3.38}$$

The next step consists in proving

$$\Big(\int_\Omega |u\varphi|^r\Big)^{2/r} \leq C \int_\Omega u^2\big(|\nabla\varphi|^2 + \varphi^2\big) \qquad \text{for all } \varphi \in C^1(\overline{\Omega}). \tag{3.39}$$

To achieve this, note that by (3.9) we have

$$\Big(\int_\Omega |u\varphi|^r\Big)^{2/r} \leq C \int_\Omega |\nabla(u\varphi)|^2 - \int_\Omega a(x)(u\varphi)^2. \tag{3.40}$$

But

$$\int_\Omega |\nabla(u\varphi)|^2 = \int_\Omega u^2|\nabla\varphi|^2 + \int_\Omega \nabla u \nabla(u\varphi^2) \tag{3.41}$$

and, multiplying (3.32) by $u\varphi^2$ and integrating we get

$$\int_\Omega \nabla u \nabla(u\varphi^2) - \int_\Omega a(x)(u\varphi)^2 = \int_\Omega c(x)u^2\varphi^2 + \int_\Omega fu\varphi^2. \tag{3.42}$$

Combining (3.40), (3.41) and (3.42) we find

$$\Big(\int_\Omega |u\varphi|^r\Big)^{2/r} \leq C \int_\Omega u^2|\nabla\varphi|^2 + \int_\Omega c(x)u^2\varphi^2 + \int_\Omega fu\varphi^2.$$

The last two terms in the right hand side can be estimated by

$$\begin{aligned} \int_\Omega c(x)u^2\varphi^2 + \int_\Omega fu\varphi^2 \ &\leq\ \|c\|_{L^\infty} \int_\Omega u^2\varphi^2 + \|f\|_{L^\infty}\Big(\int_\Omega u^2\varphi^2\Big)^{1/2}\Big(\int_\Omega \varphi^2\Big)^{1/2} \\ &\leq\ C \int_\Omega u^2\big(|\nabla\varphi|^2 + \varphi^2\big) \end{aligned}$$

by (3.38). This proves (3.39).

Finally, we interpolate (3.38) and (3.39): by Hölder's inequality

$$\int_\Omega u^s|\varphi|^q \leq \Big(\int_\Omega u^r|\varphi|^r\Big)^\lambda \Big(\int_\Omega \varphi^2\Big)^{1-\lambda}$$

if λ and s are chosen so that

$$s = \lambda r \quad \text{and} \quad r\lambda + 2(1-\lambda) = q.$$

This gives the relation (3.33) and proves the lemma. □

Step 5. We claim that

$$\zeta_0 \leq C\varphi_1.$$

Proof. This time we consider the quotient

$$w = \frac{\zeta_0}{\varphi_1}$$

which satisfies:

$$\begin{cases} -\nabla\cdot(\varphi_1^2\nabla w) &= \varphi_1 - \lambda_1\varphi_1\zeta_0 \quad \text{in } \Omega \\ \varphi_1^2\nabla w\cdot\nu &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Again we multiply this equation by $\varphi = w^{2j-1}$ to find

$$\begin{aligned} \frac{2j-1}{j^2}\int_\Omega \varphi_1^2|\nabla w^j|^2 &= \int_\Omega (\varphi_1 - \lambda_1\varphi_1\zeta_0)w^{2j-1} \\ &\leq \int_\Omega \varphi_1 w^{2j-1}. \end{aligned}$$

Here we use (3.35) to conclude that

$$\varphi_1 w^{2j-1} \leq C\zeta_0 w^{2j-1} = C\varphi_1 w^{2j}$$

so we find

$$\int_\Omega \varphi_1^2|\nabla w^j|^2 \leq Cj\int_\Omega \varphi_1 w^{2j}. \tag{3.43}$$

Letting $\varphi = w^j$ and using consecutively Hölder's inequality and Lemma 4.1 (with $u = \varphi_1$, $f \equiv 0$, $c = \lambda_1$, $s = 0$ and $q = 2$), it follows from (3.43) that

$$\begin{aligned} \int_\Omega \varphi_1^2|\nabla\varphi|^2 &\leq Cj\left(\int_\Omega \varphi_1^2\varphi^2\right)^{1/2}\left(\int_\Omega \varphi^2\right)^{1/2} \\ &\leq Cj\left(\int_\Omega \varphi_1^2\varphi^2\right)^{1/2}\left(\int_\Omega \varphi_1^2\left(\varphi^2 + |\nabla\varphi|^2\right)\right)^{1/2}. \end{aligned}$$

And by Young's inequality,

$$\int_\Omega \varphi_1^2 |\nabla\varphi|^2 \le Cj^2 \int_\Omega \varphi_1^2\varphi^2 + 1/2\left(\int_\Omega \varphi_1^2\left(\varphi^2 + |\nabla\varphi|^2\right)\right)$$

so that

$$\int_\Omega \varphi_1^2 |\nabla\varphi|^2 \le Cj^2 \int_\Omega \varphi_1^2\varphi^2. \tag{3.44}$$

Using Lemma 3.16 with $u = \varphi_1$, $f \equiv 0$, $c = \lambda_1$ and $s = 2$, we obtain a constant $q = 4\frac{r-1}{r} > 2$ and $C > 0$ so that

$$\left(\int_\Omega \varphi_1^2 w^{qj}\right)^{2/q} \le C\int_\Omega \varphi_1^2\left(|\nabla w^j|^2 + w^{2j}\right)$$

and combining with (3.44) we arrive at

$$\left(\int_\Omega \varphi_1^2 w^{qj}\right)^{2/q} \le Cj^2\int_\Omega \varphi_1^2 w^{2j}. \tag{3.45}$$

An iteration argument as in Step 2 then shows that

$$\sup_\Omega w \le C.$$

As in Step 3, we need to justify the derivation of (3.45) by an approximation argument. This time however, it is more convenient to consider

$$\Omega_\varepsilon := \left\{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon \right\},$$

let ζ_0^ε solve (3.36) and do all of the above computations in Ω_ε in place of Ω. We omit the details. □

3.5 Proof of Theorem 3.6

As in the elliptic case, using Lemma 3.15, it is enough to establish the result for bounded $a(x)$.

Let u be the solution of (3.8) and ζ_0 be the solution of (3.4). We note that $u(t) \ge c(t)\delta$ for some positive function $c(t)$ (see [4]). We will replace $u(t)$ with $u(t-\tau)$ where $\tau > 0$ is fixed, and so we can assume

$$u(t) \ge c\delta \qquad \text{for } t \in [0, T],$$

where $T > 0$ is fixed and $c > 0$ is independent of t for $t \in [0, T]$. By (3.37) we have then

$$\int_\Omega \varphi^2 \leq C \int_\Omega u(t)^2 \big(|\nabla \varphi|^2 + \varphi^2 \big) \tag{3.46}$$

for $t \in [0, T]$, with C independent of t. Since by Theorem 3.5

$$\zeta_0 \leq C\varphi_1,$$

where φ_1 denotes the first eigenfunction for $-\Delta - a(x)$, it is enough to show that for some constant C we have

$$\varphi_1 \leq Cu(t).$$

We will work with

$$v = e^{-\lambda_1 t}\varphi_1$$

which satisfies

$$\begin{cases} \partial_t v - \Delta v - a(x)v &= 0 \quad \text{in } \Omega \times (0, \infty) \\ v &= 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Set

$$w = \frac{v}{u}$$

and note that it satisfies (formally)

$$\begin{cases} u^2 w_t - \nabla \cdot \big(u^2 \nabla w \big) &= 0 \quad \text{in } \Omega \times (0, T) \\ u^2 \nabla w \cdot \nu &= 0 \quad \text{on } \partial\Omega \times (0, T). \end{cases} \tag{3.47}$$

We claim that

$$w(t) \leq Ct^{-\beta} \qquad \text{for } t \in [0, T],$$

where $\beta, C > 0$ are independent of t.

To accomplish this, we follow the idea in the paper by Brezis and Cazenave [5], which is inspired by a work of Fabes and Stroock [13]. To simplify the exposition, we first work formally with (3.47).

First, for $j \geq 1$ and $t \in [0, T]$ we define the quantity

$$\theta_j(t) = \int_\Omega u(t)^2 w(t)^j.$$

We also use the notation

$$\varphi = w^j.$$

Our first step is to derive

Claim 1.

$$\theta_{2j}'(t) + 2\|u\varphi\|_H^2 + 2\frac{j-1}{j}\int_\Omega u^2|\nabla\varphi|^2 = 0, \tag{3.48}$$

where $\|\cdot\|_H$ was defined in (3.11). *Proof.* of (3.48) Multiplying (3.47) by w^{2j-1} we find

$$\frac{1}{2j}\int_\Omega u^2\left(w^{2j}\right)_t + \frac{2j-1}{j^2}\int_\Omega u^2|\nabla w^j|^2 = 0. \tag{3.49}$$

Then writing $\varphi = w^j$, observe that

$$\frac{d}{dt}\theta_{2j}(t) = \theta_{2j}(t)' = 2\int_\Omega uu_t\varphi^2 + \int_\Omega u^2\left(\varphi^2\right)_t. \tag{3.50}$$

Hence by (3.49) and using (3.50) we obtain

$$\frac{1}{2j}\left(\theta_{2j}' - 2\int_\Omega uu_t\varphi^2\right) + \frac{2j-1}{j^2}\int_\Omega u^2|\nabla w^j|^2 = 0. \tag{3.51}$$

Now we multiply (3.8) by $u\varphi^2$ and integrate on Ω. This gives the relation

$$\int_\Omega uu_t\varphi^2 + \int_\Omega \nabla u\nabla(u\varphi^2) - \int_\Omega au^2\varphi^2 = 0.$$

Therefore

$$\begin{aligned}\int_\Omega uu_t\varphi^2 &= \int_\Omega au^2\varphi^2 - \int_\Omega \nabla u\nabla(u\varphi^2)\\ &= \int_\Omega au^2\varphi^2 - \int_\Omega |\nabla(u\varphi)|^2 + \int_\Omega u^2|\nabla\varphi|^2.\end{aligned}$$

Substituting the expression $\int uu_t\varphi^2$ from the previous equation in (3.51) yields (3.48).

Claim 2. From (3.48) immediately follows that $\theta_{2j}'(t) \le 0$ and therefore

$$\theta_j(t) \le \theta_j(0) \qquad \text{for all } t \in [0,T] \text{ and } j \ge 2. \tag{3.52}$$

Claim 3. There is constant C such that

$$\theta_{2j}'(t) + \frac{1}{C}\frac{\theta_{2j}(t)^{1+m}}{\theta_j(0)^{2m}} \le \theta_{2j}(t) \qquad \text{for } t \in [0,T], \tag{3.53}$$

where $m = \frac{3r-4}{2r-2} - 1 > 0$. *Proof.* By Hölder's inequality

$$\theta_{2j}(t) = \int_\Omega u^2\varphi^2 \leq \Big(\int_\Omega (u\varphi)^r\Big)^{\frac{2}{3r-4}} \Big(\int_\Omega \varphi^2\Big)^{\frac{r-2}{3r-4}} \Big(\int_\Omega u^2\varphi\Big)^{\frac{2r-4}{3r-4}}.$$

Now we use assumption (3.9) and (3.46) to get

$$\begin{aligned} \theta_{2j}(t) \quad &\leq C \quad \|u\varphi\|_H^{\frac{r}{3r-4}} \Big(\int_\Omega u^2\big(|\nabla\varphi|^2 + \varphi^2\big)\Big)^{\frac{r-2}{3r-4}} \Big(\int_\Omega u^2\varphi\Big)^{\frac{2r-4}{3r-4}} \\ &= \quad C\|u\varphi\|_H^{\frac{r}{3r-4}} \Big(\int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\Big)^{\frac{r-2}{3r-4}} \theta_j(t)^{\frac{2r-4}{3r-4}}. \end{aligned}$$

And by Young's inequality,

$$\theta_{2j}(t) \leq C\Big(\|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\Big)^{\frac{2r-2}{3r-4}} \theta_j(t)^{\frac{2r-4}{3r-4}}. \tag{3.54}$$

Let

$$m = \frac{3r-4}{2r-2} - 1 > 0$$

so that by (3.54) and (3.52)

$$\theta_{2j}(t)^{1+m} \leq C\Big(\|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2 + \theta_{2j}(t)\Big)\theta_j(0)^{2m}. \tag{3.55}$$

Rearranging (3.55) yields

$$\frac{1}{C}\frac{\theta_{2j}(t)^{1+m}}{\theta_j(0)^{2m}} - \theta_{2j}(t) \leq \|u\varphi\|_H^2 + \int_\Omega u^2|\nabla\varphi|^2$$

and combining the last expression with (3.48) we obtain (3.53).

Claim 4. Using (3.53) we have

$$\theta_{2j}(t) \leq Ct^{-1/m}\theta_j(0)^2 \qquad t \in [0, T]. \tag{3.56}$$

The derivation of this estimate has been formal only but, as in Step 3 of Section 4, we can make it rigorous using the same approximation argument on Ω.

Claim 5. Iterating (3.56) we find

$$\|w(t)\|_{L^\infty} \leq Ct^{-1/2m} \qquad \text{for } t \in [0, T].$$

Indeed, for $k = 1, 2, \ldots$ set $t_k = t(1 - 2^{-k+1})$ and $j_k = 2^k$. Then $t_{k+1} - t_k = 2^{-k}t$. So from (3.56) we have

$$\begin{aligned} \theta_{j_{k+1}}(t_{k+1}) \quad &= \theta_{2j_k}(t_k + 2^{-k}t) \\ &\leq C2^{k/m}t^{-1/m}\theta_{j_k}(t_k)^2. \end{aligned} \tag{3.57}$$

But recall that

$$\theta_j(t) = \int_\Omega u(t)^2 w(t)^j$$

so from (3.57) we have

$$\begin{aligned}\left(\int_\Omega u(t_{k+1})^2 w(t_{k+1})^{2^{k+1}}\right)^{1/2^{k+1}} &\leq \left(C2^{k/m}\, t^{-1/m}\right)^{1/2^{k+1}} \left(\int_\Omega u(t_k)^2 w(t_k)^{2^k}\right)^{1/2^k} \\ &\leq C' t^{-\frac{1}{m}\sum_{j=2}^{k+1} 2^{-j}} \left(\int_\Omega u(0)^2 w(0)^2\right)^{1/2}.\end{aligned}$$

Letting $k \to \infty$ we find

$$\sup_\Omega w \leq C' t^{-1/2m} \|\varphi_1\|_{L^2}.$$

□

3.6 Proof of Corollaries 3.7 and 3.8 and Theorem 3.9

Again, it is enough to reduce to the case where $a(x)$ is bounded. *Proof.* of Corollary 3.7

Step 1. A first estimate involving $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. ($S(t)$ denotes the semigroup generated by $-\Delta - a(x)$ in $L^2(\Omega)$, where $a(x)$ is now a bounded potential).

Using a fine version of the maximum principle for the heat equation (see [4] for the time dependence of the constant and Martel [14] for the dependence on the initial condition), we have that

$$u(t) \geq e^{-K/t} \left(\int_\Omega u_0 \delta\right) \delta(x) \qquad \text{for } t \in [0, T],$$

where $K = K(\Omega, T) > 0$. Let $\mu_1 > 0$ and $\psi_1 > 0$ be the first eigenvalue and eigenfunction of the Laplace operator (with zero boundary condition). Possibly increasing the constant K, it follows that

$$u(t) \geq e^{-K/t} \left(\int_\Omega u_0 \delta\right) \psi_1(x) \qquad \text{for } t \in [0, 1],$$

where $K = K(\Omega)$. Now let $v(t) = e^{\mu_1 - K} \left(\int_\Omega u_0 \delta\right) e^{-\mu_1 t} \psi_1(x)$. Then

$$\begin{cases} v_t - \Delta v &= 0 \\ v(1) &\leq u(1). \end{cases}$$

So by the maximum principle $u(t) \geq v(t)$ for $t \in (1, \infty)$ and we finally obtain

$$u(t) \geq e^{-K(t+1/t)} \left(\int_\Omega u_0 \delta \right) \delta(x) \qquad \text{for } t \in [0, \infty), \tag{3.58}$$

where $K = K(\Omega)$.

Step 2. An estimate for $u^{x_0} = S(t)\delta_{x_0}$.

First, looking carefully at the previous section, we see that if $u \geq 0$ solves (3.8) and

$$u(t) \geq \delta(x) \qquad \text{for } t \in [0, T]$$

then

$$u(t) \geq C t^\beta e^{-\lambda_1 t} \zeta_0 \qquad \text{for } t \in [0, T], \tag{3.59}$$

where C and β depend only on Ω and $\gamma(a)$.

Next, fix a ball $B \subset\subset \Omega$ and for $x_0 \in B$, let δ_{x_0} denote the Dirac mass supported by $\{x_0\}$ and u^{x_0} the solution of (3.8) with initial condition $u_0 = \delta_{x_0}$. Given $t_0 > 0$, we have by (3.58),

$$u^{x_0}(t_0) \geq \delta(x_0) e^{-K(t_0+1/t_0)} \delta(x) \geq e^{-K'(t_0+1/t_0)} \delta(x),$$

where K' depends only on Ω. Hence, for $t \in [0, T]$

$$\begin{aligned} u^{x_0}(t+t_0) &\geq e^{-K'(t_0+1/t_0)} S(t)\delta(x) \geq c e^{-K'(t_0+1/t_0)} S(t)\psi_1(x) \\ &\geq c e^{-K'(t_0+1/t_0)} e^{-\mu_1 t} \psi_1(x) \geq c e^{-K(t_0+1/t_0+T)} \delta(x), \end{aligned}$$

where $K = K(\Omega)$. Using (3.59), we obtain for $t \in [0, T]$

$$u^{x_0}(t+t_0) \geq C t^\beta e^{-\lambda_1 t} e^{-K(t_0+1/t_0+T)} \zeta_0$$

so that, choosing $t = T = t_0$

$$u^{x_0}(2t_0) \geq e^{-K''(t_0+1/t_0)} \zeta_0,$$

where K'' depends solely on Ω and $\gamma(a)$. Since $t_0 > 0$ was chosen arbitrarily, we finally obtain for all $t > 0$

$$u^{x_0}(t) \geq e^{-K''(t+1/t)} \zeta_0. \tag{3.60}$$

Step 3. Let u^B be the solution of (3.8) with initial condition $u_0 = \chi_B$. Proceeding as in the previous step, we can show that

$$u^B \geq e^{-K(t+1/t)}\zeta_0. \tag{3.61}$$

Now, let u be the solution of (3.8) with arbitrary initial condition $u_0 \geq 0$. Using (3.60), we then have for $x \in B$

$$u(t,x) = \langle \delta_x, S(t)u_0 \rangle = \int_\Omega u_0 u^x \geq e^{-K''(t+1/t)} \int_\Omega u_0 \zeta_0.$$

In other words,

$$u(t) \geq e^{-K''(t+1/t)} \left(\int_\Omega u_0 \zeta_0 \right) \chi_B.$$

Hence, using (3.61), it follows that

$$u(2t) \geq e^{-K(t+1/t)} \left(\int_\Omega u_0 \zeta_0 \right) \zeta_0$$

with $K = K(\Omega, \gamma(a), r)$, which completes the proof of Corollary 3.7. □

Proof. of Corollary 3.8 One just needs to apply Corollary 3.7 and Duhamel's principle: if u solves (3.12) then

$$u = S(1)u + \int_0^1 S(1-s) f ds \geq \left(\int_0^1 e^{-K(s+1/s)} ds \right) \left(\int_\Omega f \zeta_0 \right) \zeta_0.$$

Proof. of Theorem 3.9 Recall that we assume here that $a(x)$ satisfies (3.13) and (3.14). First we remark that, as for the case when $a(x) \geq 0$, one can reduce the proof to the case of a bounded potential by considering $a_k(x) = \min(a(x), k)$, $k > 0$, similarly to Lemma 3.15. Let $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ and u be the solution to

$$\begin{cases} \partial_t u - \Delta u - a(x)u &= 0 \quad \text{in } \Omega \times (0,\infty) \\ u &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

Let M be the constant from condition (3.14) and set $v = e^{-Mt}u$. Then v satisfies

$$v_t - \Delta v - (a(x) - M)v = 0 \quad \text{in } \Omega \times (0,\infty).$$

Observe that the potential $\tilde{a}(x) = a(x) - M$ satisfies (3.13) and (3.9). Applying Corollary 3.7 combined with Theorem 3.5 to v and the potential $\tilde{a}(x)$ we conclude that

$$v(t) \geq e^{-K(t+1/t)} \Big(\int_\Omega v(0) \varphi_1 \Big) \varphi_1.$$

(The first eigenfunctions for $-\Delta - a(x)$ and $-\Delta - \tilde{a}(x)$ are the same). Then the conclusion for u follows easily. $\square$

3.7 Further results and open problems

In this section, we question the optimality of our assumption (3.9) on the potential $a(x)$. As we shall see, potentials of the form $a(x) = c/d(x)^2$ where

$$d(x) = \mathrm{dist}(x, \Sigma)$$

is the distance function to an embedded manifold $\Sigma \subset \mathbb{R}^n$, do not necessarily satisfy our assumption (3.9) but a weaker version of it. We conjecture that some comparison result can still be obtained.

Finally, open questions on the Green's function of the operator $-\Delta - a(x)$.

We state the following generalized Hardy inequalities :

Theorem 3.17. *Let Σ be a smooth manifold of codimension $k \neq 2$ embedded in $\mathbb{R}^n$ and $d(x) = \mathrm{dist}(x, \Sigma)$. Then we have the following results:*

1) If Σ is compact then for any $\varepsilon > 0$ and $2 < r < 2n/(n-2)$, there exist $C > 0$, $\gamma > 0$ depending on Ω, Σ, r and ε, such that

$$\gamma\Big(\int_\Omega |\varphi|^r\Big)^{2/r} \leq \int_\Omega |\nabla\varphi|^2 - \frac{(k-2-\varepsilon)^2}{4}\int_\Omega \frac{\varphi^2}{d^2} + C\int_\Omega \varphi^2$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

2) If Σ is oriented then for some $r > 2$, there exist C, $\gamma > 0$ such that

$$\gamma\Big(\int_\Omega |\varphi|^r\Big)^{2/r} \leq \int_\Omega |\nabla\varphi|^2 - \frac{(k-2)^2}{4}\int_\Omega \frac{\varphi^2}{d^2} + C\int_\Omega \varphi^2$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

3) If Σ is such that $\Delta d^{k-2} \leq 0$ in $\mathcal{D}'(\Omega \setminus \Sigma)$, then for any $2 < r < 2n/(n-2)$ there exists $\gamma > 0$ such that

$$\gamma\Big(\int_\Omega |\varphi|^r\Big)^{2/r} \leq \int_\Omega |\nabla\varphi|^2 - \frac{(k-2)^2}{4}\int_\Omega \frac{\varphi^2}{d^2}$$

for all $\varphi \in C_c^\infty(\Omega \setminus \Sigma)$.

4) In particular if $\Sigma = \partial\Omega$ and Ω is convex then for any $2 < r < 2n/(n-2)$ there exists $\gamma > 0$ such that

$$\gamma\Big(\int_\Omega |\varphi|^r\Big)^{2/r} \leq \int_\Omega |\nabla\varphi|^2 - \frac{1}{4}\int_\Omega \frac{\varphi^2}{d^2}$$

for all $\varphi \in C_c^\infty(\Omega)$.

The fourth inequality was discovered with $\gamma = 0$ by Marcus, Mizel and Pinchover [15], and Matskevich and Sobolevskii [16]. It was then improved by Brezis and Marcus [2] to the case $\gamma > 0$ and $n = 2$. The general case for the third and fourth inequalities is due to Barbatis, Filippas and Tertikas [1]. We will prove the two others in a forthcoming publication.

Suppose that $a : \Omega \to [0, \infty)$ is such that for some constants $C(a)$, $\gamma(a) > 0$ and $r > 2$,

$$\gamma(a)\Big(\int_\Omega |\varphi|^r\Big)^{2/r} \leq \int_\Omega |\nabla\varphi|^2 - \int_\Omega a(x)\varphi^2 + C(a)\int_\Omega \varphi^2$$

for all $\varphi \in C_c^\infty(\Omega)$. Observe that the first two inequalities in Theorem 3.17 provide examples of such potentials.

We can then define H to be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_H^2 = \int_\Omega |\nabla u|^2 - a(x)u^2 + C(a)u^2.$$

Assume now that $u \in H$ solves for some $f \in L^\infty(\Omega)$,

$$\begin{cases} -\Delta u - a(x)u &= f \quad \text{in } \Omega \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Is it true that

$$|u| \leq C\left(\|f\|_{L^\infty} + \|u\|_{L^2}\right)\varphi_1$$

where $C = C(\Omega, C(a), \gamma(a), r)$ and $\varphi_1 \in H$ is the positive normalized eigenfunction of the operator $-\Delta - a(x)$ with respect to its first eigenvalue ?

The Green's function

Another direction interesting to pursue concerns the Green's function for the operator $-\Delta - a(x)$. We assume here that $a(x)$ satisfies (3.9). Let G_k be the Green's function for the operator $-\Delta - a_k(x)$ where $a_k(x) = \min(a(x), k)$, that is

$$\begin{cases} -\Delta_y G_k(x,\cdot) - a_k(y) G_k(x,\cdot) &= \delta_x \quad \text{in } \Omega \\ G_k(x,\cdot) &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where δ_x denotes the Dirac measure at some $x \in \Omega$. Then one can prove the following

Lemma 3.18. *We have $G_k \geq 0$ and the sequence G_k is non-decreasing and bounded in $L^1(\Omega \times \Omega)$. Therefore it converges to a function $G \in L^1(\Omega \times \Omega)$. Moreover, for any $f \in L^\infty(\Omega)$ the solution u to*

$$\begin{cases} -\Delta u - a(x) u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

can be represented as

$$u(x) = \int_\Omega G(x,y) f(y)\, dy \qquad \text{a.e. in } \Omega.$$

Then, as a consequence of the comparison result in Corollary 3.8 we have the following

Corollary 3.19. *There exists a constant $c > 0$ depending on Ω, r, $\gamma(a)$ such that*

$$G(x,y) \geq c\zeta_0(x)\zeta_0(y) \qquad \text{a.e. in } \Omega \times \Omega.$$

We have not investigated the possibility of establishing pointwise upper bounds for G. For the special case of the inverse square potential $a(x) = c/|x|^2$, in dimension $n \geq 3$ and with $0 < c < (n-2)^2/4$, Milman and Semenov [17] established upper and lower bounds for the heat kernel associated to the operator $-\Delta - a(x)$ in $\mathbb{R}^n$, from which upper bounds for the Green's function can be derived.

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Chapter 4

Semilinear elliptic PDE's with a singular potential

4.1 Introduction

4.1.1 Statement of the problem

This section focusses on the following equation :

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = f(u) + \lambda b(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_\lambda)$$

Here, Ω is a smooth bounded domain of $\mathbb{R}^n$, $\lambda > 0$ is a (small) constant and a, b, f are non-negative functions, satisfying a number of conditions listed later on. At this point, we would like to look at an example treated in [D], which motivates the study of (P_λ) and clarifies the issues at stake : take $a(x) = c/|x|^2$ where $c \in (0, (n-2)^2/4)$, $f(u) = u^p$, with $p > 1$ and $b(x) \equiv 1$. (P_λ) becomes :

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = u^p + \lambda & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

It turns out that if $0 \in \Omega$ and $n \geq 3$, there exists a critical exponent $p_0 = p_0(c, n)$ such that the above equation has no solution for any pair (p, λ) satisfying $p \geq p_0$ and $\lambda > 0$, whereas solutions exist for $p < p_0$, provided $\lambda > 0$ is chosen small enough (while no solution exist if $p < p_0$ and λ is large). It should be noted that whenever they exist, the solutions are always singular at the origin. In this work, we show that this result can be extended to a greater class of potentials, examples of which can be taken to have singularities on curves or higher dimensional submanifolds of Ω (see Section 6) : if $a(x) = c/dist(x,\Sigma)^2$, where Σ is a submanifold of codimension $k \geq 3$ of Ω, there is again a critical exponent $p_0 = p_0(c,n,k)$, which somewhat surprisingly decreases with k.

Roughly speaking, there is a better chance that (P_λ) has a solution when the potential is singular on a 'larger' set. In fact, when $\Sigma = \partial\Omega$, any power (or any nonlinearity f) is allowed. Also, this critical exponent phenomenon is just a specific case of a dichotomy between nonlinearities f that allow for existence of solutions and those that don't. We derive for this matter a sharp abstract criterium on f, in the spirit of [KV] and [BC], which is nevertheless easy to check in applications.

Even in the case of the inverse-square potential $a(x) = c/|x|^2$, this will lead us to new results complementing those of [D].

We now turn back to (P_λ) and to make all of our statements precise, list the assumptions on our data :

- $a \in L^1_{loc}(\Omega)$, $a(x) \geq 0$ a.e. and, for some > 0,

$$\int_\Omega |\nabla u|^2 - \int_\Omega a(x)u^2 \geq \int_\Omega u^2 \qquad \text{for all } u \in C_c^\infty(\Omega) \tag{0.1}$$

(0.1) states that the first eigenvalue of the operator $L = -\Delta - \frac{c}{|x|^2}$ is positive. When $c \leq c_0 := (n-2)^2/4$, $n \geq 3$ and $a(x) = c/|x|^2$, (0.1) is just the celebrated Hardy inequality (see [BV] for its proof). However, if $c > c_0$ (and $0 \in \Omega$), (0.1) fails and in fact there are no nonnegative $u \not\equiv 0$ such that $-\Delta u - \frac{c}{|x|^2}u \geq 0$, hence no solution of (P_λ) (see [BG] or [CM]). Hence (0.1) is crucial.

It also follows from (0.1) that

$$\|u\|_H^2 := \int_\Omega |\nabla u|^2 - \int_\Omega a(x)u^2$$

is (the square of) a norm on $C_c^\infty(\Omega)$. Completing $C_c^\infty(\Omega)$ with respect to this norm, we obtain a Hilbert space H. Using Lax-Milgram lemma, we then define a unique $\zeta_0 \in H$ solving

$$\begin{cases} -\Delta\zeta_0 - \dfrac{c}{|x|^2}\zeta_0 = 1 & \text{in } \Omega \\ \zeta_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{0.2}$$

in the sense that

$$(\zeta_0|\phi)_H = \langle 1, \phi\rangle_{H^*,H} \qquad \text{for all } \phi \in H. \tag{0.3}$$

Observe that given any $\epsilon > 0$, if a satisfies (0.1) then the space H associated with $a_\epsilon := (1-\epsilon)a$ coincides with $H_0^1(\Omega)$. So that in the generic case, our definition of ζ_0 reduces to the standard one. However, it was proved in [VZ] that if $a(x) = (n-2)^2/(4|x|^2)$ (this potential corresponds to the limiting case of the Hardy inequality), the associated space H contains $H_0^1(\Omega)$ as a proper subset.

- $b \in L^1_\delta(\Omega) := L^1(\Omega, dist(x, \partial\Omega)dx)$, $b \not\equiv 0$, $b(x) \geq 0$ a.e. and

$$\int_\Omega b\zeta_0 < \infty \tag{0.4}$$

where $0 \leq \zeta_0 \in H$ is the solution of (0.3). For simplicity, the reader may think of b as a smooth and bounded function. As we shall see (in Lemma 1.2), what (0.4) ensures is that there exists $\zeta_1 \geq 0$ solving (in a certain sense to be defined later on)

$$\begin{cases} -\Delta\zeta_1 - \dfrac{c}{|x|^2}\zeta_1 = b & \text{in } \Omega \\ \zeta_1 = 0 & \text{on } \partial\Omega \end{cases} \tag{0.5}$$

which is a minimum requirement, if one wants to solve (P_λ).

The following set of conditions on f, though technical, is satisfied by a wide class of nonlinearities.

- $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1, convex function with $f(0) = f'(0) = 0$ satisfying the two following growth conditions :

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty \tag{0.6}$$

$$\int_1^\infty g(s)ds < \infty \qquad \text{and} \qquad sg(s) < 1 \text{ for } s > 1 \tag{0.7}$$

where we set, for $s \geq 1$,

$$g(s) = \sup_{t>0} f(t)/f(st) \tag{0.8}$$

Clearly, g is a decreasing, nonnegative function. Moreover, $s \to sg(s)$ is nonincreasing since, by convexity, $t \to f(t)/t$ is increasing and $f(0) = 0$.

We may also assume that g is continuous. If not, since g is used in our proofs solely for comparison arguments, it suffices to replace g with a continuous function $\tilde{g} \geq g$, satisfying (0.7), such that $t \to t\tilde{g}(t)$ is nonincreasing and

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \qquad \text{is arbitrarily small}$$

We construct such a function in Lemma 2.3.

We also observe that since $g(s) \geq f(1)/f(s)$, (0.7) implies the following weaker condition, which often appears in the litterature :

$$\int^\infty \frac{1}{f(s)} ds < \infty.$$

In particular, our proofs yield no result for functions like $f(t) = t(\ln t)_+^\beta$, $\beta > 0$ for which $g(s) = 1/s$.

Examples of nonlinearities f which do satisfy our assumptions are : $f(u) = u^p$ for $p > 1$, $f(u) = e^u - u - 1$, $f(u) = u^2 - 1 + \cos(u)$,... Next, we clarify the notion of solution used in this section. We need to do so because even linear problems of the form (0.5) may not be well posed in the usual distributional or Sobolev space settings. This is shown in [D] for the potential $a(x) = c/|x|^2$. • Following [BCMR], we shall say that $u \in L^1(\Omega)$ is a **weak solution** of (P_λ) if $u \geq 0$ a.e. and if it satisfies the two following conditions :

$$\begin{cases} \int_\Omega (a(x)u + f(u))\,\delta(x) < \infty & \text{where } \delta(x) = dist(x, \partial\Omega) \\ \int_\Omega u\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) = \int_\Omega (f(u) + \lambda b)\phi & \text{for } \phi \in C^2(\bar{\Omega})\ ,\ \phi|_{\partial\Omega} = 0 \end{cases}$$

Observe that the first condition merely ensures that the integrals in the second equation make sense. Similarily, a weak solution $u \in L^1(\Omega)$ of (0.5) with $b \in L^1_\delta := L^1(\Omega, \delta(x)dx)$, is one that satisfies the equation $\int_\Omega u\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) = \int_\Omega b\phi$ (for all $\phi \in C^2(\bar{\Omega})$, $\phi|_{\partial\Omega} = 0$) with the integrability condition $\int_\Omega (a(x) + 1)|u|\delta(x) < \infty$. We will also refer to inequalities holding in the weak sense or talk about (weak) supersolutions. This means that we integrate the equation with nonnegative test functions. For example, $-\Delta u - \frac{c}{|x|^2}u \geq f$ holds **in the weak sense**, given $f \in L^1_\delta$, if $u \in L^1(\Omega)$, $a(x)u \in L^1_\delta(\Omega)$

and if

$$\int_\Omega u\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) \geq \int_\Omega f\phi \qquad \text{for all } \phi \in C^2(\bar\Omega) \text{ with } \phi \geq 0 \text{ and } \phi|_{\partial\Omega} = 0$$

These definitions are motivated by the following lemma (proved in [BCMR]) :

Lemma 0.1. *Let $f \in L^1_\delta(\Omega) := L^1(\Omega, \delta(x)dx)$. There exists a unique (weak) solution $u \in L^1(\Omega)$ of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense that

$$\int_\Omega u\,(-\Delta\phi) = \int_\Omega f\phi \qquad \text{for all} \qquad \phi \in C^2(\bar\Omega)\ ,\ \phi|_{\partial\Omega} = 0$$

Furthermore, there exists a constant $C = C(\Omega) > 0$ such that

$$\|u\|_{L^1(\Omega)} \leq C\|f\|_{L^1_\delta(\Omega)}$$

and

$$f \geq 0 \quad a.e. \quad \Longrightarrow \quad u \geq 0 \quad a.e.$$

Lemma 0.2. *Let $a(x) \in L^1_{loc}(\Omega)$, $b \in L^1_\delta(\Omega)$ and $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ be nonnegative functions. Let $\lambda > 0$. Suppose there exists a (weak) supersolution $w \geq 0$ of (P_λ) (respectively (0.2),(0.5)). Then there exists a unique weak solution $u \geq 0$ of (P_λ) (respectively (0.2),(0.5)) such that*

$$0 \leq u \leq \tilde w$$

for any (weak) supersolution $\tilde w \geq 0$ of (P_λ) (respectively (0.2),(0.5)).

u is then called **the** *minimal nonnegative weak solution of (P_λ) (respectively (0.2),(0.5)).*

Remark. The function $\zeta_0 \in H$ solving (0.3) also solves (0.2) in the weak sense. In fact, it is *the* minimal nonnegative weak solution of (0.2), so that no confusion may arise (see the remark following Lemma 1.1).

Proof. The proof is identical for all three equations (P_λ),(0.2),(0.5) so we restrict to the case where w is a supersolution of (P_λ). First if $u_1 \geq 0$ and $u_2 \geq 0$ are two weak solutions such that

$$0 \leq u_i \leq \tilde w \qquad i = 1, 2$$

for all supersolutions $\tilde{w} \geq 0$, then we must have $u_1 \leq u_2$ and $u_2 \leq u_1$, hence $u_1 = u_2$ so that the minimal solution – if it exists – is unique. Next, let $w \geq 0$ be a weak supersolution of (P_λ) and let $u_0 \in L^1(\Omega)$ be the unique solution of

$$\begin{cases} -\Delta u_0 = \lambda b & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 0.1. It follows easily from Lemma 0.1 that $0 \leq u_0 \leq w$. Next, we show by induction that there exists a unique $u_n \in L^1(\Omega)$ for $n = 1, 2, ..$ solving

$$\begin{cases} -\Delta u_n = a(x)u_{n-1} + f(u_{n-1}) + \lambda b & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

in the sense of Lemma 0.1, and such that $0 \leq u_n \leq w$. Indeed, since $0 \leq u_0 \leq w$ and w is a weak supersolution,

$$0 \leq a(x)u_0 + f(u_0) \leq a(x)w + f(w) \in L^1_\delta(\Omega).$$

So that u_1 is well defined (by Lemma 0.1) and $0 \leq u_0 \leq u_1 \leq w$ (applying Lemma 0.1 again). The same argument can be applied inductively to show that u_n is well defined (provided $0 \leq u_{n-1} \leq w$) and that

$$0 \leq u_{n-1} \leq u_n \leq w.$$

Hence $\{u_n\}_n$ is a nondecreasing sequence of nonnegative functions dominated by w. By monotone convergence, its (pointwise) limit u solves (P_λ).

Now if $\tilde{w} \geq 0$ is another supersolution, it follows easily from Lemma 0.1 that $u_0 \leq \tilde{w}$ and $u_n \leq \tilde{w}$ for all $n = 1, 2, ...$ Passing to the limit, it follows that $u \leq \tilde{w}$.

□

With these definitions in mind, we investigate the existence, uniqueness and regularity of solutions of (P_λ) :

4.1.2 Main results

Theorem 1 (existence and optimal regularity). *Assume (0.1),(0.4),(0.6),(0.7) hold and let* $\zeta_0 = G(1)$ *solve (0.2),* $\zeta_1 = G(b)$ *solve (0.5) with* $G = \left(-\Delta - \frac{c}{|x|^2}\right)^{-1}$ *defined in Lemma 1.2.*

- **Either** *there exist constants* $\epsilon > 0$, $C > 0$ *such that*

$$\int_\Omega f(\epsilon\zeta_1)\zeta_0 < \infty \qquad \textit{and} \qquad G(f(\epsilon\zeta_1)) \le C\zeta_1 \qquad \textit{a.e.} \tag{0.8}$$

Then there exists $\lambda^* > 0$ *depending on* n, $a(x)$, f, $b(x)$ *such that*

- *if* $\lambda < \lambda^*$ *then* (P_λ) *has a minimal weak solution* u.

Furthermore, for some constant $C > 0$ *independent of* $x \in \Omega$, *we have*

$$\zeta_1 \le u \le C\zeta_1 \qquad \textit{a.e. in } \Omega$$

- *if* $\lambda = \lambda^*$ *then* (P_λ) *has a minimal weak solution,*

- *if* $\lambda > \lambda^*$ *then* (P_λ) *has no solution, even in the weak sense and there is complete blow-up.*

- **Or** *(0.8) holds for no* $\epsilon > 0$, $C > 0$. *Then, given any* $\lambda > 0$, (P_λ) *has no solution, even in the weak sense, and there is complete blow-up.*

This result requires the following definitions :

Definition 0.1. *Let* $\{a_n(x)\}$, $\{b_n(x)\}$ *and* $\{f_n\}$ *be increasing sequences of bounded smooth functions converging pointwise respectively to* $a(x)$, $b(x)$ *and* f *and let* $\underline{u_n}$ *be the minimal nonnegative solution of*

$$\begin{cases} -\Delta\underline{u_n} - a_n\underline{u_n} = f_n(\underline{u_n}) + \lambda b_n & \textit{in } \Omega \\ \underline{u_n} = 0 & \textit{on } \partial\Omega \end{cases} \tag{P_n}$$

We say that there is **complete blow-up** *in* (P_λ) *if, given any such* $\{a_n(x)\}$, $\{b_n(x)\}$, $\{f_n\}$ *and* $\{\underline{u_n}\}$,

$$\frac{\underline{u_n}(x)}{\delta(x)} \to +\infty \textit{ uniformly on } \Omega,$$

where $\delta(x) := dist(x, \partial\Omega)$.

Definition 0.2. *The parameter* λ^* *is called the* **extremal parameter** *of the family of equations* $\{(P_\lambda)\}_\lambda$ *and the corresponding solution* u_{λ^*} *is called the* **extremal solution**.

Theorem 2 (uniqueness of stable solutions). *Make the same assumptions as in Theorem 1. If it exists, let u_λ denote the minimal nonnegative (weak) solution of (P_λ). If $0 < \lambda < \lambda^*$,*

- *u_λ is stable*
- *Assume that $f(\zeta_1) + \lambda b \in H^*$. Then $u_\lambda \in H$ and u_λ is the only stable (weak) solution of (P_λ) belonging to H.*

If $\lambda = \lambda^*$,

- *u_{λ^*} is stable*
- *Assume $b \in L^p$ for some $p > n$. Then u_{λ^*} is the only weak solution of (P_{λ^*}).*

Stability is defined as follows :

Definition 0.4. *We say that u is* **stable** *if the generalized first eigenvalue $\lambda_1(u)$ of the linearized operator of equation (P_λ) is nonnegative, i.e., if*

$$\lambda_1(u) := \inf\{J(\phi) : \phi \in C_c^\infty(\Omega) \setminus \{0\}\} \geq 0$$

where

$$J(\phi) = \frac{\int_\Omega |\nabla\phi|^2 - \int_\Omega a(x)\phi^2 - \int_\Omega f'(u)\phi^2}{\int_\Omega \phi^2}$$

The proof of Theorem 1 is presented in section 2 , whereas Theorem 2 is proved in section 3. Applications can be found in the remaining sections 4, 5 and 6.

4.2 Preliminary : linear theory

We construct here a few basic tools to be used later on and start out with the L^2 theory.

Lemma 1.1. *Suppose (0.1) holds and let $b \in H^* \cap L^1_\delta(\Omega)$.*

There exists a unique $u \in H$, weak solution of

$$\begin{cases} -\Delta u - a(x)u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

Furthermore,

$$\|u\|_H = \|b\|_{H^*} \tag{1.2}$$

$$b \geq 0 \textit{ in the sense of distributions } \Rightarrow u \geq 0 \textit{ a.e.} \tag{1.3}$$

Proof. By working with b^+, b^- we can assume that $b \geq 0$. It follows from Lax-Milgram lemma that there exists a unique $u \in H$ such that

$$(u|\phi)_H = \langle b, \phi \rangle_{H^*,H} \qquad \text{for all } \phi \in H.$$

Furthermore, (1.2) holds. We now show that u solves (1.1) in the weak sense : Let

$$a_k(x) = \min(a(x), k), \quad k > 0$$

and u_k be the solution to (1.1) with the potential $a(x)$ replaced by the potential $a_k(x)$. Then it is easy to check that u_k is nondecreasing in k, and converges to u in $L^2(\Omega)$. Now take $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$. Then

$$\int_\Omega u_k(-\Delta\phi) = \int_\Omega a_k(x) u_k \phi + b\phi,$$

and note that here all the integrals are finite. By taking in particular $\phi = \phi_0$ to be the solution of

$$\begin{cases} -\Delta\phi_0 = 1 \text{ in } \Omega \\ \phi_0 = 0 \text{ on } \partial\Omega \end{cases}$$

, and using Fatou's lemma, we see that $\int_\Omega a(x) u \phi_0$ exists and is finite. Given any $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$, we can find $C > 0$ so that $|\phi| \leq C\phi_0$. It follows that we can pass to the limit in the equation satisfied by u_k and conclude that u solves (1.1) in the weak sense.

Next we show that if $\tilde{u} \in H$ is another weak solution of (1.1) then $\tilde{u} = u$. By definition of H, there exits $u_n \in C_c^\infty(\Omega)$ such that $u_n \to \tilde{u}$ in H (and a fortiori in $L^1(\Omega)$). Hence for $\phi \in C_c^\infty(\Omega)$,

$$(\tilde{u}|\phi)_H = \lim_{n\to\infty} (u_n|\phi)_H$$

Using integration by parts and the fact that $u_n \to \tilde{u}$ in $L^1(\Omega)$,

$$(u_n|\phi)_H = \int_\Omega u_n \left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) \to \int_\Omega \tilde{u} \left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) = \langle b, \phi\rangle_{H^*,H}$$

So that

$$(\tilde{u}|\phi)_H = \langle b, \phi\rangle_{H^*,H} \qquad \text{for all } \phi \in C_c^\infty(\Omega).$$

By density, the equality holds for all $\phi \in H$ and $\tilde{u} = u$ by Lax-Milgram lemma.

Finally we show (1.3). Let $b \in H^*$, $b \geq 0$ (in the sense of distributions) and $u \in H$ be the corresponding solution of (1.1).

By definition of H, there exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ converging to u in H. Letting $b_n = -\Delta u_n - \frac{c}{|x|^2} u_n$, it follows that $b_n \in H^*$ and $b_n \to b$ in H^*.

Now, $u_n \in H_0^1(\Omega) \Rightarrow u_n^- \in H_0^1(\Omega)$ and integrating the equation satisfied by u_n against u_n^- yields

$$-\|u_n^-\|_H^2 = \langle b_n, u_n^-\rangle_{H*,H}$$

To pass to the limit in this last equation, we just need to prove that $\{u_n^-\}$ remains bounded in H.

$$\begin{aligned}
\|u_n^-\|_H^2 &= \int_\Omega |\nabla u_n^-|^2 - \int_\Omega a(x)(u_n^-)^2 \\
&= \int_\Omega |\nabla u_n^-|^2 - \int_\Omega a(x)u_n^2 + \int_\Omega a(x)(u_n^+)^2 \\
&\leq \int_\Omega |\nabla u_n^-|^2 - \int_\Omega a(x)u_n^2 + \int_\Omega |\nabla u_n^+|^2 = \int_\Omega |\nabla u_n|^2 - \int_\Omega a(x)u_n^2 \\
&= \|u_n\|_H^2
\end{aligned} \tag{1.4}$$

where we've used (0.1) in the inequality.

□

Remark. Observe in passing that ζ_0 solving (0.3) is the minimal nonnegative weak solution of (0.2) : by the previous lemma (and its proof), ζ_0 is indeed a weak solution of (0.2). If u denotes the minimal nonnegative weak solution of (0.2), and u_k the solution of (0.2) when a is replaced by $a_k = \min(a, k)$, it can be shown as above that $u_k \to u$ in $L^1(\Omega)$ and that $\{u_k\}_k$ remains bounded in H so that $u \in H$ and $u = \zeta_0$.

Lemma 1.2. *Let $b \in L^1(\Omega, \delta(x)dx)$ with $b \geq 0$ a.e. and $b \not\equiv 0$. The equation*

$$\begin{cases} -\Delta u - \dfrac{c}{|x|^2} u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

has a nonnegative weak solution $u \in L^1(\Omega)$ (which may not be unique) if and only if

$$\int_\Omega b(x)\zeta_0 dx < \infty \tag{1.6}$$

where ζ_0 denotes the solution of (0.3). We then denote the minimal nonnegative weak solution u of (1.5) by

$$u = G(b)$$

Proof.

Suppose first that $\int_\Omega b(x)\zeta_0 dx < \infty$ and let $b_n = \min(n, b)$ for $n \in \mathbb{N}$.

By Lemma 1.1, there exists a unique $v_n \in H$, $v_n \geq 0$, solving (1.5) with b_n in place of b and, testing with ζ_0 in (1.5) and with v_n in (0.3), we obtain

$$\int_\Omega b_n \zeta_0 = (v_n|\zeta_0)_H = \int_\Omega v_n$$

Hence,

$$\|v_n\|_{L^1} \leq \int_\Omega b\zeta_0$$

Testing with z, solving

$$\begin{cases} -\Delta z = 1 \text{ in } \Omega \\ z = 0 \text{ on } \partial\Omega \end{cases} \tag{1.7}$$

we get

$$(v_n|z)_H = \int_\Omega b_n z = \int_\Omega v_n - \int_\Omega a(x) v_n z$$

So that

$$\int_\Omega a(x) v_n \delta \leq C \int_\Omega b\zeta_0$$

Observe that Lemma 1.1 implies that v_n is nondecreasing and using a standard monotone convergence argument, it follows that $v = \lim v_n$ (weakly) solves (1.5). By Lemma

0.2, we can then construct the minimal nonnegative weak solution $u = G(b)$. Conversely, suppose $v \geq 0$ is a weak solution of (1.5) and assume for now that $b \in L^\infty$ and that v is minimal. $\|b\|_{L^\infty}\zeta_0$ is then a supersolution of (1.5), hence $v \leq C\zeta_0$. Also, as in the proof of Lemma 1.1, we can show that v is an H solution. Next, take a sequence of bounded functions Φ_n increasing pointwise to $a(x)\zeta_0$ and let v_n be the solution of

$$\begin{cases} -\Delta v_n = \Phi_n \text{ in } \Omega \\ \quad v_n = 0 \text{ on } \partial\Omega \end{cases} \tag{1.8}$$

Testing with v_n, we obtain

$$\int_\Omega b v_n = (v|v_n)_H = \int_\Omega v(\Phi_n - a(x)v_n) \tag{1.9}$$

Since $a(x)\zeta_0 \in L^1_\delta$, $\phi_n \nearrow a(x)\zeta_0$ in L^1_δ and, by Lemma 0.1, $v_n \nearrow v := \zeta_0 - z$ in L^1, with z solving (1.7).

Now, $\Phi_n \leq a(x)\zeta_0$ and $v_n \leq v \leq \zeta_0$, hence

$$|v(\Phi_n - a(x)v_n)| \leq 2a(x)\zeta_0^2$$

Suppose in addition that $\int_\Omega a(x)\zeta_0^2 < \infty$ so that we can apply Lebesgue's theorem and pass to the limit in (1.9) :

$$\int_\Omega b(\zeta_0 - z) = \int_\Omega v[a(x)\zeta_0 - a(x)(\zeta_0 - z)] = \int_\Omega a(x)vz$$

Hence,

$$\int_\Omega b\zeta_0 \leq \|z\|_{C^1}\left(\int_\Omega b\delta + \int_\Omega a(x)v\delta\right) \tag{1.10}$$

We made two auxiliary assumptions to arrive to this result. First, we assumed that $b \in L^\infty$. If this is not true, we can replace b by $b_n = \min(b, n)$, apply (1.10) to b_n and let $n \to \infty$. We also assumed that $\int_\Omega a(x)\zeta_0^2 < \infty$. If not, replace $a(x)$ by $a_\epsilon(x) := (1-\epsilon)a(x)$ and ζ_0 by ζ_ϵ the solution of (0.3) with a_ϵ in place of a. Multiply (0.1) by $(1-\epsilon)$ to obtain

$$\int_\Omega |\nabla u|^2 - \int_\Omega a_\epsilon(x)u^2 \geq \epsilon \int_\Omega |\nabla u|^2 \qquad \text{for all } u \in C_c^\infty(\Omega)$$

The space H corresponding to the potential a_ϵ is therefore good old $H^1_0(\Omega)$ and, by construction, $\zeta_\epsilon \in H^1_0(\Omega)$. Using the above inequality, it follows that

$$\int_\Omega a_\epsilon \zeta_\epsilon^2 < \infty$$

We can therefore apply (1.10) with ζ_ϵ in place of ζ_0 and let $\epsilon \to 0$. $\square$

4.3 Existence vs. complete blow-up

In this section, we will prove existence or nonexistence of weak solutions of (P_λ), using the tools we have just constructed and monotonicity arguments.

The following result, due to [BC], for which a proof in our context can be taken from [D], proves the blow-up results of Theorem 1 provided nonexistence of weak solutions is established :

Lemma 2.0. *Fix $\lambda > 0$ Suppose (P_λ) has no weak solution. Then there is complete blow-up in (P_λ)*

Next, we extend a technical result of [BC] :

Lemma 2.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a C^1, concave function such that for some $C > 0$,*

$$0 \leq \Phi' \leq C$$

Let h, $k \in L^1_{loc}(\Omega)$, h, $k \geq 0$, $k \not\equiv 0$, satisfy (1.6) so that $u = G(h)$ and $v = G(k)$ are well-defined. Then, letting $w = v\Phi(u/v)$,

$$w \in L^1(\Omega) \ , \ a(x)w \in L^1_\delta \ \textit{and}$$

$$-\Delta w - \frac{c}{|x|^2} w \geq \Phi'(u/v)(h-k) + \Phi(1)k \qquad \textit{in the weak sense} \tag{2.1}$$

Proof. Suppose first $u, v \in C^2(\bar{\Omega})$, $v > 0$ in Ω and $\Phi \in C^2(\mathbb{R})$. Applying Lemma 5.3 in [BC], it follows that a.e. in Ω,

$$\begin{aligned} -\Delta w - \frac{c}{|x|^2} w \geq & \Phi'(u/v)(-\Delta u) + \left[\Phi(u/v) - \Phi'(u/v)u/v\right](-\Delta v) - a(x)\Phi(u/v)v \\ \geq & \Phi'(u/v)h + \left[\Phi(u/v) - \Phi'(u/v)u/v\right] k \\ \geq & \Phi'(u/v)(h-k) + \left[\Phi(u/v) - \Phi'(u/v)u/v + \Phi'(u/v)\right] k \end{aligned}$$

Since Φ is concave,

$$\Phi(s) + (1-s)\Phi'(s) \geq \Phi(1) \qquad \text{for all } s \in \mathbb{R}$$

Hence,

$$-\Delta w - \frac{c}{|x|^2} w \geq \Phi'(u/v)(h-k) + \Phi(1)k \qquad \text{a.e. in } \Omega \tag{2.2}$$

Since Φ' is bounded, we see, as in [BC], that

$$|v\Phi(u/v)| = |v\left(\Phi(u/v) - \Phi(0)\right) + \Phi(0)v| \leq C(u+v) \tag{2.3}$$

Hence, w vanishes on $\partial\Omega$ and integrating by parts, (2.2) holds in the weak sense. By approximation of Φ, we can also say that (2.2) holds even when Φ is only C^1. Finally observe that all of the above computations still hold if u,v are merely $C^{1,\alpha}(\bar{\Omega})$. In the general case, let a_n,h_n,k_n be bounded functions increasing pointwise to a,h,k and u_n, v_n be the solutions of the associated equations. Also write $w_n = v_n\Phi(u_n/v_n)$. For n large enough, $k_n \not\equiv 0$ since $k \not\equiv 0$ and by the strong maximum principle, $v_n > 0$ in Ω. We can then apply (2.2) to obtain

$$-\Delta w_n - a_n(x)w_n \geq \Phi'(u_n/v_n)(h_n - k_n) + \Phi(1)k_n \qquad \text{weakly} \tag{2.4}$$

Since $-\Delta v \geq 0$ in the sense of distributions and $v \geq 0$, it follows from the mean value formula that $v > 0$ a.e. in Ω, so that $v\Phi(u/v)$ is well defined a.e. Moreover, it is clear that $u_n \nearrow u$ in L^1 and that $a_n(x)u_n(x) \nearrow a(x)u(x)$ in L^1_δ and similarly for v. So that, using (2.3) and Lebesgue's theorem

$$w_n \to w \qquad \text{in } L^1 \text{ and} \qquad a_n(x)w_n \to a(x)w \qquad \text{in } L^1_\delta$$

Since Φ' is bounded, we can also easily pass to the limit in the right-hand side of (2.4) and obtain (2.1).

□

The next lemma contains the heart of the proof.

Lemma 2.2. *Assume that $b \in L^1_{loc}(\Omega)$, $b \geq 0$ satisfies (0.4) and let $\zeta_1 = G(b)$. Suppose u is a weak solution of (P_1). Then*

$$\int_\Omega f(\zeta_1)\zeta_0 < \infty \qquad \text{and} \qquad G(f(\zeta_1)) \leq C\zeta_1 \qquad \text{where} \qquad C = \int_1^\infty g(s)ds$$

Conversely, if

$$\int_\Omega f(2\zeta_1)\zeta_0 < \infty \qquad \text{and} \qquad G(f(2\zeta_1)) \leq \zeta_1$$

then (P_1) admits a weak solution u.

Proof. Suppose first that u is a weak solution of (P_1) and, recalling (0.6), define for $t \geq 1$

$$\Phi(t) = \int_1^t g(s)ds$$

and let $w = \Phi(u/\zeta_1)\zeta_1$. Observe that u is a supersolution of the equation satisfied by ζ_1, so by minimality of ζ_1, $u \geq \zeta_1$ and one can easily check that Lemma 2.1 applies with our choice of Φ, so that

$$-\Delta w - \frac{c}{|x|^2} w \geq g(u/\zeta_1)f(u) \geq f(\zeta_1)$$

Since $w \leq C\zeta_1$, $G(f(\zeta_1))$ can be constructed e.g. by monotone iteration , hence $\int_\Omega f(\zeta_1)\zeta_0 < \infty$ and we have

$$C\zeta_1 \geq w \geq G(f(\zeta_1))$$

Conversely, suppose that $G(f(2\zeta_1)) \leq \zeta_1$ and let $w = G(f(2\zeta_1)) + \zeta_1$. Then $w \leq 2\zeta_1$ and

$$-\Delta w - \frac{c}{|x|^2} w = f(2\zeta_1) + b \geq f(w) + b$$

So w is a supersolution of (P_1) and one can construct a weak solution, using a standard argument of monotone iteration.

□

The following two lemmas are technical.

Lemma 2.3. *Fix $\epsilon \in (0,1)$ and for $t \in \mathbb{R}$ set $\tilde{g}(s) = \sup_{t \geq 1} g_t(s)$*

where

$$g_t(s) = \begin{cases} g(s) & \textit{if} \quad s \leq t \\ g(t) - \dfrac{1}{\epsilon}(s-t) & \textit{if} \quad s > t \end{cases}$$

Then $\tilde{g}(s)$ is Lipschitz continuous, satisfies (0.6), $s \to s\tilde{g}(s)$ is nonincreasing and

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \leq \epsilon/2$$

Proof. We first check the continuity of $\tilde{g}(t)$. Fix $1 \leq s < t$. We have

$$\tilde{g}(s) - \tilde{g}(t) = \sup_{r>1} g_r(s) - \tilde{g}(t).$$

Fix any $r > 1$. Then

$$g_r(s) - \tilde{g}(t) < g_r(s) - g_{\min\{r,s\}}(t).$$

For $r < s$ the right hand side is equal to $g_r(s) - g_r(t) = (t-s)/\epsilon$ and for $r \geq s$ it is equal to $g(s) - g_s(t) = (t-s)/\epsilon$ by the definition ot $g_s(t)$. So, for any $r > 1$

$$g_r(s) - \tilde{g}(t) \leq (t-s)/\epsilon,$$

hence $0 \leq \tilde{g}(s) - \tilde{g}(t) \leq (t-s)/\epsilon$, which proves the continuity of $\tilde{g}(t)$.

Now we show that

$$\int_1^\infty \tilde{g}(s)ds - \int_1^\infty g(s)ds \leq \epsilon/2.$$

Indeed, if $g(t)$ is a step function, the answer is geometrically clear : $\tilde{g}$ is then a piecewise linear map and the difference between the integrals is given by

$$\sum \frac{\epsilon}{2}[g]^2(t) \leq \frac{\epsilon}{2}g(1)\sum[g](t) = \frac{\epsilon}{2}.$$

where the sums are taken over all points t of discontinuity of g and where $[g](t)$ denotes the jump of g at t.

If g isn't a step function, since g is monotonous, we may approximate it with an increasing sequence of step functions $\{g_n(t)\}$ and denote $g_{n,s}(t)$ and $\tilde{g}_n(t)$ the corresponding functions, defined as for $g(t)$. On the one hand,

$$\int_1^\infty \tilde{g}_n(s)ds - \int_1^\infty g_n(s)ds \leq \epsilon/2$$

for any $n \in \mathbb{N}$ and on the other hand $g_{n,s}(t) \nearrow g_s(t)$, $\tilde{g}_n(t) \nearrow \tilde{g}(t)$, so the desired estimate is also true for $g(t)$.

It remains to prove the monotonicity of $s\tilde{g}(s)$. It suffices to show that for every $t > 1$ the function $sg_t(s)$ is nonincreasing. Since it is the case for $sg(s)$ we only have to check that $tg_t(t) > sg_t(s)$ for $s > t > 1$. In fact

$$sg_t(s) = tg(t) - (s-t)(g(t) - s/\epsilon) < tg(t) = tg_t(t)$$

since $\epsilon < 1$ and $s > 1$. □

Lemma 2.4. *Let $\mu \in (0,1)$. There exists a C^1, concave, bounded solution Ψ of*

$$\begin{cases} \Psi'(t) = g(t/\Psi(t)) \text{ for } t \geq 1 \\ \Psi(1) = \mu \end{cases}$$

Proof.

By the preceeding lemma, up to replacing g by $\tilde{g}$, we may assume g Lipschitz continuous. By Cauchy's theorem, there exists a unique C^1 solution Ψ, which will be globally defined if we show that it is bounded.

Setting $\varphi(t) = t/\Psi(t)$, $\gamma = \varphi(1) = \mu^{-1}$ we obtain

$$\frac{\varphi'}{\varphi - \varphi^2 g(\varphi)} = \frac{1}{t}, \qquad \text{i.e.} \qquad \int_\gamma^\varphi \frac{ds}{s - s^2 g(s)} = \log t.$$

To show that $\varphi \geq t/c$ for some $c > 0$ it suffices to see that

$$\int_\gamma^\varphi \frac{ds}{s - s^2 g(s)} \leq \log \varphi + C \text{ for } \varphi > \gamma.$$

The above is equivalent to

$$\int_\gamma^\varphi ds\left(\frac{1}{s - s^2 g(s)} - \frac{1}{s}\right) \leq C \text{ or}$$

$$\int_\gamma^\varphi ds \frac{g(s)}{1 - sg(s)} \leq C \text{ for some } C > 0.$$

Since $tg(t)$ is nonincreasing, by (0.6) we get for $\beta = 1 - \gamma g(\gamma) > 0$

$$\int_\gamma^\varphi ds \frac{g(s)}{1 - sg(s)} \leq \int_\gamma^\varphi ds \frac{g(s)}{\beta} < \infty.$$

Hence Ψ is bounded.

Finally, since g is nonincreasing, it follows from the equation that Ψ is concave if $t/\Psi(t)$ is nondecreasing, which holds true, as

$$\left(\frac{t}{\Psi}\right)' = \frac{\Psi - t\Psi'}{\Psi^2} \geq 0$$

since by (0.6), $\Psi(t)/t \geq g(t/\Psi) = \Psi'(t)$. □

This last lemma shows the estimate $u \leq C(\lambda)\zeta_1$ when $0 < \lambda < \lambda^*$.

Lemma 2.5. *Suppose there exists $\lambda^* > 0$ such that (P_{λ^*}) has a weak solution.*

Then for all $0 < \lambda < \lambda*$, (P_λ) has a solution u satisfying for some $C > 0$ (depending on λ) and for a.e. $x \in \Omega$ the following estimate

$$\zeta_1 \leq u \leq C\zeta_1.$$

Proof. We let $\mu = \lambda/\lambda^*$, define Ψ as in Lemma 2.4. and let u^* denote a weak solution of (P_{λ^*}), $v^* = \lambda^*\zeta_1$ and $w = v^*\Psi(u^*/v^*)$. Observe that $u^* \geq v^*$ by minimality of ζ_1 and apply Lemma 2.1 :

$$\begin{aligned} -\Delta w - \frac{c}{|x|^2}w \geq& \Psi'(u^*/v^*)f(u^*) + \mu\lambda^* b \\ \geq& g\left(\frac{u^*/v^*}{\Psi(u^*/v^*)}\right)f(u^*) + \lambda b \\ \geq& f(w) + \lambda b \end{aligned}$$

So the minimal solution u of (P_λ) is bounded by w and since Ψ is bounded, $u \leq Cv^* \leq C'\zeta_1$.

□

We are now in a position to prove Theorem 1.

• Suppose first that (0.8) holds. We show that (P_λ) has a weak solution for $\lambda > 0$ small. It is then standard (see e.g. [D]) to show the existence of a finite $\lambda^* > 0$ such that (P_λ) has a weak solution if and only if $\lambda \leq \lambda^*$ and by Lemmas 2.0 and 2.5, we will have proven the first part of Theorem 1.

By Lemma 2.2, (P_λ) has a solution as soon as $G(f(2\lambda\zeta_1))$ exists and

$$G(f(2\lambda\zeta_1)) \leq \lambda\zeta_1 \tag{2.5}$$

By definition of g,

$$f(2\lambda\zeta_1) \leq g\left(\frac{\epsilon}{2\lambda}\right) f(\epsilon\zeta_1)$$

By (0.8), $G(f(2\lambda\zeta_1))$ exists and, by minimality of $G(f(2\lambda\zeta_1))$,

$$G(f(2\lambda\zeta_1)) \leq g\left(\frac{\epsilon}{2\lambda}\right) G(f(\epsilon\zeta_1)) \leq Cg\left(\frac{\epsilon}{2\lambda}\right)\zeta_1$$

To prove that (2.5) holds for λ small, it is therefore enough to show that

$$\frac{1}{\lambda} g\left(\frac{\epsilon}{2\lambda}\right) \to 0 \qquad \text{as} \qquad \lambda \to 0$$

or that

$$\lim_{M\to\infty} Mg(M) = 0$$

Since $M \to Mg(M)$ is nonincreasing, the above limit is well-defined. But if we had $\lim_{M\to\infty} Mg(M) = C > 0$ then $g(M) \sim C/M$ near ∞ , contradicting $\int_1^\infty g(s)ds < \infty$.

This completes the proof of the first part of Theorem 1.

• We now prove the second part of Theorem 1 : we assume that for some $\lambda > 0$, (P_λ) has a weak solution u_λ and show that (0.8) must hold for some $C, \epsilon > 0$. By Lemma 2.2,

$$\int_\Omega f(\lambda\zeta_1)\zeta_0 < \infty \qquad \text{and} \qquad G(f(\lambda\zeta_1)) \leq C\lambda\zeta_1$$

So choosing $\epsilon = \lambda$ and $C' = C\lambda$, Theorem 1 is proved.

4.4 Proof of Theorem 2

Step 1. We first prove that u_λ is stable for $\lambda \in [0, \lambda^*]$. Consider for $n \in \mathbb{N}$ the minimal solution u_n of

$$\begin{cases} -\Delta u_n - a_n(x)u_n = f(u_n) + \lambda b_n \text{ in } \Omega \\ u_n = 0 \text{ on } \partial\Omega \end{cases} \tag{3.1}$$

where $a_n(x) = \min(a(x), n)$ and $b_n = \min(b, n)$.

On the one hand, it is known that $\lambda_1(-\Delta - a_n(x) - f'(u_n)) \geq 0$. We briefly recall the proof of this fact : fix $p > n$ and consider the functional $F : \mathbb{R} \times W^{2,p}(\Omega) \to L^p(\Omega)$ defined by

$$F(\lambda, u) = -\Delta u - a_n(x)u - f(u) - \lambda b_n$$

It follows easily from (0.1) and the implicit function theorem that there exists a unique maximal curve $\lambda \in [0, \lambda^{\#}) \to u(\lambda)$ such that

$$F(\lambda, u(\lambda)) = 0 \qquad \text{and} \qquad F_u(\lambda, u(\lambda)) \in Iso(W^{2,p}, L^p).$$

If $0 < \lambda < \lambda^{\#}$, since u_n is the minimal solution of (3.1), $u_n \leq u(\lambda)$ and it follows by elliptic regularity that u_n is in the domain of F, so that $u_n = u(\lambda)$.
If $0 < \lambda < \lambda^*$, u_n is bounded (and hence in the domain of F) so that we must have $\lambda^{\#} = \lambda^*$ (otherwise we could extend the curve $u(\lambda)$ beyond $\lambda^{\#}$, contradicting its maximality).

So $\lambda_1(F_u(\lambda, u_n))$ never vanishes for $\lambda < \lambda^*$ and since by (0.1), $\lambda_1(F_u(0,0)) > 0$, we conclude that $\lambda_1(-\Delta - a_n(x) - f'(u_n)) \geq 0$, for $0 \leq \lambda \leq \lambda^*$. On the other hand u_n increases with n to a solution of (P_λ), and since $u_n \leq u_\lambda$, the limit is the minimal solution u_λ. Now by monotone convergence we conclude

$$\lambda_1(-\Delta - a(x) - f'(u_\lambda)) \geq 0.$$

Step 2. We now show that if $f(\zeta_1) + \lambda b \in H^*$ and $\lambda < \lambda^*$ then the minimal solution u_λ of (P_λ) belongs to H. We know by Theorem 1, that $u_\lambda \leq C\zeta_1$ so that

$$0 \leq f(u_\lambda) \leq f(C\zeta_1) \leq g(1/C)f(\zeta_1)$$

Hence, for $\phi \in C_c^\infty(\Omega)$,

$$\left|\int_\Omega (f(u_\lambda) + \lambda b)\phi\right| \leq \int_\Omega (Cf(\zeta_1) + \lambda b)|\phi| \leq C\|\phi\|_H$$

So that, $G := f(u_\lambda) + \lambda b \in H^*$. Letting $a_n(x) = \min(a(x), n)$ and u_n denote the solution of

$$\begin{cases} -\Delta u_n - a_n(x)u_n = G \text{ in } \Omega \\ u_n = 0 \text{ on } \partial\Omega \end{cases}$$

We have

$$\|u_n\|_H^2 = \int_\Omega Gu_n + (a_n - a)u_n \le \int_\Omega Gu_n \le C\|u_n\|_H$$

Hence u_n is bounded in H and $u_\lambda \in H$.

Step 3. Next, following Brezis and Vazquez [BV], we prove that for $0 < \lambda < \lambda^*$ there is at most one stable solution belonging to H. Arguing by contradiction, let $u_1, u_2 \in H$ be two distinct stable solutions of (P_λ). We may suppose that $u_1 = u_\lambda$ – the minimal solution. Then by the maximum principle $u_2 - u_1 > C\delta(x)$ for some $C > 0$. The stability for u_2 writes $\lambda_1(L - f'(u_2)) \ge 0$. We test this inequality against $u_2 - u_1$ (Note that we can do it, as $u_{1,2} \in H$). We obtain

$$\int_\Omega |\nabla(u_2 - u_1)|^2 - a(x)(u_2 - u_1)^2 \ge \int_\Omega f'(u_2))(u_2 - u_1)^2.$$

Since u_1 and u_2 are solutions, we also have

$$\int_\Omega |\nabla(u_2 - u_1)|^2 - a(x)(u_2 - u_1)^2 = \int_\Omega (f(u_2) - f(u_1))(u_2 - u_1).$$

Hence

$$\int_\Omega (f(u_2) - f(u_1))(u_2 - u_1) \ge \int_\Omega f'(u_2)(u_2 - u_1)^2$$

As f is convex $f(u_2) - f(u_1) \le f'(u_2)(u_2 - u_1)$ and since $u_2 - u_1 > 0$ on Ω we arrive at

$$f(u_2) - f(u_1) = f'(u_2)(u_2 - u_1) \text{ a.e. in } \Omega \tag{3.2}$$

We claim that

$$f(t) = 0 \qquad \text{for } t \in [0, \operatorname{ess\,sup} u_2]. \tag{3.3}$$

and give two proofs for it. The first one is elementary but assumes additional regularity on f whereas the second one achieves full generality at the expense of simplicity. Note that once (3.3) is proven, we obtain a contradiction with Lemma 1.1, since u_1 and u_2 would both solve

$$\begin{cases} -\Delta u - a(x)u = b \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

• **Proof of (3.3) when f′ is Lipschitz.**

By convexity of f we conclude from (3.2) that f is affine between $u_1(x)$ and $u_2(x)$ for almost every $x \in \Omega$. Setting for $\epsilon \in (0,1)$, $v = \epsilon u_1 + (1-\epsilon)u_2$, the above implies that $f''(v(x))$ exists for a.e. $x \in \Omega$ and $f''(v(x)) = 0$ a.e. in Ω.

Using Gagliardo-Nirenberg inequalities, it is clear that $v \in W^{1,1}_{loc}(\Omega)$ since $\Delta v \in L^1_\delta(\Omega)$. So that after multiplying $f''(v(x))$ by ∇v, we may apply the chain rule and obtain $\nabla(f'(v)) = 0$ a.e. in Ω, yielding

$$f'(v) = C \qquad \text{a.e. in } \Omega$$

Repeating this procedure, we obtain

$$f(v) = Cv + D \qquad \text{a.e. in } \Omega$$

By convexity of f, this implies in turn that

$$f(t) = Ct + D \qquad \text{for } t \in [\operatorname{ess\,inf} v, \operatorname{ess\,sup} v].$$

By Lemma 3.2, $\operatorname{ess\,inf} v = 0$. Since $f(0) = f'(0) = 0$, it follows that $f \equiv 0$ between 0 and $\operatorname{ess\,sup} v$. Since $\epsilon \in (0,1)$ is arbitrary, $f(t) = 0$ for $t \in [0, \operatorname{ess\,sup} u_2]$. **Remark.** This proof works only when f' is Lipschitz. Indeed, one can construct a nonconstant, monotone and continuous function g such that $g' = 0$ a.e. (see e.g. [R] p. 144-145) and choose $f' = g$.

• **Proof of (3.3) without assuming that f′ is Lipschitz.**

Recall that we have also $u_2(x) - u_1(x) \geq C\delta(x)$ for a.e. $x \in \Omega$. Hence, by (3.2), we can fix representatives of u_1 and u_2 such that for every $x \in \Omega$ either f is affine in $(u_1(x), u_2(x))$ and $u_2(x) - u_1(x) \geq C\delta(x)$, or $u_1(x) = u_2(x) = 0$. Setting

$$A = \bigcup_{x \in \Omega} (u_1(x), u_2(x))$$

we claim that $A \supseteq (\operatorname{ess\,inf} u_1, \operatorname{ess\,sup} u_2)$. For this we need the following lemma:

Lemma 3.1. *Let $v \in W^{1,1}_{loc}(\Omega)$, where Ω is a connected domain. Then, for any representative of v, $v(\Omega)$ is dense in* $[\text{ess } \mathit{inf}\, v, \text{ess } \mathit{sup}\, v]$

Proof. Recall Stampacchia's theorem (see e.g. [GT]) asserting that if $w \in W^{1,1}_{loc}(\Omega)$ then $\nabla w = 0$ a.e. on any set where w is a constant. In particular, if $w \in W^{1,1}_{loc}(\Omega; \{0,1\})$, w is constant. Suppose by contradiction that there exists a non-void open interval $I \subset [\operatorname{ess\,inf} v, \operatorname{ess\,sup} v]$ such that $v(\Omega) \cap I = \emptyset$.

Consider a function $s : \mathbb{R} \setminus I \to \{0,1\}$ defined by $s(x) = 0$ for $x \leq \inf I$, $s(x) = 1$ elsewhere. Then $s(x)$ is regular and $s \circ v \in W^{1,1}_{loc}(\Omega, \{0,1\})$. We obtain a contradiction since $s \circ v$ is not a.e. constant due to

$$\operatorname{ess\,inf} v \leq \inf I < \sup I \leq \operatorname{ess\,sup} v.$$

□

Now we prove that $A \supseteq (\operatorname{ess\,inf} u_1, \operatorname{ess\,sup} u_2)$. Using Gagliardo-Nirenberg inequalities, we know that $\nabla u_1 \in L^1_{loc}(\Omega)$ (since $\Delta u_1 \in L^1_\delta$). Choose a sequence $\{\Omega_n\}$ of connected subdomains of Ω such that $\Omega_n \subseteq \Omega_{n+1}$, $\overline{\Omega}_n \subset \Omega$ and $\cup \Omega_n = \Omega$. Set

$$A_n = \bigcup_{x \in \Omega_n} (u_1(x), u_2(x)),$$

it suffices to show that

$$A_n \supseteq I_n = (\operatorname{ess\,inf}_{\Omega_n} u_1, \operatorname{ess\,sup}_{\Omega_n} u_2).$$

Let $c_n > 0$ be such that $u_2 - u_1 > c_n$ on Ω_n (note that such a constant exists, since $\operatorname{dist}(\Omega_n, \partial\Omega) > 0$).) It is clear that $A_n \cap I_n \neq \emptyset$.

Choose a connected component (i.e. an interval) A' of A_n such that $A' \cap I_n \neq \emptyset$. We show that $A' \supseteq I_n$. Indeed, if $\inf A' > \inf I_n$ then by Lemma 3.1 there exists $x \in \Omega$ such that $u_1(x) \in (\inf I_n, \inf A')$ and $\inf A' - u_1(x) < c_n$. Then $(u_1(x), u_2(x))$ intersects, but is not contained in A', which contradicts the maximality of the connected component A'. Hence, going back we find that f is affine in $(\operatorname{ess\,inf} u_1, \operatorname{ess\,sup} u_2)$.

Assume temporarily that $\operatorname{ess\,inf} u_1 = 0$. Since $f(0) = f'(0) = 0$, $f \equiv 0$ between 0 and $\operatorname{ess\,sup} u_2$, which completes the proof of (3.3). So it only remains to prove that $\operatorname{ess\,inf} u_1 = 0$. We prove this in the following lemma.

Lemma 3.2. *If $h \in L^1_\delta$, $u \in L^1(\Omega)$, $u \geq 0$ and for all $\phi \in C^2(\bar{\Omega})$, $\phi|_{\partial\Omega} = 0$,*

$$\int_\Omega u(-\Delta\phi) = \int_\Omega h\phi \tag{3.4}$$

$$then \qquad \text{ess}\, inf\, u = 0.$$

Proof. Assume by contradiction that $u \geq \epsilon > 0$ a.e. in Ω. Let ρ_n be a standard mollifier and, extending u and h by 0 in $\mathbb{R}^n \setminus \Omega$, let $u_n = u * \rho_n$ and $h_n = h * \rho_n$. On the one hand, there exists $\alpha > 0$ such that for n large enough

$$u_n \geq \alpha\epsilon \qquad \text{everywhere in } \Omega$$

Indeed, since Ω is smooth, there exists $\alpha > 0$ such that for $x \in \Omega$,

$$u_n(x) \geq \epsilon \int_{\Omega \cap B_{1/n}(x)} \rho_n(x-y)dy \geq \alpha\epsilon \int_{B_{1/n}(x)} \rho_n(x-y)dy = \alpha\epsilon.$$

On the other hand, since $-\Delta u = h$ in $D'(\Omega)$, given $\omega \subset\subset \Omega$, we have for n large enough

$$-\Delta u_n = h_n \qquad \text{everywhere in } \omega$$

Let ϕ solve

$$\begin{cases} -\Delta\phi = 1 & \text{in } \omega \\ \phi = 0 & \text{on } \partial\omega \end{cases} \tag{*}$$

and integrate by parts to obtain

$$\int_\omega h_n\phi - \int_\omega u_n = \int_{\partial\omega} u_n \partial_\nu\phi \leq \alpha\epsilon \int_{\partial\omega} \partial_\nu\phi = -\alpha\epsilon|\omega|$$

Now, $u_n \to u$ in L^1 and $h_n \to h$ in L^1_δ so that

$$\int_\omega h\phi - \int_\omega u \leq -\alpha\epsilon|\omega|$$

Choosing $\omega = \omega_k := \{x \in \Omega : dist(x, \partial\Omega) > 1/k\}$ with $k \to \infty$, and writing ϕ_k the corresponding solution of (*), it is clear that $\phi_k \nearrow \phi$, where ϕ solves

$$\begin{cases} -\Delta\phi = 1 & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases}$$

Passing to the limit and using (3.4) we obtain

$$0 \leq -\alpha\epsilon|\Omega|$$

and we have obtained the desired contradiction.

□

Step 4. Finally we prove uniqueness of the solution of (P_{λ_*}). For the sake of contradiction, let u_1 and u_2 be two distinct solutions of (P_{λ_*}), u_1 being the minimal solution. By the maximum principle $u_2 - u_1 > C\delta(x)$ for some $C > 0$. Set $v = \dfrac{u_1 + u_2}{2}$. Then

$$-\Delta v - \frac{c}{|x|^2} v = \frac{f(u_1) + f(u_2)}{2} + \lambda^* b \geq f\left(\frac{u_1 + u_2}{2}\right) + \lambda^* b = f(v) + \lambda^* b$$

by convexity of f. Hence v is a supersolution of (P_{λ_*}) and by Lemma 3.3, it is a solution.

Consequently, we have equality in the above expression and by convexity of f, we conclude that for almost every $x \in \Omega$, f is linear on $[u_1(x), u_2(x)]$. Arguing as before, we obtain the desired contradiction. Following the proof of Martel [M], in the next lemma we prove nonexistence of strict supersolutions for (P_{λ^*}).

Lemma 3.3. *Suppose that* $b \in L^p$ *for some* $p > N$ *and let* v *be a supersolution of* (P_{λ^*}). *Then* v *is a solution of* (P_{λ^*}).

Proof. Let $\mu \in \mathcal{D}'(\Omega)$ defined by

$$\langle \mu, \phi \rangle = \int_\Omega v\left(-\Delta\phi - \frac{c}{|x|^2}\phi\right) - (f(v) + \lambda b)\phi \qquad \text{for } \phi \in C_c^\infty(\Omega).$$

Since v is a supersolution, μ is a nonnegative Radon measure. Arguing by contradiction, suppose now that v is not a solution. This means that $\mu \not\equiv 0$. Since $a(x)v + f(v) + \lambda b \in L^1_\delta(\Omega)$, we can construct (using Lemma 0.1) the solution $\xi \in L^1(\Omega)$ of

$$\begin{cases} -\Delta\xi = \mu & \text{in } \Omega \\ \xi = 0 & \text{on } \partial\Omega \end{cases}$$

By the weak maximum principle $\xi > \epsilon\delta(x)$ for some $\epsilon > 0$. On the other hand $b \in L^p$ for some $p > N$ implies $\eta = (-\Delta)^{-1}(b) \in C^{1,\alpha}$, hence $\eta < C\delta(x)$ in Ω for some $C < \infty$. Set

$$w = v + \epsilon C^{-1}\eta - \xi$$

Then $w < v$ and

$$-\Delta w = av + f(v) + (\lambda^* + \epsilon C^{-1})b > aw + f(w) + (\lambda^* + \epsilon C^{-1})b,$$

hence w is a supersolution to $(P_{\lambda^* + \epsilon C^{-1}})$ which contradicts the extremality of λ^*.

□

4.5 Application 1 : $a(x) = c/|x|^2$, $g(u) = u^p$

This equation was extensively studied in [D] and we showed there that, in a domain Ω containing the origin, (P_λ) has a weak solution (for small $\lambda > 0$) if and only if $c \leq c_0 := (n-2)^2/4$ and

$$1 < p < p_0 \qquad \text{where} \qquad p_0 = 1 + 2/a \qquad \text{and } a = \frac{n-2-\sqrt{(n-2)^2-4c}}{2} > 0$$

In [D] we showed that for $b \in L^\infty$, $|x|^a \zeta_1 \in L^\infty$. So when $p < p_0$, $\zeta_1^p \sim |x|^{-a-2+\epsilon}$ for some $\epsilon > 0$ and $G(\zeta_1^p) \sim |x|^{-a}$ so that (0.8) is satisfied.
When $p \geq p_0$ however, ζ_1^p is at least of the order of $|x|^{-a-2}$ near 0 so that $G(\zeta_1^p)$ is at least of the order of $|x|^{-a}\ln(1/|x|)$ and (0.8) never holds. We give the details of a similar argument to prove the following new result :

Proposition 4.1. *Fix $0 < c \leq (n-2)^2/4$, $1 < p < p_0$, and $b \in L^\infty$. Let λ_p^* be the corresponding extremal parameter and $\epsilon = a(p_0 - p)$.Also define $\theta = \sqrt{(n-2)^2 - 4c}$. There exists a constant C, depending only on Ω, n, b such that*

$$C^{-1}\{\epsilon(\epsilon+\theta)\}^{\frac{1}{p-1}} \leq \lambda_p^* \leq C\{\epsilon(\epsilon+\theta)\}^{\frac{1}{p-1}}$$

Remark.

- The proposition strengthens the result of [D] by saying that solutions exist only for λ belonging to a shrinking interval $(0, \lambda_p^*]$ that eventually becomes empty when $p = p_0$.
- Though the transition is continuous, the rate of convergence of $(\lambda_p^*)^{p-1}$ to zero, jumps from ϵ when $c < c_0$ to ϵ^2 when $c = c_0$.

Proof.

To simplify notations, we write λ instead of λ_p^* and C for any constant depending only on Ω, n, b and call these constants universal. By Lemma 2.2 we have, for $C = \int_1^\infty s^{-p} ds$ (which is bounded by a universal constant since $p_0 \geq \frac{n+2}{n-2}$),

$$G((\lambda\zeta_1)^p) \leq C\lambda\zeta_1$$

Hence,

$$\lambda^{p-1} \leq C \lim_{x\to 0} \left(\zeta_1 / G(\zeta_1^p)\right)(x) \tag{4.1}$$

Restrict to the case $\Omega = B_1$, the unit ball centered at the origin and $b \equiv 1$. It is then easy to check that

$$\zeta_1 = \zeta_0 := \frac{1}{2n+c}\left(|x|^{-a} - |x|^2\right) \tag{4.2}$$

and

$$\begin{aligned} \zeta_0^p =& \frac{1}{(2n+c)^p}|x|^{-ap}\left(1-|x|^{a+2}\right)^p \geq \frac{1}{(2n+c)^p}\left(|x|^{-ap} - p|x|^{2+a-ap}\right) \\ \geq& \frac{1}{(2n+c)^p}\left(|x|^{-ap} - |x|^{\epsilon}\right) =: k \end{aligned} \tag{4.3}$$

A computation then yields

$$\begin{aligned} (2n+c)^p G(k) = &\left(\frac{1}{\epsilon(\epsilon+\theta)} - \frac{p}{(a+\epsilon)(a+\epsilon+\theta)}\right)|x|^{-a} \\ &- \frac{1}{\epsilon(\epsilon+\theta)}|x|^{-a+\epsilon} + \frac{p}{(a+\epsilon)(a+\epsilon+\theta)}|x|^{2+\epsilon} \end{aligned} \tag{4.4}$$

Combining (4.1), (4.2), (4.3) and (4.4), it follows that

$$\lambda^{p-1} \leq C(2n+c)^{p-1}\left[\frac{1}{\epsilon(\epsilon+\theta)} - \frac{p}{(a+\epsilon)(a+\epsilon+\theta)}\right]^{-1} \leq C\epsilon(\epsilon+\theta)$$

Conversely, applying Lemma 2.2, (P_λ) has a solution as soon as

$$G((2\lambda\zeta_0)^p) \leq \zeta_0$$

Hence,

$$\left(\lambda_p^*\right)^{p-1} \geq 2^{-p} \inf_{B_1} \zeta_0 / G(\zeta_0^p)$$

Now,

$$\zeta_0^p = \zeta_0^{p-1}\zeta_0 \leq C|x|^{-a(p-1)}\zeta_0 \leq Ck$$

Hence, we just need to estimate $\inf \zeta_0/G(k)$. Starting from (4.2) and (4.4), and letting $A = (\epsilon(\epsilon+\theta))^{-1}$ and $r = |x|$, it follows that

$$G(k)/\zeta_0 \leq C\frac{A(1-r^\epsilon)}{1-r^{2+a}} \leq C \cdot A$$

This inequality provides the desired lower bound on $\zeta_0/G(k)$ and hence on λ_p^*. When

$b \in L^\infty$ is arbitrary, we have, using Lemma 1.5 in [D],

$$C(\Omega)\left(\int_\Omega b\zeta_0\right)\zeta_0 \le \zeta_1 \le \|b\|_{L^\infty}\zeta_0$$

so that all of the above estimates still hold (with new constants.) For a general domain Ω, let $r, R > 0$ be such that $B_r \subset \Omega \subset B_R$ and (extending b by 0 in $B_R \setminus \Omega$) observe that

$$\lambda^*(B_R) \le \lambda^*(\Omega) \le \lambda^*(B_r)$$

This follows from the fact that if u solves (P_λ) in Ω for some $\lambda > 0$, then u is a supersolution of (P_λ) in B_r, so that a solution of (P_λ) in B_r may be constructed.

□

4.6 Application 2 : $a(x) = c/\delta(x)^2$, $\Omega = B_1$

Hardy's inequality (0.1) holds for $0 < c \le 1/4$. We show that $\zeta_0 \in L^\infty$, so that, for any perturbation $b \in L^\infty$ and any nonlinearity f satisfying our assumptions (0.5)..(0.7), (P_λ) has solutions for $\lambda > 0$ small.

Proposition 5.1. *Let $\Omega = B_1$, $b \in L^\infty(B_1)$, $0 < c \le 1/4$ and $a(x) = c/\delta(x)^2 = c/(1-|x|)^2$. Then*

$$\zeta_1 = G(b) \in L^\infty(B_1)$$

Proof. Without loss of generality, we restrict to the case $b \equiv 1$ and $c = 1/4$. By elliptic regularity, $\zeta_1 \in C^\infty(B_1)$ and $y(r) := \zeta_1(x)$ (where $r = |x|$) solves

$$y'' + \frac{n-1}{r}y' + \frac{1}{4(1-r)^2}y = -1$$

$r = 1$ is a regular singular point and the indicial equation reads :

$$s(s-1) + 1/4 = 0$$

The only root of this equation is $s = 1/2$ so by a theorem of Frobenius (see e.g. [T]), there exists a fundamental system of solutions to the homogeneous equation of the form

$$y_1 = \sqrt{1-r}A(r) \qquad y_2 = \sqrt{1-r}\ln(1-r)B(r)$$

where A and B are analytic in a neighbourhood of $r = 1$. It follows from the Wronskian method that

$$y = C_1 y_1 + C_2 y_2 + y_1 \int_r^1 \frac{y_2}{W} + y_2 \int_r^1 \frac{y_1}{W}$$

where C_1, C_2 are constants and $W = y_2' y_1 - y_1' y_2$ is the associated Wronskian. From the expression of y_1, y_2, it follows that y is bounded.

□

4.7 Application 3 : $a(x) = c/dist(x,\Sigma)^2$, $g(u) = u^p$

In this section, we let Σ be a smooth manifold of codimension $k \in \{3,..,n\}$ (with the convention that Σ is a point if $k = n$) contained in a compact subset of Ω. The letter d denotes the function

$$d(x) = dist(x,\Sigma).$$

For simplicity, we also let $b \equiv 1$.Finally we define

$$a = (k-2)/2 - \frac{1}{2}\sqrt{(k-2)^2 - 4c} \tag{6.1}$$

and

$$p_0 = 1 + 2/a \tag{6.2}$$

We will show that Hardy's inequality holds for the potential $a(x) = c/d(x)^2$ provided $c > 0$ is chosen small enough and $k \geq 3$. As mentioned in the introduction, we obtain the following critical exponent result :

Proposition 6.1. *If $1 < p < p_0$, condition (0.8) holds If $p > p_0$, condition (0.8) fails*

Remark. The case $p = p_0$ remains open.

Proof. The proof is organized as follows : we start out by constructing a system of coordinates that transforms Σ into a hyperplane and preserves $d(x) = dist(x, \Sigma)$. In that respect, since the case where Σ reduces to a single point was already treated in [D], we may assume that $k < n$.

Next, we divide the proof into several lemmas : we first prove Hardy's inequality and then compute successively Δd, ζ_0 and $G(\zeta_0^p)$. With these estimates, we can then easily prove Proposition 6.1. Since Σ is smooth, for $\beta > 0$ sufficiently small, say $\beta \leq \beta_0$, each $x \in \Omega_\beta$ has a unique projection $\pi(x) \in \Sigma$ such that $d(x) = |x - \pi(x)|$. Let $N_1, .., N_k$ be an orthonormal family of vector fields which are orthogonal to the surface Σ (they are, at least locally, well defined). Then for each $x \in \Omega_\beta$ there exists a unique $\alpha = (\alpha_1(x), .., \alpha_k(x)) \in \mathbb{R}^k$ such that

$$x = \pi(x) + \sum_{i=1}^{k} \alpha_i(x) N_i(\pi(x))$$

and letting $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^k$,

$$d(x) = |\alpha| \tag{6.3}$$

Now fix a point $\sigma_0 \in \Sigma$ and suppose for simplicity that $\sigma_0 = 0$. Let

$$\sigma : \begin{cases} W \to \Sigma \subset \mathbb{R}^n \\ y \mapsto \sigma(y) \end{cases}$$

be a parametrization of Σ near $\sigma_0 = 0$, where W is a neighbourhood of 0 in $\mathbb{R}^{n-k}$. We may choose σ so that $\left\{ \frac{\partial \sigma}{\partial y_1}|_{\sigma=0}, .., \frac{\partial \sigma}{\partial y_{n-k}}|_{\sigma=0}, N_1|_{\sigma=0}, .., N_k|_{\sigma=0} \right\}$ be a family of orthonormal vectors, which up to a rotation of $\mathbb{R}^n$ we may assume to be the canonical basis.

It follows from the above discussion that there exist $\beta_0 > 0$,V a neighbourhood of $\sigma_0 = 0$ in Ω (which may be assumed to be balanced, i.e. $\lambda V \subset V$ for all $|\lambda| < 1$), W a neighbourhood of 0 in $\mathbb{R}^{n-k}$ and a diffeomorphism

$$J : \begin{cases} V \to W \times B^k_{\beta_0} \\ x \mapsto (y, \alpha) \end{cases} \tag{6.4}$$

where $B^k_{\beta_0}$ is the ball of radius β_0 in $\mathbb{R}^k$ centered at the origin and (6.1) holds.

Observe that $J(0) = 0$. We claim that $J'(0) = Id$. Indeed if $H = J^{-1}$,

$$H(y, \alpha) = \sigma(y) + \sum_{i=1}^{k} \alpha_i N_i(\sigma(y))$$

and

$$H'(0) = \left(\frac{\partial \sigma}{\partial y_1}|_{\sigma=0}, .., \frac{\partial \sigma}{\partial y_{n-k}}|_{\sigma=0}, N_1|_{\sigma=0}, .., N_k|_{\sigma=0} \right) = Id$$

Finally define

$$l : \begin{cases} \mathbb{R}^n \to \mathbb{R}^+ \\ (x_1, .., x_n) \mapsto \left(\sum_{i=n-k+1}^{n} x_i^2 \right)^{1/2} \end{cases} \tag{6.5}$$

With these notations (6.3) reads

$$d(x) = |\alpha| = l(J(x)) \tag{6.6}$$

Lemma 6.1. Hardy's inequality. *(0.1) holds for $a(x) = c/d(x)^2$ provided $c > 0$ is chosen small enough.*

Proof. Consider first a function $\phi \in C_c^\infty(V)$ with V as in (6.4) and let $\psi = \phi \circ J$. By the standard Hardy inequality, we have

$$\int_{\mathbb{R}^k} |\nabla_\alpha \psi(y, .)|^2 d\alpha \geq \frac{(k-2)^2}{4} \int_{\mathbb{R}^k} \frac{\psi^2(y, .)}{|\alpha|^2} d\alpha$$

Integrating with respect to y, we obtain

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 \geq \frac{(k-2)^2}{4} \int_{\mathbb{R}^n} \frac{\psi^2}{l^2}$$

Changing coordinates, using (6.2) and the fact that $DJ \sim Id$ in V, we obtain

$$\int_{\mathbb{R}^n} |\nabla \phi|^2 \geq c \int_{\mathbb{R}^n} \frac{\phi^2}{d^2}$$

where c can be chosen arbitrarily close to $(k-2)^2/4$ by shrinking V.

In the general case where $\phi \in C_c^\infty(\Omega)$, one just needs to use a partition of unity adapted to a coverage of Σ by neighbourhoods V where the above computation holds. Outside of this covering, d is bounded below and we therefore have for $c > 0$ sufficiently small

$$\int_\Omega |\nabla \phi|^2 \geq c \int_\Omega \frac{\phi^2}{d^2}$$

Taking $c > 0$ even smaller, we then obtain (0.1). Also observe that the above estimates yield the following inequality, in the spirit of [BM] : for all $\epsilon > 0$ there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla\phi|^2 + \lambda \int_\Omega \phi^2 \geq \left(\frac{(k-2)^2}{4} - \epsilon\right) \int_\Omega \frac{\phi^2}{d^2}$$

The interested reader will find refined versions of the Hardy inequality in [FT] and its references. Some geometric assumptions on Ω and Σ are however required. □

Lemma 6.2. *Let* $\Omega_\beta = \{x \in \Omega : d(x) < \beta\}$. *Then*

$$\Delta d = \frac{k-1}{d}(1+\eta) \qquad \textit{in } \Omega_\beta$$

where $\eta = \eta(x;\beta) \to 0$ *uniformly in* $x \in \Omega_\beta$ *as* $\beta \to 0$.

Proof. Use the notations of (6.4) and scale the coordinates, i.e. for $\epsilon = \beta/\beta_0 > 0$, $x \in V$ let

$$\tilde{x} = \epsilon x.$$

Since $J(0) = 0$ and $J'(0) = Id$,

$$J(\tilde{x}) = \tilde{x} + h(\tilde{x}) \tag{6.7}$$

where

$$h(\tilde{x}) = \int_0^1 (1-t)(J''(t\tilde{x}) \cdot \tilde{x} \,|\, \tilde{x})dt = \epsilon^2 o(1) \qquad \text{uniformly in } \tilde{x} \in \epsilon V. \tag{6.8}$$

Using (6.6) and (6.7), we obtain

$$\frac{\partial d}{\partial \tilde{x}_i} = \frac{\partial l}{\partial z_j}\left(\delta_{ij} + \frac{\partial h_j}{\partial \tilde{x}_i}\right)$$

and

$$\frac{\partial^2 d}{\partial \tilde{x}_i^2} = \frac{\partial^2 l}{\partial z_j^2}\left(\delta_{ij} + \frac{\partial h_j}{\partial \tilde{x}_i}\right)^2 + \frac{\partial^2 h_j}{\partial \tilde{x}_i^2}\frac{\partial l}{\partial z_j}$$

With (6.8), one can show that $\nabla h = \epsilon o(1)$ and that $\nabla^2 h = o(1)$. It's also easy to check from (6.5) that $\nabla l \in L^\infty$ and $l\nabla^2 l \in L^\infty$ so that we finally obtain

$$\Delta d = \Delta l \cdot (1 + \epsilon o(1)) \qquad \text{uniformly in } \epsilon V \tag{6.9}$$

Now by a straightforward computation, we have that

$$\Delta l = \frac{k-1}{l} \tag{6.10}$$

Finally,

$$d(\tilde{x}) = l(J(\tilde{x})) = l(\tilde{x})(1 + g(\tilde{x})) \tag{6.11}$$

where

$$g(\tilde{x}) = \frac{1}{l(\tilde{x})} \int_0^1 \nabla l(\tilde{x} + th(\tilde{x})) \cdot h(\tilde{x}) dt = \epsilon o(1) \qquad \text{uniformly in } \tilde{x} \in \epsilon V,$$

as follows from (6.8) and the fact that $\nabla l \in L^\infty$. Collecting (6.9),(6.10) and (6.11), we obtain the desired result in ϵV, which remains true in a neighbourhood Ω_β of Σ by using a finite covering of Σ for which the above computations hold. □

Lemma 6.3. *For all $\epsilon > 0$, there exists $C > 0$ such that*

$$C^{-1} d^{-a+\epsilon} \delta \le \zeta_0 \le C d^{-a-\epsilon}$$

Proof. First observe that we just need to prove the estimates in a neighbourhood of Σ and apply elliptic regularity elsewhere. Define now

$$P(X) = X(X-1) + (k-1)X + c$$

and observe that $-a$ (defined in (6.1)) is the larger root of P. Next, fix $\epsilon > 0$ and define

$$w = C d^{-a-\epsilon}$$

for some constant C to be chosen later on. A simple computation and Lemma 6.2 yield

$$-\Delta w - \frac{c}{d^2} w = -CP(-a-\epsilon) d^{-a-\epsilon-2}(1+\eta)$$

By choosing $\beta > 0$ small and C large enough, it follows that

$$\begin{cases} -\Delta w - \dfrac{c}{d^2} w \ge 1 & \text{in } \Omega_\beta \\ w \ge \zeta_0 & \text{on } \partial\Omega_\beta \end{cases}$$

and by the maximum principle (apply e.g. Lemma 1.1 to $(w - \zeta_0)^-$) we obtain the desired upper bound. The lower bound is obtained in the exact same manner. □

Lemma 6.4. *For all $\epsilon > 0$ there exists $C > 0$ such that*

$$G(d^{-a-2+\epsilon}) \leq C\zeta_0 \qquad \text{and} \qquad G(d^{-a-2-2\epsilon}) \geq Cd^{-a-\epsilon}\delta$$

. The proof is analogous to that of the previous lemma and we skip it. □**Proof of**

Proposition 6.1 continued. Recall (6.2) and given $p < p_0$, fix $\epsilon > 0$ so small that $p(a + \epsilon) < a + 2 - \epsilon$. By Lemma 6.3,

$$\zeta_0^p \leq Cd^{-a-2+\epsilon}$$

And by Lemma 6.4,

$$G(\zeta_0^p) \leq C\zeta_0$$

Conversely if $p > p_0$, let $\epsilon_0 > 0$ be such that $p(a - \epsilon_0) > a + 2 + 2\epsilon_0$. By Lemma 6.3,

$$\zeta_0^p \geq Cd^{-a-2-2\epsilon_0}\delta^p$$

And by Lemma 6.4,

$$G(\zeta_0^p) \geq Cd^{-a-\epsilon_0}\delta$$

Applying Lemma 6.3 with $\epsilon < \epsilon_0$, we obtain that (0.8) can never hold. □

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Chapter 5

Hardy-type inequalities

5.1 Introduction

The well-known Hardy inequality states that for any given domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ and any $u \in C_c^\infty(\Omega)$,

$$K^2 \int_\Omega \frac{u^2}{|x|^2} \leq \int_\Omega |\nabla u|^2, \tag{5.1}$$

where $K = (n-2)/2$.

In particular, for $c < K^2$, the operator $L_0 := -\Delta - c/|x|^2$ has a positive first eigenvalue (and this fact is still true when $c = K^2$, using a refined version of (5.1) due to Brezis and Vazquez).

In this section, we first consider operators of the form $L = -\Delta - a(x)$, where $a(x) = c/d(x,\Sigma)^2$, d being the distance function and Σ a submanifold of Ω of codimension $k \neq 2$. As announced in [DD], such potentials $a(x)$ provide an example of a limiting case where some maximum principles for the associated parabolic operator $P = \partial_t - \Delta - a(x)$ still hold : roughly speaking, any positive solution of $Pu \geq 0$ can be bounded below by the first eigenfunction ϕ_1 of L, i.e. for some $c(t) > 0$,

$$u \geq c(t)\phi_1. \tag{5.2}$$

For $c \leq H^2$, where $H = (k-2)/2$, we prove indeed that the first eigenvalue of L remains finite. In fact, we provide a refinement of the analogous of inequality (5.1), so that the theory developped in [DD] still applies (see Theorem 1).

5.2 Hardy inequalities

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\Sigma \subset \Omega$ be a compact smooth manifold without boundary of codimension $k \neq 2$. Let $H = \frac{k-2}{2}$. Then there exists $p > 2$ and $C > 0$, $\gamma > 0$ independent of u such that for any $u \in C_0^\infty(\Omega \setminus \Sigma)$*

$$\gamma\Big(\int_\Omega |u|^p\Big)^{2/p} + H^2 \int_\Omega \frac{u^2}{d^2} \leq \int_\Omega |\nabla u|^2 + C\int_\Omega u^2, \tag{5.3}$$

where $d(x) = \mathrm{dist}(x, \Sigma)$.

Lemma 1. *Let $k \neq 2$ and $H = (k-2)/2$. There exists a constant $C > 0$ depending only on k such that*

$$\begin{aligned} &\int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 - H^2\frac{u^2}{r^2}\right] r^{k-1}\,dr + C\int_0^{\frac{1}{2}} u^2 r^{k-1}\,dr \geq \\ &\qquad \int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 + H^2\frac{u^2}{r^2}\right] r^k\,dr + \frac{1}{4}\int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2\,dr \end{aligned} \tag{5.4}$$

for all $u \in C_c^\infty(0, \frac{1}{2})$, and where $v(r) = r^H u(r)$.

Proof. Let $u \in C_c^\infty(0, \frac{1}{2})$ and $v(r) = r^H u(r)$. A standard computation yields

$$\left[\left(\frac{du}{dr}\right)^2 - H^2\frac{u^2}{r^2}\right] r^{k-1} = r\left(\frac{dv}{dr}\right)^2 - H\frac{d\,(v^2)}{dr}. \tag{5.5}$$

Integrating, it follows that

$$A := \int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 - H^2\frac{u^2}{r^2}\right] r^{k-1}\,dr = \int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2\,dr. \tag{5.6}$$

Similarly, using (5.5) and an integration by parts,

$$\begin{aligned} B &:= \int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 + H^2\frac{u^2}{r^2}\right] r^k\,dr \\ &= \int_0^{\frac{1}{2}} r^2\left(\frac{dv}{dr}\right)^2\,dr + 2H^2\int_0^{\frac{1}{2}} \frac{u^2}{r^2} r^k\,dr - H\int_0^{\frac{1}{2}} r\left[\frac{d\,(v^2)}{dr}\right]\,dr \\ &= \int_0^{\frac{1}{2}} r^2\left(\frac{dv}{dr}\right)^2\,dr + \left(2H^2 + H\right)\int_0^{\frac{1}{2}} v^2\,dr. \end{aligned} \tag{5.7}$$

Using integration by parts again, it follows that for given $\epsilon > 0$, there exists $C > 0$ such that

$$\begin{aligned} \int_0^{\frac{1}{2}} v^2\,dr = -2\int_0^{\frac{1}{2}} rv\frac{dv}{dr}\,dr &\leq C\int_0^{\frac{1}{2}} v^2 r\,dr + \epsilon\int_0^{\frac{1}{2}} \left(\frac{dv}{dr}\right)^2 r\,dr \\ &= C\int_0^{\frac{1}{2}} u^2 r^{k-1}\,dr + \epsilon\int_0^{\frac{1}{2}} \left(\frac{dv}{dr}\right)^2 r\,dr. \end{aligned} \tag{5.8}$$

Collecting (5.6),(5.7) and (5.8), we obtain for ϵ small enough

$$A-B \geq \int_0^{\frac{1}{2}} r(1-r-C\epsilon)\left(\frac{dv}{dr}\right)^2 dr - C\int_0^{\frac{1}{2}} u^2 r^{k-1}\, dr \geq \frac{1}{4}\int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2 dr - C\int_0^{\frac{1}{2}} u^2 r^{k-1}\, dr.$$

■

Lemma 2. *Let $k \neq 2$, $H = (k-2)/2$ and $c > \bar{c} > 0$. There exists constants $C, \tau > 0$ depending only on k and $\bar{c}$ such that*

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 - (H^2 - c)\frac{u^2}{r^2}\right] r^{k-1}\, dr + C\int_0^{\frac{1}{2}} u^2 r^{k-1}\, dr \geq$$

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 + (H^2 + c)\frac{u^2}{r^2}\right] r^{k}\, dr + \tau\int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 + c\frac{u^2}{r^2}\right] r^{k-1}\, dr \tag{5.9}$$

for all $u \in C_c^\infty(0, \frac{1}{2})$.

Proof. It follows from (5.4) that if

$$D := \int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 - (H^2 - c)\frac{u^2}{r^2}\right] r^{k-1}\, dr + C\int_0^{\frac{1}{2}} u^2 r^{k-1}\, dr \tag{5.10}$$

and

$$E := \int_0^{\frac{1}{2}} \left[\left(\frac{du}{dr}\right)^2 + (H^2 + c)\frac{u^2}{r^2}\right] r^{k}\, dr \tag{5.11}$$

then

$$\begin{aligned} D - E &\geq c\int_0^{\frac{1}{2}} \frac{u^2}{r^2}(1-r) r^{k-1}\, dr + \frac{1}{4}\int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2 dr \\ &\geq \frac{c}{2}\int_0^{\frac{1}{2}} \frac{u^2}{r^2} r^{k-1}\, dr + \frac{1}{4}\int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2 dr. \end{aligned} \tag{5.12}$$

We can also rewrite (5.5) as

$$r^{k-1}\left(\frac{du}{dr}\right)^2 = H^2\frac{u^2}{r^2} r^{k-1} + r\left(\frac{dv}{dr}\right)^2 - H\frac{d\left(v^2\right)}{dr}$$

so that if $\tau = \min(\frac{\bar{c}}{4H^2}, \frac{1}{4})$,

$$\tau\int_0^{\frac{1}{2}} r^{k-1}\left(\frac{du}{dr}\right)^2 dr \leq \frac{\bar{c}}{4}\int_0^{\frac{1}{2}} \frac{u^2}{r^2} r^{k-1}\, dr + \frac{1}{4}\int_0^{\frac{1}{2}} r\left(\frac{dv}{dr}\right)^2 dr. \tag{5.13}$$

It then follows from (5.12) and (5.13) that

$$D - E \geq \tau\int_0^{\frac{1}{2}} r^{k-1}\left(\frac{du}{dr}\right)^2 dr + \frac{c}{4}\int_0^{\frac{1}{2}} \frac{u^2}{r^2} r^{k-1}\, dr. \tag{5.14}$$

Hence (5.9) holds. ■

Lemma 3. *Let $k \neq 2$, $H = (k-2)/2$ and $\beta > 0$. Let B^k_β denote the ball of $\mathbb{R}^k$ centered at the origin and of radius β. There exist positive constants $C = C(\beta, k), \tau = \tau(k)$ and $\alpha = \alpha(\beta, k)$ such that*

$$\int_{B^k_\beta} \left(|\nabla u|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C \int_{B^k_\beta} u^2 dy \geq$$

$$\frac{1}{2\beta} \int_{B^k_\beta} |y| \left(|\nabla u|^2 + H^2 \frac{u^2}{|y|^2} \right) dy + \tau \int_{B^k_\beta} |\nabla (u - u_0)|^2 \, dy + \alpha \int_0^\beta r \left(\frac{dv_0}{dr} \right)^2 dr \quad (5.15)$$

for all $u \in C_c^\infty(B^k_\beta \setminus \{0\})$ and where $u_0(r) = u_0(|y|) = \fint_{\partial B^k_r} u d\sigma$ and $v_0(r) = r^H u_0(r)$.

Proof. Let $\{f_i\}_{i=0}^\infty$ be an orthonormal basis of $L^2(S^{k-1})$, composed of eigenvectors of the Laplace-Beltrami operator $\Delta|_{S^{k-1}}$. The corresponding eigenvalues are given by $c_{n_i} = n_i(k + n_i - 2)$, where $n_i = 0, 1, .., 1, 2, .., 2, 3, ..$ ranges over the integers, according to multiplicity of each eigenvalue (see e.g. ??).

Any $u \in C_c^\infty(B^k_{\frac{1}{2}} \setminus \{0\})$ can then be written as

$$u(x) = \sum_{i=0}^\infty u_i(r) f_i(\theta)$$

where $\frac{1}{2} > r > 0$, $\theta \in S^{k-1}$ and $x = r\theta$.

Furthermore, for $g \in C(\mathbb{R}^+, \mathbb{R})$,

$$\begin{aligned} \int_{B^k_{\frac{1}{2}}} |\nabla u|^2 g(|y|) dy &= \int_0^{\frac{1}{2}} r^{k-1} g(r) \, dr \int_{S^{k-1}} \left[\left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_\theta u|^2 \right] d\theta \\ &= \sum_{i=0}^\infty \int_0^{\frac{1}{2}} r^{k-1} g(r) \left[\left(\frac{du_i}{dr} \right)^2 + \frac{c_{n_i}}{r^2} u_i^2 \right] dr. \end{aligned} \quad (5.16)$$

For $i = 0$, it follows from (5.4) that if $v_0(r) = r^H u_0(r)$,

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du_0}{dr} \right)^2 - H^2 \frac{u_0^2}{r^2} \right] r^{k-1} \, dr + C \int_0^{\frac{1}{2}} u_0^2 r^{k-1} \, dr \geq$$

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du_0}{dr} \right)^2 + H^2 \frac{u_0^2}{r^2} \right] r^k \, dr + \frac{1}{4} \int_0^{\frac{1}{2}} r \left(\frac{dv_0}{dr} \right)^2 dr, \quad (5.17)$$

while (5.9) implies that for $i \geq 1$,

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du_i}{dr} \right)^2 - (H^2 - c_{n_i}) \frac{u_i^2}{r^2} \right] r^{k-1} \, dr + C \int_0^{\frac{1}{2}} u_i^2 r^{k-1} \, dr \geq$$

$$\int_0^{\frac{1}{2}} \left[\left(\frac{du_i}{dr} \right)^2 + (H^2 + c_{n_i}) \frac{u_i^2}{r^2} \right] r^k \, dr + \tau \int_0^{\frac{1}{2}} \left[\left(\frac{du_i}{dr} \right)^2 + c_{n_i} \frac{u_i^2}{r^2} \right] r^{k-1} \, dr. \quad (5.18)$$

Using (5.17),(5.18) and (5.16) with $g(r) \equiv 1$ for terms involving r^{k-1} and $g(r) = r$ for terms in r^k, (5.15) follows for $\beta = \frac{1}{2}$. The general case is obtained by scaling. ■

For the proof of Theorem 1 we will introduce some notation. Define

$$\Omega_\beta = \{\, x \mid \mathrm{dist}(x, \Sigma) < \beta \,\}.$$

We will work only with β small enough so that the projection $\pi : \Omega_\beta \to \Sigma$ given by $|\pi(x) - x| = \mathrm{dist}(x, \Sigma)$ is well defined and smooth.

Let $\{V_i\}_{i=1,\dots,m}$ be a family of open disjoint subsets of Σ such that

$$\Sigma = \bigcup_{i=1}^{m} \overline{V}_i, \qquad \text{and } |\overline{V}_i \cap \overline{V}_j| = 0 \quad \forall i \neq j.$$

We can also assume that:

a) $\forall i = 1, \dots, m$ there exists a smooth diffeomorphism

$$p_i : B_1^{n-k} \to U_i$$

where $U_i \subset \Sigma$ is open and $\overline{V}_i \subset U_i$;

b) $p_i^{-1}(V_i)$, which is an open set in $\mathbb{R}^{n-k}$, has a Lipschitz boundary; and

c) there is a smooth choice of unit vectors $N_1^i(\sigma), \dots, N_k^i(\sigma)$ $\forall \sigma \in U_i$ which form an orthonormal frame for Σ on $U_i \subset \mathbb{R}^n$, i.e. $\forall \sigma \in U_i$

$$N_j^i(\sigma) \in \mathbb{R}^n, \quad N_j^i(\sigma) \cdot N_k^i(\sigma) = \delta_{jk}, \qquad \text{and } N_j^i(\sigma) \cdot v = 0 \quad \forall v \in T_\sigma \Sigma.$$

Let $W_i = p_i^{-1}(V_i)$. For $z \in W_i$ we will also write (abusing the notation) $N_j^i(z) = N_j^i(p_i(z))$. Let

$$F_i(y, z) = p_i(z) + \sum_{j=1}^{k} y_j N_j^i(z),$$

where $y = (y_1, .., y_k) \in B_\beta^k$ and $z \in W_i$, so that F_i is a smooth diffeomorphism between $B_\beta^k \times W_i$ and T_β^i, where

$$T_\beta^i = \pi^{-1}(V_i) \cap \Omega_\beta. \tag{5.19}$$

It follows from the condition $|\overline{V}_i \cap \overline{V}_j| = 0$ $\forall i \neq j$ that $|\overline{T}_\beta^i \cap \overline{T}_\beta^j| = 0$ $\forall i \neq j$, and hence, for any $f \in L^1(\Omega_\beta)$ we have:

$$\begin{aligned} \int_{\Omega_\beta} f &= \sum_{i=1}^{m} \int_{T_\beta^i} f \\ &= \sum_{i=1}^{m} \int_{W_i \times B_\beta^k} f \circ F_i(y, z)\, JF_i(y, z)\, dy\, dz, \end{aligned} \tag{5.20}$$

where $JF_i(y,z)$ stands for the Jacobian of F_i at (y,z). We claim that

$$JF_i(y,z) = H_i(z)(1 + O(|y|)), \tag{5.21}$$

where $O(|y|)$ denotes a quantity bounded by $|y|$ (uniformly for $z \in W_i$) and $H_i(z)$ a smooth function, which is bounded away from zero. More precisely

$$H_i(z) = Jp_i(z) = \sqrt{\left(Dp_i(z)\right)^* Dp_i(z)}.$$

To prove (5.21) is suffices to observe that $JF_i(y,z)$ is smooth and to compute it at $y = 0$:

$$\begin{aligned} JF_i(0,z)^2 &= \det\left(DF_i(0,z)^* DF_i(0,z)\right) \\ &= \det\left(\left[D_z p_i | N_1^i, \ldots, N_k^i\right]^* \left[D_z p_i | N_1^i, \ldots, N_k^i\right]\right) \\ &= \det \begin{bmatrix} (D_z p_i)^* D_z p_i & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

Proof of Theorem 1. First, observe that it is sufficient to prove the theorem for u with support near Σ. Indeed, (the following trick is taken from Vázquez and Zuazua): let $\eta \in C_0^\infty(\mathbb{R}^n)$ so that $\eta \equiv 1$ in $\Omega_{\beta/2}$ and $\text{supp}(\eta) \subset \Omega_\beta$. Let $u \in C_0^\infty(\Omega \setminus \Sigma)$ and write $u = u_1 + u_2$ where $u_1 = \eta u$, $u_2 = (1-\eta)u$. Assume then that the conclusion of the theorem holds for u_1. Then

$$\begin{aligned} \int_\Omega |\nabla u|^2 - H^2 \frac{u^2}{d^2} &= \int_\Omega |\nabla u_1|^2 - H^2 \frac{u_1^2}{d^2} + \int_\Omega |\nabla u_2|^2 - H^2 \frac{u_2^2}{d^2} \\ &\quad + 2\int_\Omega \nabla u_1 \cdot \nabla u_2 - H^2 \frac{u_1 u_2}{d^2}. \end{aligned} \tag{5.22}$$

Since $\frac{1}{d}$ is bounded away from Σ we have

$$\int_\Omega \frac{u_2^2}{d^2} + \frac{u_1 u_2}{d^2} \leq C \int_\Omega u^2.$$

Also note that

$$\begin{aligned} \int_\Omega \nabla u_1 \cdot \nabla u_2 &= \int_\Omega \eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2 + u \nabla u \cdot \nabla \eta (1 - 2\eta) \\ &= \int_\Omega \eta(1-\eta)|\nabla u|^2 - |\nabla \eta|^2 u^2 - \frac{1}{2}\int_{\Omega_\beta \setminus \Omega_{\beta/2}} u^2 \nabla \cdot \left(\nabla \eta (1 - 2\eta)\right) \\ &\geq -C \int_\Omega u^2. \end{aligned} \tag{5.23}$$

It follows from (5.22), (5.23) that

$$\int_\Omega |\nabla u|^2 - H^2\frac{u^2}{d^2} \geq \int_\Omega |\nabla u_1|^2 - H^2\frac{u_1^2}{d^2} + \int_\Omega |\nabla u_2|^2 - C\int_\Omega u^2.$$

Using (5.3) with u_1 we thus conclude that

$$\int_\Omega |\nabla u|^2 - H^2\frac{u^2}{d^2} + C\int_\Omega u^2 \geq \gamma\Big(\int_\Omega |u_1|^p\Big)^{2/p} + \int_\Omega |\nabla u_2|^2,$$

for some $\gamma > 0$ independent of u. From here the conclusion of the theorem for u follows easily.

Let I_i denote the quantity

$$I_i = \int_{T^i_\beta} |\nabla u|^2 - H^2\frac{u^2}{d^2} + u^2,$$

where T^i_β was defined in (5.19). In what follows we will fix i and show that there is $p > 2$ and $C > 0$ independent of u such that

$$\Big(\int_{T^i_\beta} |u|^p\Big)^{2/p} \leq CI_i.$$

For simplicity, and since i is fixed, we will drop the index i from all the notation that follows.

Let us introduce some additional notation:

$$\begin{aligned}
\tilde{u}(y,z) &= u(F(y,z)) \\
\tilde{u}_0(r,z) &= \fint_{\partial B_r} \tilde{u}(y,z)\, ds(y) \\
v_0(r,z) &= r^H \tilde{u}_0(r,z).
\end{aligned}$$

Let us write

$$\nabla u = \nabla_N u + \nabla_T u$$

where $\nabla_N u$ is the gradient of u is the normal direction and $\nabla_T u$ is orthogonal to $\nabla_N u$. More precisely, for a point $x = F(y,z)$

$$\nabla_N u(x) = \sum_{j=1}^{k} \nabla u(x)\cdot N_j(z)\, N_j(z).$$

Step 1. There exists $C > 0$ independent of u such that

$$\begin{aligned} CI \geq & \int_{W\times B_\beta^k} |\nabla_y \tilde{u}|^2 |y|\, dy\, dz + \int_{W\times B_\beta^k} \left|\nabla_y\big(\tilde{u}(y,z) - \tilde{u}_0(y,z)\big)\right|^2\, dy\, dz \\ & + \int_W \int_0^\beta \Big(\frac{\partial v_0}{\partial r}\Big)^2 r\, dr\, dz + \int_{W\times B_\beta^k} |(\nabla_T u)\circ F|^2\, dy\, dz. \end{aligned} \tag{5.24}$$

First note that by (5.20), there is a constant $C > 0$ such that

$$\begin{aligned} I \geq & \int_{W\times B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 - H^2 \frac{\tilde{u}^2}{|y|^2} \right) H(z)\, dy\, dz \\ & - C\int_{W\times B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) H(z)|y|\, dy\, dz \\ & + \int_{W\times B_\beta^k} \Big(|\nabla_T u(F(y,z))|^2 + \tilde{u}^2 \Big)(1 - C|y|) H(z)\, dy\, dz. \end{aligned} \tag{5.25}$$

For fixed z we can apply Lemma 3 to the function $\tilde{u}(\cdot, z)$. Observe that

$$\frac{\partial \tilde{u}(y,z)}{\partial y_j} = \nabla u(F(y,z)) \cdot N_j(z)$$

and thus

$$|\nabla_y \tilde{u}(y,z)|^2 = |\nabla_N u(F(y,z))|.$$

Lemma 3 then yields

$$\begin{aligned} & \int_{B_\beta^k} \left(|\nabla_N u(F(y,z))|^2 - H^2 \frac{u^2}{|y|^2} \right) dy + C\int_{B_\beta^k} \tilde{u}^2\, dy \geq \\ \frac{1}{2\beta} & \int_{B_\beta^k} |y| \left(|\nabla_N u(F(y,z))|^2 + H^2 \frac{\tilde{u}^2}{|y|^2} \right) dy + \tau \int_{B_\beta^k} |\nabla_y(\tilde{u} - \tilde{u}_0)|^2\, dy + \alpha \int_0^\beta r \left(\frac{dv_0}{dr} \right)^2 dr. \end{aligned} \tag{5.26}$$

We choose (and fix once and for all) $\beta > 0$ small enough so that $1/(2\beta) \geq C + 1$. Then multiplying (5.26) by $H(z)$, integrating over W and combining the result with (5.25) we conclude that (5.24) holds.

Step 2.

$$\|\nabla v_0\|^2_{L^2(W\times B_\beta^2)} \leq CI. \tag{5.27}$$

By (5.24) the partial derivative $\frac{\partial v_0}{\partial r}$ is bounded in $L^2(W \times B_\beta^2)$ by CI. We just have to control the derivatives $\frac{\partial v_0}{\partial z_i}$, $i = 1, \ldots, n-k$. But

$$\frac{\partial v_0}{\partial z_i}(r,z) = r^H \fint_{\partial B_r} \frac{\partial \tilde{u}}{\partial z_i}(y,z)\, ds(y)$$

and

$$\frac{\partial \tilde{u}}{\partial z_i}(y,z) = \nabla u(F(y,z)) \cdot \left[\frac{\partial p}{\partial z_i} + \sum_{j=1}^{k} y_j \frac{\partial N_j}{\partial z_i} \right].$$

But note that $\frac{\partial p}{\partial z_i}$ is a tangent vector, hence

$$|\nabla_z \tilde{u}(y,z)| \leq |\nabla_T u(F(y,z))| + |y||\nabla_N u(F(y,z))|.$$

Integrating over $W \times B_\beta^k$ we have

$$\int_{W \times B_\beta^k} |\nabla_z \tilde{u}(y,z)|^2 \, dy \, dz \leq CI, \tag{5.28}$$

for some C independent of u by (5.24). It follows that

$$\begin{aligned} \int_{W \times B_\beta^k} |\nabla_z v_0|^2 \, dy \, dz &= \int_W \int_0^\beta r^{2H+1} \left| \fint_{\partial B_r} \nabla_z \tilde{u}(y,z) \, ds(y) \right|^2 dr \, dz \\ &\leq \int_W \int_0^\beta r^{k-1} \fint_{\partial B_r} |\nabla_z \tilde{u}(y,z)|^2 \, ds(y) \, dr \, dz \\ &\leq C \int_{W \times B_\beta^k} |\nabla_z \tilde{u}(y,z)|^2 \, dy \, dz \\ &\leq CI \end{aligned}$$

by (5.28).

Step 3. There is $p > 2$ such that

$$\|\tilde{u}_0\|^2_{L^p(W \times B_\beta^k)} \leq CI. \tag{5.29}$$

More precisely, for $k \geq 3$ one can take any $2 < p < p_k$ where p_k is given by

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)},$$

and for $k = 1$ one can take $2 < p \leq p_1$ where p_1 is given by

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{n+1}.$$

Using Sobolev's inequality (on $W \times B_\beta^2$) combined with (5.27) we obtain

$$\int_W \int_0^\beta |v_0|^q r \, dr \, dz \leq CI^{q/2},$$

with q given by $\frac{1}{q} = \frac{1}{2} - \frac{1}{n-k+2}$. That is, in terms of $\tilde{u}_0$ we have

$$\int_W \int_0^\beta |\tilde{u}_0|^q r^{qH+1} \, dr \, dz \leq CI^{q/2}. \tag{5.30}$$

We want an estimate for $\int |\tilde{u}_0|^p r^{k-1}\, dr\, dz$ for some suitable $2 < p < q$ and for this we use Hölder's inequality, distinguishing two cases:

Case : $k \geq 3$. We have

$$\begin{aligned}\int_W \int_0^\beta |\tilde{u}_0|^p r^{k-1}\, dr\, dz &= \int_W \int_0^\beta |\tilde{u}_0|^p r^\alpha r^{k-2-\alpha} r\, dr\, dz \\ &\leq C \left(\int_W \int_0^\beta |\tilde{u}_0|^q\, r^{\alpha q/p+1}\, dr, dz \right)^{p/q} \left(\int_0^\beta r^{\frac{k-2-\alpha}{1-p/q}+1}\, dr \right)^{1-p/q}.\end{aligned} \tag{5.31}$$

We then choose α so that

$$\frac{\alpha}{p} = H = \frac{k-2}{2}.$$

In order to have the second factor on the right hand side of (5.31) finite we need to impose

$$\frac{k-2-\alpha}{1-p/q} > -2$$

which is equivalent to the condition

$$\alpha < \frac{k}{1 + \frac{4}{q(k-2)}}.$$

Thus we need $p = \alpha/H < p_k$ where p_k is given by

$$p_k = \frac{2k}{(k-2)\left(1 + \frac{4}{q(k-2)}\right)}$$

i.e.

$$\frac{1}{p_k} = \frac{1}{2} - \frac{2}{k(n-k+2)}.$$

Observe that $p_k > 2$. Combining then (5.30) and (5.31) finishes this case.

Case: $k = 1$. In this case q is given by $\frac{1}{q} = \frac{1}{2} - \frac{1}{n+1}$, and we can choose $p = q$:

$$\begin{aligned}\int_W \int_0^\beta |\tilde{u}_0|^q r^{k-1}\, dr\, dz &= \int_W \int_0^\beta |\tilde{u}_0|^q\, dr\, dz \\ &\leq \int_W \int_0^\beta |\tilde{u}_0|^q r^{-q/2+1}\, dr\, dz \\ &= \int_W \int_0^\beta |\tilde{u}_0|^q r^{Hq+1}\, dr\, dz\end{aligned}$$

because $-q/2 + 1 < 0$.

Step 4.

$$\|\tilde{u} - \tilde{u}_0\|^2_{L^{2^*}(W \times B^k_\beta)} \le CI. \tag{5.32}$$

This is a consequence of Sobolev's inequality applied to the function $\tilde{u} - \tilde{u}_0$ on the domain $W \times B^k_\beta$. (5.24) already provides a bound in $L^2(W \times B^k_\beta)$ for $\nabla_y(\tilde{u} - \tilde{u}_0)$. Hence we only need to obtain a bound for the derivative with respect to z. For the function $\tilde{u}$ we have it already in (5.28). For $\tilde{u}_0$ it is derived by a computation very similar to that at then end of Step 2.

Conclusion. By (5.29) and (5.32) we see that

$$\|\tilde{u}\|^2_{L^p(W \times B^k_\beta)} \le CI$$

for some C independent of u. Changing variables and reintroducing the index i we have

$$\|u\|^2_{L^p(T^i_\beta)} \le C \int_{T^i_\beta} |\nabla u|^2 - H^2 \frac{u^2}{d^2} + u^2.$$

Adding these inequalities over i proves the statement of the theorem. ■

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