Sur des systèmes dynamiques dissipatifs de type gradient. Applications en Optimisation.

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Introduction

L'objet de cette thèse consiste en l'étude de divers systèmes dynamiques dissipatifs d'ordre un et deux en temps régis par des gradients de fonctions ou par des sous-différentiels de fonctions convexes. Etant donnée une fonction $f : H \mapsto \mathbb{R} \cup \{+\infty\}$ avec H espace de Hilbert et $C \subset H$ un convexe fermé, une des motivations majeures réside dans la résolution de problèmes d'optimisation du type :

$$(P) \qquad \inf_C f$$

ou encore trouver $\hat{x}$ tel que :

$$(P') \qquad \nabla f(\hat{x}) + N_C(\hat{x}) \ni 0,$$

où N_C désigne le cône normal à C en $\hat{x}$.

La première partie de la thèse est consacrée à l'étude des systèmes de types gradients de la forme :

$$(1) \qquad \dot{x}(t) + A_{x(t)} \nabla f(x(t)) = 0,\ x(0) \in \operatorname{ri} C,\ t \geq 0$$

où $A : C \times H \mapsto H$ est destiné à contenir les trajectoires de (1) dans l'intérieur relatif de C, tout en préservant l'information géométrique fournie par ∇f. Outre leur intérêt en optimisation, nous verrons que certaines généralisations de (1) en termes d'inclusions différentielles sont étroitement liées à certaines équations d'évolution en thermodynamique, offrant ainsi de nouvelles perspectives d'enrichissement mutuel.

Dans une seconde partie nous étudions une famille de systèmes inertiels - d'ordre deux en temps - de la forme :

$$(2) \qquad \gamma \ddot{x}(t) + \alpha \dot{x}(t) + \beta \nabla^2 f(x(t)) \dot{x}(t) + \nabla f(x(t)) = 0,\ x(0) \in H,\ t \geq 0$$

où α, β, γ sont des paramètres positifs. Contrairement aux systèmes de types gradients du premier ordre, (2) n'est pas une méthode de descente, ce qui lui confère un intérêt certain quant à l'exploration globale des minima locaux de f. A l'instar de (1), le système dynamique (2) fournit un modèle convaincant de certains phenomènes en mécanique unilatérale (chocs inélastiques).

PARTIE I. Les méthodes de type gradients.

En guise d'idée directrice, nous dressons tout d'abord un parallèlle entre la méthode proximale quadratique et les équations d'évolutions régies par un opérateur maximal monotone. Etant donné $A : H \rightrightarrows H$ un opérateur maximal monotone considérons l'équation d'évolution

$$(SD)_g \qquad \dot{u}(t) + Au(t) \ni 0, \;\text{ pp } t \geq 0, \; u(0) \in \text{dom}\, A.$$

Si l'on prend pour A le sous-différentiel de f supposée convexe, on obtient selon la terminologie de l'optimisation la *méthode continue de la plus grande pente* :

$$(SD) \qquad \dot{u}(t) + \partial f(u(t)) \ni 0, \;\text{ pp } t \geq 0, \; u(0) \in \text{dom}\, \partial f.$$

L'existence et l'unicité d'une solution de l'inclusion différentielle $(SD)_g$ ainsi que de nombreux autres points ont fait l'objet de quantité de travaux [37, 32, 67, 24, 40]. C'est dans [40] que l'analyse asymptotique de $(SD)_g$ fut effectuée, le résultat essentiel étant la convergence faible de $u(t)$, $t \to +\infty$ vers un équilibre $\hat{u}$ de A, c'est-à-dire tel que $A\hat{u} \ni 0$. Contrairement à certaines idées recues, les techniques développées sont loin d'être étrangères à celles utilisées en optimisation, et de ce point de vue la dynamique de (SD) se rapproche davantage de la *méthode proximale quadratique* introduite en 1970 par Martinet [101] :

$$(Prox) \qquad x^{k+1} \in \text{argmin}\{f(u) + \frac{1}{2\mu_k}|u - x^k|^2 \,, u \in H\}, \; \mu_k > 0$$

que de la *méthode de la plus grande pente discrète*

$$x^{k+1} \in x^k - \mu_k \partial f(x^k), \; \mu_k > 0.$$

Le lien étroit entre $(Prox)$ et (SD) se constate aussi bien en temps fini - voir par exemple [32] pour des aspects théoriques, que dans l'analyse asymptotique [89, 40]. Certains auteurs proposent même d'étudier conjointement les systèmes du type (SD) et leurs versions proximales.

Suite aux travaux fondamentaux de Bregman [36] et de Czisàr [50] concernant ce que l'on appelle communément les quasi-distances, le principe variationnel qui est au coeur de la méthode proximale quadratique, a ouvert la voie à de nombreux algorithmes en mimimisation convexe. Etant donnée $d : C \times C \mapsto \mathbb{R}_+$ convexe, et dans la plupart des cas intéressants de Legendre par rapport à sa première variable, cela conduit à l'étude de :

$$x^{k+1} \in \text{argmin}\{f(u) + \frac{1}{2\mu_k} d(u, x^k) \,, u \in H\}, \; \mu_k > 0,$$

avec dans les meilleurs des cas $x^k \in \text{ri}\, C, \; k \in N$.

Les nombreuses similitudes entre $(Prox)$ et (SD) sont une source majeure d'inspiration de ce qui suit. Etant donné un algorithme de minimisation, l'idée essentielle consiste à proposer, au travers d'une équation différentielle et par le jeu des discrétisations (implicites ou explicites), une ou des versions continues de cet algorithme. Ce faisant, nous verrons que certaines de ces dynamiques sont étroitement liées à certains phénomènes issus de divers domaines comme ceux de la dynamique des populations, ou de la thermodynamique.

Chapitre 1 : Opérateurs-barrières et méthodes de descentes associées.

Ici l'on suppose $H = \mathbb{R}^n$, C d'intérieur non vide et f continuement différentiable. En vue de l'usage de dynamiques de type (1) en optimisation, nous introduisons pour les applications de type $A : C \times \mathbb{R}^n \mapsto \mathbb{R}^n$ le concept d'opérateurs barrières elliptiques pour lequel nous montrons que (1) est une méthode *intérieure* de descente.

A l'aide de noyaux généraux de type $d : \mathbb{R}^n \times C \mapsto \mathbb{R}_+ \cup \{+\infty\}$, nous proposons une façon systématique de construire des opérateurs-barrières. A cet effet, et moyennant des hypothèses minimales sur d, nous introduisons la classe suivante :

$$\begin{aligned} A_x^d v &= x - \operatorname{argmin}\{\langle u, v\rangle + d(u,x) \,|u \in \mathbb{R}^n\} \\ &= x - \partial d^*(\cdot, x)(-v), \end{aligned}$$

où pour x dans C, $d^*(\cdot, x)$ est la conjuguée de Legendre-Fenchel de $d(\cdot, x)$.

Cette approche permet d'unifier l'étude de nombreuses méthodes continues de gradients : plus grande pente, gradient-projeté continu, méthodes riemanniennes, méthode de Newton continue... Soulignons aussi que, dans la perspective d'une étude conjointe des méthodes discrètes et continues, le système (1) peut se reformuler sous la forme :

$$\partial d(\dot{x}(t) + x(t), x(t)) + \nabla f(x(t)) \ni 0,\ t \geq 0.$$

Donnons quelques exemples explicites :

- Avec $d(u,x) = \frac{1}{2}|u-x|^2 + \delta_C(u)$, on obtient la méthode continue de gradient-projeté [10]
$$(CGP) \qquad \dot{x}(t) + x(t) - P_C[x(t) - \nabla f(x(t))] = 0,\ t \geq 0.$$
- Pour $C = \mathbb{R}^n_+$, $d_\varphi(u,x) = \sum_{i=1}^n x_i^2 \varphi(x_i^{-1} u_i)$ (voir [22]) où $\varphi(s) = \frac{1}{2}(s-1)^2 + (-\log s + s - 1),\ s > 0$, il vient :
$$\dot{x}_i(t) + x_i(t) + \frac{1}{2}\frac{\partial f}{\partial x_i}(x(t)) - \sqrt{\frac{1}{4}[\frac{\partial f}{\partial x_i}(x(t))]^2 + 4x_i(t)^2} = 0.$$
- Sans contraintes, pour f fortement convexe et en prenant $d(u,x) = \frac{1}{2}\langle \nabla^2 f(x)(u-x), u-x\rangle$, on obtient la méthode de Newton [8] :
$$\dot{x}(t) + \nabla^2 f(x(t))^{-1} \nabla f(x(t)) = 0.$$

Plus généralement, et grâce à des quasi-distances de types quadratiques l'approche développée permet aussi de recouvrir les méthodes riemanniennes. Nous y reviendrons plus en détail dans les chapitres suivants.

Les problèmes de convergence globale dans le cas d'un critère f convexe sont traités à l'aide d'un résultat général concernant l'existence d'une famille-type de fonctionnelles de Lyapounov. Le résultat est ensuite appliqué à plusieurs classes d'exemples couvrant de nombreuses méthodes, comme celles décrites ci-dessus. Les trajectoires $x(t)$, $t \to +\infty$ des systèmes étudiés convergent alors vers une solution de (P).

Diverses précisions et améliorations sont aussi proposées, avec en particulier des résultats de localisation du point limite.

Chapitre 2 : Sur la méthode continue de gradient-projection.

La méthode de gradient-projection déjà envisagée au chapitre précédent fait ici l'objet de diverses extensions et d'une étude fine.

Tout d'abord dans le cadre de la dimension finie et pour une condition initiale choisie dans C supposé cette fois-ci fermé, nous donnons un sens à la dynamique (CGP) pour une fonction convexe f *continue*, le résultat essentiel étant l'existence d'une solution $x \in W^{1,2}_{loc}(0, +\infty; \mathbb{R}^n)$ à l'inclusion différentielle :

$$(CGP)_g \qquad \dot{x}(t) + x(t) - P_C[x(t) - \partial f(x(t))] \ni 0,\ t \geq 0,$$

avec $x(0) \in C$. La difficulté réside ici principalement dans la non-monotonie et la non-convexité de l'opérateur multivoque $x \rightrightarrows P_C[x - \partial f(x)]$. Elle est surmontée à l'aide de systèmes dynamiques approchés obtenus par une régularisation de type Moreau-Yosida. La solution obtenue résout le problème (P) au sens où $f(x(t)) \to \inf f$, et la trajectoire converge vers une solution si (P) en admet au moins une.

Ensuite, dans le cas où H est un espace de Hilbert nous montrons cette fois-ci que les trajectoires convergent *faiblement* vers un minimum, le résultat demeurant valable pour toute condition initiale *hors de l'ensemble des contraintes* (figure 2.1 page 44.) Des conclusions analogues sont obtenues en relation avec (P') pour des critères quasi-convexes.

En vue à la fois de contraindre les orbites à la convergence forte, et d'obtenir une description de l'ensemble des solutions, un terme de contrôle de type viscosité est introduit :

$$\dot{x}(t) + x(t) - P_C[x(t) - \nabla f(x(t)) - \epsilon(t)x(t)] \ni 0,\ t \geq 0.$$

Supposant $\epsilon : \mathbb{R}_+ \mapsto \mathbb{R}_+$, décroissante et telle que $\int_{\mathbb{R}_+} \epsilon = +\infty$, la convergence de $x(t)$ devient forte et la limite obtenue est la solution de (P) *de norme minimale*. Ce genre de résultat concernant le contrôle asymptotique de systèmes dynamiques est inspiré par les travaux [15, 17, 41].

Chapitre 3 : Champs de gradients et métriques induites par les fonctions de Legendre

Les résultats et l'étude proposés s'inscrivent toujours dans l'esprit des opérateurs-barrières, mais cette fois-ci dans un cadre résolument riemannien : les applications A_x, $x \in C$ sont supposées linéaires auto-adjointes positives, faisant de $C \ni x \mapsto A_x \nabla f(x)$ un gradient riemannien. Nous supposons par la suite que $H = \mathbb{R}^n$, avec f $\mathcal{C}^1$ et nous étudions pour des métriques adéquates les trajectoires de gradient riemannien de fonctions.

Le premier résultat abstrait a pour point de départ la considération suivante : étant donnée une métrique différentiable g sur C supposé *ouvert*, nous avons, dans le cas où f

est convexe, la caractérisation variationnelle suivante :

$$\hat{x} \text{ est une solution de } (P) \Leftrightarrow g(\nabla_g f(x), x - \hat{x})_x \geq 0, \ \forall x \in C,$$

où $\nabla_g f(\cdot)$ désigne le gradient de f dans la métrique g.

Si l'on se rappelle qu'une fonction $V : C \mapsto \mathbb{R}$ est dite de Lyapounov pour $-\nabla_g f$ quand

$$g(\nabla_g f(x), \nabla_g V)_x \geq 0,$$

pour tout x dans C, il est tentant de se demander pour quelles métriques les champs de vecteurs $V^{\hat{x}} : x \mapsto x - \hat{x}$ sont des gradients.

De telles métriques sont complétement identifiées comme celles données par les *hessiens* de fonctions convexes. Si l'on prend pour g, $g_x = \langle \nabla^2 h(x) \cdot, \cdot \rangle$ avec $h : C \mapsto \mathbb{R}$ strictement convexe, les champs $V^{\hat{x}}$ sont obtenus comme des gradients par rapport à la deuxième variable de la *D-fonction* de h :

$$D_h(\hat{x}, x) = h(\hat{x}) - h(x) - \langle \nabla h(x), \hat{x} - x \rangle, \ \hat{x}, x \in C.$$

Pour obtenir des méthodes efficaces en optimisation il est alors nécessaire de choisir un noyau h qui ne puisse pas être prolongé en une fonction convexe hors de $\overline{C}$: les métriques étudiées seront donc engendrées par des *fonctions de Legendre*. Par la suite, étant donnée h une fonction de Legendre, on désigne par $\nabla_{H} f$ le gradient de f.

En restreignant les métriques obtenues aux ensembles $C \cap \mathcal{A}$, avec $\mathcal{A}$ espace affine (contraintes d'égalités), nous étudions alors :

$$(H\text{-}SD) \qquad \dot{x}(t) + \nabla_{H} f_{C \cap \mathcal{A}}(x(t)) = 0, \ x(0) \in C \cap \mathcal{A}.$$

Après avoir donné des résultats d'existence globale non triviaux, nous montrons sans plus d'hypothèses que, dans le cas d'une fonction convexe, $f(x(t)) \to \inf_{C \cap \mathcal{A}} f$, $t \to +\infty$. En supposant h de Bregman et f quasi-convexe nous prouvons la convergence des trajectoires vers une solution de (P').

Dans le cas où $\overline{C} = \mathbb{R}^n_+$ et après avoir identifié une trajectoire duale associée à $\{x(t)\}$, des résultats de convergence duale sont proposés. Le point limite est solution du problème dual associé à (P) et peut se caractériser comme solution unique d'un système hierarchique de problèmes d'optimisation. Signalons, par ailleurs, que ce résultat ne présume pas de la convergence de la trajectoire primale $\{x(t)\}$.

Un changement de coordonnées, appelé *transformation de Legendre*, ouvre de nouvelles perspectives en programmation linéaire. Il est en effet prouvé que, dans ces nouvelles coordonnées, les trajectoires de $(H\text{-}SD)$ sont des lignes droites tracées dans un cône positif. Par ailleurs diverses caractérisations de ces trajectoires sont proposées : géodésiques pour des métriques appropriées, chemins optimaux pour certaines barrières, lieux dans lesquels les algorithmes proximaux de type Bregman produisent leurs itérés, $\dot{q}$ trajectoires de Lagrangiens... Jouant sur la souplesse d'une telle méthode, nous proposons en guise d'application des estimations de la vitesse de convergence dans le cas d'un programme linéaire, et nous obtenons alors des résultats équivalents pour les dynamiques précitées.

Chapitre 4 : Sur des équations paraboliques régies par des fonctions de Legendre.

Dans ce qui suit $f : H \mapsto \mathbb{R} \cup \{+\infty\}$ est une fonction convexe propre fermée définie sur un espace de Hilbert H. L'objet essentiel de cette partie consiste en l'obtention d'une solution régulière de l'inclusion différentielle suivante :

$$\frac{d}{dt}\nabla h(x(t)) + \partial f(x(t)) \ni 0, \quad t \geq 0,\ x(0) \in \operatorname{int}\operatorname{dom} h \cap \operatorname{dom} f.$$

Les équations de ce type sont étroitement liées à des problèmes d'évolution en thermodynamique : problème de Stefan, flux dans les milieux poreux ainsi que divers problèmes issus de l'industrie [52, 83, 31].

En usant de la récente théorie des fonctions de Legendre dans les espaces de Banach réflexifs [29], nous prouvons l'existence d'une solution pour une fonctionnelle f à sous-niveaux compacts. La difficulté essentielle réside dans le traitement du comportement des solutions à l'approche de la frontière de dom h. La solution obtenue $x \in W^{1,2}(0, T^*; H)$, $T^* > 0$ évolue dans *l'intérieur du domaine* de h et satisfait à l'alternative suivante :

$$T^* = +\infty$$
$$\text{ou}$$
$$f(x(T^*)) = \inf_{\overline{\operatorname{dom} h}} f.$$

Les hypothèses de compacité concernant f s'avèrent inutiles en dimension finie, et la dynamique obtenue correspond alors à une généralisation des méthodes riemanniennes proposées au chapitre précédent.

Le comportement asymptotique des orbites est abordé dans un cadre identique à celui des conditions d'existence, il est alors montré que :

$$f(x(t)) \to \inf_{\overline{\operatorname{dom} h}} f \text{ lorsque } t \to T^*.$$

De nombreuses illustrations sont proposées. Pour $H = L^2(\Omega)$ et $g \in H$, les dynamiques etudiées permettent par exemple d'aborder des problèmes d'évolution du type :

$$\frac{\partial}{\partial t}\left(u + \ln \int_\Omega u\right) - \triangle u - g + N_C(u) \ni 0, \text{ sur } (0, T^*) \times \Omega$$

avec $u_0 \in C \cap H_0^1(\Omega)$, $\int_\Omega u_0 > 0$.
Signalons que dans des perspectives de type optimisation, la trajectoire du système ci-dessus converge fortement dans $L^2(\Omega)$ vers la solution de

$$\min \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega gu \,|u \in C \cap H_0^1(\Omega),\ \int_\Omega u \geq 0 \right\}.$$

PARTIE II . SYSTÈMES DE TYPE GRADIENT DU SECOND ORDRE EN TEMPS AVEC FROTTEMENTS DE TYPE HESSIEN.

Plusieurs motivations président à l'étude de systèmes du type (2). En optimisation l'aspect inertiel présente tout d'abord l'interêt de donner lieu à des trajectoires capables de s'affranchir de zones "pièges" correspondant à des minima locaux, tout en ouvrant la voie a une exploration globale de ces derniers.

Le système (2) correspond par ailleurs à une généralisation de dynamiques de type *boule pesante avec frottements* :

$$(HBF) \qquad \ddot{x}(t) + \alpha \dot{x}(t) + \nabla f(x(t)) = 0,\ t \geq 0$$

introduite indépendamment dans [111] et [18] et étudié selon divers point de vue dans [4, 10, 6]. Si f est supposée convexe non lisse, on aboutit formellement à

$$\ddot{x}(t) + \alpha \dot{x}(t) + \partial f(x(t)) \ni 0,\ t \geq 0,$$

signifiante en mécanique et dans le domaine des équations aux dérivées partielles [115, 120].

De façon surprenante, l'introduction d'un terme de type hessien s'avère pertinente dans les deux domaines : en optimisation il agit en stabilisant les orbites et en mécanique, il donne, par sa souplesse, la possibilité de modéliser les chocs inélastiques.

Toujours dans cette double perspective, un fait majeur concernant (2) est l'existence d'un changement de variable permettant d'éviter tout recours au hessien ; cela permet à la fois de donner un sens à des équations d'ordre deux en temps réputées difficiles, et de fournir un cadre raisonnable aux méthodes inertielles en optimisation.

Chapitre 5 : Un système dynamique inertiel avec frottements de type hessien.

Un premier aspect important de (2) en optimisation consiste en son lien avec la *méthode continue de Newton* :

$$\nabla^2 f(y(t))\dot{y}(t) + \nabla f(y(t)) = 0,\ t \geq 0.$$

Désignant par x_ϵ la solution de (2) avec $\alpha = 0, \gamma = \epsilon\ , \beta = 1$, nous prouvons, sous des hypothèses locales de forte convexité sur f, le résultat suivant :

$$\sup_{t \geq 0} |x_\epsilon(t) - y(t)| \leq C\epsilon, \text{ où } C \text{ est une constante positive.}$$

Cette estimation et quelques expériences numériques (figure 5.1 page 113) montrent qu'un choix judicieux des paramètres de (2) permet d'éviter les phénomènes d'oscillations *transversales* inhérents au système (HBF).

D'un point de vue numérique, la prise en compte du hessien peut paraître coûteuse et délicate. Pour s'affranchir de telles difficultés, nous montrons qu'à l'aide d'un changement de coordonnées adéquat, le système (2) peut s'écrire :

$$\text{(g-2)} \qquad \begin{cases} \dot{x} + \beta \nabla f(x) & +ax + by = 0 \\ \dot{y} & +ax + by = 0 \end{cases}$$

où $a = \alpha - \frac{1}{\beta}$ et $b = \frac{1}{\beta}$. La solution de (2) n'est autre que la première composante du système ci-dessus, i.e. $t \mapsto x(t)$.

L'analyse de (2) s'avère fructueuse aussi bien dans le cas d'un critère convexe que dans le cas réel analytique. Pour f convexe les résultats de [4] sont généralisés : il est en effet prouvé que les trajectoires convergent faiblement vers une solution de (P). Diverses hypothèses géométriques, telle que la parité de f, ou même topologiques, permettent par ailleurs, d'obtenir la convergence forte.

En dimension finie, l'inégalité de Lojasiewicz reliant une fonction analytique et son gradient autour d'un point critique est l'outil-clé de la convergence des orbites bornées. Ce résultat s'inscrit dans la lignée des travaux [116, 72, 80].

Pour illustrer la flexibilité de la méthode étudiée, deux applications sont proposées : en mécanique tout d'abord, puis en optimisation sous contraintes.

S'appuyant sur les travaux de Paoli-Schatzmann [107] concernant l'approximation des inclusions différentielles du type :

$$(HBF)_g \qquad \begin{cases} \ddot{x}(t) + N_C(x(t)) \ni g(t, x(t), \dot{x}(t)) \\ \dot{x}(t^+) = -e\dot{x}_N(t^-) + \dot{x}_T(t^-) \text{ pour tout } t \text{ tel que } x(t) \in \operatorname{bd} C \end{cases}$$

nous montrons au travers de diverses expériences numériques, l'interet de (2) en mécanique unilatérale.

Si l'on approche des équations du type $(HBF)_g$ par des systèmes du type :

$$\ddot{x}_\lambda(t) + \alpha\dot{x}_\lambda(t) + \theta(\lambda)\nabla^2 f(x_\lambda(t))\dot{x}_\lambda(t) + \nabla f_\lambda(x_\lambda(t)) = 0,\ \lambda \to 0,$$

avec $\theta : \mathbb{R}_+ \mapsto \mathbb{R}_+$, et où f_λ est la régularisée Moreau-Yosida de f, on obtient à la limite des trajectoires de billards dans C qui satisfont des lois de chocs inélastiques variées, conditionnées par θ (figure 5.3 page 137).

Enfin, en vue d'optimiser une fonction sous des contraintes C, nous introduisons la méthode suivante de gradient projeté :

$$\begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \beta\nabla f(x(t)) - ax(t) - by(t)] = 0 \\ \dot{y}(t) \qquad\qquad\qquad + ax(t) + by(t) = 0 \end{cases}$$

avec $x(0) \in C$, et $y(0) \in H$. En dimension finie, les valeurs d'adhérences éventuelles de x satisfont la condition (P'), et, dans le cas où f est convexe, cette fois sans restriction de dimension, on obtient convergence faible de x vers une solution de (P).

Chapitre 6 : Propriétés minimisantes d'un système inertiel en optimisation. Liens avec la méthode proximale.

La fonction $f : H \mapsto \mathbb{R}$ est supposée convexe propre et semi-continue inférieurement.

Le point de départ du travail consiste à remarquer que si $\gamma = 1$, $\alpha\beta \geq 1$, alors (2) peut s'interpréter comme une méthode de gradient pour la fonctionnelle

$$\mathcal{F}_{a,b}(x, y) = f(x) + \frac{1}{2}|ax + by|^2.$$

Outre le fait de donner un sens à (2) dans un cadre non lisse, cette approche permet de mettre en rapport divers aspects en optimisation : méthodes proximales et de relaxation, régularisations de systèmes mal posés, méthodes inertielles pour un critère non lisse.

Dans l'esprit des méthodes proximales et afin de tenir compte d'éventuelles contraintes, on introduit pour $H = \mathbb{R}^n$ et $C = \mathbb{R}^n_+$ des fonctionnelles du type :

$$\mathcal{F}_\varphi(x, y) = f(x) + \overline{d_\varphi}(x, y).$$

où $\overline{d_\varphi}$ est la régularisée semi-continue inférieure d'un noyau dit de type φ-divergence,

$$d_\varphi(x, y) = \begin{cases} \sum_{i=1}^n y_i \varphi(y_i^{-1} x_i) \ \text{ si } (x, y) \in (\mathbb{R}^n_{++})^2, \\ \\ +\infty \text{ ailleurs.} \end{cases}$$

La biconvexité de la quasi-distance d_φ permet alors de donner un sens à la méthode inertielle sous contrainte suivante :

$$\dot{x}(t) + \partial f(x(t)) + \partial_x d_\varphi(x(t), y(t)) \ni 0 \text{ p.p. sur } (0, +\infty),$$
$$\dot{y}(t) + \partial_y d_\varphi(x(t), y(t)) = 0, \ \forall t \geq 0$$

avec $x_0 \in \overline{\text{dom } f} \cap \mathbb{R}^n_+$ et $y_0 \in \mathbb{R}^n_+$.
Pour conclure nous montrons que les deux composantes (x, y) de la solution de ce système résolvent asymptotiquement le problème (P).

Première partie

LES METHODES CONTINUES DE TYPE GRADIENTS

Chapitre 1

Opérateurs-barrières et méthodes de descentes associées.

Barrier operators and associated gradient-like dynamical systems for constrained minimization problems.[1]

Jérôme Bolte and Marc Teboulle[2]

Abstract. We study some continuous dynamical systems associated with constrained optimization problems. For that purpose, we introduce the concept of elliptic barrier operators and develop a unified framework to derive and analyze the associated class of gradient-like dynamical systems, called A-Driven Descent Method (A-DM). Prominent methods belonging to this class include several continuous descent methods studied earlier in the literature such as steepest descent method, continuous gradient projection methods, Newton type methods as well as continuous interior descent methods such as Lotka-Volterra type differential equations, and Riemannian gradient methods. Related discrete iterative methods such as proximal interior point algorithms based on Bregman functions and second order homogeneous kernels can also be recovered within our framework and allow for deriving some new and interesting dynamics. We prove global existence and strong viability results of the corresponding trajectories of (A-DM) for a smooth objective function. When the objective function is convex, we analyze the asymptotic behavior at infinity of the trajectory produced by the proposed class of dynamical systems (A-DM). In particular, we derive a general criterion ensuring the global convergence of the trajectory of (A-DM) to a minimizer of a convex function over a closed convex set. This result is then applied to several dynamics built upon specific elliptic barrier operators. Throughout the paper, our results are illustrated with many examples.

Key words : Dynamical systems, continuous gradient-like systems, elliptic barrier operators, Lotka-Volterra differential equations, asymptotic analysis, viability, Lyapunov functionals, explicit and implicit discrete schemes, interior proximal algorithms, global convergence, constrained convex minimization.

1.1 Introduction

This paper proposes to study some continuous dynamical systems in relation with the constrained optimization problem

$$(\mathcal{P}) \qquad \inf\{f(x) : \; x \in \overline{C}\},$$

where C is a nonempty *open* convex subset of $\mathbb{R}^n$, $n \geq 1$, $f : \mathbb{R}^n \mapsto \mathbb{R}$ is a convex function and $\overline{C}$ denotes the closure of C.

Our first aim is to give a unified framework to smooth continuous interior descent methods studied earlier in the literature : steepest descent method, Lotka-Volterra type equations, continuous Newton method, continuous gradient projection method. Another goal of this study is to enlighten the geometrical aspects of some discrete dynamics related to $(\mathcal{P})$ (particularly proximal type algorithms), via the study of some continuous models.

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This has led us to introduce the following class of gradient-like dynamical systems

$$(A-DM) \qquad \begin{cases} \dot{x}(t) + A_{x(t)}\nabla f(x(t)) = 0, \forall t \geq 0, \\ x(0) \in C, \end{cases}$$

with

$$A : \begin{cases} C \times \mathbb{R}^n & \mapsto \quad \mathbb{R}^n \\ (x,v) & \mapsto \quad A_x v. \end{cases} \tag{1.1.1}$$

The notation $(A-DM)$ stands for A-driven descent method. To make $(A-DM)$ an interior descent method, we introduce a class of mappings of the type (1.1.1) called *elliptic barrier operators*. This is an alternative approach to the classical barrier methods, see for instance Auslender-Cominetti-Haddou [23], since the penalization does not act on the objective function f but on its gradient. Roughly speaking this implies two major requirements on the map A :

- the mapping $x \in C \rightarrow A_x\nabla f(x)$ must preserve the local optimality information given by $\nabla f(\cdot)$,
- the operator A has to vanish on $\{ (x,-\nu), x \in \overline{C},\ \nu \in N_{\overline{C}}(x)\}$, where $N_{\bar{C}}(x)$ is the normal cone to $\overline{C}$ at $x \in \overline{C}$.

In the next section, a formal definition and the basic properties of elliptic barrier operators are given. The relevance of this notion is first illustrated by the general properties of $(A-DM)$ systems. We prove existence and viability results. If ∇f is locally Lipschitz continuous, then the trajectories of $(A-DM)$ are defined for all $t \geq 0$, and remain in C. Let us emphasize the fact that, unlike in Nagumo-type theorems used in viability theory (Aubin-Cellina [21]), the trajectories never encounter the boundary of C, and thus making $(A-DM)$ an *interior* method.

In Section 1.3, we propose a general and unifying framework to generate in a systematic way elliptic barrier operators. This is achieved by developing an abstract setting, with the help of proximal-like maps involving appropriately defined distance-like functions. Given a convenient distance-like function $d : \mathbb{R}^n \times C \mapsto \mathbb{R} \cup \{+\infty\}$ closed, proper, and convex with respect of its first variable, we introduce the following class of mappings

$$A_x^d v = x - \arg\min\left\{\langle u,v\rangle + d(u,x) \mid u \in \mathbb{R}^n\right\},\ (x,v) \in C \times \mathbb{R}^n \tag{1.1.2}$$

where $\langle\cdot,\cdot\rangle$ stands for the Euclidean inner product of $\mathbb{R}^n$. Besides the fact that slight assumptions on d allow to make A^d an elliptic barrier operator, the associated A^d-driven descent method can be seen as another step towards a unified approach to both continuous and discrete gradient-like dynamics. Indeed, one of the main fact underlying the introduction of d operator is that $(A^d - DM)$ systems can be reformulated as the following differential inclusion

$$\partial_1 d(\dot{x}(t) + x(t), x(t)) + \nabla f(x(t)) \ni 0,\ t \geq 0 \tag{1.1.3}$$

where for each $t \geq 0$, $\partial_1 d(\cdot, x(t))$ denotes the subdifferential of $d(\cdot, x(t))$.
This structure is at the heart of the so called proximal-like methods, (see the examples below),

$$\partial_1 d(x^{k+1}, x^k) + \nabla f(x^{k+1}) \ni 0,\ x^0 \in C,\ k \geq 0. \tag{1.1.4}$$

For instance, with $d(u,x) = 2^{-1}|u-x|^2$, where $|\cdot|$ denotes the Euclidean norm, the inclusion (1.1.4) reduces to the proximal minimization algorithm, see e.g., Martinet [100], Lemaire [90], and references therein. Then, according to the classical idea that consists in interpreting an iterative scheme as some discretization of a continuous dynamical system, the differential inclusion (1.1.3), i.e. $(A^d - DM)$, can be proposed as a continuous model for the proximal method (1.1.4). This opens new perspectives of crossed investigations and from that viewpoint it is important to realize that the interplay between discrete and continuous dynamical systems goes far beyond the fruitful finite-time approximation aspects. For instance in Alvarez-Attouch [5], Antipin [10] crucial features of the asymptotic analysis appear also as closely related matters.

To give the reader a concrete idea on the type of operators A that will emerge in this study, we outline below some specific models.

(a) *The gradient projection operator*
The first natural example is given by

$$A^P : \left\{ \begin{array}{lll} C \times \mathbb{R}^n & \mapsto & \mathbb{R}^n \\ (x,v) & \mapsto & x - P_{\overline{C}}(x-v), \end{array} \right. \tag{1.1.5}$$

where $P_{\overline{C}}$ is the orthogonal projection on $\overline{C}$. $(A^P - DM)$ is the continuous gradient projection method as introduced in [10],

$$\dot{x}(t) + x(t) - P_{\overline{C}}[x(t) - \nabla f(x(t))] = 0,\ x(0) \in C,\ \forall t \geq 0. \tag{1.1.6}$$

The operator A^P ruling (1.1.6) can be recovered thanks to (1.1.2) with a distance-like function of the type $d : \mathbb{R}^n \times C \ni (u,x) \mapsto \frac{1}{2}|u-x|^2 + \delta_{\overline{C}}(u)$ where $\delta_{\overline{C}}$ is the indicator function of $\overline{C}$. Let us emphasize the fact that the trajectory of the continuous system (1.1.6) is interior, which is not the case for the following well known explicit discretization

$$x^{k+1} = P_{\overline{C}}[x^k - \mu_k \nabla f(x^k)],\ x^0 \in C,\ \mu_k > 0$$

see e.g., [91], [57].

(b) *The Bregman operators*
The Bregman proximal method (BPM) is obtained by replacing the quadratic kernel in the proximal minimization algorithm by a distance-like based on a Bregman function $h : \overline{C} \to \mathbb{R}$. Defining

$$\forall (x,y) \in \overline{C} \times C,\ \ D_h(x,y) = h(x) - h(y) - \langle \nabla h(y), x-y \rangle, \tag{1.1.7}$$

it leads to the scheme

$$(BPM) \qquad x^{k+1} \in \arg\min \left\{ f(x) + c_k D_h(x,x_k) | x \in \overline{C} \right\},\ c_k > 0,\ x^0 \in \overline{C}.$$

(BPM) has been studied and generalized from many viewpoints, see for instance Censor-Zenios [44], Chen-Teboulle [45], Eckstein [59], Kiwiel [84], Teboulle [119]. One of the corresponding continuous model that is proposed here is given by barrier operators A^{q_h} of the type

$$A^{q_h} : \left\{ \begin{array}{lll} C \times \mathbb{R}^n & \mapsto & \mathbb{R}^n \\ (x,v) & \mapsto & \nabla^2 h(x)^{-1} v, \end{array} \right.$$

where $\nabla^2 h(x)$ is the Hessian of some convenient Bregman function with zone C and with $q_h(u,x) = \langle \nabla^2 h(x)(u-x), u-x\rangle$, $(u,x) \in \mathbb{R}^n \times C$. The A^{q_h}-driven descent method –actually a Riemannian gradient method– is then given by

$$(A^{q_h} - DM) \qquad \dot{x}(t) + \nabla^2 h(x(t))^{-1}\nabla f(x(t)) = 0,\ x(0) \in C.$$

Besides its links with (BPM) developed in Section 4, the latter system allows to recover several dynamics. With $h_1(x) = \frac{\alpha}{2}|x|^2 + \beta\sum_{i=1..N} x_i \log x_i$, $\alpha, \beta > 0$ on $C = \mathbb{R}^n_{++} := \{x \in \mathbb{R}^n,\ x_i > 0\}$, we obtain the regularized Lotka-Volterra equation recently proposed, from a completely different viewpoint, in Attouch-Teboulle [20] :

$$(A^{q_{h_1}} - DM) \qquad \dot{x}_i(t) + \frac{x_i(t)}{\beta + \alpha x_i(t)}\frac{\partial f}{\partial x_i}(x(t)) = 0,\ \forall i = 1\ldots,n,\ x(0) \in \mathbb{R}^n_{++}, \quad (1.1.8)$$

where f is to be optimized on $\mathbb{R}^n_+$.

If $h(x) = \frac{\alpha}{2}|x|^2$ and $C = \mathbb{R}^n$, $(A^{q_h} - DM)$ is the classical continuous steepest descent method $\dot{x}(t) + \nabla f(x(t)) = 0$, $t \geq 0$, see Brézis [37].
For $h(x) = f(x)$ and $C = \mathbb{R}^n$, we obtain the continuous Newton descent method, studied in Alvarez-Pérez [8], see also [21]

$$(A^{q_f} - DM) \qquad \dot{x}(t) + \nabla^2 f(x(t))^{-1}\nabla f(x(t)) = 0. \quad (1.1.9)$$

Another surprising fact of this dynamics is to be physically meaningful in infinite-dimensional spaces. Naturally those problems are out of the scope of the present paper, but the reader interested by thermodynamical evolution equations of the form $(A^{q_h} - DM)$ is referred to Kenmochi-Pawlow [83] and references therein.

(c) *Barrier operators based on interior methods for the positive orthant*
Another line of research pursued by Auslender, Teboulle, and Ben Tiba [22] concerning proximal interior methods is based on the distance-like function

$$\forall (x,y) \in (\mathbb{R}^n_{++})^2 \ d_\varphi(x,y) = \sum_{i=1}^n y_i^2 \varphi(\frac{x_i}{y_i}), \quad (1.1.10)$$

where $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ is some relevant convex function.

The associated iterative proximal interior method is given by

$$(RIPM) \qquad x^{k+1} \in \arg\min\left\{f(x) + c_k d_\varphi(x, x_k) | x \in \mathbb{R}^n_+\right\},\ c_k > 0,\ x^0 \in \mathbb{R}^n_{++}$$

where $(RIPM)$ stands for regularized interior proximal method. Like (BPM) this algorithm can be applied to a minimize a general closed convex function. However, it enjoys stronger convergence properties, particularly when applied to a dual problem of a convex program, see [22], for further details and results.

Our continuous approach to (RIPM) is obtained by considering barrier operators of the form

$$A^{d_\varphi} : \left\{ \begin{array}{lll} \mathbb{R}^n_{++} \times \mathbb{R}^n & \mapsto & \mathbb{R}^n \\ (x,v) & \mapsto & \left(x_i - x_i(\varphi^*)'(x_i^{-1}v_i)\right)_{i=1\ldots,n} \end{array} \right.$$

where φ^* is the Legendre-Fenchel conjuguate of the function φ used in $(RIPM)$.

All these continuous models are derived and analyzed in Section 1.4. Section 1.5 of this paper is devoted to the asymptotic analysis of $(A-DM)$ in the convex case. We derive a general criterion ensuring the global convergence of the trajectories of $(A-DM)$ to a minimizer of f over $\overline{C}$. We then apply this general result to the dynamics built upon A^P, A^{q_h}, and A^{d_φ}. The proof relies on the existence of Lyapounov functionals measuring a sort of distance between the state variable and the set of equilibria. This approach is inspired at the same time by Opial's lemma [105] and the techniques used in monotone optimization algorithms. We also prove a general localisation result for the limit point of the trajectories produced by $(A-DM)$, which extends results of the same type obtained recently in [20], and in [90] for the classical continuous gradient descent scheme. Throughout this paper we give many examples exhibiting some explicit and new systems of the type $(A-DM)$. For instance with $C = \mathbb{R}^n_{++}$, one obtains the systems,

$$(A^{q_h}-DM) \qquad \dot{x}_i(t) + \frac{2x_i(t)^{3/2}}{x_i(t)^{3/2}+1}\frac{\partial f}{\partial x_i}(x(t)) = 0,\ x_i(0)>0, \quad \forall i \in \{1,\dots,n\}.$$

or

$$(A^{d_\varphi}-DM) \qquad \dot{x}_i(t) + x_i(t) + \frac{1}{2}\frac{\partial f}{\partial x_i}(x(t)) - \sqrt{\frac{1}{4}\frac{\partial f}{\partial x_i}(x(t))^2 + x_i(t)^2} = 0,$$

with $i \in \{1,\dots,n\}$, $t \geq 0$ and $x(0) \in \mathbb{R}^n_{++}$. The first equation is given by the Bregman function $h(s) = s^2/4 - 2\sqrt{s}$, $s \geq 0$ while the second one corresponds to a continuous model of the logarithmic-quadratic method [22], obtained with the choice $\varphi(s) = 1/2(s-1)^2 - \log s + s - 1,\ s > 0$.

Notations. Our notations are fairly standard. The Euclidean space $\mathbb{R}^n$ is equipped with the scalar product $\langle\cdot,\cdot\rangle$; the related norm is denoted $|\cdot|$. $N_{\overline{C}}(x)$ and $T_{\overline{C}}(x)$ denote respectively the normal cone and the tangent cone of $\overline{C}$ at $x \in \overline{C}$. We recall that $N_{\overline{C}}(x) = \{v \in \mathbb{R}^n\ \langle v, z-x\rangle \leq 0,\ \forall z \in \overline{C}\} = \{v \in \mathbb{R}^n | \forall u \in T_{\overline{C}}(x),\ \langle v,u\rangle \leq 0\}$. If $\phi : \mathbb{R}^p \to \mathbb{R}\cup\{+\infty\}$, $p \geq 1$ is a closed proper convex function, its domain is defined by $\operatorname{dom}\ \phi = \{x \in \mathbb{R}^p | \phi(x) < +\infty\}$ and its Legendre-Fenchel conjuguate, $y \in \mathbb{R}^p \to \sup\{\langle y,x\rangle - \phi(x) | x \in \mathbb{R}^p\}$, is denoted ϕ^*. If S is a closed convex subset of $\mathbb{R}^n$, the set of minimizers of ϕ on S is denoted $\operatorname{arg\,min}_S \phi$. The indicator function of $\overline{C}$ is denoted by $\delta_{\overline{C}}$. Other notations and definitions not explicitly stated here can be found in the classical book of Rockafellar [112].

1.2 Elliptic barrier operators and viability results

In this section, the definition and the first properties of elliptic barrier operators are introduced. Then, in view of constrained minimization, we study the corresponding A-driven descent methods, proving in particular that the obtained trajectories $\{x(t)\}$ are *interior* and defined for any $t \in [0,+\infty)$.

1.2.1 Elliptic barrier operators : Definition and Properties

Definition 1.2.1 $A : C \times \mathbb{R}^n \to \mathbb{R}^n$ *is an elliptic barrier operator on* C *if it satisfies :*

$(r1)$ *A is Lipschitz continuous on every compact subset of* $C \times \mathbb{R}^n$.

$(r2)$ *There exists* $\alpha > 0$, *such that for every* $(x, v) \in C \times \mathbb{R}^n$, $\langle A_x v, v\rangle \geq \alpha |A_x v|^2$.

$(r3)$ *For all* $x \in C$, $A_x v = 0$ *implies* $v = 0$.

(v) $\forall b \in \operatorname{bd} C$, $\forall \nu \in N_{\overline{C}}(b)$, $\forall M > 0$, $\exists \epsilon, K > 0$ *such that* $|x - b| < \epsilon$, $x \in C$, $|v| \leq M$ *implies*

$$\langle -A_x v, \nu\rangle \leq K\langle b - x, \nu\rangle. \tag{1.2.1}$$

This definition is motivated by the study of $(A - DM)$ systems. The *regularity* assumption $(r1)$ naturally meets the conditions of the Cauchy-Lipschitz theorem. The *ellipticity* condition $(r2)$ and the *non degeneracy* assumption $(r3)$ allow to obtain a proper descent method. An important consequence of $(r2)$ is that the term $1/\alpha$ can be seen as an upper bound for the gradient stepsize in $(A - DM)$. Indeed, it follows readily from $(r2)$ that $|A_x v| \leq \alpha^{-1}|v|$, and therefore a trajectory $x(\cdot)$ of $(A - DM)$ satisfies

$$|\dot{x}(t)| \leq \alpha^{-1}|\nabla f(x(t)|,$$

whenever $x(t)$ is defined and belongs to C.

The *normal boundary* property (v) is required to control the outwards normal impulses near the boundary of C, making the trajectories of $(A - DM)$ strongly viable, i.e., $x(t) \in C$, $t \geq 0$. The choice of the term $\langle b - x, \nu\rangle$ in (1.2.1) has also a regularizing effect. Indeed, as it will be proved in Theorem 1.2.4 below (see also Remark 1.2.1 (b)), it contributes to the fact that the trajectories of $(A - DM)$ are defined on $[0, +\infty)$.

Remark 1.2.1 (a) A natural extension of Definition 1.2.1 is obtained by replacing assumptions $(r2)$ and $(r3)$ respectively by

$(r2)'$ *For every* $(x, v) \in C \times \mathbb{R}^n$, $v \neq 0$ $\langle A_x v, v\rangle > 0$,

$(r3)'$ *For all* $x \in C$, $v = 0$ *implies* $A_x v = 0$.

Observing that $(r2)'$ and $(r3)'$ imply $(r3)$ it follows that an elliptic barrier operator satisfies this new definition. This widened concept opens new perspectives but also raises some difficulties in the study of $(A - DM)$: finite-time solutions, loss of regularity (see Theorem 2.1 in the elliptic case), no upper bound for the gradient step-sizes, etc... The study of such a class of mappings will not be carried out in the present paper, but this is certainly a matter for future research.

(b) If the left term in (1.2.1) is replaced for instance by $\langle b - x, \nu\rangle^{1-\theta}$, $\theta \in (0, 1)$ the well-posedness of $(A - DM)$ may fail : take for instance $A : (x, v) \in \mathbb{R}_+ \times \mathbb{R} \to x^{1-\theta}.v$, $\theta \in (0, 1)$, $f(x) = x + 1$ and observe that the maximal solutions of $(A - DM)$ are not defined on $[0, +\infty)$.

In what follows it is of interest to strengthen $(r1)$ by assuming the additional hypothesis,

$(r4)$ A is continuous on $\overline{C} \times \mathbb{R}^n$.

The following result shows that an elliptic barrier operator on C can be continuously extended to

$$C \times \mathbb{R}^n \cup \{(x, v) | x \in \operatorname{bd} C,\ v \in -N_{\overline{C}}(x)\},$$

by setting $A_x v = 0$, if $x \in \operatorname{bd} C$, $v \in -N_{\overline{C}}(x)$.

Proposition 1.2.1 *Let $A : C \times \mathbb{R}^n \to \mathbb{R}^n$ be an elliptic barrier operator. Assume that $(x^k, v^k), k \in N$ is a sequence in $C \times \mathbb{R}^n$ such that $x^k \to x \in \overline{C}$ and $v^k \to v \in -N_{\overline{C}}(x)$ as $k \to +\infty$. Then*
(i) $A_{x^k} v^k \to 0$ as $k \to +\infty$.
(ii) In addition, if A satisfies $(r4)$ then for all $x \in \overline{C}$ one has,

$$A_x^{-1}(\{0\}) \supset -N_{\overline{C}}(x). \tag{1.2.2}$$

Proof. If $x \in C$ the conclusion follows from $(r1)$ and $(r3)$. Else $x \in \operatorname{bd} C$. $(r2)$ and the Cauchy Schwarz inequality yield $|A_{x^k} v^k|.|v^k| \geq \alpha |A_{x^k} v^k|^2$, for all $k \in N$ and some $\alpha > 0$. Since the sequence $v^k, k \in N$ is bounded so is $A_{x^k} v^k, k \in N$. From (v) it follows that for k large enough $\langle -A_{x^k} v^k, -v \rangle \leq K \langle x - x^k, -v \rangle$ and therefore

$$\limsup_{k \to +\infty} \langle A_{x^k} v^k, v \rangle \leq 0. \tag{1.2.3}$$

On the other hand we have

$$\langle A_{x^k} v^k, v \rangle = \langle A_{x^k} v^k, v - v^k \rangle + \langle A_{x^k} v^k, v^k \rangle, \forall k \in N,$$

and since $A_{x^k} v^k, k \in N$ is bounded we obtain

$$\liminf_{k \to +\infty} \langle A_{x^k} v^k, v \rangle = \liminf_{k \to +\infty} \langle A_{x^k} v^k, v^k \rangle \geq 0. \tag{1.2.4}$$

From (1.2.3) and (1.2.4), we deduce that $\lim_{k \to +\infty} \langle A_{x^k} v^k, v \rangle = \liminf_{k \to +\infty} \langle A_{x^k} v^k, v^k \rangle = 0$, and thus by $(r2)$, $\lim_{k \to +\infty} |A_{x^k} v^k|^2 = 0$. ■

Remark 1.2.2 For simplicity, assume that f is convex, with $\arg\min_{\overline{C}} f \neq \emptyset$ and that A satisfies $(r4)$. Subdifferential calculus, (see e.g., [112]) allows to associate to $(\mathcal{P})$ the following variational characterization

$$x^* \text{ solves } (\mathcal{P}) \text{ iff } \nabla f(x^*) + N_{\overline{C}}(x^*) = 0.$$

Using (1.2.2), we know that the solutions of $(\mathcal{P})$ are contained in the set of zeros of the gradient-like map $x \in \overline{C} \to A_x \nabla f(x)$. This is only a necessary condition for optimality and it can be written,

$$\text{if } x^* \text{ solves } (\mathcal{P}) \text{ then } A_{x^*} \nabla f(x^*) = 0. \tag{1.2.5}$$

The important point here, is to realize that our approach to optimization is given *throughout $(A - DM)$ dynamics* and thus x^* is obtained as a limit point of some descent method. Indeed, as we shall see, most of the systems and examples of Section 1.4 satisfy (1.2.2) with a *strict* inclusion, yet their orbits converge to a minimizer of f on $\overline{C}$, see Section 1.5.

We conclude these introductory notions by stating a useful criterion implying assumption (v) of Definition 1.2.1.

Lemma 1.2.3 *Let $A : C \times \mathbb{R}^n \to \mathbb{R}^n$, $m > 0$, and $k : C \times \mathbb{R}^n \to [m, +\infty)$ be such that*

$$x - k(x,v)A_x v \in \overline{C},\ \forall (x,v) \in C \times \mathbb{R}^n.$$

Then A satisfies (v).

Proof. It relies on the fact that $x - k(x,v)A_x v - b \in T_{\overline{C}}(b)$ for every (x,v) in $C \times \mathbb{R}^n$ and for every $b \in \overline{C}$. By definition we have for all $\nu \in N_{\overline{C}}(b)$, $\langle x - k(x,v)A_x v - b, \nu\rangle \leq 0$, and therefore

$$\langle -A_x v, \nu\rangle \leq \frac{1}{k(x,v)}\langle b - x, \nu\rangle \leq \frac{1}{m}\langle b - x, \nu\rangle. \blacksquare$$

1.2.2 Global existence and viability results.

From now on, the function $f : \mathbb{R}^n \to \mathbb{R}$ is C^1 and satisfies
$(\mathcal{H}_1)$ ∇f is Lipschitz continuous on bounded sets,
$(\mathcal{H}_2)$ $\inf_{\overline{C}} f > -\infty$.
Observe that for the moment the function f is not supposed to be convex.

Theorem 1.2.4 *Let A be an elliptic barrier operator. Then,*
(i) The system $(A - DM)$ admits a unique C^1 solution x defined on $[0, +\infty)$. Moreover,
(ii) $\forall t \geq 0$, $x(t) \in C$.
(iii) The function $t \in [0, +\infty) \to f(x(t))$ is nonincreasing and has a limit as $t \to +\infty$,
(iv) $\dot{x} \in L^2(0, +\infty; \mathbb{R}^n)$.
(v) If A satisfies $(r4)$ and $x(\cdot)$ is bounded then $\dot{x}(t) \to 0$ as $t \to +\infty$, and all limit point x^* of $x(\cdot)$ satisfies the weak optimality condition

$$A_{x^*}\nabla f(x^*) = 0.$$

Proof. Fix $T > 0$ and consider the assertion $E(T)$:
"There exists a solution of $(A - DM)$ defined on $[0, T]$, and such that $x(t) \in C$ for all $t \in [0, T]$."
Set $T_{max} := \sup\{T \mid E(T) \text{ is satisfied}\}$. From $(r1)$, $(\mathcal{H}_1)$ and the fact that $x(0) \in C$, it follows by Cauchy-Lipschtiz Theorem that $T_{max} > 0$ and that the solution of $(A - DM)$ defined on $[0, T_{max})$ is unique.
Let us derive some a priori estimates. Let $T \in (0, T_{max})$, by the $(A - DM)$ system we have for all $t \in [0, T]$

$$\langle \dot{x}(t), \nabla f(x(t))\rangle + \langle A_{x(t)}\nabla f(x(t)), \nabla f(x(t))\ \rangle = 0,$$

and thus by $(r2)$ and $(A - DM)$ again,

$$\frac{d}{dt}f(x(t)) + \alpha|\dot{x}(t)|^2 \leq 0. \tag{1.2.6}$$

Integrating over some interval $(0, t)$, with $t \leq T$ this gives

$$f(x(t)) - f(x(0)) + \alpha\int_0^t |\dot{x}|^2 \leq 0. \tag{1.2.7}$$

Note that if $T_{max} = +\infty$, (iii) and (iv) follow from (1.2.6), (1.2.7) and $(\mathcal{H}_2)$. Let us argue by contradiction and assume that $T_{max} < +\infty$.
Using Cauchy Schwarz inequality and the fact that $\dot{x} \in L^2(0, T_{max}; \mathbb{R}^n)$, we obtain that x is a Cauchy net at T_{max}. Therefore x can be continuously extended by an application still denoted by x. Set $x(T_{max}) := b \in \overline{C}$.
By definition of T_{max}, b necessarily belongs to bd$C$. The function $t \in [0, T_{max}] \to \nabla f(x(t))$ is bounded by a positive constant M. Owing to the continuity of x and (v), there exists $t_0 \in (0, T_{max})$, $\epsilon > 0$, $K > 0$ and $\nu \in N_{\overline{C}}(b)$, $\nu \neq 0$, such that for all $t \in (t_0, T_{max})$

$$\langle -A_{x(t)} \nabla f(x(t)), \nu \rangle \leq K \langle b - x(t), \nu \rangle. \tag{1.2.8}$$

Let us project $(A-DM)$ on $\mathbb{R}\nu := \{\tau \nu \,|\tau \in \mathbb{R}\}$, this gives for all $t \in (t_0, T_{max})$

$$\frac{d}{dt} \langle x(t), -\nu \rangle + \langle A_{x(t)} \nabla f(x(t)), -\nu \rangle = 0,$$

and using (1.2.8) we obtain

$$\frac{d}{dt} \langle b - x(t), \nu \rangle + K \langle b - x(t), \nu \rangle \geq 0.$$

Multiplying the above inequality by $\exp Kt$ and integrating over (t_0, T_{max}) it follows that

$$\langle b - x(T_{max}), \nu \rangle \geq \exp[K(T_{max} - t_0)] \langle b - x(t_0), \nu \rangle.$$

Observe that by definition, $b = x(T_{max})$, hence to draw a contradiction from the latter we just have to prove that the second term of the inequality is positive. Indeed, $x(t_0) \in C$ which is open convex and $0 \neq \nu \in N_{\overline{C}}(b)$, thus there exists $\eta > 0$ such that $x(t_0) + \eta\nu \in C$, and a fortiori $x(t_0) + \eta\nu - b \in T_{\overline{C}}(b)$. This implies $\langle x(t_0) + \eta\nu - b, \nu \rangle \leq 0$ or equivalently, $\langle b - x(t_0), \nu \rangle \geq \eta |\nu|^2 > 0$, and (i) is proved.
Let us prove the last statement (v). From the boundedness property of x, along with $(r4)$ and $(\mathcal{H}_1)$, it follows that $\dot{x}$ is bounded and therefore x is a Lipschitz continuous map. The properties $(r4)$, $(\mathcal{H}_1)$ imply that $t \geq 0 \to A_{x(t)} \nabla f(x(t))$ is uniformly continuous and therefore so is $\dot{x}(\cdot)$. Combining this fact with (iv), it follows by a classical argument that $\dot{x}(t) \to 0$ as $t \to +\infty$. Using $(r4)$, it ensues that a cluster point x^* of x satisfies $A_{x^*} \nabla f(x^*) = 0$.■

1.3 A general abstract framework for dynamical systems with elliptic barrier operators

In this section, we propose with the help of proximal maps, a systematic and unifying way to generate elliptic barrier operators. We start with an informal motivation. Given a convenient distance-like function $d : \mathbb{R}^n \times C \mapsto \mathbb{R} \cup \{+\infty\}$, the idea is to realize the descent direction $-A_x \nabla f(x)$, $x \in C$ as a vector based on x and pointing on some proximal point $u^d(x, \nabla f(x))$.
Indeed, assume that d is convex with respect to its first variable, and for $x \in C$ define formally

$$u^d(x, \nabla f(x)) \in \arg\min \{\langle u, \nabla f(x) \rangle + d(u, x) | u \in \mathbb{R}^n\}. \tag{1.3.1}$$

In this definition the objective function has been replaced by its first order approximation at the point x, the constraints are supposed to be naturally taken into account by $d(\cdot,\cdot)$ and the descent direction obtained is $-A^d_x\nabla f(x) := u^d(x,\nabla f(x)) - x$. It is of interest to notice that this approach is akin to the following well known fixed point reformulation of the optimization problem $(\mathcal{P})$:

$$x^* \text{ solves } (\mathcal{P}) \text{ iff } x^* \in \arg\min \{\langle u, \nabla f(x^*)\rangle \mid u \in \overline{C}\}, \tag{1.3.2}$$

whenever f is convex. From that viewpoint, the formal definition (1.3.1), may appear as a proximal regularization of some possibly ill-posed problem. On the other hand, the corresponding A^d-driven descent method can be written as a fixed point like dynamics

$$\dot{x}(t) + x(t) = u^d[x(t), \nabla f(x(t))],\ x(0) \in C,\ \forall t \geq 0. \tag{1.3.3}$$

The solution of (1.3.3) is then expected to provide asymptotically a solution of $x^* = u^d(x^*, \nabla f(x^*))$, and when it makes sense, this last problem corresponds to another formulation of (1.3.2).

As a first example, consider $d(u,x) = 1/2|u-x|^2 + \delta_{\overline{C}}(u)$, $(u,x) \in \mathbb{R}^n \times C$. The definition of u^d writes

$$\nabla f(x) + u^d(x,\nabla f(x)) - x + N_{\overline{C}}[u^d(x,\nabla f(x))] \ni 0,$$

which in turns is equivalent to

$$u^d(x,\nabla f(x)) \in (I + N_{\overline{C}})^{-1}(x - \nabla f(x)).$$

Recalling that $(I+N_{\overline{C}})^{-1} = P_{\overline{C}}$, the proximal point is thus given by $u^d(x,\nabla f(x)) = P_{\overline{C}}(x - \nabla f(x))$. This gives rise to the descent direction $-A^d_x\nabla f(x) = P_{\overline{C}}(x - \nabla f(x)) - x$, and the projected-gradient dynamics (1.1.6) is recovered. As mentioned in the above discussion note that the reformulation of (1.3.2) throughout $d(\cdot,\cdot)$, that is $x^* = u^d(x^*, \nabla f(x^*))$, leads to the fixed point problem $x^* = P_{\overline{C}}(x^* - \nabla f(x^*))$.

Let us now develop an abstract setting that shall be illustrated in the next section with various useful kernels $d(\cdot,\cdot)$.

Let $d_0 : \mathbb{R}^n \times C \mapsto \mathbb{R}_+ \cup \{+\infty\}$ be such that
$(P1)$ d_0 is C^1 on $C \times C$,
$(P2)$ $\nabla_1 d_0(u,u) = 0$ for all $u \in C$,
$(P3)$ For every $x \in C$, the mapping $u \in \mathbb{R}^n \mapsto d_0(u,x)$ is a closed convex function.

In $(P1)$, $\nabla_1 d_0(\cdot,u)$ is the gradient of $d(\cdot,u)$; (more generally its subdifferential is denoted by $\partial_1 d_0(\cdot,u)$). Note that, since C is nonempty, $(P1)$ ensures that $u \in \mathbb{R}^n \mapsto d_0(u,x)$ is also proper.

Denote by $\mathcal{D}$ the set of mappings $d : \mathbb{R}^n \times C \mapsto \mathbb{R}_+ \cup \{+\infty\}$ that can be written

$$d(u,x) = \frac{\alpha}{2}|u-x|^2 + d_0(u,x), \tag{1.3.4}$$

with $\alpha > 0$ and with d_0 satisfying $(P1)$, $(P2)$ and $(P3)$.

Definition 1.3.1 *Let d be in $\mathcal{D}$. For all $(x,v) \in C \times \mathbb{R}^n$ set*

$$u^d(x,v) \in \arg\min \{\langle u, v\rangle + d(u,x) | u \in \mathbb{R}^n\} \tag{1.3.5}$$

and define A^d by

$$A^d_x v = x - u^d(x,v). \tag{1.3.6}$$

The following proposition justifies the second part (1.3.6) of this definition (u^d could be multivalued), and describes some of the properties of the operator A^d.

Proposition 1.3.1 *Let $d \in \mathcal{D}$.*
(i) For each $x \in C$, the map $v \in \mathbb{R}^n \mapsto u^d(x,v)$ is a single valued α^{-1}–Lipschitz continuous map.
(ii) A^d satisfies (r2), (r3), and for each $x \in C$, $v \in \mathbb{R}^n \mapsto A^d_x v$ is Lipschitz continuous.
(iii) Moreover if d satifies the property

$$(p) \quad \forall x \in C, \ \text{dom } d(\cdot\,, x) \subset \overline{C}$$

then A^d satisfies (v) of Definition 1.2.1.

Proof. Let $(x,v) \in C \times \mathbb{R}^n$. From $(P3)$ and the fact that $\alpha > 0$ it follows that $u \in \mathbb{R}^n \mapsto \langle u, v\rangle + d(u,x)$ is strongly convex and has a nonempty bounded lower level set. This implies that $u^d(x,v)$ exists and is unique. Using $(P1)$ and $(P3)$, allows to write the optimality condition in (1.3.5) as

$$v + \partial_1 d(\cdot, x)(u^d(x,v)) \ni 0,$$

and therefore by uniqueness of $u^d(x,v)$, (recalling (cf. [112]) that for any closed proper convex function F, one has $(\partial F)^{-1} = \partial F^*$), it follows that

$$u^d(x,v) = \partial_1 d^*(\cdot, x)(-v). \tag{1.3.7}$$

Denoting by I the identity map of $\mathbb{R}^n$, we observe using the definition of $d \in \mathcal{D}$ that $\partial_1 d^*(\cdot, x)$ can also be written

$$(\alpha I + \partial_1 d_0(\cdot, x) - \alpha x)^{-1}$$

or equivalently as the composition,

$$(I + \alpha^{-1}\partial_1 d_0(\cdot\,, x) - x)^{-1} \circ \alpha^{-1} I.$$

By $(P3)$, the operator $\alpha^{-1}\partial_1 d_0(\cdot\,, x) - x$ is maximal monotone and therefore by [37, Proposition 2.2], $(I + \frac{1}{\alpha}\partial_1 d_0(\cdot\,, x) - x)^{-1}$ is a contraction defined on $\mathbb{R}^n$. Recalling that $u^d(x,v) = (I + \alpha^{-1}\partial_1 d_0(\cdot\,, x) - x)^{-1} \circ \alpha^{-1} I$ and $A^d_x v = x - u^d(x,v)$, the above arguments prove (i) and the second part of statement (ii).
Assume that d complies with the property (p). By definition of u^d, this implies that $u^d(x,v) = x - A^d_x v \in \overline{C}$ and therefore (iii) is a consequence of Lemma 2.1. It remains to prove the first two assertions of (ii). Let us prove that A^d satisfies $(r3)$. Let $(x,v) \in C \times \mathbb{R}^n$, be such that $A_x v = 0$. Then by (1.3.7), $x = \partial_1 d^*(\cdot\,, x)(-v)$, which implies that $\partial_1 d(x,x) = \nabla_1 d(x,x) = -v$. Therefore, by $(P2)$ one has $v = 0$. Now to prove that $(r2)$ is also satisfied, we use the following

Lemma 1.3.1 *(Baillon-Haddad [26])*
Let H, $\langle , \rangle$ be a Hilbert space whose norm is denoted $|\,.\,|$, $\phi : H \mapsto \mathbb{R}$ a C^1 convex function and $L > 0$. The following statements are equivalent,
(i) $\forall (x,y) \in H^2$, $|\nabla\phi(x) - \nabla\phi(y)| \leq L|x-y|$
(ii) $\forall (x,y) \in H^2$, $\langle \nabla\phi(x) - \nabla\phi(y), x-y\rangle \geq \frac{1}{L}|\nabla\phi(x) - \nabla\phi(y)|^2$.

In view of (1.3.7) and (i), this result can be applied to $\phi := d^*(\cdot\,, x)$. Hence, for x fixed in C, and for all $(v_1, v_2) \in \mathbb{R}^n \times \mathbb{R}^n$ it gives

$$\langle \partial_1 d^*(\cdot\,,x)(v_1) - \partial_1 d^*(\cdot\,,x)(v_2), v_1 - v_2\rangle \geq \alpha|\partial_1 d^*(\cdot\,,x)(v_1) - \partial_1 d^*(\cdot\,,x)(v_2)|^2.$$

Now, letting $v_1 = 0$, and $v_2 = -v$ in the latter yields

$$\langle x - u^d(x,v), v\rangle \geq \alpha|x - u^d(x,v)|^2,$$

which, according to (1.3.6), is exactly $(r2)$. ■

1.4 Elliptic barrier operators and continuous models for proximal algorithms : Examples and Properties

In this section we show that for various minimization algorithms one can derive an elliptic barrier operator and construct the associated $(A - DM)$-dynamical system. It is worthwhile mentioning that many of the examples to follow will generate convergent trajectories to the minimizer of a convex function f over the closed convex set $\overline{C}$. From now on α will always denote the positive parameter involved in the definition of the class $\mathcal{D}$, cf. (1.3.4).

1.4.1 Projection-like methods

Let $h_0 : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 convex function whose gradient is Lipschitz continuous on bounded sets, and set

$$\tilde{D}_h : \begin{cases} \mathbb{R}^n \times C & \to \quad \mathbb{R}_+ \cup \{+\infty\} \\ (u,x) & \mapsto \quad D_h(u,x) + \delta_{\overline{C}}(u). \end{cases}$$

with $h(u) = \frac{\alpha}{2}|u|^2 + h_0(u)$, $u \in \mathbb{R}^n$ and where D_h is given by (cf. (1.1.7)) :

$$\forall (x,y) \in \mathbb{R}^n \times C,\;\; D_h(x,y) = h(x) - h(y) - \langle \nabla h(y), x-y\rangle. \tag{1.4.1}$$

Proposition 1.4.1 *Let $\tilde{D}_h$ as defined above. Then $A^{\tilde{D}_h}$ is an elliptic barrier operator that satisfies (r4). Moreover, we have for all $(x,v) \in C \times \mathbb{R}^n$,*

$$A_x^{\tilde{D}_h} v = x - (\nabla h + N_{\overline{C}})^{-1}(\nabla h(x) - v). \tag{1.4.2}$$

Proof. An easy computation gives $\tilde{D}_h(u,x) = \frac{\alpha}{2}|u-x|^2 + D_{h_0}(u,x) + \delta_{\overline{C}}(u)$. Letting $d_0(u,x) = D_{h_0}(u,x)+\delta_{\overline{C}}(u)$, we obtain that d_0 satisfies $(P1)$ and $(P3)$. For $(u,x) \in C\times C$, we have $\nabla_1 d_0(u,x) = \nabla h_0(u) - \nabla h_0(x)$, and as a consequence $(P2)$ is satisfied as well. Therefore $\tilde{D}_h$ is in $\mathcal{D}$, and clearly verifies (p). Now applying Proposition 1.3.1, it follows that $A^{\tilde{D}_h}$ satisfies $(r2)$,$(r3)$ and (v). The explicit formula of $A^{\tilde{D}_h}$ follows from (1.3.7). To obtain $(r1)$ and $(r4)$, we just have to observe that $(\nabla h + N_{\overline{C}})^{-1}$ and ∇h_0 are locally Lipschitz continuous on $\mathbb{R}^n$. ∎

The terminology of projection relies on the fact that (1.4.2) can be seen as some twisted projection in the Bregman sense. Indeed, defining the projection of $z \in \mathbb{R}^n$ on $\overline{C}$ by

$$P^h_{\overline{C}}(z) := \arg\min \{D_h(u,z) \mid u \in \overline{C}\},$$

we obtain that $P^h_{\overline{C}}(z) = (\nabla h + N_{\overline{C}})^{-1}(\nabla h(z))$ (recall that $\alpha > 0$) and therefore since $\nabla h^* = (\nabla h)^{-1}$, one can write

$$A^{\tilde{D}_h}_x v = x - P^h_{\overline{C}}(\nabla h^*(\nabla h(x) - v)),\ \forall (x,v) \in C \times \mathbb{R}^n.$$

It is worthwhile noticing that in the framework of convex minimization, the gradient-like map $x \mapsto A^{\tilde{D}_h}_x \nabla f(x)$ enjoys remarkable properties. As a matter of fact, assume that the objective function f is convex, and observe that the following characterization holds

$$x^* \text{ solves } (\mathcal{P}) \text{ iff } A^{\tilde{D}_h}_{x^*}\, \nabla f(x^*) = 0.$$

The associated $A^{\tilde{D}_h}$-driven descent method leads to the following differential equation

$$\dot{x}(t) + x(t) - P^h_{\overline{C}}(\nabla h^*[\nabla h(x(t)) - \nabla f(x(t))]) = 0,\ x(0) \in C,\ \forall t \geq 0. \tag{1.4.3}$$

Note that with $h_0 = 0$ and $\alpha = 1$, the corresponding dynamical system $(A^{\tilde{D}_h} - DM)$ (with corresponding operator A^P) is nothing else but the continuous gradient projection method (1.1.6), that is

$$\dot{x}(t) + x(t) - P_{\overline{C}}[x(t) - \nabla f(x(t))] = 0,\ x(0) \in C,\ \forall t \geq 0.$$

We remark that if $x(0) \notin C$ we still obtain convergent trajectories (with f convex), see [10] or Bolte [34], but the dynamical system is neither a descent, nor an interior method.

1.4.2 Continuous models for Bregman proximal minimization algorithms.

In this section, we give two quite different continuous models associated with proximal methods based on Bregman distances.

Continuous model I : A Riemannian gradient method

Our model appears as a particular case of Riemannian gradient methods on the smooth manifold C. Let us make precise the setting. Denote by $S^{++}_n(\mathbb{R})$ the cone of real definite positive symmetric matrices and let $\mathcal{T}_x C$ be the tangent space to C at $x \in C$. In the sequel we make the usual identification $\mathcal{T}_x C \simeq \mathbb{R}^n$ for all $x \in C$. If g is some differentiable metric on C, there exists a unique differentiable application $\lambda : C \to S_n^{++}(\mathbb{R})$ such that for all $(x, u, v) \in C \times \mathbb{R}^n \times \mathbb{R}^n$

$$g_x(u, v) = \langle \lambda(x) u, v \rangle.$$

The gradient of a smooth function ϕ with respect to the metric g, is then given by the formula $\nabla_g \phi(x) = \lambda(x)^{-1} \nabla \phi(x)$, $\forall x \in C$, and the corresponding gradient method is

$$\begin{cases} \dot{x}(t) + \nabla_g \phi(x(t)) = 0, \\ x(0) \in C. \end{cases} \tag{1.4.4}$$

For $C = \mathbb{R}^n$, and ϕ real analytic, a deep result of Lojasiewicz [92] allows to prove that all bounded trajectories are converging to a critical point of ϕ.
Readers interested in the use of geometric tools in optimization are referred to Bayer-Lagarias [30] in the context of Linear Programming and for more general results to the recent monograph of Helmke-Moore [73] and references therein.

We focus here on the special choice of the application $\lambda : C \to S_n^{++}(\mathbb{R})$ defined by $\lambda = \nabla^2 h$ where h is some C^3 Bregman function with zone C, see Definition 1.4.1, below. The idea is to penalize the Euclidean scalar product, rather than the objective function, and to study the corresponding Riemannian gradient method

$$\dot{x}(t) + \nabla^2 h(x(t))^{-1} \nabla f(x(t)) = 0, \tag{1.4.5}$$

or *equivalently*

$$\frac{d}{dt} \nabla h(x(t)) + \nabla f(x(t)) = 0. \tag{1.4.6}$$

When the objective function is linear this differential equation has been considered in Iusem-Svaiter-Da Cruz [78], however their approach to the asymptotic behaviour strongly relies on the linear properties of f, see Remark 1.5.7 (b) for an insight. Observe that this dynamics has, in its first form (1.4.5), the structure of A-driven descent methods. We shall see actually that most of classical Bregman functions can generate a barrier operator. Moreover, as shown below, the general framework developed in Section 1.3 allows to recover those methods by considering families of quadratic forms.

For the moment, let us compare (1.4.6) with (BPM) as given in the introduction. By an Euler implicit discretization we formally obtain

$$\frac{1}{\Delta t_k} [\nabla h(x^{k+1}) - \nabla h(x^k)] + \nabla f(x^{k+1}) = 0, \ \Delta t_k > 0. \tag{1.4.7}$$

Now observe that (BPM) has exactly the form of (1.4.7), provided that the iterates remain in C [44, 45, 59].
Before going further, we need to recall some of the basic facts concerning Bregman functions. Their definition relies mainly on their D function, as specified in (1.1.7),

Definition 1.4.1 *A function $h : \overline{C} \to \mathbb{R}$ is called a Bregman function with zone C if it satisfies the following :*

(i) h is C^1 on C.

(ii) h is continuous and strictly convex on $\overline{C}$.

(iii) For every $r \in \mathbb{R}$, the partial level subset $L_h(x_0, r) = \{y \in C | D_h(x_0, y) \leq r\}$ is bounded for every $x_0 \in \overline{C}$.

(iv) Let $(y^k, k \in N)$ be a sequence in C and $x \in \overline{C}$. If $y^k \to x$ as $k \to +\infty$, then $D_h(x, y^k) \to 0$ as $k \to +\infty$.

This definition weakens the usual definition of Bregman function proposed by Censor and Lent in [43], and is actually inspired by the more general notion of B function introduced by Kiwiel in [85]. Because of (iv) and the smoothness property of h, we have kept the terminology of Bregman function.

For the asymptotic analysis of (1.4.6) which will be developed in Section 1.5, we already record here the following useful lemma due to Kiwiel ([85, Lemma 2.16]).

Lemma 1.4.1 *Let h be a Bregman function with zone C and $x \in \overline{C}$. If y^k, $k \in N$ is a bounded sequence in C such that $D_h(x, y^k) \to 0$ as $k \to +\infty$ then $y^k \to x$ as $k \to +\infty$.*

In relation with the barrier operators to follow, let us define now a subclass of Bregman functions with zone C.

For $h : \overline{C} \to \mathbb{R}$, we consider the following assumptions :

(r_h) There exist $\alpha > 0$ and a C^3 Bregman function with zone C denoted by h_0, such that for all $x \in \overline{C}$

$$h(x) = \frac{\alpha}{2}|x|^2 + h_0(x).$$

(v_h) For every $b \in \operatorname{bd} C$ and every $\nu \in N_{\overline{C}}(b)$ there exists $K, \epsilon > 0$ such that for every $x \in C$, $|x - b| < \epsilon$,

$$|\nabla^2 h(x)^{-1}\nu| \leq K\langle b - x, \nu\rangle.$$

The set of such functions is denoted by $\mathcal{B}_C$, and for each $h \in \mathcal{B}_C$ we define a family of quadratic forms by

$$q_h : \left\{ \begin{array}{lll} \mathbb{R}^n \times C & \to & \mathbb{R}^n \\ (u, x) & \mapsto & \langle \nabla^2 h(x)(u - x), u - x\rangle. \end{array} \right.$$

Proposition 1.4.2 *For every $h \in \mathcal{B}_C$, A^{q_h} is an elliptic barrier operator on C. Moreover, for all $(x, v) \in C \times \mathbb{R}^n$ the following formula holds*

$$A_x^{q_h} v = \nabla^2 h(x)^{-1} v. \tag{1.4.8}$$

Proof. To prove that $q_h \in \mathcal{D}$, it suffices to notice that by (r_h),

$$q_h(u, x) = \alpha/2|u - x|^2 + \langle \nabla^2 h_0(x)(u - x), u - x\rangle,$$

where $\langle \nabla^2 h_0(x)(u - x), u - x\rangle$ satisfies $(P1)$,$(P2)$,$(P3)$. This implies by Proposition 1.3.1, that the operator A^{q_h} satisfies $(r2)$, $(r3)$. Note that q_h never satisfies the property (p), which precludes the use of Proposition 1.3.1 (iii).

Applying Definition 1.3.1, formula (1.4.8) can be derived easily from,

$$\nabla^2 h(x)[u^{q_h}(x,v) - x] + v = 0, \ \forall (x,v) \in C \times \mathbb{R}^n.$$

Since the mapping $M \in S_n^{++}(\mathbb{R}) \to M^{-1}$ is C^∞, we obtain by (r_h) that A^{q_h} satisfies $(r1)$. Let us prove that A^{q_h} complies with (v) of Definition 1.2.1. Take $b \in \mathrm{bd}\, C$ and ν in $N_{\overline{C}}(b)$, and let us apply (v_h). There exist $K, \epsilon > 0$ such that for every $v \in \mathbb{R}^n$, $x \in C$, $|x - b| < \epsilon$,

$$\langle -A_x^h v, \nu\rangle = -\langle \nabla^2 h(x)^{-1} v, \nu\rangle = -\langle v, \nabla^2 h(x)^{-1}\nu\rangle \le K|v|\langle b - x, \nu\rangle.$$

Therefore, if v is bounded, the latter exactly amounts to (v). ■

The next lemma gives a practical means to prove that a Bregman function is in the class $\mathcal{B}_C$.

For $a < b$ in $\overline{\mathbb{R}}$, $\varphi : (a,b) \to \mathbb{R}$ a C^2 Bregman function with zone (a,b), consider the assumptions,

(v_l) If a is finite, there exist a neighborhood U of a in $\mathbb{R}$ and a positive constant K_l such that

$$\forall u \in U \cap (a,b) \quad \varphi''(u) \ge K_l/(u-a),$$

(v_r) If b is finite, there exist a neighborhood V of b in $\mathbb{R}$ and a positive constant K_r such that

$$\forall u \in V \cap (a,b) \quad \varphi''(u) \ge K_r/(b-u).$$

Lemma 1.4.2 *Let φ_1, ..., φ_n be some C^3 Bregman functions on $\mathbb{R}$ with zones (a_1, c_1), ..., (a_n, c_n), $a_i < c_i$, $a_i, c_i \in \overline{\mathbb{R}}$, $\forall i \in \{1\dots,n\}$. Assume that φ_1,..., φ_n satisfy (v_l), (v_r) on their respective zones, and for $\alpha > 0$ set,*

$$h(x) = \frac{\alpha}{2}|x|^2 + \sum_{i=1}^{n} \varphi_i(x_i).$$

Then h belongs to $\mathcal{B}_K$, where $K = \prod_{i=1}^n (a_i,c_i)$, and A^{q_h} is an elliptic barrier operator that satisfies $(r4)$.

Proof. The fact that h is a C^3 Bregman function with zone K follows from [85, Lemma 2.8,(d)], and therefore (r_h) is satisfied.

To simplify the notations, let us assume that for all $i \in \{1,\dots,n\}$, $a_i = 0$ and $c_i = +\infty$ (which implies $K = \mathbb{R}^n_+$). For $b = (b_1,\dots,b_n) \in \mathrm{bd}\,\mathbb{R}^n_+$, set $I(b) = \{i \in \{1,\dots,n\} | b_i = 0\} \neq \emptyset$ and $J(b) = \{i \in \{1,\dots,n\} | b_i \neq 0\}$. For each $i \in I(b)$, (v_l) yields the existence of a neighborhood U_i of 0 in $\mathbb{R}$ and $K_i > 0$ such that

$$\forall u \in U_i \cap (0,+\infty) \quad \varphi''(u) \ge K_i/u. \tag{1.4.9}$$

Set $U_i = \mathbb{R}^n$ for each $i \in J(b)$, and $U = \mathbb{R}^n_{++} \cap \prod_{i=1..n} U_i$. Let $\nu \in N_{\overline{K}}(b)$, and observe that $\nu_i = 0$ for all $i \in J(b)$ and that $\nu_i < 0$ for all $i \in I(b)$. Therefore, for $x \in \mathbb{R}^n$ an easy computation gives

$$|\nabla^2 h(x)^{-1}\nu| \le \sum_{i \in I(b)} \frac{-v_i}{|\alpha + \varphi_i''(x_i)|}.$$

Now if $x \in U$, (1.4.9) implies that

$$\begin{aligned} |\nabla^2 h(x)^{-1}\nu| &\leq \sum_{i\in I(b)} -\frac{1}{K_i}\nu_i.x_i \\ &\leq \sup_{i\in I(b)} \frac{1}{K_i}\langle b-x,\nu\rangle. \end{aligned}$$

A direct computation gives for all $x \in K$, $i,j \in \{1,\dots,n\}$,

$$\left(\nabla^2 h(x)^{-1}\right)_{i,j} = \frac{\delta_{ij}}{\alpha + \varphi_i''(x_i)},$$

where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. Applying again (v_l), we see that A^{q_h} can be continuously extended on $\overline{K}$. Hence A^{q_h} satisfies $(r4)$.■

Example 1.4.1 *Bregman-based Barrier operators and their dynamics.*
The list of examples below shows thanks to Lemma 4.2 that many classical Bregman functions can be used to provide an elliptic barrier operator. In what follows α is the positive regularizing term as defined in (v_h), and β is a positive parameter. For a Bregman function h with zone $I \subset \mathbb{R}$, set $h_n(x) = \sum_{i=1}^n h(x_i)$ for all $x \in I^n$.

(a) For $\theta \in (0,1)$ consider $h(s) = \frac{\alpha}{2}s^2 - \beta\frac{s^\theta}{\theta}$, $s \in \mathbb{R}_+$. Then $h \in \mathcal{B}_{\mathbb{R}_{++}}$, $h_n \in \mathcal{B}_{\mathbb{R}^n_{++}}$ and the corresponding $(A^{q_{h_n}} - DM)$ system is

$$\dot{x}_i(t) + \frac{x_i(t)^{2-\theta}}{\alpha x_i(t)^{2-\theta} + \beta(1-\theta)}\frac{\partial f}{\partial x_i}(x(t)) = 0, \quad x_i(0) > 0,\ \forall i \in \{1,\dots,n\}. \qquad (1.4.10)$$

(b) $h(s) = \frac{\alpha}{2}s^2 + \beta s\log s$ on $\mathbb{R}_+$ is in $\mathcal{B}_{\mathbb{R}_{++}}$, $h_n \in \mathcal{B}_{\mathbb{R}^n_{++}}$ and the associated system is

$$\dot{x}_i(t) + \frac{x_i(t)}{\alpha x_i(t) + \beta}\frac{\partial f}{\partial x_i}(x(t)) = 0, \quad x_i(0) > 0,\ \forall i \in \{1,\dots,n\}.$$

This system is exactly the regularized Lotka-Volterra equation (1.1.8) recently proposed in [20]. However, it is worthwhile noticing, that (1.1.8) was introduced there as a continuous model not based on (BPM), but on the proximal-like method,

$$x^{k+1} \in \arg\min\left\{f(x) + c_k d_\varphi(x, x^k) | x \in \mathbb{R}^n_+\right\},\ c_k > 0,$$

where $\varphi(s) = s - \log s - 1$ and $d_\varphi(x,y) = \frac{\alpha}{2}|x-y|^2 + \beta\sum_{i=1}^n y_i\varphi(y_i^{-1}x_i)$ for all x, y in $\mathbb{R}^n_{++}$. For more results and applications on classical Lotka-Volterra systems see, e.g., Hofbauer-Sigmund [76].
(c) $h(s) = \frac{\alpha}{2}s^2 - \beta\sqrt{1-s^2}$ on $[-1,1]$ is in $\mathcal{B}_{(-1,1)}$, $h_n \in \mathcal{B}_{(-1,1)^n}$ and the corresponding system is

$$\dot{x}_i(t) + \frac{\left(1 - x_i(t)^2\right)^{3/2}}{\alpha\left(1 - x_i(t)^2\right)^{3/2} + \beta}\frac{\partial f}{\partial x_i}(x(t)) = 0, \quad x_i(0) \in (-1,1),\ \forall i \in \{1,\dots,n\}.$$

(d) $h(s) = \frac{\alpha}{2}s^2 - \beta\sqrt{s(1-s)}$ on $[0,1]$ is in $\mathcal{B}_{(0,1)}$, $h_n \in \mathcal{B}_{(0,1)^n}$ and the corresponding system is

$$\dot{x}_i(t) + \frac{4x_i(t)^{3/2}(1-x_i(t))^{3/2}}{4\alpha x_i(t)^{3/2}(1-x_i(t))^{3/2}+\beta}\frac{\partial f}{\partial x_i}(x(t)) = 0, \quad x_i(0) \in (0,1),\ \forall i \in \{1,\dots,n\}.$$

Remark 1.4.3 For $\epsilon, \gamma \geq 0$, and $f \in C^3(\mathbb{R}^n, \mathbb{R})$ set $h_{\epsilon,\gamma}(x) = \frac{\epsilon}{2}|x|^2 + \gamma f(x)$, $\forall x \in \mathbb{R}^n$. Then we have $h_{\epsilon,\gamma} \in \mathcal{B}_{\mathbb{R}^n}$, under one of the following assumptions :
($\star$) f is strongly convex, i.e., $\nabla^2 f - \lambda I$ is positive semi-definite, with $\lambda > 0$.
($\star$) f is convex and $\epsilon > 0$
($\star$) $\gamma = 0$, $\epsilon > 0$.

Letting $\epsilon = 0, \gamma = 1$ in the first case yields the continuous Newton descent method (1.1.9). The second version can be seen, for ϵ small, as a regularized Newton method

$$(A^{q_{h_{\epsilon,\gamma}}} - DM) \qquad \dot{x}(t) + [\epsilon I + \gamma\nabla^2 f(x(t))]^{-1}\nabla f(x(t)) = 0.$$

The last point with $\gamma = 0$, $\epsilon > 0$ gives rise to the classical steepest descent method.

In the examples just described, the $A^{q_{h_{\epsilon,\gamma}}}$ are elliptic barrier operators *on* $\mathbb{R}^n$ so that the feasible set $\overline{C}$ is the whole space $\mathbb{R}^n$, and $(v)_h$ holds vacuously. It actually raises another intereresting aspect of barrier operators : they can be used also as a geometrical means to improve convergence rate as well as well-posedness properties. This suggests, for instance, to go further in the study of the following Newton-Barrier methods

$$\dot{x}(t) + [\lambda\nabla^2 h(x(t)) + \mu\nabla^2 f(x(t))]^{-1}\nabla f(x(t)) = 0,\ t \geq 0$$

with $\lambda, \mu > 0$ and where h is a C^3 Bregman function.

Continuous Model II

The Bregman distances appearing in the definition of projection methods (Section 1.4.1), can be used in a quite different way in order to provide some other continuous model of (BPM). Indeed, replacing the kernel h_0 defined on the whole space $\mathbb{R}^n$ by some essentially smooth convex function (see definition below) allows to get rid of the normal cone and to reformulate (1.4.3) as

$$\nabla h(x(t) + \dot{x}(t)) - \nabla h(x(t)) + \nabla f(x(t)) = 0,\ \forall t \geq 0.$$

This can be discretized as follows

$$\nabla h(x_{k+1}) - \nabla h(x_k) + \nabla f(x_{k+1}) = 0,\ \forall k \in N,$$

and (BPM) is recovered with a sequence of stepsizes satisfying $c_k = 1$, $\forall k \in N$.

This model will be derived from our general framework developed in Section 1.3. First, we recall now the definition of essentially smooth convex functions, see [112].

Definition 1.4.2 *A proper convex function* $\phi : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ *is essentially smooth if it satisfies*
(i) the interior of dom ϕ *is nonempty, i.e., int* dom $\phi \neq \emptyset$.
(ii) ϕ *is differentiable on int* dom ϕ.
(iii) For all b in the boundary of int dom ϕ, *and all sequence* x_k, $k \in N$ *in int* dom ϕ *such that* $x_k \to b$ *as* $k \to +\infty$, *we have* $|\nabla \phi(x_k)| \to +\infty$ *as* $k \to +\infty$.

As in subsection 1.4.1 we study now operators of the form A^{D_h} (cf (1.4.1)) for some relevant kernels h.
Let $h_0 : \mathbb{R}^n \mapsto \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function such that,
$(i)_{h_0}$ h_0 is essentially smooth with in addition int dom $h_0 = C$,
$(ii)_{h_0}$ ∇h_0 is Lipschitz continuous on compact subsets of C.
For such a function h_0, we set $h(u) = \alpha/2|u|^2 + h_0(u)$, $\forall u \in \mathbb{R}^n$. In the following proposition, it is important to recall that D_h is an extended real function defined on the whole of $\mathbb{R}^n \times C$.

Proposition 1.4.3 *Let h be as above. Then* A^{D_h} *is an elliptic barrier operator on* C, *and for all* $(x, v) \in C \times \mathbb{R}^n$ *we have*

$$A_x^{D_h} v = x - \nabla h^*(\nabla h(x) - v). \tag{1.4.11}$$

Proof. From $(i)_{h_0}$ it ensues that $D_h \in \mathcal{D}$. Using the fact that h is essentially smooth with *int* dom $h = C$ we deduce that (p) is satisfied. By Proposition 1.3.1, we see that A^{D_h} verifies $(r2)$, $(r3)$, and (v). The formula (1.4.11), follows from (1.3.6), and $(r1)$ from $(ii)_{h_0}$.■

The associated A^{D_h}-driven descent method is thus given by

$$\dot{x}(t) + x(t) - \nabla h^*[\nabla h(x(t)) - \nabla f(x(t))] = 0,\ x(0) \in C,\ \forall t \geq 0, \tag{1.4.12}$$

or using $\nabla h^* = (\nabla h)^{-1}$ equivalently as

$$\nabla h(x(t) + \dot{x}(t)) - \nabla h(x(t)) + \nabla f(x(t)) = 0,\ x(0) \in C,\ \forall t \geq 0.$$

Example 1.4.2 Consider the regularized Burg's entropy obtained with, $g(s) = (\alpha/2)s^2 - \beta \log s$, $s > 0$, where β is a positive parameter. For $x \in \mathbb{R}^n_{++}$ set $h(x) = \sum_{i=1}^n g(x_i)$. The function h satisfies the requirements of Proposition 1.4.3. A direct computation shows that

$$(g^*)'(u) = \frac{u + \sqrt{u^2 + 4\alpha\beta}}{2\alpha},\ \forall u \in \mathbb{R}.$$

Substituting in (1.4.12), the following descent method is derived. For all $i = 1, \ldots, n$,

$$\dot{x}_i(t) + x_i(t)/2 + (2\alpha)^{-1}\left(\beta/x_i(t) + \frac{\partial f}{\partial x_i}(x(t)) - \sqrt{[\alpha x_i(t) - \beta/x_i(t) - \frac{\partial f}{\partial x_i}(x(t))]^2 + 4\alpha\beta}\right) = 0,$$

for all $t \geq 0$ and with $x_i(0) > 0$, $\forall i \in \{1, \ldots, n\}$.
It is interesting to notice that as $\alpha \to 0$ we do not recover here the Lotka-Volterra system; compare with the system given in Example 1.4.1 (b).

1.4.3 A continuous model for proximal algorithms with second order kernels

The class of operators A^{d_φ} defined in this section are built upon the kernels φ used to realize the $(RIPM)$ method introduced in [22], and which we now recall . Let $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a closed proper function whose domain dom φ is a subset of $[0, +\infty)$. Consider the following assumptions on φ

$(i)_\varphi$ φ is finite and C^2 on $(0, +\infty)$,
$(ii)_\varphi$ φ is strictly convex on $(0, +\infty)$,
$(iii)_\varphi$ $\lim_{s>0,\, s\to 0} \varphi'(s) = -\infty$,
$(iv)_\varphi$ $\varphi(1) = \varphi'(1) = 0$ and $\varphi''(1) > 0$,
$(v)_\varphi$ for all $s > 0$, $\varphi''(1)(1 - \frac{1}{s}) \leq \varphi'(s) \leq \varphi''(1)(s - 1)$.

Now for $\alpha, \beta > 0$ set

$$\varphi(s) = \frac{\alpha}{2}(s-1)^2 + \beta\varphi_0(s), \tag{1.4.13}$$

where φ_0 satisfies $(i)_\varphi - (v)_\varphi$, and denote by Φ the class of such functions. For $\varphi \in \Phi$, set

$$\forall (u, x) \in \mathbb{R}^n \times \mathbb{R}^n_{++} \quad d_\varphi(u, x) = \sum_{i=1}^{n} x_i^2 \varphi(x_i^{-1} u_i). \tag{1.4.14}$$

It is proved in [22], that the associated proximal method,

$$(RIPM) \qquad x^{k+1} \in \arg\min \left\{ f(x) + c_k d_\varphi(x, x_k) | x \in \mathbb{R}^n_+ \right\},\ c_k > 0,$$

generates a positive sequence $\{x^k\}$ provided that $x^0 \in \mathbb{R}^n_{++}$. As a consequence an equivalent formulation of $(RIPM)$ is

$$c_k \partial_1 d_\varphi(x^{k+1}, x^k) + \nabla f(x^{k+1}) = 0,\ \forall k \geq 1. \tag{1.4.15}$$

Under the additional assumptions that $\arg\min_{\mathbb{R}^n_+} f \neq \emptyset$, $\sum_{k=1}^{+\infty} c_k = \infty$ and

$$\alpha \geq \beta\varphi_0''(1), \tag{1.4.16}$$

it is proved in [22] that the sequence x^k, $k \in N$ converges to a minimizer of f.

Following the general framework developed in Section 1.3, we generate the elliptic barrier operator and dynamical system associated with $(RIPM)$.

Proposition 1.4.4 *Let $\varphi \in \Phi$. Then A^{d_φ} is an elliptic barrier operator, and one has for all $(x, v) \in \mathbb{R}^n_{++} \times \mathbb{R}^n$,*

$$\left(A_x^{d_\varphi} v\right)_i = x_i - x_i(\varphi^*)'(-x_i^{-1} v_i),\ \forall i = 1, \ldots, n. \tag{1.4.17}$$

Proof. For all $(u, x) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{++}$ we have $d_\varphi(u, x) = \alpha/2|u - x|^2 + \beta d_{\varphi_0}(u, x)$, and therefore to prove that $d_\varphi \in \mathcal{D}$, we need to show that βd_{φ_0} satisfies $(P1)$, $(P2)$, and $(P3)$. $(P1)$ follows from $(i)_\varphi$, while $(P3)$ is a consequence of the definition of φ_0. Using $(iv)_\varphi$,

we see by a direct computation that $(P2)$ is satisfied and thus that $d_\varphi \in \mathcal{D}$.
Using Definition 1.3.1 with $d := d_\varphi \in \mathcal{D}$ the optimality conditions for (1.3.5) yields

$$v_i + x_i \varphi'(u_i x_i^{-1}) = 0, \ \forall i = 1, \dots, n$$

from which formula (1.4.17) follows easily using $(\varphi^*)' = (\varphi')^{-1}$. Since dom $\varphi \subset \mathbb{R}_+$, we have for all $x \in \mathbb{R}^n_{++}$, dom $d_\varphi(., x) \subset \mathbb{R}^n_+$ and therefore by Proposition 1.3.1 A^{d_φ} satisfies $(r2)$, $(r3)$, and (v).
It remains to prove that $(r1)$ holds. Using formula (1.4.17), and since $x \in \mathbb{R}^n_{++}$, it thus suffices to show that $(\varphi^*)'$ is Lipschitz continuous. But since here φ is a smooth α–strongly convex function, one has

$$(t-s)(\varphi'(t) - \varphi'(s)) \geq \alpha (t-s)^2, \quad \forall t, s > 0,$$

and thus recalling that $(\varphi^*)' = (\varphi')^{-1}$, one easily deduces the required Lipschitz property for $(\varphi^*)'$ and $(r1)$ follows. ■

Remark 1.4.4 (a) Requirement $(v)_\varphi$ allows acute controls on d_φ in the asymptotic analysis of $(RIPM)$ and $(A^{d_\varphi} - DM)$, (see, Section 1.5, Theorem 1.5.4), and is actually not needed for the above result. Technically those controls are the reason why our operator is based on φ and not on φ^*.
(b) The assumption $(iii)_\varphi$ reduces the computation of $(\varphi^*)'$ to the inversion of $\varphi'_{|(0,+\infty)}$.
(c) Note also that A^{d_φ} does not satisfy $(r4)$ in general, but as we shall see in the next section it has no consequence on the asymptotic study of $(A^{d_\varphi} - DM)$ when f is convex.

The corresponding $(A^{d_\varphi} - DM)$ system is thus given by

$$\dot{x}_i(t) + x_i(t) - x_i(t)(\varphi^*)'(-x_i(t)^{-1} \frac{\partial f}{\partial x_i}(x(t))) = 0, \ \forall t \geq 0,$$

or equivalently as

$$x_i(t)\varphi'(\frac{\dot{x}_i(t) + x_i(t)}{x_i(t)}) + \frac{\partial f}{\partial x_i}(x(t)) = 0, \ t \geq 0.$$

To recover $(RIPM)$ by some discretization of $(A^{d_\varphi} - DM)$, the latter can be reformulated in the following way

$$\partial_1 d_\varphi (x(t) + \dot{x}(t), x(t)) + \nabla f(x(t)) = 0, \ x(0) \in \mathbb{R}^n_{++}, \ \forall t \geq 0. \tag{1.4.18}$$

Now, if we perform an implicit discretization of (1.4.18), it yields

$$\partial_1 d_\varphi \left(x^{k+1}, x^k\right) + \nabla f(x^{k+1}) = 0, \ , x^0 = x(0), \ k \in N.$$

which is exactly (1.4.15), with $c_k = 1$.

Example 1.4.3 It is a delicate matter to build a function in Φ whose Fenchel conjuguate is easily computable. As in [22] we focus on the important special choice given by a logarithmic-quadratic kernel,

$$\varphi(s) = \frac{\alpha}{2}(s-1)^2 + \beta(-\log s + s - 1),\ s > 0,$$

which admits (see [22, p.665]) an explicit conjugate $\varphi^* \in C^\infty(\mathbb{R})$, and with

$$(\varphi^*)'(s) = \frac{1}{2\alpha}[\alpha - \beta + s + \sqrt{(\alpha-\beta+s)^2 + 4\alpha\beta}],\ \forall s \in \mathbb{R}.$$

The corresponding $(A^\varphi - DM)$ system is then given by

$$\dot{x}_i(t) + \frac{\alpha+\beta}{2\alpha}x_i(t) + \frac{1}{2\alpha}\frac{\partial f(x(t))}{\partial x_i} - \sqrt{\frac{1}{4\alpha^2}[(\alpha-\beta)x_i(t) + \frac{\partial f(x(t))}{\partial x_i}]^2 + 4\alpha\beta x_i(t)^2} = 0, \tag{1.4.19}$$

with $i \in \{1,\ldots,n\}$, $t \geq 0$ and $x(0) \in \mathbb{R}^n_{++}$. An interesting fact to notice is that (1.4.19) has a sense for any $x(0) \in \mathbb{R}^n$; this suggests like in [34] a study of its properties for *non feasible* initial data.

1.5 Asymptotic analysis for a convex objective function

In the sequel f satisfies the additional assumptions

$$(\mathcal{H}') : \begin{cases} f \text{ is convex}, \\ \arg\min_{\overline{C}} f \neq \emptyset. \end{cases}$$

This section proposes a criterion concerning elliptic barrier operators to obtain the convergence of the trajectories of $(A - DM)$. It is based on Lyapounov functionals and to their (theoretical) decreasing rate. This natural approach is inspired by the classical result of Bruck [40] on the generalized steepest descent method, and by the notions of Fejer or quasi-Fejer sequences which go back to the work of Ermoliev [60] and arise in monotone and generalized gradient optimization algorithms. Such techniques have also been applied succesfully to second order in time systems by Alvarez [4], and Alvarez-Attouch [5]. Before stating the main result of this section, let us describe the typical properties of those Lyapunov functionals, sometimes called *relative entropy*, when working on systems in the nonnegative orthant, see e.g., [76]. In what follows S should be understood as the set of equilibria of some convex function.
We suggest the following general definition for viable Lyapunov functionals.

Definition 1.5.1 *Let $S \subset \overline{C}$ be a nonempty set. A family of functions $\{e_a, a \in S\}$ is Lyapunov viable if it satisfies*

$(i)_e$ For all $a \in S$, $e_a : C \to \mathbb{R}$ is C^1.

$(ii)_e$ The functions e_a are nonnegative for all $a \in S$.

$(iii)_e$ For all $a \in S$, e_a is inf bounded. That is for every $r \in \mathbb{R}$, the set $\{y \in C | e_a(y) \leq r\}$ is bounded.

$(iv)_e$ Let $x^k, k \in N$ be a sequence in C. Then for all $a \in S$,

$$e_a(x^k) \to 0 \text{ as } k \to +\infty \iff x^k \to a \text{ as } k \to +\infty.$$

The next result is a key lemma that can be used to establish convergence of trajectories of $(A-DM)$. First, we recall the following classical result (see e.g., [4, Lemma 2.2]) which will be useful to us.

Lemma 1.5.1 *Let $h : \mathbb{R} \to \mathbb{R}^+$ a C^1 function. If $(h')^+ := \max(0, h')$ is in $L^1(0, +\infty; \mathbb{R})$ then $\lim_{t\to+\infty} h(t)$ exists.*

Let us set $S := \arg\min_{\overline{C}} f$.

Lemma 1.5.2 *Let A be an elliptic barrier operator on C and f a function satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}')$. Assume that there exists $\lambda > 0$, $\mu \in \mathbb{R}$ and a family of functions $\{e_a,\ a \in S\}$ that is Lyapunov viable (i.e., satisfying $(i)_e - (iv)_e$). Suppose in addition that for all $x \in C$,*

$$\langle -A_x \nabla f(x), \nabla e_a(x)\rangle + \lambda\langle \nabla f(x), x-a\rangle \leq \mu |A_x \nabla f(x)|^2. \tag{1.5.1}$$

If $x(t)$ is the solution of $(A-DM)$, then
(i) $f(x(t)) \to \inf_{\overline{C}} f$ as $t \to +\infty$, with the estimation

$$f(x(t)) - \inf_{\overline{C}} f \leq Mt^{-1}, \text{ for some } M > 0.$$

(ii) $\dot{x}(t) \to 0$ as $t \to +\infty$.
(iii) There exists $x^* \in S$ such that $x(t) \to x^*$ as $t \to +\infty$.

Proof. Let $a \in S$, by (1.5.1) and $(A-DM)$ we obtain

$$\frac{d}{dt} e_a(x(t)) + \lambda\langle \nabla f(x(t)), x(t)-a\rangle \leq \mu |\dot{x}(t)|^2,\ t \geq 0. \tag{1.5.2}$$

From the convex inequality it follows that for all $y \in \overline{C}$,

$$0 \geq f(a) - f(y) \geq \langle \nabla f(y), a-y\rangle. \tag{1.5.3}$$

Combining (ii) of Theorem 1.2.4, (1.5.3), and (1.5.2) yields $[\frac{d}{dt} e_a(x(t))]^+ \leq \mu|\dot{x}(t)|^2$, $t \geq 0$. From $(ii)_e$ and Lemma 1.5.1, we deduce that $e_a(x(t))$ converges as $t \to +\infty$. Hence, by $(iii)_e$, $x(\cdot)$ is bounded.
Coming back to (1.5.2), we obtain for all $T \geq 0$,

$$\lambda \int_0^T \langle \nabla f(x(t)), x(t)-a\rangle dt \leq \int_0^T |\dot{x}(t)|^2 dt + e_a(x(0)) - e_a(x(T)),$$

and since $\lambda > 0$,

$$\langle \nabla f(x(\cdot)), x(\cdot)-a\rangle \in L^1(0,\infty;\mathbb{R}). \tag{1.5.4}$$

From (1.5.4), $(\mathcal{H}_1)$, and the boundedness property of x we obtain that there exist $x^* \in \overline{C}$, and a nondecreasing sequence $t_k, k \in N$ such that $\langle \nabla f(x(t_k)), x(t_k)-a\rangle \to 0$ and $x(t_k) \to x^*$ as $k \to +\infty$. Using (1.5.3) it ensues $f(x^*) \leq f(a)$ and thus $x^* \in S$.

By Theorem 1.2.4, (iii) and the continuity of f, we see that the latter argument implies $f(x(t)) \to \inf_{\overline{C}} f$ as $t \to +\infty$ and that all limit points of x are in S.
To prove the second part of (i), we first deduce from (1.5.2) and (1.5.3) that

$$\frac{d}{dt}e_a(x(t)) + \lambda(f(x(t)) - f(a)) \leq \mu|\dot{x}(t)|^2,\ t \geq 0.$$

By integration it follows from Theorem 1.2.4 (iii) that for $t \geq 0$, $t\lambda[f(x(t)) - \inf_{\overline{C}} f] \leq e_a(x(0)) - e_a(x(t)) + \mu \int_0^t |\dot{x}|^2$. Using $(iii)_e$ we obtain for all $t > 0$

$$\lambda[f(x(t)) - \inf_{\overline{C}} f] \leq \frac{1}{t}[e_a(x(0)) + \mu \int_0^t |\dot{x}|^2]. \tag{1.5.5}$$

The estimate announced in (i) is then a consequence of Theorem 1.2.4 (iv).
Let x_1^* and x_2^* be two cluster points of $x(\cdot)$ and $t_k, \tau_k, k \in N$ increasing sequences in $\mathbb{R}^+$, such that $x(t_k) \to x_1^*$, $x(\tau_k) \to x_2^*$ as $k \to +\infty$. From $(iv)_e$, we deduce $e_{x_1^*}(x(t_k)) \to 0$ as $k \to +\infty$. But since the function $e_{x_1^*}(x(\cdot))$ has a limit as $t \to +\infty$, we also have $e_{x_1^*}(x(\tau_k)) \to 0$ as $k \to +\infty$, and by applying $(iv)_e$ again we obtain $x_1^* = x_2^*$.

Let x^* be the limit point of $x(\cdot)$, it verifies the classical relation $\nabla f(x^*) \in -N_{\overline{C}}(x^*)$, and therefore $(\mathcal{H}_1)$ implies that $(x(t), \nabla f(x(t)))$ has its limit point in $\{x^*\} \times -N_{\overline{C}}(x^*)$. Applying Proposition 2.1, it follows that $\dot{x}(t) \to 0$ as $t \to +\infty$. ■

Remark 1.5.3 (a) If $\mu \leq 0$, we have by (1.5.5)

$$f(x(t)) - \inf_{\overline{C}} f \leq \frac{1}{\lambda t} e_a(x(0)),\ \forall t > 0$$

.

(b) Note that Lemma 1.5.2 allows to handle the case $\mu > 0$ in (1.5.1), which corresponds to quasi-Fejer convergence.
(c) The property $(r3)$ has not been used, but it is implicitly contained in (1.5.1).
(d) Note also that the above result holds for an elliptic barrier operator which is possibly undefined on bd $C \times \mathbb{R}^n$.

Let us apply this result to some of the operators defined in Section 1.4. In what follows *it is implicitly assumed that* $C = \mathbb{R}^n_{++}$ *when dealing with operators of the type* A^{d_φ}, $\varphi \in \Phi$, while A^P is the gradient projection operator (cf. subsection 1.4.1).

Theorem 1.5.4 *Let* $\varphi \in \Phi$ *such that* $\alpha \geq \beta\varphi_0''(1)$, $h \in \mathcal{B}_C$, *and assume that* f *satisfies* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}')$. *Then the trajectories of* $(A^P - DM)$, $(A^{q_h} - DM)$, *and* $(A^{d_\varphi} - DM)$ *converge to some minimizer of* f *on* $\overline{C}$. *Moreover, for all trajectories* x *the following properties hold :*
(i) $f(x(t)) \to \inf_{\overline{C}} f$ *as* $t \to +\infty$, *with the estimation*

$$f(x(t)) - \inf_{\overline{C}} f \leq Mt^{-1},\ \text{where } M > 0.$$

(ii) $\dot{x}(t) \to 0$ *as* $t \to +\infty$.

Proof. By Propositions 1.4.1, 1.4.2, and 1.4.4, we know that A^P, A^{q_h}, and $A^{d\varphi}$ are elliptic barrier operators. For every $a \in S$ and for all $x \in C$ set

$$\begin{aligned} e_a^P(x) &= f(x) - f(a) + \tfrac{1}{2}|x-a|^2, \\ e_a^h(x) &= D_h(a,x) = \tfrac{\alpha}{2}|x-a|^2 + D_{h_1}(a,x), \\ e_a^\varphi(x) &= f(x) - f(a) + \theta|x-a|^2, \end{aligned}$$

where $\theta = (\alpha + \varphi_0''(1))/2$. Naturally the idea is to apply Lemma 1.5.2 to the operators A^P, A^{q_h}, and A^{d_φ}. Let $a \in S$. The functions e_a^φ, e_a^h, e_a^P, satisfy clearly $(i)_e$, $(ii)_e$. To obtain $(iii)_e$, just notice that in the three cases, the structure of the functions has the following form

$$\xi_a(x) = k|x-a|^2 + \rho_a(x),\ \forall x \in C,$$

with $\rho_a \geq 0$, $k > 0$. By definition of a Bregman function and by Lemma 1.4.1, e_a^h verifies $(iv)_e$. To prove that e_a^P and e_a^φ satisfy $(iv)_e$, we just have to combine $(\mathcal{H})$, and the fact that a is a minimizer of f on $\overline{C}$. Let us prove that the property (1.5.1) holds for the couples (e_a^P, A^P), (e_a^h, A^{q_h}), and (e_a^P, A^{d_φ}).

• The continuous gradient projection method has already been studied from different viewpoints in [34], but for the sake of completeness we recall the argument. Let $x \in C$ and $a \in S$. The optimality property of the orthogonal projection operator gives for all $\xi \in \overline{C}$, $\langle x - \nabla f(x) - P_{\overline{C}}(x - \nabla f(x)), \xi - P_{\overline{C}}(x - \nabla f(x))\rangle \leq 0$. Therefore if $\xi = a$, we obtain

$$\langle -\nabla f(x) + A_x^P \nabla f(x), a - x + A_x^P \nabla f(x)\rangle \leq 0,$$

or equivalently $\langle -A_x^P \nabla f(x), x - a + \nabla f(x)\rangle + |A_x^P \nabla f(x)|^2 + \langle \nabla f(x), x - a\rangle \leq 0$, which is (1.5.1) with $\mu = -1$.

• Now, let us consider A^{q_h} where h is Bregman function that belongs to $\mathcal{B}_C$. Let us compute the gradient of e_a^h for all $a \in S$. For all $x \in C$, we have

$$\begin{aligned} \nabla e_a^h(x) &= \nabla[h(a) - h(\cdot) - \langle \nabla h(\cdot), a - \cdot\rangle](x) \\ &= \nabla^2 h(x)(x-a). \end{aligned}$$

And therefore $\langle -A_x^{q_h} \nabla f(x), \nabla e_a^h(x)\rangle = -\langle \nabla^2 h(x)^{-1} \nabla f(x), \nabla^2 h(x)(x-a)\rangle = -\langle \nabla f(x), x - a\rangle$, which verifies (1.5.1) with $\mu = 0$ and $\lambda = 1$.

• Finally, let us deal with e_a^P, A^{d_φ}. Our approach relies on the following key lemma proven in [22, Lemma 3.4]

Lemma 1.5.5 *For every $y_1 \in \mathbb{R}_+^n$ and for every $(y_1, y_2) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$, we have*

$$\langle y_1 - y_2, \partial_1 d_\varphi(y_2, y_3)\rangle \leq \theta\left(|y_1 - y_3|^2 - |y_1 - y_2|^2\right).$$

Note that it is here that the property $(v)_\varphi$, is needed. Indeed, the proof of this lemma is based on that assumption, together with the condition $\alpha \geq \beta\varphi_0''(1)$.
For all $i \in \{1\dots,n\}$ and all $x \in \mathbb{R}_{++}^n$, set $(v_x)_i = -\left(A_x^{d_\varphi}\nabla f(x)\right)_i$. The A^{d_φ}-driven descent method can be rewritten as,

$$\partial_1 d_\varphi(x(t) + v_x, x) + \nabla f(x) = 0,\ \forall x \in \mathbb{R}_{++}^n. \qquad (1.5.6)$$

Observe that $x \in \mathbb{R}^n_{++}$ implies $x + v_x \in \mathbb{R}^n_{++}$. Now for $a \in \arg\min_{\mathbb{R}^n_+} f$ and for all $x \in \mathbb{R}^n_{++}$, let us multiply (1.5.6) by $a - x - v_x$, this gives

$$\langle a - (v_x + x), \partial_1 d_\varphi(x + v_x, x) \rangle + \langle \nabla f(x), a - x - v_x \rangle = 0,$$

and therefore by Lemma 1.5.5

$$\theta \left(|a - x|^2 - |a - x - v_x|^2 \right) + \langle \nabla f(x), a - x - v_x \rangle \geq 0.$$

After direct algebra this reduces to

$$\langle v_x, 2\theta(x - a) + \nabla f(x) \rangle + \langle \nabla f(x), x - a \rangle + |v_x|^2 \leq 0, \ \forall x \in \mathbb{R}^n_{++}.$$

Recalling that $v_x = -A_x^{d_\varphi} \nabla f(x)$, we easily see that (1.5.1) is satisfied. ■

Remark 1.5.6 The convergence of the orbits generated by the other operators proposed in Section 1.4 remains an open question.

Localization of the limit point

Let A be an elliptic barrier operator, and e_a be a family of viable Lyapounov functionals satisfying (1.5.1) with $\mu \leq 0$. We assume moreover that for all a in $S \subset \overline{C}$, there exist a nonnegative convex function $\rho_a : C \mapsto \mathbb{R}$ and $k > 0$ such that

$$e_a(x) = k|x - a|^2 + \rho_a(x), \ \forall x \in C. \tag{1.5.7}$$

As in Lemaire [89], and inspired by the recent non Euclidean extension given in [20], the limit point of the trajectory produced by $(A - DM)$ can be localized.

Proposition 1.5.1 *Let A be an elliptic barrier operator on C, and let $\{e_a, a \in S\}$ be as defined in (1.5.7). Then the trajectory of $(A - DM)$, with $x(0) \in C$, converges to a minimizer x_∞ of f on $\overline{C}$, with the following estimation*

$$|x_\infty - x(0)|^2 \leq \inf\{4|x(0) - a|^2 + \frac{2}{k}\rho_a(x(0)) \mid a \in S\}$$

Proof. The convergence result of the trajectory $x(t)$ to $x_\infty \in S = \arg\min_{\overline{C}} f$ is a direct consequence of Lemma 1.5.2. To prove the estimation, let us come back to the inequality (1.5.2), proven in Lemma 1.5.2 :

$$\frac{d}{dt} e_a(x(t)) + \lambda \langle \nabla f(x(t)), x(t) - a \rangle \leq \mu |\dot{x}(t)|^2, \ t \geq 0.$$

The convexity property of f, and the fact that $\mu \leq 0$ imply that $\mathbb{R}_+ \ni t \mapsto e_a(x(t))$ is nonincreasing. Therefore, for all $a \in S$ we have $e_a(x(t)) \leq e_a(x(0))$, where $t \geq 0$. Since $\rho_a \geq 0$, by letting $t \to +\infty$, (1.5.7) yields

$$k|x_\infty - a|^2 \leq k|x(0) - a|^2 + \rho_a(x(0)). \tag{1.5.8}$$

Now for all $a \in S$, we have

$$\begin{aligned} |x_\infty - x(0)|^2 &\leq [|x_\infty - a| + |a - x(0)|]^2 \\ &\leq 2|x_\infty - a|^2 + 2|a - x(0)|^2 \\ &\leq 4|x(0) - a|^2 + \tfrac{2}{k}\rho_a(x(0)) \end{aligned}$$

where the third inequality is a consequence of (1.5.8). The desired result is then obtained by taking the infimum overall $a \in S$. ■

As a consequence, we then have

Corollary 1.5.1 *Under the assumptions of Theorem 1.5.4, we have*

$$|x_\infty - x(0)|^2 \leq 4 \inf\{|x(0)-a|^2 + \frac{1}{\alpha} D_{h_1}(a, x(0)) \mid a \in S\},$$

if $A = A^{q_h}, h(\cdot) = \alpha/2|\cdot|^2 + h_1(\cdot)$.
Defining $s : \mathbb{R}^n \mapsto \mathbb{R}$ *as* $s(y) := \inf\{|y-a|^2 \mid a \in S\}$*, then we also have*

$$|x_\infty - x(0)|^2 \leq 4\left(s(x(0)) + f(x(0)) - \inf_{\overline{C}} f\right)$$

if $A = A^P$*, and*

$$|x_\infty - x(0)|^2 \leq 4s(x(0)) + \frac{2}{\theta}\left(f(x(0)) - \inf_{\overline{C}} f\right)$$

if $A = A^{d_\varphi}$.

Proof. The families $\{e^h_a, e^P_a, e^\varphi_a,\ a \in S\}$ introduced in the beginning of the proof of Theorem 1.5.4 satisfy the assumptions of Proposition 1.5.1, and thus the claimed results follow easily. ■

Remark 1.5.7 (a) The estimations given in Corollary 1.5.1 for $A = A^{q_h}$ allow to recover the results obtained in [20, 89].
(b) Assume that f is a linear function, that is $f(x) = \langle c, x\rangle$, $\forall x \in \mathbb{R}^n$ where $c \in \mathbb{R}^n$. Take h as in Theorem 5.1. A straightforward integration of $(A^{q_h} - DM)$ in its form given in (1.4.6) yields

$$\nabla h(x(t)) - \nabla h(x(0)) + tc = 0,\ \forall t \geq 0. \tag{1.5.9}$$

As already noticed in [78], the trajectory of $(A^{q_h} - DM)$ can be viewed as an optimal path relatively to the barrier function D_h. Indeed since for all $(y,z) \in C \times C$, $\nabla_1 D_h(y,z) = \nabla h(y) - \nabla h(z)$, (1.5.9) can be reformulated as

$$x(t) \in \arg\min\{\langle c, u\rangle + \frac{1}{t} D_h(u, x(0)) \mid u \in \mathbb{R}^n\},\ t > 0.$$

The convergence techniques developed in [78], but also the viscosity methods studied in Attouch [13], allow then to fully characterize the limit point as

$$x_\infty \in \arg\min\{D_h(a, x(0)) \mid a \in S\}.$$

Chapitre 2

Sur la méthode continue de gradient projection.

On the continuous gradient projection method in Hilbert spaces.[1]

Jérôme Bolte

Abstract. This paper is concerned with the asymptotic analysis of the trajectories of some dynamical systems built upon the gradient projection method in Hilbert spaces. For a convex function with locally Lipschitz gradient, it is proved that the orbits weakly converge to a constrained minimizer (whenever it exists). This result remains valid even if the initial contidion is chosen out of the feasible set, and can be extended in some sense to quasi-convex functions. An asymptotic control result involving a Tykhonov-like regularization, shows that the orbits can be forced to strongly converge towards a well-specified minimizer. In the finite-dimensional framework, we study the differential inclusion obtained by replacing the classical gradient by the subdifferential of a continuous convex function. We prove the existence of a solution whose asymptotic properties are the same as in the smooth case.

Keywords. Gradient projection method, dynamical systems in optimization, viability, differential inclusion, asymptotic control, Lyapounov functions.
AMS classification. 37L99, 37N40, 34D05, 34H05.

2.1 Introduction

Let H be a real Hilbert space endowed with scalar product $\langle\ ,\ \rangle$ and its related norm $|\ .\ |$. If C is a closed, nonempty convex set in H, we denote by P_C the corresponding orthogonal projection, and by $N_C(x)$ the normal cone to C at x.

Our main purpose being to minimize a convex function $\phi : H \to \mathbb{R}$ on C, we study systems of the type

$$(CGP) \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\nabla\phi(x(t))] = 0, \forall t \geq 0 \\ x(0) = x_0 \in C \end{cases}$$

where (CGP) stands for *continuous gradient projection* method.

Many primal *continuous* methods to perform this kind of optimization problem consist in adding some barrier or penalty functions to ϕ, and then to study the new potential with a classical procedure like steepest descent. From a theoretical viewpoint those approaches can be seen as smooth approximations of the following problem

$$\inf_H (\phi + \delta_C), \tag{2.1.1}$$

where $\delta_C : H \mapsto \mathbb{R} \cup \{+\infty\}$ denotes the indicator function of C, ie, the real extended function with value 0 on C and $+\infty$ elsewhere.

When combining such a formulation with the steepest descent method we are led to study

$$\dot{x}(t) + \nabla\phi(x(t)) + N_C(x(t)) \ni 0, \tag{2.1.2}$$

[1]Article accepté dans *J.O.T.A.*

or equivalently, see Brézis [37],

$$\dot{x}(t^+) = P_{T_C(x(t))}[-\nabla\phi(x(t))],$$

where $T_C(x)$ is the tangent cone to C at x.

If ϕ is a proper lower semicontinuous convex function, Bruck [40] has proved that trajectories of (2.1.2) converge to a minimizer of $\phi + \delta_C$ whenever it exists. But, this solving procedure has a major drawback : the dynamics ignores the constraints until the orbit encounters the boundary of C.

This can be improved by a careful examination of the optimality condition associated to (2.1.1), indeed the two following conditions are equivalent

$$\begin{array}{ll} (\mathcal{O}) & \nabla\phi(x) + N_C(x) \ni 0, \\ (\mathcal{O}') & \exists\mu > 0,\ x = P_C\left(x - \mu\nabla\phi(x)\right). \end{array} \tag{2.1.3}$$

This reformulation of $\mathcal{O}$ is well known in discrete optimization, it has led to study algorithms of the type

$$x_{k+1} \in P_C(x_k - \mu\partial\phi(x_k))\ , x_0 \in C \tag{2.1.4}$$

where $\partial\phi$ is the subdifferential of ϕ.

For theoretical study in Hilbert spaces see Polyak [111], Mac Cormick-Tapia [97], Martinet [100], Phelps [109]; in Phelps [110] some extensions to Banach spaces are proposed. If ϕ is only assumed proper lower semicontinuous and convex the convergence of the sequence (2.1.4) is, as far as we know, an open question. In a recent work [3], Alber, Iusem, and Solodov have obtained the weak convergence of the orbits under a local boundedness assumption on $\partial\phi$.

As a continuous dynamical system (CGP) enjoys much stronger properties than its corresponding explicit discretizations, and we will see actually, that it can be considered as an ideal version of (2.1.4). (CGP) has been introduced by Antipin [10] in the finite-dimensional case with a gradient Lipschtiz continuous on the whole space H. For a second order version of (CGP), interesting results have been obtained in Alvarez- Attouch [5] and [10], but under the same strong assumptions on the gradient.

In this paper, the results obtained in [10] for the smooth case has been considerably extended. In our framework H is an Hilbert space, ϕ is a C^1 function non necessarily convex and its gradient is only supposed Lipschtiz continuous on bounded sets. Moreover, no restriction is imposed on the stepsize μ. In section 2.2, it is proved that (CGP) is a descent method generating viable trajectories, ie $\forall t \geq 0,\ x(t) \in C$.

The asymptotic behaviour of the orbits when ϕ is convex or quasi-convex is a delicate matter. In Baillon [24] one can find an example in which the trajectories of (2.1.2) do not strongly converge to an equilibrium. A key tool in the study of the convergence of the steepest descent method is the association of Fejer monotonocity with Opial lemma [105] (see also section 2.3). To be more precise, the quadratic functionals $y \in H \to 1/2|y - x^*|^2$ where x^* is some stationary point of the potential, are Lyapounov functionals for the system (2.1.2), allowing via Opial lemma to obtain weak convergence. Due to the lack of monotonocity of the operator $y \to -y + P_C(y - \mu\nabla\phi(y))$, we propose an alternative approach to the asymptotic behaviour, showing that the distance-like functions $y \to \mu\left[\phi(y) - \phi(x^*)\right] + 1/2|y - x^*|^2$ are decreasing along (CGP) trajectories. This allows to derive the weak convergence of the solution of (CGP) to a minimizer of ϕ on C (see section 2.3 and figure 2.1 below).

As noticed in [10], (CGP) conserves its optimizing properties even if the initial condition is out of C, this result is extended to infinite-dimensional spaces by use of an Opial-like lemma concerning a class of Lyapounov functionals. The figure 2.1 below gives an illustration of those results, with $\phi(x_1, x_2) = \frac{1}{2}(x_1 - x_2 - 5)^2 + \frac{1}{2}(2x_1 + x_2 - 4)^2$ and $C = \mathbb{R}_+ \times \mathbb{R}_+$. Five initial conditions have been chosen in and out of C, and three different values of μ have been used for the computations (the dashed lines delimit the lower level subsets of ϕ).

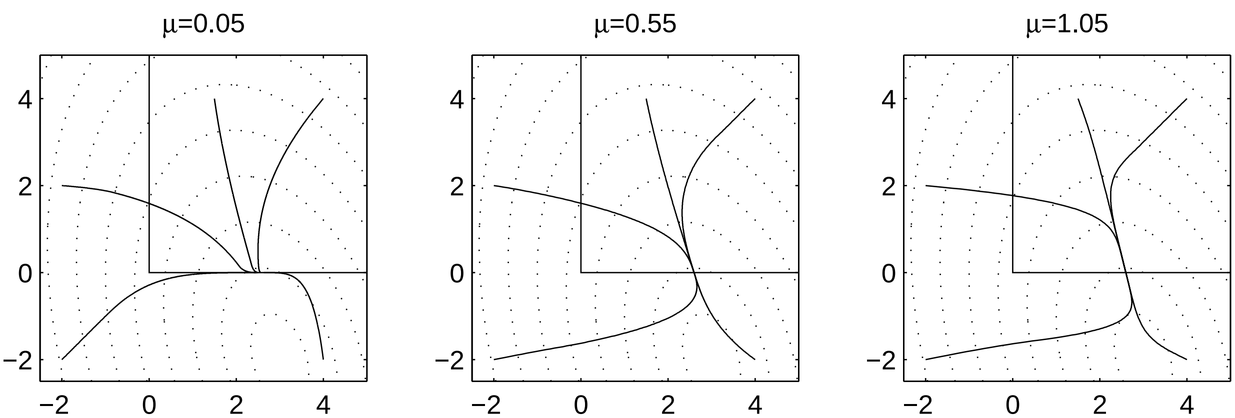

FIG. 2.1 – Some trajectories of (CGP).

This phenomenon of "nonquadratic" monotonicity appears to be relatively less studied in the framework of *continuous* dynamical systems devoted to optimization. In Attouch-Teboulle [20], inspired by entropic proximal methods, the authors study the following system

$$\dot{x}_i(t) + \frac{x_i(t)}{\alpha + \beta x_i(t)} \frac{\partial \phi}{\partial x_i}(x(t)) = 0, x_i(0) > 0, \forall t \geq 0,$$

where $i \in \{1, .., n\}$, $\alpha, \beta > 0$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ is to be optimized on the positive orthant $\mathbb{R}^n_+$. As in numerous algorithms based on proximal methods, see for instance Auslender-Teboulle-Ben-Tiba [22], Kiwiel [84], the trajectories are interior and, in some sense monotone with respect to some distance-like functionals. Those similarities with (CGP) - viability, intrinsic penalty, nonquadratic monotonicity- open new perspectives and seem to suggest the existence of a whole class of new descent methods.

In section 2.5 an asymptotic control result is obtained by considering the nonautonomous system

$$(CGP)_\varepsilon \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\nabla\phi(x(t)) - \varepsilon(t)x(t)] = 0 \\ x(0) = x_0 \in C. \end{cases}$$

where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\int_0^{+\infty} \varepsilon(s)ds = +\infty$.

The Tykhonov term $t \to \varepsilon(t)x(t)$ is used to force the orbits to attain a particular equilibrium for the strong topology. This work is inspired by Attouch-Cominetti [15], Attouch-Czarnecki [17] where the authors are concerned by steepest descent, and heavy ball with friction systems.

Section 2.6 is devoted to the following nonsmooth version of (CGP),

$$(CGP)_g \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\partial\phi(x(t))] \ni 0 \\ x(0) = x_0 \in C \end{cases}$$

where ϕ is a convex continuous function on a finite-dimensional space. Note that the multivalued vector field ruling this equation has neither the convexity, nor the regularity properties usually required in differential inclusion theory (Aubin-Cellina [21]). In order to prove the existence of a global solution, we define approximate differential systems by using Moreau-Yosida regularization; then, obtaining estimations on the approximate trajectories, we derive by compacity arguments that $(CGP)_g$ actually admits a solution. This is a classical approach to solve a nonsmooth differential inclusion see, for instance, [37] and Schatzman [115] for second order in time systems. The asymptotic properties of $(CGP)_g$ are the same as in the smooth case.

2.2 Global existence results for feasible initial data

In what follows, ϕ is a function from H into $\mathbb{R}$. For a given closed, nonempty convex subset C of H, we consider the following set of hypotheses

$$(\mathcal{H}) \quad \begin{cases} \phi \text{ is continuously differentiable,} \\ \nabla\phi \text{ is Lipschitz continuous on bounded sets,} \\ \phi \text{ is bounded from below on } C. \end{cases}$$

The continuous gradient projection method is given by

$$(CGP) \quad \begin{cases} \dot{x}(t) + x(t) - P_C\left[x(t) - \mu\nabla\phi(x(t))\right] = 0 \\ x(0) = x_0 \end{cases}$$

where $\mu > 0$ is a positive parameter.

Theorem 2.2.1 *Let us assume that ϕ satisfies $(\mathcal{H})$. Then, the following properties hold*
(i) For all $x_0 \in C$, there exists a unique solution x of (CGP) such that $x \in C^1([0,+\infty[;H)$.
(ii) The trajectory satisfies the following viability condition $\forall t \geq 0,\ x(t) \in C$.
(iii) (CGP) is a descent method, more precisely we have $\frac{d}{dt}\phi(x(t)) \leq -\frac{1}{\mu} \mid \dot{x}(t) \mid^2$.
As a consequence $\dot{x} \in L^2(0,+\infty;H)$.
(iv) If $t \to x(t)$ is bounded, then $\dot{x}(t) \to 0$ as $t \to +\infty$.

Proof. Since P_C is a Lipschitz continuous operator, the Cauchy-Lipschitz Theorem yields the existence of a unique solution of (CGP) defined on some interval $[0,T]$ with $T > 0$. Let us show that for all $t \in [0,T]$, $x(t) \in C$. (CGP) can be rewritten $\dot{x}(t) + x(t) = f(t)$, where $f(.) = P_C[x(.) - \mu\nabla\phi(x(.))\,]$ is a continuous function with values in C.

A simple integration procedure gives $x(t) = \exp(-t)x_0 + \exp(-t)\int_0^t f(s)\exp(s)ds$. Set $\mu_t = \frac{\exp(s)}{\exp(t)-1} 1_{|[0,t]} ds$; then $\mu_t([0,+\infty[) = 1$ and

$$x(t) = \exp(-t)x_0 + (1-\exp(-t))\int_0^t f(s)d\mu_t. \tag{2.2.5}$$

Since $f(s) \in C$, $\forall s \in [0,t]$, it is easy to check that $\int_0^t f(s)d\mu_t \in C$ and thus (2.2.5) shows that for all t in $[0,T]$, $\quad x(t) \in C$.

Let us now deal with (iii). For all $t \in [0,T]$, set $\xi(t) = x(t) - \mu\nabla\phi(x(t))$. Using (ii) and the optimality property of $P_C(\xi(t))$ we have $\langle x(t) - P_C(\xi(t)), \xi(t) - P_C(\xi(t))\rangle \leq 0$, and thus by (CGP), $\langle -\dot{x}(t), -\mu\nabla\phi(x(t)) - \dot{x}(t)\rangle \leq 0$. Whence, for all t in $[0,T]$

$$\mu\frac{d}{dt}\phi(x(t)) + \mid \dot{x}(t) \mid^2 \leq 0. \tag{2.2.6}$$

It is now possible, arguing by contradiction, to prove that the trajectories are defined on the whole of $\mathbb{R}_+$. Let us assume that the maximal solution of (CGP) is defined on some $[0, T_{max}[$ with $T_{max} < +\infty$. By integrating (2.2.6) on $[0,t]$ where $t < T_{max}$ we obtain $\mu\phi(x(t)) + \int_0^t \mid \dot{x}(s) \mid^2 ds \leq \mu\phi(x_0)$. Hence $\int_0^t \mid \dot{x}(s) \mid^2 ds \leq \mu\phi(x_0) - \mu \inf_C \phi$ and $\dot{x} \in L^2(0, T_{max}; H)$. Classically, this implies by Cauchy-Schwarz inequality that $\lim_{t\to T_{max}} x(t)$ exists. Since C is closed this limit belongs to C, and to obtain a contradiction we just have to use Cauchy Lipschitz Theorem at $t = T_{max}$.

In order to prove (iv), observe that if x is bounded then, with (CGP) and $(\mathcal{H})$, $\dot{x} \in L^\infty(0,+\infty; H)$. This shows that x is also a Lipschitz continuous map and using (CGP) again that $\dot{x}$ is a Lipschitz mapping by composition. Combining the fact that $\dot{x} \in L^2(0,+\infty; H)$ with its Lipschitz property, it is easy to check out that $\lim_{t\to+\infty} \dot{x}(t) = 0$. ■

2.3 Convex minimization

Asymptotic behaviour

This section is devoted to the study of (CGP) with a convex ϕ. Using new Lyapounov functionals it is proved that the trajectories enjoy nice asymptotic properties, which are very similar to those obtained for the steepest descent method. We set $S := \operatorname{argmin}_C \phi = \{x \in C \mid \phi(x) = \inf_C \phi\}$.

Theorem 2.3.1 *ϕ is supposed to be convex and to satisfy $(\mathcal{H})$. As before we assume $x_0 \in C$. Then the following properties hold*

(i) $\lim_{t\to+\infty} \phi(x(t)) = \inf_C \phi$.

Assume moreover $S \neq \emptyset$, then

(ii) there exists $M \geq 0$ such that $\phi(x(t)) - \inf_C \phi \leq M/(t+1)$, $t \geq 0$,

(iii) $x(t)$ weakly converges to some minimizer of ϕ on C as $t \to +\infty$.

The first part (i) of this Theorem is inspired by Lemaire's work on the steepest descent method [89], in which it is proved that it is not necessary to suppose $S \neq \emptyset$ to obtain a proper optimizing method. As in [40], Alvarez [4] the asymptotic analysis relies on the following

Lemma 2.3.2 *(Opial)*

Let H be a Hilbert space and $x : [0,+\infty[\to H$ an application such that there exists a nonempty set $\mathcal{S} \subset \mathcal{H}$ which satisfies

(i) All weak cluster points of x are contained in $\mathcal{S}$.

(ii) $\forall x^* \in \mathcal{S} \quad \lim_{t\to+\infty} \mid x(t) - x^* \mid$ exists.

Then $x(t)$ weakly converges as $t \to +\infty$ towards an element of $\mathcal{S}$.

Proof of Theorem 2.3.1. Let us first prove (i). Let z be an arbitrary element of C. From the convex inequality it follows that $\phi(z) \geq \phi(x(t)) + \langle \nabla\phi(x(t)), z - x(t)\rangle$, $t \geq 0$. In order to use (CGP), this inequality can be reexpressed in the following form

$$\phi(x(t)) - \phi(z) \leq \langle \nabla\phi(x(t)), \dot{x}(t) + x(t) - z\rangle - \langle \nabla\phi(x(t)), \dot{x}(t)\rangle \quad \forall t \geq 0. \tag{2.3.7}$$

But C is a convex set, thus

$$\langle x(t) - \mu\nabla\phi(x(t)) - P_C[x(t) - \mu\nabla\phi(x(t))], z - P_C[x(t) - \mu\nabla\phi(x(t))]\ \rangle \leq 0$$

From (CGP) we deduce that $\langle x(t) - \mu\nabla\phi(x(t)) - \dot{x}(t) - x(t), z - \dot{x}(t) - x(t)\rangle \leq 0$ and therefore

$$\langle \mu\nabla\phi(x(t)) + \dot{x}(t), \dot{x}(t) + x(t) - z\rangle \leq 0. \tag{2.3.8}$$

Coming back to the inequality (2.3.7) we obtain

$$\phi(x(t)) - \phi(z) \leq -\frac{1}{\mu}\langle \dot{x}(t), \dot{x}(t) + x(t) - z\rangle - \langle \nabla\phi(x(t)), \dot{x}(t)\rangle,\ t \geq 0.$$

Hence for $t \geq 0$,
$\frac{1}{\mu} \mid \dot{x}(t) \mid^2 + \frac{1}{\mu}\langle \dot{x}(t), x(t) - z\rangle + \langle \nabla\phi(x(t)), \dot{x}(t)\rangle + \ \phi(x(t)) - \phi(z) \leq 0$, from which we derive

$$\frac{d}{dt}\left(\frac{1}{2\mu} \mid x(t) - z \mid^2 + \phi(x(t))\right) + \ \phi(x(t)) - \phi(z) + \frac{1}{\mu} \mid \dot{x}(t) \mid^2 \leq 0. \tag{2.3.9}$$

Integrating (2.3.9) over $[0,t]$, we obtain

$$\frac{1}{2\mu} \mid x(t) - z \mid^2 + \phi(x(t)) + \int_0^t (\phi(x(s)) - \phi(z))ds \leq \frac{1}{2\mu} \mid x_0 - z \mid^2 + \phi(x_0). \tag{2.3.10}$$

By (iii) of Theorem 2.2.1, we know that $\phi \circ x$ is a nonincreasing function, thus (2.3.10) gives

$$\frac{1}{2\mu} \mid x(t) - z \mid^2 + \phi(x(t)) + t(\phi(x(t)) - \phi(z)) \leq \frac{1}{2\mu} \mid x_0 - z \mid^2 + \phi(x_0).$$

Hence

$$\phi(x(t)) \leq \phi(z) + \frac{1}{t}\left(\frac{1}{2\mu} \mid x_0 - z \mid^2 + \phi(x_0) - \inf_C \phi\right). \tag{2.3.11}$$

To obtain (ii), just notice that (2.3.11) is valid for all z in C. As a consequence $\lim_{t\to+\infty} \phi(x(t)) = \inf_C\ \phi$.

Assume now that $S \neq \emptyset$. The proof of (iii) is based on the fact that $t \to E(x(t), x^*) = \frac{1}{2\mu} \mid x(t) - x^* \mid^2 + \phi(x(t)) - \phi(x^*)$ is nonincreasing for all x^* fixed in S. Indeed, (2.3.9) gives, for all $z = x^*$ in S, $\frac{d}{dt}E(x(t), x^*) \leq (\inf_C \phi - \phi(x(t))) - \frac{1}{\mu} \mid \dot{x}(t) \mid^2 \leq 0$.

Combining the latter result with the fact that ϕ is bounded from below, implies that $t \to E(x(t), x^*)$ converges as $t \to +\infty$. Since $\phi(x(t))$ has a limit, it follows that, for all x^* in S, $\mid x(t) - x^* \mid$ converges as $t \to +\infty$.

Now, in order to apply Opial lemma, we need to examine the cluster points of x. Recall that x is bounded in C. Let $x^* \in C$ and $t_n \to +\infty$ such that $w - \lim_{t_n \to +\infty} x(t_n) = x^*$.

Since ϕ is convex and continuous for the strong topology, it is lower semicontinuous for the weak topology, and $\phi(x^*) \leq \liminf_{n \to +\infty} \phi(x(t_n))$.

Using (ii) it follows that $\phi(x^*) \leq \inf_C \phi$ and since, by (weak) closedness of C, $x^* \in C$, we obtain $x^* \in S$. The conclusion is now given by Opial lemma which yields $w - \lim_{t \to +\infty} x(t) = x_\infty$ with $x_\infty \in S$. ■

Trajectories starting outside the set of constraints

Our purpose in this section, is to study the trajectories of (CGP) under the following hypothesis

$$(\mathcal{H}') \quad \begin{cases} x_0 \in H, \\ \phi \text{ convex.} \end{cases}$$

Since x_0 can possibly be chosen out of C, the difficult point of the following result is to cope with a dynamics which is no more a descent method. As shown on figure 2.1 the trajectory may have all its values out of C, and the general results given by Theorem 2.2.1 do not apply any longer.

Theorem 2.3.3 *Assume that ϕ and x_0 satisfy $(\mathcal{H})$ and $(\mathcal{H}')$.*

Let us suppose moreover that $\operatorname{argmin}_C \phi \neq \emptyset$. *Then*

(i) The trajectory of (CGP) is defined on $\mathbb{R}_+$. For all $x^* \in \operatorname{argmin}_C \phi$, set

$$\mathcal{E}(t, x^*) = \frac{1}{2} \mid x(t) - x^* \mid^2 + \mu \left[\phi(x(t)) - \phi(x^*) - \langle \nabla\phi(x^*), x(t) - x^* \rangle\right],$$

then $t \mapsto \mathcal{E}(t, x^*)$ is a nonincreasing function and

$$\dot{\mathcal{E}}(t, x^*) = - \mid \dot{x}(t) \mid^2 - \mu\langle \nabla\phi(x(t)) - \nabla\phi(x^*), x(t) - x^* \rangle, \; t \geq 0. \tag{2.3.12}$$

(ii) The trajectory weakly converges towards an element x^* in $\operatorname{argmin}_C \phi$ *and $\phi(x(t))$ converges to* $\inf_C \phi$ *as $t \to +\infty$.*

Proof. As in Theorem 2.2.1, let us start by proving that the considered system is dissipative. Let $[0, T_{max}[$ be the interval corresponding to the maximal solution of (CGP).

Let $x^* \in \operatorname{argmin}_C \phi$, by convexity of ϕ the optimality condition gives

$$\forall z \in C \quad \langle z - x^*, \nabla\phi(x^*) \rangle \geq 0. \tag{2.3.13}$$

On the other hand, the convexity of C and (CGP) yield (see (2.3.8))

$$\forall z \in C \quad \langle \mu\nabla\phi(x(t)) + \dot{x}(t), z - \dot{x}(t) - x(t) \rangle \geq 0. \tag{2.3.14}$$

Take $z = x^*$ in (2.3.14), and $z = x(t) + \dot{x}(t)$ in (2.3.13). It follows that

$$\langle \mu[\nabla\phi(x(t)) - \nabla\phi(x^*)] + \dot{x}(t), x^* - x(t) - \dot{x}(t)\rangle \geq 0.$$

Thus for all t in $[0, T_{max}[$

$$\frac{d}{dt}\left(\frac{1}{2} \mid x(t) - x^* \mid^2 + \mu[\phi(x(t)) - \phi(x^*) - \langle \nabla\phi(x^*), x(t) - x^*\rangle]\right)$$
$$\leq - \mid \dot{x}(t) \mid^2 - \mu\langle \nabla\phi(x(t)) - \nabla\phi(x^*), x(t) - x^*\rangle$$

which is precisely (2.3.12).

The standard arguments developed in Theorem 2.2.1 can be applied to obtain that

- $t \mapsto x(t)$ is defined on $[0, +\infty[$,
- $\dot{x} \in L^2(0, +\infty; H)$.

Since $\phi(x(t)) - \phi(x^*) - \langle \nabla\phi(x^*), x(t) - x^*\rangle \geq 0$, for all $t \geq 0$, we have moreover

- x is bounded,
- $\lim_{t\to+\infty} \dot{x}(t) = 0$.

Let us now deal with (ii). First notice that the inequality (2.3.9) remains valid and thus

$$\forall z \in C \quad \phi(x(t)) - \phi(z) + \frac{1}{\mu} \mid \dot{x}(t) \mid^2 + \frac{1}{\mu}\langle \dot{x}(t), x(t) - z\rangle + \langle \dot{x}(t), \nabla\phi(x(t))\,\rangle \leq 0. \quad (2.3.15)$$

Since $t \mapsto x(t)$ is bounded and $\lim_{t\to+\infty} \dot{x}(t) = 0$, we infer from (2.3.15) that $\forall z \in C$, $\limsup_{t\to+\infty} \phi(x(t)) - \phi(z) \leq 0$

thus

$$\limsup_{t\to+\infty} \phi(x(t)) \leq \inf_C \phi. \quad (2.3.16)$$

Let t_n be an increasing sequence such that $\phi(x(t_n)) \mapsto \liminf_{t\to+\infty} \phi(x(t))$. Since $x(t_n)$ is a bounded sequence, it is weakly relatively compact in H. Therefore there exists $t_{n_k} \to +\infty$ and x_1 in H such that $w - \lim_{k\to+\infty} x(t_{n_k}) = x_1$. Noticing that $w - \lim_{k\to+\infty} x(t_{n_k}) = w - \lim_{k\to+\infty} \dot{x}(t_{n_k}) + x(t_{n_k}) = w - \lim_{k\to+\infty} P_C(x(t_{n_k}) - \mu\nabla\phi(x(t_{n_k})))$, we see that x_1 can be obtained as a limit of a sequence in C and thus $x_1 \in C$.

Using the weak lower semicontinuity of ϕ we obtain $\liminf_{t\to+\infty} \phi(x(t)) = \lim_{k\to+\infty} \phi(x(t_{n_k})) \geq \phi(x_1) \geq \inf_C \phi$ and by (2.3.16), it ensues that $\lim_{t\to+\infty} \phi(x(t)) = \inf_C \phi$.

Let us prove the weak convergence of the orbit x. Let x_1 and x_2 be two weak cluster points of x, and denote by $(t_n)_{n\in N}$ and $(\tau_n)_{n\in N}$ some corresponding real valued subsequences, $t_n \to \infty$, $\tau_n \to \infty$ as $n \to +\infty$.

By direct algebra

$$\mathcal{E}(t, x_1) - \mathcal{E}(t, x_2) = 2\langle x(t), x_2 - x_1\rangle + \mid x_1 \mid^2 - \mid x_2 \mid^2 - 2\mu\langle \nabla\phi(x_1) - \nabla\phi(x_2), x(t)\rangle. \quad (2.3.17)$$

The property (i) implies that $\mathcal{E}(t, x_1) - \mathcal{E}(t, x_2)$ converges as $t \to +\infty$. Replacing t, successively by t_n and τ_n and passing to the limit we obtain the equality

$$- \mid x_1 - x_2 \mid^2 - 2\mu\langle \nabla\phi(x_1) - \nabla\phi(x_2), x_1\rangle = \mid x_1 - x_2 \mid^2 - 2\mu\langle \nabla\phi(x_1) - \nabla\phi(x_2), x_2\rangle.$$

Equivalently, $2 \mid x_1 - x_2 \mid^2 + 2\mu\langle \nabla\phi(x_1) - \nabla\phi(x_2), x_1 - x_2\rangle = 0$.

By monotonicity of $\nabla\phi$, both terms of the previous equality are nonnegative, and thus $x_1 = x_2$. ■

2.4 Convergence of the trajectories for a quasi-convex function

From now on, the initial data x_0 is supposed to be in C.
Let us recall that a function $\phi : H \mapsto \mathbb{R}$ is said to be quasi-convex if its lower level sets are convex. More precisely, if we set, for all γ in $\mathbb{R}$

$$lev_\gamma \phi := \{x \in H \mid \phi(x) \leq \gamma\},$$

then the lower level set $lev_\gamma \phi$ is convex.

If ϕ is continuously differentiable then, for all $x \in H$, the following property holds

$$\forall z \in lev_{\phi(x)}, \quad \langle \nabla \phi(x), z - x \rangle \leq 0. \tag{2.4.18}$$

For a study of dissipative systems with a quasi-convex potential, see for instance Goudou [68], Kiwiel-Murty [86].

Theorem 2.4.1 *Assume that ϕ is quasi-convex, satisfies $(\mathcal{H})$, and $\inf_C \phi$ is attained. Then the solution of (CGP) converges weakly in H.*

Let x_∞ be the limit point of the trajectory and assume in addition that H is finite-dimensional. Then x_∞ satisfies the following optimality condition

$$\nabla \phi(x_\infty) \in -N_C(x_\infty). \tag{2.4.19}$$

Proof. If H is finite-dimensional and $\lim_{t\to\infty} x(t) = x_\infty$, we deduce from Theorem 2.2.1 that $x_\infty - P_C(x_\infty - \mu \nabla \phi(x_\infty)) = 0$. The inclusion (2.4.19) follows from the formula $P_C = (I + N_C)^{-1}$, where I denotes the identity map of H.

To obtain the weak convergence in the general case, let us prove, as for the convex case, that $t \mapsto E(x(t), x^*) = \frac{1}{2\mu} \mid x(t) - x^* \mid^2 + \phi(x(t)) - \phi(x^*)$ is a Lyapounov function for some well chosen x^*.

Set $m = \lim_{t\to\infty} \phi(x(t))$, and $S_m = lev_m \phi \cap C$. Note that S_m is nonempty since $\inf_C \phi$ is attained on C. For $x^* \in S_m$, let us study $t \mapsto E(x(t), x^*)$ and its first derivative

$$\dot{E}(x(t), z) = \frac{1}{\mu} \langle \dot{x}(t), \mu \nabla \phi(x(t)) + x(t) - x^* \rangle.$$

Like in Theorem 2.3.1 $\langle \mu \nabla \phi(x(t)) + \dot{x}(t), \dot{x}(t) + x(t) - x^* \rangle \leq 0$.
Hence

$$\langle \mu \nabla \phi(x(t)) + x(t) - x^*, \dot{x}(t) \rangle \leq - \mid \dot{x}(t) \mid^2 + \mu \langle \nabla \phi(x(t)), x^* - x(t) \rangle. \tag{2.4.20}$$

On the other hand, from the fact that for all $t \geq 0$, $\phi(x(t)) \geq m$, and from the quasi-convexity of ϕ we infer

$$\langle \nabla \phi(x(t)), x^* - x(t) \rangle \leq 0. \tag{2.4.21}$$

Combining (2.4.20), (2.4.21) gives, for all $t \geq 0$, $\dot{E}(x(t), x^*) \leq -\frac{1}{\mu} \mid \dot{x}(t) \mid^2$.

Whence we deduce that for all $x^* \in S_m$, $\frac{1}{2\mu} \mid x(t) - x^* \mid^2$ converges as $t \to \infty$. Observe that ϕ is lower semicontinuous for the weak topology and thus for every cluster point x_∞, $\phi(x_\infty) \leq \liminf_{t\to+\infty} \phi(x(t)) = m$. The conclusion follows again from Opial lemma. ∎

2.5 Asymptotic control

This section proposes an asymptotic control result involving a Tykhonov-like regularization. Consider the following dynamical system,

$$(CGP)_\varepsilon \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\nabla\phi(x(t)) - \varepsilon(t)x(t)] = 0 \\ x(0) = x_0 \in C \end{cases}$$

where $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function converging to zero.

The use of Tykhonov regularization as a controling means for differential inclusions has been introduced in [15] for the steepest descent method. Under reasonable assumptions, it allows both to select a particular equilibrium, and to obtain strong convergence of solutions.

In the mechanical framework, the regularizing term εx can be seen as a vanishing force, attracting the orbit towards zero. Actually, if ε goes slowly enough to zero, we shall see that all trajectories are constrained to converge strongly to the *minimal norm* solution of the problem $\min_C \phi$.

This physical interpretation finds its mathematical justification in the following,

Theorem 2.5.1 *Assume that ϕ is convex, satisfies $(\mathcal{H})$ and that $S = \operatorname{argmin}_C \phi \neq \emptyset$.*

Let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^1 nonincreasing function such that $\int_0^{+\infty} \varepsilon(s)ds = +\infty$, $\dot{\varepsilon}$ is bounded and converges to zero.

Then $(CGP)_\varepsilon$ admits a unique solution on $[0, +\infty)$, that strongly converges towards the element of minimal norm of S. In other words

$$s - \lim_{t\to\infty} x(t) = p,$$

where $p = P_S(0)$.

Proof. The arguments concerning the existence and the uniqueness of a solution are very similar to Theorem 2.2.1, we only give the main lines of the proof. As before we easily obtain that the solution x has its values in C. From this we deduce, as in Theorem 2.2.1, that

$$\frac{d}{dt}(E_\varepsilon(t)) \le -|\dot{x}(t)|^2 + \frac{\dot{\varepsilon}(t)}{2}|x(t)|^2 \tag{2.5.22}$$

where $E_\varepsilon(t) = \mu\phi(x(t)) + \frac{\varepsilon(t)|x(t)|^2}{2}$.

Adapting former arguments it follows from $(\mathcal{H})$ and (2.5.22), that the solution is defined on $[0, +\infty)$, with velocity in $L^2(0, +\infty; H)$. Moreover, if x is supposed to be bounded, it ensues that $\dot{x}(t) \to 0$ and $\phi(x(t))$ converges as $t \to \infty$. This can be summed up in

$$\begin{cases} \dot{x} \in L^2(0, +\infty; H), \\ x \quad \text{bounded} \Rightarrow \lim_{t\to+\infty} \dot{x}(t) = 0, \lim_{t\to+\infty} \phi(x(t)) \text{ exists.} \end{cases} \tag{2.5.23}$$

Let us focus on the proof of the strong convergence of the trajectory x. The nonautonomous nature of $(CGP)_\varepsilon$ gives rise to oscillating trajectories that prevents us from exhibiting a proper Lyapounov function. However an acute study of the following function allows to overcome this difficulty.

For all $t \geq 0$, set

$$F(t) = E_\varepsilon(t) + \frac{1}{2}|x(t) - p|^2. \tag{2.5.24}$$

By convexity of C : $\langle x - \mu\nabla\phi(x) - \varepsilon x - x - \dot{x}, p - x - \dot{x}\rangle \leq 0$, thus after computation

$$\frac{d}{dt}F(t) + |\dot{x}|^2 + \mu\langle\nabla\phi(x), x - p\rangle - \dot{\varepsilon}\frac{|x|^2}{2} + \varepsilon\langle x, x - p\rangle \leq 0. \tag{2.5.25}$$

The lack of monotonicity of F is due to the term $\langle x, x - p\rangle$, this leads to examine two cases. But before going further, we need a lemma.

Lemma 2.5.2 *Under assumption of Theorem 2.5.1, and if the solution x is bounded, then all the weak limit points of x are minimizers of $\phi_{|C}$.*

Proof of Lemma 5.2. x is bounded, hence $\dot{x}(t) \to 0$ as $t \to +\infty$ and $\nabla\phi(x(.))$ is bounded. Thus $\frac{d}{dt}F(t) = \langle\dot{x}(t), \nabla\phi(x(t))\rangle + \varepsilon(t)\langle\dot{x}(t), x(t)\rangle + \dot{\varepsilon}(t)\frac{|x(t)|^2}{2} + \langle\dot{x}(t), x(t) - p\rangle$ tends to zero, which combined to inequality (2.5.25) gives

$$\limsup_{t\to+\infty} \mu\langle\nabla\phi(x(t)), x(t) - p\rangle \leq 0. \tag{2.5.26}$$

This last quantity being nonnegative, it follows that $\lim_{t\to+\infty}\langle\nabla\phi(x(t)), x(t) - p\rangle = 0$.

Let x^* be a weak limit point of x relatively to an increasing sequence of positive real numbers τ_n. Using the convexity of ϕ, we obtain $\phi(p) \geq \phi(x(\tau_n)) + \langle\nabla\phi(x(\tau_n)), p - x(\tau_n)\rangle$. Passing to the inf-limit and according to the lower semicontinuity of ϕ, one obtains $\phi(p) \geq \phi(x^*)$, which exactly means that $x^* \in S$. ■

Case 1. In that part we assume that there exists $t_0 \in \mathbb{R}_+$ such that $\langle x(t), x(t) - p\rangle \geq 0$, $\forall t \geq t_0$. For simplicity we assume that $t_0 = 0$. F becomes in that case a nonincreasing function, and since E_ε is bounded, so is F (see (2.5.24)). Hence, applying (2.5.22) it follows that the functions F and $|x - p|$ have a limit as $t \to +\infty$.

Let us argue by contradiction and assume that $|x(t) - p| \to l > 0$ as $t \to +\infty$.

First we notice that $\liminf_{t\to+\infty}\langle p, x(t) - p\rangle \geq 0$. Indeed, if t_n is a sequence of real numbers realizing this inf limit, by boundedness of x and Lemma 2.5.2 there exists a subsequence t_{n_k} of t_n, such that $x(t_{n_k}) \to x^* \in S$. Therefore

$$\liminf_{t\to+\infty}\langle p, x(t) - p\rangle = \langle p, x^* - p\rangle = -\langle 0 - P_S(0), x^* - P_S(0)\rangle \geq 0.$$

Since $\langle x, x - p\rangle = |x - p|^2 + \langle p, x - p\rangle$, we can assume that there exists some $T > 0$ such that $t \geq T$ implies $\langle x(t), x(t) - p\rangle \geq \frac{l}{2}$. From (2.5.25) it ensues

$$\frac{d}{dt}F(t) + \varepsilon(t)\langle x(t), x(t) - p\rangle \leq 0$$

for all t in $\mathbb{R}_+$. Integrating over (T, t), $t \geq T$ the above inequality becomes

$$F(t) - F(T) + \frac{l}{2}\int_T^t \varepsilon \leq 0.$$

But we know that F converges whereas $\int_{\mathbb{R}_+} \varepsilon = +\infty$. This yields a contradiction.

Case 2. In that part, the function $\langle x, x-p\rangle$ is allowed to reach real negative value as time elapses, which naturally leads to examine the set $B = \{y \in H | \langle y, y-p\rangle \leq 0\}$. Observing that $\langle y, y-p\rangle = \langle y - p/2 + p/2, y - p/2 - p/2\rangle = |y-p/2|^2 - |p/2|^2$, we see that B is the $|p/2|$ radius closed ball centered at point $p/2$.

Set

$$\begin{aligned} I &= \{t \in \mathbb{R}_+ | x(t) \in B\}, \\ J &= \{t \in \mathbb{R}_+ | x(t) \notin B\}, \end{aligned}$$

and assume that I is unbounded.

I, J are respectively closed and open in $\mathbb{R}$, hence there exists a nondecreasing sequence of real numbers t_k such that $I = \bigsqcup_{k\in N}[t_{2k}, t_{2k+1}]$ and $J = \bigsqcup_{k\in N}]t_{2k+1}, t_{2k+2}[$. Note that we have implicitely assume that $x_0 \in B$, which is not restrictive in our study.

In order to tackle the most difficult problem first, we assume J to be unbounded. Before examining the asymptotic behaviour of the orbit x on I and J, let us prove that x is bounded.

If $t \in I$, then, by definition of I, $x(t) \in B$ which implies that $x_{|I}$ is bounded.

For t in J, there exists $k(t) := k$ in N such that t belongs to $]t_{2k+1}, t_{2k+2}[$. Coming back to (2.5.25), we deduce that $F_{|]t_{2k+1},t_{2k+2}[}$, and thus $F_{|[t_{2k+1},t_{2k+2}]}$ are nonincreasing functions. Since $F(t) \leq F(t_{2k+1})$ and $x(t_{2k+1}) \in B$, it follows that $F(t) \leq F(t_{2k+1}) = E_\varepsilon(t_{2k+1}) + \frac{1}{2}|x(t_{2k+1}) - p|^2 \leq E_\varepsilon(0) + \max_{y\in B}\frac{1}{2}|y-p|^2$. But since E_ε is bounded, so is $x_{|J}$.

Let us focus on the limit points of $x_{|I}(t)$ when $t \to +\infty$, $t \in I$. Observe that $B \cap S = \{p\}$, thus, by Lemma 5.2, $x_{|I}(t)$ has to weakly converge to its unique limit point p as $t \to +\infty$, $t \in I$. To obtain the strong convergence one just has to notice that $y \in B$ implies $|y| \leq |p|$, and thus $\limsup_{t\to+\infty}|x_{|I}(t)|^2 \leq |p|^2$.

Now we prove an equivalent result for $x_{|J}$. If $t \in J$, let $k(t)$ be as above, and set $\tau_t = t_{2k(t)+1}$. Observe that τ_t belongs to I and therefore $x(\tau_t)$ strongly converges to p as $t \to +\infty$, $t \in J$. Besides we know that $F(t) \leq F(\tau_t)$, and thus

$$\mu\phi(x(t)) + \frac{\varepsilon(t)|x(t)|^2}{2} + \frac{1}{2}|x(t)-p|^2 \leq \mu\phi(x(\tau_t)) + \frac{\varepsilon(\tau_t)|x(\tau_t)|^2}{2} + \frac{1}{2}|x(\tau_t)-p|^2.$$

Passing to the sup-limit, (2.5.23) yields

$$\mu \lim_{t\to+\infty,\, t\in J}\phi(x(t)) + \limsup_{t\to+\infty,\, t\in J}\frac{1}{2}|x(t)-p|^2 \leq \mu \lim_{t\to+\infty,\, t\in J}\phi(x(\tau_t)). \tag{2.5.27}$$

We finally deduce from (2.5.27), that $x(t)$, $t \in J$ strongly converges to p as $t \to +\infty$. The case for which J is bounded can be solved with similar ideas. ∎

2.6 The gradient projection method for a continuous convex criterion.

In the sequel, H is supposed to be finite-dimensional, and ϕ to be convex continuous on H. Classically, the subdifferential of ϕ at $y_0 \in H$ is the convex subset $\partial\phi(y_0)$ of H

characterized by the following property

$$z \in \partial\phi(y_0) \Leftrightarrow \forall y \in H,\ \phi(y) \geq \phi(y_0) + \langle z, y - y_0\rangle. \qquad (2.6.28)$$

We propose to establish the existence of an absolutely continuous solution to the following differential inclusion

$$(CGP)_g \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\partial\phi(x(t))] \ni 0 \quad \text{ae in } (0, +\infty), \\ x(0) = x_0 \in C. \end{cases}$$

Defining $A : H^2 \to H$ by $A_x(v) = x - P_C(x - \mu v)$ for all (x, v) in H^2, it is easy to see that, if x is fixed, then $A_x : H \to H$ is a maximal monotone operator. Therefore $(CGP)_g$ can be reexpressed in the following form

$$A^{-1}_{x(t)}(\dot{x}(t)) + \partial\phi(x(t)) \ni 0, \ \text{ ae in } (0, +\infty).$$

This formulation is akin to doubly nonlinear problems arising in PDE, see Colli- Visintin [46], Kenmochi-Pawlow [83] and references therein. As in [46], where $B\dot{x}(t) + \partial\phi(x(t)) \ni 0$, B maximal monotone, is considered, we have not been able to prove the uniqueness of the solution and as far as we know, it is still an open question.

To obtain a solution of $(CGP)_g$, let us define for any positive λ, the approximate problems

$$(CGP)_\lambda \quad \begin{cases} \dot{x}_\lambda(t) + x_\lambda(t) - P_C[x_\lambda(t) - \mu\nabla\phi_\lambda(x_\lambda(t))] = 0 \\ x_\lambda(0) = x_{0\lambda} \in C \end{cases} \qquad (2.6.29)$$

where $x_{0\lambda}$ is a sequence in C such that $\lim_{\lambda\to 0} x_{0\lambda} = x_0$, and ϕ_λ is the Moreau-Yosida approximation of ϕ.

The general results concerning the Moreau-Yosida approximate can be found in [37], or in Rockafellar-Wets [114]. Let us recall that, for any positive λ, ϕ_λ is defined as the episum of ϕ and of the quadratic kernel $y \in H \to \frac{1}{2\lambda}|y|^2$, that is

$$\forall y \in H \quad \phi_\lambda(y) = \inf_{z\in H}\{\phi(z) + \frac{1}{2\lambda}|y - z|^2\}.$$

ϕ_λ is a C^1 function from H into $\mathbb{R}$, whose gradient $\nabla\phi_\lambda$ is Lipschitz continuous. Moreover, for any y in H

$$\sup_{\lambda>0}\phi_\lambda(y) = \lim_{\lambda\to 0}\phi_\lambda(y) = \phi(y). \qquad (2.6.30)$$

Set $\partial\phi^o(y) = \inf_{z\in\partial\phi(y)}|z|$ where $y \in dom\,\partial\phi$, then

$$|\nabla\phi_\lambda(y)| \leq \partial\phi^o(y). \qquad (2.6.31)$$

Let us state the central result of this section

Theorem 2.6.1 *Assume that ϕ is convex, continuous on H, and bounded from below.*

Then there exists an absolutely continuous solution $t \in [0, +\infty) \to x(t) \in H$ satisfying

$$(CGP)_g \quad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\partial\phi(x(t))] \ni 0 \quad ae \ in \ (0,+\infty) \\ x(0) = x_0 \in C \end{cases} . \tag{2.6.32}$$

Moreover,
(i) $\forall t \geq 0, \ x(t) \in C$.
(ii) The function $t \in [0,+\infty) \to \phi(x(t))$ *is absolutely continuous with*

$$\frac{d}{dt}\phi(x(t)) \leq -|\dot{x}(t)|^2 \ ae \ in \ (0,+\infty).$$

Therefore $\dot{x} \in L^2(0,+\infty; H)$.
(iii) $\lim_{t\to+\infty} \phi(x(t)) = \inf_C \phi$.
(iv) If $\operatorname{argmin}_C \phi \neq \emptyset$*, then* $x(t)$ *converges to some element of* $\operatorname{argmin}_C \phi$*, as* $t \to +\infty$. *Besides, there exists a nonnegative constant* M *such that* $[\phi(x(t)) - \inf_C \phi] \leq M/t+1, \forall t \geq 0$.

Fix $T > 0$, and denote by $\mathcal{D}(]0,T[)$ the set of C^∞ real functions with compact support in $]0,T[$. We shall need the following useful lemmas.

Lemma 2.6.2 *If* ϕ *is a continuous convex function on a finite-dimensional space* H*, then* $\partial\phi$ *is bounded on bounded sets. More precisely if* B *is bounded in* H*, then there exists some* $M > 0$ *such that*

$$\forall y \in B \quad \forall z \in \partial\phi(y) \quad |z| \leq M. \tag{2.6.33}$$

Proof. Recall that H is finite-dimensional and see for instance Rockafellar [112], (th. 24.7, p237). ■

Lemma 2.6.3 *Let* $t \in [0,+\infty) \to u(t) \in H$ *be an absolutely continuous function, and assume that* $t \in [0,+\infty) \to \phi(u(t))$ *is also absolutely continuous.*

Let D *be the subset of* $\mathbb{R}_+$ *on which* $t \to \phi(u(t))$ *and* $t \to u(t)$ *are derivable. Then,* dt *being the Lebesgue measure on* $\mathbb{R}$*,* $dt(\mathbb{R}_+ \backslash D) = 0$ *and*

$$\forall t \in D \quad \forall z \in \partial\phi(u(t)) \quad \frac{d}{dt}\phi(u(t)) = \langle \dot{u}(t), z \rangle.$$

Proof. The fact that an absolute continuous function is derivable almost everywhere is a classical result, and as a consequence $dt(\mathbb{R}_+ \backslash D) = 0$.

Fix t in D. We have, for convenient positive ε

$$\phi(u(t+\varepsilon)) - \phi(u(t)) \geq \langle u(t+\varepsilon) - u(t), z \rangle,$$

where $z \in \partial\phi(u(t)) \neq \emptyset$ (recall that ϕ is continuous). Divide by ε and let $\varepsilon \to 0$, this gives at the limit $\frac{d}{dt}\phi(u(t)) \geq \langle \dot{u}(t), z \rangle$. Replacing ε by $-\varepsilon$, yields the converse inequality. ■

Proof of Theorem 2.6.1. First, some uniform estimations relying on the solutions of $(CGP)_\lambda$ are established on a bounded time interval $[0,T]$. Then arguing by compacity, we pass to the limit to obtain (2.6.32) on $[0,T]$. When no confusion can occur, the time variable t will be dropped down. For the sake of legibility, *all subsequences of* $x_\lambda, \dot{x_\lambda}$... are still denoted $x_\lambda, \dot{x_\lambda}$...

Estimations

• Owing to Theorem 2.2.1, the solution x_λ of the approximation scheme $(CGP)_\lambda$ satisfies for all t in $[0,T]$

$$\phi_\lambda(x_\lambda(t)) - \phi_\lambda(x_{0\lambda}) + \int_0^t |\dot{x}_\lambda|^2 \leq 0. \tag{2.6.34}$$

By (2.6.30) and the fact that ϕ is bounded from below, we obtain that $\dot{x}_\lambda$ is a bounded sequence in $L^2(0,T;H)$. Thus one can extract from $\dot{x}_\lambda$ a subsequence that weakly converges in $L^2(0,T;H)$ to some function $v:(0,T)\to H$.

• From the formula $x_\lambda(t) - x_\lambda(\tau) = \int_\tau^t \dot{x}_\lambda$, we deduce that for all $t \geq \tau$ in $[0,T]$ $|x_\lambda(t) - x_\lambda(\tau)| \leq \sqrt{t-\tau}\sqrt{\int_\tau^t |\dot{x}_\lambda|^2}$.

It ensues that x_λ is an equicontinuous bounded sequence in $C([0,T],H)$ equipped with the supremum norm, and therefore Ascoli Theorem yields the existence of a cluster point $x \in C([0,T],H)$ to the sequence x_λ. Moreover by Theorem 2.2.1, $x_\lambda([0,T]) \subset C$ and therefore, C being closed,

$$\forall t \in [0,T],\ x(t) \in C. \tag{2.6.35}$$

• The preceding two points yield the existence of a subsequence x_λ such that

$$x_\lambda \to x \text{ in } C([0,T],H), \tag{2.6.36}$$

$$\dot{x}_\lambda \to v \text{ in } w - L^2(0,T;H). \tag{2.6.37}$$

Whence, we have $\dot{x}_\lambda \to \dot{x}$ and $\dot{x}_\lambda \to v$ in the sense of distributions in $]0,T[$. Identifying both limits, we see that $\dot{x}$ belongs to $L^2(0,T;H)$ which implies that x is absolutely continuous.

• By (2.6.31), we have for all t in $[0,T]$, $|\nabla\phi_\lambda(x_\lambda(t))| \leq |\partial\phi^o(x(t))|$. Now Lemma 2.6.2, and the continuity property of x implies that $\nabla\phi_\lambda(x_\lambda(.))$ is a bounded sequence in $L^\infty(0,+\infty,H)$. In particular, it is relatively compact in $w - L^2(0,T;H)$, with at least some cluster point, say $g \in L^2(0,T;H)$. Therefore, after extraction,

$$\nabla\phi_\lambda(x_\lambda(.)) \to g \quad \text{in} \quad w - L^2(0,T;H). \tag{2.6.38}$$

• Let us study the sequence $\phi_\lambda(x_\lambda(.))$. For all $t \geq \tau$ in $[0,T]$

$$|\phi_\lambda(x_\lambda(t)) - \phi_\lambda(x_\lambda(\tau))| \leq \int_\tau^t |< \nabla\phi_\lambda(x_\lambda), \dot{x}_\lambda >| ds \leq M\sqrt{t-\tau}\sqrt{\int_\tau^t |\dot{x}_\lambda|^2} \tag{2.6.39}$$

where M is a bound of $|\nabla\phi_\lambda(x_\lambda(.))|$ on $[0,T]$. This shows by Ascoli Theorem that a subsequence of $\phi_\lambda(x_\lambda(.))$ converges uniformly on $[0,T]$ to an element ψ of $C([0,T],H)$.

Let us prove that, for all $t \in [0,T]$, $\psi(t) = \phi(x(t))$. Take t in $[0,T]$. If $\lambda_0 > \lambda > 0$, it follows from (2.6.30) that $\phi_{\lambda_0}(x_\lambda(t)) \leq \phi_\lambda(x_\lambda(t))$, and letting $\lambda \to 0$, (2.6.36) yields $\phi_{\lambda_0}(x(t)) \leq \psi(t)$. Using (2.6.30) again, it ensues $\phi(x(t)) \leq \psi(t)$.
Let $M > 0$ be a bound of $\nabla\phi_\lambda(x_\lambda(.))$. The convex inequality gives

$$\phi_\lambda(x(t)) \geq \phi_\lambda(x_\lambda(t)) - M|x_\lambda(t) - x(t)|. \tag{2.6.40}$$

From (2.6.30), and (2.6.36), we finally deduce $\phi(x(t)) \geq \psi(t)$, and thus

$$\phi_\lambda(x_\lambda(t)) \to \phi(x(t)) \text{ in } C([0,T],H). \tag{2.6.41}$$

Besides, $\dot{x}_\lambda$ and $\nabla\phi_\lambda(x_\lambda(.))$ being respectively bounded sequences in $L^2(0,T;H)$ and $L^\infty(0,T;H)$, it follows that $\frac{d}{dt}\phi_\lambda(x_\lambda(.))$ is bounded in $L^2(0,T;H)$. This implies that the first derivative in the sense of distributions of $t \in (0,T) \to \phi(x(t))$ is in $L^2(0,T;H)$, and in particular that $\phi(x(.))$ is absolutely continuous.

• Let us identify g. Fix $\theta \geq 0$ in $\mathcal{D}(]0,T[)$. Integrating the convex inequality, we obtain for all $y \in H$

$$\int_0^T \theta(t)[\phi_\lambda(y) - \phi_\lambda(x_\lambda(t)) - \langle\nabla\phi_\lambda(x_\lambda(t)), y - x_\lambda(t)\rangle]dt \geq 0. \tag{2.6.42}$$

From (2.6.36), (2.6.38), (2.6.30) and (2.6.41), we obtain

$$\int_0^T \theta(t)[\phi(y) - \phi(x(t)) - \langle g(t), y - x(t)\rangle]dt \geq 0.$$

The latter being true for all $\theta \geq 0$ in $\mathcal{D}(]0,T[)$, it follows that

$$\phi(y) \geq \phi(x(t)) + \langle g(t), y - x(t)\rangle \text{ ae in } [0,T].$$

This implies by definition of the subdifferential, that

$$g(t) \in \partial\phi(x(t)) \;\text{ ae in } [0,T]. \tag{2.6.43}$$

All the above results are gathered in the following statement

$$\begin{aligned}
x_\lambda &\to x \text{ in } C([0,T],H), && (2.6.44)\\
\dot{x}_\lambda &\to \dot{x} \text{ in } w - L^2(0,T;H), && (2.6.45)\\
\phi_\lambda(x_\lambda(t)) &\to \phi(x(.)) \text{ in } C([0,T],H), && (2.6.46)\\
\nabla\phi_\lambda(x_\lambda(.)) &\to g(t) \in \partial\phi(x(t)) \text{ in } w - L^2(0,T;H). && (2.6.47)
\end{aligned}$$

Passing to the limit

The (sub)sequence x_λ verifies

$$\begin{cases} \dot{x}_\lambda(t) + x_\lambda(t) - P_C[x_\lambda(t) - \mu\nabla\phi_\lambda(x_\lambda(t))] = 0,\\ \forall t \in [0,T], x_\lambda(0) = x_{0\lambda}, \end{cases}$$

which is equivalent to

$$\begin{cases} \langle -\mu\nabla\phi_\lambda(x(t)) - \dot{x}_\lambda(t), \xi - x_\lambda(t) - \dot{x}_\lambda(t)\rangle \leq 0,\\ \forall t \in [0,T], \forall\xi \in C, x_\lambda(0) = x_{0\lambda}. \end{cases}$$

This is also equivalent to

$$\begin{cases} \int_0^T \langle -\mu\nabla\phi_\lambda(x_\lambda(t)) - \dot{x}_\lambda(t), \xi - x_\lambda(t) - \dot{x}_\lambda(t)\rangle\theta(t)dt \leq 0,\\ \forall\theta \geq 0 \in \mathcal{D}(]0,T[), \forall\xi \in C, x_\lambda(0) = x_{0\lambda}, \end{cases}$$

and thus to

$$\begin{cases} \int_0^T \langle \mu\nabla\phi_\lambda(x_\lambda) + \dot{x}_\lambda, x_\lambda - \xi\rangle\theta + \int_0^T \mu\theta\frac{d}{dt}\phi_\lambda(x_\lambda) + \int_0^T |\dot{x}_\lambda|^2\theta \le 0, \\ \forall\theta \ge 0 \in \mathcal{D}(]0,T[), \forall\xi \in C, x_\lambda(0) = x_{0\lambda}. \end{cases}$$

Now, in order to take the inf-limit of each term in the previous inequality fix $\theta \ge 0$, and ξ in C.

By (weak) lower semicontinuity of the semi-norm $\int_0^T |\,.\,|^2\theta(s)ds$ on $L^2(0,T;H)$ and (2.6.45),

$$\int_0^T |\dot{x}|^2\theta \le \liminf_{\lambda\to 0}\int_0^T |\dot{x}_\lambda|^2\theta.$$

By combining (2.6.44), (2.6.45), and (2.6.47) one obtains that

$$\lim_{\lambda\to 0}\int_0^T \theta\langle\mu\nabla\phi_\lambda(x_\lambda) + \dot{x}_\lambda, x_\lambda - \xi\rangle = \int_0^T \theta\langle\mu g + \dot{x}, x - \xi\rangle.$$

From Lemma 2.6.3 and the fact that $t \to \phi(x(t))$ is absolutely continuous it follows that the first derivative almost everywhere, and the derivative in the sense of distributions on $]0,T[$ of $t \to \phi(x(t))$ are both equal to $t \to \langle\dot{x}(t), g(t)\rangle$. From (2.6.46) we deduce $\frac{d}{dt}\phi_\lambda(x_\lambda(.)) \to \frac{d}{dt}\phi(x(.))$ in the sense of distributions on $]0,T[$, and thus

$$\lim_{\lambda\to 0}\int_0^T \theta(s)\frac{d}{dt}\phi_\lambda(x_\lambda(s))ds = \int_0^T \theta(s)\frac{d}{dt}\phi(x(s))ds = \int_0^T \theta\langle\dot{x}, g\rangle.$$

Combining the last three results yields

$$\begin{cases} \int_0^T \langle\mu g + \dot{x}, x - \xi\rangle\theta + \int_0^T \mu\langle g, \dot{x}\rangle + \int_0^T |\dot{x}|^2\theta \le 0, \\ \forall\theta \ge 0 \in \mathcal{D}(]0,T[), \forall\xi \in C, x(0) = x_0, \end{cases}$$

and after rearranging terms

$$\begin{cases} \int_0^T \theta\langle x - \mu g - x - \dot{x}, \xi - x - \dot{x}\rangle \le 0, \\ \forall\theta \in \mathcal{D}(]0,T[), \forall\xi \in C, x(0) = x_0. \end{cases} \tag{2.6.48}$$

In order to use the variational characterization of $P_C(x(.) - \mu g(.))$ in (2.6.48), let us prove that $\dot{x}(t) + x(t) \in C$ ae on [0,T]. Consider the following subset of $L^2(0,T;H)$, $\mathcal{C} = \{f \in L^2(0,T;H) | f(t) \in C$ ae on (0,T)$\}$. Clearly $\mathcal{C}$ is closed in $L^2(0,T;H)$ for the strong topology, and since $\mathcal{C}$ is convex, it is also closed for the weak topology. By $(CGP)_\lambda$, we have $\dot{x}_\lambda + x_\lambda \in \mathcal{C}$. Whence, from (2.6.44), (2.6.45), and the weak closedness property of $\mathcal{C}$, it follows that $\dot{x} + x \in \mathcal{C}$. Using (2.6.48) we obtain that $\dot{x}(t) + x(t) - P_C[x(t) - \mu g(t)] = 0$ ae on $(0,T)$, and thus by (2.6.43), it follows that

$$\dot{x}(t) + x(t) - P_C[x(t) - \mu\partial\phi(x(t))] \ni 0 \text{ ae in } (0,T), \tag{2.6.49}$$

with

$$x(0) = x_0. \tag{2.6.50}$$

To obtain a solution of $(CGP)_g$ defined on $[0,+\infty[$, let us observe that (2.6.30) and (2.6.34) imply that the sequence $\dot{x}_\lambda$ is actually bounded in $L^2(0,+\infty;H)$. Combining this fact with those obtained above yields the existence of a global solution x satisfying the claimed properties. Indeed we deduce from (2.6.49), (2.6.50) that x satisfies (2.6.32). The viability property (i) of x, is guaranteed by (2.6.35).

The proofs of (ii),(iii), an (iv) rely on the absolute continuity of x and $\phi(x(.))$. This allows by use of Lemma 2.6.3, to reproduce the arguments of Theorems 2.2.1-2.3.1 with nearly no change. ■

Remarks (a) Denote by $ri\,C$ the relative interior of C. By adapting the argument of Theorem 2.2.1 (ii) it follows easily that

$$x_0 \in ri\,C \Rightarrow \forall t \geq 0,\ x(t) \in ri\,C.$$

In other words $(CGP)_g$ is an *interior method* as soon as the initial value is strictly feasible.
(b) On the implementation of the method.
Given some sequences $\mu_k, \Delta t_k > 0$, an explicit discretization of $(CGP)_g$ gives

$$\frac{x_{k+1} - x_k}{\Delta t_k} + x_k - P_C[x_k - \mu_k \partial\phi(x_k)] \ni 0,\ k \in N,$$

which can be reformulated as

$$x_{k+1} \in (1 - \Delta t_k)x_k + \Delta t_k P_C[x_k - \mu_k \partial\phi(x_k)],\ k \in N. \tag{2.6.51}$$

This approach for solving approximatively dynamical systems is well-known, and of course, for $\mu_k = \mu$ and $\Delta t_k = 1$ the usual gradient-projected method (2.1.4) is recovered. By the above remark (a) we know that $(CGP)_g$ is an interior method whenever x_0 belongs to $ri\,C$; this suggests that a "good" discrete approximation should also enjoy this property. Very simple examples show that it is not the case of (2.1.4), however if we assume that the steptime parameters of (2.6.51) satisfy $\Delta t_k < 1$, an easy induction implies that the sequences $x_k,\ k \in N$ complying with (2.6.51) also verify

$$x_0 \in ri\,C \Rightarrow \forall k \in N,\ x_k \in ri\,C.$$

Besides, much like as in convex feasibility problems (see Censor-Eggermont-Gordon [42] and references therein), the form of (2.6.51) suggests to interpret Δt_k, $k \in N$ as a sequence of relaxation parameters and to study (2.6.51) within that perspective. Such a study is out of the scope of the present paper, but it is certainly a matter for future research.

Chapitre 3

Champs de gradients et métriques induites par les fonctions de Legendre.

Gradient flows associated with Hessian Riemannian structures induced by Legendre functions in constrained optimization

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Abstract. Motivated by a class of constrained minimization problems, we study the gradient flows with respect to Hessian Riemannian metrics induced by convex functions of Legendre type. The first result characterizes Hessian Riemannian structures on convex sets as those metrics that have a specific integration property with respect to variational inequalities, giving a new motivation for the introduction of Bregman-type distances. Then, the general evolution problem is introduced and a differential inclusion reformulation is given. Some explicit examples of these gradient flows are discussed. A general existence result is proved and global convergence is established under quasi-convexity conditions, with interesting refinements in the case of convex minimization. Dual trajectories are identified and sufficient conditions for dual convergence are examined for a convex program with positivity and equality constraints. Some convergence rate results are established. In the case of a linear objective function, several optimality characterizations of the orbits are given : optimal path of viscosity methods, continuous-time model of Bregman-type proximal algorithms, geodesics for some adequate metrics and projections of $\dot{q}$-trajectories of some Lagrange equations and completely integrable Hamiltonian systems. These results are based on a change of coordinates which is called Legendre transform coordinates and is studied in a general setting. Some of these results unify and generalize several previous works.

Keywords. Gradient flow, Hessian Riemannian metric, Legendre type convex function, existence, global convergence, Bregman distance, Liapounov functional, quasi-convex minimization, convex and linear programming, Legendre transform coordinates, geodesics, Lagrange and Hamilton equations.

3.1 Introduction

The aim of this paper is to study the existence, global convergence and geometric properties of gradient flows with respect to a specific class of Hessian Riemannian metrics on convex sets. Our work is indeed deeply related to the constrained minimization problem

$$(P) \qquad \min\{f(x) \mid x \in \overline{C},\ Ax = b\},$$

where $\overline{C}$ is the closure of a nonempty, *open* and convex subset C of $\mathbb{R}^n$, A is a $m < n$ real matrix with $m \leq n$, $b \in \mathbb{R}^m$ and $f \in C^1(\mathbb{R}^n)$. A strategy to solve (P) consists in endowing C with a Riemannian structure $(\cdot,\cdot)^H$, to restrict it to the relative interior of the feasible set $\mathcal{F} := C \cap \{x \mid Ax = b\}$, and then to consider the trajectories generated by the steepest descent vector field $-\nabla_H f_{|\mathcal{F}}$. This leads to the initial value problem

$$(H\text{-}SD) \qquad \dot{x}(t) + \nabla_H f_{|\mathcal{F}}(x(t)) = 0,\ x(0) \in \mathcal{F},$$

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where (H-SD) stands for H-steepest descent. We focus on those metrics that are induced by the Hessian $H = \nabla^2 h$ of a *Legendre type* convex function h defined on C (cf. Def. 3.3.1).

The use of Riemannian methods in optimization has increased recently : in relation with Karmarkar algorithm and linear programming see Karmarkar [82], Bayer-Lagarias [30] ; for continuous-time models of proximal type algorithms and related topics see Iusem-Svaiter-Da Cruz [78], Bolte-Teboulle [35]. For a systematic dynamical system approach to constrained optimization based on double bracket flows, see Brockett [38, 39], the monograph of Helmke-Moore [73] and the references therein. On the other hand, the structure of (H-SD) is also at the heart of some important problems in applied mathematics. For connections with population dynamics and game theory see Hofbauer-Sygmund [76], Akin [2], Attouch-Teboulle [20]. We will see that (H-SD) can be reformulated as the differential inclusion $\frac{d}{dt}\nabla h(x(t)) + \nabla f(x(t)) \in \mathrm{Im}\, A^T$, $x(t) \in \mathcal{F}$, which is formally similar to some evolution problems in infinite dimensional spaces arising in thermodynamical systems, see for instance Kenmochi-Pawlow [83] and Blanchard-Francfort [33].

A classical approach in the asymptotic analysis of dynamical systems consists in exhibiting attractors of the orbits by using Liapounov functionals. Our choice of Hessian Riemannian metrics is based on this idea. In fact, we consider first the important case where f is convex, a condition that permits us to reformulate (P) as a variational inequality problem : find $a \in \overline{\mathcal{F}}$ such that $(\nabla_{_H} f_{|\mathcal{F}}(x), x-a)_x^H \geq 0$ for all x in $\mathcal{F}$. In order to identify a suitable Liapounov functional, this variational problem is met through the following integration problem :*find the metrics* $(\cdot,\cdot)^H$ *for which the vector fields* $V^a : \mathcal{F} \to \mathbb{R}^n$, $a \in \mathcal{F}$, *defined by* $V^a(x) = x - a$, *are* $(\cdot,\cdot)^H$*-gradient vector fields.* Our first result (cf. Theorem 3.3.1) establishes that such metrics are given by the Hessian of strictly convex functions, and in that case the vector fields V^a appear as gradients with respect to the second variable of some distance-like functions that are called D-functions. Indeed, if $(\cdot,\cdot)^H$ is induced by the Hessian $H = \nabla^2 h$ of $h : \mathcal{F} \mapsto \mathbb{R}$, we have for all a, x in $\mathcal{F}$: $\nabla_{_H} D_h(a,.)(x) = x - a$, where $D_h(a,x) = h(a) - h(x) - dh(x)(x-a)$. For another characterization of Hessian metrics, see Duistermaat [56].

Motivated by the previous result and with the aim of solving (P), we are then naturally led to consider Hessian Riemannian metrics that cannot be smoothly extended out of $\mathcal{F}$. Such a requirement is fulfilled by the Hessian of a Legendre type convex function h, whose definition is recalled in section 3.3. We give then a differential inclusion reformulation of (H-SD), which permits to show that in the case of a linear objective function f, the flow of $-\nabla_{_H} f_{|\mathcal{F}}$ stands at the crossroad of many optimization methods. In fact, following [78], we prove that viscosity methods and Bregman proximal algorithms produce their paths or iterates in the orbit of (H-SD). The D-function of h plays an essential role for this. In section 3.3.4 it is given a systematic method to construct Legendre functions based on barrier functions for convex inequality problems, which is illustrated with some examples ; relations to other works are discussed.

Section 3.4 deals with global existence and convergence properties. After having given a non trivial well-posedness result (cf. Theorem 3.4.1), we prove in section 3.4.2 that $f(x(t)) \to \inf_{\overline{\mathcal{F}}} f$ as $t \to +\infty$ whenever f is convex. A natural problem that arises is the trajectory convergence to a critical point. Since one expects the limit to be a (local) solution to (P), which may belong to the boundary of C, the notion of critical point must be understood in the sense of the optimality condition for a local minimizer a of f over $\overline{\mathcal{F}}$:

$$(\mathcal{O}) \qquad \nabla f(a) + N_{\overline{\mathcal{F}}}(a) \ni 0,\ a \in \overline{\mathcal{F}},$$

where $N_{\overline{\mathcal{F}}}(a)$ is the normal cone to $\overline{\mathcal{F}}$ at a, and ∇f is the Euclidean gradient of f. This involves an asymptotic singular behavior that is rather unusual in the classical theory of dynamical systems, where the critical points are typically supposed to be in the manifold. In section 3.4.3 we assume that the Legendre type function h is a *Bregman function with zone* C and prove that under a quasi-convexity assumption on f, the trajectory converges to some point a satisfying $(\mathcal{O})$. When f is convex, the preceding result amounts to the convergence of $x(t)$ toward a global minimizer of f over $\overline{\mathcal{F}}$. We also give a variational characterization of the limit and establish an abstract result on the rate of convergence under uniqueness of the solution. We consider in section 3.4.4 the case of linear programming, for which asymptotic convergence as well as a variational characterization are proved without the Bregman-type condition. Within this framework, we also give some estimates on the convergence rate that are valid for the specific Legendre functions commonly used in practice. In section 3.4.5, we consider the interesting case of positivity and equality constraints, introducing a *dual* trajectory $\lambda(t)$ that, under some appropriate conditions, converges to a solution to the dual problem of (P) whenever f is convex, even if primal convergence is not ensured.

Finally, inspired by the seminal work [30], we define in section 3.5 a change of coordinates called *Legendre transform coordinates*, which permits to show that the orbits of $(H\text{-}SD)$ may be seen as straight lines in a positive cone. This leads to additional geometric interpretations of the flow of $-\nabla_H f_{|\mathcal{F}}$. On the one hand, the orbits are geodesics with respect to an appropriate metric and, on the other hand, they may be seen as $\dot{q}$-trajectories of some Lagrangian, with consequences in terms of integrable Hamiltonians.

Notations. $\operatorname{Ker} A = \{x \in \mathbb{R}^n \mid Ax = 0\}$. The orthogonal complement of $\mathcal{A}_0$ is denoted by $\mathcal{A}_0^{\perp}$, and $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product of $\mathbb{R}^n$. Let us denote by $\mathbf{S}_+^n$ the cone of real symmetric definite positive matrices. Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f : \Omega \to \mathbb{R}$ is differentiable then ∇f stands for the Euclidean gradient of f. If $h : \Omega \mapsto \mathbb{R}$ is twice differentiable then its Euclidean Hessian at $x \in \Omega$ is denoted by $\nabla^2 h(x)$ and is defined as the endomorphism of $\mathbb{R}^n$ whose matrix in canonical coordinates is given by $[\frac{\partial^2 h(x)}{\partial x_i \partial x_j}]_{i,j \in \{1,..,n\}}$. Thus, $\forall x \in \Omega$, $d^2 h(x) = \langle \nabla^2 h(x) \cdot, \cdot \rangle$.

3.2 Preliminaries

3.2.1 The minimization problem and optimality conditions

Given a positive integer $m < n$, a full rank matrix $A \in \mathbb{R}^{m \times n}$ and $b \in \operatorname{Im} A$, let us define

$$\mathcal{A} = \{x \in \mathbb{R}^n \mid Ax = b\}. \tag{3.2.1}$$

Set $\mathcal{A}_0 = \mathcal{A} - \mathcal{A} = \operatorname{Ker} A$. Of course, $\mathcal{A}_0^{\perp} = \operatorname{Im} A^T$ where A^T is the transpose of A. Let C be a nonempty, open and convex subset of $\mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ a $\mathcal{C}^1$ function. Consider the constrained minimization problem

$$(P) \qquad \inf\{f(x) \mid x \in \overline{C},\ Ax = b\}.$$

The set of optimal solutions of (P) is denoted by $S(P)$. We call f the *objective function* of (P). The *feasible set* of (P) is given by $\overline{\mathcal{F}} = \{x \in \mathbb{R}^n \mid x \in \overline{C},\ Ax = b\} = \overline{C} \cap \mathcal{A}$, and $\mathcal{F}$ stands for the *relative interior* of $\overline{\mathcal{F}}$, that is

$$\mathcal{F} = \operatorname{ri} \overline{\mathcal{F}} = \{x \in \mathbb{R}^n \mid x \in C,\ Ax = b\} = C \cap \mathcal{A}. \tag{3.2.2}$$

Throughout this article, we assume that

$$\mathcal{F} \neq \emptyset \text{ and } \inf\{f(x) \mid x \in \overline{\mathcal{F}}\} > -\infty. \tag{3.2.3}$$

It is well known that a necessary condition for a to be locally minimal for f over $\overline{\mathcal{F}}$ is $(\mathcal{O}) : -\nabla f(a) \in N_{\overline{\mathcal{F}}}(a)$, where $N_{\overline{\mathcal{F}}}(x) = \{\nu \in \mathbb{R}^n \mid \forall y \in \overline{\mathcal{F}},\ \langle y - x, \nu\rangle \leq 0\}$ is the *normal cone* to $\overline{\mathcal{F}}$ at $x \in \overline{\mathcal{F}}$ ($N_{\overline{\mathcal{F}}}(x) = \emptyset$ when $x \notin \overline{\mathcal{F}}$) ; see for instance [114, Theorem 6.12]. By [112, Corollary 23.8.1], $N_{\overline{\mathcal{F}}}(x) = N_{\overline{C} \cap \mathcal{A}}(x) = N_{\overline{C}}(x) + N_{\mathcal{A}}(x) = N_{\overline{C}}(x) + \mathcal{A}_0^{\perp}$, for all $x \in \overline{\mathcal{F}}$. Therefore, the necessary optimality condition for $a \in \overline{\mathcal{F}}$ is

$$-\nabla f(a) \in N_{\overline{C}}(a) + \mathcal{A}_0^{\perp}. \tag{3.2.4}$$

If f is convex then this condition is also sufficient for $a \in \overline{\mathcal{F}}$ to be in $S(P)$.

3.2.2 Riemannian gradient flows on the relative interior of the feasible set

Let M be a smooth manifold. The tangent space to M at $x \in M$ is denoted by T_xM. If $f : M \mapsto \mathbb{R}$ is a $\mathcal{C}^1$ function then $df(x)$ denotes its differential or tangent map $df(x) : T_xM \to \mathbb{R}$ at $x \in M$. A $\mathcal{C}^k$ metric on M, $k \geq 0$, is a family of scalar products $(\cdot,\cdot)_x$ on each T_xM, $x \in M$, such that $(\cdot,\cdot)_x$ depends in a $\mathcal{C}^k$ way on x. The couple $M, (\cdot,\cdot)_x$ is called a $\mathcal{C}^k$ Riemannian manifold. This structure permits one to define a gradient vector field of f in M, which is denoted by $\nabla_{(\cdot,\cdot)} f$ and is uniquely determined by the following conditions :

(g_1) Tangency condition : for all $x \in M$, $\nabla_{(\cdot,\cdot)} f(x) \in T_xM$.

(g_2) Compatibility condition : for all $x \in M$, $v \in T_xM$, $df(x)(v) = (\nabla_{(\cdot,\cdot)} f(x), v)_x$.

If N is a submanifold of M then $T_xN \subset T_xM$ for all $x \in N$ and the metric $(\cdot,\cdot)_x$ on M induces a metric on N by restriction. We refer the reader to [54, 88] for further details.

Let us return to the minimization problem (P). Since C is open, we can take $M = C$, which is a smooth submanifold of $\mathbb{R}^n$ with the usual identification $T_xC \simeq \mathbb{R}^n$ for every $x \in C$. Given a continuous mapping $H : C \to \mathbf{S}^n_{+}$, the scalar product defined by

$$\forall x \in C,\ \forall u, v \in \mathbb{R}^n,\ (u, v)^H_x = \langle H(x)u, v\rangle, \tag{3.2.5}$$

endows C with a $\mathcal{C}^0$ Riemannian structure. The corresponding Riemannian gradient vector field of the objective function f restricted to C, which we denote by $\nabla_H f_{|C}$, is given by

$$\nabla_H f_{|C}(x) = H(x)^{-1}\nabla f(x). \tag{3.2.6}$$

Next, take $N = \mathcal{F} = C \cap \mathcal{A}$, which is a smooth submanifold of C with $T_x\mathcal{F} \simeq \mathcal{A}_0$ for each $x \in \mathcal{F}$. Definition (3.2.5) induces a metric on $\mathcal{F}$ for which the gradient of the restriction $f_{|\mathcal{F}}$ is denoted by $\nabla_H f_{|\mathcal{F}}$. Conditions (g_1) and (g_2) imply that for all $x \in \mathcal{F}$

$$\nabla_H f_{|\mathcal{F}}(x) = P_x H(x)^{-1}\nabla f(x), \tag{3.2.7}$$

where, given $x \in C$, $P_x : \mathbb{R}^n \to \mathcal{A}_0$ is the orthogonal projection onto the linear subspace $\mathcal{A}_0$ with respect to the scalar product $(\cdot,\cdot)_x^H$. Since A has full rank, it is easy to see that

$$P_x = I - H(x)^{-1}A^T(AH(x)^{-1}A^T)^{-1}A, \tag{3.2.8}$$

and we conclude that for all $x \in \mathcal{F}$

$$\nabla_H f_{|\mathcal{F}}(x) = H(x)^{-1}[I - A^T(AH(x)^{-1}A^T)^{-1}AH(x)^{-1}]\nabla f(x). \tag{3.2.9}$$

It is usual to say that $a \in \mathcal{F}$ is a *critical point* of $f_{|\mathcal{F}}$ when $\nabla_H f_{|\mathcal{F}}(a) = 0$. Remark that $\nabla_H f_{|\mathcal{F}}(a) = 0$ iff $\nabla f(a) \in \mathcal{A}_0^\perp$, where the latter is exactly the optimality condition (3.2.4) when a belongs to $\mathcal{F}$ (recall that $N_{\overline{C}}(x) = \{0\}$ when $x \in C$ because C is open).

Given $x \in \mathcal{F}$, the vector $-\nabla_H f_{|\mathcal{F}}(x)$ can be interpreted as that direction in $\mathcal{A}_0$ such that f decreases the most steeply at x with respect to the metric $(\cdot,\cdot)_x^H$. The *steepest descent method* for the (local) minimization of f on the Riemannian manifold $\mathcal{F}, (\cdot,\cdot)_x^H$ consists in finding the solution trajectory $x(t)$ of the vector field $-\nabla_H f_{|\mathcal{F}}$ with initial condition $x^0 \in \mathcal{F}$:

$$\begin{cases} \dot{x} + \nabla_H f_{|\mathcal{F}}(x) = 0, \\ x(0) = x^0 \in \mathcal{F}. \end{cases} \tag{3.2.10}$$

When x^0 is a critical point of $f_{|\mathcal{F}}$, the solution to (3.2.10) is stationary, i.e. $x(t) \equiv x^0$. Otherwise, it is natural to expect $x(t)$ to approach the set of local minima of f on $\overline{\mathcal{F}}$. General results and interesting examples concerning the existence and asymptotic behavior of solutions to Riemannian gradient flows can be found in [73].

3.3 Legendre gradient flows in constrained optimization

3.3.1 Liapounov functionals, variational inequalities and Hessian metrics.

This section is intended to motivate the particular class of Riemannian metrics that is studied in this paper in view of the asymptotic convergence of the solution to (3.2.10).

Let us consider the minimization problem (P) and assume that C is endowed with some Riemannian metric $(\cdot,\cdot)_x^H$ as defined in (3.2.5). We say that $V : \mathcal{F} \mapsto \mathbb{R}$ is a *Liapounov functional* for the vector field $-\nabla_H f_{|\mathcal{F}}$ if $\forall x \in \mathcal{F}$, $(-\nabla_H f_{|\mathcal{F}}(x), \nabla_H V(x))_x^H \leq 0$. If $x(t)$ is a solution to (3.2.10) in some interval (a,b) and V is a Liapounov functional for $-\nabla_H f_{|\mathcal{F}}$ then the mapping $(a,b) \ni t \mapsto V(x(t))$ is non-increasing. Although $f_{|\mathcal{F}}$ is indeed a Liapounov functional for $-\nabla_H f_{|\mathcal{F}}$, this does not ensure the convergence of $x(t)$. Even in the Euclidean case $(\cdot,\cdot)_x^H = \langle\cdot,\cdot\rangle$, the convergence of $x(t)$ may fail without additional geometrical or topological assumptions on f ; see for instance the counterexample of Palis-De Melo [106].

Suppose that the objective function f is convex. For simplicity, we also assume that $A = 0$ so that $\mathcal{F} = C$. In the framework of convex minimization, the set of minimizers of f over $\overline{C}$, denoted by $\operatorname{argmin}_{\overline{C}} f$, is characterized in variational terms as follows :

$$a \in \operatorname{argmin}_{\overline{C}} f \quad \Leftrightarrow \quad \forall x \in \overline{C},\ \langle \nabla f(x), x - a\rangle \geq 0 \tag{3.3.11}$$

In the Euclidean case, we define $q_a(x) = \frac{1}{2}|x-a|^2$ and observe that $\nabla q_a(x) = x - a$. By (3.3.11), for every $a \in \mathrm{argmin}_{\overline{C}} f$, q_a is a Liapounov functional for $-\nabla f$. This key property allows one to establish the asymptotic convergence as $t \to +\infty$ of the corresponding steepest descent trajectories ; see [40] for more details in a very general non-smooth setting. To use the same kind of arguments in a non Euclidean context, observe that by (3.2.6) together with the continuity of ∇f, the following variational Riemannian characterization holds

$$a \in \mathrm{argmin}_{\overline{C}} f \quad \Leftrightarrow \quad \forall x \in C,\ (\nabla_H f(x), x-a)^H_x \geq 0. \tag{3.3.12}$$

We are thus naturally led to the problem of *finding the Riemannian metrics on C for which the mappings $C \ni x \mapsto x - y \in \mathbb{R}^n$, $y \in C$, are gradient vector fields.*

The next result gives a complete answer to this question. For simplicity, we restrict our attention here to those metrics that are differentiable. In the sequel, the set of metrics complying with the previous requirements is denoted by $\mathcal{M}$, that is, $(\cdot,\cdot)^H_x \in \mathcal{M} \Leftrightarrow H \in \mathcal{C}^1(C;\mathbf{S}^n_+)$ and $\forall y \in C$, $\exists \varphi_y \in \mathcal{C}^1(C;\mathbb{R})$, $\nabla_H \varphi_y(x) = x - y$.

Theorem 3.3.1 *If $(\cdot,\cdot)^H_x \in \mathcal{M}$ then there exists a strictly convex function $h \in \mathcal{C}^3(C)$ such that $\forall x \in C$, $H(x) = \nabla^2 h(x)$. Besides, defining $D_h : C \times C \mapsto \mathbb{R}$ by*

$$D_h(y,x) = h(y) - h(x) - \langle \nabla h(x), x - y \rangle, \tag{3.3.13}$$

we obtain $\nabla_H D_h(y,\cdot)(x) = x - y$.

Proof. Let $(x_1,..,x_n)$ denote the canonical coordinates of $\mathbb{R}^n$ and write $\sum_{i,j} H_{ij}(x)dx_i dx_j$ for $(\cdot,\cdot)^H_x$. By (3.2.6), the mappings $x \mapsto x-y$, $y \in C$, define a family of $(\cdot,\cdot)^H_x$ gradients iff $k_y : x \mapsto H(x)(x-y)$, $y \in C$, is a family of Euclidean gradients. Setting $\alpha^y(x) = \langle k_y(x), \cdot \rangle$, $x, y \in C$, the problem amounts to find necessary (and sufficient) conditions under which the 1-forms α^y are all exact. Let $y \in C$. Since C is convex, the Poincaré lemma [88, Theorem V.4.1] states that α^y is exact iff it is closed. In canonical coordinates we have $\alpha^y(x) = \sum_i \left(\sum_k H_{ik}(x)(x_k - y_k)\right) dx_i,\ x \in C$, and therefore α^y is exact iff for all $i, j \in \{1,..,n\}$ we have $\frac{\partial}{\partial x_j} \sum_k H_{ik}(x)(x_k - y_k) = \frac{\partial}{\partial x_i} \sum_k H_{jk}(x)(x_k - y_k)$, which is equivalent to $\sum_k \frac{\partial}{\partial x_j} H_{ik}(x)(x_k - y_k) + H_{ij}(x) = \sum_k \frac{\partial}{\partial x_i} H_{jk}(x)(x_k - y_k) + H_{ji}(x)$. Since $H_{ij}(x) = H_{ji}(x)$, this gives the following condition : $\sum_k \frac{\partial}{\partial x_j} H_{ik}(x)(x_k - y_k) = \sum_k \frac{\partial}{\partial x_i} H_{jk}(x)(x_k - y_k)$, $\forall i, j \in \{1,..,n\}$. If we set $V_x = (\frac{\partial}{\partial x_j} H_{i1}(x), .., \frac{\partial}{\partial x_j} H_{in}(x))^T$ and $W_x = (\frac{\partial}{\partial x_i} H_{j1}(x), .., \frac{\partial}{\partial x_i} H_{jn}(x))^T$, the latter can be rewritten $\langle V_x - W_x, x - y \rangle = 0$, which must hold for all $(x,y) \in C \times C$. Fix $x \in C$. Let $\epsilon_x > 0$ be such that the open ball of center x with radius ϵ_x is contained in C. For every ν such that $|\nu| = 1$, take $y = x + \epsilon_x/2\nu$ to obtain that $\langle V_x - W_x, \nu \rangle = 0$. Consequently, $V_x = W_x$ for all $x \in C$. Therefore, $(\cdot,\cdot)^H_x \in \mathcal{M}$ iff

$$\forall x \in C,\ \forall i,j,k \in \{1,..,n\},\ \frac{\partial}{\partial x_i} H_{jk}(x) = \frac{\partial}{\partial x_j} H_{ik}(x). \tag{3.3.14}$$

Lemma 3.3.2 *If $H : C \mapsto \mathbf{S}^n_+$ is a differentiable mapping satisfying* (3.3.14)*, then there exists $h \in \mathcal{C}^3(C)$ such that $\forall x \in C$, $H(x) = \nabla^2 h(x)$. In particular, h is strictly convex.*

Proof of Lemma 3.3.2. For all $i \in \{1,..,n\}$, set $\beta^i = \sum_k H_{ik} dx_k$. By (3.3.14), β^i is closed and therefore exact. Let $\phi_i : C \mapsto \mathbb{R}$ be such that $d\phi_i = \beta^i$ on C, and set $\omega = \sum_k \phi_k dx_k$. We have that $\frac{\partial}{\partial x_j}\phi_i(x) = H_{ij}(x) = H_{ji}(x) = \frac{\partial}{\partial x_i}\phi_j(x)$, $\forall x \in C$. This proves that ω is

closed, and therefore there exists $h \in \mathcal{C}^2(C, \mathbb{R})$ such that $dh = \omega$. To conclude we just have to notice that $\frac{\partial}{\partial x_i} h(x) = \phi_i$, and thus $\frac{\partial^2 h}{\partial x_j \partial x_i}(x) = H_{ji}(x)$, $\forall x \in C$. ■
To finish the proof, remark that taking $\varphi_y = D_h(y, \cdot)$ with D_h being defined by (3.3.13), we obtain $\nabla \varphi_y(x) = \nabla^2 h(x)(x-y)$, and therefore $\nabla_H \varphi_y(x) = x - y$ in virtue of (3.2.6). ■

Remark 3.3.3 (a) In the theory of Bregman proximal methods for convex optimization, the distance-like function D_h defined by (3.3.13) is called the *D-function* of h. Theorem 3.3.1 is a new motivation for the introduction of D_h in relation with variational inequality problems.
(b) For a theoretical approach to Hessian Riemannian structures the reader is referred to the recent work of Duistermaat [56].

Theorem 3.3.1 suggests to endow C with a Riemannian structure associated with the Hessian $H = \nabla^2 h$ of a strictly convex function $h : C \mapsto \mathbb{R}$. As we will see under some additional conditions, the D-function of h is essential to establish the asymptotic convergence of the trajectory. On the other hand, if it is possible to replace h by a sufficiently smooth strictly convex function $h' : C' \mapsto \mathbb{R}$ with $C' \supset\supset C$ and $h'_{|C} = h$, then the gradient flows for h and h' are the same on C but the steepest descent trajectories associated with the latter may leave the feasible set of (P) and in general they will not converge to a solution of (P). We shall see that to avoid this drawback it is sufficient to require that $|\nabla h(x^j)| \to +\infty$ for all sequences (x^j) in C converging to a boundary point of C. This may be interpreted as a sort of *penalty technique*, a classical strategy to enforce feasibility in optimization theory.

3.3.2 Legendre type functions and the $(H\text{-}SD)$ dynamical system

In the sequel, we adopt the standard notations of convex analysis theory; see [112]. Given a closed convex subset S of $\mathbb{R}^n$, we say that an extended-real-valued function $g : S \mapsto \mathbb{R} \cup \{+\infty\}$ belongs to the class $\Gamma_0(S)$ when g is lower semicontinuous, proper ($g \not\equiv +\infty$) and convex. For such a function $g \in \Gamma_0(S)$, its *effective domain* is defined by $\operatorname{dom} g = \{x \in S \mid g(x) < +\infty\}$. When $g \in \Gamma_0(\mathbb{R}^n)$ its *Legendre-Fenchel conjugate* is given by $g^*(y) = \sup\{\langle x, y\rangle - g(x) \mid x \in \mathbb{R}^n\}$, and its *subdifferential* is the set-valued mapping $\partial g : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ given by $\partial g(x) = \{y \in \mathbb{R}^n \mid \forall z \in \mathbb{R}^n,\ f(x) + \langle y, z - x\rangle \le f(z)\}$. We set $\operatorname{dom} \partial g = \{x \in \mathbb{R}^n \mid \partial g(x) \neq \emptyset\}$.

Definition 3.3.1 [112, Chapter 26] *A function* $h \in \Gamma_0(\mathbb{R}^n)$ *is called :*
(i) essentially smooth, *if* h *is differentiable on* $\operatorname{int}\operatorname{dom} h$, *with moreover* $|\nabla h(x^j)| \to +\infty$ *for every sequence* $(x^j) \subset \operatorname{int}\operatorname{dom} h$ *converging to a boundary point of* $\operatorname{dom} h$ *as* $j \to +\infty$*;*
(ii) of Legendre type *if* h *is essentially smooth and strictly convex on* $\operatorname{int}\operatorname{dom} h$.

Remark that by [112, Theorem 26.1], $h \in \Gamma_0(\mathbb{R}^n)$ is essentially smooth iff $\partial h(x) = \{\nabla h(x)\}$ if $x \in \operatorname{int}\operatorname{dom} h$ and $\partial h(x) = \emptyset$ otherwise; in particular, $\operatorname{dom} \partial h = \operatorname{int}\operatorname{dom} h$.

Consider the minimization problem (P). Motivated by the results of section 3.3.1, we define a Riemannian structure on C by introducing a function $h \in \Gamma_0(\mathbb{R}^n)$ such that :

$$(H_0) \qquad \begin{cases} \text{(i)} & h \text{ is of Legendre type with } \operatorname{int}\operatorname{dom} h = C. \\ \text{(ii)} & h_{|C} \in \mathcal{C}^2(C; \mathbb{R}) \text{ and } \forall x \in C, \nabla^2 h(x) \in \mathbf{S}^n_{+}. \\ \text{(iii)} & \text{The mapping } C \ni x \mapsto \nabla^2 h(x) \text{ is locally Lipschitz continuous.} \end{cases}$$

Here and subsequently, we take $H = \nabla^2 h$ with h satisfying (H_0). The Hessian mapping $C \ni x \mapsto H(x)$ endows C with the (locally Lipschitz continuous) Riemannian metric

$$\forall x \in C,\ \forall u, v \in \mathbb{R}^n,\ (u,v)^H_x = \langle H(x)u, v\rangle = \langle \nabla^2 h(x)u, v\rangle, \tag{3.3.15}$$

and we say that $(\cdot,\cdot)^H_x$ is the *Legendre metric* on C induced by the Legendre type function h, which also defines a metric on $\mathcal{F} = C \cap \mathcal{A}$ by restriction. In addition to $f \in \mathcal{C}^1(\mathbb{R}^n)$, we suppose that the objective function satisfies

$$\nabla f \text{ is locally Lipschitz continuous on } \mathbb{R}^n. \tag{3.3.16}$$

The corresponding steepest descent method in the manifold $\mathcal{F}, (\cdot,\cdot)^H_x$, which we refer to as $(H\text{-}SD)$ for short, is then the following continuous dynamical system

$$(H\text{-}SD) \qquad \begin{cases} \dot{x}(t) + \nabla_{H} f_{|\mathcal{F}}(x(t)) = 0,\ t \in (T_m, T_M), \\ x(0) = x^0 \in \mathcal{F}, \end{cases}$$

where $\nabla_{H} f_{|\mathcal{F}}$ is given by (3.2.9) with $H = \nabla^2 h$ and $T_m < 0 < T_M$ define the interval corresponding to the unique *maximal solution* of $(H\text{-}SD)$.

Definition 3.3.2 *Given an initial condition $x^0 \in \mathcal{F}$, we say that $(H\text{-}SD)$ is* well-posed *when its maximal solution satisfies $T_M = +\infty$.*

In section 3.4.1 we will give some sufficient conditions ensuring the well-posedness of $(H\text{-}SD)$.

3.3.3 Differential inclusion formulation of $(H\text{-}SD)$ and some consequences

It is easily seen that the solution $x(t)$ of $(H\text{-}SD)$ satisfies :

$$\begin{cases} \dfrac{d}{dt}\nabla h(x(t)) + \nabla f(x(t)) & \in & \mathcal{A}_0^\perp \text{ on } (T_m, T_M), \\ x(t) & \in & \mathcal{F} \text{ on } (T_m, T_M), \\ x(0) & = & x^0 \in \mathcal{F}. \end{cases} \tag{3.3.17}$$

This differential inclusion problem makes sense even when $x \in W^{1,1}_{loc}(T_m, T_M; \mathbb{R}^n)$, the inclusions being satisfied almost everywhere on (T_m, T_M). Actually, the following result establishes that $(H\text{-}SD)$ and (3.3.17) describe the same trajectory.

Proposition 3.3.1 *Let $x \in W^{1,1}_{loc}(T_m, T_M; \mathbb{R}^n)$. Then, x is a solution of* (3.3.17) *iff x is the solution of $(H\text{-}SD)$. In particular,* (3.3.17) *admits a unique solution of class $\mathcal{C}^1$.*

Proof. Assume that x is a solution of (3.3.17), and let I' be the subset of (T_m, T_M) on which $t \mapsto (x(t), \nabla h(x(t))$ is derivable. We may assume that $x(t) \in \mathcal{F}$ and $\frac{d}{dt}\nabla h(x(t)) + \nabla f(x(t)) \in \mathcal{A}_0^\perp$, $\forall t \in I'$. Since x is absolutely continuous, $\dot{x}(t) + H(x(t))^{-1}\nabla f((x(t)) \in H(x(t))^{-1}\mathcal{A}_0^\perp$ and $\dot{x}(t) \in \mathcal{A}_0$, $\forall t \in I'$. But the orthogonal complement of $\mathcal{A}_0$ with respect to the inner product $\langle H(x)\cdot,\cdot\rangle$ is exactly $H(x)^{-1}\mathcal{A}_0^\perp$ when $x \in \mathcal{F}$. It follows that $\dot{x} + P_x H(x)^{-1}\nabla f(x) = 0$ on I'. This implies that x is the $\mathcal{C}^1$ solution of $(H\text{-}SD)$. ■

Suppose that f is convex. On account of Proposition 3.3.1, (H-SD) can be interpreted as a continuous-time model for a well-known class of iterative minimization algorithms. In fact, an implicit discretization of (3.3.17) yields the following iterative scheme : $\nabla h(x^{k+1}) - \nabla h(x^k) + \mu_k \nabla f(x^{k+1}) \in \operatorname{Im} A^T$, $Ax^{k+1} = b$, where $\mu_k > 0$ is a step-size parameter and $x^0 \in \mathcal{F}$. This is the optimality condition for

$$x^{k+1} \in \operatorname{argmin} \left\{ f(x) + 1/\mu_k D_h(x, x^k) \mid Ax = b \right\}, \tag{3.3.18}$$

where D_h is given by

$$D_h(x,y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle,\ x \in \operatorname{dom} h,\ y \in \operatorname{dom} \partial h = C. \tag{3.3.19}$$

When $C = \mathbb{R}^n$ and $h(x) = \frac{1}{2}|x|^2$, (3.3.18) corresponds to the *proximal point* algorithm introduced by Martinet in [101]. With the aim of incorporating constraints, the Euclidean distance is replaced by the D-function of a Bregman function (see Definition 3.4.1), and in that case, (3.3.18) is referred to as the *Bregman proximal minimization* method. For more details on this method, see for instance [44, 45, 77, 84]. When applied to dual problems, (3.3.18) may be viewed as an *augmented Lagrangian multiplier* algorithm ; see [59, 119]. Next, assume that $f(x) = \langle c, x \rangle$ for some $c \in \mathbb{R}^n$. As already noticed in [30, 64, 96] for the log-metric and in [78] for a fairly general h, in this case the (H-SD) gradient trajectory can be viewed as a *central optimal path.* Indeed, integrating (3.3.17) over $[0,t]$ we obtain $\nabla h(x(t)) - \nabla h(x^0) + tc \in \mathcal{A}_0^\perp$. Since $x(t) \in \mathcal{A}$, it follows that

$$x(t) \in \operatorname{argmin} \left\{ \langle c, x \rangle + 1/t D_h(x, x^0) \mid Ax = b \right\}, \tag{3.3.20}$$

which corresponds to the so-called *viscosity method* relative to $g(x) = D_h(x, x^0)$. Generally speaking, the viscosity method for solving (P) is based on the introduction of a strictly convex function $g \in \Gamma_0(\mathbb{R}^n)$ such that $\overline{\operatorname{dom} g} = \overline{C}$, then solve

$$\widetilde{x}(\varepsilon) \in \operatorname{argmin} \{ f(x) + \varepsilon g(x) | Ax = b \},\ \varepsilon > 0,$$

and finally let $\varepsilon \to 0$. When $\operatorname{dom} g = \overline{C}$, general conditions ensure that $\widetilde{x}(\varepsilon)$ converges to the unique $x^* \in S(P)$ that minimizes g on $S(P)$; see [13, 16, 78] and Corollary 3.4.1. This result can be extended to some specific cases where $\operatorname{dom} g = C$; see [16, Theorem 3.4], [78, Theorem 2] and Proposition 3.4.3. Remark that for a linear objective function, (3.3.18) and (3.3.20) are essentially the same : the sequence generated by the former belongs to the optimal path defined by the latter. Indeed, setting $t_0 = 0$ and $t_{k+1} = t_k + \mu_k$ for all $k \geq 0$ ($\mu_0 = 0$) and integrating (3.3.17) over $[t_k, t_{k+1}]$, we obtain that $x(t_{k+1})$ satisfies the optimality condition for (3.3.18). The following result summarizes the previous discussion.

Proposition 3.3.2 *Assume that f is linear and that the corresponding (H-SD) dynamical system is well-posed. Then, the viscosity optimal path $\widetilde{x}(\varepsilon)$ relative to $g(x) = D_h(x, x^0)$ and the sequence (x^k) generated by (3.3.18) exist and are unique, with in addition $\widetilde{x}(\varepsilon) = x(1/\varepsilon)$, $\forall \varepsilon > 0$, and $x^k = x(\sum_{l=0}^{k-1} \mu_l)$, $\forall\ k \geq 1$, where $x(t)$ is the solution of (H-SD).*

Remark 3.3.4 In order to ensure asymptotic convergence for proximal-type algorithms, it is usually required that the step-size parameters satisfy $\sum \mu_k = +\infty$. By Proposition 3.3.2, this is necessary for the convergence of (3.3.18) in the sense that when (H-SD) is well-posed, if x^k converges to some $x^* \in S(P)$ then either $x^0 = x^*$ or $\sum \mu_k = +\infty$.

We finish this section with another interesting consequence of (3.3.17). Recall that, by [112, Theorem 26.5], h is of Legendre type iff its Fenchel conjugate h^* is of Legendre type. Moreover, the gradient mapping $\nabla h : \operatorname{int}\operatorname{dom} h \to \operatorname{int}\operatorname{dom} h^*$ is one-to-one with

$$(\nabla h)^{-1} = \nabla h^*. \tag{3.3.21}$$

When $\mathcal{A}_0 = \mathbb{R}^n$, we obtain the following differential equation for $y = \nabla h(x)$:

$$\dot{y} + \nabla f(\nabla h^*(y)) = 0. \tag{3.3.22}$$

In general, this equation does not appear to be simpler to study than the original one. Nevertheless, in the specific case of $f(x) = \langle c, x\rangle$, it reduces $(H\text{-}SD)$ to the trivial equation $\dot{y} + c = 0$; see section 3.3.4 for some examples. Section 3.5 is devoted to the general case for which the change of coordinates is given by $y = \Pi_{\mathcal{A}_0}\nabla h(x)$, where

$$\Pi_{\mathcal{A}_0} = I - A^T(AA^T)^{-1}A \tag{3.3.23}$$

is the Euclidean orthogonal projection onto $\mathcal{A}_0$.

3.3.4 Examples : interior point flows in convex programming

Let $p \geq 1$ be an integer and set $I = \{1, \dots, p\}$. Let us assume that to each $i \in I$ there corresponds a $\mathcal{C}^3$ concave function $g_i : \mathbb{R}^n \to \mathbb{R}$. We also assume that

$$\exists x^0 \in \mathbb{R}^n,\ \forall i \in I,\ g_i(x^0) > 0. \tag{3.3.24}$$

Suppose that the open convex set C is given by

$$C = \{x \in \mathbb{R}^n \mid g_i(x) > 0, i \in I\}. \tag{3.3.25}$$

By (3.3.24) we have that $C \neq \emptyset$ and $\overline{C} = \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i \in I\}$. Let us introduce a class of convex functions of Legendre type $\theta \in \Gamma_0(\mathbb{R})$ satisfying

$$(H_1)\quad \begin{cases} \text{(i) } (0,\infty) \subset \operatorname{dom}\theta \subset [0,\infty). \\ \text{(ii) } \theta \in \mathcal{C}^3(0,\infty) \text{ and } \lim_{s\to 0^+}\theta'(s) = -\infty. \\ \text{(iii) } \forall s > 0,\ \theta''(s) > 0. \\ \text{(iv) Either } \theta \text{ is non-increasing or } \forall i \in I, g_i \text{ is an affine function.} \end{cases}$$

Proposition 3.3.3 *Under* (3.3.24) *and* (H_1)*, the function* $h \in \Gamma_0(\mathbb{R}^n)$ *defined by*

$$h(x) = \sum_{i\in I}\theta(g_i(x)). \tag{3.3.26}$$

is essentially smooth with $\operatorname{int}\operatorname{dom} h = C$ *and* $h \in \mathcal{C}^3(C)$*, where* C *is given by* (3.3.25)*. If we assume in addition the following non-degeneracy condition :*

$$\forall x \in C,\ \operatorname{span}\{\nabla g_i(x) \mid i \in I\} = \mathbb{R}^n, \tag{3.3.27}$$

then $H = \nabla^2 h$ *is positive definite on* C*, and consequently* h *satisfies* (H_0)*.*

Proof. Define $h_i \in \Gamma_0(\mathbb{R}^n)$ by $h_i(x) = \theta(g_i(x))$. We have that $\forall i \in I$, $C \subset \operatorname{dom} h_i$. Hence $\operatorname{intdom} h = \bigcap_{i\in I} \operatorname{intdom} h_i \supseteq C \neq \emptyset$, and by [112, Theorem 23.8], we conclude that $\partial h(x) = \sum_{i\in I} \partial h_i(x)$ for all $x \in \mathbb{R}^n$. But $\partial h_i(x) = \theta'(g_i(x))\nabla g_i(x)$ if $g_i(x) > 0$ and $\partial h_i(x) = \emptyset$ if $g_i(x) \leq 0$; see [75, Theorem IX.3.6.1]. Therefore $\partial h(x) = \sum_{i\in I} \theta'(g_i(x))\nabla g_i(x)$ if $x \in C$, and $\partial h(x) = \emptyset$ otherwise. Since ∂h is a single-valued mapping, it follows from [112, Theorem 26.1] that h is essentially smooth and $\operatorname{int}\operatorname{dom} h = \operatorname{dom} \partial h = C$. Clearly, h is of class $\mathcal{C}^3$ on C. Assume now that (3.3.27) holds. For $x \in C$, we have $\nabla^2 h(x) = \sum_{i\in I} \theta''(g_i(x))\nabla g_i(x)\nabla g_i(x)^T + \sum_{i\in I} \theta'(g_i(x))\nabla^2 g_i(x)$. By (H_1)(iv), it follows that for any $v \in \mathbb{R}^n$, $\sum_{i\in I} \theta'(g_i(x))\langle \nabla^2 g_i(x)v, v\rangle \geq 0$. Let $v \in \mathbb{R}^n$ be such that $\langle \nabla^2 h(x)v, v\rangle = 0$, which yields $\sum_{i\in I} \theta''(g_i(x))\langle v, \nabla g_i(x)\rangle^2 = 0$. According to (H_1)(iii), the latter implies that $v \in \operatorname{span}\{\nabla g_i(x) | i \in I\}^\perp = \{0\}$. Hence $\nabla^2 h(x) \in \mathbf{S}^n_+$ and the proof is complete. ■

If h is defined by (3.3.26) with $\theta \in \Gamma_0(\mathbb{R})$ satisfying (H_1), we say that θ is the *Legendre kernel* of h. Such kernels can be divided into two classes. The first one corresponds to those kernels θ for which $\operatorname{dom} \theta = (0, \infty)$ so that $\theta(0) = +\infty$, and are associated with *interior barrier* methods in optimization. Examples :

- Log barrier : $\theta_1(s) = -\ln(s)$, $s > 0$.
- Inverse barrier : $\theta_2(s) = 1/s$, $s > 0$.

The kernels θ belonging to the second class satisfy $\theta(0) < +\infty$, and are connected with the notion of *Bregman function* in proximal algorithms theory (see Section 3.4.3). Examples :

- Boltzmann-Shannon entropy [36] : $\theta_3(s) = s\ln(s) - s$, $s \geq 0$ (with $0 \ln 0 = 0$).
- Kiwiel [85] : $\theta_4(s) = -\frac{1}{\gamma}s^\gamma$ with $\gamma \in (0, 1)$, $s \geq 0$.
- Teboulle [119] : $\theta_5(s) = (\gamma s - s^\gamma)/(1-\gamma)$ with $\gamma \in (0, 1)$, $s \geq 0$.
- The "$x \log x$" entropy : $\theta_6(s) = s \ln s$, $s \geq 0$.

In order to illustrate the type of dynamical systems given by $(H\text{-}SD)$, consider the case of positivity constraints where $p = n$ and $g_i(x) = x_i$, $i \in I$. Thus $C = \mathbb{R}^n_{++}$ and $\overline{C} = \mathbb{R}^n_+$. Let us assume that $\exists x^0 \in \mathbb{R}^n_{++}$, $Ax = b$. The corresponding minimization problem is

$$\min\{f(x) \mid x \geq 0,\ Ax = b\}. \tag{3.3.28}$$

Take first the kernel θ_3 from above. The associated Legendre function (3.3.26) is given by

$$h(x) = \sum_{i=1}^{n} x_i \ln x_i - x_i,\ x \in \mathbb{R}^n_+, \tag{3.3.29}$$

thus $H(x) = \operatorname{diag}(1/x_1, ..., 1/x_n) \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n_{++}$, the induced Riemannian metric is

$$\forall x \in \mathbb{R}^n_{++},\ \forall u, v \in \mathbb{R}^n,\ (u, v)^H_x = \langle X^{-1}u, v\rangle, \tag{3.3.30}$$

where

$$X = \operatorname{diag}(x_1, ..., x_n), \tag{3.3.31}$$

and the differential equation in $(H\text{-}SD)$ is given by

$$\dot{x} + [I - XA^T(AXA^T)^{-1}A]X\nabla f(x) = 0. \tag{3.3.32}$$

If $f(x) = \langle c, x\rangle$ for some $c \in \mathbb{R}^n$ and in absence of linear equality constraints, then (3.3.32) is $\dot{x} + Xc = 0$. The change of coordinates $y = \nabla h(x) = (\ln x_1, ..., \ln x_n)$ gives

$\dot{y} + c = 0$. Hence, $x(t) = (x_1^0 e^{-c_1 t}, ..., x_n^0 e^{-c_n t})$, $t \in \mathbb{R}$, where $x^0 = (x_1^0, ..., x_n^0) \in \mathbb{R}^n_{++}$. If $c \in \mathbb{R}^n_+$ then $\inf_{x \in \mathbb{R}^n_+} \langle c, x \rangle = 0$ and $x(t)$ converges to a minimizer of $f = \langle c, \cdot \rangle$ on $\mathbb{R}^n_+$; if $c_{i_0} < 0$ for some i_0, then $\inf_{x \in \mathbb{R}^n_+} \langle c, x \rangle = -\infty$ and $x_{i_0}(t) \to +\infty$ as $t \to +\infty$. Next, take $A = (1, \ldots, 1) \in \mathbb{R}^{1 \times n}$ and $b = 1$ so that the feasible set of (3.3.28) is given by $\overline{\mathcal{F}} = \Delta_{n-1} = \{x \in \mathbb{R}^n \mid x \geq 0, \sum_{i=1}^n x_i = 1\}$, that is the $(n-1)$-dimensional simplex. In this case, (3.3.32) corresponds to $\dot{x} + [X - xx^T]\nabla f(x) = 0$, or componentwise

$$\dot{x}_i + x_i \left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j} \right) = 0, \quad i = 1, \ldots, n. \tag{3.3.33}$$

For suitable choices of f, this is a *Lotka-Volterra* type equation that naturally arises in population dynamics theory and, in that context, (3.3.30) is usually referred to as the *Shahshahani* metric ; see [2, 76] and the references therein.The figure 3.1 gives a numerical illustration of system (3.3.33) for $n = 3$ and with $f(x) = x_3 - x_2$.

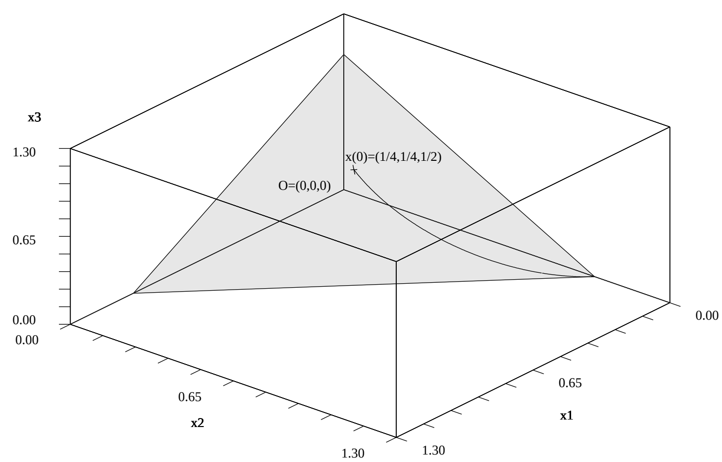

FIG. 3.1 – A trajectory of (3.3.33).

Karmarkar studied (3.3.33) in [82] for a quadratic objective function as a continuous model of the interior point algorithm introduced by him in [81]. Equation (3.3.32) is studied by Faybusovich in [61, 62, 63] when (3.3.28) is a linear program, establishing connections with completely integrable Hamiltonian systems and exponential convergence rate, and by Herzel et al. in [74], who prove quadratic convergence for an explicit discretization. See [73] for more general linear inequality constraints and for connections with the *double Lie bracket* flow introduced by Brockett in [38, 39].

Take now the log barrier kernel θ_1 and $h(x) = -\sum_{i=1}^n \ln x_i$. Since $\nabla^2 h(x) = X^{-2}$ with X defined by (3.3.31), the associated differential equation is

$$\dot{x} + [I - X^2 A^T (AX^2 A^T)^{-1} A] X^2 \nabla f(x) = 0. \tag{3.3.34}$$

This equation was considered by Bayer and Lagarias in [30] for a linear program. In the particular case $f(x) = \langle c, x \rangle$ and without linear equality constraints, (3.3.34) amounts to

$\dot{x} + X^2 c = 0$, or $\dot{y} + c = 0$ for $y = \nabla h(x) = -X^{-1}e$ with $e = (1, \cdots, 1) \in \mathbb{R}^n$, which gives $x(t) = (1/(1/x_1^0 + c_1 t), ..., 1/(1/x_n^0 + c_n t))$, $T_m \leq t \leq T_M$, with $T_m = \max\{-1/x_i^0 c_i \mid c_i > 0\}$ and $T_M = \min\{-1/x_i^0 c_i \mid c_i < 0\}$ (see [30, pag. 515]). To study the associated trajectories for a general linear program, it is introduced in [30] the *Legendre transform coordinates* $y = \Pi_{\mathcal{A}_0} \nabla h(x) = [I - A^T(AA^T)^{-1}A]X^{-1}e$, which still linearizes (3.3.34) when f is linear (see section 3.5 for an extension of this result), and permits to establish some remarkable analytic and geometric properties of the trajectories. A similar system was considered in [64, 96] as a continuous log-barrier method for nonlinear inequality constraints and with $\mathcal{A}_0 = \mathbb{R}^n$.

New systems may be derived by choosing other kernels. For instance, taking $h(x) = -1/\gamma \sum_{i=1}^n x_i^\gamma$ with $\gamma \in (0,1)$, $A = (1, \ldots, 1) \in \mathbb{R}^{1\times n}$ and $b = 1$, we obtain

$$\dot{x}_i + \frac{x_i^{2-\gamma}}{1-\gamma}\left(\frac{\partial f}{\partial x_i} - \sum_{j=1}^n \frac{x_j^{2-\gamma}}{\sum_{k=1}^n x_k^{2-\gamma}} \frac{\partial f}{\partial x_j}\right) = 0, \quad i = 1, \ldots, n. \tag{3.3.35}$$

Different Legendre kernels may be combined by taking $h(x) = \sum_{i=1}^p h_i(g_i(x))$. The case of *box* constraints $L_i \leq x_i \leq U_i$ may be handled with a term of the form $h_i(x_i) = \theta(x_i - L_i) + \theta(U_i - x_i)$; taking for instance the $x \log x$-entropy and $0 \leq x_i \leq 1$, we obtain the *Fermi-Dirac* entropy $h_i(x_i) = x_i \ln x_i + (1 - x_i)\ln(1 - x_i)$.

3.4 Global existence and asymptotic analysis

3.4.1 Well-posedness of $(H\text{-}SD)$

In this section we establish the well-posedness of $(H\text{-}SD)$ (see Definition 3.3.2) under three different conditions. In order to avoid any confusion, we say that a set $E \subset \mathbb{R}^n$ is *bounded* when it is so for the usual Euclidean norm $|y| = \sqrt{\langle y, y\rangle}$. First, we propose the condition :

(WP_1) The lower level set $\{y \in \overline{\mathcal{F}} \mid f(y) \leq f(x^0)\}$ is bounded.

Notice that (WP_1) is weaker than the classical assumption imposing f to have bounded lower level sets in the H metric sense. Next, let D_h be the D-function of h that is defined by (3.3.19) and consider the following condition :

(WP_2)

$$\begin{cases} \text{(i) dom } h = \overline{C} \text{ and } \forall a \in \overline{C},\ \forall \gamma \in \mathbb{R},\ \{y \in \mathcal{F} \mid D_h(a,y) \leq \gamma\} \text{ is bounded.} \\ \text{(ii) } S(P) \neq \emptyset \text{ and } f \text{ is quasi-convex (i.e. the lower level sets of } f \text{ are convex).} \end{cases}$$

When $\overline{\mathcal{F}}$ is unbounded (WP_1) and (WP_2) involve some a priori properties on f. This is actually not necessary for the well-posedness of $(H\text{-}SD)$. Consider :

(WP_3) $\exists\, K \geq 0$, $L \in \mathbb{R}$ such that $\forall x \in C$, $||H(x)^{-1}|| \leq K|x| + L$.

This property is satisfied by relevant Legendre type functions; take for instance (3.3.29).

Theorem 3.4.1 *Assume that* (3.2.3), (3.3.16) *and* (H_0) *hold and additionally that either* (WP_1), (WP_2) *or* (WP_3) *is satisfied. Then the dynamical system* $(H\text{-}SD)$ *is well-posed. Consequently, the mapping* $t \mapsto f(x(t))$ *is non-increasing and convergent as* $t \to +\infty$.

Proof. When no confusion may occur, we drop the dependence on the time variable t. By definition, $T_M = \sup\{T > 0 | \exists!$ solution x of $(H\text{-}SD)$ on $[0,T)$ s.t. $x([0,T)) \subset \mathcal{F}\}$. We have that $T_M > 0$. The definition (3.2.8) of P_x implies that for all $y \in \mathcal{A}_0$, $(H(x)^{-1}\nabla f(x) + \dot{x}, y + \dot{x})_x^H = 0$ on $[0, T_M)$ and therefore

$$\langle \nabla f(x) + H(x)\dot{x}, y + \dot{x} \rangle = 0 \text{ on } [0, T_M). \tag{3.4.36}$$

Letting $y = 0$ in (3.4.36), yields

$$\frac{d}{dt} f(x) + \langle H(x)\dot{x}, \dot{x} \rangle = 0. \tag{3.4.37}$$

By (3.2.3)(ii), $f(x(t))$ is convergent as $t \to T_M$. Moreover

$$\langle H(x(\cdot))\dot{x}(\cdot), \dot{x}(\cdot) \rangle \in L^1(0, T_M; \mathbb{R}). \tag{3.4.38}$$

Suppose that $T_M < +\infty$. To obtain a contradiction, we begin by proving that x is bounded. If (WP_1) holds then x is bounded because $f(x(t))$ is non-increasing so that $x(t) \in \{y \in \overline{\mathcal{F}} \,|\, f(y) \leq f(x^0)\}$, $\forall t \in [0, T_M)$. Assume now that f and h comply with (WP_2), and let $a \in \overline{\mathcal{F}}$. For each $t \in [0, T_M)$ take $y = x(t) - a$ in (3.4.36) to obtain $\langle \nabla f(x) + \frac{d}{dt}\nabla h(x), x - a + \dot{x} \rangle = 0$. By (3.4.37), this gives $\langle \frac{d}{dt}\nabla h(x), x - a \rangle + \langle \nabla f(x), x - a \rangle = 0$, which we rewrite as

$$\frac{d}{dt} D_h(a, x(t)) + \langle \nabla f(x(t)), x(t) - a \rangle = 0, \ \forall t \in [0, T_M). \tag{3.4.39}$$

Now, let $a \in \overline{F}$ be a minimizer of f on $\overline{\mathcal{F}}$. From the quasi-convexity property of f, it follows that $\forall t \in [0, T_M)$, $\langle \nabla f(x(t)), x(t) - a \rangle \geq 0$. Therefore, $D_h(a, x(t))$ is non-increasing and (WP_2)(ii) implies that x is bounded. Next, suppose that (WP_3) holds and fix $t \in [0, T_M)$, we have $|x(t) - x^0| \leq \int_0^t |\dot{x}(s)| ds \leq \int_0^t ||\sqrt{H(x(s))^{-1}}|| |\sqrt{H(x(s))}\, \dot{x}(s)| ds \leq (\int_0^t ||H(x(s))^{-1}|| ds)^{1/2} (\int_0^t \langle H(x(s))\dot{x}(s), \dot{x}(s) \rangle ds)^{1/2}$. The third inequality is a consequence of the Cauchy-Schwartz inequality and the fact that $\|H(x)\|^2$ is the biggest eigenvalue of $H(x)$. Hence, $|x(t) - x^0| \leq 1/2[\int_0^t ||H(x(s))^{-1}|| ds + \int_0^t \langle H(x(s))\dot{x}(s), \dot{x}(r) \rangle ds]$. Combining (WP_3) and (3.4.38), Gronwall's lemma yields the boundedness of x.

Let $\omega(x^0)$ be the set of limit points of x, and set $K = x([0, T_M)) \cup \omega(x^0)$. Since x is bounded, $\omega(x^0) \neq \emptyset$ and K is compact. If $K \subset C$ then the compactness of K implies that x can be extended beyond T_M, which contradicts the maximality of T_M. Let us prove $K \subset C$. We argue again by contradiction. Assume that $x(t_j) \to x^*$, with $t_j < T_M$, $t_j \to T_M$ as $j \to +\infty$ and $x^* \in \text{bd}\, C = \overline{C} \setminus C$. Since h is of Legendre type, we have $|\nabla h(x(t_j))| \to +\infty$, and we may assume that $\nabla h(x(t_j))/|\nabla h(x(t_j))| \to \nu \in \mathbb{R}^n$ with $|\nu| = 1$.

Lemma 3.4.2 *If $(x^j) \subset C$ is a sequence such that $x^j \to x^* \in \text{bd}C$ and $\nabla h(x^j)/|\nabla h(x^j))| \to \nu \in \mathbb{R}^n$, h being a function of Legendre type with $C = \text{int dom}\, h$, then $\nu \in N_{\overline{C}}(x^*)$.*

Proof of Lemma 3.4.2. By convexity of h, $\langle \nabla h(x^j) - \nabla h(y), x^j - y \rangle \geq 0$ for all $y \in C$. Dividing by $|\nabla h(x^j)|$ and letting $j \to +\infty$, we get $\langle \nu, y - x^* \rangle \leq 0$ for all $y \in C$, which holds also for $y \in \overline{C}$. Hence, $\nu \in N_{\overline{C}}(x^*)$. ■

Therefore, $\nu \in N_{\overline{C}}(x^*)$. Let $\nu_0 = \Pi_{\mathcal{A}_0}\nu$ be the Euclidean orthogonal projection of ν onto $\mathcal{A}_0$, and take $y = \nu_0$ in (3.4.36). Using (3.4.37), integration gives

$$\langle \nabla h(x(t_j)), \nu_0 \rangle = \langle \nabla h(x^0) - \int_0^{t_j} \nabla f(x(s)) ds, \nu_0 \rangle. \tag{3.4.40}$$

By (H_0) and the boundedness property of x, the right-hand side of (3.4.40) is bounded under the assumption $T_M < +\infty$. Hence, to draw a contradiction from (3.4.40) it suffices to prove $\langle \nabla h(x(t_j)), \nu_0\rangle \to +\infty$. Since $\langle \nabla h(x(t_j))/|\nabla h(x(t_j))|, \nu_0\rangle \to |\nu_0|^2$, the proof of the result is complete if we check that $\nu_0 \neq 0$. This is a direct consequence of the following

Lemma 3.4.3 *Let C be a nonempty open convex subset of $\mathbb{R}^n$ and $\mathcal{A}$ an affine subspace of $\mathbb{R}^n$ such that $C \cap \mathcal{A} \neq \emptyset$. If $x^* \in (\operatorname{bd} C) \cap \mathcal{A}$ then $N_{\overline{C}}(x^*) \cap \mathcal{A}_0^\perp = \{0\}$ with $\mathcal{A}_0 = \mathcal{A} - \mathcal{A}$.*

Proof of Lemma 3.4.3. Let us argue by contradiction and suppose that we can pick some $v \neq 0$ in $\mathcal{A}_0^\perp \cap N_{\overline{C}}(x^*)$. For $y_0 \in C \cap \mathcal{A}$ we have $\langle v, x^* - y_0\rangle = 0$. For $r \geq 0$, $z \in \mathbb{R}^n$, let $B(z, r)$ denote the ball with center z and radius r. There exists $\epsilon > 0$, such that $B(y_0, \epsilon) \subset C$. Take w in $B(0, \epsilon)$ such that $\langle v, w\rangle < 0$, then $y_0 + w \in C$, yet $\langle v, x^* - (y_0 + w)\rangle = \langle v, w\rangle < 0$. This contradicts the fact that v is in $N_{\overline{C}}(x^*)$. ■

Remark 3.4.4 We have proved that under either (WP_1) or (WP_2), $x(t)$ is bounded, and if (WP_2) holds then for each $a \in S(P)$, $D_h(a, x(t))$ is a Liapounov functional for $(H\text{-}SD)$.

3.4.2 Value convergence for a convex objective function

As a first result concerning the asymptotic behavior of $(H\text{-}SD)$, we have the following :

Proposition 3.4.1 *If $(H\text{-}SD)$ is well-posed and f is convex then $\forall a \in \mathcal{F}$, $\forall t > 0$, $f(x(t)) \leq f(a) + \frac{1}{t} D_h(a, x^0)$, where D_h is defined by (3.3.19), and therefore $\lim_{t\to+\infty} f(x(t)) = \inf_{\overline{\mathcal{F}}} f$.*

Proof. We begin by noticing that $f(x(t))$ converges as $t \to +\infty$ (see Theorem 3.4.1). Fix $a \in \mathcal{F}$. By (3.4.39), we have that the solution $x(t)$ of $(H\text{-}SD)$ satisfies $\frac{d}{dt} D_h(a, x(t)) + \langle \nabla f(x(t)), x(t) - a\rangle = 0$, $\forall t \geq 0$. The convex inequality $f(x) + \langle \nabla f(x), x - a\rangle \leq f(a)$ yields $D_h(a, x(t)) + \int_0^t [f(x(s)) - f(a)]ds \leq D_h(a, x^0)$. Using that $D_h \geq 0$ and since $f(x(t))$ is non-increasing, we get the estimate. Letting $t \to +\infty$, it follows that $\lim_{t\to+\infty} f(x(t)) \leq f(a)$. Since $a \in \mathcal{F}$ was arbitrary chosen, the proof is complete. ■

Under the assumptions of Proposition 3.4.1 together with (WP_1) or (WP_2), $x(t)$ is bounded (see Remark 3.4.4) and its cluster points belong to $S(P)$. However, the convergence of $x(t)$ as $t \to +\infty$ is more delicate and we will require additional conditions on h or (P).

3.4.3 Bregman metrics and trajectory convergence

In this section we establish the convergence of $x(t)$ under some additional properties on the D-function of h. Let us begin with a definition.

Definition 3.4.1 *A function $h \in \Gamma_0(\mathbb{R}^n)$ is called* Bregman function with zone C *when the following conditions are satisfied :*
(i) $\operatorname{dom} h = \overline{C}$, *$h$ is continuous and strictly convex on $\overline{C}$ and $h_{|C} \in \mathcal{C}^1(C; \mathbb{R})$.*
(ii) $\forall a \in \overline{C}$, $\forall \gamma \in \mathbb{R}$, *$\{y \in C | D_h(a, y) \leq \gamma\}$ is bounded, where D_h is defined by* (3.3.19).
(iii) $\forall y \in \overline{C}$, $\forall y^j \to y$ *with $y^j \in C$, $D_h(y, y^j) \to 0$.*

Observe that this notion slightly weakens the usual definition of Bregman function that was proposed by Censor and Lent in [43] ; see also [36]. Actually, a Bregman function in the sense of Definition 3.4.1 belongs to the class of B-functions introduced by Kiwiel (see [85, Definition 2.4]). Recall the following important asymptotic separation property :

Lemma 3.4.5 [85, Lemma 2.16] *If h is a Bregman function with zone C then $\forall y \in \overline{C}$, $\forall (y^j) \subset C$ such that $D_h(y, y^j) \to 0$, we have $y^j \to y$.*

In relation with the examples given in section 3.3.4, the Legendre kernels θ_i, $i = 3, ..., 6$, are all Bregman functions with zone $\mathbb{R}_+$. For $i = 3, 6$ we obtain the so called *Kullback-Liebler divergence* : $D_{\theta_i}(r, s) = r \ln(r/s) + s - r$, $r \geq 0$ and $s > 0$, while for $i = 4, 5$ with $\gamma = 1/2$ we have $D_{\theta_i}(r, s) = (r^{1/2} - s^{1/2})/s^{1/2}$, $r \geq 0$ and $s > 0$. Concerning the induced Legendre function (3.3.26), we have the following elementary result :

Lemma 3.4.6 *Let C be as in* (3.3.25) *and $\theta \in \Gamma_0(\mathbb{R})$ satisfy (H_1) with θ being a Bregman function with zone $\mathbb{R}_+$. Under* (3.3.27)*, the corresponding Legendre function h defined by* (3.3.26) *is indeed a Bregman function with zone C.*

Theorem 3.4.7 *Suppose that (H_0) holds with h being a Bregman function with zone C. If f is quasi-convex satisfying* (3.2.3), (3.3.16) *and $S(P) \neq \emptyset$ then $(H\text{-}SD)$ is well-posed and its solution $x(t)$ converges as $t \to +\infty$ to some $x^* \in \overline{\mathcal{F}}$ with $-\nabla f(x^*) \in N_{\overline{C}}(x^*) + \mathcal{A}_0^\perp$. If in addition f is convex then $x(t)$ converges to a solution of (P).*

Proof. Notice first that (WP_2) is satisfied. By Theorem 3.4.1, $(H\text{-}SD)$ is well-posed, $x(t)$ is bounded and for each $a \in S(P)$, $D_h(a, x(t))$ is non-increasing and hence convergent (see Remark 3.4.4). Set $f_\infty = \lim_{t \to +\infty} f(x(t))$ and define $L = \{y \in \overline{\mathcal{F}} \mid f(y) \leq f_\infty\}$. The set L is nonempty and closed. Since f is supposed to be quasi-convex, L is convex, and similar arguments as in the proof of Theorem 3.4.1 under (WP_2) show that $D_h(a, x(t))$ is convergent for all $a \in L$. Let $x^* \in L$ denote a cluster point of $x(t)$ and take $t_j \to +\infty$ such that $x(t_j) \to x^*$. Then, by (iii) in Definition 3.4.1, $\lim_t D_h(x^*, x(t)) = \lim_j D_h(x^*, x(t_j)) = 0$. Therefore, $x(t) \to x^*$ thanks to Lemma 3.4.5. Let us prove that x^* satisfies the optimality condition $-\nabla f(x^*) \in N_{\overline{C}}(x^*) + \mathcal{A}_0^\perp$. Fix $z \in \mathcal{A}_0$, and for each $t \geq 0$ take $y = -\dot{x}(t) + z$ in (3.4.36) to obtain $\langle \frac{d}{dt} \nabla h(x(t)) + \nabla f(x(t)), z \rangle = 0$. This gives

$$\frac{1}{t} \int_0^t \langle \nabla f(x(s)), z \rangle ds = \langle s(t), z \rangle, \tag{3.4.41}$$

where $s(t) = [\nabla h(x^0) - \nabla h(x(t))]/t$. If $x^* \in \mathcal{F}$ then $\nabla h(x(t)) \to \nabla h(x^*)$, hence $\langle \nabla f(x^*), z \rangle = \lim_{t \to +\infty} \frac{1}{t} \int_0^t \langle \nabla f(x(s)), z \rangle ds = \lim_{t \to +\infty} \langle s(t), z \rangle = 0$. Thus, $\Pi_{\mathcal{A}_0} \nabla f(x^*) = 0$. But $N_{\overline{\mathcal{F}}}(x^*) = \mathcal{A}_0^\perp$ when $x^* \in \mathcal{F}$, which proves our claim in this case. Assume now that $x^* \notin \mathcal{F}$, which implies that $x^* \in \partial C \cap \mathcal{A}$. By (3.4.41), we have that $\langle s(t), z \rangle$ converges to $\langle \nabla f(x^*), z \rangle$ as $t \to +\infty$ for all $z \in \mathcal{A}_0$, and therefore $\Pi_{\mathcal{A}_0} s(t) \to \Pi_{\mathcal{A}_0} \nabla f(x^*)$ as $t \to +\infty$. On the other hand, by Lemma 3.4.2, we have that there exists $\nu \in -N_{\overline{C}}(x^*)$ with $|\nu| = 1$ such that $\nabla h(x(t_j))/|\nabla h(x(t_j))| \to \nu$ for some $t_j \to +\infty$. Since $N_{\overline{C}}(x^*)$ is positively homogeneous, we deduce that $\exists\ \bar{\nu} \in -N_{\overline{C}}(x^*)$ such that $\Pi_{\mathcal{A}_0} \nabla f(x^*) = \Pi_{\mathcal{A}_0} \bar{\nu}$. Thus, $-\nabla f(x^*) \in -\Pi_{\mathcal{A}_0} \bar{\nu} + \mathcal{A}_0^\perp \subseteq N_{\overline{C}}(x^*) + \mathcal{A}_0^\perp$, which proves the theorem. ∎

Following [78], we remark that when f is linear, the limit point can be characterized as a sort of "D_h-projection" of the initial condition onto the optimal set $S(P)$. In fact, we have :

Corollary 3.4.1 *Under the assumptions of Theorem* 3.4.7, *if f is linear then the solution $x(t)$ of $(H\text{-}SD)$ converges as $t \to +\infty$ to the unique optimal solution x^* of*

$$\min_{x \in S(P)} D_h(x, x^0). \tag{3.4.42}$$

Proof. Let $x^* \in S(P)$ be such that $x(t) \to x^*$ as $t \to +\infty$. Let $\bar{x} \in S(P)$. Since $x(t) \in \mathcal{F}$, the optimality of $\bar{x}$ yields $f(x(t)) \geq f(\bar{x})$, and it follows from (3.3.20) that $D_h(x(t), x^0) \leq D_h(\bar{x}, x^0)$. Letting $t \to +\infty$ in the last inequality, we deduce that x^* solves (3.4.42). Noticing that $D_h(\cdot, x^0)$ is strictly convex due to Definition 3.4.1(i), we conclude the result. ∎

We finish this section with an abstract result concerning the rate of convergence under uniqueness of the optimal solution. We will apply this result in the next section. Suppose that f is convex and satisfies (3.2.3) and (3.3.16), with in addition $S(P) = \{a\}$. Given a Bregman function h complying with (H_0), consider the following growth condition :

$$(GC) \qquad f(x) - f(a) \geq \alpha D_h(a, x)^\beta, \ \forall x \in U_a \cap \overline{C},$$

where U_a is a neighborhood of a and with $\alpha > 0$, $\beta \geq 1$. infinity. The next abstract result gives an

Proposition 3.4.2 *Assume that f and h satisfy the above conditions an let $x : [0, +\infty) \to \mathcal{F}$ be the solution of $(H\text{-}SD)$. Then we have the following estimations :*
- *If $\beta = 1$ then there exists $K > 0$ such that $D_h(a, x(t)) \leq Ke^{-\alpha t}$, $\forall t > 0$.*
- *If $\beta > 1$ then there exists $K' > 0$ such that $D_h(a, x(t)) \leq K'/t^{\frac{1}{\beta-1}}$, $\forall t > 0$.*

Proof. The assumptions of Theorem 3.4.7 are satisfied, this yields the well-posedness of $(H\text{-}SD)$ and the convergence of $x(t)$ to a as $t \to +\infty$. Besides, from (3.4.39) it follows that for all $t \geq 0$, $\frac{d}{dt} D_h(a, x(t)) + \langle \nabla f(x(t)), x(t) - a \rangle = 0$. By convexity of f, we have $\frac{d}{dt} D_h(a, x(t)) + f(x(t)) - f(a) \leq 0$. Since $x(t) \to a$, there exists t_0 such that $\forall t \geq t_0$, $x(t) \in U_a \cap \mathcal{F}$. Therefore by combining (GC) and the last inequality it follows that

$$\frac{d}{dt} D_h(a, x(t)) + \alpha D_h(a, x(t))^\beta \leq 0, \ \forall t \geq t_0. \tag{3.4.43}$$

In order to integrate this differential inequality, let us first observe that we have the following equivalence : $D_h(a, x(t)) > 0$, $\forall t \geq 0$ iff $x^0 \neq a$. Indeed, if $a \in \overline{\mathcal{F}} \setminus \mathcal{F}$ then the equivalence follows from $x(t) \in \mathcal{F}$ together with Lemma 3.4.5; if $a \in \mathcal{F}$ then the optimality condition that is satisfied by a is $\Pi_{\mathcal{A}_0} \nabla f(a) = 0$, and the equivalence is a consequence of the uniqueness of the solution $x(t)$ of $(H\text{-}SD)$. Hence, we can assume that $x^0 \neq a$ and divide (3.4.43) by $D_h(a, x(t))^\beta$ for all $t \geq t_0$. A simple integration procedure then yields the result. ∎

3.4.4 Linear programming

Let us consider the specific case of a linear program

$$(LP) \qquad \min_{x \in \mathbb{R}^n} \{\langle c, x \rangle \mid Bx \geq d, \ Ax = b\},$$

where A and b are as in section 3.2.1, $c \in \mathbb{R}^n$, B is a $p \times n$ full rank real matrix with $p \geq n$ and $d \in \mathbb{R}^p$. We assume that the optimal set satisfies

$$S(LP) \text{ is nonempty and bounded,} \tag{3.4.44}$$

and there exists a *Slater point* $x^0 \in \mathbb{R}^n$, $Bx^0 > d$ and $Ax^0 = b$. Take the Legendre function

$$h(x) = \sum_{i=1}^{n} \theta(g_i(x)), \quad g_i(x) = \langle B_i, x \rangle - d_i, \tag{3.4.45}$$

where $B_i \in \mathbb{R}^n$ is the ith-row of B and the Legendre kernel θ satisfies (H_1). By (3.4.44), (WP_1) holds and therefore $(H\text{-}SD)$ is well-posed due to Theorem 3.4.1. Moreover, $x(t)$ is bounded (see Remark 3.4.4) and all its cluster points belong to $S(LP)$ by Proposition 3.4.1. The variational property (3.3.20) ensures the convergence of $x(t)$ and gives a variational characterization of the limit as well. Indeed, we have the following result :

Proposition 3.4.3 *Let h be given by* (3.4.45) *with θ satisfying* (H_1). *Under* (3.4.44), $(H\text{-}SD)$ *is well-posed and $x(t)$ converges as $t \to +\infty$ to the unique solution x^* of*

$$\min_{x \in S(LP)} \sum_{i \notin I_0} D_\theta(g_i(x), g_i(x^0)), \tag{3.4.46}$$

where $I_0 = \{i \in I \mid g_i(x) = 0$ for all $x \in S(LP)\}$.

Proof. Assume that $S(LP)$ is not a singleton, otherwise there is nothing to prove. The relative interior ri $S(LP)$ is nonempty and moreover ri $S(LP) = \{x \in \mathbb{R}^n \mid g_i(x) = 0 \text{ for } i \in I_0,\ g_i(x) > 0 \text{ for } i \notin I_0,\ Ax = b\}$. By compactness of $S(LP)$ and strict convexity of $\theta \circ g_i$, there exists a unique solution x^* of (3.4.46). Indeed, it is easy to see that $x^* \in$ ri (LP). Let $\bar{x} \in S(LP)$ and $t_j \to +\infty$ be such that $x(t_j) \to \bar{x}$. It suffices to prove that $\bar{x} = x^*$. When $\theta(0) < +\infty$, the latter follows by the same arguments as in Corollary 3.4.1. When $\theta(0) = +\infty$, the proof of [16, Theorem 3.1] can be adapted to our setting (see also [78, Theorem 2]). Set $x^*(t) = x(t) - \bar{x} + x^*$. Since $Ax^*(t) = b$ and $D_h(x, x^0) = \sum_{i=1}^m D_\theta(g_i(x), g_i(x^0))$, (3.3.20) gives

$$\langle c, x(t) \rangle + \frac{1}{t} \sum_{i=1}^{m} D_\theta(g_i(x(t)), g_i(x^0)) \leq \langle c, x^*(t) \rangle + \frac{1}{t} \sum_{i=1}^{m} D_\theta(g_i(x^*(t)), g_i(x^0)). \tag{3.4.47}$$

But $\langle c, x(t) \rangle = \langle c, x^*(t) \rangle$ and $\forall i \in I_0$, $g_i(x^*(t)) = g_i(x(t)) > 0$. Since $x^* \in \text{ri}\, S(LP)$, for all $i \notin I_0$ and j large enough, $g_i(x^*(t_j)) > 0$. Thus, the right-hand side of (3.4.47) is finite at t_j, and it follows that $\sum_{i \notin I_0} D_\theta(g_i(\bar{x}), g_i(x^0)) \leq \sum_{i \notin I_0} D_\theta(g_i(x^*), g_i(x^0))$. Hence, $\bar{x} = x^*$. ∎

We turn now to the case where there is no equality constraint so that the linear program is

$$\min_{x \in \mathbb{R}^n} \{\langle c, x \rangle \mid Bx \geq d\}. \tag{3.4.48}$$

We assume that (3.4.48) admits a unique solution a and we study the rate of convergence when θ is a Bregman function with zone $\mathbb{R}_+$. To apply Proposition 3.4.2, we need :

Lemma 3.4.8 *Set $C = \{x \in \mathbb{R}^n | Bx > d\}$. If* (3.4.48) *admits a unique solution $a \in \mathbb{R}^n$ then $\exists k_0 > 0$, $\forall y \in \overline{C}$, $\langle c, y - a \rangle \geq k_0 \mathcal{N}(y - a)$, where $\mathcal{N}(x) = \sum_{i \in I} |\langle B_i, x \rangle|$ is a norm on $\mathbb{R}^n$.*

Proof. Set $I_0 = \{i \in I \mid \langle B_i, a\rangle = d_i\}$. The optimality conditions for a imply the existence of a *multiplier* vector $\lambda \in \mathbb{R}^p_+$ such that $\lambda_i[d_i - \langle B_i, a\rangle] = 0$, $\forall i \in I$, and $c = \sum_{i\in I} \lambda_i B_i$. Let $y \in \overline{C}$. We deduce that $\langle c, y-a\rangle = N(y-a)$ where $N(x) = \sum_{i\in I_0} \lambda_i |\langle B_i, x\rangle|$. By uniqueness of the optimal solution, it is easy to see that $\text{span}\{B_i \mid i \in I_0\} = \mathbb{R}^n$, hence N is a norm on $\mathbb{R}^n$. Since $\mathcal{N}(x) = \sum_{i\in I} |\langle B_i, x\rangle|$ is also a norm on $\mathbb{R}^n$ (recall that B is a full rank matrix), we deduce that $\exists k_0$ such that $N(x) \geq k_0 \mathcal{N}(x)$. ■

The following lemma is a sharper version of Proposition 3.4.2 in the linear context.

Lemma 3.4.9 *Under the assumptions of Proposition* 3.4.3*, assume in addition that* θ *is a Bregman function with zone* $\mathbb{R}_-$ *and that there exist* $\alpha > 0$, $\beta \geq 1$ *and* $\varepsilon > 0$ *such that*

$$\forall s \in (0, \varepsilon),\ \alpha D_\theta(0, s)^\beta \leq s. \tag{3.4.49}$$

Then there exists positive constants K, L, M *such that for all* $t > 0$ *the trajectory of (H-SD) satisfies* $D_h(a, x(t)) \leq Ke^{-Lt}$ *if* $\beta = 1$, *and* $D_h(a, x(t)) \leq M/t^{\frac{1}{\beta-1}}$ *if* $\beta > 1$.

Proof. By Lemma 3.4.8, there exists k_0 such that for all $t > 0$,

$$\langle c, x(t) - a\rangle \geq \sum_{i\in I} k_0 |\langle B_i, x(t)\rangle - \langle B_i, a\rangle|. \tag{3.4.50}$$

Now, if we prove that $\exists \lambda > 0$ such that

$$|\langle B_i, x(t)\rangle - \langle B_i, a\rangle| \geq \lambda D_\theta(\langle B_i, a\rangle - d_i, \langle B_i, x(t)\rangle - d_i) \tag{3.4.51}$$

for all $i \in I$ and for t large enough, then from (3.4.50) it follows that $f(\cdot) = \langle c, \cdot\rangle$ satisfies the assumptions of Proposition 3.4.2 and the conclusion follows easily. Since $x(t) \to a$, to prove (3.4.51) it suffices to show that $\forall r_0 \geq 0$, $\exists \eta, \mu > 0$ such that $\forall s$, $|s - r_0| < \eta$, $\mu D_\theta(r_0, s)^\beta \leq |r_0 - s|$. The case where $r_0 = 0$ is a direct consequence of (3.4.49). Let $r_0 > 0$. An easy computation yields $\frac{d^2}{ds^2} D_\theta(r_0, s)_{|s=r_0} = \theta''(r_0)$, and by Taylor's expansion formula

$$D_\theta(r_0, s) = \frac{\theta''(r_0)}{2}(s - r_0)^2 + o(s - r_0)^2 \tag{3.4.52}$$

with $\theta''(r_0) > 0$ due to (H_1)(iii). Let η be such that $\forall s$, $|s - r_0| < \eta$, $s > 0$, $D_\theta(r_0, s) \leq \theta''(r_0)(s - r_0)^2$ and $D_\theta(r_0, s) \leq 1$; since $\beta \geq 1$, $D_\theta(r_0, s)^\beta \leq D_\theta(r_0, s) \leq \theta''(r_0)|s - r_0|$. ■

To obtain Euclidean estimates, the functions $s \mapsto D_\theta(r_0, s)$, $r_0 \in \mathbb{R}_+$ have to be locally compared to $s \mapsto |r_0 - s|$. By (3.4.52) and the fact that $\theta'' > 0$, for each $r_0 > 0$ there exists $K, \eta > 0$ such that $|r_0 - s| \leq K\sqrt{D_\theta(r_0, s)}$, $\forall s$, $|r_0 - s| < \eta$. This shows that, in practice, the Euclidean estimate depends only on a property of the type (3.4.49). Examples :

- The Boltzmann-Shannon entropy $\theta_3(s) = s\ln(s) - s$ and $\theta_6(s) = s\ln s$ satisfy $D_{\theta_i}(0, s) = s$, $s > 0$; hence for some $K, L > 0$, $|x(t) - a| \leq Ke^{-Lt}$, $\forall t \geq 0$.
- With either $\theta_4(s) = -s^\gamma/\gamma$ or $\theta_5(s) = (\gamma s - s^\gamma)/(1 - \gamma)$, $\gamma \in (0, 1)$, we have $D_{\theta_i}(0, s) = (1 + 1/\gamma)s^\gamma$, $s > 0$; hence $|x(t) - a| \leq K/t^{\frac{\gamma}{2-2\gamma}}$, $\forall t > 0$.

3.4.5 Dual convergence

In this section we focus on the case $C = \mathbb{R}^n_{++}$, so that the minimization problem is

$$(P) \qquad \min\{f(x) \mid x \geq 0,\ Ax = b\}.$$

We assume

$$f \text{ is convex and } S(P) \neq \emptyset, \tag{3.4.53}$$

together with the Slater condition

$$\exists x^0 \in \mathbb{R}^n,\ x^0 > 0,\ Ax^0 = b. \tag{3.4.54}$$

In convex optimization theory, it is usual to associate with (P) the *dual* problem given by

$$(D) \qquad \min\{p(\lambda) \mid \lambda \geq 0\},$$

where $p(\lambda) = \sup\{\langle \lambda, x\rangle - f(x) \mid Ax = b\}$. For many applications, dual solutions are as important as primal ones. In the particular case of a linear program where $f(x) = \langle c, x\rangle$ for some $c \in \mathbb{R}^n$, writing $\lambda = c + A^T y$ with $y \in \mathbb{R}^m$ the linear dual problem may equivalently be expressed as $\min\{\langle b, y\rangle \mid A^T y + c \geq 0\}$. Thus, λ is interpreted as a vector of *slack* variables for the dual inequality constraints. In the general case, $S(D)$ is nonempty and bounded under (3.4.53) and (3.4.54), and moreover $S(D) = \{\lambda \in \mathbb{R}^n \mid \lambda \geq 0,\ \lambda \in \nabla f(x^*) + \operatorname{Im} A^T,\ \langle \lambda, x^*\rangle = 0\}$, where x^* is any solution of (P); see for instance [75, Theorems VII.2.3.2 and VII.4.5.1].
Let us introduce a Legendre kernel θ satisfying (H_1) and define

$$h(x) = \sum_{i=1}^{n} \theta(x_i). \tag{3.4.55}$$

Suppose that $(H\text{-}SD)$ is well-posed. Integrating the differential inclusion (3.3.17), we obtain

$$\lambda(t) \in c(t) + \operatorname{Im} A^T, \tag{3.4.56}$$

where $c(t) = \frac{1}{t}\int_0^t \nabla f(x(\tau))d\tau$ and $\lambda(t)$ is the *dual trajectory* defined by

$$\lambda(t) = \frac{1}{t}[\nabla h(x^0) - \nabla h(x(t))]. \tag{3.4.57}$$

Assume that $x(t)$ is bounded. From (3.4.53), it follows that ∇f is constant on $S(P)$, and then it is easy to see that $\nabla f(x(t)) \to \nabla f(x^*)$ as $t \to +\infty$ for any $x^* \in S(P)$. Consequently, $c(t) \to \nabla f(x^*)$. By (3.4.57) together with (3.3.21), we have $x(t) = \nabla h^*(\nabla h(x^0) - t\lambda(t))$, where the Fenchel conjugate h^* is given by $h^*(\lambda) = \sum_{i=1}^n \theta^*(\lambda_i)$. Take any solution $\widetilde{x}$ of $A\widetilde{x} = b$. Since $Ax(t) = b$, we have $\widetilde{x} - \nabla h^*(\nabla h(x^0) - t\lambda(t)) \in \operatorname{Ker} A$. On account of (3.4.56), $\lambda(t)$ is the unique optimal solution of

$$\lambda(t) \in \operatorname{argmin}\left\{ \langle \widetilde{x}, \lambda\rangle + \frac{1}{t}\sum_{i=1}^{n} \theta^*(\theta'(x_i^0) - t\lambda_i) \mid \lambda \in c(t) + \operatorname{Im} A^T \right\}. \tag{3.4.58}$$

By (H_1)(iii), θ' is increasing in $\mathbb{R}_{++}$. Set $\eta = \lim_{s\to+\infty} \theta'(s) \in (-\infty, +\infty]$. Since θ^* is a Legendre type function, $\operatorname{int} \operatorname{dom} \theta^* = \operatorname{dom} \partial\theta^* = \operatorname{Im} \partial\theta = (-\infty, \eta)$. From $(\theta^*)' = (\theta')^{-1}$, it follows that $\lim_{u\to-\infty} (\theta^*)'(u) = 0$ and $\lim_{u\to\eta^-} (\theta^*)'(u) = +\infty$. Consequently, (3.4.58) can be interpreted as a *penalty approximation scheme* of the dual problem (D), where the dual positivity constraints are penalized by a separable strictly convex function. Similar schemes have been treated in [16, 48, 77]. Consider the additional condition

$$\text{Either } \theta(0) < \infty, \text{ or } S(P) \text{ is bounded, or } f \text{ is linear.} \tag{3.4.59}$$

As a direct consequence of [77, Propositions 10 and 11], we obtain that under (3.4.53), (3.4.54), (3.4.59) and (H_1), $\{\lambda(t) \mid t \to +\infty\}$ is bounded and its cluster points belong to $S(D)$. The convergence of $\lambda(t)$ is more difficult to establish. In fact, under some additional conditions on θ^* (see [48, Conditions (H_0)-(H_1)] or [77, Conditions (A7) and (A8)]) it is possible to show that $\lambda(t)$ converges to a particular element of the dual optimal set (the "θ^*-center" in the sense of [48, Definition 5.1] or the $D_h(\cdot, x^0)$-center as defined in [77, pag. 616]), which is characterized as the unique solution of a *nested hierarchy* of optimization problems on the dual optimal set. We will not develop this point here. Let us only mention that for all the examples of section 3.3.4, θ_i^* satisfies such additional conditions and consequently :

Proposition 3.4.4 *Under* (3.4.53), (3.4.54) *and* (3.4.59)*, for each of the explicit Legendre kernels given in section* 3.3.4*,* $\lambda(t)$ *given by* (3.4.55) *converges to a particular dual solution.*

3.5 Legendre transform coordinates

3.5.1 Legendre functions on affine subspaces

The first objective of this section is to slightly generalize the notion of Legendre type function to the case of functions whose domains are contained in an affine subspace of $\mathbb{R}^n$. We begin by noticing that the Legendre type property does not depend on canonical coordinates.

Lemma 3.5.1 *Let* $g \in \Gamma_0(\mathbb{R}^r)$*,* $r \geq 1$*, and* $T : \mathbb{R}^r \to \mathbb{R}^r$ *an affine invertible mapping. Then* g *is of Legendre type iff* $g \circ T$ *is of Legendre type.*

Proof. The proof is elementary and is left to the reader. ■

From now on, $\mathcal{A}$ is the affine subspace defined by (3.2.1), whose dimension is $r = n - m$.

Definition 3.5.1 *A function* $g \in \Gamma_0(\mathcal{A})$ *is said to be of* Legendre type *if there exists an affine invertible mapping* $T : \mathcal{A} \to \mathbb{R}^r$ *such that* $g \circ T^{-1}$ *is a Legendre type function in* $\Gamma_0(\mathbb{R}^r)$*.*

By Lemma 3.5.1, the previous definition is consistent.

Proposition 3.5.1 *Let* $h \in \Gamma_0(\mathbb{R}^n)$ *be a function of Legendre type with* $C = \operatorname{int} \operatorname{dom} h$*. If* $\mathcal{F} = C \cap \mathcal{A} \neq \emptyset$ *then the restriction* $h_{|\mathcal{A}}$ *of* h *to* $\mathcal{A}$ *is of Legendre type and moreover* $\operatorname{int}_{\mathcal{A}} \operatorname{dom} h_{|\mathcal{A}} = \mathcal{F}$ *(*$\operatorname{int}_{\mathcal{A}} B$ *stands for the interior of* B *in* $\mathcal{A}$ *as a topological subspace of* $\mathbb{R}^n$*).*

Proof. From the inclusions $\mathcal{F} \subset \operatorname{dom} h_{|\mathcal{A}} \subset \overline{\mathcal{F}} = \overline{C} \cap \mathcal{A}$ and since $\operatorname{ri} \overline{\mathcal{F}} = \mathcal{F}$, we conclude that $\operatorname{int}_{\mathcal{A}} \operatorname{dom} h_{|\mathcal{A}} = \mathcal{F} \neq \emptyset$. Let $T : \mathbb{R}^r \to \mathcal{A}$ be an invertible transformation with $Tz = Lz + x^0$ for all $z \in \mathbb{R}^r$, where $x^0 \in \mathcal{A}$ and $L : \mathbb{R}^r \to \mathcal{A}_0$ is a nonsingular linear mapping. Define $k = h_{|\mathcal{A}} \circ T$. Clearly, $k \in \Gamma_0(\mathbb{R}^r)$. Let us prove that k is essentially smooth. We have $\text{dom}\, k = T^{-1}\text{dom}\, h_{|\mathcal{A}}$ and therefore $\text{int}\,\text{dom}\, k = T^{-1}\mathcal{F}$. Since h is differentiable on C, we conclude that k is differentiable on $\text{int}\,\text{dom}\, k$. Now, let $(z^j) \in \text{int}\,\text{dom}\, k$ be a sequence that converges to a boundary point $z \in \text{bd}\,\text{dom}\, k$. Then, $Tz^j \in \text{int}_{\mathcal{A}}\text{dom}\, h_{|\mathcal{A}}$ and $Tz^j \to Tz \in \text{bd}_{\mathcal{A}}\text{dom}\, h_{|\mathcal{A}} \subset \text{bd}\,\text{dom}\, h$. Since h is essentially smooth, $|\nabla h(Tz^j)| \to +\infty$. Thus, to prove that $|\nabla k(z^j)| \to +\infty$ it suffices to show that there exists $\lambda > 0$ such that $|\nabla k(z^j)| \geq \lambda |\nabla h(Tz^j)|$ for all j large enough. Note that $\nabla k(z^j) = \nabla[h_{|\mathcal{A}} \circ T](z^j) = L^*\nabla h_{|\mathcal{A}}(Tz^j) = L^*\Pi_{\mathcal{A}_0}\nabla h(Tz^j)$, where $L^* : \mathcal{A}_0 \to \mathbb{R}^r$ is defined by $\langle z, L^*x\rangle = \langle Lz, x\rangle$, $\forall (z,x) \in \mathbb{R}^r \times \mathcal{A}_0$. Of course, L^* is linear with $\text{Ker}\, L^* = \{0\}$. Therefore $\frac{\nabla k(z^j)}{|\nabla h(Tz^j)|} = L^*\Pi_{\mathcal{A}_0}\frac{\nabla h(Tz^j)}{|\nabla h(Tz^j)|}$. Let ω denote the nonempty and compact set of cluster points of the normalized sequence $\nabla h(Tz^j)/|\nabla h(Tz^j)|$, $j \in N$. By Lemma 3.4.2, we have that $\omega \subset \{\nu \in N_{\overline{C}}(Tz) \mid |\nu| = 1\}$, and consequently Lemma 3.4.3 yields $\Pi_{\mathcal{A}_0}\omega \cap \{0\} = \emptyset$. By compactness of ω, we obtain $\liminf_{j\to+\infty} |\Pi_{\mathcal{A}_0}\nabla h(Tz^j)|/|\nabla h(Tz^j)| > 0$, which proves our claim. Finally, the strict convexity of k on $\text{dom}\,\partial k = \text{int}\,\text{dom}\, k = T^{-1}\mathcal{F}$ is a direct consequence of the strict convexity of h in $\mathcal{F}$. ■

3.5.2 Legendre transform coordinates

As we have already recalled in section 3.3.2, the prominent fact of Legendre functions theory is that $h \in \Gamma_0(\mathbb{R}^n)$ is of Legendre type iff its Fenchel conjugate h^* is of Legendre type, and $\nabla h : \text{int}\,\text{dom}\, h \to \text{int}\,\text{dom}\, h^*$ is onto with $(\nabla h)^{-1} = \nabla h^*$. In the case of Legendre functions on affine subspaces, we have the following generalization :

Proposition 3.5.2 *If $g \in \Gamma_0(\mathcal{A})$ is of Legendre type in the sense of Definition* 3.5.1, *then $\nabla g(\text{int}_{\mathcal{A}}\text{dom}\, g)$ is a nonempty, open and convex subset of $\mathcal{A}_0$. In addition, ∇g is a one-to-one continuous mapping from $\text{int}_{\mathcal{A}}\text{dom}\, g$ onto its image.*

Proof. Let $Tx = Lx + z_0$ with $L : \mathcal{A}_0 \to \mathbb{R}^r$ being a linear invertible mapping and $z_0 \in \mathbb{R}^p$. Set $k = g \circ T^{-1} \in \Gamma_0(\mathbb{R}^r)$, which is of Legendre type. We have $\text{dom}\, k = T\text{dom}\, g$. Define $L^* : \mathbb{R}^r \to \mathcal{A}_0$ by $\langle L^*z, x\rangle = \langle z, Lx\rangle$, $\forall (z,x) \in \mathbb{R}^r \times \mathcal{A}_0$. We have that $\nabla g(x) = \nabla[k \circ T](x) = L^*\nabla k(Tx)$ for all $x \in \text{int}_{\mathcal{A}}\text{dom}\, g$. Therefore $\nabla g(\text{int}_{\mathcal{A}}\text{dom}\, g) = L^*\nabla k(T\text{int}_{\mathcal{A}}\text{dom}\, g) = L^*\nabla k(\text{int}_{\mathbb{R}^r}\text{dom}\, k) = L^*\text{int}_{\mathbb{R}^r}\text{dom}\, k^*$. Since $\text{int}_{\mathbb{R}^r}\text{dom}\, k^*$ is a nonempty, open and convex subset of $\mathbb{R}^r$ and L^* is an invertible linear mapping, then $L^*\text{int}_{\mathbb{R}^r}\text{dom}\, k^*$ is an open and nonempty subset of $\mathcal{A}_0$. Moreover, by [112, Theorem 6.6], we have $L^*\text{int}_{\mathbb{R}^r}\text{dom}\, k^* = \text{ri}\, L^*\text{dom}\, k^*$. Consequently, $\nabla g(\text{int}_{\mathcal{A}}\text{dom}\, g) = \text{ri}\, L^*\text{dom}\, k^* = \text{int}_{\mathcal{A}_0}L^*\text{dom}\, k^* \neq \emptyset$. Finally, since $\nabla k : \text{int}_{\mathbb{R}^r}\text{dom}\, k \to \text{int}_{\mathbb{R}^r}\text{dom}\, k^*$ is one-to-one and continuous, the same result holds for $\nabla g = L^* \circ \nabla k \circ T$ on $\text{int}_{\mathcal{A}}\text{dom}\, g$. ■

In the sequel, we assume that h satisfies the basic condition (H_0) and $\mathcal{F} = C \cap \mathcal{A} \neq \emptyset$. The *Legendre transform coordinate mapping* on $\mathcal{F}$ associated with h is defined by

$$\begin{array}{rccl} \phi_h : & \mathcal{F} & \to & \mathcal{F}^* = \phi_h(\mathcal{F}) \\ & x & \mapsto & \phi_h(x) = \nabla(h_{|\mathcal{A}}) = \Pi_{\mathcal{A}_0}\nabla h(x). \end{array} \tag{3.5.60}$$

This definition retrieves the Legendre transform coordinates introduced by Bayer and Lagarias in [30] for the particular case of the log-barrier on a polyhedral set.

Theorem 3.5.2 *Under the above definitions and assumptions, $\mathcal{F}^*$ is a convex, (relatively) open and nonempty subset of $\mathcal{A}_0$, ϕ_h is a $\mathcal{C}^1$ diffeomorphism from $\mathcal{F}$ to $\mathcal{F}^*$, and for* *all* $x \in \mathcal{F}$, $d\phi_h(x) = \Pi_{\mathcal{A}_0} H(x)$ *and* $d\phi_h(x)^{-1} = \sqrt{H(x)^{-1}}\Pi_{\sqrt{H(x)}\mathcal{A}_0}\sqrt{H(x)^{-1}}$, *where* $H(x) = \nabla^2 h(x)$.

Proof. By Propositions 3.5.1 and 3.5.2, $\mathcal{F}^*$ is a convex, open and nonempty subset of $\mathcal{A}_0$ and ϕ_h is a continuous bijection. By (H_0)(ii), ϕ_h is of class $\mathcal{C}^1$ on $\mathcal{F}$ and we have for all $x \in \mathcal{F}$, $d\phi_h(x) = \Pi_{\mathcal{A}_0}\nabla^2 h(x) = \Pi_{\mathcal{A}_0} H(x)$. Let $v \in \mathcal{A}_0$ be such that $d\phi_h(x)v = 0$. It follows that $H(x)v \in \mathcal{A}_0^\perp$ and in particular $\langle H(x)v, v\rangle = 0$. Hence, $v = 0$ thanks to (H_0)(iii). The implicit function theorem implies then that ϕ_h is a $\mathcal{C}^1$ diffeomorphism. The formula concerning $d\phi_h(x)^{-1}$ is a direct consequence of the next lemma.

Lemma 3.5.3 *Define the linear operators* $L_i : \mathbb{R}^n \to \mathbb{R}^n$ *by* $L_1 = \Pi_{\mathcal{A}_0} H(x)$ *and* $L_2 = \sqrt{H(x)^{-1}}\Pi_{\sqrt{H(x)}\mathcal{A}_0}\sqrt{H(x)^{-1}}$. *Then* $L_2 L_1 v = v$ *for all* $v \in \mathcal{A}_0$.

This follows by the same method as in [30], pag. 545; we leave the proof to the reader. ■

Similarly to the classical Legendre type functions theory, the inverse of ϕ_h can be expressed in terms of Fenchel conjugates. For that purpose, we notice that inverting ϕ_h is a minimization problem. Indeed, given $y \in \mathcal{A}_0$, the problem of finding $x \in \mathcal{F}$ such that $y = \Pi_{\mathcal{A}_0}\nabla h(x)$ is equivalent to $x = \mathrm{argmin}\{h(z) - \langle y, z\rangle | z \in \mathcal{A}\}$, or equivalently

$$x = \mathrm{argmin}\{(h + \delta_{\mathcal{A}})(z) - \langle y, z\rangle\}, \tag{3.5.61}$$

where $\delta_{\mathcal{A}}$ is the *indicator* of $\mathcal{A}$, i.e. $\delta_{\mathcal{A}}(z) = 0$ if $z \in \mathcal{A}$ and $+\infty$ otherwise. Let us recall the definition of *epigraphical sum* of two functions $g_1, g_2 \in \Gamma_0(\mathbb{R}^n)$, which is given by $(g_1 \blacksquare g_2)(y) = \inf\{g_1(u) + g_2(v) | u + v = y\}$, $\forall y \in \mathbb{R}^n$. We have $g_1 \blacksquare g_2 \in \Gamma_0(\mathbb{R}^n)$ and if g_1 and g_2 satisfy $\mathrm{ri}\,\mathrm{dom}\, g_1 \cap \mathrm{ri}\,\mathrm{dom}\, g_2 \neq \emptyset$ then $(g_1 + g_2)^* = g_1^* \blacksquare g_2^*$ (see [112]).

Proposition 3.5.3 *We have that* $\phi_h^{-1} : \mathcal{F}^* \to \mathcal{F}$ *is given by* $\phi_h^{-1}(y) = \nabla[h^* \blacksquare (\delta_{\mathcal{A}_0^\perp} + \langle \cdot, \widetilde{x}\rangle)](y)$, *for any* $\widetilde{x} \in \mathcal{A}$, *and moreover* $\mathcal{F}^* = \Pi_{\mathcal{A}_0}\mathrm{int}\,\mathrm{dom}\, h^*$.

Proof. The optimality condition for (3.5.61) yields $y \in \partial(h + \delta_{\mathcal{A}})(x)$. Thus, $x \in \partial(h + \delta_{\mathcal{A}})^*(y)$. From $\mathcal{F} \neq \emptyset$, we conclude that the function $g \in \Gamma_0(\mathbb{R}^n)$ defined by $g = (h + \delta_{\mathcal{A}})^*$ satisfies $g = h^* \blacksquare \delta_{\mathcal{A}}^* = h^* \blacksquare (\delta_{\mathcal{A}_0^\perp} + \langle \cdot, \widetilde{x}\rangle)$ with $\widetilde{x} \in \mathcal{A}$. Moreover, by [112, Corollary 26.3.2], g is essentially smooth and we deduce that indeed $x = \nabla g(y)$. Since g is essentially smooth, $\mathrm{dom}\,\partial g = \mathrm{int}\,\mathrm{dom}\, g$. By definition of epigraphical sum, $g(y) = \inf\{h^*(u) + \delta_{\mathcal{A}_0^\perp}(v) + \langle v, \widetilde{x}\rangle | u + v = y\}$, and consequently we have that $y \in \mathrm{dom}\, g$ iff $y \in \mathrm{dom}\, h^* + \mathcal{A}_0^\perp$. Hence, $\mathrm{int}\,\mathrm{dom}\, g = \mathrm{int}\,\mathrm{dom}\, h^* + \mathcal{A}_0^\perp$ (see for instance [112, Corollary 6.6.2]). Recalling that $\mathcal{F}^*$ is a relatively open subset of $\mathcal{A}_0$, we deduce that $\mathcal{F}^* = \Pi_{\mathcal{A}_0}\mathrm{dom}\,\partial g = \Pi_{\mathcal{A}_0}\mathrm{int}\,\mathrm{dom}\, h^*$. ■

Motivated by [30, 118], we finish this section with the following definition.

Definition 3.5.2 *A point* $x_h \in \mathcal{F}$ *is called the* h-center *of* $\overline{\mathcal{F}}$ *when* $\phi_h(x_h) = 0$.

Remark that, when the h-center exists, x_h is the minimizer of h on $\mathcal{F}$.

3.5.3 Polyhedral sets in Legendre transform coordinates

In this section, we take $C = \{x \in \mathbb{R}^n | Bx > d\}$ with B a $p \times n$ full rank matrix with $p \geq n$.

Proposition 3.5.4 *Suppose that* h *is of the form* (3.4.45) *with* θ *satisfying* (H_1), *and let* $\eta = \lim_{s\to+\infty}\theta'(s) \in (-\infty, +\infty]$. *If* $\eta < +\infty$ *then* $\overline{\mathrm{dom}\, h^*} = \{y \in \mathbb{R}^n \mid y + B^T\lambda = 0,\ \lambda_i \geq -\eta\}$, *and* $\mathrm{dom}\, h^* = \mathbb{R}^n$ *when* $\eta = +\infty$.

Proof. By [114, Theorem 11.5], $\overline{\text{dom}\, h^*} = \{y \in \mathbb{R}^n \mid \langle y, d\rangle \leq h^\infty(d) \text{ for all } d \in \mathbb{R}^n\}$, where h^∞ is the *recession* function, also known as *horizon* function, of h. The recession function is defined by $h^\infty(d) = \lim_{t\to+\infty} \frac{1}{t}[h(\bar{x}+td) - h(\bar{x})]$, $d \in \mathbb{R}^n$, where $\bar{x} \in \text{dom}\, h$; this limit does not depend of $\bar{x}$ and eventually $h^\infty(d) = +\infty$ (see also [112]). In this case, it is easy to verify that $h^\infty(d) = \sum_{i=1}^p \theta^\infty(\langle B_i, d\rangle)$. Clearly, $\theta^\infty(-1) = +\infty$ and $\theta^\infty(1) = \lim_{s\to+\infty} \theta'(s) = \eta$. In particular, if $\eta = +\infty$ then $\overline{\text{dom}\, h^*} = \mathbb{R}^n$. If $\eta < +\infty$ then $y \in \overline{\text{dom}\, h^*}$ iff for all $d \in \mathbb{R}^n$ such that $Bd \geq 0$, $\langle y, d\rangle \leq h^\infty(d) = \sum_{i=1}^p \eta\langle B_i, d\rangle$, that is $\langle y - \eta B^T e, d\rangle \leq 0$ with $e = (1, \cdots, 1)$. Thus, by the Farkas lemma, $y \in \overline{\text{dom}\, h^*}$ iff $\exists \mu \geq 0$, $y - \eta B^T e + B^T \mu = 0$. ■

Corollary 3.5.1 *Under the assumptions of Proposition* 3.5.4*, if* $\eta = 0$ *then* $\mathcal{F}^*$ *is a positive convex cone and if* $\eta = +\infty$ *then* $\mathcal{F}^* = \mathcal{A}_0$.

Proof. This is a direct consequence of Propositions 3.5.3 and 3.5.4. ■

3.5.4 $(H\text{-}SD)$-trajectories in Legendre transform coordinates

The *push forward* vector field of $\nabla_H f_{|\mathcal{F}}$ by ϕ_h is defined for every $y \in \mathcal{F}^*$ by $[(\phi_h)_* \nabla_H f_{|\mathcal{F}}](y) = d\phi_h(\phi_h^{-1}(y))\nabla_H f_{|\mathcal{F}}(\phi_h^{-1}(y))$. In the sequel, we assume that $f(x) = \langle c, x\rangle$ for some $c \in \mathbb{R}^n$.

Proposition 3.5.5 *For all* $y \in \mathcal{F}^*$, $[(\phi_h)_* \nabla_H f_{|\mathcal{F}}]\,(y) = \Pi_{\mathcal{A}_0} c$.

Proof. Let $y \in \mathcal{F}^*$. Setting $x = \phi_h^{-1}(y)$, by Theorem 3.5.2 we get $[(\phi_h)_* \nabla_H f_{|\mathcal{F}}]\,(y) = d\phi_h(x)\nabla_H f_{|\mathcal{F}}(x) = \Pi_{\mathcal{A}_0} H(x) H(x)^{-1}[I - A^T(AH(x)^{-1}A^T)^{-1}AH(x)^{-1}]c = \Pi_{\mathcal{A}_0} c - \Pi_{\mathcal{A}_0} A^T z$, where $z = [(AH(x)^{-1}A^T)^{-1}AH(x)^{-1}]c$. Since Im $A^T = \mathcal{A}_0^\perp$, the conclusion follows. ■

We will give two optimality characterizations of the orbits of $(H\text{-}SD)$, extending thus to the general case the results of [30] for the log-metric.

Geodesic curves. First, we claim that the orbits of $(H\text{-}SD)$ can be regarded as geodesics curves with respect to some appropriate metric on $\mathcal{F}$. To this end, we endow $\mathcal{F}^* = \phi_h(\mathcal{F})$ with the Euclidean metric, which allows us to define on $\mathcal{F}$ the metric

$$(\cdot,\cdot)^{H^2} = (\phi_h)_* \langle\cdot,\cdot\rangle, \tag{3.5.62}$$

that is, $\forall (x, u, v) \in \mathcal{F}\times\mathbb{R}^n\times\mathbb{R}^n$, $(u, v)_x^{H^2} = \langle d\phi_h(x)u, d\phi_h(x)v\rangle = \langle \Pi_{\mathcal{A}_0} H(x)u, \Pi_{\mathcal{A}_0} H(x)v\rangle$. For each initial condition $x^0 \in \mathcal{F}$, and for every $c \in \mathbb{R}^n$ we set

$$v = d\phi_h(x^0)^{-1}\Pi_{\mathcal{A}_0} c = \sqrt{H(x^0)^{-1}}\,\Pi_{\sqrt{H(x^0)}\mathcal{A}_0} \sqrt{H(x^0)^{-1}}\,\Pi_{\mathcal{A}_0} c. \tag{3.5.63}$$

Theorem 3.5.4 *Let* $(x^0, c) \in \mathcal{F} \times \mathbb{R}^n$*, set* $f(x) = \langle c, x\rangle$, $\forall x \in C$ *and define* v *as in* (3.5.63)*. If* $\mathcal{F}$ *is endowed with the metric* $(\cdot,\cdot)^{H^2}$ *given by* (3.5.62)*, then the solution* $x(t)$ *of* $(H\text{-}SD)$ *is the unique geodesic passing through* x^0 *with velocity* v.

Proof. Since $\mathcal{F}$, $(\cdot,\cdot)^{H^2}$ is isometric to the Euclidean Riemannian manifold $\mathcal{F}^*$, the geodesic joining two points of $\mathcal{F}$ exists and is unique. Let us denote by $\gamma : J \subset \mathbb{R} \mapsto \mathcal{F}$ the geodesic passing through x^0 with velocity v. By definition of $(\cdot,\cdot)^{H^2}$, $\phi_h(\gamma)$ is a geodesic in $\mathcal{F}^*$. Whence $\phi_h(\gamma(t)) = \phi_h(x^0) + td\phi_h(x^0)v$, where $t \in J$. In view of (3.5.63), this can be rewritten $\phi_h(\gamma(t)) = \phi_h(x^0) + t\Pi_{\mathcal{A}_0} c$. By Proposition 3.5.5 we know that $(\phi_h)_* \nabla_H f_{|\mathcal{F}} = \Pi_{\mathcal{A}_0} c$, and therefore $\phi_h^{-1}(\phi_h(\gamma)) = \gamma$ is exactly the solution of $(H\text{-}SD)$.■

Remark 3.5.5 A Riemannian manifold is called *geodesically complete* if the maximal interval of definition of every geodesic is $\mathbb{R}$. When $\Pi_{\mathcal{A}_0}c \neq 0$ and $\mathcal{F}^*$ is not an affine subspace of $\mathbb{R}^n$, the Riemannian manifold $\mathcal{F}$, $(\cdot,\cdot)^{H^2}$ is not complete in this sense.

Lagrange equations. Following the ideas of [30], we describe the orbits of (H-SD) as orthogonal projections on $\mathcal{A}$ of $\dot{q}-$trajectories of a specific *Lagrangian system.* Recall that given a real-valued mapping $\mathcal{L}(q,\dot{q})$ called the Lagrangian, where $q=(q_1,\ldots,q_n)$ and $\dot{q}=(\dot{q}_1,\ldots,\dot{q}_n)$, the associated Lagrange equations of motion are the following

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}=\frac{\partial \mathcal{L}}{\partial q_i}, \quad \frac{d}{dt}q_i=\dot{q}_i, \quad \forall i=1\ldots n. \tag{3.5.64}$$

Their solutions are C^1-piecewise paths $\gamma : t \longmapsto (q(t),\dot{q}(t))$, defined for $t\in J\subset \mathbb{R}$, that satisfy (3.5.64), and appear as extremals of the functional $\widehat{\mathcal{L}}(\gamma)=\int_J \mathcal{L}(q(t),\dot{q}(t))dt$. Notice that in general, the solutions are not unique, in the sense that they do not only depend on the initial condition $\gamma(0)$. Let us introduce the Lagrangian $\mathcal{L}:\mathbb{R}^n\times C\to\mathbb{R}$ defined by

$$\mathcal{L}(q,\dot{q})=\langle \Pi_{\mathcal{A}_0}c,q\rangle - h(\Pi_{\mathcal{A}}\dot{q}), \tag{3.5.65}$$

where $\Pi_{\mathcal{A}}$ is the orthogonal projection onto $\mathcal{A}$, i.e. $\Pi_{\mathcal{A}}x=\widetilde{x}+\Pi_{\mathcal{A}_0}(x-\widetilde{x})$ for any $\widetilde{x}\in\mathcal{A}$.

Theorem 3.5.6 *For any solution* $\gamma(t)=(q(t),\dot{q}(t))$ *of the Lagrangian dynamical system* (3.5.64) *with Lagrangian given by* (3.5.65)*, the projection* $x(t)=\Pi_{\mathcal{A}}\dot{q}(t)$ *is the solution of (H-SD) with initial condition* $x^0=\Pi_{\mathcal{A}}\dot{q}(0)$.

Proof. It is easy to verify that $\nabla(h\circ\Pi_{\mathcal{A}})(x)=\Pi_{\mathcal{A}_0}\nabla h(\Pi_{\mathcal{A}}x)$ for any $x\in\mathbb{R}^n$. Given a solution $\gamma(t)=(q(t),\dot{q}(t))$ of (3.5.65) defined on J, we set $p(t)=(p_1(t),\ldots,p_n(t))=\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}(\gamma(t)),\ldots,\frac{\partial\mathcal{L}}{\partial\dot{q}_n}(\gamma(t))\right)$. We have $p(t)=\nabla(h\circ\Pi_{\mathcal{A}})(\dot{q}(t))=\Pi_{\mathcal{A}_0}\nabla h(\Pi_{\mathcal{A}}\dot{q}(t))=\phi_h(\Pi_{\mathcal{A}}\dot{q}(t))$. Equations of motion become $\frac{d}{dt}p(t)=\Pi_{\mathcal{A}_0}c$, that is, $\frac{d}{dt}\phi_h(\Pi_{\mathcal{A}}\dot{q}(t))=\Pi_{\mathcal{A}_0}c$. Since $\phi_h:\mathcal{F}\to\mathcal{F}^*$ is a diffeomorphism, the latter means, according to Proposition 3.5.5, that $\Pi_{\mathcal{A}}\dot{q}(t)$ is a trajectory for the vector field $\nabla_H f_{|\mathcal{F}}$. Notice that C being convex, as soon as $\dot{q}(0)\in C$, $\Pi_{\mathcal{A}}\dot{q}(0)\in C\cap\mathcal{A}=\mathcal{F}$, and what precedes forces $\Pi_{\mathcal{A}}\dot{q}(t)$ to stay in $\mathcal{F}$ for any $t\in J$. ■

Completely integrable Hamiltonian systems. In the sequel, all mappings are supposed to be at least of class $\mathcal{C}^2$. Let us first recall the notion of Hamiltonian system. Given an integer $r\geq 1$ and a real-valued mapping $\mathcal{H}(q,p)$ on $\mathbb{R}^{2r}$ with coordinates $(q,p)=(q_1,\ldots,q_r,p_1,\ldots,p_r)$, the *Hamiltonian vector field* $X_{\mathcal{H}}$ *associated with* $\mathcal{H}$ is defined by

$$X_{\mathcal{H}}=\sum_{i=1}^{r}\frac{\partial\mathcal{H}}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial\mathcal{H}}{\partial q_i}\frac{\partial}{\partial p_i}.$$

The trajectories of the dynamical system induced by $X_{\mathcal{H}}$ are the solutions to

$$\begin{cases} \dot{p}_i(t)=-\frac{\partial}{\partial q_i}\mathcal{H}(q(t),p(t)),\ i=1,\ldots,r, \\ \dot{q}_i(t)=\frac{\partial}{\partial p_i}\mathcal{H}(q(t),p(t)),\ i=1,\ldots,r. \end{cases} \tag{3.5.66}$$

Following a standard procedure, Lagrangian functions $\mathcal{L}(q,\dot{q})$ are associated with Hamiltonian systems by means of the so-called Legendre transform

$$\Phi:\begin{cases} \mathbb{R}^{2r} & \longrightarrow & \mathbb{R}^{2r} \\ (q,\dot{q}) & \longmapsto & (q,\frac{\partial\mathcal{L}}{\partial\dot{q}}(q,\dot{q})) \end{cases}$$

In fact, when Φ is a diffeomorphism, the Hamiltonian function $\mathcal{H}$ associated with the Lagrangian $\mathcal{L}$ is defined on $\Phi(\mathbb{R}^{2r})$ by

$$\mathcal{H}(p,q) = \sum_{i=1}^{r} p_i \dot{q}_i - \mathcal{L}(q,\dot{q}) = \langle p, \psi^{-1}(q,p)\rangle - \mathcal{L}(q,\psi^{-1}(q,p)),$$

where $(q,\psi^{-1}(q,p)) := \Phi^{-1}(q,p)$. With these definitions, Φ sends the trajectories of the corresponding Lagrangian system on the trajectories of the Hamiltonian system (3.5.66).

In general, the Lagrangian (3.5.65) does not lead to an invertible Φ on $\mathbb{R}^{2n}$. However, we are only interested in the projections $\Pi_{\mathcal{A}}\dot{q}$ of the trajectories, which, according to Theorem 3.5.6, take their values in $\mathcal{F}$. Moreover, notice that for any differentiable path $t \mapsto q^{\perp}(t)$ lying in $\mathcal{A}_0^{\perp}$, $t \mapsto (q(t),\dot{q}(t))$ is a solution of (3.5.64) iff $t \mapsto (q(t)+q^{\perp}(t), \dot{q}(t)+\dot{q}^{\perp}(t))$ is. This legitimates the idea of restricting $\mathcal{L}$ to $\mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F}$. Hence and from now on, $\mathcal{L}$ denotes the function :

$$\mathcal{L} : \left\{ \begin{array}{ccc} \mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F} & \longrightarrow & \mathbb{R} \\ (q,\dot{q}) & \longmapsto & \mathcal{L}(q,\dot{q}). \end{array} \right.$$

Taking $(q_1,\ldots,q_r)$, with $r = n-m$, a linear system of coordinates induced by an Euclidean orthonormal basis for $\mathcal{A}_0$, we easily see that this "new" Lagrangian has trajectories $(q(t),\dot{q}(t))$ lying in $\mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F}$, whose projections $\Pi_{\mathcal{A}}\dot{q}(t)$ are exactly the $(H\text{-}SD)$ trajectories. Moreover, an easy computation yields

$$\frac{\partial \mathcal{L}}{\partial \dot{q}}(q,\dot{q}) = \Pi_{\mathcal{A}_0}\nabla h(\Pi_{\mathcal{A}_0}\dot{q}) = [\phi_h \circ \Pi_{\mathcal{A}}](\dot{q}),$$

which is a diffeomorphism by Proposition 3.5.2. The Legendre transform is then given by

$$\Phi : \left\{ \begin{array}{ccc} \mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F} & \longrightarrow & \mathcal{A}_0 \times \mathcal{F}^* \\ (q,\dot{q}) & \longmapsto & (q, [\phi_h \circ \Pi_{\mathcal{A}}](\dot{q})), \end{array} \right.$$

and therefore, $\mathcal{L}$ is converted into the Hamiltonian system associated with

$$\mathcal{H} : \left\{ \begin{array}{ccc} \mathcal{A}_0 \times \mathcal{F}^* & \longrightarrow & \mathbb{R} \\ (q,p) & \longmapsto & \langle p, [\phi_h \circ \Pi_{\mathcal{A}}]^{-1}(p)\rangle - \mathcal{L}(q, [\phi_h \circ \Pi_{\mathcal{A}}]^{-1}(p)). \end{array} \right. \tag{3.5.67}$$

Let us now introduce the concept of completely integrable Hamiltonian system. The Poisson bracket of two real valued functions f_1, f_2 on $\mathbb{R}^{2r}$ is given by

$$\{f_1, f_2\} = \sum_{i=1}^{r} \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q_i} - \frac{\partial f_1}{\partial q_i}\frac{\partial f_2}{\partial p_i}.$$

Notice that, from the definitions, we have $\{f_1,f_2\} = X_{f_1}(f_2)$ and $X_{\{f_1,f_2\}} = [X_{f_1}, X_{f_2}]$, where $[\cdot,\cdot]$ is the standard *bracket product* of vector fields [88]. Now, the system (3.5.66) is called *completely integrable* if there exist n functions $f_1,\ldots,f_r$ with $f_1 = \mathcal{H}$, satisfying

$$\left\{ \begin{array}{l} \{f_i, f_j\} = 0, \quad \forall i,j = 1,\ldots,r. \\ df_1(x),\ldots,df_r(x) \text{ are linearly independent at any } x \in \mathbb{R}^{2r}. \end{array} \right.$$

As a motivation for completely integrable systems, we will just point out the following : the functions f_i are called *integrals of motions* because $X_{\mathcal{H}}(f_i) = \{h, f_i\} = 0$, which means that any trajectory of $X_{\mathcal{H}}$ lies on the level sets of each f_i (the same holds for all X_{f_j}). Also, the trajectory passing through (q_0, p_0) lies in the set $\bigcap_{i=1\ldots r} f_i^{-1}(\{f_i(q_o, p_0)\})$. Besides, $[X_{f_i}, X_{f_j}] = 0$ implies that we can find, at least locally, coordinates $(x_1, \ldots, x_r)$ on this set such that $X_{\mathcal{H}} = \frac{\partial}{\partial x_1}, X_{f_2} = \frac{\partial}{\partial x_2}, \ldots, X_{f_r} = \frac{\partial}{\partial x_r}$, that is, in these coordinates, the trajectories of X_{f_i} are straight lines.

Theorem 3.5.7 *Suppose $\Pi_{\mathcal{A}_0} c \neq 0$. The Lagrangian system on $\mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F}$ associated with*

$$\mathcal{L} : \begin{cases} \mathcal{A}_0 \times \Pi_{\mathcal{A}_0}\mathcal{F} & \longrightarrow \quad \mathbb{R} \\ (q, \dot{q}) & \longmapsto \quad \mathcal{L}(q, \dot{q}) \end{cases}$$

gives rise, by the Legendre transform, to a completely integrable Hamiltonian system on $\mathcal{A}_0 \times \mathcal{F}^*$ with Hamiltonian

$$\mathcal{H} : \begin{cases} \mathcal{A}_0 \times \mathcal{F}^* \longrightarrow R \\ (q, p) \longmapsto \langle p, [\phi_h \circ \Pi_{\mathcal{A}}]^{-1}(p) \rangle - \mathcal{L}(q, [\phi_h \circ \Pi_{\mathcal{A}}]^{-1}(p)). \end{cases}$$

Proof. There only remains to prove the complete integrability of the system. To this end, we adapt the proof of [30, Theorem II.12.2] to our abstract framework. Take the integrals of motion to be $f_1 = \mathcal{H}$, $f_i(q, p) = \langle c_i, p \rangle$, $i = 2, \ldots, r$ where $r = n - m$ and $\{\Pi_{\mathcal{A}_0} c,\, c_2, \ldots, c_r\}$ is chosen as to be an orthonormal basis of $\mathcal{A}_0$. For any $i, j \in \{2, \ldots, r\}$, $\{f_i, f_j\}$ is zero since f_i and f_j only depend on p. Let $\phi_{h,l}^{-1}(q, p)$ (resp. $(\Pi_{\mathcal{A}_0} c)_l$) stand for the l-th component of $\phi_h^{-1}(q, p)$ (resp. the l-th component of $\Pi_{\mathcal{A}_0} c$) and take some $k \in \{1, ..., r\}$. Since

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial q_k}(q, p) &= \frac{\partial(\sum_{l=1}^{r} p_l \phi_{h,l}^{-1})}{\partial q_k}(q, p) - \frac{\partial(\mathcal{L} \circ \Phi^{-1})}{\partial q_k}(q, p) \\ &= \sum_{l=1}^{r} p_l \frac{\partial \phi_{h,l}^{-1}}{\partial q_k}(p, q) - \frac{\partial \mathcal{L}}{\partial q_k}(q, \phi_h^{-1}(q, p)) - \sum_{l=1}^{r} \frac{\partial \mathcal{L}}{\partial \dot{q}_l}(q, \phi_h^{-1}(q, p)) \frac{\partial \phi_{h,l}}{\partial q_k}(q, p) \\ &= -(\Pi_{\mathcal{A}_0} c)_k \end{aligned}$$

we deduce that for all $i \in \{2, ..., r\}$, $\{\mathcal{H}, f_i\} = \sum_{k=1}^{r} -\frac{\partial f_i}{\partial p_k}\frac{\partial \mathcal{H}}{\partial q_k} = \langle \Pi_{\mathcal{A}_0} c, c_i \rangle = 0$. The second condition for complete integrability is satisfied too, as the $r \times 2r$ matrix

$$\left([\frac{\partial f_i}{\partial q_1}, \ldots, \frac{\partial f_i}{\partial q_r}, \frac{\partial f_i}{\partial p_1}, \ldots, \frac{\partial f_i}{\partial p_r}] \right)_{i=1,\ldots,r} = \begin{pmatrix} \Pi_{\mathcal{A}_0} c^T & \star \\ & c_1^T \\ 0 & \ldots \\ & c_r^T \end{pmatrix}$$

is full rank. ∎

Chapitre 4

Sur des équations paraboliques régies par des fonctions de Legendre

On doubly nonlinear evolution equations with Legendre type functions in a Hilbert space.

Jérôme Bolte

Abstract. Given two closed proper convex functions f, h on a Hilbert space, we study the existence and the properties of the solutions of the following differential inclusion

$$\frac{d}{dt}\nabla h(x(t)) + \partial f(x(t)) \ni 0,\ t \geq 0.$$

The function h is assumed to be of Legendre type, and typically its domain is different from the whole space. An existence result provides a solution satisfying a selection law and enjoying strong asymptotic properties.

This kind of dynamical system is closely connected with some thermodynamical evolution processes and with constrained convex minimization problems : several examples are given.

Keywords. Legendre functions, nonlinear parabolic equations, convex minimization in Hilbert spaces, Riemannian subdifferential flow.

4.1 Introduction

The Legendre function theory in finite-dimensional spaces has had many applications in optimization from both a theoretical and a practical viewpoint : barrier methods, proximal algorithms, proximal regularization, projection methods [112, 84, 28, 44, 119]... Recently Bauschke-Borwein-Combettes [29] have extended this notion to reflexive Banach spaces, opening the road to new applications and perspectives in optimization and in functional analysis.

Given $\mathcal{H}$, $\langle\cdot,\cdot\rangle$ a real Hilbert space, and $f, h : \mathcal{H} \mapsto \mathbb{R} \cup \{+\infty\}$ two closed proper convex functions, with h of Legendre type (see Definition 4.2.1), this paper is concerned with the following type of differential inclusions

$$\frac{d}{dt}\nabla h(x(t)) + \partial f(x(t)) \ni 0,\ t \geq 0, \tag{4.1.1}$$

with $x(0) := x_0 \in \operatorname{int} \operatorname{dom} h \cap \operatorname{dom} f$.

Evolution equations of the type (4.1.1) have been tackled by many authors, most of time in relation with some thermodynamical problems : multiphase Stefan equation, flows in porous media, and more generally problems arising from industry. For instance in Kenmochi-Pawlow [83] the authors consider time-dependent equations of the type (4.1.1) in relation with flows in porous media. From a technical viewpoint, their approach strongly relies on the fact that the operator ∇h is Lipschitz continuous and defined on the whole of $\mathcal{H}$, which, as we will see, is not required for our main results.

In relation with the Stefan problem, Damlamian-Kenmochi [52] study the following problem

$$\frac{d}{dt}\beta(u) - \triangle u = 0 \text{ on } (0,T) \times \Omega$$

where β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, and Ω is an open domain of $\mathbb{R}^n$, $n \geq 1$. Their results allow to consider the previous equation with $\operatorname{dom} \beta \neq \mathbb{R}$, as an abstract evolution equation governed by a subdifferential operator in the dual space of $H^1(\Omega)$. This is an improvement of [51] where some growth conditions concerning β^{-1} were needed, implying in particular that $\operatorname{dom} \beta = \mathbb{R}$. For other approaches to this type of equations and some extensions, the reader is referred to Bénilan [31], Blanchard-Francfort [33] in reflexive Banach spaces.

Even if they involve some regularity assumptions on h, the results proposed here are different and new ; in particular they allow to handle the case where $\operatorname{dom} h \neq \mathcal{H}$. For a f with compact lower level sets, and without any growth condition, we prove the existence of a solution $x \in W^{1,2}(0, T^*; \mathcal{H})$ satisfying (4.1.1) almost everywhere on $(0, T^*)$, with in addition

$$\forall t \in [0, T^*),\ x(t) \in \operatorname{int} \operatorname{dom} h,$$
$$f(x(T^*)) = \inf_{\operatorname{dom} h} f \text{ if } T^* < \infty.$$

It is worthwhile pointing out that those results hold in finite-dimensional spaces for a general closed proper convex function. This allows to extend most of the results given in Chapter 3, with clear applications to convex minimization.

Another interesting aspect we develop consists in interpreting the differential inclusion (4.1.1) as a Riemannian subdifferential method. By setting $H(x) := \nabla^2 h(x)$ for all $x \in \operatorname{int} \operatorname{dom} h$, the smooth manifold $\operatorname{int} \operatorname{dom} h$ is naturally endowed with a Riemannian structure $(\cdot, \cdot)_x = \langle H(x) \cdot, \cdot \rangle$. Through an adequate definition of the subdifferential of f in $\operatorname{int} \operatorname{dom} h$, $(\cdot, \cdot)$, the system can be then reformulated as

$$\dot{x}(t) + \partial_H f(x(t)) \ni 0,\ t \geq 0$$

with $\partial_H f(x) = H(x)^{-1} \partial f(x)$ for all $x \in \operatorname{int} \operatorname{dom} h \cap \operatorname{dom} \partial f$. The relevance of this viewpoint is confirmed by the proof of a selection law generalizing the well-known "lazy" aspects of usual subdifferential equations.

Concerning the asymptotic behaviour of the orbits of (4.1.1), several results are given ; the most important is that

$$f(x(t)) \to \inf_{\operatorname{dom} h} f \text{ as } t \to T^*.$$

In the final section some examples of Legendre functions and of specific dynamical systems of the type (4.1.1) are given. For instance,

$$\frac{\partial}{\partial t}\left(u + \frac{u}{\sqrt{1 - \int_\Omega u^2}} \right) - \triangle u - g = 0, \quad \text{on } (0, T^*) \times \Omega$$

with $g \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$, $\int_\Omega u_0^2 < 1$, generates a trajectory satisfying $\int_\Omega u^2(t) < 1$, $\forall t \in [0, T^*)$, and solves asymptotically

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega gu \;|u \in H_0^1(\Omega),\ \int_\Omega u^2 \leq 1 \right\}.$$

Notations. They are rather standard. The norm in $\mathcal{H}$ is denoted by $|\cdot|$; the interior, the closure and the boundary of a subset S of $\mathcal{H}$ are respectively denoted by $\operatorname{int} S$, $\overline{S}$ and $\operatorname{bd} S$. The set of closed proper convex functions on $\mathcal{H}$ is denoted by $\Gamma_0(\mathcal{H})$. For $g \in \Gamma_0(\mathcal{H})$ its domain, its Legendre-Fenchel conjugate and its subdifferential are respectively denoted by $\operatorname{dom} g$, g^*, ∂g. The domain of the set valued mapping ∂g is denoted by $\operatorname{dom} \partial g$, and we have $\operatorname{dom} \partial g = \{x \in \mathcal{H} \,|\partial g(x) \neq \emptyset\}$. If S is a subset of $\mathcal{H}$, its indicator function is denoted δ_S and defined by

$$\delta_S(x) = \begin{cases} 0 \text{ if } x \in S, \\ +\infty \text{ elsewhere.} \end{cases}$$

4.2 Legendre functions and Legendre type metrics

Following the lines of [29], let us recall the notion of essential smoothness, essential strict convexity and the definition of Legendre function in a Hilbert space.

Definition 4.2.1 *Let h be in $\Gamma_0(\mathcal{H})$. The function h is called*
(i) essentially smooth, if ∂h is both single valued and locally bounded on its domain.
(ii) essentially strictly convex, if ∂h^{-1} is locally bounded on its domain and h is strictly convex on every convex subset of $\operatorname{dom} \partial h$.
(iii) Legendre, if it satisfies both (i) and (ii).

By [29, Theorem 5.4], the notions introduced in $(i), (ii)$ are dual to each other, i.e. h is essentially smooth if and only if h^* is essentially strictly convex; and therefore

$$h \text{ is of Legendre type if and only if } h^* \text{ is of Legendre type.}$$

A first important result is given by

Theorem 4.2.1 *Assume that $h \in \Gamma_0(\mathcal{H})$ is Legendre. Then*

$$\nabla h : \operatorname{int} \operatorname{dom} h \mapsto \operatorname{int} \operatorname{dom} h^*$$

is bijective with inverse $\nabla h^{-1} = \nabla h^* : \operatorname{int} \operatorname{dom} h^* \mapsto \operatorname{int} \operatorname{dom} h.$

Proof. Again one is referred to [29, Theorem 5.10]. ■

An important feature of essentially smooth functions is their "blow-up" property near the boundary of their domain. Indeed as in finite-dimensional spaces we have

Proposition 4.2.1 *Let h be in $\Gamma_0(\mathcal{H})$. h is essentially smooth if and only if* $\operatorname{int}\operatorname{dom} h \neq \emptyset$, *$h$ is Frechet differentiable on* $\operatorname{int} \operatorname{dom} h$, *with $|\nabla h(x_n)| \to +\infty$ for each sequence x_n in* $\operatorname{int} \operatorname{dom} h$ *converging to a boundary point of* $\operatorname{dom} h$.

Proof. See [29, Theorem 5.6], . ■

This fact and some of its consequences are crucial to our approach of the differential inclusion (4.1.1). For an insight of the interest of such a property in various domains of optimization, one is referred to [119], [44]. To give the reader a concrete idea of its importance in our context, let us now prove a result involving Proposition 4.2.1 for a smooth version of (4.1.1).

Let $h \in \Gamma_0(\mathcal{H})$ be a Legendre function and consider the assumptions
$(L1)$ h is C^2 on $\operatorname{int}\operatorname{dom} h$,
$(L2)$ The mapping $x \in \operatorname{int}\operatorname{dom} h \mapsto H(x) := \nabla^2 h(x) \in \mathcal{L}(\mathcal{H})$ is Lipschitz continuous on the bounded subsets of $\mathcal{H}$ whose closures are contained in $\operatorname{int}\operatorname{dom} h$,
$(L3)$ There exists a positive constant α such that

$$\forall x \in \operatorname{int}\operatorname{dom} h,\ \forall u \in \mathcal{H},\ \ \langle H(x)u, u\rangle \geq \alpha |u|^2.$$

As a direct consequence we have

Proposition 4.2.2 *Assume that $h \in \Gamma_0(\mathcal{H})$ satisfies $(L1)$, $(L2)$ and $(L3)$. For any x in $\operatorname{int}\operatorname{dom} h$, $H(x) : \mathcal{H} \mapsto \mathcal{H}$ is a self-adjoint linear isomorphism.*

Proof. Take x in $\operatorname{int}\operatorname{dom} h$. $(L3)$ clearly implies that $H(x)$ is one-to-one, and if we set

$$H_1(x) = H(x) - \frac{\alpha}{2} Id,$$

$H_1(x)$ is readily seen to be maximal monotone. According to a classical result $H(x) = \frac{\alpha}{2} Id + H_1(x)$ is onto. Since by definition $H(x)$ is self-adjoint the proof is complete. ■

Let $f : \mathcal{H} \to \mathbb{R}$ be a differentiable function whose gradient is Lipschitz continuous on bounded subsets of $\mathcal{H}$, and for $x_0 \in \operatorname{int}\operatorname{dom} h$ consider the following dynamical system

$$\begin{cases} \dot{x}(t) + H(x(t))^{-1} \nabla f(x(t)) = 0,\ t \geq 0, \\ x(0) = x_0. \end{cases} \tag{4.2.2}$$

Proposition 4.2.3 *The system (4.2.2) has a unique solution x defined on $[0, +\infty)$ that satisfies :*
(i) $\forall t \geq 0,\ x(t) \in \operatorname{int}\operatorname{dom} h$
(ii) The function $f(x(t))$ is nonincreasing.

Proof. The arguments are very close to those of Theorem 3.4.1, Chapter 3, and we only sketch the main lines. The existence of a solution defined on some $[0, T_{max})$ and satisfying $x(t) \in \operatorname{int}\operatorname{dom} h$ for all t in $[0, T_{max})$, follows from $(L2)$, and Proposition 4.2.2. Using (4.2.2), we deduce that

$$\frac{d}{dt} f((x(t)) + \langle H(x(t))\dot{x}(t), \dot{x}(t)\rangle = 0,\ \forall t \in [0, T_{max})$$

and therefore by $(L3)$, $\dot{x}(\cdot) \in L^2(0, T_{max}; \mathcal{H})$.

If $T_{max} < +\infty$ then x is Cauchy net at $t = T_{max}$ for the norm topology. Let us argue by contradiction and assume that $x(T_{max}) \in \operatorname{bd}\operatorname{dom} h$.

Integrating (4.2.2) over $(0, t)$, $t < T_{max}$ gives

$$\nabla h(x(t)) - \nabla h(x(0)) + \int_0^t \nabla f(x(s)) ds = 0,$$

and thus by Proposition 4.2.1 we have $|\nabla h(x(t))| \to +\infty$ as $t \to T_{max}$ while $\int_0^t \{\nabla f(x(s))ds\}$ remains bounded. Necessarily $x(T_{max}) \in \operatorname{int}\operatorname{dom} h$, so that the solution can be extended beyond T_{max}. ■

Let us recall now the notion of *D-function* or *Bregman distance*. Given $h \in \Gamma_0(\mathcal{H})$ of Legendre type, we define the D-function of h by

$$D_h : \begin{array}{rcl} \mathcal{H} \times \operatorname{int}\operatorname{dom} h & \to & \mathbb{R} \\ (x,y) & \mapsto & D_h(x,y) = h(x) - h(y) - \langle \nabla h(y), x-y \rangle \end{array}$$

The above functions have a great interest in many fields of convex minimization, see Bregman [36] for his seminal works on convex feasibility problems, Kiwiel [84] for proximal methods of Bregman type, and also Chapter 3.

Here are some of the elementary properties of D_h.

Proposition 4.2.4 *(i)* $D_h \geq 0$
(ii) For all $(x,y) \in \operatorname{int}\operatorname{dom} h \times \operatorname{int}\operatorname{dom} h$, $D_h(x,y) = 0$ *iff* $x = y$.
(iii) For all $(x,y) \in \operatorname{int}\operatorname{dom} h \times \operatorname{int}\operatorname{dom} h$, $D_h(x,y) = D_{h^*}(\nabla f(y), \nabla f(x))$.
(iv) For each sequence y_n *in* $\operatorname{int}\operatorname{dom} h$ *converging to a point in* $\operatorname{int}\operatorname{dom} h$, *we have* $D_h(y, y_n) \to 0$ *as* $n \to +\infty$.
(v) $D(\cdot, y)$ *is coercive for all* $y \in \operatorname{int}\operatorname{dom} h$, *i.e.*

$$\lim_{|x| \to +\infty} D_h(x,y) = +\infty$$

Proof. See [29, Lemma 7.3].

The following Lemma will be very useful in the study of $(H\text{-}SD)_g$.

Lemma 4.2.2 *Let* $y_n \in \operatorname{int}\operatorname{dom} h$ *be a sequence converging to a boundary point of* $\operatorname{dom} h$ *and* $x \in \operatorname{int}\operatorname{dom} h$. *Then* $D_h(x, y_n) \to +\infty$ *as* $n \to +\infty$.

Proof. By Theorem 4.2.1 and Proposition 4.2.4 (iii) we have respectively $\nabla h(x) \in \operatorname{int}\operatorname{dom} h^*$ and $D_h(x, y_n) = D_{h^*}(\nabla h(y_n), \nabla h(x))$. Since h is a Legendre function, so is h^*. From Proposition 4.2.4 (v) it ensues that $D_{h^*}(\cdot, \nabla h(x))$ is coercive. Since h is of Legendre type we have $|\nabla h(y_n)| \to +\infty$ as $n \to +\infty$ and thus the coerciveness of $D_{h^*}(\cdot, \nabla h(x))$ yields the desired result.■

Legendre Metrics. $\mathcal{H}$ is assumed to be finite-dimensional. Let h be a Legendre function that satisfies $(L1) - (L3)$, and set $S := \operatorname{int}\operatorname{dom} h$. As in Chapter 3, the smooth manifold S can be endowed with the metric induced by the hessian of h. For all $x \in \operatorname{int}\operatorname{dom} h$, set

$$\forall u, v \in T_x S \simeq \mathcal{H}, \ \ (u,v)_x = \langle H(x)u, v \rangle,$$

where $T_x S$ denotes the tangent space to S at x. The gradient of a smooth function f in the Riemannian manifold S, $(\cdot,\cdot)$ is then given by

$$\nabla_H f(x) = H(x)^{-1} \nabla f(x) \text{ for any } x \text{ in } S.$$

The metric $(\cdot,\cdot)$ is called the Legendre metric induced by h. Following the ideas of the preceding chapter, the dynamical system (4.2.2) can be rewritten as a Riemannian gradient method

$$\begin{cases} \dot{x}(t) + \nabla_H f(x(t)) = 0, \ t \geq 0, \\ x(0) = x_0. \end{cases} \tag{4.2.3}$$

The section to follow will show that under a minimal qualification assumption, the dynamical system (4.1.1) can also be reformulated as *Riemannian subdifferential method.*

4.3 The dynamical system $(H\text{-}SD)_g$

From now on, h denotes a Legendre function that satisfies the requirements $(L1), (L2)$ *and* $(L3)$.

Let $f \in \Gamma_0(\mathcal{H})$ be such that

$(f1)$ $\operatorname{dom} f \cap \operatorname{int} \operatorname{dom} h \neq \emptyset$,

$(f2)$ Either f has compact lower levels
or $\mathcal{H}$ is finite-dimensional with $\inf\{f(x) \,|x \in \overline{\operatorname{dom} h}\} > -\infty$.

Observe that if f has compact lower levels we also have

$$\inf\{f(x) \,|x \in \overline{\operatorname{dom} h}\} > -\infty.$$

For $x_0 \in \operatorname{int} \operatorname{dom} h \cap \operatorname{dom} f$ we are concerned by the following evolution equation

$$(H\text{-}SD)_g \qquad \dot{x}(t) + H(x(t))^{-1} \nabla f(x(t)) \ni 0, t \geq 0$$

with initial condition $x(0) = x_0$ and where $(H\text{-}SD)_g$ stands for generalized H-steepest descent.
x is said to be a solution of $(H\text{-}SD)_g$ if there exists T^* such that

- $x \in W^{1,1}_{loc}(0, T^*; \mathcal{H})$
- $x(0) = x_0$
- $\forall t \in [0, T^*), \ x(t) \in \operatorname{int} \operatorname{dom} h$
- The equation $(H\text{-}SD)_g$ is satisfied almost everywhere on $(0, T^*)$.

Before giving the main result of this paper, let us give an interpretation of $(H\text{-}SD)_g$ in term of a Riemannian subdifferential method. Let $g \in \Gamma_0(\mathcal{H})$. Consider the restriction of g to $\operatorname{int} \operatorname{dom} h$ and define the multivalued operator

$$\partial_H g \ : \operatorname{int} \operatorname{dom} h \rightrightarrows \mathcal{H}$$

by

$$\begin{aligned} &\forall x \in \operatorname{int} \operatorname{dom} h, \\ &u \in \partial_H g(x) \Leftrightarrow \forall y \in \operatorname{int} \operatorname{dom} h, \ g(y) \geq g(x) + (u, y - x)_x \end{aligned}$$

and its domain by $\operatorname{dom} \partial_H g = \{x \in \operatorname{int} \operatorname{dom} h | \partial_H g(x) \neq \emptyset\}$.

Note that this definition does not imply any a priori information on g *out of the interior of* $\operatorname{dom} h$. As for the smooth case in finite-dimensional spaces $\partial_H g$ and its domain can be expressed thanks to the usual subdifferential of convex analysis. For this we need the following useful Lemma

Lemma 4.3.1 *For $g : \mathcal{H} \mapsto \mathbb{R} \cup \{+\infty\}$ proper and convex, let us denote by $\overline{g}$ its lower semicontinuous regularized. If g satisfies (Q) : $\operatorname{dom} g \cap \operatorname{int} \operatorname{dom} h \neq \emptyset$ then*

$$\overline{g + \delta_{\operatorname{int} \operatorname{dom} h}} = g + \delta_{\overline{\operatorname{dom} h}}.$$

Proof. Classically, we have $(g + \delta_{\operatorname{int} \operatorname{dom} h})^{**} = \overline{g + \delta_{\operatorname{int} \operatorname{dom} h}}$. For two given real extended valued functions $g_1, g_2 : \mathcal{H} \mapsto \mathbb{R} \cup \{+\infty\}$, we recall that their epigraphical sum is defined by $(g_1 +_e g_2)\ (x) = \inf\{g_1(z) + g_2(x - z)\ |z \in \mathcal{H}\}$. Combining (Q) together with the continuity property of $\delta_{\operatorname{int} \operatorname{dom} h}$ on $\operatorname{int} \operatorname{dom} h$, a well-known result implies that

$$(g + \delta_{\operatorname{int} \operatorname{dom} h})^* = g^* +_e (\delta_{\operatorname{int} \operatorname{dom} h})^*.$$

The conclusion follows from the classical duality relation

$$(g^* +_e (\delta_{\operatorname{int} \operatorname{dom} h})^*)^* = g^{**} + (\delta_{\operatorname{int} \operatorname{dom} h})^{**} = g + \delta_{\overline{\operatorname{dom} h}}. \quad \blacksquare$$

We can now state the following

Proposition 4.3.1 *Let $g \in \Gamma_0(\mathcal{H})$ such that* $\operatorname{dom} g \cap \operatorname{int} \operatorname{dom} h \neq \emptyset$. *Then*

$$\operatorname{dom} \partial_H g = \operatorname{dom} \partial g \cap \operatorname{int} \operatorname{dom} h,$$

and for all $x \in \operatorname{dom} \partial_H g$, we have

$$\partial_H g(x) = H(x)^{-1} \partial g(x).$$

Proof. Let $x \in \operatorname{dom} \partial_H g$ and $u \in \partial_H g(x)$. By definition, we have for all y in $\operatorname{int} \operatorname{dom} h$, $g(y) \geq g(x) + (u, y - x)_x$. Using Lemma 4.3.1 it follows that the last inequality holds for any y in $\overline{\operatorname{dom} h}$; in other words

$$[g + \delta_{\overline{\operatorname{dom} h}}](y) \geq [g + \delta_{\overline{\operatorname{dom} h}}](x) + \langle H(x)u, y - x \rangle, \ \forall y \in \mathcal{H}$$

Hence

$$\begin{aligned} H(x)u &\in \partial(g + \delta_{\overline{\operatorname{dom} h}})(x) \\ &= \partial g(x) + N_{\overline{\operatorname{dom} h}}(x) \\ &= \partial g(x) \end{aligned}$$

where the second inequality is a consequence of the qualification assumption $\operatorname{dom} g \cap \operatorname{int} \operatorname{dom} h \neq \emptyset$. The end of the proof follows easily. $\blacksquare$

This viewpoint allows to reformulate $(H\text{-}SD)_g$ as

$$(H\text{-}SD)_g \qquad \dot{x}(t) + \partial_H f(x(t)) \ni 0, t \geq 0$$

with initial condition $x(0) = x_0 \in \operatorname{int} \operatorname{dom} h \cap \operatorname{dom} f$.

Beyond the fact that the usual subdifferential evolution equation is generalized, the interest of defining a subdifferential with respect to some "metric" allows technically to give *a variational characterization* of the multivalued operator governing our differential inclusion which could be of interest in further studies. On the other hand, we shall see that the regular solutions of $(H\text{-}SD)_g$ are subject to a specific selection law which admits a very simple formulation in terms of the metric $(\cdot,\cdot)$.

Let us now state the main result of this paper

Theorem 4.3.2 *The function f is assumed to satisfy $(f1)$, $(f2)$ and the Legendre function h is chosen to verify $(L1)-(L3)$. The differential inclusion $(H\text{-}SD)_g$ admits a solution x such that*
(i) $x \in W^{1,2}(0,T^*;\mathcal{H})$,
(ii) The function $t \ni [0,T^*) \mapsto f(x(t))$ is absolutely continuous and nonincreasing,
(iii) If $T^* < +\infty$ then $x(T^*) \in \mathrm{bd\,dom}\, h$ with

$$f(x(T^*)) = \inf_{\mathrm{dom}\, h} f. \tag{4.3.4}$$

Remark 4.3.3 Let x be the solution obtained above. Using the regularity properties of h and x the dynamical system $(H\text{-}SD)_g$ can be rewritten

$$\frac{d}{dt}\nabla h(x(t)) + \partial f(x(t)) \ni 0, \quad \text{ae on } (0,T^*).$$

The uniqueness of the solution is as far as we know an open question, however the solutions of $(H\text{-}SD)_g$ have a specific property which corresponds to a non Euclidean version of the well-known *selection law* [37]

$$\dot{x}(t^+) + \partial f^0(x(t)) = 0, \; \forall t \geq 0$$

where $\partial f^0(x) = \Pi_{\partial f(x)}(0)$ and with $\Pi_{\partial f(x)}$ being the $\langle\cdot,\cdot\rangle$-orthogonal projection onto the closed subset $\partial f(x)$.

Observing that for x fixed in $\mathrm{dom}\, \partial_{_H} f$, $\partial_{_H} f(x)$ is also a nonempty closed convex subset (cf Proposition 4.3.1) and that $(\cdot,\cdot)_x$ is some inner product on $\mathcal{H}$, we can define the $(\cdot,\cdot)_x$-orthogonal projection onto $\partial_{_H} f(x)$, namely $P_{\partial_{_H} f(x)}$. Let us set

$$\partial_{_H} f^{0H}(x) = P_{\partial_{_H} f(x)}(0).$$

We have

Proposition 4.3.2 *Let x be a solution of $(H\text{-}SD)_g$ on $(0,T^*)$, $T^* > 0$ that satisfies $x \in W^{1,2}(0,T^*,\mathcal{H})$, then*
(i) $t \in [0,T] \mapsto f(x(t))$, is absolutely continuous for all $0 < T < T^*$.
(ii) The following selection law holds

$$\dot{x}(t) + \partial_{_H} f^{0H}(x(t)) = 0, \quad \textit{ae on } (0,T^*).$$

4.4 Existence of a solution. Main proofs.

In order to obtain some smooth approximations of $(H\text{-}SD)_g$, let us recall the definition and the basic properties of the Moreau-Yosida approximate.

For each $\lambda > 0$, define for all $x \in \mathcal{H}$

$$f_\lambda(x) = \inf\{f(y) + \frac{1}{2\lambda}|x-y|^2 \,|y \in \mathcal{H}\}.$$

The proximal mapping of index λ is defined by $J_\lambda := (I + \lambda\partial f)^{-1}$. J_λ is a contraction defined on the whole of $\mathcal{H}$, and we have moreover $f_\lambda(x) = f(J_\lambda x) + \frac{1}{2\lambda}|x - J_\lambda x|^2$.

The sequence of functions f_λ has the following properties

$(MY1)$ f_λ is C^1 and convex,

$(MY2)$ ∇f_λ is Lipschitz continuous,

$(MY3)$ $\forall x \in \mathcal{H}$, $f_\lambda(x) \to f(x)$, and for all $\mu > \lambda$, $f_\lambda \leq f_\mu$.

$(MY4)$ For all closed subset C of $\mathcal{H}$, we have $\inf_C f_\lambda \to \inf_C f$ as $\lambda \to 0$.

For the first three classical results see for instance [12, 37], for the last one is referred for instance to [12]. The approximate dynamical systems are then given by

$$(H\text{-}SD)_\lambda \quad \begin{cases} \dot{x_\lambda}(t) + H(x_\lambda(t))^{-1}\nabla f_\lambda(x_\lambda(t)) = 0,\ t \geq 0, \\ x_\lambda(0) = x_0. \end{cases}$$

By Lemma 4.3.1 these approximate equations can also be rewritten

$$(H\text{-}SD)_\lambda \quad \begin{cases} \dot{x_\lambda}(t) + \nabla_{H} f_\lambda(x_\lambda(t)) = 0,\ t \geq 0, \\ x_\lambda(0) = x_0. \end{cases}$$

The fact that the solution of $(H\text{-}SD)_\lambda$ is defined for all $t \geq 0$ is a consequence of Proposition 4.2.3. Following a standard procedure, we derive from $(H\text{-}SD)_\lambda$ several estimates, and then using compactness arguments we prove the existence of a solution to $(H\text{-}SD)_g$.

For simplicity the subsequences of x_λ, $\dot{x_\lambda}$... are still denoted x_λ, $\dot{x_\lambda}$... When no confusion can occur, the time variable t is dropped out. We set $C := \overline{\text{dom}\, h}$.

Proof of Theorem 4.3.2

Estimates on the sequence x_λ

From $(H\text{-}SD)_\lambda$ it ensues that

$$\frac{d}{dt} f_\lambda(x_\lambda) + (\dot{x_\lambda}, \dot{x_\lambda})_{x_\lambda} = 0. \tag{4.4.5}$$

By using $(L3)$, we obtain successively

$$\begin{aligned} \int_0^t |\dot{x_\lambda}|^2 &\leq 1/\alpha(f_\lambda(x_0) - f_\lambda(x_\lambda(t)) \\ &\leq 1/\alpha(f_\lambda(x_0) - \inf_C f_\lambda),\ \forall t \geq 0 \end{aligned}$$

Hence by $(MY4)$ there exists a constant M_1 such that

$$\sup_{\lambda>0} \int_0^{+\infty} |\dot{x_\lambda}|^2 \leq M_1. \tag{4.4.6}$$

On the other hand, $\inf_C f_\lambda \leq f_\lambda(x_\lambda) \leq f_\lambda(x_0)$, $\forall t \geq 0$, and thus by $(MY4)$ again

$$\sup_{\lambda>0} \sup_{t\geq 0} |f_\lambda(x_\lambda(t))| < +\infty. \tag{4.4.7}$$

From (4.4.5) we deduce that there exists a constant M_2 such that $\int_0^{+\infty} (\dot{x_\lambda}, \dot{x_\lambda})_{x_\lambda} \leq M_2$ and therefore $\sup_{\lambda>0} \int_0^{+\infty} |\frac{d}{dt} f_\lambda(x_\lambda)| < +\infty$. Combining the last inequality with (4.4.7) we obtain that

$$f_\lambda(x_\lambda(\cdot)) \text{ is bounded in } W^{1,1}(0,T),\ \forall T > 0. \tag{4.4.8}$$

Using the Cauchy-Swarz inequality and (4.4.6), it follows that

$$\forall t \geq s \geq 0,\ |x_\lambda(t) - x_\lambda(s)| \leq \sqrt{t-s}\, M_1 \tag{4.4.9}$$

and thus

$$x_\lambda \text{ is bounded in } L^\infty(0,T;\mathcal{H}),\ \forall T > 0. \tag{4.4.10}$$

Let us now examine the properties of the sequence $J_\lambda x_\lambda$. Since J_λ is a contraction on $\mathcal{H}$, we deduce from (4.4.9) that $J_\lambda x_\lambda$ is equicontinuous on $[0,T]$, $\forall T > 0$. On the other hand, and according to (4.4.7) there exists M_3 such that

$$f_\lambda(x_\lambda) = f(J_\lambda x_\lambda) + \frac{1}{2\lambda}|x_\lambda - J_\lambda x_\lambda|^2 \leq M_3. \tag{4.4.11}$$

By $(f2)$, f has compact lower levels so that (4.4.11) implies that for all $t \geq 0$, $J_\lambda x_\lambda(t)$ is relatively compact in $\mathcal{H}$. Resorting to Ascoli Theorem it follows that for all $T > 0$

$$J_\lambda x_\lambda \text{ is relatively compact in } C([0,T];\mathcal{H}) \text{ equipped with the supremum norm.} \tag{4.4.12}$$

Relative compactness properties of the sequence x_λ.

From (4.4.6) and (4.4.10), we deduce the existence of x and v in $L^2(0,+\infty;\mathcal{H})$ such that

$$x_\lambda \to x \text{ in } w - L^2(0,T;\mathcal{H}),\ \forall T > 0 \tag{4.4.13}$$

$$\dot{x_\lambda} \to v \text{ in } w - L^2(0,T;\mathcal{H}),\ \forall T > 0 \tag{4.4.14}$$

with $v = \dot{x}$ in the distributional derivative sense.
Fix $T > 0$. Let us prove that x_λ converges strongly in $C([0,T];\mathcal{H})$ to x. The property (4.4.12) yields the existence of a Cauchy subsequence of $J_\lambda x_\lambda$ in $C([0,T];\mathcal{H})$, still denoted $J_\lambda x_\lambda$. On the other hand, it follows from (4.4.11) and $(MY4)$ that for all $t \geq 0$,

$$\begin{aligned} |x_\lambda(t) - J_\lambda x_\lambda(t)|^2 &\leq \lambda\,(M_3 - f_\lambda(x_\lambda(t))) \\ &\leq \lambda\left(M_3 - \inf_C f_\lambda\right) \leq \lambda M_4 \end{aligned} \tag{4.4.15}$$

where M_4 is a positive constant. Therefore for all $\lambda, \mu > 0$, $t \geq 0$ we have

$$\begin{aligned} |x_\lambda(t) - x_\mu(t)| &\leq |x_\lambda(t) - J_\lambda x_\lambda(t)| + |J_\lambda x_\lambda(t) - J_\mu x_\mu(t)| + |J_\mu x_\mu(t) - x_\mu(t)| \\ &\leq (\sqrt{\lambda} + \sqrt{\mu})\sqrt{M_4} + |J_\lambda x_\lambda(t) - J_\mu x_\mu(t)| \end{aligned}$$

Recalling that $J_\lambda x_\lambda$ is a Cauchy sequence in $C([0,T];\mathcal{H})$, the above inequality entails

$$x_\lambda \to x \text{ in } C([0,T];\mathcal{H}),\ \forall T > 0. \tag{4.4.16}$$

Consequently x is absolutely continuous and $v = \dot{x}$ ae on $(0, +\infty)$.

The sequence $f_\lambda(x_\lambda)$ is bounded in variation on $[0,T]$, cf (4.4.8) ; applying Helly's first Theorem (cf [104] p.222), we may assert the existence of $\psi \in BV_{loc}(0,T)$ such that

$$f_\lambda(x_\lambda(t)) \to \psi(t) \text{ pointwise on } [0,T]. \tag{4.4.17}$$

To identify ψ let us first notice that, according to the epiconvergence properties of the Moreau-Yosida approximate, see [12], (4.4.17) and (4.4.16) imply that

$$\psi(t) = \liminf_{\lambda\to 0} f_\lambda(x_\lambda(t)) \geq f(x(t)),\ \forall t \in [0,T]$$

Now, take $\lambda_0 > 0$, we have by $(MY3)$ $f_{\lambda_0}(x_\lambda(t)) \leq f_\lambda(x_\lambda(t))$, $\forall \lambda < \lambda_0$ and letting $\lambda \to 0$, (4.4.17) and $(MY1)$ give $f_{\lambda_0}(x(t)) \leq \psi(t)$. By $(MY3)$, we have $f(x(t)) \leq \psi(t)$ which finally yields

$$f_\lambda(x_\lambda(t)) \to f(x(t)) \text{ pointwise on } [0,T],\ \forall T > 0. \tag{4.4.18}$$

The boundary time

x is a continuous function from $[0, +\infty)$ to $\mathcal{H}$, with in addition $x(0) = x_0 \in \operatorname{int}\operatorname{dom} h$. Set

$$T^* = \sup\{t \geq 0\ |x(t) \in \operatorname{int}\operatorname{dom} h\}.$$

For all $0 \leq t < T^*$ we have $x(t) \in \operatorname{int}\operatorname{dom} h$, and if T^* is finite we have

$$x(T^*) \in \operatorname{bd}\operatorname{dom} h. \tag{4.4.19}$$

Equation satisfied by the limit

From $(H\text{-}SD)_\lambda$ and (4.4.14) we deduce that

$$\nabla_{H} f_\lambda(x_\lambda(t)) \to -\dot{x}(t) \text{ in } w - L^2(0,T;\mathcal{H}),\ \forall T > 0. \tag{4.4.20}$$

Fix $T \in [0, T^*)$ and $y \in \mathcal{H}$. Let us prove that $-\dot{x}(t) \in \partial_{H} f(x(t))$ ae on $(0,T)$. By convexity of f_λ we have

$$f_\lambda(y) \geq f_\lambda(x_\lambda(t)) + (\nabla_{H} f_\lambda(x_\lambda(t)), y - x_\lambda(t))_{x_\lambda(t)},\ \forall t \in [0,T].$$

Therefore for all $\theta \geq 0$ in $\mathcal{D}(0,T)$ - the set of C^∞ functions with compact support in $(0,T)$- it ensues that

$$\int_0^T \theta(s)\,[f_\lambda(y) - f_\lambda(x_\lambda(s)) + \langle \nabla_{H} f_\lambda(x_\lambda(s)), H(x_\lambda(s))[y - x_\lambda(s)]\,\rangle\,]\,ds \geq 0. \tag{4.4.21}$$

Set $K = \cup_{\lambda>0} x_\lambda([0,T]) \cup x([0,T])$, by using (4.4.16), Proposition 4.2.3, and the fact that $T < T^*$, we see that K is a compact set contained in int dom h. In view of $(L2)$, $H_{|_K}$ is Lipschitz continuous and thus,

$$H(x_\lambda(\cdot))(y - x_\lambda(\cdot)) \to H(x(\cdot))(y - x(\cdot)) \text{ strongly in } C([0,T];\mathcal{H}) \tag{4.4.22}$$

Using the Lebesgue dominated convergence theorem together with (4.4.18), we deduce from (4.4.21), $(MY3)$, (4.4.20) and (4.4.22) that

$$\int_0^T \theta(s) \left[f(y) - f(x(s)) + (\dot{x}(s), y - x(s))_{x(s)} \right] ds \geq 0. \tag{4.4.23}$$

Since θ and y have been arbitrarily chosen, it follows that

$$-\dot{x}(t) \in \partial_H f(x(t)) \text{ ae on } (0, T^*).$$

The case for which $T^* < +\infty$.

We must prove that necessarily $f(x(T^*)) = \inf_C f$ where $C = \overline{\text{dom}\, h}$. Let us argue by contradiction and assume that there exists $a \in C$ such that

$$f(a) < f(x(T^*)) < +\infty. \tag{4.4.24}$$

Note first that by Lemma 4.3.1 we can assume that $a \in \text{int}\,\text{dom}\, h$, with (4.4.24) being still satisfied. By (4.4.18) and $(MY3)$, we know that $f_\lambda(x_\lambda(T^*)) \to f(x(T^*))$ and $f_\lambda(a) \to f(a)$. Therefore (4.4.24) implies that there exists λ_0 such that

$$f_\lambda(a) \leq f(a) < f_\lambda(x(T^*)), \;\; \forall \lambda \leq \lambda_0. \tag{4.4.25}$$

Since the f_λ are convex, the convex inequality together with (4.4.25) and Lemma 4.2.3 (ii) yields

$$\langle \nabla f_\lambda(x_\lambda(t)), x_\lambda(t) - a \rangle \geq f_\lambda(x_\lambda(t)) - f_\lambda(a) \geq f_\lambda(x(T^*)) - f_\lambda(a) > 0, \;\; \forall t \in [0, T^*]$$

Now, let us notice that since $H(y)$, $y \in$ int dom h is self-adjoint, $(H\text{-}SD)_\lambda$ and (4.4.25) imply

$$\begin{aligned} \frac{d}{dt} D_h(a, x(t)) &= \langle \dot{x_\lambda}(t), H(x_\lambda(t))(x_\lambda(t) - a) \rangle \\ &= -\langle \nabla f_\lambda(x_\lambda(t)), x_\lambda(t) - a \rangle \leq 0, \;\; \forall t \in [0, T^*] \end{aligned}$$

and in particular

$$0 \leq D_h(a, x_\lambda(T^*)) \leq D_h(a, x_\lambda(0)) = D_h(a, x_0). \tag{4.4.26}$$

Recalling that $x_\lambda(T^*)$ is a sequence in int dom h converging to a boundary point of dom h (cf (4.4.16) and (4.4.19)), we have by Lemma 4.2.2, $D_h(a, x_\lambda(T^*)) \to +\infty$ as $\lambda \to 0$. This contradicts (4.4.26) and therefore $x(T^*)$ is a minimizer of f.

Regularity property of $f(x(\cdot))$

We recall the following Lemma ([37, Proposition 3.3]) :

Lemma 4.4.1 *Let $g \in \Gamma_0(\mathcal{H})$, $T > 0$ and $y \in W^{1,2}(0,T;\mathcal{H})$ with*

$$y(t) \in \text{dom } \partial g \text{ ae on } (0,T).$$

Assume that there exists $\gamma \in L^2(0,T;\mathcal{H})$ such that $\gamma(t) \in \partial g(t)$ ae on $(0,T)$. Then $t \mapsto g(x(t))$ is absolutely continuous with

$$\frac{d}{dt} g(y(t)) = \langle \dot{y}(t), h \rangle, \quad \forall h \in \partial g(y(t)) \text{ ae on } (0,T).$$

In order to use this result, we notice that by Proposition 4.3.1, $\dot{x}(t) \in -\partial_{H} f(x(t))$ ae on $(0,T^*)$ can be rewritten $H(x(t))\dot{x}(t) \in -\partial f(x(t))$ ae on $(0,T^*)$. Fix T in $[0,T^*)$. Since $x([0,T])$ is compact and contained in $\text{int}\,\text{dom}\,h$, the continuous function $|H|_{|x([0,T])}$ attains its upper bound $\beta > 0$. Therefore, recalling that $\dot{x} \in L^2(0,T;\mathcal{H})$, it ensues that $H(x(t))\dot{x}(t) \in L^2(0,T;\mathcal{H})$, and the previous Lemma yields (i). ■

Proof of Proposition 4.3.2

As in the proof just above, the fact that $\dot{x} \in L^2(0,T^*)$ implies that $f(x(\cdot))$ is absolutely continuous and (i) is proved.

Resorting to Lemma 4.4.1, we have for almost all t in $(0,T^*)$

$$\langle \dot{x}(t), \gamma_1 \rangle = \langle \dot{x}(t), \gamma_2 \rangle, \quad \forall \gamma_1, \gamma_2 \in \partial f(x(t)). \tag{4.4.27}$$

Let us set for all such t in $[0,T^*)$

$$-\dot{x}(t) = g_H(t)$$

with $g_H(t) \in \partial_{H} f(x(t))$. The equation (4.4.27) can be then rewritten

$$(g_H(t), \gamma_1')_{x(t)} = (g_H(t), \gamma_1')_{x(t)},\ \forall \gamma_1', \gamma_2' \in \partial_{H} f(x(t))$$

and we have in particular

$$(g_H(t), g_H(t) - \gamma')_{x(t)} \leq 0,\ \forall \gamma' \in \partial_{H} f(x(t)).$$

The last inequality entirely characterizes $\partial_{H} f^{0H}(x(t))$, so that (ii) is proved. ■

4.5 On the asymptotic behaviour

The convergence of the trajectories towards an equilibrium of f over the closed convex subset $\overline{\text{dom}\,h}$ is as far as we know an open question, however we have the following general result :

Theorem 4.5.1 *The assumptions are those of Theorem 4.3.2 Assume that $T^* = +\infty$, then*

$$f(x(t)) \to \inf_{\overline{\text{dom}\,h}} f \text{ as } t \to +\infty$$

Proof. Let $a \in \operatorname{int}\operatorname{dom} h \cap \operatorname{dom} f$.

The function $t \in [0,+\infty) \mapsto D_h(a,x(t))$ is absolutely continuous and we have for almost all $t \geq 0$

$$\frac{d}{dt}D_h(a,x(t)) = \langle H(x(t))[(x(t))-a], \dot{x}(t)\rangle. \tag{4.5.28}$$

By $(H\text{-}SD)_g$, there exists $g(t) \in \partial f(x(t))$ ae on $(0,+\infty)$ such that $\dot{x}(t) = -H(x(t))^{-1}g(t)$. Using (4.5.28), the fact that $H(x)$ is self-adjoint, and the convex inequality for f, we obtain successively

$$\begin{aligned}\frac{d}{dt}D_h(a,x(t)) &= \langle H(x(t))[(x(t))-a], -H(x(t))^{-1}g(t)\rangle \\ &= -\langle x(t)-a, g(t)\rangle \\ &\leq f(a) - f(x(t)) \text{ ae on}(0,+\infty).\end{aligned}$$

Integrating over $(0,t)$, $t>0$, one obtains $D_h(a,x(t)) + \int_0^t [f(x(s))-f(a)]ds \leq D_h(a,x_0)$ and since $f(x(\cdot))$ is nonincreasing we have by Proposition 4.2.4 (i)

$$\begin{aligned} t[f(x(t))-f(a)] &\leq D_h(a,x_0) \\ f(x(t)) &\leq \frac{1}{t}D_h(a,x_0) + f(a). \end{aligned} \tag{4.5.29}$$

Finally $\lim_{t\to+\infty} f(x(t)) \leq f(a)$, for all $a \in \operatorname{int}\operatorname{dom} h \cap \operatorname{dom} f$, and the conclusion follows from Lemma 4.3.1. ∎

Corollary 4.5.1 Assume that $\mathcal{H}$ is infinite-dimensional with f as specified in $(f2)$. The weak cluster points of $\{x(t)\}$ are contained in $\operatorname{argmin}_{\overline{\operatorname{dom} h}} f$. If f has a unique minimizer a over $\overline{\operatorname{dom} h}$ then

$$x(t) \text{ weakly converges to } a \text{ as } t \to +\infty$$

Corollary 4.5.2 If h is finite at the boundary and if $\inf_{\overline{\operatorname{dom} h}} f$ is attained, the following estimate holds

$$f(x(t)) - \inf_{\overline{\operatorname{dom} h}} f \leq \frac{1}{t}D_h(a,x_0), \tag{4.5.30}$$

where a is a minimizer of f over $\overline{\operatorname{dom} h}$.

Proofs. For the 4.5.1 claim, we just recall that f has compact lower levels, which implies by Theorem 4.3.2 (ii) that $x(t)$ is bounded and has at least one weak cluster point. If x^* is such a point, let $t_n \to +\infty$, $n \to +\infty$ be such that $x(t_n) \to x^*$ weakly in $\mathcal{H}$ as $t_n \to +\infty$. Since $f \in \Gamma_0(\mathcal{H})$, we have $f(x^*) \leq \liminf_{n\to+\infty} f(x(t_n))$ and the conclusion follows from Theorem 4.5.1. To prove Corollary 4.5.2, we notice that since f has compact lower level, the infimum over $\overline{\operatorname{dom} h}$ is attained. Using (4.5.29), we obtain the desired estimate.

Remark 4.5.2 When $\mathcal{H}$ is finite-dimensional with h being a Bregman function, one can prove that the trajectories of $(H\text{-}SD)_g$ converge to a minimizer of f over $\overline{\operatorname{dom} h}$. This is an adaptation of the results of Theorem 3.4.7, Chapter 3.

4.6 Examples of Legendre functions and of $(H\text{-}SD)_g$ dynamical systems

We first propose several types of Legendre functions complying with the requirements $(L1)$, $(L2)$ and $(L3)$, and then give some examples of associated $(H\text{-}SD)_g$ dynamical systems.

Examples of Legendre functions
Take $\alpha, \beta > 0$.

Example 1.

$$h_1(x) = \begin{cases} \frac{\alpha}{2}|x|^2 - \beta\sqrt{1-|x|^2} \text{ if } |x| \leq 1 \\ \\ +\infty \text{ otherwise.} \end{cases}$$

Denote respectively by $\mathbf{B}$ and $\overline{\mathbf{B}}$ the unit open ball of $\mathcal{H}$ and its closure. We have

$$\text{dom}\, h_1 = \overline{\mathbf{B}},\ \text{int dom}\, h_1 = \text{dom}\, \partial h_1 = \mathbf{B},$$

and

$$\nabla h_1(x) = \alpha x + \beta \frac{x}{\sqrt{1-|x|^2}}\ ,\ \forall x \in \text{int dom}\, h_1.$$

The fact that h_1 is essentially smooth follows for instance of Proposition 4.2.1. To prove that h_1 is essentially strictly convex the most difficult point is to prove that ∂h_1^{-1} is locally bounded on its domain. For that, we notice that $\partial h_1^{-1} = [\alpha Id + A]^{-1}$ with A maximal monotone, so that $\partial h_1^* = \partial h_1^{-1}$ is defined and Lipschitz continuous on $\mathcal{H}$. The fact that h_1 is Legendre is now obvious. To see that $(L2)$ is satisfied it suffices to use the fact that the hessian of h is C^∞ and locally Lipschitz continuous as a composition of Lipschitz mappings. The property $(L3)$ is clearly satisfied.

Example 2.

$$h_2(x) = \begin{cases} \frac{\alpha}{2}|x|^2 - \beta\frac{1}{1-|x|^2} \text{ if } |x| < 1 \\ \\ +\infty \text{ otherwise.} \end{cases}$$

We have

$$\text{dom}\, h_2 = \text{int dom}\, h_2 = \text{dom}\, \partial h_2 = \mathbf{B},$$

and

$$\nabla h_2(x) = \alpha x - \beta \frac{2\beta}{(1-|x|^2)^2}\ ,\ \forall x \in \text{int dom}\, h_2.$$

To see that h is a Legendre function which satisfies $(L1)$, $(L2)$ and $(L3)$, the arguments are of the same type as above.

Example 3. Let $\theta : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ be a Legendre function such that $(0, +\infty) \subset \text{dom}\, \theta \subset [0, +\infty)$.

For $x^* \neq 0$ in $\mathcal{H}$, define

$$h^{\theta,x^*}(x) = \frac{\alpha}{2}|x|^2 + \beta\theta(\langle x, x^*\rangle).$$

If $\theta(0) = +\infty$ then dom h^{θ,x^*} is the closed half plane $\overline{\mathcal{P}} := \{y \in \mathcal{H} \,|\, \langle y, x^*\rangle \geq 0\}$. Else dom $h^{\theta,x^*} = \mathcal{P} = \{y \in \mathcal{H} \,|\, \langle y, x^*\rangle > 0\}$. For all $x \in \mathcal{P}$ we have

$$\nabla h^{\theta,x^*}(x) = \alpha x + \beta\theta'(\langle x, x^*\rangle)x^*.$$

To see that h^{θ,x^*} is essentially smooth just apply Proposition 4.2.1 together with the Legendre property of θ. The essential strict convexity, $(L1)$, $(L2)$ and $(L3)$ follows as in Example 1.

Example 4. Assume that $\mathcal{H}$ is finite-dimensional and denote by $x_1, .., x_n$ the canonical coordinates of $\mathcal{H} = \mathbb{R}^n$, $n \geq 1$. For θ as in the example above, we set

$$h_\theta(x) = \frac{\alpha}{2}|x|^2 + \sum_{i=1}^{n} \theta(x_i).$$

As in Chapter 3, h_θ is of Legendre type with

$$\text{int dom } h_\theta = \mathbb{R}^n_{++} := \{x \in \mathbb{R}^n \,|\, x_i > 0, \ \forall i \in \{1, ..n\}\}.$$

Some associated $(H\text{-}SD)_g$ dynamical systems

When $\mathcal{H}$ is infinite-dimensional, the state variable is classically denoted by u and for simplicity the dynamics are written in the following form :

$$\frac{d}{dt}\nabla h(u(t)) + \partial f(u(t)) \ni 0, \ \text{ ae on } (0, T^*)$$

with $u(0) = u_0 \in \text{int dom } h \cap \text{dom } f$.

Let Ω be a nonempty bounded open subset of $\mathbb{R}^n$, $n \geq 1$ with a regular boundary and set $\mathcal{H} := L^2(\Omega)$. Given $g \in L^2(\Omega)$, f is defined as the Dirichlet integral, that is,

$$f : L^2(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$$

with

$$f(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega gu \text{ if } u \in H^1_0(\Omega), \\ \\ +\infty \text{ otherwise.} \end{cases}$$

Let us recall that $\text{dom}\partial f = H^2(\Omega) \cap H^1_0(\Omega)$, with $\partial f(u) = -\Delta u - g$ whenever $u \in \text{dom}\partial f$.

1. With the kernel h_1 we obtain

$$\frac{\partial}{\partial t}\left(\alpha u + \beta\frac{u}{\sqrt{1 - \int_\Omega u^2}}\right) - \Delta u - g = 0, \ \text{ on } (0, T^*) \times \Omega$$

with $u_0 \in H_0^1(\Omega)$, $\int_\Omega u_0^2 < 1$. This evolution equation solves asymptotically the following problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega gu \,|u \in H_0^1(\Omega),\ \int_\Omega u^2 \leq 1\right\}.$$

2. With the kernel h_2 we obtain

$$\frac{\partial}{\partial t}\left(\alpha u - \frac{2\beta}{(1-\int_\Omega u^2)^2}\right) - \triangle u - g = 0, \quad \text{on } (0, T^*) \times \Omega$$

with $u_0 \in H_0^1(\Omega)$, $\int_\Omega u_0^2 < 1$.

3. To give an example with a Legendre function of the type h^{θ, x^*}, we take $\theta(s) = s \ln s$, $s \geq 0$, $+\infty$ otherwise and with the convention $0 \ln 0 = 0$. The linear form x^* is given by the constant function, 1_Ω, i.e. $1_\Omega(y) = 1$, $\forall y \in \Omega$. This gives

$$\frac{\partial}{\partial t}\left(\alpha u + \beta \ln \int_\Omega u\right) - \triangle u - g = 0, \quad \text{on } (0, T^*) \times \Omega$$

with $u_0 \in H_0^1(\Omega)$, $\int_\Omega u_0 > 0$.

Let us finish this section by giving an example in finite-dimensional space. The function θ is chosen as above, and taking a kernel of the type h_θ (cf Example 4), we obtain the following dynamical system

$$\dot{x}(t) + \text{diag}\left[\frac{x_i}{\beta + \alpha x_i}\right] \partial f(x(t)) \ni 0, \quad \text{ae on } (0, T^*)$$

with $\forall i \in \{1, .., n\}$, $(x_0)_i > 0$. For a smooth criterion f, this equation was already proposed in [20]. Using Theorem 4.3.2 and Theorem 4.5.1, we see that the above dynamical system solves asymptotically the following convex minimization problem

$$\inf\{f(x) \,|x \in \mathbb{R}_+^n\}.$$

Deuxième partie

SYSTEMES-GRADIENTS DU SECOND ORDRE EN TEMPS AVEC FROTTEMENTS DE TYPE HESSIEN.

Chapitre 5

Un système dynamique inertiel avec frottements de type hessien

A second-order gradient-like dissipative dynamical system with hessian driven damping. Application to Optimization and Mechanics.[1]

F. ALVAREZ [2] , H. ATTOUCH [3], J. BOLTE [3] AND P. REDONT. [3]

Résumé. Nous étudions le système dynamique

$$\text{(DIN)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$$

où $\Phi : H \to \mathbb{R}$ est une fonctionnelle de classe $\mathcal{C}^2$, H un espace de Hilbert réel, et α, β des paramètres > 0. Le terme inertiel $\ddot{x}(t)$ peut être vu comme une perturbation singulière mais aussi une régularisation de la méthode de Newton continue $\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$.

Le système (DIN) est bien posé. La dissipativité confère aux trajectoires des propriétés intéressantes pour l'optimisation de Φ. Par exemple, si Φ est convexe et $\operatorname{argmin}\Phi \neq \emptyset$, toute trajectoire converge faiblement vers un minimum de Φ. En dimension finie, si Φ est analytique, toute trajectoire converge vers un point critique de Φ.

De façon remarquable, (DIN) est équivalent à un système du premier ordre où le hessien $\nabla^2\Phi$ ne figure pas

$$\begin{cases} \dot{x}(t) + c\nabla\Phi(x(t)) + ax(t) + by(t) = 0 \\ \dot{y}(t) \qquad\qquad\qquad\;\; + ax(t) + by(t) = 0 \end{cases}$$

Il est donc possible de donner un sens à (DIN) losque Φ est de classe $\mathcal{C}^1$, ou même soumise à des contraintes. Nous en donnons deux illustrations : 1) un système dynamique de type gradient projeté avec des trajectoires inertielles viables et des propriétés de minimisation ; 2) une approche du rebond inélastique en mécanique.

Abstract. Given H a real Hilbert space and $\Phi : H \to \mathbb{R}$ a smooth $\mathcal{C}^2$ function, we study the dynamical system

$$\text{(DIN)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$$

where α and β are positive parameters. The inertial term $\ddot{x}(t)$ acts as a singular perturbation and, in fact, regularization of the possibly degenerate classical Newton continuous dynamical system $\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$.

We show that (DIN) is a well-posed dynamical system. Due to their dissipative aspect, trajectories of (DIN) enjoy remarkable optimization properties. For example, when Φ is convex and $\operatorname{argmin}\Phi \neq \emptyset$, then each trajectory of (DIN) weakly converges to a minimizer of Φ. If Φ is real analytic, then each trajectory converges to a critical point of Φ.

A remarkable feature of (DIN) is that one can produce an equivalent system which is first-order in time and with no occurrence of the Hessian, namely

$$\begin{cases} \dot{x}(t) + c\nabla\Phi(x(t)) + ax(t) + by(t) = 0 \\ \dot{y}(t) \qquad\qquad\qquad\;\; + ax(t) + by(t) = 0 \end{cases}$$

where a, b, c are parameters which can be explicitly expressed in terms of α and β. This allows to consider (DIN) when Φ is $\mathcal{C}^1$ only, or more generally, non-smooth or subject to constraints. This is first illustrated

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by a gradient projection dynamical system exhibiting both viable trajectories, inertial aspects, optimization properties, and secondly by a mechanical system with impact.

Keywords : continuous Newton method, dissipative dynamical systems, asymptotic behaviour, gradient-like dynamical systems, optimal control, second-order in time dynamical system, shocks in mechanics, gradient-projection methods.
AMS classification : 37Bxx, 37Cxx, 37Lxx, 37N40, 47H06.

5.1 Introduction

Let H be a real Hilbert space and $\Phi : H \to \mathbb{R}$ a smooth function whose gradient and Hessian are respectively denoted by $\nabla\Phi$ and $\nabla^2\Phi$. Our purpose is to study the following dynamical system

$$\text{(DIN)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0,$$

where α and β are positive parameters. We use the following notations : t is the time variable, $x \in H$ is the state variable, trajectories in H are functions $t \mapsto x(t)$ whose first and second time derivatives are respectively denoted by $\dot{x}(t)$ and $\ddot{x}(t)$.

The above dynamical system will be referred to as the *Dynamical Inertial Newton-like* system, or (DIN) for short. This evolution problem comes naturally into play in various domains like optimization (minimization of Φ), mechanics (non-elastic shocks), control theory (asymptotic stabilization of oscillators) and PDE theory (damped wave equation). The terminology reflects the fact that (DIN) is a second order in time dynamical system, the acceleration $\ddot{x}(t)$ being associated with inertial effects, while Newton's dynamics refers to the action of the Hessian operator $\nabla^2\Phi(x(t))$ on the velocity vector $\dot{x}(t)$ (see (CN) below).

This paper focuses on the study of (DIN) as a dissipative dynamical system ; accordingly, the investigation relies on Lyapounov methods (for facts on dissipative systems see [69, 71, 120]). The convergence of the trajectories of (DIN), as the time t goes to $+\infty$, is established under various assumptions on Φ : Φ analytic (Theorem 5.4.1), Φ convex (Theorem 5.5.1). Indeed, by following the trajectories of (DIN) as t goes to $+\infty$, one expects to reach local minima of Φ (global minima when Φ is convex), with clear applications to optimization and mechanics.

Let us discuss some motivations for the introduction of the (DIN) system.

In recent years, numerous papers have been devoted to the study of dynamical systems that overcome some of the drawbacks of the classical steepest descent method

$$\text{(SD)} \qquad \dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

For instance, Alvarez and Pérez study in [8] the *Continuous Newton* method

$$\text{(CN)} \qquad \nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$$

as a tool in optimization and show how to combine this dynamics with an approximation of Φ by smooth functions Φ_ε, when Φ is nonsmooth. On the other hand, Attouch, Goudou and Redont study in [18] the heavy ball with friction dynamical system

$$\text{(HBF)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \nabla\Phi(x(t)) = 0,$$

where $\alpha > 0$ can be interpreted as a viscous friction parameter. This dissipative dynamical system, which was first introduced by Polyak [111] and Antipin [10] enjoys remarkable optimization properties. For example, when Φ is convex, the trajectories of (HBF) weakly converge in H as $t \to +\infty$ to minimizers of Φ. This result, proved by Alvarez in [4], may be seen as an extension of the celebrated Bruck Theorem for (SD) [40] to a second-order in time differential dynamical system ; see also [5] for an implicit discrete proximal version of their result.

There is a drastic difference between (SD) and (HBF). By contrast with (SD), (HBF) is no more a descent method : the function $\Phi(x(t))$ does not decrease along the trajectories in general ; it is the energy $E(t) := \frac{1}{2}|\dot{x}(t)|^2 + \Phi(x(t))$ that is decreasing. This confers to this system interesting properties for the exploration of local minima of Φ, see [18] for more details.

Both the Newton and the heavy ball with friction methods can be seen as second order extensions of (SD), the latter in time (with $\ddot{x}$ in addition to $\dot{x}$) and the former in space (with $\nabla^2\Phi$ in addition to $\nabla\Phi$). Each one improves (SD) in some respects, but they also raise some new difficulties. In (CN), $\nabla^2\Phi(x(t))$ may be degenerate and (CN) is no more defined as a dynamical system, moreover $\nabla^2\Phi(x(t))$ may be complicated to compute. In (HBF), the trajectories may exhibit oscillations which are not desirable for a numerical optimization purpose.

If one combines the continuous Newton dynamical system with the heavy ball with friction system, the system so obtained,

$$\text{(DIN)} \qquad \ddot{x} + \alpha\dot{x} + \beta\nabla^2\Phi(x)\dot{x} + \nabla\Phi(x) = 0,$$

inherits most of the advantages of the two preceding systems and corrects both of the above-mentioned drawbacks : the term $\nabla^2\Phi(x(t))\dot{x}(t)$ is a clever geometric damping term, while the acceleration term $\ddot{x}(t)$ makes (DIN) a well-posed dynamical system, even if $\nabla^2\Phi(x(t))$ is degenerate ; see Attouch and Redont [19] for a first study of this question.

The relative roles of the damping terms $\alpha\dot{x}$ and $\beta\nabla^2\Phi(x)\dot{x}$ are illustrated below on Rosenbrock's function, $\Phi(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$, which possesses a global minimum at point $(1, 1)$ at the bottom of a flat long winding valley ; see figure 5.1. When the geometric damping is low ($\beta = 10^{-3}$) the trajectory is prone to large oscillations, transversal to the valley axis, and is quite similar to a (HBF) trajectory ($\beta = 0$, see [18]). When the geometric damping is effective ($\beta = 1$), but with a low viscous damping ($\alpha = 10^{-3}$), the trajectory is forced to the bottom of the valley. While transversal oscillations are suppressed, longitudinal oscillations remain important, due to the Hessian being nearly zero in the direction of the valley. As can be seen in the lower plot, a combination of viscous and geometric damping ($\alpha = 1$, $\beta = 1$) puts down any oscillations and produces a trajectory converging regularly to the minimum.

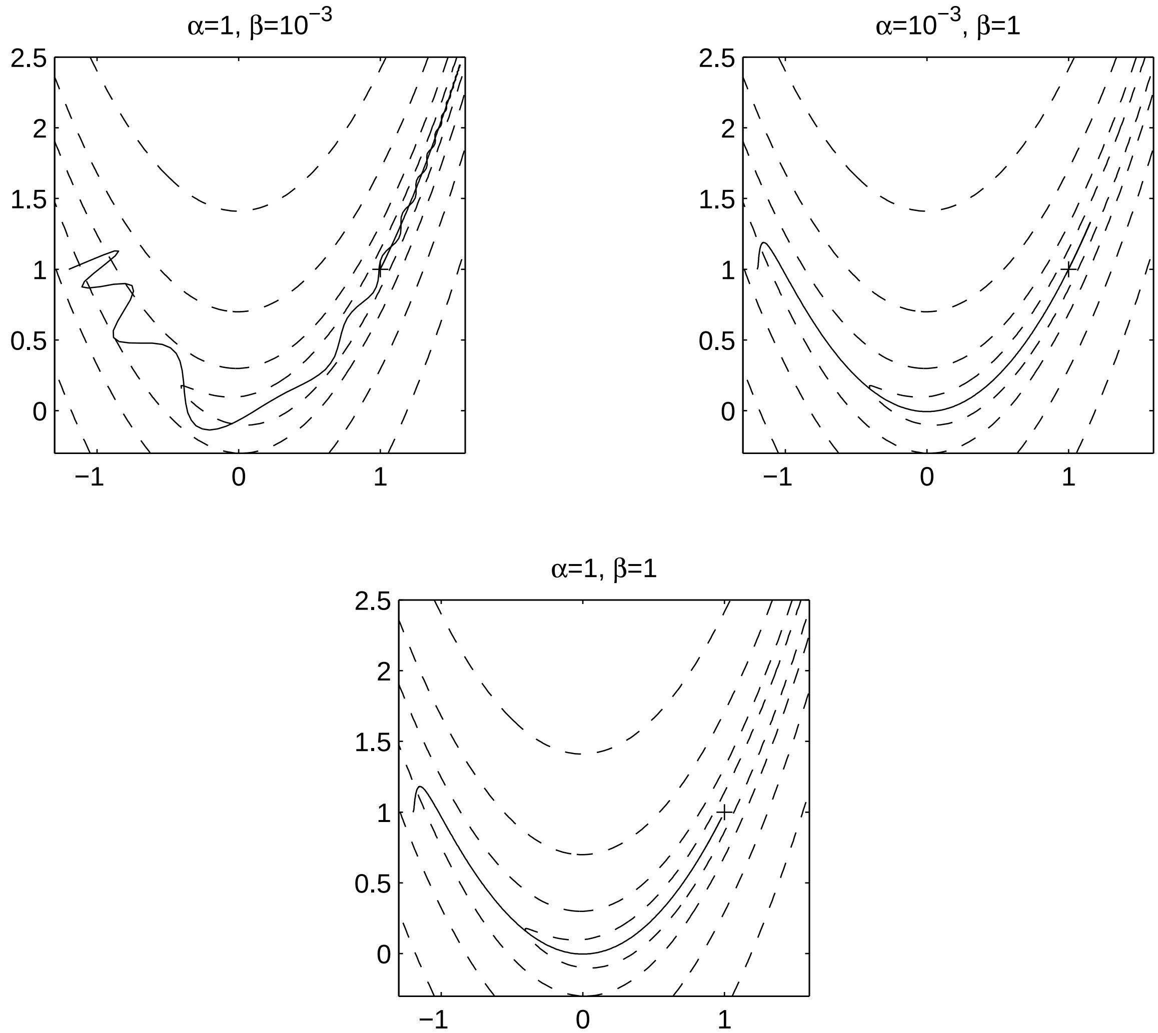

FIG. 5.1 – Versatility of (DIN).

We stress the fact that (DIN) is a second order system both in time (because of the acceleration term $\ddot{x}(t)$) and in space ($\nabla^2\Phi(x(t))$ is the Hessian). The central point of this paper is that, surprisingly, one can "integrate" in some sense this system, and exhibit an equivalent first order system *in time and space* in $H \times H$ which involves no Hessian (section 5.6.3, Theorem 5.6.2)

$$\begin{cases} \dot{x}(t) + c\nabla\Phi(x(t)) + ax(t) + by(t) = 0 \\ \dot{y}(t) \qquad\qquad\qquad\quad + ax(t) + by(t) = 0 \end{cases}$$

This result opens new interesting perspectives : it allows to consider (DIN) for nonsmooth functions, possibly only lower semicontinuous or involving constraints, with clear applications to mechanics and PDE's (wave equations, shocks). For example, when taking $H = L^2(\Omega)$ and Φ being equal to the Dirichlet integral with domain $H_0^1(\Omega)$, the system (DIN) provides the following wave equation with higher order damping, which has been considered by Aassila in [1].

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \alpha \dfrac{\partial u}{\partial t} - \beta \Delta (\dfrac{\partial u}{\partial t}) - \Delta u = 0 & \text{in } \Omega \times]0, +\infty[\\ u = 0 & \text{on } \partial\Omega \times]0, +\infty[\\ u(0) = u_0, \ \dfrac{\partial u}{\partial t}(0) = u_1 & \text{in } \Omega. \end{cases}$$

Another interesting situation corresponds to the case where Φ is proportional to the square of the distance function to a convex set K : $\Phi(x) = \Psi_{K,\lambda}(x) = \frac{1}{2\lambda}\text{dist}^2(x, K)$, $\lambda > 0$ (which is also the Moreau-Yosida approximation of the indicator function of K). In that case, (DIN), written under the form

$$\ddot{x}_\lambda + 2\varepsilon\sqrt{\lambda}\nabla^2\Psi_{K,\lambda}(x)\dot{x}_\lambda + \nabla\Psi_{K,\lambda}(x) = -\alpha\dot{x}_\lambda,$$

is closely related to a dynamical system introduced by Paoli and Schatzman [107] to model non-elastic shocks in mechanics.

Let us finally mention that the formulation of (DIN) as a first-order dynamical system which only involves the gradient of Φ, naturally suggests a way to define the second-order subdifferential $\partial^2\Phi$ of non-smooth functions Φ. It is certainly worthwile comparing this new aproach to $\partial^2\Phi$ *via* dynamical systems, with the recent studies of R. T. Rockafellar [114], Mordukhovich-Outrata [102] and Kummer [87].

Clearly, a precise study of these quite involved questions is out of the scope of the present article. We just mention them in order to stress the importance and the versatility of the (DIN) system.

The paper is organized as follows. Section 5.2 gives the existence and the basic properties of the solution to (DIN). In section 5.3, we justify the terminology *Dynamical Inertial Newton* method by showing that (DIN) may be considered as a perturbation of the continuous Newton method. The next two sections deal with the asymptotic behaviour of the (DIN) trajectories : convergence to a critical point is proved for an analytic function Φ (section 5.4), and convergence to a minimizer is proved for a convex function (section 5.5). Section 5.6 presents a first-order in time and space system that is equivalent to (DIN). In section 5.7, constraints are introduced in that new system, which gives rise to a continuous gradient-projection system ; the trajectories are shown to be viable and to enjoy optimizing properties. Section 5.8 concludes the paper with an illustration in impact dynamics.

5.2 Global existence

Throughout this paper, H is a real Hilbert space with scalar product and norm denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ respectively. Let $\Phi : H \to \mathbb{R}$ be a mapping satisfying :

$$(\mathcal{H}) \begin{cases} \Phi \text{ is bounded from below on } H, \\ \Phi \text{ is twice continuously differentiable on } H, \\ \text{the Hessian } \nabla^2\Phi \text{ is Lipschitz continuous on the bounded subsets of } H. \end{cases}$$

Given two parameters $\alpha > 0$ and $\beta > 0$ consider the following second order in time system in H

$$\text{(DIN)} \qquad \ddot{x} + \alpha\dot{x} + \beta\nabla^2\Phi(x)\dot{x} + \nabla\Phi(x) = 0.$$

Along every trajectory of (DIN), and for $\lambda > 0$ define

$$E_\lambda(t) = \lambda\Phi(x(t)) + \frac{1}{2}|\dot{x}(t) + \beta\nabla\Phi(x(t))|^2. \tag{5.2.1}$$

In particular, we will write for short

$$E(t) = E_{\alpha\beta+1}(t) = (\alpha\beta + 1)\Phi(x(t)) + \frac{1}{2}|\dot{x}(t) + \beta\nabla\Phi(x(t))|^2. \tag{5.2.2}$$

Theorem 5.2.1 *Let Φ satisfy $(\mathcal{H})$. Then the following properties hold for* (DIN)*, provided $\alpha > 0$ and $\beta > 0$:*

- **(i)** *For each $(x_0, \dot{x}_0) \in H \times H$, there exists a unique global solution $x(t)$ of* (DIN) *satisfying the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, with $x \in \mathcal{C}^2([0, +\infty[; H)$.*
- **(ii)** *For every trajectory $x(t)$ of* (DIN) *and $\lambda \in [(1 - \sqrt{\alpha\beta})^2, (1 + \sqrt{\alpha\beta})^2]$, the scalar function E_λ defined by (5.2.1) is bounded from below and decreasing on $[0, +\infty[$, hence it converges as $t \to +\infty$. Moreover*
  - *$\dot{x}$ and $\nabla\Phi(x)$ belong to $L^2(0, +\infty; H)$,*
  - *$\lim_{t\to+\infty} \Phi(x(t))$ exists,*
  - *$\lim_{t\to+\infty}(\dot{x}(t) + \beta\nabla\Phi x(t)) = 0$.*
- **(iii)** *Assuming moreover that $x \in L^\infty(0, +\infty; H)$, we have*
  - *$\dot{x}$, $\ddot{x}$, $\nabla\Phi(x)$ and $\nabla^2\Phi(x)$ are bounded on $[0, +\infty[$,*
  - *$\lim_{t\to+\infty} \nabla\Phi(x(t)) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \ddot{x}(t) = 0$.*

Proof. **(i)** For any choice of initial conditions $(x_0, \dot{x}_0) \in H \times H$, the existence and uniqueness of a classic local solution to (DIN) follow from the Cauchy-Lipschitz Theorem applied to the equivalent first order in time system in the phase space $H \times H$, $\dot{Y} = F(Y)$, with

$$Y(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \quad \text{and} \quad F(u, v) = \begin{pmatrix} v \\ -\alpha v - \beta\nabla^2\Phi(u)v - \nabla\Phi(u) \end{pmatrix}.$$

Let x denote the maximal solution defined on some interval $[0, T_{max}[$ with $0 < T_{max} \le +\infty$. The regularity assumptions on Φ imply that $x \in \mathcal{C}^2([0, T_{max}[; H)$. Suppose, contrary to our claim, that $T_{max} < +\infty$. Differentiating $E(t)$ (see (5.2.2)) and using (DIN), we successively obtain

$$\begin{aligned} \dot{E}(t) &= (\alpha\beta + 1)\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle + \langle\ddot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t), \dot{x}(t) + \beta\nabla\Phi(x(t))\rangle \\ &= (\alpha\beta + 1)\langle\nabla\Phi(x(t)), \dot{x}(t)\rangle - \langle\alpha\dot{x}(t) + \nabla\Phi(x(t)), \dot{x}(t) + \beta\nabla\Phi(x(t))\rangle \\ &= -\alpha|\dot{x}(t)|^2 - \beta|\nabla\Phi(x(t))|^2. \end{aligned} \tag{5.2.3}$$

Hence $E(t)$ is a Lyapounov function for the trajectory x. Further, for all $t \in [0, T_{max}[$

$$(\alpha\beta + 1)\Phi(x(t)) + \frac{1}{2}|\dot{x}(t) + \beta\nabla\Phi(x(t))|^2 + \alpha\int_0^t |\dot{x}(\tau)|^2 d\tau + \beta\int_0^t |\nabla\Phi(x(\tau))|^2 d\tau = E(0). \tag{5.2.4}$$

Since Φ is bounded from below and $\alpha, \beta > 0$, we obtain that $\dot{x}$ and $\nabla\Phi(x)$ belong to $L^2(0, T_{max}; H)$. Therefore, for all $0 \le s \le t < T_{max}$

$$|x(t) - x(s)| \le \int_s^t |\dot{x}(\tau)| d\tau \le \sqrt{t-s} \sqrt{\int_s^t |\dot{x}(\tau)|^2 d\tau} \le \sqrt{t-s} \parallel \dot{x} \parallel_{L^2(0,T_{max};H)},$$

which shows that $\lim_{t\to T_{max}} x(t)$ exists. As a consequence, x is bounded on $[0, T_{max}[$ and so is $\nabla^2\Phi(x)$ in view of the Lipschitz continuity of $\nabla^2\Phi$. Thus $\ddot{x} = -\alpha\dot{x} + \beta\nabla^2\Phi(x)\dot{x} - \nabla\Phi(x)$ belongs to $L^2(0, T_{max}; H)$, and we have for all $0 \le s \le t < T_{max}$

$$|\dot{x}(t) - \dot{x}(s)| \le \int_s^t |\ddot{x}(\tau)| d\tau \le \sqrt{t-s} \parallel \ddot{x} \parallel_{L^2(0,T_{max};H)}$$

so that $\lim_{t\to T_{max}} \dot{x}(t)$ exists. Applying the Cauchy-Lipschitz local existence Theorem to (DIN) with initial data at T_{max} given by $(\lim_{t\to T_{max}} x(t), \lim_{t\to T_{max}} \dot{x}(t))$, we can extend the maximal solution to an interval strictly larger than $[0, T_{max}[$, which contradicts the maximality of the solution. Consequently, $T_{max} = +\infty$.

(ii) The point here is to realize that there is a whole family of Lyapounov functions for the trajectory x. Indeed, setting for short (recall (5.2.1))

$$E_\pm(t) = E_{1\pm\sqrt{\alpha\beta}} = (1 \pm \sqrt{\alpha\beta})^2\Phi(x(t)) + \frac{1}{2}|\dot{x}(t) + \beta\nabla\Phi(x(t))|^2,$$

we obtain

$$\dot{E}_\pm(t) = -|\sqrt{\alpha}\dot{x}(t) \mp \sqrt{\beta}\nabla\Phi(x(t))|^2.$$

Hence E_+ and E_- are two Lyapounov functions for x, as well as any convex combination of them. As a result, for any λ in $[(1-\sqrt{\alpha\beta})^2, (1+\sqrt{\alpha\beta})^2]$, E_λ is decreasing on $[0, +\infty[$, (*e.g.* $E = E_{\alpha\beta+1} = \frac{1}{2}(E^+ + E^-)$). Further we have

$$(1 \pm \sqrt{\alpha\beta})^2\Phi(x(t)) + \frac{1}{2}|\dot{x}(t) + \beta\nabla\Phi(x(t))|^2 - E_\pm(0) = -\int_0^t |\sqrt{\alpha}\dot{x}(\tau) \mp \sqrt{\beta}\nabla\Phi(x(\tau))|^2 d\tau.$$

Since Φ is bounded from below, we obtain that both $|\sqrt{\alpha}\dot{x} - \sqrt{\beta}\nabla\Phi(x)|$ and $|\sqrt{\alpha}\dot{x} + \sqrt{\beta}\nabla\Phi(x)|$ belong to $L^2(0, +\infty)$ and hence $\dot{x}$ and $\nabla\Phi(x)$ are in $L^2(0, +\infty; H)$. Now, since E_+ and E_- are decreasing and bounded from below, $\lim_{t\to+\infty} E_+(t)$ and $\lim_{t\to+\infty} E_-(t)$ exist. Therefore, $\Phi(x(t)) = \frac{1}{4\sqrt{\alpha\beta}}(E_+(t) - E_-(t))$ admits a limit as $t \to +\infty$. As a consequence, $|\dot{x}(t) + \beta\nabla\Phi(x(t))|$ has a limit as $t \to +\infty$, which is zero because $|\dot{x}(t) + \beta\nabla\Phi(x(t))| \in L^2(0, +\infty)$.

(iii) We now assume that x is in $L^\infty(0, +\infty; H)$. Then, by $(\mathcal{H})$, $\nabla^2\Phi(x)$ and $\nabla\Phi(x)$ are bounded on $[0, +\infty[$; and so are $\dot{x} = (\dot{x} + \beta\nabla\Phi(x)) - \beta\nabla\Phi(x)$ and $\ddot{x} = -\alpha\dot{x} + \beta\nabla^2\Phi(x)\dot{x} - \nabla\Phi(x)$. Set $h(t) = \frac{1}{2}|\nabla\Phi(x(t))|^2$ and note that $h \in L^1(0, +\infty)$ and $\dot{h} = \langle\nabla^2\Phi(x)\dot{x}, \nabla\Phi(x)\rangle \in L^\infty(0, +\infty)$; then, by a standard argument, $\lim_{t\to+\infty} h(t) = 0$. Likewise, if we set $k(t) = \frac{1}{2}|\dot{x}(t)|^2$ then $\lim_{t\to+\infty} k(t) = 0$. It follows that $\ddot{x}(t) \to 0$ as $t \to +\infty$. ■

Corollary 5.2.1 *Assume that* $\Phi : H \to \mathbb{R}$ *satisfies* $(\mathcal{H})$ *and is coercive, i.e.* $\lim_{|x|\to+\infty} \Phi(x) = +\infty$. *Then the solution* x *of* (DIN) *is in* $L^\infty(0,+\infty;H)$. *In particular, the properties in Theorem* 5.2.1(iii) *hold.*

Proof. It suffices to observe that (5.2.4) gives $(\alpha\beta+1)\Phi(x(t)) \le E(0)$. This estimate and the coerciveness of Φ imply that the trajectory x remains bounded. ■

5.3 (DIN) as a singular perturbation of Newton's method

In this section we assume that Φ belongs to $\mathcal{C}^2(H)$, with a Hessian Lipschitz continuous on bounded subsets, and that Φ is coercive with $\nabla\Phi$ strongly monotone on bounded subsets of H. More precisely, it is required that $\forall R > 0$, $\exists \beta_R > 0$ such that $\forall x, y \in H$

$$\max\{|x|, |y|\} < R \Rightarrow \langle \nabla\Phi(x) - \nabla\Phi(y), x - y\rangle \ge \beta_R |x-y|^2. \tag{5.3.5}$$

In particular, Φ is strictly convex and for all $x \in H$ the Hessian operator $\nabla^2\Phi(x)$ is positive definite. Indeed, (5.3.5) yields $\forall R > 0$, $\exists \beta_R > 0 : \forall x \in H$, if $|x| < R$ then $\forall h \in H$, $\langle \nabla^2\Phi(x)h, h\rangle \ge \beta_R|h|^2$. On the other hand, when $H = \mathbb{R}^n$ and $\nabla^2\Phi(x)$ is positive definite for every $x \in \mathbb{R}^n$, (5.3.5) holds with β_R being a positive lower bound for the eigenvalues of $\nabla^2\Phi(x)$ over the ball $B(0,R)$.

For simplicity, take $\alpha = 0$ and $\beta = 1$ and, for each $\varepsilon > 0$, consider a solution $x_\varepsilon \in C^2([0,\infty[;H)$ to the initial value problem (x_ε does exist, see [19])

$$(\varepsilon\text{-DIN}) \qquad \begin{cases} \varepsilon\ddot{x}_\varepsilon + \nabla^2\Phi(x_\varepsilon)\dot{x}_\varepsilon + \nabla\Phi(x_\varepsilon) = 0, & t > 0, \\ x_\varepsilon(0) = x_0, \dot{x}_\varepsilon(0) = \dot{x}_0, \end{cases}$$

where $x_0, \dot{x}_0 \in H$ are given. We are interested in the asymptotic behaviour of x_ε as $\varepsilon \to 0$. Observe that (ε-DIN) may be considered as a singular perturbation of the following evolution equation

$$(\text{CN}) \qquad \begin{cases} \nabla^2\Phi(x)\dot{x} + \nabla\Phi(x) = 0, & t > 0, \\ x(0) = x_0. \end{cases}$$

This is the *Continuous Newton* method for the minimization of Φ, which is a continuous version of the well-known Newton iteration

$$\nabla^2\Phi(x^k)(x^{k+1} - x^k) + \nabla\Phi(x^k) = 0.$$

The unique solution $x \in C^2([0,\infty[;H)$ of (CN) satisfies

$$\frac{d}{dt}[\nabla\Phi(x(t))] = -\nabla\Phi(x(t)),$$

which yields the following remarkable property of Newton's trajectories

$$\nabla\Phi(x(t)) = e^{-t}\nabla\Phi(x_0). \tag{5.3.6}$$

Moreover, since Φ is coercive, it follows from (5.3.5) and (5.3.6) that for an appropriate $\beta_R > 0$, $|x(t) - \widehat{x}| \le \frac{e^{-t}}{\beta_R}|\nabla\Phi(x_0)|$, where $\widehat{x}$ is the unique minimizer of Φ. We refer the reader to [8, 21] for fuller treatments of the continuous Newton method.

Proposition 5.3.1 *There exists a constant $C > 0$ such that $\forall t \geq 0$, $|x_\varepsilon(t) - x(t)| \leq C\sqrt{\varepsilon}$. Therefore, $x_\varepsilon \to x$ uniformly on $[0, +\infty[$.*

Proof. Let us introduce the ε-energy

$$U_\varepsilon(t) := \frac{\varepsilon}{2}|\dot{x}_\varepsilon(t)|^2 + \Phi(x_\varepsilon(t)),$$

which satisfies

$$\dot{U}_\varepsilon(t) = -\langle \nabla^2\Phi(x_\varepsilon(t))\dot{x}_\varepsilon(t), \dot{x}_\varepsilon(t)\rangle \leq 0.$$

Hence

$$U_\varepsilon(t) \leq U_\varepsilon(0) = \frac{\varepsilon}{2}|\dot{x}_0|^2 + \Phi(x_0), \tag{5.3.7}$$

and consequently

$$\sup_{0<\varepsilon\leq 1}\sup_{t\geq 0}\Phi(x_\varepsilon(t)) \leq \frac{1}{2}|\dot{x}_0|^2 + \Phi(x_0) =: \alpha.$$

Since Φ is coercive, the sublevel set $\Gamma_\alpha(\Phi) := \{x \in H : \Phi(x) \leq \alpha\}$ is bounded and then $\sup_{0<\varepsilon\leq 1}\sup_{t\geq 0}|x_\varepsilon(t)| < R$ for a suitable constant $R > 0$. Similarly, we obtain that the solution $x(t)$ of (CN) satisfies $\{x(t) : t \geq 0\} \subset \Gamma_{\Phi(x_0)}(\Phi) \subset \Gamma_\alpha(\Phi)$, so that we may assume that $\sup_{t\geq 0}|x(t)| < R$. By (5.3.5), we have

$$\forall t > 0,\ |x_\varepsilon(t) - x(t)| \leq \frac{1}{\beta_R}|\nabla\Phi(x_\varepsilon(t)) - \nabla\Phi(x(t))|. \tag{5.3.8}$$

Notice that the differential equation in (ε-DIN) may be rewritten

$$\frac{d}{dt}\left[\varepsilon\dot{x}_\varepsilon(t) + \nabla\Phi(x_\varepsilon(t))\right] + \nabla\Phi(x_\varepsilon(t)) = 0.$$

Setting $\omega_\varepsilon(t) := \varepsilon\dot{x}_\varepsilon(t) + \nabla\Phi(x_\varepsilon(t))$, we obtain the nonhomogeneous initial value problem

$$\begin{cases} \dot{\omega}_\varepsilon + \omega_\varepsilon = \varepsilon\dot{x}_\varepsilon(t), t > 0, \\ \omega_\varepsilon(0) = \varepsilon\dot{x}_0 + \nabla\Phi(x_0), \end{cases}$$

whose solution is given by

$$\omega_\varepsilon(t) = e^{-t}(\varepsilon\dot{x}_0 + \nabla\Phi(x_0)) + \varepsilon\int_0^t e^{-(t-\tau)}\dot{x}_\varepsilon(\tau)d\tau.$$

Thus

$$\nabla\Phi(x_\varepsilon(t)) = e^{-t}(\varepsilon\dot{x}_0 + \nabla\Phi(x_0)) - \varepsilon\dot{x}_\varepsilon(t) + \varepsilon\int_0^t e^{-(t-\tau)}\dot{x}_\varepsilon(\tau)d\tau.$$

By (5.3.6) together with (5.3.8), we have

$$|x_\varepsilon(t) - x(t)| \leq \frac{1}{\beta_R}\left(\varepsilon|\dot{x}_0| + \varepsilon|\dot{x}_\varepsilon(t)| + \int_0^t e^{-(t-\tau)}\varepsilon|\dot{x}_\varepsilon(\tau)|d\tau\right).$$

On the other hand, from the energy estimate (5.3.7), it follows that $\sup_{0<\varepsilon\leq 1}\sup_{t\geq 0}\varepsilon|\dot{x}_\varepsilon(t)| \leq \sqrt{2\varepsilon(\alpha - \inf\Phi)}$. Consequently,

$$|x_\varepsilon(t) - x(t)| \leq \frac{1}{\beta_R}\left(\varepsilon|\dot{x}_0| + 2\sqrt{2\varepsilon(\alpha - \inf\Phi)}\right) \leq \frac{\sqrt{\varepsilon}}{\beta_R}\left(|\dot{x}_0| + 2\sqrt{2(\alpha - \inf\Phi)}\right),$$

which completes the proof. ■

5.4 Convergence of the trajectories : Φ analytic

Since $\lim_{t\to+\infty} \nabla\Phi(x(t)) = 0$, it is natural to expect that for a sufficiently smooth Φ, trajectories will converge towards a critical point of that function. Actually we show, in the finite-dimensional case, that if Φ is real analytic, x will finally converge to $x_\infty \in H$, with $\nabla\Phi(x_\infty) = 0$. The proof of this convergence result relies on an inequality due to Lojasiewicz [94], linking Φ and $\nabla\Phi$ in a neighbourhood of critical points. Lojasiewicz applied it in [93] to study the asymptotic behaviour of a gradient-like system. More recently, Haraux and Jendoubi [72] showed that bounded trajectories of HBF with an analytic potential converge towards critical points. This analyticity hypothesis is also useful for infinite dimensional systems with analytic nonlinearities, see Simon's work [116] for the heat equation and Haraux [70] and Jendoubi [80] for the damped wave equation.

Let us recall the definition of a real analytic function.

Definition 5.4.1 *Let Ω be an open subset of $\mathbb{R}^N$. A function $\Phi : \Omega \mapsto \mathbb{R}$ is real analytic (in Ω), if for every point $\xi = (\xi_1, \dots, \xi_N)$ in Ω there exist a neighbourhood $U \subseteq \Omega$ of ξ and real coefficients $(c_{\nu_1,\dots,\nu_N})_{(\nu_1,\dots,\nu_N)\in\mathbf{N}^N}$ such that*

$$x = (x_1, \dots, x_N) \in U \quad \Rightarrow \quad \Phi(x) = \sum_{(\nu_1,\dots,\nu_N)\in\mathbf{N}^N} c_{\nu_1,\dots,\nu_N}(x_1 - \xi_1)^{\nu_1} \dots (x_N - \xi_N)^{\nu_N}.$$

Lemma 5.4.1 *(Lojasiewicz) Let $\Phi : \mathbb{R}^N \to \mathbb{R}$ be a function which is supposed to be analytic in a neighbourhood of a critical point a. Then, there exist $\sigma > 0$ and $\theta \in]0, \frac{1}{2}[$ such that* [4]

$$|x - a| < \sigma \Rightarrow |\Phi(x) - \Phi(a)|^{1-\theta} \leq |\nabla\Phi(x)|.$$

The next corollary extends the Lemma to a compact connected set of critical points.

Corollary 5.4.1 *Let $\Phi : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}$ be a function which is supposed to be analytic in the open set Ω. Let A be a nonempty subset of Ω such that $\nabla\Phi(a) = 0$ for all a in A*

1. *If A is connected then Φ assumes a constant value on A, say Φ_A.*
2. *If A is connected and compact, then there exist $\sigma > 0$ and $\theta \in]0, \frac{1}{2}[$ such that*

$$dist(x, A) < \sigma \Rightarrow |\Phi(x) - \Phi_A|^{1-\theta} \leq |\nabla\Phi(x)|.$$

Proof. 1. Pick some a in A. After the Lemma there exist $\sigma > 0$ and $\theta \in]0, \frac{1}{2}[$ such that

$$|x - a| < \sigma \Rightarrow |\Phi(x) - \Phi(a)|^{1-\theta} \leq |\nabla\Phi(x)|.$$

Hence if x belongs to $A \cap B(a, \sigma)$ where $B(a, \sigma)$ is the open ball with center a and radius σ, then $|\Phi(x) - \Phi(a)| = 0$. As a consequence the set $\{x \in A/\Phi(x) = \Phi(a)\}$ is open in A; as it is obviously closed in A and non-void it is equal to A.

2. Without restriction we may assume that Φ vanishes on A. According to Lojasiewicz's Lemma and owing to the compactness of A, there exists a finite family $(a_i, \sigma_i, \theta_i)_{i\in\{1,\dots,n\}}$ with $a_i \in A$, $\sigma_i > 0$, $\theta_i \in]0, \frac{1}{2}[$ such that

[4] Originally ([94, p. 92]), the Lemma states that θ lies in $]0, 1[$; but it is harmless to suppose that σ satisfies $|x - a| < \sigma \Rightarrow |\Phi(x) - \Phi(a)| \leq 1$, which, together with $0 < \theta < 1$, entails $|\Phi(x) - \Phi(a)|^{1-\theta/2} \leq |\Phi(x) - \Phi(a)|^{1-\theta}$; this justifies the assertion $\theta \in]0, \frac{1}{2}[$.

- the balls $B(a_i, \sigma_i)$, build a finite open cover of A,
- $x \in \Omega, |x - a_i| < \sigma_i \Rightarrow |\Phi(x)|^{1-\theta_i} \leq |\nabla\Phi(x)|$.

Resorting once more to the compactness of A, and to the continuity of Φ, we assert the existence of some $\sigma > 0$ such that

$$\operatorname{dist}(x, A) < \sigma \Rightarrow x \in \Omega, x \in \bigcup_{i=1}^{n} B(a_i, \sigma_i), |\Phi(x)| \leq 1.$$

If we set $\theta = \min \theta_i$, then any x complying with $\operatorname{dist}(x, A) < \sigma$ verifies $x \in \Omega$ and $x \in B(a_i, \sigma_i)$ for some $i \in \{1, \dots, n\}$; hence $|\Phi(x)|^{1-\theta} \leq |\Phi(x)|^{1-\theta_i} \leq |\nabla\Phi(x)|$. ■

Theorem 5.4.1 *Let x be a bounded solution of* (DIN) *and assume that $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is analytic. Then $\dot{x}$ belongs to $L^1(0, +\infty; H)$ and $x(t)$ converges towards a critical point of Φ as $t \to \infty$.*

Proof. Let $\omega(x)$ denote the ω-limit set of x. Classically ([71] *e. g.*), $\omega(x)$ is a compact connected set which consists of critical points of Φ. Moreover from Theorem 5.2.1**(ii)**, Φ assumes a constant value on $\omega(x)$, which we may suppose to be 0. Further, $\operatorname{dist}(x(t), \omega(x)) \to 0$ as $t \to \infty$.

After the corollary 5.4.1, there exist some $T > 0$ and some $\theta \in]0, \frac{1}{2}[$ such that

$$t \geq T \Rightarrow |\Phi(x(t))|^{1-\theta} \leq |\nabla\Phi(x(t))|. \tag{5.4.9}$$

The proof of the convergence of x relies on the equality

$$-\frac{d}{dt} E(t)^{\theta} = -\dot{E}(t) E(t)^{\theta-1}$$

and on lower bounds for $-\dot{E}(t)$ and $E(t)^{\theta-1}$ involving $|\dot{x}(t)|$; recall that the energy E is defined by (5.2.2).

First, we have (recall (5.2.3))

$$-\dot{E}(t) \geq \frac{1}{2} \min(\alpha, \beta)\{|\dot{x}(t)| + |\nabla\Phi(x(t))|\}^2. \tag{5.4.10}$$

Further, for $C = \max(\alpha\beta + 1, \beta^2)$, we have (recall (5.2.2))

$$E(t) \leq C\{|\Phi(x(t))| + |\dot{x}(t)|^2 + |\nabla\Phi(x(t))|^2\}.$$

Hence (using the inequality $(r+s)^{1-\theta} \leq r^{1-\theta} + s^{1-\theta}$)

$$E(t)^{1-\theta} \leq C^{1-\theta}\{|\Phi(x(t))|^{1-\theta} + |\dot{x}(t)|^{2(1-\theta)} + |\nabla\Phi(x(t))|^{2(1-\theta)}\}.$$

Using (5.4.9), we have for $t \geq T$

$$E(t)^{1-\theta} \leq C^{1-\theta}\{|\nabla\Phi(x(t))| + |\dot{x}(t)|^{2(1-\theta)} + |\nabla\Phi(x(t))|^{2(1-\theta)}\}.$$

Since $|\nabla\Phi(x(t))|$ and $|\dot{x}(t)|$ tend to zero as $t \to \infty$ and since $2(1-\theta) > 1$, the quantities $|\nabla\Phi(x(t))|^{2(1-\theta)}$ and $|\dot{x}(t)|^{2(1-\theta)}$ are negligible with respect to $|\nabla\Phi(x(t))|$ and $|\dot{x}(t)|$. Therefore, there is some constant $D > 0$ such that for $t \geq T$

$$E(t)^{1-\theta} \leq D\{|\nabla\Phi(x(t))| + |\dot{x}(t)|\}. \tag{5.4.11}$$

If $|\nabla\Phi(x(t))| + |\dot{x}(t)|$ happens to vanish at some time $t_1 \geq T$, then owing to the unicity of the solution to (DIN), $x(t)$ is equal to $x(t_1)$ for $t \geq t_1$, and the Theorem is proved.

Else from (5.4.10) and (5.4.11) we obtain for $t \geq T$

$$-\frac{d}{dt}E(t)^{\theta} \geq \frac{1}{2D}\min(\alpha,\beta)\{|\nabla\Phi(x(t))| + |\dot{x}(t)|\}.$$

Since $\lim_{t\to\infty} E(t)$ exists, $|\dot{x}|$ belongs to $L^1([0,+\infty[)$ and consequently $\lim_{t\to\infty} x(t)$ exists. ■

5.5 Convergence of the trajectories : Φ convex

5.5.1 Weak convergence in the general convex case.

The proof of the asymptotic convergence in the convex case relies on the following Lemma, which is essentially due to Opial [105].

Lemma 5.5.1 *(Opial) Let H be a Hilbert space and $x : [0,+\infty[\mapsto H$ a function such that there exists a nonempty set $S \subseteq H$ verifying :*

(a) if $x(t_n) \rightharpoonup \bar{x}$ weakly in H for some $t_n \to +\infty$ then $\bar{x} \in S$,

(b) $\forall z \in S$, $\lim_{t\to+\infty} |x(t) - z|$ exists.

Then, $x(t)$ weakly converges as $t \to +\infty$ to an element of S.

Theorem 5.5.1 *Let Φ be a convex function satisfying $(\mathcal{H})$ and assume that argmin $\Phi \neq \emptyset$. Let x be a solution of* (DIN). *Then for all $z \in$ argmin Φ, $\lim_{t\to+\infty} |x(t) - z|$ exists, and $x(t)$ weakly converges to a minimum point of Φ as $t \to +\infty$.*

Proof. Write $S = \operatorname{argmin} \Phi$ and pick some z in S. In order to prove the existence of $\lim_{t\to+\infty} |x(t) - z|$, we introduce an auxiliary energy

$$E_\varepsilon(t) = E(t) + \varepsilon(\frac{\alpha}{2} \mid x(t) - z \mid^2 + \langle \dot{x}(t) + \beta\nabla\Phi(x(t)), x(t) - z\rangle) \tag{5.5.12}$$

where E is the energy defined by (5.2.2) and ε is a positive parameter. Let us show that, by choosing ε small enough, E_ε is a Lyapounov function for (DIN). Thanks to (DIN) and (5.2.3), we have

$$\dot{E}_\varepsilon(t) = -(\alpha - \varepsilon) \mid \dot{x}(t) \mid^2 - \beta \mid \nabla\Phi(x(t)) \mid^2 - \varepsilon\langle \nabla\Phi(x(t)), x(t) - z\rangle + \varepsilon\langle \beta\nabla\Phi(x(t)), \dot{x}(t)\rangle.$$

Using the Young inequality for the last term, we obtain

$$\dot{E}_\varepsilon(t) \leq -(\alpha - \frac{3\varepsilon}{2}) \mid \dot{x}(t) \mid^2 - \beta(1 - \frac{\varepsilon\beta}{2}) \mid \nabla\Phi(x(t)) \mid^2 - \varepsilon\langle \nabla\Phi(x(t)), x(t) - z\rangle. \tag{5.5.13}$$

Take ε so small that each term in the previous expression is nonpositive (for the last term, use the fact that $\nabla\Phi$ is monotone and $z \in S$) ; then E_ε is nonincreasing and we readily obtain

$$\langle \dot{x}(t) + \beta\nabla\Phi(x(t)), x(t) - z\rangle + \frac{\alpha}{2} \mid x(t) - z \mid^2 \leq \frac{1}{\varepsilon}(E_\varepsilon(0) - E(t)).$$

Since $E(t)$ is bounded from below, because so is Φ, there exists some constant M such that

$$\langle \dot{x}(t) + \beta \nabla \Phi(x(t)), x(t) - z \rangle + \frac{\alpha}{2} \mid x(t) - z \mid^2 \leq M.$$

As $\dot{x} + \beta \nabla \Phi(x)$ is bounded by Theorem 5.2.1(ii), $\mid x(t) - z \mid$ is bounded. Hence $E_\varepsilon(t)$, which is bounded from below and decreasing, admits a limit as $t \to +\infty$. Moreover, Theorem 5.2.1(ii-iii) asserts the following : $\lim_{t \to +\infty} E(t)$ exists and $\lim_{t \to +\infty} \dot{x}(t) = \lim_{t \to +\infty} \nabla \Phi(x(t) = 0$; hence, after (5.5.12), $\lim_{t \to +\infty} |x(t) - z|$ exists.

In order to apply the Opial Lemma we need to prove that the weak cluster points of the trajectory x are in S. Let $\bar{x} \in H$ and $t_n \to +\infty$ be such that $x(t_n) \rightharpoonup \bar{x}$. Thanks to the convexity inequality, we have for any $z \in S$

$$\Phi(z) = \min \Phi \geq \Phi(x(t_n)) + \langle \nabla \Phi(x(t_n)), z - x(t_n) \rangle.$$

Since $\nabla \Phi(x(t_n)) \to 0$ and Φ is lower semicontinuous, we obtain

$$\min \Phi \geq \liminf_{n \to +\infty} \Phi(x(t_n)) \geq \Phi(\bar{x}),$$

which means that $\bar{x} \in S$. The Opial Lemma then applies, ensuring the weak convergence of x, and we also deduce that $\Phi(x(t)) \to \min \Phi$ as $t \to \infty$. ■

5.5.2 Strong convergence under int(argmin Φ) $\neq \emptyset$.

A counterexample due to Baillon [24] for the steepest descent equation $\dot{x} + \nabla \Phi(x) = 0$ suggests that, likely, convexity alone is not sufficient for the trajectories of (DIN) to converge strongly in H. Nevertheless, a result of Brézis [37, Theorem 3.13] shows that the steepest descent trajectories do strongly converge under the additional hypothesis int(argmin Φ) $\neq \emptyset$. This property also holds for (DIN) trajectories.

Proposition 5.5.1 *Under the hypotheses of Theorem* 5.5.1, *if moreover int(argmin* Φ) $\neq \emptyset$ *then every trajectory of* (DIN) *converges to a minimizer of* Φ *with respect to the strong topology of* H.

Proof. Fix $z \in$ int(argmin Φ) so that there exists $\rho > 0$ such that for every $z' \in H$ with $|z' - z| < \rho$ then $z' \in$ int(argmin Φ) and consequently $\nabla \Phi(z') = 0$. By monotonicity of $\nabla \Phi$, we have

$$\langle \nabla \Phi(y), y - z \rangle \ \geq \ \langle \nabla \Phi(y), z' - z \rangle$$

for all $y \in H$ and $z' \in H$ with $\nabla \Phi(z') = 0$. Thus, for every $y \in H$

$$\langle \nabla \Phi(y), y - z \rangle \ \geq \ \rho |\nabla \Phi(y)|.$$

Specialize y to $x(t)$ to obtain for all $t \geq 0$ and all $z \in$ int(argmin Φ)

$$\langle \nabla \Phi(x(t)), x(t) - z \rangle \ \geq \ \rho |\nabla \Phi(x(t))|. \tag{5.5.14}$$

Now, for $\varepsilon > 0$ small enough, the inequality (5.5.13) may be simplified to

$$0 \ \leq \ \varepsilon \langle \nabla \Phi(x(t)), x(t) - z \rangle \ \leq \ -\dot{E}_\varepsilon(t);$$

integrating the latter yields

$$0 \leq \varepsilon \int_0^t \langle \nabla\Phi(x(s)), x(s) - z\rangle ds \leq E_\varepsilon(0) - E_\varepsilon(t).$$

Since $\lim_{t\to+\infty} E_\varepsilon(t)$ exists, after the proof of Theorem 5.5.1, we deduce that $\langle \nabla\Phi(x), x-z\rangle$ belongs to $L^1(0,+\infty)$, and so does $|\nabla\Phi(x)|$ in view of (5.5.14). If we now integrate (DIN)

$$\dot{x}(t) + \alpha x(t) + \beta\nabla\Phi(x(t)) + \int_0^t \nabla\Phi(x(s))ds = \dot{x}_0 + \alpha x_0 + \beta\nabla\Phi(x_0),$$

we see that $\lim_{t\to+\infty} x(t)$ exists in H, since $\lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} \nabla\Phi(x(t)) = 0$, after Theorem 5.2.1(iii). ■

5.5.3 Strong convergence under the symmetry property $\Phi(y) = \Phi(-y)$.

Bruck [40] has shown that the convexity of Φ together with the symmetry assumption $\Phi(y) = \Phi(-y)$ entails the strong convergence of the steepest descent trajectories. This result has been extended by Alvarez [4] to (HBF) trajectories and we extend it now to (DIN) trajectories.

Proposition 5.5.2 *Under the hypotheses of Theorem* 5.5.1*, if moreover* Φ *is supposed to be even, i.e.* $\forall y \in H, \Phi(y) = \Phi(-y)$*, then every trajectory of* (DIN) *converges to a minimizer of* Φ *with respect to the strong topology of* H*.*

Proof. Let us successively consider the case $\alpha\beta \leq 1$ and the case $\alpha\beta > 1$.

1. Case $\alpha\beta \leq 1$. Fix $t_0 > 0$ and define $g_{t_0} : [0, t_0] \mapsto \mathbb{R}$ by

$$g_{t_0}(t) = |x(t)|^2 - |x(t_0)|^2 - \frac{1}{2}|x(t) - x(t_0)|^2.$$

We have $\dot{g}_{t_0}(t) = \langle \dot{x}(t), x(t) + x(t_0)\rangle$ and $\ddot{g}_{t_0}(t) = \langle \ddot{x}(t), x(t) + x(t_0)\rangle + |\dot{x}(t)|^2$. From this we obtain

$$\begin{aligned}
\ddot{g}_{t_0}(t) + \alpha\dot{g}_{t_0}(t) &= \langle -\beta\nabla^2\Phi(x(t))\dot{x}(t) - \nabla\Phi(x(t)), x(t) + x(t_0)\rangle + |\dot{x}(t)|^2 \\
&= \tfrac{d}{dt}\langle -\beta\nabla\Phi(x(t)), x(t) + x(t_0)\rangle + \langle \beta\nabla\Phi(x(t)), \dot{x}(t)\rangle \\
&\quad + \tfrac{1}{\beta}\langle -\beta\nabla\Phi(x(t)), x(t) + x(t_0)\rangle + |\dot{x}(t)|^2 \\
&= e^{-\frac{1}{\beta}t}\tfrac{d}{dt}e^{\frac{1}{\beta}t}\langle -\beta\nabla\Phi(x(t)), x(t) + x(t_0)\rangle + \langle \dot{x}(t) + \beta\nabla\Phi(x(t)), \dot{x}(t)\rangle.
\end{aligned}$$

Set $f(t) = \langle \dot{x}(t) + \beta\nabla\Phi(x(t)), \dot{x}(t)\rangle$. Since $\dot{x}$ and $\nabla\Phi(x)$ are in $L^2(0,+\infty; H)$, f belongs to $L^1(0,+\infty)$. We have

$$\frac{d}{dt}[e^{\alpha t}\dot{g}_{t_0}(t)] = e^{(\alpha-1/\beta)t}\frac{d}{dt}e^{\frac{1}{\beta}t}\langle -\beta\nabla\Phi(x(t)), x(t) + x(t_0)\rangle + e^{\alpha t}f(t)$$

and so, for every $t \in]0, t_0]$

$$e^{\alpha t}\dot{g}_{t_0}(t) - \dot{g}_{t_0}(0) = \int_0^t e^{(\alpha-1/\beta)\tau}\frac{d}{ds}[\beta e^{s/\beta}\omega_{t_0}(s)]_{s=\tau}d\tau + \int_0^t e^{\alpha\tau}f(\tau)d\tau,$$

with $\omega_{t_0}(s) = \langle -\nabla\Phi(x(s)), x(s) + x(t_0)\rangle$. An integration by parts yields

$$\int_0^t e^{(\alpha-1/\beta)\tau}\frac{d}{ds}[\beta e^{s/\beta}\omega_{t_0}(s)]_{s=\tau}d\tau = \beta e^{\alpha t}\omega_{t_0}(t) - \beta\omega_{t_0}(0) + (1-\alpha\beta)\int_0^t e^{\alpha\tau}\omega_{t_0}(\tau)d\tau.$$

We conclude that

$$\dot g_{t_0}(t) = \langle \dot x_0 + \beta\nabla\Phi(x_0), x_0 + x(t_0)\rangle e^{-\alpha t} + \beta\omega_{t_0}(t) + \int_0^t e^{-\alpha(t-\tau)}[(1-\alpha\beta)\omega_{t_0}(\tau) + f(\tau)]d\tau.$$

Set $F(t) = \frac{1}{2}|\dot x(t)|^2 + \Phi(x(t))$, which is nonincreasing because Φ is convex (in fact, $\dot F(t) = -\alpha|\dot x(t)|^2 - \beta\langle\nabla^2\Phi(x(t))\dot x(t), \dot x(t)\rangle \le 0$). Then, for all $t \in [0, t_0]$

$$F(t) \ge F(t_0) = \frac{1}{2}|\dot x(t_0)|^2 + \Phi(x(t_0)) = \frac{1}{2}|\dot x(t_0)|^2 + \Phi(-x(t_0)).$$

By convexity of Φ

$$\Phi(-x(t_0)) \ge \Phi(x(t)) + \langle\nabla\Phi(x(t)), -x(t_0) - x(t)\rangle$$

and consequently

$$\omega_{t_0}(t) = \langle -\nabla\Phi(x(t)), x(t) + x(t_0)\rangle \le \frac{1}{2}|\dot x(t)|^2.$$

Therefore

$$\dot g_{t_0}(t) \le \langle \dot x_0 + \beta\nabla\Phi(x_0), x_0 + x(t_0)\rangle e^{-\alpha t} + \frac{\beta}{2}|\dot x(t)|^2 + \int_0^t e^{-\alpha(t-\tau)}h(\tau)d\tau,$$

where $h(t) = \frac{1-\alpha\beta}{2}|\dot x(t)|^2 + |f(t)| \in L^1(0,\infty)$. Hence, for all $t \in [0, t_0]$

$$g_{t_0}(t_0) - g_{t_0}(t) \le \frac{1}{\alpha}\langle \dot x_0 + \beta\nabla\Phi(x_0), x_0 + x(t_0)\rangle(e^{-\alpha t} - e^{-\alpha t_0}) + \frac{\beta}{2}\int_t^{t_0}|\dot x(\tau)|^2 d\tau + \int_t^{t_0}\int_0^\theta e^{-\alpha(\theta-\tau)}h(\tau)d\tau d\theta$$

which gives

$$\frac{1}{2}|x(t_0) - x(t)|^2 \le |x(t)|^2 - |x(t_0)|^2 + \frac{1}{\alpha}\langle \dot x_0 + \beta\nabla\Phi(x_0), x_0 + x(t_0)\rangle(e^{-\alpha t} - e^{-\alpha t_0}) + \int_t^{t_0} p(\theta)d\theta,$$

where $p \in L^1(0,\infty)$. We know that $x(t) \rightharpoonup x_\infty$ as $t \to \infty$ where $x_\infty \in$ argminΦ. Moreover, for all $z \in$ argminΦ there exists some $l_z \in \mathbb{R}$ such that $|x(t) - z|^2 \to l_z$, as $t \to \infty$ (see Theorem 5.5.1). Since Φ is even, 0 is a minimizer of Φ so that there is some $l_0 \in \mathbb{R}$ such that $\lim_{t\to\infty}|x(t)|^2 = l_0$. From the inequality above it follows that $\{x(t) : t \to \infty\}$ is a Cauchy net in H, hence $x(t) \to x_\infty$ strongly in H.

2. Case $\alpha\beta > 1$. The conclusion follows in this case from a well-known result of Bruck [40] applied to an equivalent gradient-type first order system defined on $H \times H$ (see section 5.6.3). ■

Remark. If $\Phi(x) = \frac{1}{2}\langle Ax, x\rangle$ where $A : H \mapsto H$ is a positive self-adjoint and bounded linear operator, then argmin$\Phi =$ Ker$A = \{z \in H : Az = 0\}$ and $x(t)$ strongly converges in H to the projection of $x_0 + \frac{1}{\alpha}\dot{x}_0$ on Ker$A$. Indeed, for every $z \in$ Ker$A$ and $t > 0$ we have

$$\begin{aligned}\langle \dot{x}(t) + \alpha x(t) - \dot{x}_0 - \alpha x_0, z\rangle &= \int_0^t \langle -\beta\nabla^2\Phi(x(\tau))\dot{x}(\tau) - \nabla\Phi(x(\tau)), z\rangle d\tau \\ &= \int_0^t \langle -\beta A\dot{x}(\tau) - Ax(\tau), z\rangle d\tau \\ &= \int_0^t \langle -\beta\dot{x}(\tau) - x(\tau), Az\rangle d\tau = 0.\end{aligned}$$

Since $\dot{x}(t) \to 0$ and $x(t) \to x_\infty \in$ Ker$A$ strongly, we deduce that $\langle x_\infty - x_0 - \frac{1}{\alpha}\dot{x}_0, z\rangle = 0$ for all $z \in$ Ker$A$, which proves our claim.

5.6 (DIN) as a first order in time gradient-like system

This part is devoted to establishing two remarkable properties of (DIN) :

- actually (DIN) proves to be equivalent to a system of first order in time with no occurrence of the Hessian of Φ,
- further, if the positive parameters α and β satisfy $\alpha\beta > 1$, then (DIN) is a gradient system.

5.6.1 (DIN) as a system of first order in time and with no occurrence of the Hessian of Φ

In this section, the requirements on the constants α, β and on the function Φ in (DIN) may be relaxed to $\beta \neq 0$ and $\Phi \in \mathcal{C}^2(H)$ only.

Let x be a solution of (DIN), and define the function y by

$$\dot{x} + \beta\nabla\Phi(x) + (\alpha - \frac{1}{\beta})x + \frac{1}{\beta}y = 0. \tag{5.6.15}$$

Differentiate (5.6.15) to obtain

$$\beta[\ddot{x} + \beta\nabla^2\Phi(x)\dot{x} + (\alpha - \frac{1}{\beta})\dot{x}] + \dot{y} = 0,$$

which, in view of (DIN), yields

$$\beta[-\nabla\Phi(x) - \frac{1}{\beta}\dot{x}] + \dot{y} = 0. \tag{5.6.16}$$

Adding (5.6.15) and (5.6.16) gives

$$(\alpha - \frac{1}{\beta})x + \dot{y} + \frac{1}{\beta}y = 0. \tag{5.6.17}$$

Collecting (5.6.15) and (5.6.17) gives the first-order system

$$\begin{cases} \dot{x} + \beta\nabla\Phi(x) & +(\alpha - \frac{1}{\beta})x + \frac{1}{\beta}y = 0 \\ \dot{y} & +(\alpha - \frac{1}{\beta})x + \frac{1}{\beta}y = 0 \end{cases} \tag{5.6.18}$$

Conversely, let (x, y) be a solution of (5.6.18). Combining the two lines of (5.6.18) yields

$$\dot{y} = \dot{x} + \beta\nabla\Phi(x)$$

while differentiating the first equation yields

$$\ddot{x} + \beta\nabla^2\Phi(x)\dot{x} + (\alpha - \frac{1}{\beta})\dot{x} + \frac{1}{\beta}\dot{y} = 0.$$

Substituting the value of $\dot{y}$ in the above equation gives (DIN) again. Thus (DIN) is equivalent to (5.6.18).

It is natural now to introduce the following first-order sytem (where g stands for *generalized*)

$$\text{(g-DIN)} \quad \begin{cases} \dot{x} + \beta\nabla\Phi(x) & +ax + by = 0 \\ \dot{y} & +ax + by = 0 \end{cases}$$

which is a slight generalization of (5.6.18) ; indeed (g-DIN) is (5.6.18) if we set

$$a = \alpha - \frac{1}{\beta}, \quad b = \frac{1}{\beta} \tag{5.6.19}$$

The following Theorem summarizes the above computation, and emphasizes the equivalence of (DIN), which is of second order in time and involves the Hessian of Φ, with a system which is of first order in time and with no occurrence of the Hessian.

Theorem 5.6.1 *Suppose $\Phi \in \mathcal{C}^2(H)$, and let the constants α, β, a, b satisfy $\beta \neq 0$ and (5.6.19). The systems (DIN) and (g-DIN) are equivalent in the sense that x is a solution of (DIN) if and only if there exists $y \in \mathcal{C}^2([0, +\infty[, H)$ such that (x, y) is a solution of (g-DIN).*

5.6.2 Existence and asymptotic behaviour of the solutions of (g-DIN)

Beyond being of first order in time, the system (g-DIN) is interesting because it does not involve the Hessian of Φ. As a first consequence, the numerical solution of (DIN) is highly simplified, since it may be performed on (g-DIN) and only requires approximating the gradient of Φ. As a second consequence, (g-DIN) allows to give a sense to (DIN) when Φ is of class $\mathcal{C}^1$ only, or when Φ is nonsmooth or involves constraints, provided that a notion of generalized gradient is available (*e.g.* the subdifferential set for a convex function Φ). But that remark would be of little utility if (g-DIN) did not have good existence and convergence properties under the sole assumption $\Phi \in \mathcal{C}^1(H)$; recall that (DIN), as studied in the previous sections, requires $\Phi \in \mathcal{C}^2(H)$. Actually (g-DIN) enjoys the same properties as (DIN) does, at least if $\Phi \in \mathcal{C}^{1,1}(H)$, and Theorems similar to Theorems 5.2.1 and 5.5.1 can be stated about (g-DIN).

Theorem 5.6.2 *Assume that $\Phi : H \mapsto \mathbb{R}$ is bounded from below, differentiable with $\nabla\Phi$ Lipschitz continuous on the bounded subsets of H ; assume further $\beta > 0$, $b > 0$, $b + a > 0$ in* (g-DIN). *Then the following properties hold :*

(i) *For each (x_0, y_0) in $H \times H$, there exists a unique solution (x, y) of* (g-DIN) *defined on the whole interval $[0, +\infty[$, which belongs to $\mathcal{C}^1(0, \infty; H) \times \mathcal{C}^2(0, \infty; H)$ and satisfies the initial conditions $x(0) = x_0$ and $y(0) = y_0$.*

(ii) *For any $\lambda \in [\beta(\sqrt{a+b} - \sqrt{b})^2, \beta(\sqrt{a+b} + \sqrt{b})^2]$ the function $F_\lambda : (x, y) \in H \times H \mapsto \lambda\Phi(x) + \frac{1}{2}|ax + by|^2$ is a Lyapounov function of (g-DIN); for every solution (x, y) the energy $F_\lambda(x(t), y(t))$ is decreasing on $[0, +\infty[$, bounded from below and hence it converges to some real value as $t \to +\infty$. Moreover*

- *$\dot{x}$ and $\nabla\Phi(x)$ belong to $L^2(0, +\infty; H)$,*
- *$\lim_{t\to+\infty} \Phi(x(t))$ exists,*
- *$\lim_{t\to+\infty}(\dot{x}(t) + \beta\nabla\Phi x(t)) = 0$.*

(iii) *Assuming moreover that x is in $L^\infty(0, +\infty; H)$, then we have*

- *$\dot{x}$, $\nabla\Phi(x)$ are bounded on $[0, +\infty[$,*
- *$\lim_{t\to+\infty} \nabla\Phi(x(t)) = \lim_{t\to+\infty} \dot{x}(t) = 0$.*

Theorem 5.6.3 *In addition to the hypotheses of Theorem* 5.6.2, *assume that Φ is convex, and that argmin Φ, the set of minimizers of Φ on H, is nonempty. Then for any solution $(x, y))$ of* (g-DIN), *$x(t)$ weakly converges to a minimizer of Φ on H as t goes to infinity.*

The proof follows the lines of Theorems 5.2.1 and 5.5.1 and will not be given. Besides, a more general situation will be examinated in section 5.7 (cf. Theorems 5.7.1 and 5.7.2).

Theorem 5.2.1 is a mere corollary of Theorems 5.6.1 and 5.6.2. Indeed suppose that Φ and α, β meet the assumptions of Theorem 5.2.1 : Φ satisfies $(\mathcal{H})$ and $\alpha > 0$, $\beta > 0$. Then $\nabla\Phi$ is Lipschitz continuous on the bounded subsets of H, and the constants $a = \alpha - 1/\beta$ and $b = 1/\beta$ satisfy $a + b > 0$, $b > 0$. So the assumptions of Theorem 5.6.2 are met ; in view of the equivalence between (DIN) and (g-DIN) given by Theorem 5.6.1, the conclusions of Theorem 5.6.2 apply to (DIN).

Further if $\Phi \in \mathcal{C}^2(H)$ meets the assumptions of Theorem 5.6.2, the system (DIN) makes sense but Theorem 5.2.1 does not apply since $\nabla^2\Phi$ need not be Lipschitz continuous. Yet we can resort to Theorems 5.6.1 and 5.6.2 to assert the existence of a solution to (DIN) enjoying the properties stated in Theorem 5.6.2. Consequently the assumptions of Theorem 5.2.1 may be weakened, while its conclusions remain valid, as far as $\ddot{x}$ and $\nabla^2\Phi$ are not concerned.

Likewise Theorem 5.5.1 is a corollary of Theorems 5.6.1 and 5.6.3 and its hypotheses may be weakened.

5.6.3 (DIN) as a gradient system if $\alpha\beta > 1$

Suppose $\Phi \in \mathcal{C}^1(H)$ and $a > 0$, $b > 0$ in (g-DIN). Rescaling the variable y by $y = \sqrt{\frac{a}{b}}z$ transforms (g-DIN) into the equivalent system

$$\begin{cases} \dot{x} + \beta\nabla\Phi(x) & +ax + \sqrt{ab}z = 0 \\ \dot{z} & +\sqrt{ab}x + bz = 0 \end{cases} \tag{5.6.20}$$

We note that (5.6.20) is exactly the gradient system

$$\dot{X} + \nabla \mathcal{E}(X) = 0 \tag{5.6.21}$$

where $X = (x, z)$ and $\mathcal{E} : H \times H \mapsto \mathbb{R}$ is defined by

$$\mathcal{E}(X) = \beta\Phi(x) + \frac{1}{2}|\sqrt{a}x + \sqrt{b}z|^2.$$

Suppose now that Φ belongs to $\mathcal{C}^2(H)$ and let us turn to (DIN) which we know is equivalent to (g-DIN) with $a = \alpha - 1/\beta$, $b = 1/\beta$. If α, β satisfy $\alpha\beta > 1$ in addition to $\alpha > 0$, $\beta > 0$, then a, b satisfy $a > 0$, $b > 0$. As a consequence (DIN) is equivalent to the gradient system (5.6.20) ; using the parameters α, β the expression of $\mathcal{E}$ is

$$\mathcal{E}(X) = \mathcal{E}(x, z) = \beta\Phi(x) + \frac{1}{2\beta}|\sqrt{\alpha\beta - 1}x + z|^2 \tag{5.6.22}$$

We state as a Proposition that remarkable property of (DIN).

Proposition 5.6.1 *Suppose $\Phi \in \mathcal{C}^2(H)$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta > 1$. The system (DIN) is equivalent to the gradient system (5.6.21) with $\mathcal{E}$ given by (5.6.22).*

Since the functional $\mathcal{E}$ equals $\beta\Phi$ plus a positive quadratic form, it inherits most of the eventual properties of Φ : boundedness from below, coercivity, regularity, analyticity, convexity...Moreover if $(\bar{x}, \bar{z})$ is a critical (or minimum) point of $\mathcal{E}$ then $\bar{x}$ is a critical (or minimum) point of Φ. Thus the equivalence of (DIN) with the gradient system (5.6.21) allows properties of gradient systems to pass to (DIN).

For example, if Φ is analytic then so is $\mathcal{E}$. Further, if x is a bounded solution of (DIN) then $\dot{x}$ is bounded (Theorem 5.2.1(iii)) and (x, z) is a bounded solution of (5.6.21) which is known to converge to a critical point of $\mathcal{E}$ [116, 93]. Hence x converges to a critical point of Φ.

Likewise, in the convex case, Theorem 5.5.1, and Propositions 5.5.1 and 5.5.2 are consequences of Theorems of Bruck [40] and Brezis [37] ; that remark completes the proof of Proposition 5.5.2 where the case $\alpha\beta > 1$ was pending.

5.6.4 Remarks

1. Structure of (DIN) when $\alpha\beta < 1$. Suppose $\Phi \in \mathcal{C}^1(H)$ and $a < 0$, $b > 0$ in (g-DIN). Rescaling the variable y by $y = \sqrt{\frac{-a}{b}}z$ transforms (g-DIN) into the equivalent system

$$\begin{cases} \dot{x} + \beta\nabla\Phi(x) & + ax + \sqrt{-ab}z = 0 \\ \dot{z} & -\sqrt{-ab}x + bz = 0 \end{cases} \tag{5.6.23}$$

Set $X = (x, z)$ and define the functional $\mathcal{F} : H \times H \mapsto \mathbb{R}$ by $\mathcal{F}(X) = \beta\Phi(x) + \frac{1}{2}(a|x|^2 + b|z|^2)$, and the linear operator $J : H \times H \mapsto H \times H$ by $J(X) = \sqrt{-ab}(z, -x)$. Then (5.6.23) can be written

$$\dot{X} + \nabla\mathcal{F}(X) + J(X) = 0 \tag{5.6.24}$$

which appears as a gradient system perturbed by the monotone operator J. Unfortunately, properties such as convexity or boundedness from below do not pass from Φ to $\mathcal{F}$ since the quadratic form $\frac{1}{2}(a|x|^2 + b|z|^2)$ is not positive.

As to (DIN), if we suppose $\Phi \in \mathcal{C}^2(H)$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta < 1$, then the equivalent (g-DIN) system verifies $a < 0$, $b > 0$, and (DIN) turns to be equivalent to (5.6.24) too.

The system (g-DIN) can be given another equivalent form if we suppose $a < 0$ and $a + b > 0$ [5]. Indeed make the change of variable $y = \frac{1}{b}(\sqrt{-a(a+b)}z - ax)$; then (g-DIN) becomes

$$\begin{cases} \dot{x} \quad + \beta\nabla\Phi(x) \quad + \sqrt{-a(a+b)}z = 0 \\ \dot{z} - \beta\sqrt{\dfrac{-a}{a+b}}\nabla\Phi(x) \quad + (a+b)z \quad = 0 \end{cases} \tag{5.6.25}$$

Introduce the function $\mathcal{G}(X) = \mathcal{G}(x, z) = \beta\Phi(x) + \frac{1}{2}|z|^2$ and the linear monotone operator $J(x, z) = \sqrt{\frac{-a}{a+b}}(z, -x)$, then (5.6.25) becomes

$$\dot{X} + (1 + J)\nabla\mathcal{G}(X) = 0. \tag{5.6.26}$$

Turning back to (DIN), if we suppose $\Phi \in \mathcal{C}^2(H)$, $\alpha > 0$, $\beta > 0$ and $\alpha\beta < 1$, then we have $a < 0$ and $a + b > 0$ in the system (g-DIN) associated *via* (5.6.19), and hence (DIN) is equivalent to (5.6.26).

Unfortunately, systems (5.6.24) and (5.6.26) are not easy to deal with, and when $\alpha\beta < 1$ in (DIN) (or $a < 0$ in (g-DIN)) the only results remain those given in sections 5.2, 5.4, 5.5 (or by Theorems 5.6.2, 5.6.3).

2. The change of coordinates in (5.6.15), which allows to transform (DIN) into the first order system (g-DIN), may appear as a trick. Yet, when investigating the minimum (or critical) points of Φ, there often appears a function of the form $\Psi(x, y) = \Phi(x) + \frac{1}{2}|ax + by|^2$ (x, y in H and a, b real) the decrease of which lies at the root of the analysis. One recognizes in Ψ the energy functional of (DIN) or (HBF), and perhaps more subtly the function $(x, y) \mapsto \Phi(x) + \frac{1}{2\lambda}|x - y|^2$ ($\lambda > 0$) which occurs in the minimization of Φ by the proximal algorithm [75]

$$x_{n+1} = \text{argmin}_{x \in H}\{\Phi(x) + \frac{1}{2\lambda}|x - x_n|^2\}.$$

Applying the continuous steepest descent method to Ψ is then tempting ; it yields a first order system such as (g-DIN), and eliminating y gives (DIN). Performing the computations backward, and generalizing them, leads to the developments of sections 5.6.1 and 5.6.2.

3. (DIN) can be written as an integro-differential equation

$$\dot{x}(t) + \beta\nabla\Phi(x(t)) = (\alpha\beta - 1)\int_0^t \nabla\Phi(x(s)) \exp\left(\alpha(s - t)\right) ds + (\dot{x}_0 + \beta\nabla\Phi(x_0)) \exp(-\alpha t).$$

Thus, if $\alpha\beta = 1$, one obtains the non autonomous first order gradient system

$$\dot{x}(t) + \beta\nabla\Phi(x(t)) = (\dot{x}_0 + \beta\nabla\Phi(x_0)) \exp(-\alpha t).$$

[5] We are indebted to our colleague X. Goudou for pointing out this fact to us.

5.7 Application to constrained optimization.

The equivalence between (DIN) and (g-DIN) suggests a method to solve constrained optimization problems with the help of a dynamical system like (g-DIN); that is the subject of this section.

Fix C a non empty closed convex set of H. In the following we suppose that Φ is $\mathcal{C}^1$ with $\nabla\Phi$ Lipschitz continuous on bounded sets and we consider the following problem

$$(\mathcal{P}) \qquad \inf_C \Phi.$$

When we want to solve $(\mathcal{P})$ with a second order in time dynamical system, we have to face a major difficulty : how can we both force the orbits starting in C to lie in C and to keep their inertial aspects ? In many practical cases such a *viability property* is of interest. Those problems of viability are easier to handle when we deal with first order systems. If we consider, for example, the following system initiated by Antipin [9, 10]

$$(\text{S1}) \qquad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \mu\nabla\Phi(x(t))] = 0 \\ x(0) = x_0 \in C \end{cases}$$

where P_C is the projection on C and $\mu > 0$, then the viability property is obvious since the corresponding vector field enters the set of constraints. This dynamics provides moreover orbits that enjoy nice asymptotic properties : if we suppose Φ to be convex then trajectories weakly converge towards a minimum of Φ on C, even if we only assume $x_0 \in C$. This system has also been studied in its second order in time form, namely

$$(\text{S2}) \qquad \begin{cases} \ddot{x}(t) + \alpha\dot{x}(t) + x(t) - P_C[x(t) - \mu\nabla\Phi(x(t))] = 0 \\ x(0) = x_0 \in C, \quad \dot{x}(0) = \dot{x}_0 \in H \end{cases}$$

but in that case the viability property is no longer maintained. This naturally leads to strong hypotheses on the potential Φ to obtain a proper optimizing system, see for example [10, 11, 6].

We propose in the following Theorem to combine (g-DIN) and (S1) to solve $(\mathcal{P})$. More precisely, given real parameters β, a and b such that $\beta > 0$, $a \neq 0$, $b > 0$ and $b + a > 0$, we consider the first order system in $H \times H$

$$(\text{c-DIN}) \qquad \begin{cases} \dot{x}(t) + x(t) - P_C[x(t) - \beta\nabla\Phi(x(t)) - ax(t) - by(t)] = 0 \\ \dot{y}(t) + ax(t) + by(t) = 0 \end{cases}$$

with initial conditions

$$x(0) = x_0 \in C, \quad y(0) = y_0 \in H. \tag{5.7.27}$$

Of course, (c-DIN) reduces to (g-DIN) if $C = H$. The functional Φ is required to satisfy the following hypotheses

$$(\mathcal{H}\text{-c}) \quad \begin{cases} \Phi \text{ is defined and } C^1 \text{ on an open neighbourhood of the closed convex set } C \\ \Phi \text{ is bounded from below on } C \\ \text{the gradient } \nabla\Phi \text{ is Lipschitz continuous on the bounded subsets of } C. \end{cases}$$

If (x, y) is a solution to (c-DIN), and for $\lambda > 0$, let us define

$$E_\lambda(t) = \lambda\Phi(x(t)) + \frac{1}{2}|ax(t) + by(t)|^2. \tag{5.7.28}$$

A Theorem similar to Theorem 5.2.1 can be stated and proved for (c-DIN).

Theorem 5.7.1 *Let Φ satisfy the hypotheses ($\mathcal{H}$-c) and assume $\beta > 0$, $a \neq 0$, $b > 0$ and $b + a > 0$. Then the following properties hold :*

(i) *For each $(x_0, y_0) \in C \times H$, there exists a unique solution $(x(t), y(t))$ of* (c-DIN) *defined on the whole interval $[0, +\infty[$ which satisfies the initial conditions $x(0) = x_0, y(0) = y_0$; (x, y) belongs to $\mathcal{C}^1(0, +\infty; H) \times \mathcal{C}^2(0, +\infty; H)$ and x is viable, that is $x(t)$ lies in C for all $t \geq 0$.*

(ii) *For every trajectory $(x(t), y(t))$ of* (c-DIN) *and for $\lambda \in [\beta(\sqrt{b} - \sqrt{b+a})^2, \beta(\sqrt{b} + \sqrt{b+a})^2]$, the energy E_λ is decreasing on $[0, +\infty[$, bounded from below and hence converges to some real value as $t \to +\infty$. Moreover*

- *$\dot{x}$ and $\dot{y}$ belong to $L^2(0, +\infty; H)$,*
- *$\lim_{t\to+\infty} \Phi(x(t))$ exists,*
- *$\lim_{t\to+\infty} \dot{y}(t) = 0$.*

(iii) *Assuming in addition that x is in $L^\infty(0, +\infty; H)$, we have*

- *$\nabla\Phi(x)$, y, $\dot{x}$ are bounded on $[0, +\infty[$,*
- *$\lim_{t\to+\infty} \dot{x}(t) = 0$.*

The proof essentially goes along the same lines as in Theorem 5.2.1. The nonlinearity caused by the projection P_C is compensated by the characteristic inequality $\langle v - P_C u, u - P_C u\rangle \leq 0$ for all (u, v) in $H \times C$. The natural quantities upon which the calculations rely are $\dot{x}$ and $\dot{y}$ (rather than $\dot{x}$ and $\nabla\Phi(x)$ in the proof of Theorem 5.2.1).

Proof of Theorem 5.7.1.

(i) Since the projection P_C is a Lipschitz continuous operator, the local existence and the uniqueness of a solution to (c-DIN) with initial conditions (5.7.27) follow from the Cauchy-Lipschitz Theorem. Let (x, y) denote the maximal solution defined on some interval $[0, T_{max}[$ with $0 \leq T_{max} \leq +\infty$.

First let us show that x is viable for $t \in [0, T_{max}[$. Define $p : [0, T_{max}[\mapsto C$ by $p(t) = P_C[x(t) - \beta\nabla\Phi(x(t)) - ax(t) - by(t)]$ and integrate the equation $\dot{x} + x = p$ on $[0, t] \subset [0, T_{max}[$

$$x(t) = \int_0^t e^{-(t-s)} p(s) ds + e^{-t} x_0.$$

Observe that $\xi(t) = \int_0^t \frac{e^{-(t-s)}}{1-e^{-t}} p(s) ds$ belongs to C, as the weight function $s \mapsto \frac{e^{-(t-s)}}{1-e^{-t}}$ is positive and its integral over $[0, t]$ is 1. Now writing $x(t) = (1 - e^{-t})\xi(t) + e^{-t} x_0$ shows that $x(t)$ belongs to C.

Next, the viability of x and the convexity of C are used to derive the following inequality on $[0, T_{max}[$

$$\langle x - P_C(x - \beta\nabla\Phi(x) + \dot{y}), x - \beta\nabla\Phi(x) + \dot{y} - P_C(x - \beta\nabla\Phi(x) + \dot{y})\rangle \leq 0,$$

which, in view of (c-DIN), successively reduces to

$$\langle -\dot{x}, -\dot{x} - \beta\nabla\Phi(x) + \dot{y}\rangle \leq 0,$$

$$\beta\langle \dot{x}, \nabla\Phi(x)\rangle \leq -|\dot{x}|^2 + \langle \dot{x}, \dot{y}\rangle. \tag{5.7.29}$$

Further, in order to apply classical energy arguments, we show that E_λ, defined by (5.7.28), is decreasing along the trajectory (x, y), at least for some value of λ. Indeed we have (using the second equation in (c-DIN))

$$\dot{E}_\lambda = \lambda\langle \dot{x}, \nabla\Phi(x)\rangle - b|\dot{y}|^2 - a\langle \dot{x}, \dot{y}\rangle.$$

Taking (5.7.29) into account, we obtain

$$\dot{E}_\lambda \leq -\frac{\lambda}{\beta}|\dot{x}|^2 - b|\dot{y}|^2 + (\frac{\lambda}{\beta} - a)\langle \dot{x}, \dot{y}\rangle. \tag{5.7.30}$$

In particular, if we choose $\lambda = \beta(a + 2b)$ (this last quantity is positive), we have

$$\dot{E}_{\beta(a+2b)} \leq -(a+b)|\dot{x}|^2 - b|\dot{x} - \dot{y}|^2. \tag{5.7.31}$$

Integrating this inequality over $[0, t] \subset [0, T_{max}[$ we obtain

$$\beta(a+2b)\Phi(x(t)) + \frac{1}{2}|ax(t) + by(t)|^2 + (a+b)\int_0^t |\dot{x}(\tau)|^2 d\tau + b\int_0^t |\dot{x}(\tau) - \dot{y}(\tau)|^2 d\tau$$
$$\leq \beta(a+2b)\Phi(x_0) + \frac{1}{2}|ax_0 + by_0|^2 \tag{5.7.32}$$

Finally, to prove that (x, y) is defined over $[0, +\infty[$, we suppose that $T_{max} < +\infty$ and argue by contradiction. Since x is viable and Φ is bounded from below, (5.7.32) shows that $\dot{y} = -(ax + by)$ is bounded on $[0, T_{max}[$; hence $\lim_{t\to T_{max}} y(t)$ exists. As a consequence y and $x = -\frac{1}{a}(\dot{y} + by)$ are bounded, and so is $\nabla\Phi(x)$ in view of ($\mathcal{H}$-c). Then (c-DIN) shows that $\dot{x}$ is bounded too. Hence $\lim_{t\to T_{\max}} x(t)$ exists. This classically yields a contradiction, and T_{max} must be equal to $+\infty$.

The last assertion, $(x, y) \in \mathcal{C}^1(0, +\infty; H) \times \mathcal{C}^2(0, +\infty; H)$, immediately follows from (c-DIN).

(ii) Set $q(\lambda) = -\frac{\lambda}{\beta}|\dot{x}|^2 - b|\dot{y}|^2 + (\frac{\lambda}{\beta} - a)\langle \dot{x}, \dot{y}\rangle$, $\lambda_{min} = \beta(\sqrt{b} - \sqrt{b+a})^2$, and $\lambda_{max} = \beta(\sqrt{b} + \sqrt{b+a})^2$. The inequality (5.7.30) yields

$$\dot{E}_{\lambda_{min}} \leq q(\lambda_{min}) = -|(\sqrt{b} - \sqrt{b+a})\dot{x} + \sqrt{b}\dot{y}|^2$$
$$\dot{E}_{\lambda_{max}} \leq q(\lambda_{max}) = -|(\sqrt{b} + \sqrt{b+a})\dot{x} - \sqrt{b}\dot{y}|^2$$

Since q is an affine function of λ, for every $\lambda \in [\lambda_{min}, \lambda_{max}]$, $\dot{E}_\lambda$ lies between $q(\lambda_{min})$ and $q(\lambda_{max})$ and hence is nonpositive. The energy E_λ is then decreasing on $[0, +\infty[$ and converges since Φ is bounded from below on C.

The inequality (5.7.32) shows that $\dot{x}$ and $\dot{y}$ belong to $L^2(0, +\infty; H)$.

Now, considering two different values λ, λ' in $[\lambda_{min}, \lambda_{max}]$ shows that $\Phi(x) = \frac{1}{\lambda' - \lambda}(E_{\lambda'} - E_\lambda)$ admits a limit as $t \to +\infty$.

Hence $|\dot{y}|^2 = |ax + by|^2 = 2(E_\lambda - \lambda\Phi(x))$ also admits a limit which necessarily is zero since $|\dot{y}|$ belongs to $L^2(0, +\infty; H)$.

(iii) If x is bounded, then $\nabla\Phi(x)$ is bounded (after $(\mathcal{H}-c)$), and $y = -\frac{1}{b}(ax+\dot{y})$ is bounded (recall $\dot{y} \to 0, t \to +\infty$). Further $\dot{x}$ is bounded in view of (c-DIN). Since $\dot{x}$ and $\dot{y}$ are bounded, x and y are Lipschitz continuous, which shows, in view of (c-DIN), that $\dot{x}$ itself is Lipschitz continuous. But $\dot{x}$ belongs to $L^2(0,+\infty;H)$, hence, according to a classical argument, $\dot{x}(t) \to 0$ as $t \to +\infty$. ■

Theorem 5.7.2 *In addition to the hypotheses of Theorem* 5.7.1, *assume that* Φ *is convex, and that* $\operatorname{argmin}_C\Phi$, *the set of minimizers of* Φ *on* C, *is nonempty. Then for any solution* $(x(t),y(t))$ *of* (c-DIN), $x(t)$ *weakly converges to a minimizer of* Φ *on* C *as* t *goes to infinity.*

Proof. First, let us establish some useful inequalities. Let x^* be a minimizer of Φ on C. Use the characteristic inequality for P_C to write (it is implicit that the time variable t varies in $[0,+\infty[$ in the following)

$$\langle x^* - P_C(x-\beta\nabla\Phi(x)+\dot{y}), x-\beta\nabla\Phi(x)+\dot{y}-P_C(x-\beta\nabla\Phi(x)+\dot{y})\rangle \le 0.$$

In view of (c-DIN) we derive

$$\langle x^*-x-\dot{x}, -\dot{x}-\beta\nabla\Phi(x)+\dot{y}\rangle \le 0,$$

$$\langle x^*-x, \dot{y}-\dot{x}\rangle + \beta\langle \dot{x}, \nabla\Phi(x)\rangle \le \langle x^*-x, \beta\nabla\Phi(x)\rangle - |\dot{x}|^2. \tag{5.7.33}$$

But $\langle x^*-x, \nabla\Phi(x^*)-\nabla\Phi(x)\rangle$ is nonnegative since Φ is convex; and $\langle x^*-x, -\nabla\Phi(x^*)\rangle$ is nonnegative because x^* is a minimizer of Φ on C. Hence $\langle x^*-x, -\nabla\Phi(x)\rangle$ is nonnegative and (5.7.33) entails

$$\langle x^*-x, \dot{y}-\dot{x}\rangle + \beta\langle \dot{x}, \nabla\Phi(x)\rangle \le -|\dot{x}|^2. \tag{5.7.34}$$

Our aim now is to introduce an energy functional involving the term $|x^*-x|$. Set

$$F(t) = \langle x^*-x(t), ax(t)+by(t)\rangle + \frac{1}{2}(b+a)|x^*-x(t)|^2 + b\beta\Phi(x(t)).$$

We have

$$\dot{F} = b(\langle x^*-x, \dot{y}-\dot{x}\rangle + \langle \dot{x}, \beta\nabla\Phi(x)\rangle) + \langle \dot{x}, \dot{y}\rangle.$$

And in view of (5.7.34) we obtain

$$\dot{F} \le \langle \dot{x}, \dot{y}\rangle - b|\dot{x}|^2 \le -(b-\frac{3}{2})|\dot{x}|^2 + \frac{1}{2}|\dot{y}-\dot{x}|^2. \tag{5.7.35}$$

In view of (5.7.31) and (5.7.35) we may fix some $\varepsilon > 0$ so small that the function $\mathcal{E} : \mathbb{R} \mapsto H$ defined by

$$\mathcal{E} = E_{a+2b} + \varepsilon F = (a+2b+\varepsilon b\beta)\Phi(x) + \frac{1}{2}|ax+by|^2 + \varepsilon\langle x^*-x, ax+by\rangle + \frac{\varepsilon}{2}(a+b)|x^*-x|^2$$

is decreasing and hence bounded above. Since $\Phi(x)$ is bounded from below on C, the quantity $-|ax+by||x^*-x| + \frac{1}{2}(b+a)|x-x^*|^2$, which is less than $\langle x^*-x, ax+by\rangle + \frac{1}{2}(b+a)|x^*-x|^2$, is bounded from above; hence $|x^*-x|$ is bounded because $\dot{y} = ax+by$ is bounded (Theorem (5.7.1)**(ii)**). From that we deduce that $\mathcal{E}$ is bounded below and admits a limit as $t \to +\infty$. Now in the expression of $\mathcal{E}$ the first three terms are known to have a limit, as $t \to +\infty$, hence $|x^*-x|$ has a limit.

In order to apply Opial's Lemma, we now show that any weak limit point x_∞ of x belongs to $\text{argmin}_C \Phi$. Let x^* be an element of $\text{argmin}_C \Phi$. Invoking the convexity of Φ and inequality (5.7.33) we have

$$\Phi(x^*) \geq \Phi(x(t)) + \langle x^* - x, \nabla\Phi(x)\rangle$$

$$\Phi(x^*) \geq \Phi(x(t)) + \frac{1}{\beta}\langle x^* - x, \dot{y} - \dot{x}\rangle + \frac{1}{\beta}\langle \dot{x}, \dot{x} + \beta\nabla\Phi(x)\rangle.$$

Since $|\dot{x}| + |\dot{y}| \to 0$ as $t \to +\infty$, and since $(x^* - x)$ and $(\dot{x} + \beta\nabla\Phi(x))$ are bounded we have

$$\langle x^* - x, \dot{y} - \dot{x}\rangle + \langle \dot{x}, \dot{x} + \beta\nabla\Phi(x)\rangle \to 0,\ t \to +\infty.$$

So, if t_n is a sequence going to infinity such that $x(t_n)$ weakly converges to x_∞, we have $\Phi(x^*) \geq \liminf \Phi(x(t_n)) \geq \Phi(x_\infty)$. Hence x_∞ is a minimizer of Φ on C, and Opial's Lemma entails that $x(t)$ weakly converges to x_∞. ■

The inertial aspect and the effect of the constraints in (c-DIN) are illustrated by a two-dimensional example (fig. 5.2) : $\Phi(x_1, x_2) = \frac{1}{2}\{(x_1 + x_2 + 1)^2 + 4(x_1 - x_2 - 1)^2\}$, $C = \mathbb{R}^{+2}$.
- the trajectories of (c-DIN) (continuous lines) converge to point $(3/5, 0)$, the minimum of Φ on C,
- in the absence of constraints, the trajectories (dashed lines) converge to $(0, -1)$, the minimum of Φ on $\mathbb{R}^2$.

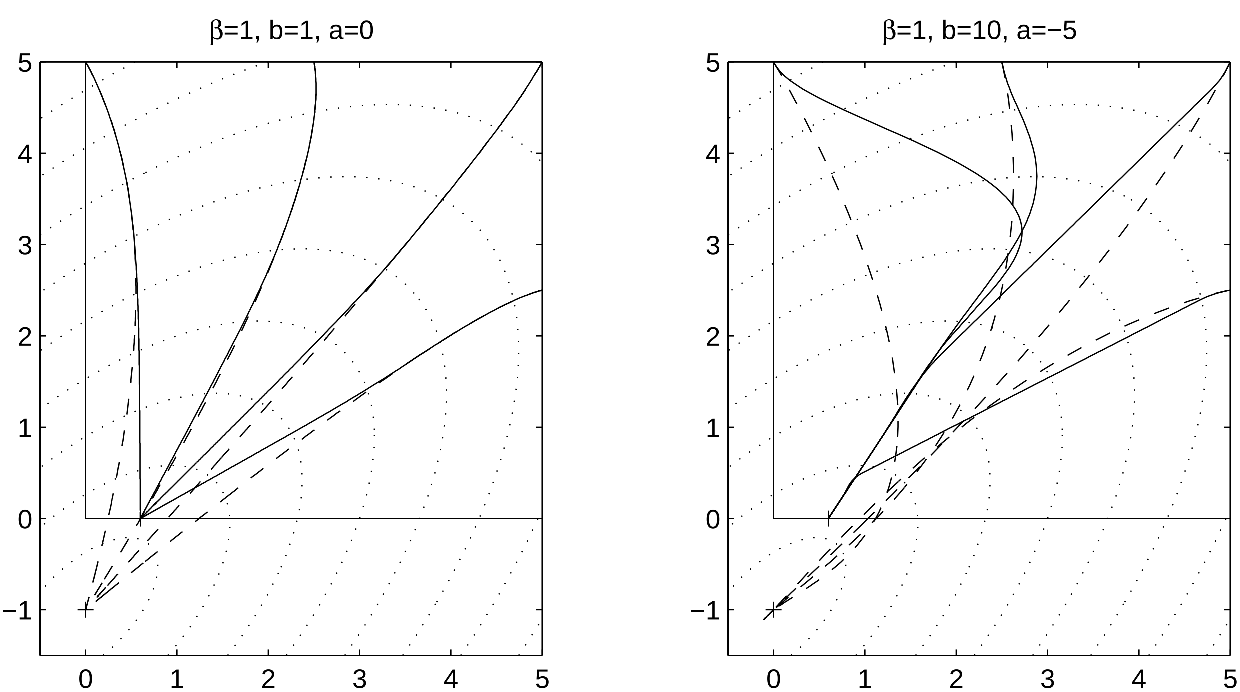

FIG. 5.2 – A few trajectories of *(c-DIN)*.

5.8 Application to impact dynamics

In [107], Paoli and Schatzman have studied the system

$$\begin{cases} \ddot{x}(t) + \partial\Psi_K(x(t)) \ni f(t, x(t), \dot{x}(t)) \\ \dot{x}(t^+) = -e\dot{x}_N(t^-) + \dot{x}_T(t^-) \text{ for any } t \text{ such that } x(t) \in \partial K \end{cases} \tag{5.8.36}$$

where K is a closed convex subset of a finite-dimensional Hilbert space H, and $\partial\Psi_K$ is the subgradient set of the indicator function Ψ_K ($\Psi_K(x) = 0$ if $x \in K$ and $\Psi_K(x) = +\infty$ elsewhere). The first equation models the evolution of a mechanical system under the action of the force f, with state $x(t)$ subject to remain in K. The second equation models the instantaneous change in the system whenever its representative point $x(t)$ hits the boundary of K : the tangential velocity is conserved, while the normal velocity is reversed and multiplied by the *restitution coefficient* $e \in]0, 1]$; this rule accounts for a possible loss of energy at the impact.

Owing to Ψ_K being a definitely non-smooth function, Paoli and Schatzman have to define a notion of solution to (5.8.36), and in order to prove the existence they introduce a regularized version obtained by a penalty method

$$\ddot{x}_\lambda(t) + \frac{2\varepsilon}{\sqrt{\lambda}} G(\nabla\Psi_{K,\lambda}(x_\lambda(t)), \dot{x}_\lambda(t)) + \nabla\Psi_{K,\lambda}(x_\lambda(t)) = f(t, x_\lambda(t), \dot{x}_\lambda(t)). \tag{5.8.37}$$

The function $\Psi_{K,\lambda}(x) = \frac{1}{2\lambda}\text{dist}^2(x, K)$ is the usual Moreau-Yosida regularization of Ψ_K with parameter $\lambda > 0$, and the operator $G : H \times H \mapsto H$ is defined by $G(w, 0) = 0$ and $G(w, v) =< w, \frac{v}{|v|} > \frac{v}{|v|}$ if $v \neq 0$. The constant $\varepsilon \in [0, +\infty[$ is related to e by $\varepsilon = -\frac{\log e}{\sqrt{\pi^2 + \log^2 e}}$. Passing to the limit $\lambda \to 0$ in (5.8.37) then yields a solution to (5.8.36).

We propose below a slightly different, and hopefully simpler, approach to (5.8.36). If K is a whole half-space, then it is not difficult to realize that $\frac{1}{\lambda}G(\nabla\Psi_{K,\lambda}(x), v)$ is exactly the Hessian $\nabla^2\Psi_{K,\lambda}(x)$ applied to v, except if x belongs to ∂K in which case $\nabla^2\Psi_{K,\lambda}(x)$ is not defined. When K is arbitrary, a formal, and bold, linearization of the boundary of K leads to replace $G(\nabla\Psi_{K,\lambda}(x_\lambda(t)), \dot{x}_\lambda(t))$ in (5.8.37) by $\lambda\nabla^2\Psi_{K,\lambda}(x_\lambda(t))\dot{x}_\lambda(t)$, which gives

$$\ddot{x}_\lambda(t) + 2\varepsilon\sqrt{\lambda}\nabla^2\Psi_{K,\lambda}(x_\lambda(t))\dot{x}_\lambda(t) + \nabla\Psi_{K,\lambda}(x_\lambda(t)) = f(t, x_\lambda(t), \dot{x}_\lambda(t)).$$

For simplicity, assume henceforth that the exterior force reduces to a viscous friction : $f(t, x_\lambda(t), \dot{x}_\lambda(t)) = -\alpha\dot{x}_\lambda(t)$, $\alpha \geq 0$. The preceding equation becomes

$$\ddot{x}_\lambda + \alpha\dot{x}_\lambda + 2\varepsilon\sqrt{\lambda}\nabla^2\Psi_{K,\lambda}(x)\dot{x}_\lambda + \nabla\Psi_{K,\lambda}(x) = 0.$$

This is (DIN) with $\beta = 2\varepsilon\sqrt{\lambda}$. But this equation has to be given a sense since $\Psi_{K,\lambda}$ is not twice differentiable everywhere. The cure is to write it in the form (g-DIN) which is of first order in time and space (recall $\beta = 2\varepsilon\sqrt{\lambda}$)

$$\begin{cases} \dot{x}_\lambda + \beta\nabla\Psi_{K,\lambda}(x_\lambda) + (\alpha - \frac{1}{\beta})x_\lambda + \frac{1}{\beta}y_\lambda = 0 \\ \dot{y}_\lambda \qquad\qquad\qquad + (\alpha - \frac{1}{\beta})x_\lambda + \frac{1}{\beta}y_\lambda = 0 \end{cases} \tag{5.8.38}$$

This system is numerically solvable as it stands. A few numerical experiments are reported in figure 5.3 : K is the unit disk, $\alpha = 0$, $\lambda = 10^{-4}$, the system representative point starts from position (0.5, 0) with velocity (0, 0.1) ; the coefficient $\beta = 2\varepsilon\sqrt{\lambda}$ runs through $\{0.02, 0.01, 0.008, 0.006, 0.004, 0.002, 0.001, 0.0001, 10^{-7}\}$, and correspondingly the restitution coefficient e runs through $\{0, 0.16, 0.25, 0.37, 0.53, 0.73, 0.85, 0.98, 0.99998\}$.

The experiments display the whole range of possible shocks :
- completely anelastic shocks for $\beta = 0.02$: after the first shock the trajectory follows the boundary,
- nearly perfectly elastic shocks for $\beta = 10^{-7}$ (the theoretical trajectory in the disk - without penalization - is an equilateral triangle),
- shocks with partial restitution of energy for intermediate values of β.

The purpose of these experiments is to illustrate the behaviour of the solutions of (5.8.38) and to suggest the latter as a theoretical regularization of (5.8.36). The numerical solution of (5.8.38) is prone to stiffness as λ becomes smaller (see [108] in this respect).

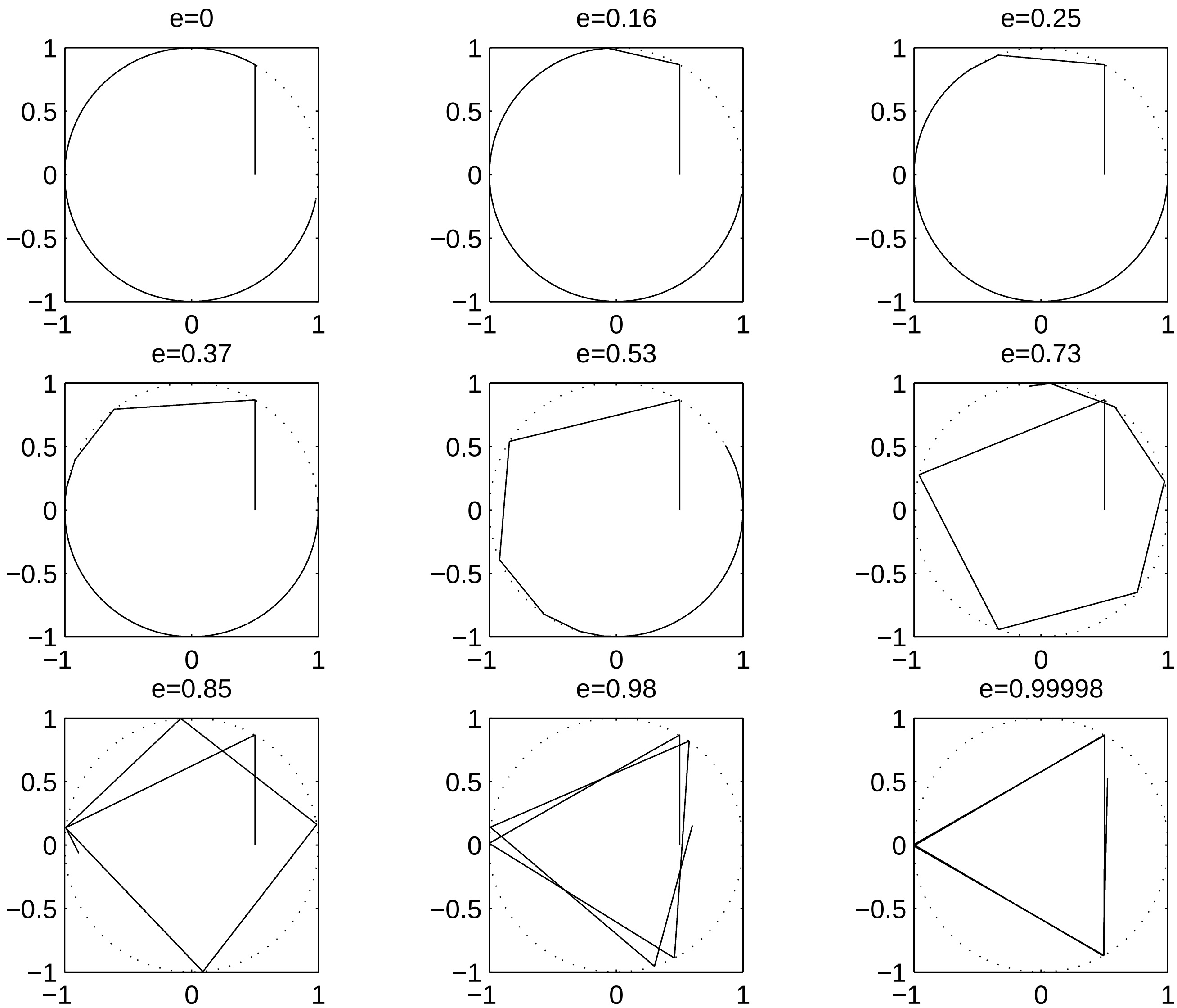

FIG. 5.3 – Impacts in a disk.

Chapitre 6

Propriétés minimisantes d'un système inertiel en optimisation. Liens avec la méthode proximale.

Optimizing Properties of an Inertial Dynamical System with Geometric Damping. Link with Proximal Methods[1]

H. Attouch [2], J. Bolte [2] and P. Redont [2]

Abstract. The second-order dynamical system $\ddot{x} + \alpha\dot{x} + \beta\nabla^2\Phi(x)\dot{x} + \nabla\Phi(x) = 0$, $\alpha > 0$, $\beta > 0$, where the Hessian $\nabla^2\Phi(x)$ acts as a geometric damping, is introduced, mainly in view of the minimization of Φ. Minimizing Φ is a problem equivalent to the minimization of the functional $\Psi_{a,b}(x,y) = \frac{1}{b^2}\Phi(x) + \frac{1}{2}|ax + by|^2$, $a > 0$, $b > 0$. The latter naturally appears in the proximal regularization of Φ ; it may also be viewed as an energy. The continuous steepest descent method applied to $\Psi_{a,b}$ yields a first-order system, which proves to be equivalent to the above-mentioned second-order system, when Φ is of class $\mathcal{C}^2$.
Keywords. Dynamical systems in optimization, proximal regularization method, steepest descent method, entropic methods in optimization.
AMS classification : 37N40, 90Cxx, 65Kxx.

6.1 Introduction

Let H be a real Hilbert space and $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semicontinuous, convex function. Consider the convex minimization problem

$$(\mathcal{P}) \qquad \inf\{\Phi(x) : x \in H\}$$

and let $S :=$ argmin Φ denote the solution set of $(\mathcal{P})$.

In relation with $(\mathcal{P})$, we wish to introduce a new dynamical system, called (DIN), which naturally arises and enjoys remarkable properties in convex optimization (its range of applications is much wider indeed). When Φ is a smooth $\mathcal{C}^2$ function, (DIN) assumes the following form

$$\text{(DIN)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$$

where $\nabla^2\Phi$ is the Hessian of Φ and where α and β are positive parameters.

This dynamical system can be viewed from different perspectives.

The second derivative $\ddot{x}(t)$ (which induces inertial effects) may be considered as a singular perturbation, and in fact regularization, of the possibly degenerate classical continuous Newton dynamical system

$$\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

That is the origin of the terminology : (DIN) stands in short for Dynamical Inertial Newton-like system.

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The system (DIN) also naturally derives from the Heavy Ball with Friction dynamical system (see Poliak [111], Antipin [10], Attouch-Goudou-Redont [18])

$$\text{(HBF)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

The damping term $\alpha\dot{x}(t)$ confers optimizing properties on (HBF), but it acts isotropically and ignores the geometry of Φ. Adding a geometric damping term like $\beta\nabla^2\Phi(x(t))\dot{x}(t)$ puts down the possible oscillations of the trajectories and gives rise to (DIN).

Lastly, the system (DIN) is closely related to the minimization of the function

$$(x, y) \in H \times H \mapsto \psi(x, y) = \Phi(x) + \frac{1}{2\lambda}|x - y|^2$$

where λ is some fixed positive parameter. Indeed the Continuous Steepest Descent method applied to ψ yields

$$\begin{cases} \dot{x}(t) + \nabla\Phi(x(t)) + \frac{1}{\lambda}(x(t) - y(t)) = 0 \\ \dot{y}(t) \qquad\qquad\quad + \frac{1}{\lambda}(y(t) - x(t)) = 0. \end{cases}$$

Eliminating y, we obtain the following (DIN) system

$$\ddot{x}(t) + \frac{2}{\lambda}\dot{x}(t) + \nabla^2\Phi(x(t))\dot{x}(t) + \frac{1}{\lambda}\nabla\Phi(x(t)) = 0.$$

Introducing the function ψ is no contrived idea, since it naturally appears in two circumstances at least.

First, the proximal regularization method applied to $(\mathcal{P})$ (see Moreau [103], Martinet [99], Rockafellar [113]) is nothing else than the iterated minimization of ψ alternatively with respect to the x and y variable. This point of view is set out in section

In many situations of practical importance, the minimization problem $(\mathcal{P})$ is not well-posed, see for example Dontchev and Zolezzi [55] for a thorough exposition of the notions of well-posedness and the presentation of various situations occurring in mathematical programming, calculus of variations, statistics, control theory, inverse problems, where well-posedness fails to be satisfied.

To regularize the problem $(\mathcal{P})$, a fruitful idea is to add a positive definite quadratic term, typically $\varepsilon|x|^2$, to $\Phi(x)$. This leads to various methods, like Tikhonov approximation method, but in that case the conditioning becomes worse and worse as the approximation parameter ε goes to zero. By contrast, proximal regularization methods allow to preserve the conditioning away from zero.

The basic idea which lies behind the proximal methods is the following : take some $x^* \in S = \text{argmin}\ \Phi$ and some $\lambda > 0$. Then consider the minimization problem

$$(\mathcal{P}_*) \qquad \min\{\Phi(x) + \frac{1}{2\lambda}|x - x^*|^2 : x \in H\}.$$

Clearly, $(\mathcal{P}_*)$ is a well-posed convex minimization problem with x^* as unique solution and $\inf(\mathcal{P}) = \inf(\mathcal{P}_*)$. Unfortunately this method is not constructive, since it makes use of some $x^* \in S$, which is unknown. Nevertheless, from a theoretical point of view, this method has proved to be quite fruitful. It was used by Barbu [27] in the optimal control of variational inequalities, then J.-L. Lions [95] made a systematic use of it in the study of singular distributed control problems, in order to obtain optimality conditions.

The proximal method which has been developed for numerical purposes consists in solving $(\mathcal{P}_*)$ not as a minimization problem (which is impossible, x^* is unknown) but as a *fixed point problem*. Indeed, for any $y \in H$ let us introduce the minimization problem

$$(\mathcal{P}_y) \qquad \min\{\Phi(x) + \frac{1}{2\lambda}|x-y|^2 : x \in H\}.$$

whose unique solution is denoted by $J_\lambda^\Phi(y)$. Clearly, x^* is a solution of $(\mathcal{P})$ if and only if $J_\lambda^\Phi(x^*) = x^*$. Taking advantage of $J_\lambda^\Phi(y)$ being a contraction (indeed a firmly nonexpansive mapping), the proximal method consists in solving this fixed point problem by the successive approximation method. One obtains the following classical algorithm

$$(\mathcal{P}_k) \quad \begin{array}{l} x_0 \text{ given} \\ x_k \to x_{k+1} = \operatorname{argmin}\{\Phi(x) + \frac{1}{2\lambda_k}|x-x_k|^2 : x \in H\}. \end{array}$$

This method, first introduced by Martinet [99] in convex optimization, has been developed in a general framework by Rockafellar [113] (see Lemaire [90] for a thorough exposition and further references). When writing the optimality condition for $(\mathcal{P}_k)$ one obtains

$$\lambda_k^{-1}(x_{k+1} - x_k) + \partial\Phi(x_{k+1}) \ni 0$$

which can be interpreted as the implicit discretization of the generalized continuous steepest descent method

$$\dot{x}(t) + \partial\Phi(x(t)) \ni 0.$$

Note that, in this continuous-discrete interaction, the property $\sum_{k=1}^{+\infty} \lambda_k = +\infty$ corresponds to $t \to +\infty$ (since $x(t_k) = x_k$, and $\lambda_k = t_{k+1} - t_k$). It is a remarkable property that both systems (discrete and continuous) enjoy a very similar asymptotical behaviour. In both cases, with Opial Lemma one can prove that the trajectories converge weakly in H to an optimal solution. In the continuous case, this result has been obtained by Bruck [40].

Let us notice too that the continuous dynamical system allows to treat parabolic PDEs, like (nonlinear) heat equations, see H. Brézis [37].

Let us now come to the original aspect of our approach. To that end, let us give a different formulation of the proximal regularization method. We are going to interpret it as a relaxation method applied to an energy-like function. Indeed, as we have already observed, the function of two variables

$$\begin{array}{rl} \psi : H \times H & \mapsto \mathbb{R} \cup \{+\infty\} \\ (x,y) & \mapsto \psi(x,y) := \Phi(x) + (2\lambda)^{-1}|x-y|^2 \end{array}$$

plays a central role in the above results. In order to get some flexibility we introduce two other parameters :

Definition 6.1.1 *Let $a, b \in \mathbb{R}$ be two real parameters, with $b \neq 0$. We define*

$$\begin{array}{rl} \psi_{a,b} : H \times H & \mapsto \mathbb{R} \cup \{+\infty\} \\ (x,y) & \mapsto \psi_{a,b}(x,y) \end{array}$$

by the following formula

$$\psi_{a,b}(x,y) = \Phi(x) + \frac{1}{2}|ax + by|^2.$$

It is called the energy function attached to the convex minimization problem $(\mathcal{P})$, *(with parameters* a *and* b*). The energy minimization problem* $(\mathcal{P}_{a,b})$ *is defined by*

$$(\mathcal{P}_{a,b}) \qquad \inf\{\Phi(x) + \frac{1}{2}|ax + by|^2 : (x,y) \in H\}.$$

Let us notice that we are not in the classical perturbation theory for convex problems since $\psi_{a,b}(x,0) = \Phi(x) + \frac{1}{2}|ax|^2$ is not equal to the original function Φ (unless $a = 0$). Let us make precise the connection between $(\mathcal{P}_{a,b})$ and $(\mathcal{P})$.

Proposition 6.1.1 *For any values of* $a, b \in \mathbb{R}$, $b \neq 0$ *the following equalities hold :*
i) $\inf\{\Phi(x) : x \in H\} = \inf\{\psi_{a,b}(x,y) : (x,y) \in H \times H\}$.
ii) *If* x^* *is an optimal solution of* $(\mathcal{P})$, *then* $(x^*, -\frac{a}{b}x^*)$ *is an optimal solution of* $(\mathcal{P}_{a,b})$.
iii) *Conversely, if* $(x^*, y^*) \in H \times H$ *is an optimal solution of* $(\mathcal{P}_{a,b})$, *then* $y^* = -\frac{a}{b}x^*$, *and* x^* *is an optimal solution of* $(\mathcal{P})$.

Proof. The statements are easy consequences of the following facts

$$\forall (x,y) \in H \times H, \Phi(x) \leq \psi_{a,b}(x,y),$$
$$\Phi(x) = \psi_{a,b}(x,y) \Leftrightarrow y = -\frac{a}{b}x. \blacksquare$$

As a consequence, solving $(\mathcal{P})$ is equivalent to solving $(\mathcal{P}_{a,b})$. Note that $(\mathcal{P}_{a,b})$ is only partially well-conditioned. It is not globally well-conditioned because of the direction $y = -\frac{a}{b}x$ along which the quadratic form is degenerate.

Indeed, the strategy of the proximal method consists in minimizing $\psi_{a,b}$ by using a relaxation method making only use of directions where $(\mathcal{P}_{a,b})$ is well-conditioned, namely the x- and y-subspaces. Let us make this precise in the following statement

Proposition 6.1.2 *The proximal method is the relaxation minimization method applied to* $\psi_{a,-a}$, *for* $a = \frac{1}{\sqrt{\lambda}}$. *More precisely*

$$(x_k, y_k = x_k) \to (x_{k+1}, y_{k+1}) : \quad \begin{aligned} x_{k+1} &= \mathit{argmin}\{\psi_{a,-a}(x, y_k) : x \in H\} \\ y_{k+1} &= \mathit{argmin}\{\psi_{a,-a}(x_{k+1}, y) : y \in H\}. \end{aligned}$$

Proof. By definition of the proximal method, by taking $a^2 = \frac{1}{\lambda}$

$$\begin{aligned} x_{k+1} &= \operatorname{argmin}\{\Phi(x) + \tfrac{1}{2\lambda}|x - x_k|^2 : x \in H\} \\ &= \operatorname{argmin}\{\psi_{a,-a}(x, y_k) : x \in H\} \end{aligned}$$

since $y_k = x_k$. Next, when considering y_{k+1} as

$$y_{k+1} = \operatorname{argmin}\{\Phi(x_{k+1}) + \frac{a^2}{2}|x_{k+1} - y|^2 : y \in H\}$$

one clearly gets $y_{k+1} = x_{k+1}$. And so on. $\blacksquare$

From a numerical point of view it is tempting to minimize $\psi_{a,b}$ using better descent directions than those *a priori* given by the x and y directions. A natural candidate is the steepest descent method. Let us decribe it when it is applied to $\psi_{a,b}$. Indeed it is convenient to consider the function

$$\Psi_{a,b}(x,y) = \frac{1}{b^2}\Phi(x) + \frac{1}{2}|ax+by|^2$$

in order to obtain a quite simple formulation (note that replacing Φ by $\frac{1}{b^2}\Phi$ does not change anything to the minimization problem $(\mathcal{P})$ and, like $\psi_{a,b}$, $\Psi_{a,b}$ may be called an energy associated to Φ).

Theorem 6.1.1 *Let $\Phi : H \mapsto \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous, proper function. Let a, b be real constants with $b \neq 0$.*
a) *The generalized continuous steepest descent method when applied to*

$$\Psi_{a,b}(x,y) = \frac{1}{b^2}\Phi(x) + \frac{1}{2}|ax+by|^2$$

provides the following system (energetical steepest descent)

$$(ESD) \qquad \begin{cases} \dot{x}(t) + \frac{1}{b^2}\partial\Phi(x(t)) + a[ax(t)+by(t)] \ni 0 & (ESD1) \\ \dot{y}(t) + b[ax(t)+by(t)] = 0 & (ESD2) \end{cases}$$

b) *When Φ is a smooth $\mathcal{C}^2$ function, and $a \neq 0$, the above system (ESD) can equivalently be written (by eliminating the variable y)*

$$\ddot{x}(t) + (a^2+b^2)\dot{x}(t) + \frac{1}{b^2}\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0. \tag{6.1.1}$$

c.1) *For any initial condition $x_0 \in \overline{dom\, \Phi}$ and $y_0 \in H$, there exists a unique solution (x,y) of (ESD) in the following sense*
. *$x : [0,+\infty[\mapsto H$ is a continuous function, with $x(t) \in dom\,\Phi$ $\forall t > 0$, Lipschitz continuous on $[\delta,+\infty[$ for every $\delta > 0$,*
. *$y : [0,+\infty[\mapsto H$ is a $\mathcal{C}^1$ function, with a Lipschitz continuous derivative on $[\delta,+\infty[$ for every $\delta > 0$,*
. *(ESD1) is satisfied almost everywhere on $]0,+\infty[$,*
. *(ESD2) is satisfied for every $t \in]0,+\infty[$,*
. *$x(0) = x_0$ and $y(0) = y_0$.*
c.2) *As $t \to +\infty$, $\Phi(x(t))$ converges to $\inf\Phi$, whether the latter be finite or not.*
c.3) *If $S = argmin\ \Phi \neq \emptyset$, then x and y weakly converge as $t \to +\infty$: $x(t) \overset{w-H}{\longrightarrow} x_\infty \in S$ and $y(t) \overset{w-H}{\longrightarrow} -\frac{a}{b}x_\infty$.*
c.4) *If in addition Φ is even, then x and y converge strongly as $t \to +\infty$.*

Proof. a) The generalized continuous steepest descent (see Brézis [37]) applied to $\Psi_{a,b}$ reads

$$(\dot{x}(t),\dot{y}(t)) + \partial\Psi_{a,b}(x(t),y(t)) \ni 0. \tag{6.1.2}$$

Making the inclusion above explicit yields (ESD).

b) Let (x, y) be a solution of (ESD). Since Φ is $\mathcal{C}^2$ we have

$$\dot{x} + \frac{1}{b^2}\nabla\Phi(x) + a[ax + by] = 0 \tag{6.1.3}$$

$$\dot{y} + b[ax + by] = 0. \tag{6.1.4}$$

Differentiate (6.1.3) to get

$$\ddot{x} + a^2\dot{x} + \frac{1}{b^2}\nabla^2\Phi(x)\dot{x} + ab\dot{y} = 0. \tag{6.1.5}$$

Perform a linear combination of (6.1.3), (6.1.4), (6.1.5) with b^2, $-ab$, 1 as coefficients to obtain (6.1.1).

Conversely, let x satisfy (6.1.1); define y by (6.1.3), which is legal since $ab \neq 0$. Differentiating (6.1.3) yields (6.1.5) as above. Perform a linear combination of (6.1.1), (6.1.3), (6.1.5) with -1, b^2, 1 as coefficients to obtain $ab\dot{y} + ab^2[ax + by] = 0$, which is equation (6.1.4).

c.1) The function $\Psi_{a,b}$ is proper, lower semicontinuous and convex; the point (x_0, y_0) belongs to $\overline{\text{dom }\Phi} \times H = \overline{\text{dom }\Psi_{a,b}}$. A Theorem of Brézis [37, th. 3.2] then asserts the existence and uniqueness of a continuous function $(x, y) : [0, +\infty[\mapsto H \times H$, with $(x(t), y(t)) \in \text{dom }\Psi_{a,b}$ for any t, $x(0) = x_0$, $y(0) = y_0$, which is Lipschitz continuous on $[\delta, +\infty[$ for every $\delta > 0$, and which satisfies (6.1.2) almost everywhere. This result readily entails the assertions.

c.2) After [90, cor. 2.1] we have : $\Psi_{a,b}(x(t), y(t)) \to \inf \Psi_{a,b}$, as $t \to +\infty$. The inequalities $\inf \frac{1}{b^2}\Phi = \inf \Psi_{a,b} \leq \frac{1}{b^2}\Phi(x(t)) \leq \Psi_{a,b}(x(t), y(t))$ then entail the asserted convergence result.

c.3) If argmin $\Phi \neq \emptyset$ then argmin $\Psi_{a,b} \neq \emptyset$. It is now a Theorem of Bruck [40] which asserts the weak convergence of (x, y) towards a minimum point $(x_\infty, y_\infty) = (x_\infty, -\frac{a}{b}x_\infty)$ of $\Psi_{a,b}$ as $t \to +\infty$.

c.4) If Φ is even, then so is $\Psi_{a,b}$. Resorting once more to a Theorem of [40] yields the strong convergence. ∎

To keep with clarity, let us briefly sum up how (DIN) has been derived.

By analogy with the proximal regularization method, the minimization of the convex function Φ is replaced by the minimization of the convex function $\Psi_{a,b}(x, y) = \frac{1}{b^2}\Phi(x) + \frac{1}{2}|ax + by|^2$.

To that end, the continuous steepest descent method is applied to $\Psi_{a,b}$, which gives rise to system (ESD). Any solution (x, y) of the latter is such that $x(t)$ weakly converges to a minimum point of Φ as $t \to +\infty$.

If Φ is $\mathcal{C}^2$, then (ESD) is equivalent to a (DIN) system with $\alpha\beta > 1$ ($\alpha = a^2 + b^2$ and $\beta = \frac{1}{b^2}$, indeed).

6.2 Optimizing properties of (DIN) in general

In this part, the optimizing properties of (DIN) are examined with more generality than before, *i.e.* Φ need not be convex and $\alpha\beta > 1$ need not hold. Facts are stated without proofs, which may be found in [7].

Let α, β, A, B, C be real constants, arbitrary for the moment. The system (DIN), which we recall

$$\text{(DIN)} \qquad \ddot{x}(t) + \alpha\dot{x}(t) + \beta\nabla^2\Phi(x(t))\dot{x}(t) + \nabla\Phi(x(t)) = 0$$

bears a strait relation with the following first order system

$$\text{(g-DIN)} \quad \begin{cases} \dot{x} + C\nabla\Phi(x) & +Ax + By = 0 \\ \dot{y} & +Ax + By = 0 \end{cases}$$

as the next proposition shows. In spite of their resemblance, (g-DIN) is not a gradient system (except if $A = B$) while (ESD) is. But the equivalence between (DIN) and (g-DIN) is more general than the equivalence between (DIN) and (ESD) which requires $\alpha\beta > 1$.

Proposition 6.2.1 *Suppose $\Phi \in \mathcal{C}^2(H)$, and let the constants α, β, A, B, C satisfy*

$$\beta \neq 0, \; A = \alpha - \frac{1}{\beta}, \; B = \frac{1}{\beta}, \; C = \beta.$$

The systems (DIN) and (g-DIN) are equivalent in the sense that x is a solution of (DIN) if and only if there exists $y \in \mathcal{C}^2([0, +\infty[, H)$ such that (x, y) is a solution of (g-DIN).

Beyond being of first order in time, the system (g-DIN) is interesting because it does not involve the Hessian of Φ. As a first consequence, the numerical solution of (DIN) is highly simplified, since it may be performed on (g-DIN) and only requires approximating the gradient of Φ. As a second consequence, (g-DIN) allows to give a sense to (DIN) when Φ is of class $\mathcal{C}^1$ only, or when Φ is nonsmooth or involves constraints, provided that a notion of generalized gradient is available (*e.g.* the subdifferential set for a convex function Φ). But that remark would be of little utility if (g-DIN) did not have good existence and asymptotic convergence properties as $t \to +\infty$, under the sole assumption $\Phi \in \mathcal{C}^1(H)$. Actually (g-DIN) retains some of the optimizing properties of (DIN), at least if $\Phi \in \mathcal{C}^{1,1}(H)$.

Theorem 6.2.1 *(optimizing properties of* (g-DIN)*)*
Assume that $\Phi : H \mapsto \mathbb{R}$ is bounded from below, differentiable with $\nabla\Phi$ Lipschitz continuous on the bounded subsets of H ; assume further $C > 0$, $B > 0$, $B + A > 0$ in (g-DIN)*. Then the following properties hold :*

- **(i)** *For each (x_0, y_0) in $H \times H$, there exists a unique solution (x, y) of* (g-DIN) *defined on the whole interval $[0, +\infty[$, which belongs to $\mathcal{C}^1(0, \infty; H) \times \mathcal{C}^2(0, \infty; H)$ and satisfies the initial conditions $x(0) = x_0$ and $y(0) = y_0$.*
- **(ii)** • *$\dot{x}$ and $\nabla\Phi(x)$ belong to $L^2(0, +\infty; H)$,*
  • *$\lim_{t\to+\infty} \Phi(x(t))$ exists,*
  • *$\lim_{t\to+\infty}(\dot{x}(t) + C\nabla\Phi(x(t)) = 0$.*
- **(iii)** *Assuming moreover that x is in $L^\infty(0, +\infty; H)$, then we have*
  • *$\dot{x}$, $\nabla\Phi(x)$ are bounded on $[0, +\infty[$,*
  • *$\lim_{t\to+\infty} \nabla\Phi(x(t)) = \lim_{t\to+\infty} \dot{x}(t) = 0$.*

In view of proposition 6.2.1, when Φ belongs to $\mathcal{C}^2(H)$, the conditions $C > 0$, $B > 0$, $B + A > 0$ for (g-DIN) are easily seen to be equivalent to $\alpha > 0$, $\beta > 0$ for (DIN). This readily implies the following corollary of Theorem 6.2.1.

Corollary 6.2.1 *(optimizing properties of* (DIN)*)*
Assume that $\Phi : H \mapsto \mathbb{R}$ is bounded from below, twice differentiable with $\nabla^2\Phi$ Lipschitz continuous on the bounded subsets of H ; assume further $\alpha > 0$, $\beta > 0$ in (DIN). *Then the following properties hold :*

(i) *For each $(x_0, \dot{x}_0)$ in $H \times H$, there exists a unique solution x of* (DIN) *defined on the whole interval $[0, +\infty[$, which belongs to $\mathcal{C}^2(0, \infty; H)$ and satisfies the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.*

(ii) • *$\dot{x}$ and $\nabla\Phi(x)$ belong to $L^2(0, +\infty; H)$,*
• $\lim_{t\to+\infty} \Phi(x(t))$ *exists,*
• $\lim_{t\to+\infty}(\dot{x}(t) + \beta\nabla\Phi x(t)) = 0$.

(iii) *Assuming moreover that x is in $L^\infty(0, +\infty; H)$, then we have*
• *$\dot{x}$, $\nabla\Phi(x)$ are bounded on $[0, +\infty[$,*
• $\lim_{t\to+\infty} \nabla\Phi(x(t)) = \lim_{t\to+\infty} \dot{x}(t) = 0$.

Let us finally state two convergence results ([7]).

Theorem 6.2.2 *In addition to the hypotheses of Theorem* 6.2.1, *assume that Φ is convex, and that* argmin Φ, *the set of minimizers of Φ on H, is nonempty. Then for any solution (x, y) of* (g-DIN), *$x(t)$ weakly converges to a minimizer of Φ on H as t goes to infinity.*

Theorem 6.2.3 *Assume that $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ is analytic, and let x be a bounded solution of* (DIN) *with $\alpha > 0$, $\beta > 0$. Then $\dot{x}$ belongs to $L^1(0, +\infty; H)$ and $x(t)$ converges towards a critical point of Φ as $t \to \infty$.*

6.3 An entropy-like version of the system (ESD)

From now on, H is assumed to be finite-dimensional, that is $H = \mathbb{R}^N$, $N \geq 1$.

A common feature in constrained optimization consists in replacing the quadratic kernel in the proximal point algorithm by a distance-like functional that forces, in good cases, the iterates to remain in the interior of the feasible set. If C is a non empty closed convex subset of $\mathbb{R}^N$, this leads to dynamics of the type

$$x^{k+1} \in \text{argmin}\left\{\Phi(x) + \lambda_k d(x, x^k) : x \in C\right\},\ \lambda_k > 0,$$

where $d : C \times C \to \mathbb{R} \cup \{+\infty\}$ is strictly convex with respect to its first variable. Let us mention, for instance, the comprehensive survey of Kiwiel [84] on generalized Bregman distances, the entropy-like algorithm using φ-divergences proposed in Iusem-Svaiter-Teboulle [79], and also the recent logarithmic quadratic method of Auslender-Ben Tiba-Teboulle [22].

Inspired by those fruitful ideas, and motivated by the properties of (DIN), we devote this section to the construction of an inertial method of the type (ESD), but with a φ divergence kernel - see formula (6.3.7) below- instead of the quadratic term $(x, y) \to \frac{1}{2}|ax + by|^2$.

The choice of this particular kernel is suggested by its remarkable jointly convex property, which naturally fits our energy-like descent method approach.

Let us now specify the setting. Consider the problem

$$(\mathcal{P}_+) \quad \inf\{\Phi(x) : x \in \mathbb{R}^N_+\},$$

where the objective function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is assumed to be lower semicontinuous and convex with

$$\text{dom } \Phi \cap \mathbb{R}^N_{++} \neq \emptyset, \ \ \mathbb{R}^N_{++} = \left\{x \in \mathbb{R}^N | x_i > 0, \ \forall i \in \{1..N\}\right\}. \tag{6.3.6}$$

φ divergences are generated by the functions $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ satisfying the following properties

$$(H)_\varphi \qquad \left\{ \begin{array}{l} (i)_\varphi \ \varphi \text{ is continuous and nonnegative on } \mathbb{R}_{++}, \\ (ii)_\varphi \ \varphi \text{ is strictly convex}, \\ (iii)_\varphi \ \varphi(1) = 0. \end{array} \right.$$

Define the φ divergence $d_\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ as

$$d_\varphi(x,y) = \left\{ \begin{array}{l} \sum_{i=1}^N y_i \varphi(y_i^{-1} x_i) \text{ if } (x,y) \in (\mathbb{R}^N_{++})^2, \\ +\infty \text{ elsewhere.} \end{array} \right. \tag{6.3.7}$$

Example. As in [79], where many other examples are given, a particularly interesting example is provided by

$$\varphi_0(s) = s \log s - s + 1, \ s \geq 0,$$

with the convention $0 \log 0 = 0$. The associated φ_0 divergence is the Kullback-Liebler entropy, that is

$$d_{\varphi_0}(x,y) = \sum_{i=1}^N x_i \log \frac{x_i}{y_i} + y_i - x_i, \ \forall (x,y) \in \mathbb{R}^N_+ \times \mathbb{R}^N_{++}.$$

It is worthwile pointing out that d_{φ_0} can also be viewed as the D function of the Bregman function $\mathbb{R}^N_+ \ni x \to \sum_{i=1..N} x_i \log x_i$. This has relevant consequences in the asymptotic analysis of the proximal-like dynamics associated to d_{φ_0}, [79, 20].

The φ divergence d_φ need not be lower semicontinuous. In order to meet the classical assumptions in minimization problems, we introduce the lower semicontinuous regularization of d_φ, denoted by $\overline{d_\varphi}$. It is characterized by the following properties

(a) For all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, and for all sequences satisfying $(x^k, y^k) \to (x,y)$ as $k \to +\infty$,

$$\liminf_{k \to +\infty} d_\varphi(x^k, y^k) \geq \overline{d_\varphi}(x,y),$$

(b) For all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, there exists a sequence satisfying $(x^k, y^k) \to (x,y)$ as $k \to +\infty$, such that

$$\limsup_{k \to +\infty} d_\varphi(x^k, y^k) \leq \overline{d_\varphi}(x,y).$$

We have the following

Lemma 6.3.1 *Let φ and d_φ satisfy $(H)_\varphi$ and (6.3.7). Then*

(i) $\overline{d_\varphi}$ is a proper, lower semicontinuous, convex function,

(ii) $\overline{d_\varphi} \geq 0$,

(iii) For all $(x,y) \in \mathbb{R}^N_{++} \times \mathbb{R}^N_{++}$, $\overline{d_\varphi}(x,y) = d_\varphi(x,y)$,

(iv) For all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, the following separation property holds, $\overline{d_\varphi}(x,y) = 0 \Leftrightarrow x = y \,, x \in \mathbb{R}^N_+$.

Proof. The convexity of d_φ, and therefore (i), comes from $(ii)_\varphi$ and the following fact :

$$g : R_{++} \to R \text{ is convex if and only if } (r,s) \in (\mathbb{R}_{++})^2 \to sg(s^{-1}r) \text{ is convex.}$$

One recognizes in $(r,s) \in (\mathbb{R}_{++})^2 \to sg(s^{-1}r)$ Hörmander's perspective function of g ; for a reference and further developments on the topic, see Maréchal [98]. The property (ii) follows from the fact that $d_\varphi \geq 0$, while (iii) is a consequence of $(i)_\varphi$.

To deal with (iv), let us examine the values of $\overline{d_\varphi}$. If $(x,y) \notin (\mathbb{R}^N_+)^2$ then, obviously, $\overline{d_\varphi}(x,y) = +\infty$ and by (iii) $\overline{d_\varphi}(x,y) = d_\varphi(x,y)$ as soon as $(x,y) \in (\mathbb{R}^N_{++})^2$.

To cope with the case $(x,y) \in \text{bd}\ (\mathbb{R}^N_+)^2$, where bd $(\mathbb{R}^N_+)^2$ denotes the boundary of $(\mathbb{R}^N_+)^2$, let us first notice that the definition of d_φ, allows to restrict the requirement (a) to nonnegative sequences. Besides, in order to compute $\liminf_{k\to+\infty} d_\varphi(x^k, y^k)$, where (x^k, y^k) is a nonnegative sequence, observe that the structure of d_φ permits to argue on *each* coordinates, and thus it can be assumed, without restriction, that $N = 1$.

For $(x,y) \in \text{bd}\ \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2_+ | xy = 0\}$, let (x^k, y^k) $x^k, y^k > 0$ be a sequence converging to (x,y) as $k \to +\infty$. Three cases are distinguished,

• $x = 0, y \neq 0$. From $(i)_\varphi$, $(ii)_\varphi$ and $(iii)_\varphi$ it ensues that φ is non increasing on $(0,1)$ and achieves its minimum at $s = 1$. Therefore $d_\varphi(x^k, y^k) \to y \lim_{s\to0^+} \varphi(s) > 0$, as $k \to +\infty$.

• $x \neq 0$ and $y = 0$. Fix $s_0 > 1$, and let us apply the convex inequality to φ, this gives for all $s \in \mathbb{R}_{++}$ and for all $g \in \partial\varphi(s_0)$

$$\varphi(s) \geq \varphi(s_0) + g.(s - s_0) \geq g.(s - s_0). \tag{6.3.8}$$

Observe that $(i)_\varphi$, $(ii)_\varphi$ and $(iii)_\varphi$ imply that all subgradients contained in $\partial\varphi(s_0)$ are positive. Hence (6.3.8) yields

$$d_\varphi(x^k, y^k) = y^k \varphi(\frac{x^k}{y^k}) \geq gx^k - gy^k s_0,$$

where $g \in \partial\varphi(s_0)$, $g > 0$, and thus $\liminf_{k\to+\infty} d_\varphi(x^k, y^k) \geq gx > 0$.

• $x = y = 0$. Just notice that $d_\varphi(\frac{1}{k}, \frac{1}{k}) = 0$ for all $k \geq 1$.

Applying the above results together with the properties $(i)_\varphi$, $(ii)_\varphi$, we easily deduce (iv). ■

Take d_φ as above and define for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\Psi_\varphi(x,y) = \Phi(x) + \overline{d_\varphi}(x,y). \tag{6.3.9}$$

By Lemma 6.3.1 and (6.3.6) this gives rise to a proper lower semicontinuous convex function. The optimality properties of Ψ_φ and Φ are linked in the following way :

Lemma 6.3.2 *Let φ, d_φ and Ψ_φ satisfy $(\mathcal{H}_\varphi)$, (6.3.7) and (6.3.9). Then*

$$\inf_{\mathbb{R}^N \times \mathbb{R}^N} \Psi_\varphi = \inf_{\mathbb{R}^N_+} \Phi$$
$$\text{argmin}\left\{\Psi_\varphi(x,y) : (x,y) \in \mathbb{R}^N \times \mathbb{R}^N\right\} = \left\{(x,x) : x \in \text{argmin}_{\mathbb{R}^N_+} \Phi\right\}.$$

Proof. It relies on Lemma 6.3.1, and on the relations

$$\Psi_\varphi(x,x)=\Phi(x) \qquad \forall x\in \mathbb{R}^N_+,$$
$$\Psi_\varphi(x,y)\geq \Phi(x) \qquad \forall (x,y)\in \mathbb{R}^N_+\times\mathbb{R}^N. \ \blacksquare$$

For each $x,y\in\mathbb{R}^N$ let us set $X=(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$. Following the lines of Section 6.2, let us define the dynamical system

$$(ESD)_\varphi \begin{cases} \dot{X}(t)+\partial\Psi_\varphi(X(t))\ni 0 \text{ a.e. on } [0,+\infty[\\ X(0)=X_0 \end{cases}$$

where $X_0=(x_0,y_0)\in\overline{\text{dom }\Psi_\varphi}$ and $X(.)$ is the unique continuous solution ([37]). The properties (6.3.6) and *(iii)* of Lemma 6.3.1 allow to apply to Ψ_φ a classical Theorem concerning the subdifferential of a sum ([58, th. 5.6]). Hence (ESD)$_\varphi$ can be rewritten

$$(ESD)_\varphi \begin{cases} \dot{x}(t)+\partial\Phi(x(t))+\partial_x d_\varphi(x(t),y(t))\ni 0 \text{ a.e. on } [0,+\infty[, \\ \dot{y}(t)+\partial_y d_\varphi(x(t),y(t))=0, \ \forall t\geq 0, \end{cases}$$

with $x_0\in\overline{\text{dom }\Phi}\cap\mathbb{R}^N_+$ and $y_0\in\mathbb{R}^N_+$.

The dynamical system $(ESD)_\varphi$ presents the advantage of taking the constraints $\mathbb{R}^N_+$ into account without penalizing Φ, more precisely we have the following

Theorem 6.3.1 *Let d_φ be a φ divergence, and Ψ_φ as in (6.3.9). Let $t\to X(t)=(x(t),y(t))$ be a solution of $(ESD)_\varphi$ then*

(i) $\Phi(x(t))\to\inf\left\{\Phi(x)|x\in\mathbb{R}^N_+\right\}$ as $t\to+\infty$.

(ii) If moreover $S_+=\text{argmin}\left\{\Phi(x)|x\in\mathbb{R}^N_+\right\}$ is non empty, then there exists $x^*\in S_+$ such that $(x(t),y(t))\to(x^*,x^*)$ as $t\to+\infty$.

Proof. It is a consequence of the previous Lemma and of the results proved in [90] for *(i)*, and in [37, 40] for *(ii)*. $\blacksquare$

Remarks. 1. Parallelizing the derivation of (ESD) and (ESD)$_\varphi$ from $\Psi_{a,b}$ and Ψ_φ, respectively, via the continuous gradient method, we could also derive a nonautonomous version of (ESD) by considering the following family of functions : $\Psi_t(x,y)=\frac{1}{b^2(t)}\Phi(x)+\frac{1}{2}|a(t)x+b(t)y|^2$, where a and b are positive functions of t. This would lead to the following differential inclusion : $(\dot{x}(t),\dot{y}(t))+\partial\Psi_t(x(t),y(t))\ni 0$ (see [25, 67] for facts about this type of problems).

2. It would be interesting to know if (ESD)$_\varphi$ is a dynamical interior point method. Indeed its numerical treatment may be delicate if a trajectory happens to touch the boundary of $(\mathbb{R}^N_{++})^2$, since $\bar{d}_\varphi$ is liable to singularity there ; choosing numerically good functions $\bar{d}_\varphi$ is not so easy. Yet we presume that, under fairly general assumptions on Φ and φ, each trajectory starting from $(x_0,y_0)\in(\text{dom }\Phi\cap\mathbb{R}^N_{++})\times\mathbb{R}^N_{++}$ remains in the interior of the constraints. Certainly this question deserves further study.

Bibliographie

[1] AASSILA M., *A new approach of strong stabilization of distributed systems*, Differential and Integral Equations **11**, n° 2 (1998), 369-376.

[2] AKIN E., "The geometry of population genetics", Lecture Notes in Biomathematics 31, Springer-Verlag, Berlin, 1979.

[3] ALBER,Y.I., IUSEM, A.N., and SOLODOV, M.V., *On the projected subgradient method for nonsmooth convex optimization in a Hilbert space*, Mathematical Programming, **81**, 23-25, 1998.

[4] ALVAREZ, F., *On the minimizing property of a second order dissipative system in Hilbert space*, SIAM J. of Control and Optimization, 38 (2000), N.4, pp. 1102-1119.

[5] ALVAREZ, F., ATTOUCH, H., *An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping*, Wellposedness in Optimization and related topics (Gargagno, 1999), Set Valued Analysis, 9 (2001), N.1-2, pp 3-11.

[6] ALVAREZ, F., ATTOUCH, H., *The heavy ball with friction dynamical system for convex constrained minimization problems*, Optimization (Namur, 1998), 25–35, Lecture Notes in Econom. and Math. Systems, **481**, Springer, Berlin, 2000.

[7] ALVAREZ F., ATTOUCH H., BOLTE J., REDONT P., *A second-order gradient-like dissipative dynamical system with Hessian driven damping*, Journal de Mathématiques Pures et Appliquées, to appear.

[8] ALVAREZ, F., PÉREZ, J.M., *A dynamical system associated with Newton's method for parametric approximations of convex minimization problems*, Appl. Math. Optm., 38 (1998), N.2, pp. 193- 217.

[9] ANTIPIN A.S., *Linearization method*, in Nonlinear dynamic systems : Qualitative analysis and control (in Russian), *Proc. ISA Russ. Acad. Sci.*, n° 2, pp. 4-20, 1994 ; (english translation : Computational Mathematics and Modeling, n° 1, **8**, pp. 1-15, Jan. 1997).

[10] ANTIPIN, A.S., *Minimization of convex functions on convex sets by means of differential equations*, Differential Equations, 30 (1994), pp.

[11] ANTIPIN A.S., NEDICH A., *Continuous second-order linearization method for convex programming problems*, Computational Mathematics and Cybernetics, Moscow University, **2**, pp. 1-9, 1996.

[12] ATTOUCH, H., "Variational convergence for Functions and Operators", Pitman Advance Publishing Program, 1984.

[13] ATTOUCH, H., *Viscosity solutions of minimization problems*, SIAM J.Optim, 6 (1996), N.3, pp 769-806.

[14] ATTOUCH H., CABOT A., REDONT P., *The dynamics of elastic shocks via epigraphical regularization of a differential inclusion. Barrier and penalty approximations*, to appear in Advances in Mathematical Sciences and Applications, **12**, n° 1 (2002).

[15] ATTOUCH H., COMINETTI R., *A dynamical approach to convex minimization coupling approximation with the steepest descent method*, J. Differential Equations, **128** (2), (1996), 519-540.

[16] A. Auslender, R. Cominetti and M. Haddou, *Asymptotic analysis for penalty and barrier methods in convex and linear programming*, Math. Oper. Res., 22 (1997), pp. 43-62.

[17] ATTOUCH, H., and CZARNECKI, M.O., *Asymptotic control and stabilization of nonlinear oscillators with non isolated equilibria*, to appear in J. of Differential Equations, 2001.

[18] ATTOUCH H., GOUDOU X., REDONT P., *The heavy ball with friction method, I. The continuous dynamical system : global exploration of the global minima of a real-valued function by asymptotic analysis of a dissipative dynamical system*, Communications in Contemporary Mathematics, **2**, n° 1 (2000), 1-34.

[19] ATTOUCH H., REDONT P., *The second-order in time continuous Newton method*, Approximation Optimization and Mathematical Economics, M. Lassonde ed., Proceedings of the Fifth International Conference on Approximation and Optimization in the Caribbean, Physica-Verlag, 2001, 25-36.

[20] ATTOUCH, H., TEBOULLE, M., *A regularized Lotka Volterra dynamical system as a continuous proximal-like method in optimization*, December 2001. Submitted for publication.

[21] AUBIN, J.P., CELLINA, A., "Differential inclusions", Springer Verlag, 1984.

[22] AUSLENDER, A., TEBOULLE, M., BEN-TIBA, S., *Interior proximal and multiplier method based on second order homogeneous kernels*, Mathematics of Operation Research, 24 (1999), pp 645-668.

[23] AUSLENDER A., COMINETTI R., HADDOU M., *Asymptotic analysis for penalty and barrier methods in convex and linear programming*, Mathematics of Operations Research, 22 (1997), N.1, pp 43-62.

[24] BAILLON J.-B., *Un exemple concernant le comportement asymptotique de la solution du problème* $du/dt + \partial\varphi(u) \ni 0$, Journal of Functional Analysis, **28**, 369-376 (1978).

[25] BAILLON J.-B., COMINETTI R., *A Convergence Result for Subgradient Evolution Equations and its Application to the Steepest Descent Exponential Penalty Trajectory in Linear Programming*, Journal of Functional Analysis, **187**, 2001, 263-273.

[26] BAILLON J.-B., HADDAD G., *Quelques propriétés des opérateurs angle-bornés et n-cycliquement monotones*, Israel J. of Mathematiques, 26 (1977), pp 137-150.

[27] BARBU V., *Necessary conditions for distributed control problems governed by parabolic variational inequalities*, SIAM J. on Control and Optimization, **19**(1) (1981), 64-86.

[28] Bauschke, H.H., Borwein J.M., *Legendre functions and the method of random projections,* J. Convex An., 4(1), 1997, pp 27-67.

[29] Bauschke, H.H., Borwein J.M., and Combettes P.L., *Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces*, *Comm. in Comtemp. Math.*, N.4, 3(2001), pp 615-647.

[30] Bayer, D., A., Lagarias, J.C., *The nonlinear geometry of linear programming : Part I, Affine and projective scaling trajectories; Part II, Legendre Transform coordinates and central trajectories*, Trans. of AMS, 314, (1989), pp 499-526 and pp 527-581.

[31] Bénilan P., *Equations d'évolutions dans un espace de Banach quelconque et applications,* Thèse d'état, Université Paris-Sud, 1972.

[32] Bénilan P., Crandall M.G., Pazy A., "Nonlinear evolution equations in Banach spaces", Preprint book.

[33] Blanchard, D., and Francfort G., *A few results on a class of degenerate parabolic equations*, Ann. Scu. Norm. Pisa, 8 2, (1991), pp 213-249.

[34] Bolte, J., *On the continuous gradient projection method in Hilbert spaces*, (2001), to appear in J. of Opt. Theory and Applications.

[35] Bolte J., Teboulle M., *Barrier operators and associated gradient-like dynamical systems for constrained minimization problems*, submitted (June 2002).

[36] Bregman L.M, *The relaxation method for finding the common point of convex sets and its application to the solution of problems in convex programming*, Zh. Vychisl. Mat. i Mat. Fiz., 7 (1967), pp. 620-631 (in Russian). English transl. in U.S.S.R. Comput. Math. and Math. Phys., Vol. 7 (1967), pp. 200-217.

[37] Brézis, H., "Opérateurs maximaux monotones", Lecture notes N.5, North Holland, 1973.

[38] Brockett R.W., *Dynamical systems that sort lists and solve linear programming problems*, Proc. IEEE Conf. Decision and Control, Austin, Texas, 1988, pp. 779-803.

[39] Brockett R.W., *Dynamical systems that sort lists, diagonalize matrices and solve linear programming problems*, Linear Alg. Appl., 146 (1991), pp. 79-91.

[40] Bruck, R.E., *Asymptotic convergence of nonlinear contraction semi-groups in Hilbert space*, Journal of Functional Analysis, 18 (1974), pp. 15-26.

[41] Cabot A., "Etude de quelques systèmes dissipatifs. Application à l'Optimisation et à la Mécanique. Stabilisation de systèmes oscillants avec équilibres multiples", Thèse (2002), Université de Montpellier II.

[42] Censor, Y., Eggermont, P.P.B., and Gordon, D., *Strong underrelaxation in Kaczmarz's method for inconsistent systems*, Numerische Mathematik, **41**, 83-92, 1983.

[43] Censor, Y., Lent, A., *An iterative row-action method for interval convex programming,* J. Opt. Theory and Applications, 34 (1981), N.3, pp. 321-353.

[44] Censor, Y., Zenios, S., *The proximal minimization algorithm with D-functions*, J. of Optim. Theory and Applications, 73, (1992), pp. 451-464.

[45] CHEN, G., TEBOULLE, M., *Convergence analysis of a proximal-like minimization algorithm using Bregman functions*, SIAM J. Optimization, 3, (1993), pp. 538-543.

[46] COLLI, P., and VISINTIN, A., *On a class of doubly nonlinear evolution problems*, Communications in Partial Differential Equations, **15**, 737-756, 1990.

[47] COMINETTI R., *Coupling the proximal point algorithm with approximation methods*, J. Optim. Theory Appl., **95** (1997), n° 3, 581-600.

[48] COMINETTI R., *Nonlinear average and convergence of penalty trajectories in convex programming*, Ill-posed variational problems and regularization techniques (Trier, 1998), 65-78, Lecture notes in Econom. and Math systems, 477, Springer, Berlin, 1999.

[49] COMINETTI R., SAN MARTÍN J., *Asymptotic Analysis of the Exponential Penalty Trajectory in Linear Programming*, Math. Programming, Vol. 67 (1994), pp. 169-187.

[50] CSIZÁR I., *Why least squares and maximum entropy ? An axiomatic approach to inference for linear inverse problems*, Ann. Statist., 19(1991), N4, pp 2032-2066.

[51] DAMLAMIAN A., *Some results on the multiphase Stefan problem*, Comm. P.D.E., 2(1977), pp 1017-144.

[52] DAMLAMIAN A., and KENMOCHI, N., *Evolution equations generated by subdifferentials in the dual space of $H^1(\Omega)$*, Discrete and Continuous Dynamical Systems, 5, N.2, (1999), pp 269-278.

[53] DEN HERTOG D., ROOS C., TERLAKY T., *Inverse Barrier Method for Linear Programming*, Report 91-27, Faculty of Technical Mathematics and Informatics, Delft University of Technology, The Netherlands (1991).

[54] DO CARMO M.P., "Riemannian Geometry" (Mathematics, Theory and Applications), Birkhäuser, 1992.

[55] DONTCHEV A.L., T. ZOLEZZI T., *Well-Posed optimization problems*, Lectures Notes in Mathematics, **1543**, Springer, 1993.

[56] DUISTERMAAT J.J., *On Hessian Riemannian structures*, Asian J. Math., 5 (2001), no 1, 79-91.

[57] DUNN, J.C., *Global and asymptotic convergence rate estimates for a class of projected gradient processes*, SIAM J. on Control, 19, (1981), pp. 368-400.

[58] EKELAND I., TEMAM R., "Analyse convexe et problèmes variationnels", Dunod,Paris, 1973.

[59] ECKSTEIN, J., *Nonlinear proximal point algorithms using Bregman functions, with application to convex programming,* Mathematics of Operations Research, 18 (1993), N.1, pp. 202-226.

[60] ERMOLIEV, Y., M., *On the method of generalized stochastic gradients and Quasi-Fejer sequences*, Cybernetics, 5, (1969), pp. 208-220.

[61] FAYBUSOVICH L.E., *Dynamical systems which solve optimization problems with linear constraints*, IMA J. Math. Control and Inf., Vol. 8 (1991), pp. 135-149.

[62] FAYBUSOVICH L.E., *Hamiltonian structure of dynamical systems which solve linear programming problems*, Physica, Vol. D 53 (1991), pp. 217-232.

[63] FAYBUSOVICH L.E., *Interior point methods and entropy*, Proc. IEEE Conf. Decision and Control, Tucson, Arizona (1992), pp. 1626-1631.

[64] FIACCO A.V, *Perturbed variations of penalty function methods. Example : Projective SUMT*, Annals of Operations Research 27, (1990) 371-380.

[65] FRISCH K.R., *The Logarithmic Potential Method of Convex Programming*, Memorandum of May 13, 1955, University Institute of Economics, Oslo.

[66] FLAM S., . HORVATH CH., *Network games; adaptations to Nash-Cournot equilibrium*, Annals of Operations Research, **64** (1996), 179-195.

[67] FURUYA H., MIYASHIBA K., KENMOCHI N., *Asymptotic Behavior of Solutions to a Class of Nonlinear Evolution Equations*, Journal of Differential Equations, **62**, 1986, 73-94.

[68] GOUDOU, X., *The heavy ball with friction system for a smooth quasi-convex function*, working paper, Université Montpellier II, France.

[69] HALE J.K., "Asymptotic behavior of dissipative systems", Mathematical Surveys and Monographs **25**, A.M.S., Providence, R. I. (1987).

[70] HARAUX A., *Semilinear hyperbolic problems in bounded domains*, Mathematical Reports (J. Dieudonné ed.) **3**, Part I, Harwood Academic, Reading, UK, 1987.

[71] HARAUX A., "Systèmes dynamiques dissipatifs et applications", RMA 17, Masson, Paris, (1991).

[72] HARAUX A., M. JENDOUBI M., *Convergence of solutions of second-order gradient-like systems with analytic nonlinearities*, Journal of Differential Equations, **144**, n° 2 (1998).

[73] HELMKE, U., MOORE, J.B., "Optimization and Dynamical Systems", Springer-Verlag, London, 1994.

[74] HERZEL S., RECCHINI M.C. AND ZIRILLI F., *A quadratically convergent method for linear programming*, Linear Algebra Appl., Vol. 151 (1991), pp. 255-290.

[75] HIRIART-URRUTY J.-B., LEMARÉCHAL C., "Convex Analysis and Minimization Algorithms II : Advanced Theory and Bundle Methods", Springer-Verlag, Berlin, 1993.

[76] HOFBAUER, J., SIGMUND, K., "The Theory of Evolution and Dynamical Systems", Cambridge University Press, 1988.

[77] IUSEM A.N., MONTEIRO R.D.C., *On Dual Convergence of the Generalized Proximal Point Method with Bregman Distances*, Mathematics of Operations Research, Vol. 25, No. 4 (2000), pp. 606-624.

[78] IUSEM A.N., SVAITER B.F., DA CRUZ J.X., *Central paths, generalized proximal point methods, and Cauchy trajectories in Riemannian manifolds*, SIAM J.Control Optim, Vol.37, N.2 (1999), pp. 566-588.

[79] IUSEM A.N, SVAITER B.F., TEBOULLE M., *Entropy-like proximal methods in convex programming*, Mathematics of Operation Research, **19**, 4, 790-814, 1994.

[80] JENDOUBI M., *Convergence of bounded and global solutions of the wave equation with linear dissipation and analytic nonlinearity*, J. Differential Equations, **144** (1998), 302-312.

[81] KARMARKAR N., *A new polynomial time algorithm for linear programming*, Combinatorica 4 (1984), pp. 373-395.

[82] KARMARKAR N., *Riemannian geometry underlying interior point methods for linear programming*, in "Mathematical Developments Arising from Linear Programming", Contemporary Mathematics 114, J.C. Lagarias and M.J. Todd (eds.), AMS, Providence, RI, 1990, pp. 51-76.

[83] KENMOCHI, N., PAWLOW, I., *A class of doubly nonlinear elliptic-parabolic equations with time dependent constraints*, Nonlinear Analysis, 10 (1986),pp 1181-1202.

[84] KIWIEL, K.C., *Proximal mimization methods with generalized Bregman functions*, SIAM J. of Control and Optimization, 35 (1997), N.4, pp. 1142-1168.

[85] KIWIEL, K.C., *Free-steering relaxation methods for problems with strictly convex costs,*, M. of Operations Research. , 22 (1997), N.2, pp. 326-349.

[86] KIWIEL, K.C., and MURTY, K., *Convergence of the steepest descent method for minimizing quasiconvex functions*, J. of Optimization Theory and Applications, **vol.89**, N.1, 221-226, 1996.

[87] KUMMER B., *Generalized Newton and NCP Methods : convergence, regularity, actions*, Discussiones Mathematicae, Differential Inclusions, Control and Optimization, **20** (2000), 209-244.

[88] LANG, S., "Differential and Riemannian Manifolds", Springer-Verlag, New-York, 1995.

[89] LEMAIRE, B., *An asymptotical variational principle associated with the steepest descent method for a convex function*, J.Convex Analysis, 3, 1996, pp. 63-70.

[90] LEMAIRE, B., *The proximal point algorithm*, International series of numerical mathematics 1987 (J.P. Penot, Ed.), Birkhauser, Verlag, Basel, pp. 73-87.

[91] LEVITIN, E. S., POLYAK, B.T., *Constrained minimization methods*, USSR Comp. Math. and Phys., 6 (1965), pp. 787-823.

[92] LOJASIEWICZ, S., *Une propriété topologique des sous-ensembles analytiques réels*, Les équations aux dérivées partielles (Paris 1962), pp. 87-89, Editions du centre National de la Recherche Scientifique, Paris.

[93] LOJASIEWICZ, *Une propriété topologique des sous-ensembles analytiques réels, in Colloques Internationaux du C.N.R.S. Les équations aux dérivées partielles*, **117** (1963).

[94] LOJASIEWICZ S., *Ensembles semi-analytiques*, I.H.E.S. Notes, (1965).

[95] LIONS J.-L., *Contrôle des systèmes distribués singuliers*, Méthodes Mathématiques de l'Informatique **13**, Gauthier-Villars, Paris, 1983.

[96] MAC CORMICK G.P., *The continuous Projective SUMT method for convex programming*, Math. Op. Res. 14(2), (1989), 203-223.

[97] MAC CORMICK, G.P, and TAPIA, R.A., *The gradient projection method under mild differentiability conditions*, SIAM J. of control and Optimization, **vol. 10**, 93-98, 1972.

[98] MARÉCHAL P., *On the convexity of the multiplicative potential and penalty functions and related topics*, Mathematical Programming, Ser. A **89**, 505-516, 2001.

[99] MARTINET B., *Détermination approchée d'un point fixe d'une application pseudo-contractante. Cas de l'application prox.* Comptes-Rendus de l'Académie des Sciences de Paris, **274**, (1972), 163-165.

[100] MARTINET, B., *Perturbation des méthodes d'optimisation, application*, R.A.I.R.O, Analyse numérique, 12 (1978), pp. 153-171.

[101] MARTINET B., *Régularisation d'inéquations variationnelles par approximations succesives*, RAIRO Rech. Opér., Vol. 4 (1970), pp. 154-158.

[102] MORDUKHOVICH B., OUTRATA J., *On second-order subdifferentials and their applications*, SIAM J. Optim., **12**, N° 1, pp. 139-169.

[103] MOREAU J.-J, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France, **93**, (1965), 273-299.

[104] NATANSON I.P., "Theory of functions of a real variable", Frederick Ungar, New york, 1955.

[105] OPIAL, Z., *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. of the American Mathematical Society, 73 (1967), pp 591-597.

[106] PALIS J., DE MELO W., "Geometric theory of dynamical systems", Springer, 1982.

[107] PAOLI L., SCHATZMAN M., *Mouvement à un nombre fini de degrés de liberté avec contraintes unilatérales : cas avec perte d'énergie*, Mod. Math. Anal. Num., **27** (1993), 673-717.

[108] PAOLI L., SCHATZMAN M., *Ill-posedness in vibro-impact and its numerical consequences*, ECOMAS conference 2000, Proceedings, to appear.

[109] PHELPS, R.R., *The gradient projection method using Curry's steplength*, SIAM J. of control and Optimization, **vol. 24**, N.4, 692-699, 1986.

[110] PHELPS, R.R., *Metric projections and the gradient projection method in Banach spaces*, SIAM J. of control and Optimization, **vol. 23**, 973-977, 1985.

[111] POLYAK B.T., "Introduction to Optimization", Optimization Software, New York, 1987.

[112] ROCKAFELLAR, R.T., "Convex Analysis", Princeton University Press, 1970.

[113] ROCKAFELLAR R.T., *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, **14**, n° 5 (1976), 877-898.

[114] ROCKFELLAR R.T., WETS R., "Variational Analysis", Springer, Berlin, 1998.

[115] SCHATZMAN, M., *A class of nonlinear differential equations of second order in time*, Nonlinear Analysis, **vol.2**, 355-373, 1978.

[116] SIMON L., *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math., **118**, (1983), 525-571.

[117] SMALE S., *A convergent process of price adjustment and global Newton methods*, Journal of Mathematical Economics, **3** (1976), 107-120.

[118] SONNEVEND G., *An "analytical centre" for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming*, Proc. 12th IFIP Conf. System Modelling, Budapest, 1985, Lecture Notes in Computer Science, 1986.

[119] TEBOULLE M., *Entropic proximal mappings with applications to nonlinear programming,* Mathematics of Operations Research, 17 (1992), N.3, pp. 670-690.

[120] TEMAM R., "Infinite dimensional dynamical systems in mechanics and physics", Applied mathematical sciences, **68**, Springer, 1997.

Résumé.
L'étude et l'introduction de nouveaux systèmes dynamiques de type gradient sont l'objet central de cette thèse. Le caractère dissipatif de telles dynamiques est au coeur de nombreux domaines en mathématiques : optimisation, mécanique, équations d'évolutions en dimension infinie.

Dans une première partie, les champs de gradients (ou de sous-différentiels de fonction convexe) sont contrôlés à l'aide d'*opérateurs-barrières*. La motivation essentielle est d'obtenir des méthodes *intérieures de descente* en vue d'optimiser une fonction sous des contraintes convexes. Le cadre d'étude proposé permet d'unifier dans un même formalisme de nombreuses méthodes continues. Toujours dans cette perspective, les fonctions de Legendre jouent un rôle crucial : elles permettent d'une part de donner lieu à des structures riemanniennes possédant de nombreuses propriétés, et d'autre part, elles fournissent en dimension infinie un cadre intéressant pour l'étude de certaines équations d'évolution de type parabolique.

La deuxième partie est consacrée à l'étude de systèmes dynamiques du second ordre en temps avec une dissipation géométrique de type hessien. Outre leur intérêt en optimisation et leurs liens avec les méthodes de type Newton, ces systèmes sont d'une grande souplesse et permettent d'approcher certains phénomènes non-lisses en mécanique unilatérale.

L'une des préoccupations majeures de cette thèse est la question de la convergence des orbites des sytèmes étudiés. Dans le cadre de la minimisation convexe, quasi-convexe, ou analytique, de nombreux résultats sont proposés.

Mots-clés : système dynamique dissipatif, système de type gradient, analyse asymptotique, méthode de Newton continue, minimisation convexe, chocs inélastiques, gradient-projeté, fonctions de Legendre, opérateurs barrières, équations paraboliques, métriques hessiennes, fonctions de Lyapounov, méthodes proximales.

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