Renormalisation perturbative et T-dualite - Nouvelles metriques d'Einstein et super-espace harmonique

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Part I

T-dualité

Chapter 1

Introduction à la T-dualité non abelienne

La T-dualité, ou dualité sur espace cible, est un sujet qui a été très étudié au cours de ces dernières années, comme en témoigne la très vaste littérature qui y est consacrée. De nombreux articles couvrent aussi bien la dualité abelienne que la dualité non-abelienne, plus récente, ainsi que ses applications à la théorie des cordes et à la physique statistique [AAGBL94, AAGL95, GPR94, Bus87, Bus88]. La T-dualité fournit ainsi une équivalence entre différentes théories des cordes. Ses aspects géométriques sont décrits par O. Alvarez dans [Alv00a, Alv00b]. Découvert initialement pour des modèles sigma possédant une isométrie abelienne, le concept de T-dualité a ensuite été étendu à des théories non-abeliennes [AAGL94a, Loz95, dlOQ93, Sfe96]. L'étude que nous en avons faite se situe strictement dans un cadre de théorie des champs et de renormalisation perturbative.

Afin de cerner la problématique et donner un état des lieux de la T-dualité non-abelienne, nous allons tout d'abord étudier un exemple simple, aisément généralisable, de T-dualisation : celui du modèle sigma $\dfrac{SU(2)\times SU(2)}{SU(2)}$.

1.1 Les modèles sigma avec torsion

De façon générale, nous étudierons différentes théories qui seront toutes des modèles sigma. Ceux-ci sont définis par une action pouvant s'écrire sous la forme suivante :

$$S[\phi] = \frac{1}{2T}\int d^2x\ [g_{ij}\,\eta^{\mu\nu} + h_{ij}\,\epsilon^{\mu\nu}]\,\partial_\mu\phi^i\partial_\nu\phi^j\ .$$

Les variables ϕ^i sont les champs de notre théorie et T est la constante de couplage. Les champs agissent sur un espace bidimensionnel Minkowskien défini par $\eta^{\mu\nu}$ et prennent leurs valeurs sur une variété Riemannienne M appelée espace cible et munie de la métrique g_{ij}. Les modèles sigma obtenus par T-dualité seront en outre très souvent dotés d'une torsion, définie par le potentiel de torsion h_{ij}. A ce potentiel, on associe le tenseur de torsion T_{ijk} :

$$T_{ijk} = \frac{3}{2} \partial_{[i} h_{jk]} \ .$$

Le potentiel de torsion, antisymétrique, n'est défini qu'à un rotationnel près. Sa présence aura de nombreuses conséquences, notamment sur les conditions de renormalisabilité des théories que nous étudierons car il détruit par exemple la symétrie du tenseur de Ricci, entraînant ainsi des équations de champ plus nombreuses et plus complexes. Nous donnons dans l'Annexe A les conventions utilisées pour la connexion de Levi-Civita, la connexion avec torsion, les tenseurs de Riemann et de Ricci, *etc...*

1.2 Un exemple fondamental : le modèle sigma SU(2)

Le plus simple des modèles que nous étudierons est celui du modèle principal chiral $SU(2)$. Historiquement, c'est à partir de lui que la T-dualité non-abelienne a été introduite. Nous le rappelons ici brièvement.

Le groupe $SU(2)$ peut être représenté par les matrices 2×2 unitaires de déterminant un. L'algèbre de Lie $su(2)$ associée est formée des matrices antihermitiennes 2×2 de trace nulle dont une base est l'ensemble $\{\frac{\tau_i}{2\,i}\}$ où le triplet $\vec{\tau}$ est formé des matrices de Pauli. Les relations de commutation et d'anti-commutation correspondantes sont donc :

$$\left[\frac{\tau_i}{2\,i}, \frac{\tau_j}{2\,i}\right] = \epsilon_{ijk} \frac{\tau_k}{2\,i} \,, \qquad \left\{\frac{\tau_i}{2\,i}, \frac{\tau_j}{2\,i}\right\} = -\frac{1}{2} \delta_{ij} \mathcal{I} \,.$$

L'action associée au modèle principal chiral $SU(2)$ s'écrit

$$S = \frac{1}{T} \int d^2x \, \mathrm{Tr} \left[\partial_\mu U^{-1}(x) \, \partial^\mu U(x)\right] \ , \qquad U \in SU(2) \,, \tag{1.1}$$

et fournit, grâce aux équations du mouvement

$$\partial_\mu \left(U^{-1}(x) \partial^\mu U(x)\right) = 0 \,, \tag{1.2}$$

un courant conservé $J_\mu = U^{-1}(x) \partial_\mu U(x)$. Ce courant appartient à l'algèbre de Lie $su(2)$ et peut donc s'écrire sous la forme $J_\mu = \vec{J}_\mu . \frac{\vec{\tau}}{2\,i}$. L'action devient alors, en fonction des courants,

$$S = \frac{1}{2\,T} \int d^2x \, \vec{J}_\mu . \vec{J}^\mu \ . \tag{1.3}$$

En choisissant d'écrire U sous la forme $U = \rho \mathcal{I} + i \, \vec{\pi}.\vec{\tau}$ où $\rho^2 + \vec{\pi}^2 = 1$, on peut développer le courant $\frac{1}{2} \vec{J}_\mu = \vec{\pi} \, \partial_\mu \rho - \rho \, \partial_\mu \vec{\pi} - \vec{\pi} \wedge \partial_\mu \vec{\pi}$ en fonction des champs $\vec{\pi}$. On obtient alors l'action (1.1) sous la forme d'un modèle sigma sans torsion :

$$S = \frac{1}{2\,T} \int d^2x \, (4\, g_{ij}) \, \partial_\mu \pi^i \, \partial_\mu \pi^j \ , \qquad g_{ij} = \delta_{ij} + \frac{\pi^i \, \pi^j}{1 - \vec{\pi}^2} \ . \tag{1.4}$$

Le tenseur g_{ij} est le tenseur métrique de la sphère S^3 et la théorie associée à la métrique $(4\,g_{ij})$ est donc à la fois renormalisable à tous les ordres et asymptotiquement libre. Dans le cadre de la renormalisation dimensionnelle en dimension $d = 2 - \varepsilon$, on obtient alors pour l'action renormalisée à une boucle, au premier ordre en $1/\varepsilon$:

$$S_R = \frac{1}{2\,T} \int d^{2-\varepsilon}x \, \left[1 + \frac{\hbar T}{4\,\pi\,\varepsilon}\right] \, (4\,g_{ij})\, \partial_\mu \pi^i \, \partial_\mu \pi^j \ .$$

On en déduit alors la fonction β associée à la constante de couplage : $\beta_T = -\dfrac{T^2}{4\,\pi}$.

Notons par ailleurs que la métrique est Einstein, le tenseur de Ricci étant proportionnel à la métrique :

$$ric_{ij} = \lambda\,(4\,g_{ij})\,, \qquad \text{où } \lambda = \frac{1}{2}.$$

Nous concluons ce rappel par une description des symétries. De par la forme de l'action (1.1), on voit qu'il est possible de définir une action à gauche et une action à droite :

- L'action à gauche est définie par la transformation infinitésimale de paramètre $\vec{\epsilon}_L : U \longrightarrow g\,U$ où $(g - \mathcal{I}) \in su(2)$, $g = \mathcal{I} + \vec{\epsilon}_L . \dfrac{\vec{\tau}}{2\,i}$. Cette transformation laisse le lagrangien inchangé et on peut lui associer le courant de Noether $\vec{K}_\mu$ tel que $\vec{K}_\mu . \dfrac{\vec{\tau}}{2\,i} = \partial_\mu U\, U^{-1}$ et $\frac{1}{2}\,\vec{K}_\mu = \vec{\pi}\,\partial_\mu \rho - \rho\,\partial_\mu \vec{\pi} + \vec{\pi} \wedge \partial_\mu \vec{\pi}$. Remarquons que cette transformation laisse le courant J_μ invariant ($\delta \vec{J}_\mu = 0$) et que le lagrangien peut aussi s'écrire sous la forme : $\mathcal{L} = \frac{1}{2} \vec{K}_\mu \vec{K}^\mu$.
- De la même façon, on définit une action à droite de paramètre $\vec{\epsilon}_R$ par $U \longrightarrow U\,d$ où $(d - \mathcal{I}) \in su(2)$ et $d = \mathcal{I} + \vec{\epsilon}_R . \dfrac{\vec{\tau}}{2\,i}$. Le courant de Noether associé est maintenant $\vec{J}_\mu$, et la transformation est telle que $\delta \vec{K}_\mu = 0$.

Dans cette section, nous avons vu qu'il existe deux façons de définir le modèle principal chiral $SU(2)$:

- par l'action (1.1) où interviennent les éléments U du groupe $SU(2)$, définis ainsi par trois paramètres (le champ $\vec{\pi}$ dans la paramétrisation historique).
- par l'action "courant-courant" (1.3) où apparaissent des éléments de l'algèbre $su(2)$ via les courants $\vec{J}_\mu$. Ces courants étant définis par six composantes, ils doivent vérifier trois contraintes afin qu'il n'y ait véritablement que trois degrés de liberté.

Ces contraintes, nécessaires pour définir notre modèle chiral à partir de l'action "courant-courant" (1.3), doivent traduire l'appartenance des matrices U au groupe $SU(2)$: elles sont données par l'identité de Bianchi

$$M^i_{\mu\nu}(J) \equiv \partial_\mu J^i_\nu - \partial_\nu J^i_\mu + \epsilon_{ijk}\, J^j_\mu J^k_\nu = 0 \iff \epsilon^{\mu\nu} M^i_{\mu\nu}(J) = 0 \ . \qquad (1.5)$$

Afin d'utiliser l'action (1.3), cette identité est supposée être toujours vérifiée algébriquement, sans l'aide des équations du mouvement. On peut alors se poser la question de l'existence d'une nouvelle théorie où les rôles respectifs des identités de Bianchi et des équations du mouvement seraient inversés.

1.3 Une théorie duale "naïve"

Cette première construction d'une nouvelle théorie à partir du modèle $SU(2)$ a été élaborée par Chiara R. Nappi [Nap80] en 1980. Elle consiste à trouver

- une redéfinition de J_μ telle que (1.2) soit toujours vérifié,
- un lagrangien dont les équations du mouvement redonnerait (1.5).

Pour cela, Nappi propose d'écrire le courant sous la forme

$$J_\mu = \epsilon_{\mu\nu}\, \partial^\nu \psi \text{ où } \psi \in su(2) \; .$$

La conservation de J_μ est alors évidente, et l'identité de Bianchi (1.5) se réécrit

$$\partial^2 \psi - \frac{\epsilon^{\mu\nu}}{2} \left[\partial_\mu \psi, \partial_\nu \psi\right] = 0 \; .$$

Cette dernière équation est l'équation du mouvement fournie par le lagrangien

$$\mathcal{L}_{\text{naïf}} = -\frac{1}{T} \text{Tr} \left(\partial_\mu \psi \, \partial^\mu \psi + \frac{1}{3} \epsilon^{\mu\nu} \, \psi \, \left[\partial^\mu \psi, \partial^\nu \psi\right] \right) \; .$$

Si l'on écrit maintenant ψ sous la forme $\psi = \vec{\psi}.\frac{\vec{\tau}}{2\, i}$, on trouve l'action correspondante à $\mathcal{L}_{\text{naïf}}$ sous la forme d'un modèle sigma :

$$S_{\text{naïve}} = \frac{1}{2\,T} \int d^2x \left[\delta_{ij} \eta^{\mu\nu} + \frac{1}{3} \epsilon_{ijk}\, \psi^k \epsilon^{\mu\nu} \right] \partial_\mu \psi^i \partial_\nu \psi^j \; .$$

Cette théorie redonne bien le même système {Equations du mouvements + Identité de Bianchi}, et d'une façon inversée par rapport à la théorie initiale. Cependant, ce processus de "dualisation" ne peut être formulé sous la forme d'une transformation canonique et l'on ne peut dons pas parler d' "équivalence classique" entre les deux théories. Par ailleurs, bien qu'homogène et donc renormalisable à tous les ordres [B$^+$88], le couplage ne se renormalise plus de la même manière : les fonctions β sont différentes d'un modèle à l'autre. En effet, ce qui va généralement (*cf.* Annexe A) renormaliser la constante de couplage T, c'est le facteur de proportionnalité λ – la constante cosmologique – entre le tenseur de Ricci et la métrique. On a alors, pour la fonction β :

$$ric_{ij} = \lambda\, g_{ij} \qquad \Longrightarrow \qquad \beta_T = -\frac{\lambda\, T^2}{2\,\pi} \; .$$

Pour notre théorie duale "naïve", on a $\lambda = -\frac{1}{2}$ alors que dans le modèle de départ on avait $\lambda = \frac{1}{2}$. Notons que le signe négatif qui apparaît fait que ce modèle n'est d'ailleurs plus asymptotiquement libre.

1.4 La T-dualité non-abelienne

Une autre dualisation du modèle sigma $SU(2)$ a été proposé par B.E. Fridling et A. Jevicki [FJ84], sur le modèle de la T-dualisation abelienne de modèles sigma. La T-dualité non-abelienne consiste dans un premier temps à imposer dans l'action (1.3) les contraintes fournies par l'identité de Bianchi, via l'introduction de facteurs de Lagrange ϕ^i.

Dans le cadre de notre exemple $SU(2)$, on définit donc tout d'abord une action $\tilde{S}$ par

$$\tilde{S} = \frac{1}{2T} \int d^2x \left[\eta^{\mu\nu}\, \delta_{ij} J^i_\mu\, J^j_\nu - \phi^i\, \epsilon^{\mu\nu}\, M^i_{\mu\nu}(J)\right] \ . \tag{1.6}$$

Par construction, on voit que les équations du mouvement en ϕ^i redonnent immédiatement les identités de Bianchi du modèle $SU(2)$. L'action $\tilde{S}$ se réécrit dans les coordonnées du cône de lumière sous la forme

$$\tilde{S} = \frac{1}{T} \int d^2x \left[(\mathcal{I} + A.\phi)_{ij}\, J^i_+ J^j_- - \phi^i(\partial_+ J^i_- - \partial_- J^i_+)\right] \ ,$$

où $(A.\phi)_{ij} = -\epsilon_{ijk}\, \phi^k$.

Dans un deuxième temps, on considère les facteurs de Lagrange ϕ^i comme les véritables champs de notre théorie duale, les $\vec{J}_\pm$ n'étant plus alors que des champs auxiliaires que l'on va éliminer. Ainsi, les équations du mouvement en $J^i_\pm$ donnent :

$$J^i_- = (\mathcal{I} + A.\phi)^{is}\, \partial_- \phi^s \quad \text{et} \quad J^i_+ = \partial_+ \phi^s\, (\mathcal{I} + A.\phi)^{si} \, ,$$

avec $(\mathcal{I} + A.\phi)^{is}\, (\mathcal{I} + A.\phi)_{sj} = \delta^i{}_j$. Après une intégration par partie sur ϕ^i, et en remplaçant alors les courants dans $\tilde{S}$, on obtient l'action duale :

$$S_{\text{dual}} = \frac{1}{T} \int d^2x\, \partial_+ \phi^i\, J^i_- = \frac{1}{T} \int d^2x\, (\mathcal{I} + A.\phi)^{ij}\, \partial_+ \phi^i\, \partial_- \phi^j \ .$$

Enfin, en retournant aux coordonnées Minkowskiennes, on obtient

$$S_{\text{dual}} = \frac{1}{2T} \int d^2x\, G_{ij}\, (\eta^{\mu\nu} + \epsilon^{\mu\nu})\, \partial_\mu \phi^i\, \partial_\nu \phi^j \ ,$$

où $G^{ij} = (\mathcal{I} + A.\phi)_{ij}$ et $G_{is}\, G^{sj} = \delta_i{}^j$. On retrouve encore un modèle sigma avec torsion dont la métrique vaut maintenant $g_{ij} = G_{(ij)}$ et le potentiel de torsion $h_{ij} = G_{[ij]}$.

Plusieurs remarques peuvent être faites sur le processus de T-dualisation :

- Tout d'abord, la T-dualisation n'est possible que pour des théories dont l'action initiale peut se mettre sous la forme “courant-courant” : à une intégration par partie près, il faut que l'action $\tilde{S}$ (1.6) soit **algébrique** en $\vec{J}_\mu$ si l'on veut pouvoir éliminer ces derniers au profit des nouveaux champs ϕ^i.

- Comme dans la version "naïve" les équations du mouvement de la nouvelle théorie entraînent les identités de Bianchi initiales (1.5). Cependant, il n'y a plus vérification des équations du mouvement originelles ($\partial_\mu \vec{J}^\mu = 0$) que lorsque les **nouvelles** équations du mouvement sont réalisées. La T-dualité n'assure donc plus comme le faisait la dualité "naïve" un véritable échange identités de Bianchi–Equations du mouvements.
- Le fait que l'on ait $\partial_\mu \vec{J}^\mu = 0$ grâce aux équations du mouvement montre que les symétries droites du modèle $SU(2)$ ont été conservées. La transformation infinitésimale de paramètre $\vec{\epsilon}$ associée à cette symétrie est celle qui fait tourner le triplet $\vec{\phi}$:
$$\vec{\phi} \longrightarrow \vec{\phi} + \vec{\epsilon} \wedge \vec{\phi} \ .$$
On peut vérifier que $\vec{J}_\mu$ est le **seul** courant local de la nouvelle théorie : durant le processus de dualisation, les symétries gauches qui laissent $\vec{J}_\mu$ inchangé ont disparu. La T-dualité non-abelienne ne peut donc être inversible, comme cela était le cas pour la dualité abelienne ; le terme "dualité", bien que consacré, est ainsi excessif. Cependant, un espoir, bien que jusqu'ici sans suite, reste de retrouver la trace de ces symétries gauches qui pourrait se manifester par la présence de courants non locaux.
- Comme nous l'avons remarqué lors de la description des symétries du modèle principal chiral $SU(2)$, le lagrangien peut aussi s'écrire en fonction des courants à gauches : $\mathcal{L} = \frac{1}{2}\vec{K}_\mu \vec{K}^\mu$. Il est alors tout aussi possible de dualiser par rapport aux $\vec{K}_\mu$, ce qui fournirait une théorie duale où les symétries gauches seraient conservées tandis que les symétries droites disparaîtraient. Pour cela, il faut utiliser les identités de Bianchi relatives aux courants $\vec{K}_\mu$, qui ne diffèrent des précédentes que par un signe :
$$M'^{\,i}_{\mu\nu}(K) \equiv \partial_\mu K^i_\nu - \partial_\nu K^i_\mu - \epsilon_{ijk}\, K^j_\mu K^k_\nu = 0 \ .$$
Ce signe se retrouverait alors dans la théorie duale via la matrice G qui deviendrait $G' = (\mathcal{I} - A.\phi)^{-1}$.

Le premier avantage de la T-dualité réside dans le fait que celle-ci est une transformation **canonique** [AAGL94a, GRV89, CZ94]. En conséquence, théorie initiale et théorie duale sont **classiquement équivalentes**.

Par ailleurs, dans tous les exemples testés jusqu'à aujourd'hui, dont ceux que nous allons étudier par la suite, les constantes de couplage se renormalisent de la même façon dans les deux théories. Pour toute la suite, on parlera dans de tels cas d' **équivalence au premier ordre quantique**. Notons dès maintenant que ceci n'a pas encore été démontré de façon générale. Notons aussi que ce que nous appellerons "équivalence quantique" n'en est en fait qu'une condition *nécessaire*, la véritable équivalence quantique imposant en plus l'égalité des matrices S, ici au premier ordre en $\hbar$.

La T-dualité non-abelienne fut d'abord appliquée à des théories possédant des symétries basées sur des groupes de Lie. En particulier, l'égalité des fonctions β à une boucle a pour la première fois été démontrée pour le groupe $SU(2)$ [FJ84, FT85]. On trouve en effet pour ce cas particulier une métrique duale quasi-Einstein (avec torsion) :

$$Ric_{ij} = \lambda\, G_{ij} + D_j v_i\,, \qquad \lambda = \frac{1}{2}\,, \qquad \vec{v} = \vec{\nabla}\left[\ln(1+\phi^2) - \frac{\phi^2}{2}\right]\,. \tag{1.7}$$

La constante cosmologique λ est identique à celle du modèle principal chiral $SU(2)$, entraînant ainsi l'égalité des fonctions β. Tyurin a généralisé cette équivalence à une boucle pour tous les modèles basés sur un groupe de Lie [Tyu95]. Cependant, ainsi qu'il fut noté dans [B$^+$98], Tyurin ne considère que des modèles possédant une invariance explicite sous l'action à droite. Un premier pas vers des modèles ne possédant pas une symétrie à droite maximale fût étudié récemment dans [B$^+$98, HKP97] où les auteurs examinèrent la théorie duale du modèle principal chiral $(SU(2)_L \times SU(2)_R)/SU(2)_D$ après avoir brisé $SU(2)_R$ en $U(1)$.

A travers le vecteur $\vec{v}$ de l'équation (1.7), qui renormalise les champs (*cf.* Annexe A), apparaît un phénomène important dans le cadre de la théorie des cordes. En effet, lorsque l'on veut interpréter une théorie duale comme une nouvelle théorie des cordes, l'invariance conforme de la théorie initiale doit être préservée. Ceci sera le cas, du moins jusqu'au premier ordre en perturbations, si les divergences des champs peuvent être réabsorbées par la renormalisation d'un champ propre à la théorie des cordes, le dilaton. Pour cela, il faut que les contre-termes associés aux champs puissent s'écrire sous la forme d'un gradient, ce qui est bien le cas ici. On appellera par la suite cette exigence sous le nom de **propriété dilatonique**. Comme il a été démontré dans [GRV93] par une analyse du modèle principal chiral Bianchi V, celle-ci peut ne pas survivre au processus de dualisation. Une "anomalie dilatonique" peut en effet apparaître dans le cas de groupes de Lie qui ne sont pas semi-simples [AAGL94b, GRV93] : dans ces cas-là, la trace non nulle des constantes de structure de l'algèbre empêche la réécriture du vecteur $\vec{v}$ sous la forme d'un gradient.

Le Chapitre 2 porte sur les propriétés à une boucle de certains modèles T-duaux. Dans notre premier article, "*Quantum structure of T-dualised models with symmetry breaking*" (*cf.* Annexe C.1), nous avons montré de façon générale l'équivalence quantique à une boucle de tout modèle dual basé sur un groupe de Lie, **quelle que soit la brisure des symétries droites**. Comme les symétries gauches disparaissent durant la dualisation, on peut ainsi construire des théories au moins renormalisables à une boucle ne possédant plus aucune symétrie ! Par ailleurs, par une étude complète des modèles de Bianchi à trois dimensions, nous montrons qu'un choix de brisure approprié peut faire disparaître l'anomalie dilatonique de certains cas non semi-simples. Nous étudierons aussi dans ce chapitre la dualisation de modèles fondés sur des métriques inhomogènes en nous appuyant sur l'article "*Renormalisability of non-homogeneous T-dualised sigma-models*" (*cf.* Annexe C.2). La dualisation de métriques non-homogènes telles que celles du trou noir de Schwarzschild ou de Taub-NUT a été entreprise dans [dlOQ93, AAGL94b, Hew96]. Nous faisons dans cet article une étude de toute la classe des théories basées sur des métriques possédant une symétrie $SU(2) \times U(1)$ de cohomogénéité un. Pour toute cette classe, l'équivalence quantique à une boucle perdure alors que ces modèles initiaux ne sont pas – sous-cas homogènes exceptés – renormalisables à deux boucles dans le schéma dimensionnel minimal. Nous montrerons pour ces modèles inhomogènes que, lorsqu'elle existe, la propriété Kähler de la théorie de départ est conservée à travers le processus de dualisation.

C'est l'équivalence quantique à deux boucles qui sera abordée dans le Chapitre 3, basé sur l'article "*Dualised σ-models at the two loop order*" (*cf.* Annexe C.3). L'absence de renormalisabilité à deux boucles de la théorie duale, dans le schéma dimensionnel minimal, du modèle $SU(2)$ ayant été démontré [ST96, BFHP96], nous en refaisons une étude plus approfondie en permettant cette fois des contre-termes **finis** au premier ordre en $\hbar$ à la métrique elle-même. Tout en forçant cette déformation à vérifier les symétries droites de la théorie, nous montrons qu'il est toujours possible de définir une théorie duale renormalisable à deux boucles.

Chapter 2

Equivalence à une boucle

2.1 Dualisation des modèles $(G \times G)/G$

2.1.1 Théorie initiale, théorie duale

Nous allons ici généraliser le modèle de départ $SU(2)$ en considérant dans un premier temps une algèbre de Lie quelconque. Dans un second temps, nous briserons de la façon la plus générale possible les symétries droites de la théorie.

Pour cela, nous considérons une algèbre de Lie $\mathcal{G} = \{X_i, i = 1, \cdots, \nu\}$ définie par ses constantes de structure ${f_{ij}}^s$:

$$[X_i, X_j] = {f_{ij}}^s \, X_s \ .$$

On notera $[\mathrm{ad}(X_i)]_j{}^k = -{f_{ij}}^k$ les matrices associées aux générateurs X_i dans leur représentation adjointe. Nous pouvons alors passer par exponentiation aux éléments du groupe de Lie associé qui s'écrivent $g = \exp(z.X)$. On peut ensuite définir les courants par l'égalité

$$g^{-1}\partial_\mu g = J^i_\mu \, X_i \ .$$

Ces courants sont algébriquement contraints par les identités de Bianchi de la théorie :

$$M^i_{\mu\nu}(J) \equiv \partial_\mu J^i_\nu - \partial_\nu J^i_\mu + {f_{jk}}^i \, J^j_\mu J^k_\nu = 0 \iff \epsilon^{\mu\nu} M^i_{\mu\nu}(J) = 0 \ . \tag{2.1}$$

L'action du modèle initial s'écrit simplement

$$S = \frac{1}{2} \int d^2x \, B_{ij} \, \eta^{\mu\nu} \, J^i_\mu \, J^j_\nu \ . \tag{2.2}$$

La matrice B_{ij} doit être symétrique et inversible. Elle contient la constante de couplage T ainsi que les constantes de couplage qui brisent les symétries droites. En effet, sous l'action des symétries gauches, droites et diagonales sur g, on obtient

$$\left\{ \begin{array}{lcl} g & \longrightarrow & h_L \, g \, h_R^{-1} \\ g & \longrightarrow & h_D \, g \, h_D^{-1} \end{array} \right. \qquad \Longrightarrow \qquad g^{-1}\partial_\mu g \longrightarrow h_R \, g^{-1}\partial_\mu g \, h_R^{-1} \ .$$

Si on ne s'occupe que des transformations infinitésimales, cela donne

$$\left\{\begin{array}{lll} h_R & \approx & \mathcal{I}+\epsilon^i_R\,\mathrm{ad}(X_i) \\ h_L & \approx & \mathcal{I}+\epsilon^i_L\,\mathrm{ad}(X_i) \\ h_D & \approx & \mathcal{I}+\epsilon^i_D\,\mathrm{ad}(X_i) \end{array}\right. \Longrightarrow \delta J^k_\mu = f_{ij}{}^k\,\epsilon^i_R\,J^j_\mu \Longrightarrow \delta\mathcal{L}_{\text{dual}} = \frac{1}{2}\eta^{\mu\nu}N_{ijk}J^i_\mu J^j_\nu \epsilon^k_R\ ,$$

où l'on a posé $N_{ijk} = B_{is}\,f_{kj}{}^s + B_{js}\,f_{ki}{}^s = -2[\mathrm{ad}(X_k).B]_{(ij)}$. Par conséquent, l'action (2.2) est bien invariante sous l'action des symétries gauches tandis que B_{ij} brisera dans la plupart des cas les symétries droites en un sous-groupe dont les générateurs X_k vérifient $N_{ijk}=0,\ \forall\ i,j$.

Le processus de dualisation qui suit est en fait strictement le même que celui du cas particulier $SU(2)$ étudié dans le Chapitre 1. Seuls deux éléments ont changé :

- la matrice B_{ij} dans l'action initiale du modèle général (2.2) a pris la place de la matrice $\frac{1}{T}\delta_{ij}$ qui contractait les courants dans l'action (1.4),
- les constantes de structure $f_{ij}{}^k$ présentes dans les identités du Bianchi du cas général (2.1) ont remplacé les ϵ_{ijk} de $SU(2)$.

Ainsi, en appliquant les règles

$$\left\{\begin{array}{ccc} \mathcal{I} & \longrightarrow & B_{ij} \\ T & \longrightarrow & 1 \\ \epsilon_{ijk} & \longrightarrow & f_{ij}{}^k \end{array}\right.$$

on trouve l'action duale de notre modèle (2.2) :

$$S_{\text{dual}} = \frac{1}{2}\int d^2x\, G_{ij}\,(\eta^{\mu\nu}+\epsilon^{\mu\nu})\,\partial_\mu\phi^i\partial_\nu\phi^j \tag{2.3}$$

avec $G^{-1} = B + A.\phi$ et $(A.\phi)_{ij} = -f_{ij}{}^k\,\phi^k$. Comme pour le modèle $SU(2)$, on définit la nouvelle métrique et le potentiel de torsion par

$$g_{ij} = G_{(ij)}\ , \qquad\qquad h_{ij} = G_{[ij]}\ .$$

On notera par la suite $f_{ij,k} = f_{ij}{}^s\,B_{ks}$.

On peut vérifier que les symétries gauches ont ici aussi disparu après dualisation.

2.1.2 Equivalence quantique à une boucle

Rappelons que par "équivalence quantique", nous entendons "égalité des fonctions β pour les couplages entre théorie initiale et théorie duale". La démonstration de cette équivalence se ramène à un problème géométrique. Nous devons d'abord caractériser les théories initiales qui sont effectivement renormalisables au premier ordre en perturbation. Les divergences à une boucle de la théorie initiale s'écrivent

$$Div^1_{ij} = -\frac{\hbar}{4\,\pi\varepsilon}\int d^2x\, ric_{ij}\,\eta^{\mu\nu}\,J^i_\mu\,J^j_\nu \qquad\qquad d = 2-\varepsilon\,.$$

Au sens strict de la théorie des champs, la renormalisation de la théorie initiale implique alors que ces divergences puissent être ré-absorbées par une redéfinition des constantes de couplage ρ_s ($T = \rho_1$) et/ou du champ.

Bien que le calcul du tenseur de Ricci ric_{ij} soit assez compliqué, on peut démontrer que celui-ci ne dépend pas des champs lorsqu'il est exprimé dans la base du vierbein ($e_i = J^i_\mu$), mais uniquement de la matrice de brisure B_{ij} et des constantes de structure $f_{ij}{}^k$. Ceci implique que tout terme de la forme $D_{(i}v_{j)}$ est exclu dans ric_{ij}. Dans un tel cas, la théorie initiale est renormalisable à une boucle si et seulement si il existe des fonctions $\chi_s(\rho)$ ne dépendant **que** des constantes de couplage ρ_s, de façon à ce que l'on ait

$$ric_{ij} = \chi_s(\rho) \, \frac{\partial B_{ij}}{\partial \rho_s} \; . \tag{2.4}$$

Cette renormalisabilité à une boucle est assurée de façon certaine pour les deux cas suivants :

- Celui où la théorie initiale est basée sur une algèbre semi-simple équipée de sa métrique bi-invariante (celle qui maximise les symétries) [Tyu95].
- Celui où la brisure des symétries droites est maximale, chaque terme de la matrice B_{ij} étant alors formé d'une constante de couplage dont la redéfinition permet d'absorber le terme correspondant dans ric_{ij}.

Pour ce qui est des cas intermédiaires, où les symétries droites sont partiellement brisées, l'égalité (2.4) peut ne pas être vérifiée, celle-ci mélangeant des conditions sur la matrice B_{ij} et les constantes de structure du groupe.

En ce qui concerne la théorie duale, en présence de torsion, la condition de renormalisabilité à une boucle s'écrit

$$Ric_{ij} = \chi_s(\rho) \, \frac{\partial G_{ij}}{\partial \rho_s} + D_j v_i + \partial_{[i} w_{j]} \; . \tag{2.5}$$

Le vecteur $\vec{v}$ renormalisera les champs ϕ^i (*cf.* Annexe A). La présence du terme $\partial_{[i} w_{j]}$ est due au fait que le potentiel de torsion n'est défini qu'à un rotationnel près. Les constantes de couplage ρ_s sont les mêmes que dans le modèle initial, même si elles apparaissent maintenant de façon non triviale dans la nouvelle théorie. Les deux théories ne pourront être équivalentes au premier ordre quantique que si les fonctions χ_s sont les mêmes que celles de la théorie initiale.

Une relation très importante, clef de voûte de l'équivalence, donne le tenseur de Ricci du modèle dual (noté *Ric*) en fonction du tenseur de Ricci du modèle original (noté *ric*) :

$$Ric_{ij} = -(G.ric.G)_{ij} + D_j v_i \, , \qquad\qquad v_i = -2\, G_{it}\, f_{st}{}^s - \partial_i \ln(\sqrt{\det g}) \; . \tag{2.6}$$

Une telle relation liant deux tenseurs de Ricci construits sur des métriques différentes peut sembler étrange : elle n'est évidemment possible que parce que la matrice ric_{ij} ne dépend pas des champs de la théorie initiale.

Il est maintenant possible de démontrer l'équivalence : après avoir supposé que la théorie initiale était renormalisable à une boucle, on peut insérer la relation (2.4) dans l'égalité (2.6) :

$$Ric_{ij} = -\chi_s\,(G.\frac{\partial B}{\partial \rho_s}.G)_{ij} + D_j v_i\ .$$

A partir de $G^{-1} = B + A.\phi$, on peut aussi écrire

$$\frac{\partial G}{\partial \rho_s} = -G.\frac{\partial B}{\partial \rho_s}.G\ .$$

Ces deux relations permettent alors d'obtenir la forme (2.5) recherchée :

$$Ric_{ij} = \chi_s\,\frac{\partial G_{ij}}{\partial \rho_s} + D_j v_i\,, \qquad v_i = -2\,G_{it}\,f_{st}{}^{s} - \partial_i \ln(\sqrt{\det g})\ . \tag{2.7}$$

L'équivalence quantique à une boucle entre théorie initiale et théorie duale est alors démontrée par le fait que ce sont les mêmes facteurs χ_s qui vont renormaliser les constantes de couplage ρ_s.

Remarques :

- Nous avons démontré l'équivalence quantique à une boucle indépendamment du schéma de brisure caché dans la matrice B_{ij}. Si l'on suppose que l'on brise toutes les symétries droites, on aboutit à une théorie duale qui n'en possède plus du tout, les symétries gauches ayant été perdues durant la dualisation. Ces cas donnent ainsi un exemple de théorie non-homogène avec torsion dont la renormalisabilité à une boucle ne peut être expliquée par aucune symétrie locale. Il semble peu probable que ce fait soit accidentel et une explication théorique plus complète de la T-dualisation reste encore à faire. Celle-ci devrait alors retrouver la trace des symétries droites perdues, peut-être sous la forme de symétries non-locales, expliquant ainsi la renormalisation à une boucle.
- Le même genre de démonstration permet de généraliser cette équivalence à une boucle à des théories initiales possédant une torsion **constante**, avec un lagrangien de la forme $\mathcal{L} = \frac{1}{2}(B_{ij}\,\eta^{\mu\nu} + C_{ij}\,\epsilon^{\mu\nu})\,J^i_\mu\,J^j_\nu$.
- La situation à deux boucles sera étudiée en détail dans le Chapitre 3. Notons que dans le schéma dimensionnel minimal, on peut prouver que la théorie duale du modèle $SU(2)$ n'est pas renormalisable à deux boucles, contrairement au modèle de départ.[ST96, BFHP96]

2.1.3 Théorie duale sans torsion

Le calcul explicite dans la théorie duale de la torsion T_{ijk} en fonction de la matrice B_{ij} et des constantes de structure permet de poser la question de son éventuelle nullité. En effet, lorsque T est l'unique constante de couplage, la renormalisabilité à une boucle entraîne automatiquement pour la métrique duale la propriété d'être quasi-Einstein. Trouver les conditions sur B_{ij} et sur les $f_{ij}{}^k$ assurant la nullité de la torsion transforme donc la T-dualité en une méthode originale pour construire des métriques quasi-Einstein sans torsion avec peu de symétries. L'idéal étant bien sûr de pouvoir en trouver parmi celles-ci certaines qui soient vraiment Einstein – nous rappelons au lecteur que **toutes** les métriques d'Einstein connues à ce jour possèdent des symétries. Nous avons été amené à nous intéresser à ce phénomène en nous rendant compte que le dual du modèle principal chiral Bianchi V était sans torsion, chose que bizarrement les auteurs Gasperini, Ricci et Veneziano ne semblaient pas avoir remarquée dans [GRV93]. Un seul autre cas du même type a été, à notre connaissance, noté dans la littérature : le dual du modèle non-homogène de Schwarzschild dans [dlOQ93, Hew96].

Les conditions nécessaires et suffisantes pour annuler la torsion des modèles duaux des théories $(G \times G)/G$ sont :

$$f_{[ij,k]} = 0, \qquad \text{et} \qquad B^{st}\, f_{rs}{}^{(u}\, f_{t[k}{}^{v)}\, f_{ij]}{}^{r} = 0, \quad \forall\, [ijk]\;, \quad \forall\, (uv). \tag{2.8}$$

Pour une algèbre semi-simple sans brisure de symétrie ($B_{ij} = \delta_{ij}$), la première condition n'est, de façon évidente, jamais réalisée. Bien que particulièrement difficiles à résoudre, ces deux conditions possèdent pourtant des solutions, comme le modèle Bianchi V déjà cité. Elles permettent aussi de construire des algèbres qui les satisferont, et ce quelque soit le choix de B_{ij}. Un exemple est donné par l'algèbre de Lie dont les générateurs $\{X_i, i = 1, \cdots, \eta\}$ vérifient

$$[X_1, X_i] = X_i, \quad i = 2, \cdots, \eta\;, \qquad [X_i, X_j] = 0, \quad 1 < i < j\;.$$

Cette algèbre donnera un modèle dual sans torsion.

2.1.4 Propriété dilatonique et algèbres de Bianchi

Afin que la propriété dilatonique soit vérifiée, (2.7) montre qu'il faut pouvoir écrire $G_{it}\, f_{st}{}^s$ sous la forme d'un gradient :

$$K_i = G_{it}\, f_{st}{}^s = \partial_i \Phi \Longleftrightarrow \partial_{[i} K_{j]} = 0 \Longleftrightarrow G_{su}\, f_{vu}{}^v (f_{st}{}^i\, G_{jt} - f_{st}{}^j\, G_{it}) = 0. \tag{2.9}$$

A part les cas où la trace sur les constantes de structure est nulle, comme par exemple pour les algèbres semi-simples, (2.9) n'est générallement pas vérifiée. C'est ce qui se passe en particulier pour l'algèbre de Lie Bianchi V [GRV93, EGR$^+$95].

Les algèbres de Lie à trois dimensions ont été étudiées et classées en 1897 par Bianchi. Dans leur représentation moderne [EWB68, EM69], ces algèbres sont décrites à l'aide d'un paramètre a et d'un vecteur $\vec{n} = \{n_1, n_2, n_3\}$ selon la table de commutation suivante

$$[X_1, X_2] = a\, X_2 + n_3\, X_3\,, \qquad [X_2, X_3] = n_1\, X_1\,,$$
$$[X_3, X_1] = n_2\, X_2 - a\, X_3\,.$$

L'identité de Jacobi entraîne la relation $a.n_1 = 0$, tandis que la trace sur les constantes de structure s'écrit $f_{st}{}^{s} = -2a\,\delta_{t1}$. Les algèbres de Lie à trois dimensions sont ainsi séparées en deux classes, suivant la nullité de a.

Pour la classe A ($a = 0$), la propriété dilatonique est vérifiée.

Pour la classe B ($a \neq 0$), la brisure des symétries droites va conditionner la vérification de la propriété dilatonique. Ainsi, si on prend la matrice B_{ij} la plus générale,

$$B_{ij} = \begin{pmatrix} r_1 & s_3 & s_2 \\ s_3 & r_2 & s_1 \\ s_2 & s_1 & r_3 \end{pmatrix}_{ij} \quad ,$$

la relation (2.9) est alors équivalente au système suivant :

$$\begin{cases} \nu & \equiv & n_2\,r_2 + n_3\,r_3 & = 0\ , \\ \mu & \equiv & s_1^2 - r_2\,r_3 & = 0\ . \end{cases} \tag{2.10}$$

Il est toujours possible de choisir une matrice de brisure de déterminant non nul telle que le système (2.10) soit vérifié, excepté dans le cas de l'algèbre Bianchi VII$_a$ ($n_2 = n_3 = 1$).

Pour ces algèbres de Bianchi de classe B, il est remarquable que les conditions (2.10) assurent aussi une torsion duale nulle. En effet, (2.8) se réécrit alors

$$f_{[ij,k]} = \frac{\nu}{3}, \qquad\qquad B^{st}\, f_{rs}{}^{(u}\, f_{t[k}{}^{v)}\, f_{ij]}{}^{r} = \frac{\mu}{3}\,\frac{(a^2 + n_2 n_3)}{\det B}\, n_{\underline{u}}\,\delta_{uv}\,\epsilon_{ijk}\ .$$

2.2 Dualisation du modèle inhomogène $SU(2)\times U(1)$

Nous n'avons jusqu'ici envisagé que des théories fondées sur des modèles homogènes. Il est aussi possible de dualiser des théories inhomogènes en rajoutant par exemple une coordonnée "spectatrice" aux modèles précédemment étudiés. Ceci avait déjà été fait pour quelques modèles construits sur des métriques comme celles de Eguchi-Hanson, de Taub-Nut et de Schwarzschild [dlOQ93, AAGL94b, Hew96]. Nous proposons ici d'étendre la démonstration de l'équivalence à une boucle à toute la classe des métriques quasi-Einstein à quatre dimensions de co-homogénéité un sous les isométries $SU(2) \times U(1)$. Ainsi qu'il est montré en Annexe B, il s'agit d'une classe très vaste comportant de nombreuses métriques connues . Ces théories, quasi-homogènes, sont aussi intéressantes parce qu'aucune d'elles n'est renormalisable à deux boucles dans le schéma dimensionnel minimal. Nous parlerons aussi de la conservation d'une propriété appartenant à certaines d'entre elles : l'existence de structures complexes.

2.2.1 Théories initiales, théories duales

De façon générale, les métriques de co-homogénéité un sous des symétries $SU(2)\times U(1)$ peuvent s'écrire sous la forme

$$g = \alpha(t)\, dt^2 + \beta(t)\, ({\sigma_1}^2 + {\sigma_2}^2) + \gamma(t)\, {\sigma_3}^2 \ , \tag{2.11}$$

où les σ_i sont les 1-formes invariantes de $SU(2)$ qui dépendent des angles d'Euler $\{\theta, \varphi, \psi\}$. Les identités de Bianchi se traduisent pour ces 1-formes par les identités de Maurer-Cartan

$$d\sigma_i = \varepsilon\, \frac{1}{2}\, \epsilon_{ijk}\, \sigma_j \wedge \sigma_k, \qquad \varepsilon = \pm 1 \ .$$

Dans le cas où $\varepsilon = +1$, une transformation infinitésimale de $su(2)_L \oplus su(2)_R$ agit sur le triplet $\vec{\sigma}$ par

$$\delta\vec{\sigma} = \vec{\epsilon}_R \wedge \vec{\sigma} \ .$$

Ainsi, $\vec{\sigma}$ est un singlet pour $SU(2)_L$ et un triplet pour $SU(2)_R$. Si dans (2.11), les fonctions $\beta(t)$ et $\gamma(t)$ sont différentes, la symétrie $SU(2)_R$ est brisée en un $U(1)$ et le groupe de symétrie final sera $SU(2)_L \times U(1)$. Dans le cas où $\varepsilon = -1$, le groupe de symétrie final sera $SU(2)_R \times U(1)$ puisque les transformations infinitésimales de $su(2)_L \oplus su(2)_R$ donneront alors

$$\delta\vec{\sigma} = \vec{\epsilon}_L \wedge \vec{\sigma} \ .$$

Dans tous les cas, si on a $\beta(t) = \gamma(t)$, le groupe de symétrie est élargi à $SU(2)_R \times SU(2)_L$ et la métrique g est alors conformément plate.

On peut alors, à partir de la métrique g, définir le modèle sigma correspondant :

$$S = \frac{1}{T} \int d^2x\, g_{ij} \partial_+ \phi^i \partial_- \phi^j \ .$$

Par commodité, on a posé $\{\phi^0 = t, \phi^1 = \theta, \phi^2 = \varphi, \phi^3 = \psi\}$ et $g = g_{ij}\, d\phi^i\, d\phi^j$.

La méthode la plus simple pour dualiser la théorie initiale par rapport aux symétries $SU(2)$ est d'utiliser le fait que celle-ci est de co-homogénéité un : on dualise la métrique homogène à trois dimensions

$$g_3 = \beta(t)\,({\sigma_1}^2 + {\sigma_2}^2) + \gamma(t)\,{\sigma_3}^2 \ ,$$

en oubliant momentanément le terme $\alpha(t)\, dt^2$ que l'on rajoutera à la fin. Les 1-formes σ_i jouent alors le rôle des courants J^i_μ ($\varepsilon = 1$) ou K^i_μ ($\varepsilon = -1$), que l'on avait dans les dualisations précédentes [Hew96]. Les angles d'Euler $\{\theta, \varphi, \psi\}$ de la théorie initiale vont alors être remplacés dans la théorie duale par de nouvelles coordonnées[1] $\{r, y, z\}$. On obtient pour la métrique finale

$$\hat{g} = \alpha(t)\, dt^2 + \frac{r^2 + \beta(t)^2}{\Delta} \left(dr + \frac{r\, y}{r^2 + \beta(t)^2}\, dy \right)^2 + \frac{\beta(t)}{r^2 + \beta(t)^2}\, dy^2 + \frac{y^2\, \beta(t)\, \gamma(t)}{\Delta}\, dz^2 \tag{2.12}$$

où

$$\Delta = y^2\, \beta(t) + \left(r^2 + \beta(t)^2 \right)\, \gamma(t) \ .$$

La 2-forme associée au potentiel de torsion s'écrit

$$H = -\frac{y^2\, \beta(t)}{\Delta}\, dr \wedge dz + \frac{r\, y\, \gamma(t)}{\Delta}\, dy \wedge dz \ . \tag{2.13}$$

La métrique $\hat{g}$ a perdu sa symétrie $SU(2)$ et ne possède plus qu'un $U(1)$ associé au Killing ∂_z.

On définit les matrices $\hat{g}_{ij}$, $\hat{h}_{ij}$ et $\hat{G}_{ij}$ usuelles par $H = \frac{1}{2}\, \hat{h}_{ij}\, d\hat{\phi}^i \wedge d\hat{\phi}^j$, $\hat{g} = \hat{g}_{ij}\, d\hat{\phi}^i\, d\hat{\phi}^j$, et $\hat{G}_{ij} = \hat{g}_{ij} + \hat{h}_{ij}$ avec $\{\hat{\phi}^0 = t, \hat{\phi}^1 = r, \hat{\phi}^2 = y, \hat{\phi}^3 = z\}$. On notera ric_{ij} le tenseur de Ricci de la théorie initiale, et $\hat{R}ic_{ij}$ celui de la théorie duale.

2.2.2 Equivalence à une boucle

De nombreux paramètres peuvent être cachés dans les fonctions $\alpha(t)$, $\beta(t)$ et $\gamma(t)$. Nous ne nous intéresserons ici qu'aux théories dont la renormalisabilité à une boucle ne dépend pas d'une renormalisation éventuelle de ces paramètres. Puisque T est alors le seul couplage renormalisé, nous ne prendrons en compte que des métriques quasi-Einstein. Ces métriques n'étant pas toutes connues, ce seront les équations découlant de cette dernière propriété qui définiront les théories initiales. En absence de torsion, la propriété quasi-Einstein des théories initiales s'écrit simplement

$$ric_{ij} = \lambda\, g_{ij} + D_{(i} v_{j)} \ .$$

[1] Les coordonnées $\{r, y, z\}$ que nous utilisons n'apparaissent en fait qu'après un changement de coordonnées adéquat sur les multiplicateurs de Lagrange de la dualisation. Cette redéfinition permet de simplifier la métrique duale.

Les isométries de la théorie initiale forcent, au Killing ∂_ψ près, la forme de v_i à $v_i = \delta_{it} f(t)$.

Un travail purement algébrique [Cas02] sur les trois équations différentielles traduisant le caractère quasi-Einstein de la théorie initiale permet alors de démontrer l'équivalence

$$ric_{ij} = \lambda\, g_{ij} + D_{(i} v_{j)} \Longleftrightarrow \hat{R}ic_{ij} = \hat{\lambda}\, \hat{G}_{ij} + D_j \hat{v}_i + \partial_{[i} \hat{w}_{j]} \ , \tag{2.14}$$

où

$$\begin{cases} \hat{\lambda} & = \lambda \ , \\ \hat{v}_i & = -2\,\lambda\, \hat{g}_{is}\, X^s + \partial_i \mathrm{Log}\Delta + v_i \ , \\ \hat{w}_i & = -2\,\lambda\, X^s\, \hat{G}_{si} \ , \end{cases} \tag{2.15}$$

X^s étant défini par $X = r\,\partial_r + y\,\partial_y$.

Tout d'abord, l'égalité $\hat{\lambda} = \lambda$ montre que la constante de couplage T se renormalise exactement de la même façon pour la théorie initiale et la théorie duale, ce que nous nous étions proposé de démontrer.

Ensuite, le fait que (2.14) soit une équivalence implique, en général, l'unicité de la solution (2.15) puisque, en général, λ et v sont eux-mêmes définis de manière unique[2]. Seul l'espace plat échappe, dans notre classe de modèle, à cette règle : on peut dans ce cas écrire $\lambda\, g_{ij} + D_{(i} v_{j)} = 0$ où $v = -2\,\lambda\, dt$, pour tout $\lambda \in \mathbb{R}$.

Enfin, notons la présence du v_i de la théorie initiale dans le vecteur $\hat{v}_i$: on retrouve dans les contre-termes à une boucle du champ de la théorie duale ceux de la théorie initiale.

2.2.3 Propriété dilatonique

Si l'on suppose que, dans la théorie initiale, la propriété dilatonique est vérifiée, c'est à dire qu'il est possible d'écrire $v_i = \partial_i \Psi(t)$, qu'advient-il de cette propriété pour la théorie duale ? Si $\lambda = 0$, alors la forme de $\hat{v}$ permet de conclure immédiatement à la conservation de la propriété dilatonique. Sinon, il faut que le terme $\hat{g}_{is}\, X^s$ puisse s'écrire sous la forme d'un gradient. Dans [Cas02], il est démontré que cela n'est jamais possible. Nous avons donc d'abord conclu, pour $\lambda \neq 0$, à l'existence d'une nouvelle forme d'anomalie dilatonique, non liée cette fois à une trace non nulle sur les constantes de structure de l'algèbre dualisée.

En fait, l'apparition du terme $-2\,\lambda\, \hat{g}_{is}\, X^s$ dans $\hat{v}_i$ est lié au fait que nous considérons que la constante de couplage T est extérieure à la métrique g_{ij} dans la théorie initiale. Cela n'a pas d'influence sur la renormalisation de la constante de couplage elle-même, mais va changer la renormalisation du champ dans la théorie duale en faisant

[2] En fait, $\hat{v}$ (v) et $\hat{w}$ ne sont bien sûr définis qu'au Killing ∂_z (∂_ψ) près, ainsi qu'à un gradient près pour $\hat{w}$.

disparaître le terme $-2\,\lambda\,\hat{g}_{is}\,X^s$ de $\hat{v}_i$. En effet, lorsque le couplage est dans la métrique, alors ce n'est plus le fait que celle-ci soit quasi-Einstein qui assure la renormalisabilité à une boucle de la théorie, mais la propriété géométrique, ici donnée pour la théorie duale,

$$\hat{R}ic_{ij} = \chi_T \,\frac{\partial}{\partial T}\hat{G}_{ij}(T) + D_j\hat{v}_i + \partial_{[i}\hat{w}_{j]} \ .$$

Pour toute métrique Einstein ou quasi-Einstein à constante cosmologique non nulle, il est toujours possible de faire un changement de coordonnées permettant d'écrire la métrique sous la forme

$$g_{ij}(\lambda) = \frac{t_{ij}}{\lambda} \ ,$$

où t_{ij} est une métrique de constante cosmologique égale à l'unité. On voit ainsi, avec un bon système de coordonnées, qu'il est assez simple de rentrer le couplage dans la métrique initiale :

$$\frac{1}{T}\,g_{ij}(\lambda) = \frac{1}{\lambda T}\,t_{ij} = g_{ij}(\lambda\,T) \ .$$

Pour pouvoir continuer à utiliser les résultats de la section précédente, il suffit donc d'oublier le facteur $\frac{1}{T}$ global et de faire le remplacement $\lambda \longrightarrow \lambda\,T$ dans (2.14) et (2.15). On peut par ailleurs poser dans (2.11)

$$\alpha(t) = \frac{a(t)}{\lambda\,T} \ , \qquad \beta(t) = \frac{b(t)}{\lambda\,T} \ , \qquad \gamma(t) = \frac{c(t)}{\lambda\,T} \ ,$$

où les fonctions $a(t)$, $b(t)$ et $c(t)$ sont indépendantes de λ et de T. On va donc obtenir une métrique duale $\hat{G}_{ij}(T)$ qui dépend du couplage. Il est alors possible de démontrer l'égalité

$$\hat{G}_{ij} - 2\,D_j(\hat{g}_{is}\,X^s) - 2\,\partial_{[i}X^s\,\hat{G}_{sj]} \ = -T\,\frac{\partial}{\partial T}\hat{G}_{ij},$$

ce qui donne pour le tenseur de Ricci de la théorie duale

$$\hat{R}ic_{ij} = \chi_T\frac{\partial}{\partial T}\hat{G}_{ij} + D_j(v_i + \partial_i \mathrm{Log}\Delta) \ , \qquad \chi_T = -\lambda\,T^2.$$

Cette dernière relation démontre à la fois l'équivalence à une boucle et la conservation de la propriété dilatonique. Notons que, pour la théorie initiale, on a bien

$$ric_{ij} = \lambda\,T\,g_{ij} + D_jv_i = t_{ij} + D_jv_i = -\lambda\,T^2\frac{\partial}{\partial T}\left(\frac{t_{ij}}{\lambda\,T}\right) + D_jv_i = \chi_T\frac{\partial}{\partial T}g_{ij} + D_jv_i \ .$$

Dans cette perspective où le couplage est inclu dans la métrique initiale, la propriété dilatonique est bien vérifiée : les divergences supplémentaires du champ de la théorie duale sont éliminées par la redéfinition du dilaton

$$\hat{\Psi} = \Psi + \mathrm{Log}\,\Delta \ .$$

2.2.4 La conservation des structures complexes

Dans un cadre d'applications super-symétriques, Bakas et Sfetsos ont décrit comment se transforment les structures complexes lorsque l'on dualise des métriques hyper-Kähler [BS95]. Nous allons envisager ici certains cas où la métrique initiale $SU(2)\times U(1)$ est Kähler, et démontrer que cette propriété existe encore pour leur métrique duale.

Nous supposerons qu'il existe un choix de coordonnées holomorphes pour lesquelles les symétries $SU(2) \times U(1)$ agissent linéairement. Une telle hypothèse implique l'intégrabilité des structures complexes. Afin de définir celles-ci, nous définissons, pour la métrique initiale (2.11), le vierbein

$$\begin{aligned} e_0 &= \sqrt{\alpha(t)}\, dt \ , \quad e_1 = \sqrt{\beta(t)}\, \sigma_1 \ , \\ e_2 &= \sqrt{\beta(t)}\, \sigma_2 \ , \quad e_3 = \sqrt{\gamma(t)}\, \sigma_3 \ . \end{aligned}$$

Lorsque $\beta(t)$ n'est pas une constante, il est toujours possible de redéfinir dans (2.11) la coordonnée t de façon à ce que l'on ait $\beta(t) = t$. Une condition suffisante pour que la théorie de départ soit Kähler est alors donnée par l'identité

$$\gamma(t) = \frac{1}{\alpha(t)} \ . \tag{2.16}$$

Cette condition fournit en effet la forme de Kähler suivante :

$$\rho_1 = e_0 \wedge e_3 + e_1 \wedge e_2 = dt \wedge \sigma_3 + t\, d\sigma_3 = d(t\, \sigma_3) \ .$$

Qu'advient-il alors de la structure complexe associée à ρ_1 après dualisation ? Si on pose

$$\hat{\sigma}_i = -\hat{G}_{si}\, d\hat{\phi}^s, \qquad i, s \in [\![1, 3]\!],$$

on peut écrire la métrique duale sous la forme

$$\hat{g} = \frac{1}{\gamma(t)}\, dt^2 + t\,(\hat{\sigma_1}^2 + \hat{\sigma_2}^2) + \gamma(t)\, \hat{\sigma_3}^2 \ ,$$

et vérifier que la 1-forme

$$\hat{\rho}_1 = dt \wedge \hat{\sigma}_3 + t\, \hat{\sigma}_1 \wedge \hat{\sigma}_2 = \frac{1}{2} \hat{J}_{1ij}\, d\hat{\phi}^i \wedge d\hat{\phi}^j$$

est bien une forme de Kähler **avec torsion**. On a en effet, pour la structure presque complexe $\hat{J}_1$ associée, les propriétés

$$\begin{cases} \hat{J}_{1is}\, {J_1}^{sj} = -{\delta_i}^j \ , \\ \hat{J}_{1(ij)} = 0 \ , \\ D_i \hat{J}_{1jk} = 0 \ , \end{cases}$$

où D est la derivée covariante avec torsion. Notons que la présence de torsion fait que l'on n'a plus la fermeture de la forme de Kähler, mais la relation [Cha97]

$$d\hat{\rho}_1 = (\star\, dH) \wedge \hat{\rho}_1 \ ,$$

H étant défini par (2.13).

Par ailleurs, lorsque qu'en plus de (2.16), on a

$$\gamma(t) = t + \frac{a}{t} \ ,$$

la métrique initiale (2.11) s'identifie alors à la métrique d'Eguchi-Hanson [EH78] et devient donc hyper-Kähler. Les structures complexes auto-duales de la théorie initiale fournissent alors, par simple changement $\sigma_i \longrightarrow \hat{\sigma}_i$, celles de la théorie duale :

$$\begin{cases} \hat{\rho}_2 = \sqrt{\frac{t}{\gamma(t)}}\, dt \wedge \hat{\sigma}_1 + \sqrt{t\,\gamma(t)}\, \hat{\sigma}_2 \wedge \hat{\sigma}_3 = \frac{1}{2} \hat{J}_{2ij}\, d\hat{\phi}^i \wedge d\hat{\phi}^j \ , \\ \hat{\rho}_3 = \sqrt{\frac{t}{\gamma(t)}}\, dt \wedge \hat{\sigma}_2 + \sqrt{t\,\gamma(t)}\, \hat{\sigma}_3 \wedge \hat{\sigma}_1 = \frac{1}{2} \hat{J}_{3ij}\, d\hat{\phi}^i \wedge d\hat{\phi}^j \ . \end{cases}$$

On vérifie alors que les trois structures presque complexes $\hat{J}_a$ satisfont aux conditions

$$\begin{cases} \hat{J}_{ais}\, \hat{J}_b^{sj} = -\delta_{ab}\, \delta_i{}^j - \epsilon_{abc}\ \hat{J}_c^{ij}, \\ \hat{J}_{a(ij)} = 0 \ , \\ D_i \hat{J}_{ajk} = 0 \ , \end{cases}$$

ce qui démontre la propriété hyper-Kähler de la théorie duale d'Eguchi-Hanson.

Chapter 3

Renormalisabilité à deux boucles du dual du modèle $SU(2)$

Dans le schéma dimensionnel minimal, la renormalisabilité à deux boucles pour le dual du modèle chiral principal $SU(2)$ est perdue [ST96, BFHP96]. Nous avons démontré dans [CV00] qu'il en va de même lorsqu'on dualise le modèle Bianchi V (l'absence de torsion du modèle dual permettant alors une simplification des calculs). Nous nous sommes alors demandé s'il était possible de rétablir une renormalisabilité à deux boucles en permettant une redéfinition **finie** de la métrique elle-même. Le travail effectué dans l'article "*Dualised σ-models at the two loop order*" [BC01] a permis de répondre à cette question par l'affirmative dans le cas de la théorie $SU(2)$. La métrique déformée fait apparaître deux paramètres dont l'origine a pu être expliquée par la suite.

Le lecteur doit avoir à l'esprit que, en toute généralité, pour des corrections quantiques à l'action classique, tous les termes n'étant pas spécifiquement interdits pour une raison quelconque (comptage de puissance, symétries, lois de conservations, *etc...*) doivent apparaître dans l'action. Dans notre cas, toute action $O(3)$-invariante est ainsi possible. Le problème réside ici dans le fait qu'une partie des symétries d'origine a disparu lors de la dualisation et qu'ainsi la théorie duale n'est plus définie par un nombre suffisant d'identités de Ward. Pour l'instant, à notre connaissance, ces contraintes qui définiraient de façon inéquivoque la théorie duale ne sont pas connues. Dans [BD85], G. Bonneau et F. Delduc déforment dans un tel esprit le modèle sigma non-linéaire "Complex Sine-Gordon". Ils assurent la renormalisabilité à deux boucles de ce modèle en rajoutant comme contrainte la propriété classique de factorisation et de non-production. Ce sont des contraintes de ce type qui nous manquent ici. Celles-ci sont probablement liées à la dimension 2 de l'espace-temps, notamment à cause du potentiel de torsion et de son facteur $\epsilon^{\mu\nu}$: cela pourait expliquer pourquoi une renormalisation dimensionnelle minimale échoue puisque l'on sait que lorsque le processus de renormalisation ne respecte pas toutes les propriétés définissant la métrique, de nouveaux contre-termes finis sont nécessaires [Bon90].

3.1 L'action nue à deux boucles

L'action du modèle dual de la théorie $SU(2)$ s'écrit

$$S = \frac{1}{T}\int d^2x\ G_{ij}\ \ \partial_+\phi^i\partial_-\phi^j\ ,$$

où

$$G_{(ij)} \equiv g_{ij} = \frac{1}{1+\phi^2}\left[\delta_{ij}+\phi^i\phi^j\right] \quad \text{et} \quad G_{[ij]} \equiv h_{ij} = \frac{1}{1+\phi^2}\epsilon_{ij}{}^k\phi^k\ .$$

Le vecteur réel $\vec{\phi}$ est une représentation de $SU(2)$ et donc $\vec{\phi}^2$ est $SO(3)$ invariant. On note que la torsion brise la parité, mais que celle-ci est retrouvée par les changements simultanés $\vec{\phi} \longrightarrow -\vec{\phi}$ et $\epsilon_{ijk} \longrightarrow -\epsilon_{ijk}$. Ce sont les seules symétries globales de notre modèle.

Afin d'analyser la renormalisabilité à deux boucles, nous avons d'abord examiné les différentes façons possibles de réabsorber les divergences en développant l'action nue à deux boucles. Aux contre-termes usuels insuffisants, finis et infinis, qui vont renormaliser la constante de couplage et les champs, nous avons rajouté une déformation **finie** d'ordre $\hbar$ de la métrique classique g_{ij} et du potentiel de torsion h_{ij}. Ceci se fait bien sûr en conservant, pour la métrique déformée, l'invariance $SO(3)$.

Cette action nue s'écrit alors

$$S^o = \frac{1}{T^o}\int d^2x\ G^o_{ij}\ \partial_+\phi^{oi}\partial_-\phi^{oj}\ , \tag{3.1}$$

avec

$$\begin{cases} \dfrac{1}{T^o} = \dfrac{1}{T}\left[1+\dfrac{\hbar T}{2\pi}\left(\dfrac{\Lambda_1}{\varepsilon}+b\right)+\left(\dfrac{\hbar T}{2\pi}\right)^2\left(\dfrac{c}{\varepsilon^2}+\dfrac{\Lambda_2}{\varepsilon}+d\right)+\cdots\right], \\ \vec{\phi}^o = \vec{\phi}+\dfrac{\hbar T}{2\pi}\left(\dfrac{\vec{v}_1(\vec{\phi})}{\varepsilon}+\vec{w}_1(\vec{\phi})\right)+\left(\dfrac{\hbar T}{2\pi}\right)^2\left(\dfrac{\vec{v}_2(\vec{\phi})}{\varepsilon^2}+\dfrac{\vec{w}_2(\vec{\phi})}{\varepsilon}+\vec{x}_2(\vec{\phi})\right)+\cdots, \\ G^o_{ij} = G_{ij}+\dfrac{\hbar T}{2\pi}\ \widetilde{G}_{ij}+\left(\dfrac{\hbar T}{2\pi}\right)^2\widetilde{\widetilde{G}}_{ij}+\cdots \end{cases} \tag{3.2}$$

Parmi tous ces contre-termes introduits, certains vont de façon évidente être redondants car les déformations finies de la métrique peuvent contenir les redéfinitions finies du couplage et des champs. On peut ainsi faire disparaître ces dernières en les rentrant dans la déformation $\widetilde{G}_{ij}$. Ceci se fait par le biais d'une redéfinition des paramètres d'ordre $\hbar^2$. En posant

$$\begin{cases} \hat{\Lambda}_2 &= \Lambda_2 - b\,\Lambda_1\ , \\ \hat{\vec{w}}_2 &= \vec{w}_2 - v_1^k\partial_k\vec{w}_1\ , \\ \widehat{G}_{ij} &= \widetilde{G}_{ij} + \underset{\vec{w}_1}{\mathcal{L}}\left(G_{ij}\right) + b\,G_{ij}\ , \end{cases} \tag{3.3}$$

où $\mathcal{L}$ désigne la dérivée de Lie, on fait disparaître b et $\vec{w}_1$ de l'action nue. Il est donc possible de supposer ceux-ci nuls. La résolution des équations entraînant la renormalisabilité à deux boucles les fera de toute façon ré-apparaître sous la forme de paramètres arbitraires dans $\widetilde{G}_{ij}$.

Le développement de l'action nue (3.1) selon ses différents ordres en $\hbar$ nous donne alors la forme des contre-termes possibles à deux boucles :

$$CT_{ij}^2 = \frac{\hbar^2\, T}{4\pi^2\varepsilon}\left(\Lambda_1\, \widetilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\widetilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{w}_2}{\mathcal{L}}(G_{ij})\right)\ . \tag{3.4}$$

Par ailleurs, la relation (1.7) du Chapitre 1 fournit les contre-termes infinis à une boucle Λ_1 et v_1.

3.2 Renormalisation à deux boucles

Nous avons utilisé les divergences à deux boucles, notées Div_{ij}^2, trouvées par Hull et Townsend [HT87] dans le schéma dimensionnel minimal. Si celles-ci peuvent être compensées par les contre-termes (3.4), la théorie sera renormalisable jusqu'à deux boucles. Il faut pour cela vérifier l'équation

$$-\frac{\hbar^2\, T}{4\pi^2\varepsilon}\Delta_{ij} + \frac{\hbar^2\, T}{4\pi^2\varepsilon}\left(\Lambda_1\widetilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\widetilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{w}_2}{\mathcal{L}}(G_{ij})\right) = -Div_{ij}^2\ . \tag{3.5}$$

Le terme Δ_{ij} traduit des divergences en $\hbar^2$ induites par les divergences à une boucle de la métrique déformée : un terme en $\hbar$ dans la métrique entraîne automatiquement un terme du même ordre dans le tenseur de Ricci. A cause de la symétrie de la théorie, on peut écrire les inconnues dépendantes des champs sous la forme

$$\widetilde{G}_{ij} = \alpha(\tau)\delta_{ij} + \beta(\tau)\phi^i\phi^j + \gamma(\tau)\epsilon_{ijk}\phi^k \quad , \quad \vec{w}_2 = w_2(\tau)\vec{\phi}\ ,$$

où $\tau = \vec{\phi}^2$. En développant le système matriciel (3.5) selon δ_{ij}, $\phi^i\phi^j$ et $\epsilon_{ijk}\,\phi^k$, le problème devient alors celui de la résolution d'un système de trois équations différentielles assez compliquées, dont les inconnues sont les fonctions $\alpha(\tau)$, $\beta(\tau)$, $\gamma(\tau)$, $w_2(\tau)$ et la constante Λ_2. Une résolution exacte, mais partielle, de (3.5) montre que ces inconnues dépendent de deux fonctions $\bar{w}_1(\tau)$ et $\Gamma(\tau)$, et d'une constante $\bar{b}$. La fonction $\bar{w}_1(\tau)$ et la constante $\bar{b}$ sont en fait totalement arbitraires : nous retrouvons ici l'ambiguité de la déformation $\widetilde{G}$ qui peut être vue comme une façon d'introduire des contre-termes finis au couplage et aux champs. La fonction $\Gamma(\tau)$ est la solution d'une équation différentielle linéaire d'ordre quatre dont il est possible de donner les premiers termes de son développement en série grâce aux méthodes de Frobenius et de variation des constantes. Ceux-ci sont donnés dans [BC01]. Cette fonction dépend de quatre nouveaux paramètres arbitraires λ_0^o, λ_1^o, $\lambda_{-\frac{1}{2}}^o$ et $\lambda_{-\frac{3}{2}}^o$. Mais comme on effectue un développement en perturbation, il faut que les solutions soient régulières en $\tau = \vec{\phi}^2$ et cela impose l'égalité $\lambda_{-\frac{1}{2}}^o = \lambda_{-\frac{3}{2}}^o = 0$.

Nous pouvons donc conclure que la théorie duale $SU(2)$ est bien renormalisable à deux boucles, à condition de rajouter une déformation finie d'ordre $\hbar$ à la métrique classique. Cette déformation dépend de deux nouveaux paramètres, λ_0^o et λ_1^o. Notons par ailleurs qu'il est impossible de trouver un schéma de renormalisation (via un choix de $\bar{b}$ et $\bar{w}_1$) permettant d'assurer la nullité de $\widetilde{G}$. Ceci re-démontre la non-renormalisabilité du modèle $SU(2)$ en absence de déformation.

Nous allons maintenant discuter de l'origine de ces deux nouveaux paramètres. Celle-ci nous est apparue après la parution de [BC01]

3.3 Origine des "nouveaux" paramètres

En utilisant le fait que Δ_{ij} est proportionnel à la différence des tenseurs de Ricci obtenus avec et sans déformation, il est possible de réécrire (3.5) sous la forme intéressante

$$Ric\left(G+\frac{\hbar T}{2\pi}\widetilde{G}\right)-\left(\Lambda_1+\frac{\hbar T}{2\pi}\Lambda_2\right)\left(G+\frac{\hbar T}{2\pi}\widetilde{G}\right)-\underset{\vec{v}_1+\frac{\hbar T}{2\pi}\vec{w}_2}{\mathcal{L}}\left(G+\frac{\hbar T}{2\pi}\widetilde{G}\right) = -\frac{2\pi\varepsilon}{\hbar}Div^2+\mathcal{O}(\hbar^2)\ .$$

Cette écriture montre que les solutions des équations homogènes (le terme de gauche) donnent exactement les déformations possibles de la métrique qui conservent le caractère quasi-Einstein de la métrique initiale jusqu'au premier ordre en $\hbar$. Ainsi, les paramètres libres qui vont apparaître dans la solution générale ne sont en fait que l'expression d'un schéma de renormalisation à une boucle vis à vis des déformations possible de la métrique : ces nouvelles constantes renormalisent des paramètres cachés dans la métrique duale qui ont été fixés par le processus de dualisation. Nous retrouvons ici le fait mentionné dans l'introducion de ce chapitre : tout terme non explicitement interdit devrait apparaître dans le lagrangien. Supposons par exemple que l'on connaisse la métrique **générale** $\widehat{G}_{ij}[\mu]$ quasi-Einstein à trois dimensions et de symétrie $SU(2)$ et qu'elle ne dépende, par simplicité, que d'un paramètre[1] μ. Si on note $\lambda_\mu\,\widetilde{G}_\mu$ la solution de l'équation homogène associée à (3.5), et que l'on suppose que $\widehat{G}[\mu=0]=G$, l'action renormalisée à une boucle s'écrit

$$S_{R^1}=\underbrace{\frac{1}{T}\left(1+\frac{\hbar T}{2\pi\varepsilon}\Lambda_1\right)}_{\displaystyle\frac{1}{T_{R^1}}}\int d^2x\ \underbrace{\left(G_{ij}+\frac{\hbar T}{2\pi}\lambda_\mu\,\widetilde{G}_\mu\right)}_{\displaystyle\widehat{G}_{ij}\left[\mu_{R^1}=0+\frac{\hbar T}{2\pi}\lambda_\mu\right]}\ \partial_+\underbrace{(\phi^i+\frac{\hbar T}{2\pi\varepsilon}v^i)}_{\displaystyle\phi^i_{R^1}}\,\partial_-\underbrace{(\phi^j+\frac{\hbar T}{2\pi\varepsilon}v^j)}_{\displaystyle\phi^j_{R^1}}\ .$$

On voit bien ainsi que tout nouveau paramètre λ_μ de la solution générale de (3.5) ne concerne que la renormalisation à une boucle du couplage μ. C'est donc uniquement

[1] Les paragraphes précédents montrent que ces paramètres sont en fait au nombre de quatre.

une solution particulière de l'équation avec second membre (3.5) qui va donner la déformation de la métrique permettant de retrouver la renormalisabilité à deux boucles. Dit autrement, **il n'y a pas plus de nouveaux paramètres à une boucle qu'il n'y en avait déjà dans la théorie duale classique**! Ceci est très important puisque l'on sait que la renormalisabilité d'une théorie est non seulement fondée sur une absorption des divergences par des contre-termes locaux, mais surtout sur le fait que l'on ne doit pas être obligé d'introduire indéfiniment de nouveaux paramètres.

3.4 Conclusion

Si la transformation canonique et l'approche par intégrale fonctionnelle assurent l'équivalence classique entre les théories initiales et leurs théories duales, la quantification des théories duales reste un problème ouvert. Ainsi qu'il était noté dans l'introduction de ce chapitre, les théories duales ne sont pas suffisamment définies. Certes, à l'ordre un en perturbation, pour toutes les théories testées, le couplage se renormalise de façon identique pour les deux théories, ce qui suggère une équivalence au premier ordre quantique. Cependant, l'exemple du modèle chiral principal $SU(2)$ montre qu'il existe dans le dual des paramètres cachés μ_i dont la présence resurgit lors des renormalisations aux ordres supérieurs à un. On ne sait pas, dans le processus de dualisation, ce qui fixe ces paramètres au niveau classique, mais ceux-ci empêchent de définir de façon inéquivoque les déformations en $\hbar$ de la métrique nécessaires à la renormalisabilité à deux boucles : qu'est-ce qui, dans une telle déformation, relève d'une renormalisation finie d'ordre $\hbar$ des μ_i et qu'est-ce qui n'en relève pas ? Il manque une condition qui exprimerait la façon dont les paramètres μ_i sont fixés classiquement, condition dont on peut penser qu'elle fixerait aussi, ordre par ordre, les déformations possible de la métrique.

Enfin, revenons sur la propriété qu'ont certaines théories, telle celle du modèle chiral principal Bianchi V, de donner un dual sans torsion. Si on brise les isométries droites, la théorie duale ne possède plus aucune symétrie. Il semble ainsi possible de construire des métriques quasi-Einstein sans Killing. Cela serait un pas intéressant vers la construction de métriques d'Einstein sans isométries, construction qui prend tout son sens après la découverte récente d'une constante cosmologique non nulle. La T-dualité deviendrait dans cette perspective un outil original de construction géométrique.

Part II

Super-espace harmonique et métriques d'Einstein

Chapter 4

La super-symétrie : un outil géométrique

Nous nous proposons dans ce chapitre de rappeler comment, a partir de théories super-symétriques, il est possible de construire des métriques hyper-Kähler en dimension $4n$. Cette vision de la super-symétrie comme un outil géométrique a été rendu possible grâce à une certaine technique et à un certain formalisme. La technique est celle du "quotient hyper-Kähler", voire du "quotient quaternionique", qui permet d'élaborer des métriques complexes à partir d'une métrique simple, généralement celle de l'espace plat. Le formalisme est celui du super-espace harmonique qui simplifie, dans notre perspective de construction, les calculs nécessaires, et permet d'imposer, de façon plus ou moins simple mais toujours claire, les isométries de la métrique recherchée. Après un certain nombre d'étapes, que nous expliciterons, cette métrique se réduit au secteur bosonique de la théorie super-symétrique.

4.1 Extension super-symétrique d'un modèle sigma

Nous allons dans un premier temps rappeler comment les modèles sigma super-symétriques sont reliés aux géométries riemanniennes Kählériennes, hyper-Kähleriennes et quaternioniques.

Pour cela, nous considèrons un modèle sigma à quatre dimensions d'espace-temps, d'action

$$S = \int d^4x \, \eta^{\mu\nu} \, g_{ij}(\phi) \, \partial_\mu \phi^i \partial_\nu \phi^j \ ,$$

et nous nous proposons d'en étudier ses éventuelles extensions super-symétriques. Pour que de telles extensions existent, la métrique g doit vérifier certaines propriétés.

Si N désigne le nombre de générateurs de super-symétrie, il a été démontré que

- pour une super-symétrie $N = 1$ globale, la métrique g doit être Kähler [Zum79], c'est à dire qu'il existe $J^i{}_j$ vérifiant

$$\left\{ \begin{array}{ll} J \text{ est une structure presque complexe} & : J^2 = -\mathcal{I} \ , \\ J \text{ est hermitique par rapport à } g_{ij} & : J_{ij} = J_{[ij]} = g_{ik}\, J^k{}_j \ , \\ D_i J_{jk} = 0 \ . & \end{array} \right.$$

  La dernière condition entraîne l'intégrabilité de $J^i{}_j$.

- pour une super-symétrie globale $N = 2$, la métrique g doit être hyper-Kähler [AGF81], c'est à dire qu'il existe un triplet de structures complexes (*i.e.* presque complexes et intégrables) $J_{a=1,2,3}$ vérifiant

$$\left\{ \begin{array}{l} J_a J_b = -\delta_{ab}\mathcal{I} + \epsilon_{abc} J_c \ , \\ D_i (J_a)_{jk} = 0 \ . \end{array} \right.$$

  La condition hyper-Kähler entraîne la nullité du tenseur de Ricci :

$$\boldsymbol{Ric(g) = 0} \ .$$

- pour coupler le modèle sigma à la super-gravité $N = 2$, il faut que g soit Einstein à Weyl auto-dual ($D = 4$), ou quaternionique ($D = 4n \geq 8$) [BW83]. Dans la littérature mathématique, on utilise la terminologie "quaternion-Kähler" pour désigner ces métriques. La quaternionicité se traduit sur les structures complexes par le fait que la condition $D_i(J_a)_{jk} = 0$ devient

$$D_i (J_a)_{jk} = \epsilon_{abc}\, (A_b)_i\, (J_c)_{jk} \ ,$$

  la table de multiplication quaternionique restant inchangée. La métrique g est alors Einstein :

$$\boldsymbol{Ric(g) = \lambda\, g} \ , \qquad \boldsymbol{\lambda \neq 0} \ .$$

Ainsi, les extensions super-symétriques $N = 1$ et $N = 2$ d'un modèle sigma imposent des conditions **géométriques** sur la métrique g. Inversement, il est possible, à partir d'une théorie super-symétrique, de construire de nouvelles métriques hyper-Kähler ou quaternioniques après extraction du secteur bosonique.

4.2 Premiers résultats, méthode du "quotient"

Ce sont Curtright et Freedman qui, dans leur article [CF80], ont pour la première fois utilisé cette idée. Ils ont ainsi pu obtenir un modèle hyper-Kähler et un modèle quaternionique.

4.2.1 Un modèle hyper-Kähler

On considère deux multiplets de champs complexes φ_1 et φ_2 à $(n+1)$ composantes qui se transforment selon un doublet de $SU(2)_H$:

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} , \qquad \delta\phi = \vec{\varepsilon} \cdot \frac{\vec{\tau}}{2\,i}\,\phi \,,$$

et selon la représentation fondamentale de $SU(n+1)_T$:

$$\delta\varphi_1 = \vec{\varepsilon} \cdot \frac{\vec{\lambda}}{2\,i}\,\varphi_1 \,, \qquad \delta\varphi_2 = \vec{\varepsilon} \cdot \frac{\vec{\lambda}}{2\,i}\,\varphi_2 \,,$$

où les $\vec{\lambda}$ sont les matrices de Gell-Mann $(n+1) \times (n+1)$ hermitiques de trace nulle.

Curtright et Freedman partent de la métrique la plus simple, celle de l'espace plat : le secteur bosonique du lagrangien s'écrit alors $\partial_\mu \phi^+ . \partial_\mu \phi$. La construction du modèle hyper-Kähler se fait alors en deux temps :

1. On jauge tout d'abord le $U(1)$, de champ de jauge A_μ, défini par les transformations infinitésimales :

   $$\delta\varphi_1 = i\,\varepsilon(x)\,\varphi_1 \,, \qquad \delta\varphi_2 = i\,\varepsilon(x)\,\varphi_2 \,, \qquad \delta A_\mu = \partial_\mu \varepsilon(x) \,.$$

   Les dérivées partielles ∂_μ sont alors remplacées par les dérivées covariantes $D_\mu = \partial_\mu - i\,A_\mu$, et, après fixation de la jauge, le secteur bosonique du lagrangien s'écrit

   $$L = \partial_\mu \bar{\varphi}_i \,\partial_\mu \varphi_i + \frac{(\bar{\varphi}_i \partial_\mu \varphi_i - \varphi_i \partial_\mu \bar{\varphi}_i)^2}{4\,\bar{\varphi}_i\,\varphi_i} \,. \qquad (4.1)$$

   Ce lagrangien donne la distance par simple substitution $\partial_\mu \longrightarrow d$.

2. On impose ensuite la contrainte vectorielle

   $$\phi^+\,\vec{\tau}\,\phi = \vec{b} \,. \qquad (4.2)$$

   L'origine super-symétrique de cette contrainte sera explicitée plus loin dans le cadre du super-espace harmonique. Avec le choix $\vec{b} = (0,0,1)$, celle-ci se traduit pour les multiplets par

   $$\bar{\varphi}_1\,\varphi_2 = 0 \,, \qquad \bar{\varphi}_1\,\varphi_1 - \bar{\varphi}_2\,\varphi_2 = 1 \,.$$

   La première égalité fournit ainsi deux contraintes réelles, tandis que la deuxième n'en fournit qu'une.

Remarques :

- Nous sommes partis d'une théorie possédant $4(n+1)$ paramètres réels. En jaugeant un Killing et en fixant trois contraintes réelles, on aboutit donc à une théorie finale à $4n$ coordonnées.

- La contrainte (4.2) brise le $SU(2)_H$ en un $U(1)$, exprimé par les rotations autour de $\vec{b}$. Les isométries finales de la théorie seront donc $U(1)_H \times SU(n+1)_T$.
- Curtright et Freedman n'ont pas obtenu la métrique explicite de ce modèle. En effet, une bonne partie du travail reste à faire puisqu'il faut encore trouver une redéfinition des multiplets φ_1 et φ_2 afin d'obtenir des coordonnées indépendantes X^i qui devront satisfaire la contrainte (4.2) : c'est ce qu'on appelle la **résolution des contraintes**. Cette redéfinition faite, et après report dans (4.1), la distance sera obtenue en remplaçant les termes $\partial_\mu X^i$ par dX^i.
- Ce type de construction, dégagée de son origine super-symétrique, a été formulé de façon mathématique dans la construction dite du "**quotient hyper-Kähler**" [HKLR87]. Elle consiste, en partant d'une métrique simple (ici l'espace plat), à diminuer petit à petit le nombre de dimensions via des fixations de jauge et des contraintes. Comme les métriques hyper-Kähler sont forcément de dimension $4n$, le nombre total de Killing jaugés et de contraintes réelles doit être un multiple de 4.

4.2.2 Un modèle quaternionique

A partir du même contenu bosonique $\partial_\mu\phi^+ .\, \partial_\mu\phi$, on construit cette fois une théorie de jauge non-abelienne $SU(2)$, de champ de jauge $\vec{V}_\mu$, définie par les transformations infinitésimale :

$$\delta\phi = \vec{\varepsilon}(x).\frac{\vec{\tau}}{2\,i}\,\phi\,, \qquad \delta\vec{V}_\mu = \partial_\mu\vec{\varepsilon}(x) + \vec{\varepsilon}(x)\wedge\vec{V}_\mu\,.$$

Les dérivées partielles ∂_μ sont ainsi remplacées par les dérivées covariantes $D_\mu = \partial_\mu - i\,\frac{\vec{\tau}}{2}\cdot\vec{V}_\mu$. Après élimination des trois degrés de liberté de $\vec{V}_\mu$ grâce aux équations du champ, on obtient la distance

$$d\phi^+ .\, d\phi + \frac{(\phi^+\vec{\tau}\,d\phi - d\phi^+\vec{\tau}\,\phi)^2}{4\,\phi^+ .\,\phi}\ . \tag{4.3}$$

La super-symétrie impose cette fois la contrainte scalaire

$$\phi^+ .\,\phi = 1\ . \tag{4.4}$$

Remarques :

- Ici, on jauge trois degrés de liberté, et on impose une contrainte réelle : encore une fois, la métrique finale sera de dimension $4n$. A la fin de leur article, Curtright et Freedman suggèrent de combiner les deux processus précédents en partant d'une action à $4(n+2)$ paramètres afin d'obtenir, après les deux fixations de jauge et les quatres contraintes, une métrique à $4n$ dimensions. Bien que non explorée, on pense que cette idée permet d'aboutir à l'extension quaternionique de la métrique d'Eguchi-Hanson.

- Les isométries de la théorie finale sont au moins $SU(2)_H \times SU(n+1)_T$. En fait, on peut les étendre à $Sp(n+1)$, comme on le verra plus loin.
- Ce type de construction a lui aussi été formulé de façon mathématique dans la construction dite du "**quotient quaternionique**" [Gal87a]. De nombreux exemples quaternioniques obtenus par cette méthode sont donnés dans [Gal87b, Gal92], mais là encore, les métriques explicites n'ont pas été données.

4.3 Le problème de la résolution des contraintes

Il restait, pour les deux modèles précédents, à redéfinir les $4n$ paramètres restants dans le secteur bosonique, de façon à résoudre les contraintes. Le travail est alors purement **algébrique**. Par exemple, si on part du lagrangien

$$L = \partial_\mu \Phi^T \, \partial_\mu \Phi \qquad \text{où } \Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} ,$$

les φ_i étant réels, et que l'on impose la contrainte $\Phi^T \, \Phi = 1$, on voit bien qu'un choix de coordonnées reste à faire. On peut ainsi, à partir de la contrainte, décider d'exprimer φ_3 en fonction de φ_1 et φ_2, que l'on reporte alors dans le lagrangien :

$$L = \partial_\mu \varphi_i \, \partial_\mu \varphi_i + \frac{(\varphi_i \, \partial_\mu \varphi_i)^2}{1 - \varphi_i \varphi_i} , \qquad i = 1, 2 .$$

La substitution formelle $\partial_\mu \longrightarrow d$ permet ainsi d'obtenir une distance sur la sphère S^2, avec les coordonnées φ_1 et φ_2. Les cas où la résolution des contraintes est aussi simple sont malheureusement rares.

Pour le modèle hyper-Kähler (4.1), Alvarez-Gaumé et Freedman ont obtenu un système de coordonnées satisfaisant la contrainte (4.2), pour $\vec{b} = (0, 2, 0)$, dans [AGF80]. Celui-ci est donné par

$$\begin{cases} \varphi_1 & = f \, (1 + \bar{u}.u)^{-1/2} \, (u, 1) \ , \\ \varphi_2 & = f \, (1 + \bar{v}.v)^{-1/2} \, (\bar{v}, 1) \ , \\ f & = \left[\dfrac{(1 + \bar{u}.u)(1 + \bar{v}.v)}{(1 + u.v)^2} \right]^{1/4} \end{cases} ,$$

où u et v sont des vecteurs à n composantes complexes. Dans ces coordonnées, la métrique est explicitement Kähler avec pour potentiel de Kähler

$$K = \left[\frac{(1 + \bar{u}.u)(1 + \bar{v}.v)}{(1 + u.v)(1 + \bar{u}.\bar{v})} \right]^{1/2} .$$

Ils ont pu vérifier la nullité du tenseur de Ricci, mais n'ont pu démontrer la propriété hyper-Kähler de la métrique. Cependant, guidés par diverses considérations, ils ont conjecturé que celle-ci était la métrique de Calabi [Cal79]. Cette conjecture n'a été démontrée que quatorze ans plus tard par G. Valent dans [Val94]. Auparavant, le cas $n = 1$ avait été résolu dans [FG81], où il était démontré que cette métrique d'isométries $U(2)$ s'identifiait à celle d'Eguchi-Hanson [EH78], cas particuler des métriques de Calabi pour la dimension 4.

Rappelons que la métrique de Calabi était définie à partir des coordonnées z_α complexes ($\alpha = 1, \cdots, n$) sur CP^n, pour lesquelles la métrique de Fubini-Study s'écrit

$$g_{\alpha\bar\beta} = \frac{\partial^2 K}{\partial z_\alpha \, \partial z_{\bar\beta}} \,, \qquad K = \ln(1 + \bar z . z) \,.$$

On complétait alors celles-ci par les coordonnées ζ_α complexes ($\alpha = 1, \cdots, n$) avec

$$t = g^{\alpha\bar\beta}(z, \bar z) \, \zeta_\alpha \, \bar\zeta_\beta \,.$$

La métrique hyper-Kähler de Calabi découlait alors du potentiel de Kähler $\mathcal{K}$, en coordonnées holmorphes (z_α, ζ_α), donné par

$$\mathcal{K} = \ln(1 + \bar z . z) - \ln(1 + \sqrt{1 + 4\, t}) \,.$$

L'identification avec (4.1), et avec le vecteur $\vec b = (0, 0, 1)$, est donné dans [Val94] :

$$\varphi_1 = \frac{1}{\sqrt{(1 + \bar z . z)\, h}} \, (z, 1) \,, \qquad \varphi_2 = \sqrt{(1 + \bar z . z)\, h} \, (\bar\zeta, -\bar z \, \bar\zeta) \,, \qquad h = \frac{2}{1 + \sqrt{1 + 4\, t}} \,.$$

Comme le montre cet exemple, non seulement il n'est pas aisé de trouver de "bonnes" coordonnées qui simplifieront la résolution des contraintes, mais de surcroît, celles-ci ne seront généralement pas les "bonnes" coordonnées qui donneront au final une métrique explicite simple, voire reconnaissable.

En ce qui concerne le modèle quaternionique trouvé par Curtright et Freedman, l'identification de la métrique obtenue ne semble pas avoir été abordée dans la littérature. Il s'agit en fait du modèle $\mathbb{HP}^n$, modèle quaternionique compact correspondant au quotient $\dfrac{Sp(n+1)}{Sp(1) \times Sp(n)}$. On le démontre en posant

$$\varphi_1^\alpha = q_0^\alpha - i \, q_3^\alpha \,, \qquad \varphi_2^\alpha = q_2^\alpha - i \, q_1^\alpha \,, \qquad \alpha = 1, \cdots, n+1 \,,$$

où q_0^α et q_i^α ($i = 1, 2, 3$) sont réels. On peut alors définir un vecteur à $n+1$ composantes quaternioniques :

$$\zeta^\alpha = q_0^\alpha \, e_0 + \sum_{r=1}^{3} q_r^\alpha \, e_r \,, \qquad \begin{cases} e_0 \, e_r = e_r \, e_0 = e_r \\ e_r \, e_s = -\delta_{rs} \, e_0 + \epsilon_{rst} \, e_t \end{cases} .$$

La contrainte (4.4) devient alors

$$\hat\zeta^\alpha \, \zeta^\alpha = 1 \,, \qquad \text{où } \hat\zeta^\alpha = q_0^\alpha \, e_0 - \sum_{r=1}^{3} q_r^\alpha \, e_r \,,$$

tandis que la distance (4.3) s'identifie à

$$d\hat{\zeta}^\alpha\, d\zeta^\alpha - (\hat{\zeta}^\alpha\, d\zeta^\alpha)(d\hat{\zeta}^\beta\, \zeta^\beta)\,.$$

La contrainte peut alors s'éliminer en posant

$$\zeta^\alpha = \frac{\chi^\alpha}{\sqrt{1+\rho}} \quad \text{pour } \alpha = 1, \cdots, n\,, \qquad \zeta^{n+1} = \frac{1}{\sqrt{1+\rho}}\,, \qquad \rho = \hat{\chi}^\alpha\, \chi^\alpha\,.$$

On retrouve alors pour la métrique l'écriture de type Fubini-Study correspondante à $\mathbb{HP}^n$:

$$ds^2 = \frac{d\hat{\chi}^\alpha\, d\chi^\alpha}{1+\rho} - \frac{(\hat{\chi}^\alpha\, d\chi^\alpha)(d\hat{\chi}^\beta\, \chi^\beta)}{(1+\rho)^2}\,.$$

Pour $n = 1$, on obtient la sphère S^4, d'isométries $Sp(2) \sim SO(5)$.

4.4 Métriques hyper-Kähler et super-espace harmonique

La découverte de nouvelles métriques dans le cadre super-symétrique n'a pu être réalisée concrètement que grâce à l'introduction d'un formalisme simple et puissant, celui du super-espace harmonique dans lequel la super-symétrie $N = 2$ est automatiquement réalisée. Celui-ci a fait son apparition dans [GIK$^+$84]. Outre de nombreuses applications, ce formalisme de super-champs a permis de mettre en oeuvre la construction explicite de nouvelles métriques hyper-Kähler à partir du secteur bosonique de modèles sigma dans le cadre de la super-symétrie $N = 2$. Tout en fournissant une méthode systématique pour construire de telles métriques, il permet de lire directement sur le lagrangien les symétries que possédera la métrique finale.

Dans le secteur analytique, *i.e.* stable sous l'action de la super-symétrie $N = 2$, les coordonnées du super-espace harmonique sont

$$(\zeta) = \{x_A^m = x^m - 2\, i\, \theta^+\, \sigma^m\, \bar{\theta}^+, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}, u_i^\pm\}\,. \tag{4.5}$$

Le super-espace harmonique tire son nom des coordonnées $u_i^\pm$ qui sont en fait les harmoniques sphériques sur la sphère S^2 :

$$\begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^- \end{pmatrix} = \begin{pmatrix} \cos\theta & i\,\sin\theta\, e^{-i\phi} \\ i\,\sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix}\,, \qquad \text{avec} \begin{cases} u^{+i}\, u_i^- = 1 \\ u_i^+\, u_j^- - u_i^-\, u_j^+ = \epsilon_{ij} \end{cases}\,.$$

Les indices $i, j \in [\![1, 2]\!]$ sont montés et baissés à l'aide du tenseur anti-symétrique ϵ_{ij}, avec la convention $\epsilon_{12} = \epsilon^{21} = 1$. Il est facile de vérifier que tout terme de la forme X^{+i} se projette sur les harmoniques $u^{\pm i}$ selon

$$X^{+i} = (u_s^-\, X^{+s})\, u^{+i} - (u_s^+\, X^{+s})\, u^{-i}\,.$$

Les θ^+ et $\bar{\theta}^+$ sont des variables grassmaniennes, et l'on a

$$\theta^+_\alpha = \theta^i_\alpha \, u^+_i \ , \ \bar{\theta}^+_{\dot{\alpha}} = \bar{\theta}^i_{\dot{\alpha}} \, u^+_i \ .$$

On peut maintenant écrire la forme générale des super-champs dans le secteur analytique :

$$Q^+(x_A, \theta^+, \bar{\theta}^+, u^\pm_i) = F^+ + \theta^{+2} \, M^- + \bar{\theta}^{+2} \, N^- + i \, \theta^+ \, \sigma^m \bar{\theta}^+ \, A^-_m + \theta^{+2} \, \bar{\theta}^{+2} \, D^{-3} + \text{fermions} \ .$$

Les composantes F^+, M^-, N^-, A^-_m et D^{-3} sont des séries, généralement infinies, en $u^\pm_i$. On a donc généralement une infinité de champs auxiliaires. Comme on ne s'intéresse qu'au secteur bosonique du lagrangien, on ne tiendra jamais compte des composantes fermioniques des super-champs. On définit par ailleurs la dérivée covariante

$$D^{++} = \partial^{++} - 2 \, i \, \theta^+ \, \sigma^m \, \bar{\theta}^+ \, \partial_m \ , \qquad \text{où } \partial^{\pm\pm} = u^\pm_i \, \frac{\partial}{\partial u^{\mp i}} \ .$$

Ensuite, notons que le super-espace (4.5) est réel vis à vis de la conjugaison généralisée $\tilde{}$, produit de la conjugaison complexe et de la projection antipodale sur S^2. Ainsi, on a

$$\widetilde{u^\pm_i} = u^{\pm i} \ , \qquad \widetilde{\theta^+} = \bar{\theta}^+ \ , \qquad \widetilde{\bar{\theta}^+} = -\theta^+ \ , \qquad \widetilde{D^{++}} = D^{++} \ .$$

Enfin, en plus du $SU(2)_S$ super-symétrique agissant sur les indices i, j, on introduit un $SU(2)_{PG}$ de Pauli-Gürsey sur des indices a, b en définissant le doublet Q^+_a par

$$Q^+_{a=1} = Q^+ \qquad \text{et} \qquad Q^+_{a=2} = -\widetilde{Q^+} \ .$$

Ici aussi, les indices a et b sont montés et baissés à l'aide de ϵ_{ab}.

Il est alors possible d'énoncer le résultat de [GIK+84] :

Le lagrangien le plus général d'un modèle sigma non-linéaire à super-symétrie globale N=2, en dimension 4, est donné par

$$\boldsymbol{L_{HK} = \frac{1}{2} \int [d^2\theta^+ d^2\bar{\theta}^+][du] \ \{ L^+_a(Q^+, u^\pm) \, D^{++} Q^{+a} + L^{+4}(Q^+, u^\pm) \} \ .} \quad (4.6)$$

L'intégration sur $[d^2\theta^+ d^2\bar{\theta}^+]$ extrait les termes en $\theta^{+2} \, \bar{\theta}^{+2}$, tandis que l'intégration sur $[du]$ ne garde que le singlet sur S^2 des séries en $u^\pm_i$. En pratique, nous prendrons toujours

$$L^+_a(Q^+, u^\pm) = Q^+_a \ .$$

La métrique correspondante à (4.6) ne doit dépendre que de quatres "coordonnées". Celles-ci sont le plus souvent données par le terme d'ordre un en u^+_i de $Q^+_a|_{\theta=0}$ que l'on note F^i_a. Pour obtenir la distance, il faut donc d'abord calculer tous les champs auxiliaires (M^-_a, N^-_a, $A^-_{a,m}$, D^{-3}_a et les termes d'ordre supérieur à un en u^+_i de F^+_a), ne garder que le singlet sur S^2 du terme en $\theta^{+2} \, \bar{\theta}^{+2}$, puis effectuer la substitution $\partial_m \longrightarrow d$. Les métriques obtenues seront alors toutes hyper-Kähler.

4.5 Pré-potentiels, exemples

Le terme $L^{+4}(Q^+, u^\pm)$ de (4.6) est appelé **pré-potentiel** car cet objet contient toutes les informations sur la métrique finale. La correspondance entre le pré-potentiel et la métrique finale n'est pas connue à priori, celle-ci n'étant explicite qu'une fois les calculs (élimination des champs auxiliaires plus éventuellement fixation des jauges et résolution des contraintes) effectués. Il s'agit donc de construire une correspondance entre pré-potentiels et métriques finales. Cependant, comme nous le verrons par la suite, les isométries de la métrique finale apparaissent clairement dans le pré-potentiel. C'est dans ce dernier que seront introduit des termes brisant explicitement le $SU(2)_{PG}$ ou le $SU(2)_S$. Dans [GORV88], il a été démontré que tout Killing de $SU(2)_S$ était holomorphe tandis que tout Killing de $SU(2)_{PG}$ était tri-holomorphe. Ainsi, toute métrique obtenue à partir du lagrangien (4.6) qui ne brise pas totalement le $SU(2)_{PG}$ donnera une métrique multicentre (*cf.* Annexe B).

4.5.1 Taub-NUT

La première correspondance entre le pré-potentiel $L^{+4}(Q^+, u^\pm)$ et une métrique hyper-Kähler en dimension 4 a été donnée dans [GIOS86]. Cet article fondateur marque le début de la recherche des actions du super-espace harmonique qui sont associées aux métriques multicentres. Ses auteurs y ont établi que la densité lagrangienne

$$\mathcal{L}_{TN} = \widetilde{Q^+}\, D^{++} Q^+ + \frac{\lambda}{2}\left(\widetilde{Q^+}\, Q^+\right)^2 \ , \tag{4.7}$$

correspond au multicentre Taub-NUT, de potentiel (*cf.* Annexe B)

$$V_{TN} = \lambda + \frac{1}{|\vec{X}|} \ .$$

On comprend mieux les isométries de (4.7) si on définit le triplet $a^{(ab)}$ tel que

$$a^{11} = a^{22} = 0\,, \qquad a^{12} = \frac{i\,\sqrt{\lambda}}{2} \ .$$

On peut alors réécrire, à un facteur (-2) près, (4.7) sous la forme

$$Q^+_a\, D^{++} Q^{+a} + \left(a^{ab}\, Q^+_a\, Q^+_b\right)^2 \ .$$

Celle-ci montre explicitement la brisure du $SU(2)_{PG}$ en un $U(1)_{PG}$ par le triplet a^{ab}. Comme ce dernier lagrangien ne comporte aucune dépendance explicite en $u^\pm_i$, le $SU(2)_S$ est préservé. Les isométries finales sont donc $SU(2)_S \times U(1)_{PG}$. Le super-espace harmonique permet de voir directement sur le lagrangien les isométries de la métrique finale.

4.5.2 Eguchi-Hanson

Le pré-potentiel donnant Eguchi-Hanson a été trouvé dans [GIOT86] en utilisant, dans le super-espace harmonique, l'approche du quotient hyper-Kähler utilisé par Curtright et Friedman. Pour cela, on construit d'abord une théorie de jauge avec un doublet de $SO(2)$ supplémentaire d'indices $A, B = 1,2$: les super-champs s'écrivent maintenant Q_A^{+a}. Le lagrangien de départ est alors

$$L_{EH} = \frac{1}{2}\int [d^2\theta^+ d^2\bar{\theta}^+][du]\ \left\{Q^+_{aA} D^{++} Q_A^{+a} + W^{++}\ \left(\epsilon_{AB}\, Q_A^{+a}\, Q^+_{aB} + c^{++}\right)\right\}\ , \qquad (4.8)$$

avec $D^{++}c^{++} = 0$.

Cette fois-ci, le $SU(2)_{PG}$ est conservé, tandis que le terme $c^{++} = c^{(ij)}u_i^+ u_j^+$ brise explicitement le $SU(2)_S$ en un $U(1)_S$. L'invariance de jauge locale est donnée par

$$\delta Q_A^{+a} = \varepsilon\, \epsilon_{AB}\, Q_B^{+a}\ , \qquad \delta W^{++} = D^{++}\varepsilon\ .$$

Le champ de jauge W^{++} joue le rôle d'un multiplicateur de Lagrange qui introduit une contrainte sur les champs Q_A^{+a}.

On définit les composantes des super-champs par

$$\begin{cases} Q^+_{aA} = F^+_{aA} + i\,\theta^+\,\sigma^m\bar{\theta}^+\,A^-_{m,aA} + \theta^{+2}\,\bar{\theta}^{+2}\,D^{-3}_{aA} + \text{fermions}\ , \\ W^{++} = i\,\theta^+\,\sigma^m\bar{\theta}^+\,W_m + \theta^{+2}\,\bar{\theta}^{+2}\,q^{-2} + \text{fermions}\ . \end{cases}$$

On peut en effet démontrer que les termes en θ^{+2} et $\bar{\theta}^{+2}$ disparaissent à la fin des calculs. Par ailleurs, on a utilisé la jauge de Wess-Zumino pour le super-vecteur W^{++} : on a alors $W^{++}|_{\theta=0} = 0$.

Après développement, et extraction des termes en $\theta^{+2}\,\bar{\theta}^{+2}$, on obtient l'écriture du lagrangien (4.8) sous la forme

$$L_{EH} = \frac{1}{2}\int [du]\ \Big\{2D^{-3}_{aA}\,\partial^{++}F_A^{+a} + 2\,A^-_{m,aA}\,\partial_m F_A^{+a} - \frac{1}{2}\,A^-_{m,aA}\,\partial^{++}A^{-a}_{m,A}$$
$$+\,\epsilon_{AB}\,q^{-2}\,F_A^{+a}\,F^+_{aB} + \epsilon_{AB}\,W_m\,A^-_{m,aA}\,F_B^{+a} + q^{-2}\,c^{++}\Big\}\ .$$

Il faut maintenant éliminer les champs auxiliares, obtenir les contraintes, et fixer la jauge sur $SO(2)$. Tout ceci se fait grâce aux équations du mouvement. Ces dernières s'écrivent :

- par rapport à D^{-3}_{aA} :

$$\partial^{++}F_A^{+a} = 0 \qquad \Longrightarrow \qquad F_A^{+a} \equiv F_A^{ia}\,u_i^+\ .$$

Ainsi, seul le premier terme en u_i^+ dans le développement de F_A^{+a} n'est pas nul, ce qui permet d'éliminer une infinité de champs auxiliaires ! Les F_A^{ia} seront les futures coordonnées contraintes de la métrique recherchée.

- par rapport à q^{-2} :

$$\epsilon_{AB}\, F_A^{+a}\, F_{aB}^{+} + c^{++} = 0 \ .$$

Il s'agit d'un triplet de contraintes sur les F_A^{ia} : les contraintes sont donc données par le facteur du champ de jauge W^{++} dans le lagrangien initial (4.8).

- par rapport à $A^{-}_{m,aA}$:

$$\partial^{++} A^{-\ a}_{m,\ A} = 2\,\partial_m F_A^{+a} + W_m\, \epsilon_{AB}\, F_B^{+a} \ . \tag{4.9}$$

Comme F_A^{+a} est limité à son unique terme en u_i^{+}, cette relation s'intègre aisément et fournit

$$A^{-\ a}_{m,\ A} = 2\,\partial_m F_A^{-a} + W_m\, \epsilon_{AB}\, F_B^{-a} \ , \qquad F_A^{-a} = F_A^{ia}\, u_i^{-} \ .$$

Avant de fixer la jauge grâce aux équations du mouvement en W_m, il est plus simple de reporter d'abord cette dernière expression dans le lagrangien et d'intégrer sur les harmoniques $u\pm_i$. On obtient alors

$$L_{EH} = \frac{1}{2}\partial_m F_A^{ia}\, \partial_m F_{iaA} + K_m\, W_m + \alpha\, {W_m}^2 \ ,$$

où

$$\alpha = \frac{1}{8}F^2 \, , \qquad F^2 = F_A^{ia}\, F_{iaA} \, , \qquad K_m = -\frac{1}{2}\epsilon_{AB}\, F_A^{ia}\, \partial_m F_{iaB} \ .$$

On peut maintenant fixer la jauge $SO(2)$, les équations du mouvement en W_m donnant

$$W_m = -\frac{K_m}{2\,\alpha} \ .$$

On reste alors avec la distance

$$L_{EH} = L_0 + L_{\text{vec}} \, , \qquad \text{où} \quad \left\{ \begin{array}{rcl} L_0 & = & \frac{1}{2}\partial_m F_A^{ia}\, \partial_m F_{iaA} \\ L_{\text{vec}} & = & -\frac{{K_m}^2}{4\,\alpha} \end{array} \right. \ ,$$

et le triplet de contraintes

$$\epsilon_{AB}\, F_A^{ia}\, F_{aB}^{j} + c^{ij} = 0 \ .$$

Si on identifie $F_1^{ia} \equiv 2\,i\,\varphi_1$ et $F_2^{ia} \equiv 2\,i\,\varphi_2$, on retrouve, à un facteur près, le lagrangien (4.1) et la contrainte (4.2) donnés par Curtright et Friedman dans le cas $n = 1$: il s'agit bien de la métrique d'Eguchi-Hanson.

4.5.3 Double Taub-NUT avec masses différentes

Dans [GIOT86], les auteurs ont proposé de mélanger les deux pré-potentiels précédents :

$$\mathcal{L}_{dTN} = Q^{+}_{aA} D^{++} Q^{+a}_{A} + W^{++} \left(\epsilon_{AB}\, Q^{+a}_{A}\, Q^{+}_{aB} + c^{++}\right) + \left(a^{ab}\, Q^{+}_{aA}\, Q^{+}_{bA}\right)^2 \ .$$

Les isométries finales sont alors $U(1)_S \times U(1)_{PG}$ à cause de la double brisure c^{++} et a^{ab}. Il a été démontré dans [GORV88] que l'on obtient ainsi la métrique du double Taub-NUT, de potentiel (*cf.* Annexe B) :

$$V = \lambda + \frac{1}{|\vec{X} - \vec{\xi}|} + \frac{1}{|\vec{X} + \vec{\xi}|} \ .$$

La constante λ apparaît dans le pré-potentiel via le triplet a^{ab}, le vecteur $\vec{\xi}$ via le triplet c^{++}.

Il manquait encore le paramètre permettant d'associer une masse différente à chacun des centres (le paramètre ρ dans (B.6)), ceci afin d'obtenir la métrique à deux centres la plus générale. Nous avons résolu ce problème dans l'article "*$U(1) \times U(1)$ quaternionic metrics from harmonic superspace*" (*cf.* Annexe C.5) . Une façon de procéder est de partir du pré-potentiel

$$L^{+4}_{dTN+} = W^{++} \left(\epsilon_{AB}\, Q^{+a}_{A}\, Q^{+}_{aB} - \beta_0\, a^{ab}\, Q^{+}_{aA}\, Q^{+}_{bA} + c^{++}\right) + \left(a^{ab}\, Q^{+}_{aA}\, Q^{+}_{bA}\right)^2 \ ,$$

avec une invariance de jauge $SO(2)$ locale modifiée :

$$\delta Q^{+a}_{A} = \varepsilon \left(\epsilon_{AB}\, Q^{+a}_{B} + \beta_0\, a^{ab}\, Q^{+}_{bA}\right) \ , \qquad \delta W^{++} = D^{++}\varepsilon \ .$$

C'est la constante β_0 qui donnera une masse différente aux centres $+\vec{\xi}$ et $-\vec{\xi}$. Cependant, cette présentation est en fait peu satisfaisante car on y note une dissymétrie profonde entre son terme "Eguchi-Hanson", quadratique en Q^{+a}_{A}, et son terme "Taub-NUT", quartique en Q^{+a}_{A}. Par ailleurs, on voit que la métrique d'Eguchi-Hanson s'obtient par une théorie de jauge, contrairement à celle de Taub-NUT. Il est possible d'unifier ces deux métriques, ainsi qu'il a été démontré dans [IV00], de façon à n'avoir que des termes quadratiques dans les super-champs. Pour cela, on rajoute un super-champ g^{+}_{r}, de façon à partir de la densité lagrangienne

$$\begin{aligned}\mathcal{L}_{dTN++} = Q^{+}_{aA} D^{++} Q^{+a}_{A} + g^{+}_{r} D^{++} g^{+r} + W^{++} \left(\epsilon_{AB}\, Q^{+a}_{A}\, Q^{+}_{aB} - \beta_0\, a^{ab}\, Q^{+}_{aA}\, Q^{+}_{bA} + c^{++}\right) \\ + V^{++} \left(2\, u^{+}_{r}\, g^{+r} - a^{ab}\, Q^{+}_{aA}\, Q^{+}_{bA}\right) \ .\end{aligned} \tag{4.10}$$

Il faut alors jauger les deux $U(1)$ commutants suivants :

1. $\delta_\varepsilon Q^{+a}_{A} = \varepsilon \left(\epsilon_{AB}\, Q^{+a}_{B} + \beta_0\, a^{ab}\, Q^{+}_{bA}\right) \ , \ \ \delta_\varepsilon g^{+}_{r} = 0 \ , \ \ \delta_\varepsilon W^{++} = D^{++}\varepsilon \ , \ \ \delta_\varepsilon V^{++} = 0 \ .$
2. $\delta_\varphi Q^{+a}_{A} = \varphi\, a^{ab}\, Q^{+}_{bA} \ , \ \ \delta_\varphi g^{+}_{r} = \varphi\, u^{+}_{r} \ , \ \ \delta_\varphi W^{++} = 0 \ , \ \ \delta_\varphi V^{++} = D^{++}\varphi \ .$

La transformation de jauge (2) permet de restreindre g^+_r par la condition $u^-_r\, g^{+r} = 0$, soit

$$g^{+r} = -\left(u^+_s\, g^{+s}\right)\, u^{-r}\ . \tag{4.11}$$

La variation par rapport à V^{++} fournit la contrainte

$$u^+_r\, g^{+r} = \frac{1}{2} a^{ab}\, Q^+_{aA}\, Q^+_{bA}\ ,$$

qui donne alors pour le terme cinétique

$$g^+_r\, D^{++} g^{+r} = \left(u^+_r\, g^{+r}\right)^2 = \frac{1}{4}\left(a^{ab}\, Q^+_{aA}\, Q^+_{bA}\right)^2\ .$$

On retrouve bien, à un facteur près, le terme quartique du pré-potentiel de taub-NUT.

C'est par cette méthode de double jauge que la métrique du double Taub-NUT avec des masses différentes a été obtenue dans l'article [CIV02] ; elle y apparait en tant que limite hyper-Kähler d'une nouvelle métrique d'Einstein à Weyl auto-dual dont la densité lagrangienne se réduit bien à (4.10) lorsque l'on fait tendre la constante d'Einstein vers zéro. Ceci sera développé plus en détail dans le Chapitre 5.

Si on pose $g^{+r} = g^{ri}\, u^+_i + \cdots$, les contraintes que l'on obtient se lisent sur le lagrangien :

$$\begin{aligned}
&\epsilon_{AB}\, F^{a(i}_A\, F^{j)}_{aB} - \beta_0\, a^{ab}\, F^i_{aA}\, F^j_{bA} + c^{(ij)} = 0\\
&g^{(ij)} - \frac{1}{2} a^{ab}\, F^i_{aA}\, F^j_{bA} = 0
\end{aligned}$$

Notons ici que le choix de jauge (4.11) a permis de réduire g^{ij} à un triplet $g^{(ij)}$. Ainsi, il faut partir d'une métrique à 12 dimensions (8 pour les F^+_{aA} et 4 pour g^+_r). Après avoir jaugé deux Killing et obtenu 3+3=6 contraintes réelles, on obtient bien une métrique à 4 dimensions. Les isométries finales sont $U(1)_S \times U(1)_{PG}$. Celles-ci peuvent être augmentées dans deux, et seulement deux, cas :

1. $a^{(ab)} = 0$: on obtient alors Taub-NUT,
2. $c^{(ij)} = 0$ et $\beta_0 = 0$: on obtient alors Eguchi-Hanson.

De façon plus générale, le pré-potentiel associé à un multicentre quelconque a été donné dans [GORV88]. Une dérivation plus transparente et plus simple se trouve dans [G+01].

Notons enfin que la métrique d'Atiyah-Hitchin, qui ne possède aucun Killing tri-holomorphe (*cf.* Annexe B), n'a pas encore pu être déduite du super-espace harmonique, même si son pré-potentiel est connu. En effet, si nous considérons

$$\mathcal{L}_{AH} = \widetilde{Q^+}\, D^{++} Q^+ + \frac{\lambda}{2}\left(\widetilde{Q^+}\, Q^+\right)^2 + \frac{\mu}{2}\left(\widetilde{Q^+}^2 + Q^{+2}\right)^2\ , \tag{4.12}$$

on voit que le couplage en μ brise le $U(1)_{PG}$ qui subsiste avec le couplage en λ. Il ne reste alors plus comme isométries que celles, typiques de la métrique d'Atiyah-Hitchin, du $SU(2)_S$. Malheureusement, à cause des termes quartiques du lagrangien, les équations sur la sphère S^2 lors du calcul des champs auxiliaires sont d'une telle complexité que personne n'a pu jusqu'ici descendre à la métrique explicite. En effet, les champs auxiliaires sont fixés par les équations du mouvement qui abaissent d'un ordre le degré du lagrangien dans les super-champs. Ainsi, si celui-ci est au plus de degré deux, on peut être sûr que l'on ne pourra avoir que des équations différentielles **linéaires** à résoudre. C'est exactement ce qui se passait dans la section précédente (*cf.* équation (4.9)). Notons que dans le cas du pré-potentiel quartique de Taub-NUT (4.7), la présence du $U(1)_{PG}$ restant permet de retrouver cette linéarisabilité et d'aboutir à la métrique.

Chapter 5

Métriques quaternioniques et super-espace harmonique

Les métriques hyper-Kähler en dimension 4 avec au moins un Killing tri-holomorphe étant connues, ainsi qu'il est rappelé dans l'Annexe B, ce chapitre montre comment il est possible d'en obtenir des extensions quaternioniques, toujours dans le cadre du super-espace harmonique. En effet, à moins de briser totalement le $SU(2)_{PG}$, on est certain de retrouver une métrique multicentre dans le cadre des super-symétries globales $N = 2$ en dimension 4.

5.1 Lagrangien général

E. Ivanov et G. Valent ont pu extraire dans [IV00] le secteur bosonique des supergravités obtenues dans [GIOS87]. Leur résultat est qu'a toute métrique hyper-Kähler de lagrangien (4.6) correspond une métrique quaternionique dont le lagrangien est donné par

$$\begin{aligned} L_Q &= \frac{1}{2}\int [d^2\theta^+ d^2\bar\theta^+][du]\ \left\{-q_i^+\, D^{++} q^{+i} + \kappa^2\,(u_i^-\, q^{+i})^2\left[Q_a^+\, D^{++}Q^{+a} + L^{+4}(Q^+, v^\pm)\right]\right\} \\ &\quad + \frac{1}{2}\kappa^2\,\mathcal{V}_m^{(ij)}\,\mathcal{V}_{m(ij)}\ , \end{aligned} \tag{5.1}$$

où

$$\mathcal{V}_m^{(ij)} = 3\int [du]\, u^{-i}u^{-j}\left[f^{+k}\partial_m f_k^+ - \kappa^2 (u_k^-\, f^{+k})^2\, F^{+a}\partial_m F_a^+\right]\ .$$

La constante κ^2 est proportionnelle à la constante d'Einstein. Les super-champs supplémentaires q_i^+ sont appelés "compensateurs". On a posé

$$f_i^+ = q_i^+|_{\theta=0} \qquad \text{et} \qquad F_a^+ = Q_a^+|_{\theta=0}\ .$$

Les $u_i^{\pm}$ sont remplacés dans L^{+4} par de nouvelles harmoniques $v_i^{\pm}$ définies par

$$v^{+i} = \frac{q^{+i}}{(u_t^- \, q^{+t})} = u^{+i} - \frac{(u_s^+ \, q^{+s})}{(u_t^- \, q^{+t})} \, u^{-i} \qquad \text{et} \qquad v^{-i} = u^{-i} \; .$$

Par ailleurs, la partie bosonique f^{+i} du compensateur est contrainte par la relation

$$\int [du] \; \left\{ f^{+i} \partial^{--} f_i^+ - \kappa^2 \, (u_i^- \, f^{+i})^2 \, F^{+a} \partial^{--} F_a^+ \right\} = \frac{1}{\kappa^2} \; . \tag{5.2}$$

La limite hyper-Kähler, c'est à dire la limite $\kappa^2 \longrightarrow 0$, se traduit sur les compensateurs par

$$q^{+i} \longrightarrow \frac{u^{+i}}{|\kappa|} \; .$$

Dans cette limite, on a alors

$$\kappa^2 \, (u_i^- \, q^{+i})^2 \longrightarrow (u_i^- \, u^{+i})^2 = 1 \; , \quad D^{++} q_i^+ \longrightarrow 0 \quad \text{et} \quad (u_i^+ \, q^{+i}) \longrightarrow 0 \Longrightarrow v^{\pm i} \longrightarrow u^{\pm i} \; ,$$

ce qui entraîne la convergence du lagrangien quaternionique (5.1) vers le lagrangien hyper-Kähler (4.6) lorsque $\kappa^2 \longrightarrow 0$.

En pratique, tous les calculs s'opèrent dans la jauge

$$f^{+i} = f^j{}_i \, u_j^+ \qquad \text{avec} \qquad f^j{}_i = \omega \, \delta^j{}_i \; ,$$

ce qui fournit les simplifications suivantes :

$$\begin{cases} u_i^+ \, f^{+i} = 0 \; , \\ u_i^- \, f^{+i} = \omega \; , \\ f^{ij} \, f_{ij} = 2 \, \omega^2 \; . \end{cases}$$

Ce choix permet de réécrire la contrainte (5.2) sous la forme

$$\kappa^2 \, \omega^2 = \frac{1}{1 - \frac{\kappa^2}{2} F^2} \; , \qquad \text{où} \quad F^2 = F_A^{ia} \, F_{iaA} \; .$$

Ainsi, à un signe près qui ne change pas le lagrangien, ω est fixé : les compensateurs n'introduisent pas de degrés de liberté supplémentaires. Ce choix de jauge donne pour le vecteur $\mathcal{V}_m^{(ij)}$ l'écriture

$$\mathcal{V}_m^{(ij)} = -\kappa^2 \, \omega^2 \, F_{aA}^{(i} \, \partial_m F_A^{j)a} \; .$$

Enfin, lorsque $L^{+4}(Q^+, v^{\pm})$ est une forme bi-linéaire dans les super-champs et dans les harmoniques v^+, comme par exemple $\epsilon_{AB} \, Q_A^{+a} \, Q_{aB}^+ + 2 \, v_r^+ \, g^{+r} + c^{(ij)} \, v_i^+ \, v_j^+$, une simplification très appréciable apparaît si on pose pour les super-champs :

$$\hat{Q}^+ = |\kappa| \, (u_i^- \, q^{+i}) \, Q^+ \; . \tag{5.3}$$

La densité lagrangienne de (5.1) s'écrit alors :

$$\mathcal{L}_Q = -q_i^+ \, D^{++} q^{+i} + \hat{Q}_a^+ \, D^{++} \hat{Q}^{+a} + L^{+4}(\hat{Q}^+, |\kappa| \, q^+) \; ,$$

où les harmoniques v_i^+ présentes dans L^{+4} ont été remplacées par $|\kappa| \, q_i^+$.

5.2 Extension quaternionique d'Eguchi-Hanson

A partir du pré-potentiel (4.8) donné au chapitre 4 de la métrique d'Eguchi-Hanson, et afin d'obtenir l'extension quaternionique de cette dernière, on sait donc qu'il faut maintenant utiliser la densité lagrangienne

$$\mathcal{L}_{QEH} = -q_i^+ D^{++} q^{+i} + \kappa^2 (u_i^- q^{+i})^2 \left[Q_{aA}^+ D^{++} Q_A^{+a} + W^{++} \left(\epsilon_{AB} Q_A^{+a} Q_{aB}^+ + c^{++} \right) \right] ,$$

avec cette fois

$$c^{++} = c^{ij} v_i^+ v_j^+ , \qquad D^{++} c^{++} = 0 .$$

En utilisant la redéfinition (5.3), cette densité se simplifie en

$$\mathcal{L}_{QEH} = -q_i^+ D^{++} q^{+i} + \hat{Q}_{aA}^+ D^{++} \hat{Q}_A^{+a} + W^{++} \left(\epsilon_{AB} \hat{Q}_A^{+a} \hat{Q}_{aB}^+ + \kappa^2 c_{ij} q_i^+ q_j^+ \right) .$$

On peut alors vérifier l'invariance de cette dernière expression sous l'action du $SO(2)$ local tel que

$$\delta \hat{Q}_{aA}^+ = \varepsilon \, \epsilon_{AB} \hat{Q}_{aB}^+ , \qquad \delta q^{+i} = \varepsilon \, \kappa^2 c^{ij} q_j^+ , \qquad \delta W^{++} = D^{++} \varepsilon .$$

Notons que par rapport au cas hyper-Kähler, la jauge doit être définie de façon différente : à cause du fait que c^{++} est maintenant multiplié par un terme nouveau contenant le compensateur, celui-ci doit obligatoirement subir une déformation infinitésimale si on veut maintenir $W^{++} = D^{++}\varepsilon$.

La descente vers la métrique de l'extension quaternionique d'Eguchi-Hanson donne alors lieu à des calculs tout à fait comparables à ceux de la section (4.5.2). Ceux-ci permettent d'aboutir au lagrangien

$$L = \frac{\kappa^2 \omega^2}{2} \left[\nabla_m F_A^{ia} \nabla_m F_{iaA} - \frac{\kappa^2}{4} \hat{c}^2 W_m{}^2 \right] + \frac{\kappa^2}{2} \mathcal{V}_m^{(ij)} \mathcal{V}_{m(ij)} ,$$

où

$$\hat{c}^2 = c^{ij} c_{ij} \qquad \text{et} \qquad \nabla_m F_A^{ia} = \partial_m F_A^{ia} + \frac{1}{2} W_m \, \epsilon_{AB} F_B^{ia} .$$

Les équations de champ par rapport au champ de jauge W_m donnent alors

$$W_m = 2 \frac{\epsilon_{AB} F_A^{ia} \partial_m \hat{F}_{iaB}}{\hat{F}^2 - \hat{c}^2 \kappa^2} .$$

Cette métrique est complétée par le *même* triplet de contraintes que pour Eguchi-Hanson :

$$\epsilon_{AB} F_A^{a(i} F_{aB}^{j)} + c^{(ij)} = 0 .$$

Nous n'avons présenté ici que la moitié du travail : il faut encore résoudre les contraintes et donner de "vraies" coordonnées pour la métrique finale de dimension 4. Une bonne écriture de celle-ci est

$$g = \frac{1}{4(1-\kappa^2 s)^2} \left\{ \frac{s - \kappa^2 c^2}{s^2 - c^2} ds^2 + (s - \kappa^2 c^2)({\sigma_1}^2 + {\sigma_2}^2) + \frac{s^2 - c^2}{s - \kappa^2 c^2} {\sigma_3}^2 \right\} ,$$

où les 1-formes σ_i vérifient $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k$. Celles-ci sont invariantes sous l'action de $SU(2)_{PG}$ et tournent comme un vecteur sous l'action de $SU(2)_S$. Ainsi, les isométries sont bien $SU(2)_{PG} \times U(1)_S$, le paramètre c décrivant la brisure de $SU(2)_S$. Notons que alors que dans le cas hyper-Kähler la limite $c \longrightarrow 0$ redonne l'espace plat, on retrouve ici la sphère S^4.

5.3 Extension quaternionique du double Taub-NUT

L'extension quaternionique de la métrique du double Taub-NUT a été obtenue de façon tout à fait similaire dans la lettre "*Quaternionic extension of the double Taub-NUT metric*" (*cf.* Annexe C.4). On obtient alors une métrique d'Einstein à Weyl auto-dual, invariante sous l'action des symétries $U(1)_{PG} \times U(1)_S$. Cette métrique dépend de deux paramètres a et c qui dans la limite hyper-Kähler donnent respectivement le potentiel à l'infini et la distance entre les deux centres de la métrique du double Taub-NUT (*cf.* Annexe B). Dans l'article [CIV02], plus détaillé, il a été rajouté le paramètre β_0 de la section (4.5.3) qui donne, dans la limite $\kappa \longrightarrow 0$, des masses différentes au deux centres. Il a aussi été possible de rajouter un quatrième paramètre α_0 qui disparaît dans la limite hyper-Kähler. Ainsi, des métriques quaternioniques différentes peuvent avoir la même limite hyper-Kähler.

La densité lagrangienne de départ se déduit (presque) directement de celle qui correspond au cas hyper-Kähler (4.10). Rappelons qu'il faut introduire un super-champ supplémentaire g_r^+ afin de briser le $SU(2)_{PG}$ sans terme quartique en Q^+. Nous écrivons ici cette densité sous sa forme simplifiée en $\hat{Q}^+$, $\hat{g}^+$:

$$\begin{aligned}\mathcal{L}_{QdTN+} \;=\;& -q_i^+\mathcal{D}^{++}q^{+i} + \hat{Q}^+_{aA}\mathcal{D}^{++}\hat{Q}^{+a}_A + \hat{g}^+_r\mathcal{D}^{++}\hat{g}^{+r} \\ & + W^{++}\left[\epsilon_{AB}\,\hat{Q}^{+a}_A\hat{Q}^+_{aB} - \beta_0\, a^{(ab)}\hat{Q}^+_{aA}\hat{Q}^+_{bA} + \kappa^2 c^{(ij)}\left(q_i^+\,q_j^+ - \hat{g}_i^+\,\hat{g}_j^+\right)\right] \\ & + V^{++}\left[2|\kappa|\left(q_r^+\,\hat{g}^{+r}\right) - a^{(ab)}\hat{Q}^+_{aA}\hat{Q}^+_{bA} + \alpha_0\,\kappa^2 c^{(ij)}\left(q_i^+\,q_j^+ - \hat{g}_i^+\,\hat{g}_j^+\right)\right].\end{aligned} \tag{5.4}$$

Ce lagrangien est invariant sous les deux transformations de jauge $U(1)$ qui commutent :

$$\left\{\begin{aligned} \delta_\varepsilon\hat{Q}^{+a}_A &= \varepsilon\,\epsilon_{AB}\,\hat{Q}^{+a}_B + \varepsilon\,\beta_0\,a^{(ab)}\,\hat{Q}^+_{bA} \\ \delta_\varepsilon\hat{g}^{+r} &= \varepsilon\,\kappa^2\,c^{(rs)}\hat{g}^+_s \\ \delta_\varepsilon q^{+i} &= \varepsilon\,\kappa^2\,c^{(ij)}\,q_j^+ \\ \delta_\varepsilon W^{++} &= D^{++}\varepsilon \\ \delta_\varepsilon V^{++} &= 0 \end{aligned}\right. \quad \text{et} \quad \left\{\begin{aligned} \delta_\varphi\hat{Q}^{+a}_A &= \varphi\,a^{(ab)}\hat{Q}^+_{bA} \\ \delta_\varphi\hat{g}^{+r} &= \varphi\,|\kappa|\,q^{+r} + \varphi\,\alpha_0\,\kappa^2\,c^{(rs)}\,\hat{g}^+_s \\ \delta_\varphi q^{+i} &= \varphi\,|\kappa|\,\hat{g}^{+i} + \varphi\,\alpha_0\,\kappa^2\,c^{(ij)}q_j^+ \\ \delta_\varepsilon W^{++} &= 0 \\ \delta_\varepsilon V^{++} &= D^{++}\varphi \end{aligned}\right. .$$

Notons que la présence d'un terme en $\hat{g}_i^+\,\hat{g}_j^+$ dans le facteur de W^{++}, absente dans le cas hyper-Kähler, est ici rendu nécessaire pour que ce facteur soit bien invariant sous la transformation de jauge de paramètre φ. Dans le cas contraire les deux $U(1)$ jaugés ne commuteraient pas, ce qui entrainerait par la suite des contraintes supplémentaires qui trivialiseraient la métrique. Ce terme nouveau par rapport à (4.10) entraîne à son tour une redéfinition des transformations de jauge de paramètres ε : on n'a plus $\delta_\varepsilon g^{+r} = 0$ comme dans le cas hyper-Kähler.

Les paramètres supplémentaires α_0 et β_0 sont rendus possibles par le fait qu'il existe des libertés quant à la façon de jauger deux $U(1)$ indépendants sur le lagrangien L_{QdTN+}. Le paramètre α_0 ne peut pas être introduit dans le cas hyper-Kähler car il repose sur l'existence d'un $U(1)$ supplémentaire pour le lagrangien quaternionique (5.4) par rapport au lagrangien hyper-Kähler correspondant (4.10). En effet, on voit très bien dans (5.4) la similarité entre les termes en q^{+i} et ceux en $\hat{g}^{+r}$: le lagrangien quaternionique est invariant sous la rotation hyperbolique de q^{+i} et $\hat{g}^{+r}$,

$$\delta \hat{g}^{+r} = \varphi \, |\kappa| \, q^{+r} \ , \qquad \delta q^{+i} = \varphi \, |\kappa| \, \hat{g}^{+r} \ .$$

Encore une fois, les deux triplets de contraintes se lisent sur la densité lagrangienne (5.4) :

$$\epsilon_{AB} \, F_A^{a(i} \, F_{aB}^{j)} - \beta_0 \, a^{(ab)} \, F_{aA}^i \, F_{bA}^j - \kappa^2 \, c_{(kl)} \, g^{(ki)} \, g^{(lj)} + c^{(ij)} = 0 \ ,$$

et

$$2 \, g^{(ij)} - a^{(ab)} \, F_{aA}^i \, F_{bA}^j + \alpha_0 \, \left[c^{(ij)} - \kappa^2 \, c_{(kl)} \, g^{(ki)} \, g^{(lj)} \right] = 0 \ .$$

A partir de L_{QdTN+}, nous avons pu descendre en composantes, résoudre les contraintes et calculer la métrique finale. Celle-ci peut s'écrire sous la forme contractée

$$4D^2 \, \boldsymbol{g} = \frac{P}{A} \left(d\phi + \frac{Q}{4 \, P} d\alpha \right)^2 + A \left(\boldsymbol{g}_0 + \frac{1 + a^2 \lambda \, \rho^2}{P} \, X^2 \, d\alpha^2 \right)$$

où

$$\boldsymbol{g}_0 = \frac{dY^2 + dX^2 + a^2 \lambda \, (X \, dY - Y \, dX)^2}{1 + a^2 \lambda \, \rho^2}$$

est la métrique sur la sphère S^2 ($a^2\lambda < 0$), l'espace plat ($a^2\lambda = 0$), ou le plan hyperbolique ($a^2\lambda > 0$). On a posé $\rho^2 = X^2 + Y^2$. La constante d'Einstein vaut $\Lambda = -16 \, \lambda$ avec $\lambda = \frac{\kappa^2}{4}$. Les fonctions D, A, P et Q ne dépendent que des coordonnées X et Y, ainsi que des paramètres a, c, α_0, β_0 et de la constante d'Einstein. Les deux Killing ∂_α et ∂_ϕ rendent alors l'isométrie $U(1) \times U(1)$ transparente dans l'écriture de $\boldsymbol{g}$.

Les quatres fonctions qui apparaissent dans $\boldsymbol{g}$ sont très compliquées, ce qui fait qu'il n'a pas pu être possible de vérifier par un calcul direct, hormis les cas $\alpha_0 = \beta_0 = 0$, que cette métrique était bien Einstein et que son tenseur de Weyl était auto-dual. Pour cela, nous avons utilisé une approche due à Przanowski [Prz91] et Tod [Tod95]. Ces auteurs ont montré que pour toute métrique d'Einstein à Weyl auto-dual qui possède un Killing ∂_ϕ, il existe un système de coordonnées dans lequel la métrique peut s'écrire sous la forme

$$g = \frac{1}{w^2} \left[\frac{1}{\mathcal{W}} (d\phi + \Theta)^2 + \mathcal{W} (e^v (d\nu^2 + d\mu^2) + dw^2) \right] . \qquad (5.5)$$

La métrique g est alors Einstein à Weyl auto-dual si et seulement si

$$\begin{cases} \text{(a)} \qquad -2\,\dfrac{\Lambda}{3}\,\mathcal{W} \;=\; 2 - w\,\partial_w v\ , \\ \text{(b)} \quad \left(\partial_\nu^2 + \partial_\mu^2\right) v + \partial_w^2 \left(e^v\right) \;=\; 0\ , \\ \text{(c)} \quad -d\Theta = \partial_\nu \mathcal{W}\, d\mu \wedge dw \;+\; \partial_\mu \mathcal{W}\, dw \wedge d\nu + \partial_w \left(\mathcal{W}\, e^v\right) d\nu \wedge d\mu\ . \end{cases}$$

Il a été possible d'écrire $\boldsymbol{g}$ sous la forme donné par Przanowski et Tod, et de vérifier cet ensemble de trois conditions. La métrique que nous avons obtenue est bien une métrique d'Einstein à Weyl auto-dual.

Par ailleurs, il a été possible de démontrer que $\boldsymbol{g}$ était, au facteur conforme w^2 de (5.5) près, une nouvelle solution aux équations couplées d'Einstein-Maxwell :

$$\begin{cases} Ric_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho} g^{\rho\sigma} F_{\nu\sigma} - \frac{1}{4}\, g_{\mu\nu}\, F_{\rho\sigma}\, F^{\rho\sigma}\right). \\ dF^- = 0\ , \qquad dF^+ = 0\ . \end{cases}$$

En effet, une équivalence due à Flaherty [Fla78] assure que toute métrique à Weyl auto-dual et solution des équations couplées Einstein-Maxwell est une métrique Kähler à courbure scalaire nulle, et réciproquement. De plus, toute métrique Einstein à Weyl auto-dual avec au moins un Killing est conforme à une métrique Kähler à courbure scalaire nulle, le facteur conforme étant w^2. Il en résulte que $w^2\boldsymbol{g}$ est une solution des équations couplées Einstein-Maxwell. Il restait à comparer celle-ci aux solutions connues dans la littérature : les métriques de Perjès-Israel-Wilson [Per71, IW72] et les métriques de Plebanski-Demianski [PD76]. Nous avons vérifié que $w^2\boldsymbol{g}$ n'appartient à aucune de ces deux classes.

Les différentes limites de $\boldsymbol{g}$ sont détaillées sur la figure (5.1).

5.4 Conclusion et perspectives

La preuve qu'il était possible de trouver explicitement de nouvelles métriques d'Einstein grâce au super-espace harmonique a été donnée. En dimension 4, les métriques obtenues sont de plus conformes à des solutions des équations couplées Einstein-Maxwell

De façon générale, la résolution des équations d'Einstein fait apparaître des équations différentielles couplées non-linéaires du deuxième ordre. Grâce au super-espace harmonique, il est possible d'obtenir des classes de solutions quaternioniques ne nécessitant que la résolution d'équations différentielles du **premier ordre**. Celles-ci apparaissent lors du calcul des champs auxiliaires. La plupart du temps, elles sont de plus **linéaires** : il faut pour cela que le lagrangien de départ soit au plus quadratique dans les super-champs. En contre-partie, la réelle difficulté devient **algébrique** : il s'agit de

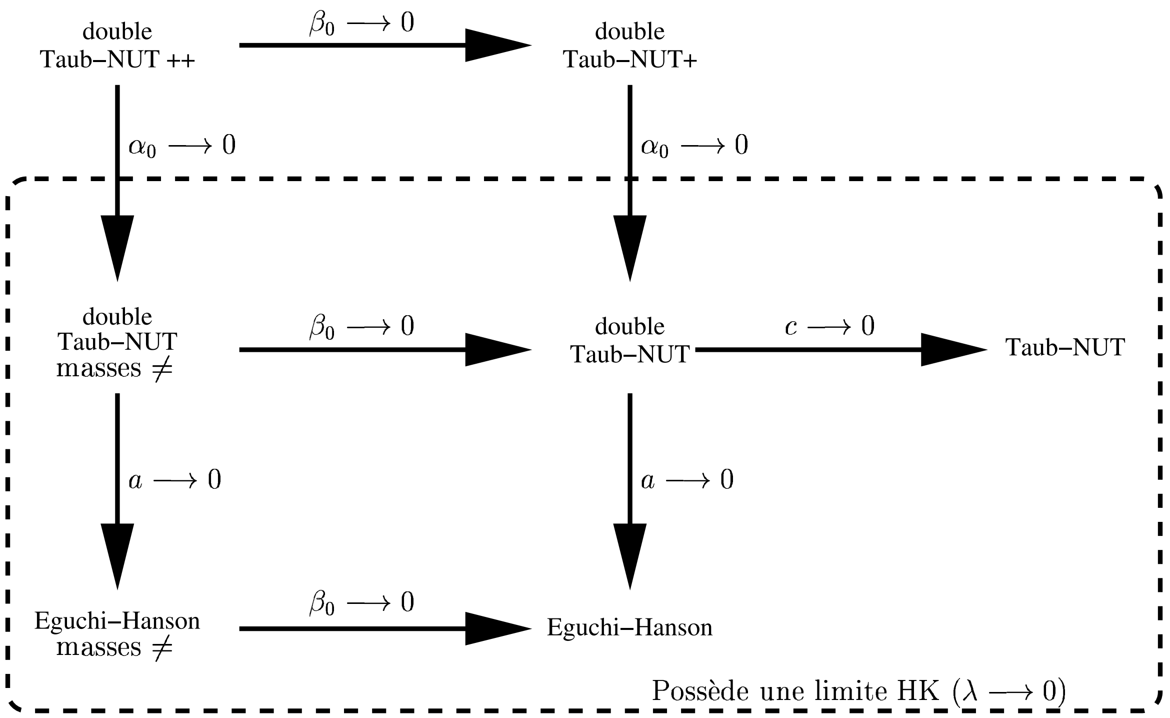

Figure 5.1: Cas limites

la résolution des contraintes, d'une part, et de la mise en forme de la métrique d'autre part afin de regrouper les éléments matriciels g_{ij}. Cette mise en forme nécessite, dès qu'il y a quelques paramètres, l'utilisation de l'outil informatique. Ainsi, la forme brute de la métrique $\boldsymbol{g}$ après résolution des contraintes tient sur cinq pages d'impression, sa forme développée sur plus de cent pages ! Trois changements de coordonnées successifs ont été nécessaires afin de pouvoir écrire la métrique sous une forme plus simple. Il n'en reste pas moins que les divers termes qui apparaissent, et en particulier la fonction Q, sont encore relativement compliqués. Il manque au super-espace harmonique une méthode qui lui permetrait de fournir des coordonnées plus "naturelles" dans lesquelles la métrique obtenue serait plus simple.

Alors que nous écrivions l'article [CIV02], Calderbank et Pedersen ont trouvé [CP01] la linéarisation correspondante aux métriques d'Einstein à Weyl auto-dual et d'isométries $U(1) \times U(1)$ à 4 dimensions : ils ont pu écrire celles-ci sous une forme compacte qui ne dépend que d'une fonction. Cette fonction, comparable au potentiel V des multicentres, doit être solution d'une équation différentielle *linéaire* du deuxième ordre dans les coordonnées. Cependant, la méthode qu'ils ont employée dans leur recherche ne leur permet pas d'aller plus loin vers des métriques d'isométries plus faibles. Le super-espace harmonique ne possède pas une telle limitation, et il semble possible de rajouter des paramètres qui, par exemple, briseraient le $U(1)_S$ restant en rajoutant une troisième "masse" dans la limite hyper-Kähler. Pour cela, trois paramètres et la constante d'Einstein suffisent.

D'autres voies prometteuses restent à explorer. Il s'agit des extensions quaternioniques des métriques de Calabi et des extensions hyper-Kähler ou quaternioniques de la métrique de Taub-NUT vers des espaces de dimension $4n$. Par ailleurs, s'il est possible de briser totalement le $SU(2)_{PG}$, on peut espérer obtenir, dans le cas hyper-Kähler, de nouvelles métriques sans Killing tri-holomorphe.

Appendix A

Renormalisation à une boucle d'un modèle sigma non-linéaire

On considère un modèle sigma non-linéaire d'action

$$S = \frac{1}{2T} \int d^2x \; [g_{ij}(\phi)\, \eta^{\mu\nu} + h_{ij}(\phi)\, \epsilon^{\mu\nu}] \; \partial_\mu \phi^i \partial_\nu \phi^j \ . \tag{A.1}$$

Les variables ϕ^i sont des champs agissant sur un espace bi-dimensionnel Euclidien ou Minkowskien et à valeurs sur une variété riemmannienne M que l'on appelle espace cible. Cet espace cible est muni d'une métrique $g_{ij}(\phi)$ dans les coordonnées locales ϕ^i. Le tenseur g_{ij} est symétrique et son inverse est noté g^{ij}. La distance sur M est alors donnée par

$$ds^2 = g_{ij}\, d\phi^i\, d\phi^j \ .$$

Par ailleurs, notre modèle sigma possède un terme de couplage de Wess-Zumino-Witten (WZW) de par la présence du potentiel de torsion h_{ij}. Le tenseur h_{ij} est anti-symétrique et permet d'interpréter la géométrie sur l'espace cible comme une géometrie avec torsion. Celle-ci, automatiquement fermée, est définie par

$$T_{ijk} = \frac{3}{2}\, \partial_{[i} h_{jk]} = \frac{1}{2}\, (\partial_i h_{jk} + \partial_j h_{ki} + \partial_k h_{ij}) \ .$$

Le potentiel de torsion n'est défini qu'à un rotationnel près. En effet, il est évident que tout terme de la forme $\partial_{[i}\Phi_{j]}$ dans h_{ij} ne change pas la torsion. Une intégration par partie le fait de même disparaître de l'action S.

A.1 Géométrie avec torsion

Dans ce cadre de géométrie avec torsion, nous définirons les symboles de Christoffel par

$$\Gamma^i_{jk} = \gamma^i_{jk} + T^i{}_{jk} \ ,$$

où γ^i_{jk} est la connexion symétrique de Levi-Civita habituelle des géométries sans torsion :

$$\gamma^i_{jk} = \frac{1}{2}\, g^{is}\left(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk}\right) \ .$$

La connexion Γ^i_{jk} n'est par conséquent plus symétrique $(j \leftrightarrow k)$. Si on note

$$G_{ij} = g_{ij} + h_{ij} \ ,$$

alors la connexion avec torsion s'écrit aussi

$$\Gamma^i_{jk} = \frac{1}{2}\, g^{is}\left(\partial_j G_{ks} + \partial_k G_{sj} - \partial_s G_{kj}\right) \ .$$

Les dérivées covariantes avec torsion sont alors données par

$$\begin{cases} D_i A^j = \partial_i A^j + \Gamma^j_{is}\, A^s = \nabla_i\, A^j + T^j{}_{is}\, A^s \ , \\ D_i A_j = \partial_i A_j - \Gamma^s_{ij}\, A_s = \nabla_i\, A_j - T^s{}_{ij}\, A_s \ , \end{cases}$$

où ∇ désigne la dérivée covariante sans torsion. On définit ensuite le tenseur de Riemann avec torsion par

$$[D_k, D_l]\, v^i = R^i{}_{j,kl}\, v^j - 2\, T^s{}_{kl}\, D_s v^i \ ,$$

ce qui donne :

$$R^i{}_{j,kl} = \partial_k \Gamma^i_{lj} + \Gamma^i_{ks}\Gamma^s_{lj} - (k \leftrightarrow l) \ .$$

Enfin, le tenseur de Ricci et la courbure s'écrivent

$$Ric_{ij} = R^s{}_{i,sj} \ , \qquad R = Ric^s{}_s.$$

Le tenseur de Riemann avec torsion $R_{ij,kl}$ n'est plus symétrique par l'échange des couples d'indices $(ij) \leftrightarrow (kl)$ mais reste antisymétrique sur les indices $(i \leftrightarrow j)$ et $(k \leftrightarrow l)$. Le tenseur de Ricci avec torsion n'est donc plus symétrique :

$$Ric_{[ij]} = -D_s T^s{}_{ij} = -\nabla_s T^s{}_{ij} \ .$$

Nous aurons aussi besoin de la dérivée de Lie $\mathcal{L}$. Pour tout tenseur S_{ij} défini sur M, on a, dans un changement de coordonnées $\vec{\phi} \longrightarrow \vec{\phi}^o$,

$$S^o_{ij}(\vec{\phi}^o)\, \partial_\mu \phi^{oi}\, \partial_\nu \phi^{oj} = S_{ij}(\vec{\phi})\, \partial_\mu \phi^i\, \partial_\nu \phi^j \ .$$

Si $\vec{\phi}^o = \vec{\phi} + \eta\, \vec{v}$, la dérivée de Lie de S^o_{ij} par rapport à $\vec{v}$ est définie par

$$S_{ij}(\vec{\phi}) = S^o_{ij}(\vec{\phi}) - \eta \underset{\vec{v}}{\mathcal{L}}\left(S^o_{ij}(\vec{\phi})\right) + \mathcal{O}(\eta^2) \ ,$$

et l'on a alors

$$\underset{\vec{v}}{\mathcal{L}}\,(S_{ij}) = v^s \nabla_s S_{ij} + S_{sj} \nabla_i v^s + S_{is} \nabla_j v^s \ . \tag{A.2}$$

A.2 Renormalisabilité à une boucle

Les divergences à une boucle du modèle sigma non linéaire (A.1) ont été données par Friedan [Fri85], et l'action renormalisée à une boucle en dimension $d = 2 - \varepsilon$ s'écrit

$$S_R^1 = \frac{1}{2T} \int d^2x \left[G_{ij} + \frac{\hbar T}{2\pi\varepsilon} Ric_{ij} \right] (\eta^{\mu\nu} + \epsilon^{\mu\nu})\, \partial_\mu \phi^i \partial_\nu \phi^j \ .$$

Les contre-termes possibles au couplage et aux champs sont donnés par le développement en $\hbar$ de l'action nue. Dans le schéma dimensionnel minimal, si on définit la constante λ et le vecteur $\vec{v}$ par

$$\begin{cases} \dfrac{1}{T^o} = \dfrac{1}{T} \left(1 + \dfrac{\hbar T}{2\pi\varepsilon} \lambda + \mathcal{O}(\hbar^2) \right) \ , \\ \vec{\phi}^o = \vec{\phi} + \dfrac{\hbar T}{4\pi\varepsilon} \vec{v} + \mathcal{O}(\hbar^2) \ , \end{cases}$$

où T^o et $\vec{\phi}^o$ sont respectivement le couplage nu et les champs nus, on obtient une deuxième écriture de l'action renormalisée à une boucle :

$$\begin{aligned} S_R^{1\prime} &= \frac{1}{2T} \int d^2x \left(1 + \frac{\hbar T \lambda}{2\pi\varepsilon} \right) G_{ij}\ (\eta^{\mu\nu} + \epsilon^{\mu\nu})\, \partial_\mu \left(\phi^i + \frac{\hbar T v^i}{4\pi\varepsilon} \right) \partial_\nu \left(\phi^j + \frac{\hbar T v^j}{4\pi\varepsilon} \right) \\ &= \frac{1}{2T} \int d^2x \left[G_{ij} + \frac{\hbar T}{2\pi\varepsilon} \left(\lambda G_{ij} + \frac{1}{2} \underset{\vec{v}}{\mathcal{L}} (G_{ij}) \right) \right] (\eta^{\mu\nu} + \epsilon^{\mu\nu})\, \partial_\mu \phi^i \partial_\nu \phi^j \ . \end{aligned}$$

Il reste à comparer S_R^1 à $S_R^{1\prime}$ afin d'en déduire la condition de renormalisabilité à une boucle de la théorie (A.1). En utilisant $\nabla_i g_{jk} = 0$ et en faisant l'identification $S_{ij} \equiv G_{ij}$ dans (A.2), on démontre l'égalité

$$\underset{\vec{v}}{\mathcal{L}}(G_{ij}) = 2 D_j v_i + \partial_{[i} \zeta_{j]} \ , \qquad \zeta_i = 2\, k^s\, G_{si} \ .$$

Tout terme de la forme $\partial_{[i} \zeta_{j]}$ disparaît de l'action grâce à une intégration par parties. En tenant compte de cette liberté, on obtient la condition de renormalisabilité à une boucle du modèle sigma non-linéaire :

$$Ric_{ij} = \lambda\, G_{ij} + D_j v_i + \partial_{[i} w_{j]} \ .$$

La renormalisabilité à une boucle de la théorie (A.1) se traduit donc par une contrainte géométrique : **la métrique doit être quasi-Einstein avec torsion**. En séparant partie symétrique et partie anti-symétrique, cette condition se réécrit

$$\begin{cases} Ric_{(ij)} = \lambda\, g_{ij} + D_{(i} v_{j)} \ , \\ Ric_{[ij]} = \lambda\, h_{ij} + v_s\, T^s{}_{ij} + \partial_{[i} (w - v)_{j]} \ . \end{cases}$$

Appendix B

Métriques usuelles en dimension 4

B.1 Métriques quasi-Einstein $SU(2) \times U(1)$

Nous considérons ici les métriques à 4 dimensions de co-homogénéité un sous les isométries $SU(2) \times U(1)$. Celles-ci peuvent s'écrire sous la forme

$$g = \alpha(t)\, dt^2 + \beta(t)\, ({\sigma_1}^2 + {\sigma_2}^2) + \gamma(t)\, {\sigma_3}^2 \ ,$$

où les σ_i sont les 1-formes invariantes de $SU(2)$ qui dépendent des angles d'Euler $\{\theta, \varphi, \psi\}$:

$${\sigma_1}^2 + {\sigma_2}^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \,, \qquad \sigma_3 = d\psi + \cos\theta\, d\varphi \ .$$

Les identités de Maurer-Cartan s'écrivent

$$d\sigma_i = \varepsilon\, \frac{1}{2}\, \epsilon_{ijk}\, \sigma_j \wedge \sigma_k, \qquad \varepsilon = \pm 1 \ .$$

La propriété quasi-Einstein est alors donnée par

$$ric_{ij} = \lambda\, g_{ij} + D_{(i} v_{j)} \ , \qquad v = v_0(t)\, dt \ .$$

La forme particulière de v est imposée par les symétries. Celui-ci n'est en fait défini qu'au Killing ∂_ψ près dont la 1-forme correspondante s'écrit $\gamma(t)\, \sigma_3$.

Sauf dans le cas particulier où $\beta(t) = cste$, une redéfinition de la coordonnée t permet toujours de prendre $\beta(t) = t$. La condition quasi-Einstein est alors équivalente à trois équations différentielles non-linéaires portant sur $\alpha(t)$, $\gamma(t)$ et $v_0(t)$:

$$\left\{ \begin{array}{rcl} \dfrac{1}{t^2} + \left(\dfrac{1}{t} + \dfrac{\gamma'(t)}{2\,\gamma(t)}\right) \dfrac{\alpha'(t)}{\alpha(t)} + \dfrac{\gamma'(t)^2}{2\,\gamma(t)^2} - \dfrac{\gamma''(t)}{\gamma(t)} & = & 2\,\lambda\,\alpha(t) + 2\,v_0'(t) - \dfrac{v_0(t)}{\alpha(t)} \\ 2\left(2 - \dfrac{\gamma(t)}{t}\right) \alpha(t) + \dfrac{\alpha'(t)}{\alpha(t)} - \dfrac{\gamma'(t)}{\gamma(t)} & = & 4\,\lambda\, t\, \alpha(t) + 2\, v_0(t) \\ -\dfrac{2}{t} + \dfrac{2}{t^2}\, \dfrac{\gamma(t)^2}{\gamma'(t)}\, \alpha(t) + \dfrac{\alpha'(t)}{\alpha(t)} + \dfrac{\gamma'(t)}{\gamma(t)} - \dfrac{2\,\gamma''(t)}{\gamma'(t)} & = & 4\,\lambda\, \dfrac{\gamma(t)}{\gamma'(t)}\, \alpha(t) + 2\, v_0(t) \end{array} \right. \qquad \text{(B.1)}$$

Il est possible d'éliminer $\alpha(t)$ et $v_0(t)$ de ce système, mais on aboutit alors à une équation différentielle hautement non-linéaire d'ordre quatre en $\gamma(t)$, pour l'instant non résolue. Celle-ci montre cependant que la métrique générale quasi-Einstein à quatre dimensions $SU(2) \times U(1)$ doit dépendre de quatre paramètres indépendants. La résolution est cependant possible si l'on impose certaines conditions afin de restreindre la recherche au cas des métriques d'Einstein ou au cas de certaines métriques Kähler.

B.1.1 Métriques d'Einstein

Si l'on impose la condition $v_0(t) = 0$, les système (B.1) est totalement résolu et la solution, qui dépend de deux paramètres A et B, est donnée par

$$\begin{cases} \alpha(t) &= \dfrac{1}{1+A\,t} \cdot \dfrac{1}{\gamma(t)} \,, \\[2ex] \gamma(t) &= \dfrac{4\,t}{\left(1+\sqrt{1+A\,t}\right)^2} - \dfrac{4\,\lambda\,t^2}{3} \dfrac{3+\sqrt{1+A\,t}}{\left(1+\sqrt{1+A\,t}\right)^3} + \dfrac{B}{t}\sqrt{1+A\,t} \,. \end{cases}$$

Si $A = 0$, g s'identifie à l'extension Kähler-Einstein de la métrique d'Eguchi-Hanson. Le changement de coordonnées $t \equiv s^2$ et $B = -a^4$ permet de retrouver sa forme usuelle :

- Extension Kähler-Einstein de Eguchi-Hanson :

$$g = \frac{4}{F}\,ds^2 + s^2\left({\sigma_1}^2 + {\sigma_2}^2 + F\,{\sigma_3}^2\right) \,, \qquad F = 1 - \frac{a^4}{s^4} - \frac{2\,\lambda}{3}\,s^2 \,. \tag{B.2}$$

Si $A \neq 0$, la métrique s'identifie à la large classe de métriques d'Einstein découverte par Carter [Car68]. En effectuant le changement de coordonnées

$$t \longrightarrow t^2 - n^2 \qquad \text{avec } A = \frac{1}{n^2} \text{ et } B = -8(M-n)\,n^3 \,,$$

on en obtient l'écriture suivante, plus simple,

$$\begin{cases} g &= \dfrac{t^2-n^2}{f(t)}\,dt^2 + (t^2-n^2)\,({\sigma_1}^2+{\sigma_2}^2) + \dfrac{4\,n^2}{t^2-n^2}\,f(t)\,{\sigma_3}^2 \,, \\[2ex] f(t) &= t^2 - 2\,M\,t + n^2 - \dfrac{\lambda}{3}\,(t-n)^3(t+3\,n) \,. \end{cases} \tag{B.3}$$

Les constantes A et B étant réelles, M, n et t doivent être simultanément réels ou imaginaires purs. Cette classe de métriques de Carter contient, entre autres,

- Taub-NUT $(M = n$ et $\lambda = 0)$:

$$g = \frac{t+n}{t-n}\,dt^2 + (t^2-n^2)({\sigma_1}^2+{\sigma_2}^2) + 4\,n^2\,\frac{t-n}{t+n}\,{\sigma_3}^2 \,,$$

- Schwarzschild avec constante cosmologique $\left(d\psi \equiv \dfrac{d\Psi}{2\,n}\right.$ puis $\left.n \longrightarrow 0\right)$:

$$g = \frac{1}{G}\,dt^2 + t^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2) + G\,d\Psi^2\ , \qquad G = 1 - \frac{2\,M}{t} - \frac{\lambda}{3}\,t^2\ , \tag{B.4}$$

- Page sur $P_2(\mathbb{C})\#\overline{P_2(\mathbb{C})}$ $\left(t = n\,\nu\,s,\ M = -\dfrac{4\,\lambda}{3}n^3,\ n = i\,\sqrt{3}\sqrt{\dfrac{1+\nu^2}{\lambda\ (3+6\,\nu^2-\nu^4)}}\right)$:

$$g = \frac{3}{\lambda}\,\left(1+\nu^2\right)\left\{H\,ds^2 + \frac{1-\nu^2\,s^2}{3+6\,\nu^2-\nu^4}\,({\sigma_1}^2+{\sigma_2}^2) + \frac{4\,\nu^2}{H\,(3+6\,\nu^2-\nu^4)^2}\,{\sigma_3}^2\right\}\ , \tag{B.5}$$

où $H = \dfrac{1-\nu^2\,s^2}{(1-s^2)\ (3-\nu^2-\nu^2\ (1+\nu^2)\ s^2)}$ et où ν est l'unique solution dans $[0,1]$ de l'équation algébrique $4\,\nu\,(3+\nu^2) = 3+6\,\nu^2-\nu^4$.

B.1.2 Métriques Kähler quasi-Einstein

La condition $\alpha(t) = \dfrac{1}{\gamma(t)}$ permet aussi de résoudre le système (B.1), et l'on obtient alors certaines métriques $SU(2) \times U(1)$ qui sont Kähler. On suppose ici qu'il existe un choix de coordonnées holomorphes pour lesquelles les isométries agissent linéairement, ce qui impose l'intégrabilité des structures complexes. La forme de Kähler est donnée par

$$\mathcal{K} = d(t\,\sigma_3)\ .$$

La nouvelle solution dépend aussi de deux paramètres, C et D, et s'écrit :

$$g = \frac{1}{\gamma(t)}\,dt^2 + t\,\left({\sigma_1}^2+{\sigma_2}^2\right) + \gamma(t)\,{\sigma_3}^2\,, \qquad v_0(t) = -C\,,$$

où

$$\gamma(t) = \frac{D\,e^{C\,t}}{t} + t + \frac{2}{C^2\,t}\left(1-\frac{2\,\lambda}{C}\right)\left(e^{C\,t} - 1 - C\,t - \frac{1}{2}\,C^2\,t^2\right)\ .$$

Cette classe de métriques, découverte par T. Chave et G. Valent [CV96], est, à notre connaissance, le seul exemple de métriques d'isométries $U(2)$ quasi-Einstein en $D=4$ de la littérature. Dans la limite $C \longrightarrow 0$, on a $v_0 = 0$ et on retrouve donc la métrique Einstein avec le potentiel de Kähler $\mathcal{K}$. Il s'agit de l'extension Kähler-Einstein de la métrique d'Eguchi-Hanson. Pour retrouver (B.2), il faut effectuer le changement de coordonnées $t \equiv s^2$. La correspondance entre les paramètres est $D = B = -a^4$. Celle-ci montre que les quatres paramètres de la solution *g*énérale de (B.1) ne peuvent pas être A, B, C et D puisqu'ils ne sont pas indépendants.

B.1.3 Métriques homogènes

Parmi les métriques précédemment citées, certaines sont homogènes. Elles correspondent toutes à des symétries plus importantes que $SU(2)\times U(1)$:

- L'espace plat $\mathbb{R}^4$:

$$g = \frac{1}{t}\,dt^2 + t\,\left({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right)\,.$$

- $S^1\times S^3$:

$$g = \frac{1}{2\,\lambda}\left(dt^2 + {\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right)\,,\qquad v = -\frac{t}{2}\,dt\,.$$

Il s'agit de l'unique métrique quasi-Einstein avec $\beta(t) = cste$.

- $S^2\times S^2$ $\quad(d\psi \equiv \dfrac{d\Psi}{2\,\nu}$ puis $\nu\to 0$ dans (B.5)) :

$$g = \frac{1}{\lambda}\left\{\frac{ds^2}{1-s^2} + \left(1-s^2\right)\,d\Psi^2 + {\sigma_1}^2 + {\sigma_2}^2\right\}\,.$$

- de Sitter S^4 $\quad(M^2 = n^2 = -\dfrac{3}{4\,\lambda}$ et $t^2 = -\dfrac{3}{4\,\lambda} + \dfrac{s^2}{4\left(1+\frac{s^2\,\lambda}{12}\right)^2}$ dans (B.3)) :

$$g = \frac{1}{\left(1+\frac{\lambda}{12}\,s^2\right)^2}\left\{ds^2 + \frac{s^2}{4}\,\left({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right)\right\}.$$

On peut aussi l'obtenir en annulant M dans (B.4). Si $\lambda < 0$, il s'agit de la métrique non-compacte anti-de Sitter correspondante.

- Fubini-Study sur $P_2(\mathbb{C}) = \frac{SU(3)}{U(2)}$ $\quad(s \equiv \dfrac{r}{2\sqrt{1+\frac{\lambda}{6}\,r^2}}$ puis $a\to 0$ dans (B.2)) :

$$g = \frac{1/4}{1+\frac{\lambda}{6}\,r^2}\left\{\frac{4\,dr^2 + r^2\,{\sigma_3}^2}{1+\frac{\lambda}{6}\,r^2} + r^2\,\left({\sigma_1}^2 + {\sigma_2}^2\right)\right\}\,.$$

Si $\lambda < 0$, il s'agit de son partenaire non compact sur $\frac{SU(2,1)}{U(2)}$.

B.2 Métriques hyper-Kähler

Toutes les métriques hyper-Kähler en dimension 4 connues possèdent au moins *un* Killing, que l'on notera $K = \partial_t$. Boyer et Finley ont montré dans [BF82] qu'il était alors possible de classer ces métriques en deux groupes distincts. En effet, il y a une dichotomie pour les métriques considérées de symétrie minimale $U(1)$ suivant qu'elles possèdent ou non un Killing tri-holomorphe. Rappelons par ailleurs que ces métriques ont toutes un tenseur de Ricci nul.

B.2.1 Métriques avec au moins un Killing tri-holomorphe

Nous considérons ici les métriques pour lesquelles la dérivée de Lie des structures complexes J_i selon K est nulle :

$$\underset{K}{\mathcal{L}} J_i = 0\,, \qquad i \in [\![1,3]\!].$$

Ces métriques ont été découvertes plusieurs fois par différents auteurs dans [KSD74, GH78, Hit79], mais dans des coordonnées différentes, ce qui fait que l'on n'a pas su tout de suite qu'il s'agissait en fait des mêmes.

Il s'agit des **multicentres** pour lesquelles la distance peut toujours s'écrire sous la forme

$$ds^2 = \frac{1}{V}\,(dt+\Theta)^2 + V\,h\,, \qquad h = d\vec{X}.d\vec{X}\ ,$$

où V et les composantes de la 1-forme Θ, définies par $\Theta = \Theta_i\, dX^i$, sont indépendantes de t. La propriété hyper-Kähler découle de la relation fondamentale des multicentres

$$\underset{h}{\star}\, d\Theta = \pm dV\ .$$

Cette relation entraîne la nullité du laplacien du potentiel V :

$$\underset{h}{\Delta}\, V = 0\ .$$

Ainsi, à toute fonction V harmonique dans $\mathbb{R}^3$, il est possible d'associer une métrique hyper-Kähler (euclidienne) en dimension 4. Le nom de “multicentres” donné à ces métriques vient du fait que l'on considère généralement des potentiels ayant pour forme

$$V = \lambda + \sum_i \frac{\mu_i}{\left|\vec{X}-\vec{X}_i\right|}\ .$$

On peut ainsi visualiser ces métriques par un ensemble de “masses” μ_i dispersées en certains points $\vec{X}_i$ (les “centres”), le potentiel à l'infini valant par ailleurs λ.

Il est possible de démontrer que le tenseur de courbure et le tenseur de Weyl possèdent une auto-dualité opposée à celle des structures complexes. C'est d'ailleurs le fait que le tenseur de courbure soit (anti-)auto-dual qui entraîne la nullité du tenseur de Ricci.

Parmi les multicentres, en dehors de l'espace plat ($V = \lambda$ ou $V = \frac{1}{|\vec{X}|}$), il y en a deux qui possèdent l'isométrie maximale $SU(2) \times U(1)$ et que l'on a déja vu dans la section précédente :

- Taub-NUT :
$$V = \lambda + \frac{1}{|\vec{X}|}\ .$$
Le Killing associé à l'isométrie $U(1)$ est alors tri-holomorphe, tandis que les rotations spatiales, qui laissent V invariant, donnent lieu à des isométries holomorphes.
- Eguchi-Hanson :
$$V = \frac{1}{|\vec{X}-\vec{\xi}|} + \frac{1}{|\vec{X}+\vec{\xi}|}\ .$$
Le Killing associé à l'isométrie $U(1)$ est maintenant holomorphe et correspond à la symétrie de rotation autour de $\vec{\xi}$. Alors que la métrique correspondante ne semble posséder qu'une symétrie $U(1)_T \times U(1)_H$, il a été démontré dans [EH78] que des symétries "cachées" augmentaient les symétries tri-holomorphes jusqu'à un $SU(2)_T$.
C'est Prasad, dans [Pra79], qui a démontré que la métrique obtenue par Eguchi et Hanson était bien un multicentre.

En combinant ces deux potentiels, il est possible d'obtenir une métrique d'isométries $U(1)_T \times U(1)_H$. Il s'agit du

- double Taub-NUT :
$$V = \lambda + \frac{1}{|\vec{X}-\vec{\xi}|} + \frac{1}{|\vec{X}+\vec{\xi}|}\ .$$

Ces trois métriques sont complètes. Cependant, si l'on se limite au cas à deux centres, le potentiel le plus général à isométries $U(1)_T \times U(1)_H$ possède trois paramètres puisque l'on peut encore dans le double Taub-NUT donner des masses différentes à chacun des centres. On a alors pour le potentiel

$$V = \lambda + \frac{1}{|\vec{X}-\vec{\xi}|} + \frac{\rho}{|\vec{X}+\vec{\xi}|}\ . \tag{B.6}$$

B.2.2 Métriques sans Killing tri-holomorphe

Lorsque tous les Killing sont holomorphes, c'est à dire que l'on a pour les structures complexes

$$\underset{K}{\mathcal{L}}\ J_3 = 0\,, \qquad \text{et} \qquad \begin{cases} \mathcal{L}_K\ J_1 = J_2 \\ \mathcal{L}_K\ J_2 = -J_1 \end{cases}\ ,$$

on ne connait pas explicitement toutes les métriques à 4 dimensions d'isométrie minimale $U(1)_H$. Cependant, Boyer et Finley on montré que celles-ci pouvaient toujours s'écrire sous la forme [BF82] :

$$ds^2 = \frac{1}{V}\ \left(dt^2 + \Theta\right)^2 + V\ \left[dz^2 + e^u\left(dx^2+dy^2\right)\right]\,,$$

avec les relations

$$\begin{cases} \partial_x^2 u + \partial_y^2 u + \partial_z^2 e^u = 0 \ , \\ V = \partial_z u \quad \text{et} \quad \pm\, d\Theta = \partial_x V \, dy \wedge dz + \partial_y V \, dz \wedge dx + \partial_z \left(V \, e^u\right) dx \wedge dy \ . \end{cases}$$

Le seul exemple explicite que l'on connaisse est la métrique de Atiyah-Hitchin qui possède un $SU(2)_H$ de Killing holomorphes. Les relations entre les coordonnées données par les auteurs et les coordonnées x, y, z ont été obtenues dans [Oli91]. Sa forme usuelle s'écrit [AH88] :

- Atiyah-Hitchin :

$$ds^2 = A^2 B^2 C^2 \left(\frac{dk}{k\,(1-k^2)\,K^2} \right)^2 + A^2\,\sigma_1^2 + B^2\,\sigma_2^2 + C^2\,\sigma_3^2 \ ,$$

où $K(k)$ et $E(k)$ sont respectivement les intégrales elliptiques complètes de première et seconde espèce :

$$K(k) = \int_0^{\pi/2} \frac{dq}{\sqrt{1 - k^2 \sin^2 q}} \quad \text{et} \quad E(k) = \int_0^{\pi/2} dq \, \sqrt{1 - k^2 \sin^2 q} \ ,$$

et où

$$\begin{aligned} A\,B &= -K\,\left[E - K\right] \ , \\ B\,C &= -K\,\left[E - (1-k^2)\,K\right] \ , \\ A\,C &= -K\,E \ . \end{aligned}$$

Appendix C

Articles

C.1 Quantum structure of T-dualized models with symmetry breaking

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Quantum structure of T-dualized models with symmetry breaking

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Abstract

We study the principal σ-models defined on any group manifold $G_L \times G_R/G_D$ with breaking of G_R, and their T-dual transforms. For arbitrary breaking we can express the torsion and Ricci tensor of the dual model in terms of the frame geometry of the initial principal model. Using these results we give necessary and sufficient conditions for the dual model to be torsionless and prove that the one-loop renormalizability of a given principal model is inherited by its dual partner, who shares the same β functions. These results are shown to hold also if the principal model is endowed with torsion. As an application we compute the β functions for the full Bianchi family and show that for some choices of the breaking parameters the dilaton anomaly is absent: for these choices the dual torsion vanishes. For the dualized Bianchi V model (which is torsionless for any breaking), we take advantage of its simpler structure, to study its two-loops renormalizability. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The subject of classical versus quantum equivalence of T-dualized σ-models has been strongly studied in recent years, and extensive reviews covering abelian, non-abelian dualities and their applications to string theory and statistical physics are available [2,5, 20]. More recent developments on the geometrical aspects of duality can be found in [1].

The interpretation of T-duality as a canonical transformation, for constant backgrounds, was first given by [21]. Its more general formulation [4] was applied to the non-abelian case in [25,27].

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After the settling of the classical equivalence, the most interesting problem was its study at the quantum level. This was done mostly for dualizations of Lie groups, with emphasis put on $SU(2)$. For this model the one-loop equivalence was established in [16,18]. The way towards the general case was cleaned up with the derivation of the classical structure of the non-abelian dual for any group [2,3,18,22] and for non-inhomogeneous geometries in [26]. However, the analysis of Bianchi V in [19] revealed that for some renormalizable dual theories the divergences could not be absorbed by a re-definition of the dilaton field! It was further realized that this phenomenon occurs for non semi-simple Lie groups with traceful structure constants ($f_{si}^{s} = 0$), and that it can be interpreted as a mixed gravitational-gauge anomaly [3].

A further decisive progress was made by Tyurin [28], who generalized the one-loop equivalence to an arbitrary Lie group and derived the general structure of the dilaton anomaly. However, as pointed out in [7], his analysis considers only models with explicit invariance under the left group action (whose existence is crucial for the dualization process) leaving aside the right action and the possible symmetry breaking schemes for it. The one-loop equivalence problem in this more general setting has been examined recently [7,23] for the group manifold $SU(2)_L \times SU(2)_R/SU(2)_D$, where $SU(2)_R$ is broken down to a $U(1)$. The renormalizability and dilatonic properties do survive despite the lowering of the right isometries. It is the purpose of the present article to analyze the geometry of the dualized model for a large class of models built on $G_L \times G_R/G_D$, with arbitrary breaking of G_R. While in [28] supersymmetry considerations à la Busher [8,9] were convenient to derive the dualized geometry, we will show that a direct computation in local coordinates is fairly efficient to extract the Ricci tensor in the presence of symmetry breaking.

The content of this article is the following: after setting the notations, in Section 2 we study the geometry of the group manifold $(G_L \times G_R)/G_D$. This is most conveniently done using frames and, despite symmetry breaking, one obtains a manageable form for the Ricci tensor. In Section 3 the dualized theory is examined and its torsion and Ricci tensor are computed, exhibiting their dependence with respect to the geometrical quantities of the principal model. The possibility of torsionless dualized models is discussed. In Section 4 we use the previous results to show that the one-loop renormalizability of the principal model is inherited by its T-dual. In Section 5 we generalize the previous analyses to deal with a principal model endowed with torsion. In Section 6 we examine the models in the Bianchi class, compute their beta functions, and for the non semi-simple algebras discuss the dilaton anomaly. For some breaking choices this anomaly may vanish and in these cases the dual models are torsionless. Since any dualized Bianchi V model is torsionless, we study in Section 7, for the simplest breaking, its two-loops renormalizability.

2. Geometry of the broken principal models

Since we have in view perturbative applications, our considerations will be of a local nature. Let us consider a Lie algebra $\mathcal{G} = \{X_i,\ i = 1, \ldots, \nu\}$ with structure constants

$$[X_i, X_j] = f_{ij}^{\ s} X_s.$$

Denoting by z^i the local coordinates in a neighbourhood of the origin, we exponentiate to the group by $g = \exp(z \cdot T)$, and define

$$g^{-1}\partial_\mu g = J_\mu^i X_i. \tag{1}$$

For further use we introduce the adjoint representation by

$$(T_i)_j^{\ k} \equiv (\mathrm{ad}\, X_i)_j^{\ k} = -f_{ij}^{\ k}, \tag{2}$$

which allows to write the Jacobi identity

$$[T_i, T_j] = f_{ij}^{\ s} T_s, \quad i, j, s = 1, \ldots, \nu = \dim(\mathcal{G}). \tag{3}$$

Then the action of the corresponding principal model can be written

$$S = \frac{1}{2}\int d^2x\, B_{ij}\eta^{\mu\nu} J_\mu^i J_\nu^j, \tag{4}$$

where the matrix B is symmetric and invertible. For field theoretic applications one should add the restriction that B is positive definite [6], while this does not seem to be necessary for stringy applications. This restriction implies, in the semi-simple case that its simple components have to be compact. Our analysis will not make use of this positivity hypothesis.

Taking the curl of the first relation in (1) gives the Bianchi identity

$$M_{\mu\nu}^i(J) \equiv \partial_\mu J_\nu^i - \partial_\nu J_\mu^i + f_{st}^{\ i} J_\mu^s J_\nu^t = 0 \Longleftrightarrow \epsilon^{\mu\nu} M_{\mu\nu}^i(J) = 0. \tag{5}$$

2.1. Isometries

Let us proceed to a discussion of the isometries of the action (4). The groups $G_L \times G_R$ and G_D act on g according to

$$g \longrightarrow g' = G_L\, g\, G_R^{-1}, \qquad g \longrightarrow g' = G_D\, g\, G_D^{-1}. \tag{6}$$

As a consequence

$$g^{-1}\partial_\mu g \longrightarrow G_R\, g^{-1}\partial_\mu g\, G_R^{-1},$$

and specializing to infinitesimal transformations one gets

$$G_R \approx \mathbb{I} + \epsilon_R^i T_i \Longrightarrow \delta J_\mu^k = f_{ij}^{\ k} \epsilon_R^i J_\mu^j. \tag{7}$$

It follows that the action (4) is invariant under G_L, while the matrix B_{ij} will generally break G_R down to some subgroup H (possibly trivial). Denoting by $\{T_s,\ s = 1, \ldots, h\}$ the generators of its Lie algebra $\mathcal{H}$, these should satisfy

$$(T_s)_i^{\ k} B_{kj} + (T_s)_j^{\ k} B_{ik} = 0, \quad \forall\, T_s \in \mathcal{H}. \tag{8}$$

Let us emphasis that the metric B can be freely chosen (as far as it is symmetric and invertible!), but, if $\mathcal{G}$ is simple, the most symmetric choice is given by the bi-invariant metric

$$B_{ij} = \frac{1}{\rho} g_{ij}, \qquad g_{ij} = \mathrm{Tr}(T_i T_j) = \tilde{\rho}\ \mathrm{Tr}(t_i t_j), \tag{9}$$

where g_{ij} is the Killing metric and the t_i the defining representation of the simple algebra under consideration. In the simple compact case we have

	$so(n)$	$su(n)$	$sp(n)$
$\tilde\rho$	$(n-2)$	$2n$	$2(n+1)$

and with the standard normalization of the generators $\mathrm{Tr}(t_i t_j) = -2\delta_{ij}$, we see that the choice $\rho = -2\tilde\rho$ gives $B_{ij} = \delta_{ij}$. In the simple non-compact case the same choice of ρ gives $B_{ij} = \eta_{ij}$, which is diagonal, with $\eta_{ii} = +1$ for a compact generator t_i and $\eta_{ii} = -1$ for a non-compact one.

The bi-invariant metric has for isometry group the full $G_L \times G_R$ because $(T_s)_i{}^k g_{kl} = -f_{sil}$ is fully skew-symmetric and therefore (8) is true for all the generators of G_R.

For a semi-simple $\mathcal{G}$ the situation is not very different, since it can be split into a direct sum of simple algebras

$$\mathcal{G} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k, \qquad [\mathcal{S}_i, \mathcal{S}_j] = 0, \quad i \neq j.$$

2.2. Geometry of frames

In order to have a better insight into the geometry of the principal models with action (4), it is convenient to use a vielbein formalism, through the identification

$$B_{ij}\eta^{\mu\nu} J^i_\mu J^j_\nu \longleftrightarrow B_{ij} e^i e^j,$$

and now the Bianchi identities appear as the Maurer–Cartan equations

$$de^i + \frac{1}{2} f_{st}{}^i e^s \wedge e^t = 0. \tag{10}$$

We follow the notations of [12] and define the spin-connection $\omega^i{}_j$ by

$$de^i + \omega^i{}_s \wedge e^s = 0, \qquad \omega^i{}_j = \omega^i{}_{j,s} e^s.$$

The frame indices are lowered or raised using the metric B_{ij} and its inverse $B^{ij} = B^{-1}_{ij}$. A straightforward computation gives

$$2\omega_{ij,k} = f_{ij,k} + f_{ik,j} - f_{jk,i}, \quad f_{ij,k} = f_{ij}{}^s B_{sk}. \tag{11}$$

For further use let us point out two consequences

$$\omega^i{}_{j,k} - \omega^i{}_{k,j} = -f_{jk}{}^i, \quad \omega^s{}_{i,s} = -f_{is}{}^s. \tag{12}$$

The curvature and the Ricci tensor are defined by

$$R^i{}_j = d\omega^i{}_j + \omega^i{}_s \wedge \omega^s{}_j = \frac{1}{2} R^i{}_{j,st}\, e^s \wedge e^t, \quad ric_{ij} = R^s{}_{i,sj}.$$

It follows that

$$R^i{}_{j,st} = -\omega^i{}_{j,a} f_{st}{}^a - \omega^i{}_{a,t}\,\omega^a{}_{j,s} + \omega^i{}_{a,s}\,\omega^a{}_{j,t}. \tag{13}$$

In the Ricci tensor the first two terms are gathered using (12) and give

$$ric_{ij} = -\omega^s{}_{i,t}\,\omega^t{}_{j,s} + \omega^t{}_{s,t}\,\omega^s{}_{i,j}. \tag{14}$$

The $i \leftrightarrow j$ symmetry of the first term is obvious while for the second it follows from

$$\omega^t{}_{s,t}\big(\omega^s{}_{i,j} - \omega^s{}_{j,i}\big) = f_{st}{}^t\, f_{ij}{}^s = 0, \tag{15}$$

where the last equality is obtained by taking the trace of the Jacobi identity (3).

One can give the following explicit form of the Ricci tensor

$$\begin{aligned} ric_{ij} &= \frac{1}{2}B_{st}\big(A^s B^{-1} A^t\big)_{ij} - \frac{1}{4}B_{is}\,\mathrm{Tr}\big(B^{-1}A^s B^{-1}A^t\big)B_{tj} \\ &\quad - \frac{1}{2}\mathrm{Tr}(T_i T_j) + \frac{1}{2}\mathrm{Tr}(T_s)\big(f^s{}_{i,j} + f^s{}_{j,i}\big), \quad f^s{}_{i,j} = \big(B^{-1}\big)_{st} f_{ti,j}, \end{aligned} \tag{16}$$

which exhibits that it is an homogeneous function of degree 0 in the breaking matrix B. The scalar curvature $R = (B^{-1})_{ij}\,ric_{ij}$ is a constant, as it should for homogeneous spaces.

A drastic simplification takes place for the bi-invariant metric (9), for which we have

$$ric_{ij} = -\frac{\rho}{4}\,B_{ij}. \tag{17}$$

The metric is therefore Einstein, and such a simple structure will have a counterpart in the dualized theory.

2.3. Dualization

For the reader's convenience we present a quick derivation [2,18] of the dualized model. The essence of the dualization process is to switch from the coordinates on the group, which parametrize g, to new coordinates ψ_i defined as the Lagrange multipliers of the Bianchi identities. Concretely this transformation is carried out starting from the action

$$S = \frac{1}{4}\int d^2x\big\{B_{ij}\eta^{\mu\nu}J^i_\mu J^j_\nu - \epsilon^{\mu\nu}\psi_i M^i_{\mu\nu}(J)\big\}.$$

Using light-cone coordinates, with the following conventions

$$x_\pm = \frac{x^0 \pm x^1}{\sqrt{2}}, \qquad \epsilon_{01} = 1, \qquad \epsilon^{\mu\sigma}\epsilon_{\sigma\nu} = \delta^\mu_\nu, \qquad J_\pm = \frac{J_0 \pm J_1}{\sqrt{2}},$$

one has

$$S = \frac{1}{2}\int d^2x\big\{(B + A\cdot\psi)_{ij}\,J^i_+ J^j_- - \psi_i\big(\partial_+ J^i_- + \partial_- J^i_+\big)\big\}, \tag{18}$$

with

$$(A^s)_{ij} = (T_i)_j^{\ s} = -f_{ij}^{\ \ s}, \qquad (A\cdot\psi)_{ij} = (A^s)_{ij}\psi_s. \tag{19}$$

The field equations obtained from the variations with respect to the currents $J^i_\pm$ give

$$\begin{aligned} &J^i_- = (B + A\cdot\psi)^{is}\,\partial_-\psi_s, \qquad J^i_+ = -\partial_+\psi_s\,(B + A\cdot\psi)^{si}, \\ &(B + A\cdot\psi)^{is}(B + A\cdot\psi)_{sj} = \delta^i_j. \end{aligned}$$

Using minkowskian coordinates on the worldsheet one has

$$J^{\mu i} = B^{ij}\epsilon^{\mu\nu}\big(\partial_\nu\psi_j - (A\cdot\psi)_{jk}J^k_\nu\big), \qquad B^{is}B_{sk} = \delta^i_k.$$

Using this relation, the action (18) can be written, up to total derivatives

$$S = \frac{1}{2}\int d^2x\,\partial_+\psi_i J^i_- = \frac{1}{2}\int d^2x\,\partial_+\psi_i(B + A\cdot\psi)^{ij}\partial_-\psi_j. \tag{20}$$

Comparing this action with the one given in relation (4.16) of [28] we see that in this reference only the unbroken case $B_{ij} = \delta_{ij}$ has been considered.

Let us emphasize the following points:

1. Before dualization, all the field dependence on the coordinates chosen to parametrize G must be hidden in expressions involving solely the currents J^i_μ. If this is not the case the dualization process is not possible.

2. The dualized action is completely defined by the breaking matrix B and the field matrix $A\cdot\psi \in so(\nu)$. There are as many coordinates as generators in $\mathcal{G}$.

3. In the process of dualization the isometries corresponding to G_L (which leave the J^i_μ invariant) are lost. This has for consequence that starting from an homogeneous metric, we are led to a non-homogeneous one.

3. Geometry of the dualized theory

In (20) we come back to standard notations and change the coordinates ψ_i to ψ^i. Let us write the dual action

$$S = \frac{1}{2}\int d^2x\,G_{ij}\partial_+\psi^i\partial_-\psi^j, \qquad G_{ij} = (B + A\cdot\psi)^{-1}_{ij}. \tag{21}$$

For further use we define the matrices

$$G^{\pm} = (B \pm A\cdot\psi)^{-1}, \qquad G \equiv G^+, \qquad \Gamma^{\pm} = B \pm A\cdot\psi, \qquad (A\cdot\psi)_{ij} = -f_{ij}^{\ \ s}\,\psi^s.$$

Writing the dual action (21) in minkowskian coordinates

$$S = \frac{1}{2}\int d^2x\big\{g_{ij}\eta^{\mu\nu}\partial_\mu\psi^i\partial_\nu\psi^j + h_{ij}\epsilon^{\mu\nu}\partial_\mu\psi^i\partial_\nu\psi^j\big\}, \tag{22}$$

gives for metric and torsion potential

$$g_{ij} = \frac{1}{2}(G_{ij} + G_{ji}), \qquad h_{ij} = \frac{1}{2}(G_{ij} - G_{ji}), \qquad G_{ij} = g_{ij} + h_{ij}.$$

Using matrix notations we have

$$g = G^+BG^- = G^-BG^+, \qquad h = -g(A\cdot\psi)B^{-1}, \tag{23}$$

and for the inverse metric:

$$g^{-1} = \Gamma^+B^{-1}\Gamma^- = \Gamma^-B^{-1}\Gamma^+. \tag{24}$$

The determinant of the metric is

$$\det g = \frac{\det B}{(\det\Gamma^{\pm})^2} = \det B\cdot\big(\det G^{\pm}\big)^2.$$

3.1. Connection

We work with the standard conventions

$$\Gamma^i_{jk} = \gamma^i_{jk} + T^i_{jk}, \quad T^i_{jk} = g^{is} T_{sjk},$$
$$T_{ijk} = \frac{1}{2}(\partial_i h_{jk} + \partial_k h_{ij} + \partial_j h_{ki}), \quad \partial_i \equiv \frac{\partial}{\partial \psi^i}, \tag{25}$$

or using differential forms

$$H = \frac{1}{2!} h_{ij}\, d\psi^i \wedge d\psi^j, \qquad T = \frac{1}{3!} T_{ijk}\, d\psi^i \wedge d\psi^j \wedge d\psi^k = \frac{1}{2}\, dH.$$

The torsion potential is not uniquely defined since the following gauge transformation leaves invariant the torsion:

$$H \to H + dA, \quad A = A_i\, d\psi^i \quad \Longleftrightarrow \quad h_{ij} \to h_{ij} + \partial_{[i} A_{j]}. \tag{26}$$

The connection is given by

$$\Gamma^i_{jk} = \frac{1}{2}\big(g^{-1}\big)_{is}\big(\partial_j G_{ks} + \partial_k G_{sj} - \partial_s G_{kj}\big). \tag{27}$$

Using the relation

$$\partial_i G_{jk} = f_{st}{}^{i} G_{js} G_{tk} = -\big(G A^i G\big)_{jk}, \tag{28}$$

one gets

$$\Gamma^i_{jk} = -\frac{1}{2} f_{st}{}^{j}\big(\Gamma^+ B^{-1}\big)_{is} G_{kt} - \frac{1}{2} f_{st}{}^{k}\big(B^{-1}\Gamma^+\big)_{ti} G_{sj} + \frac{1}{2}\big(g^{-1}\big)_{iu} f_{st}{}^{u} G_{sj} G_{kt}. \tag{29}$$

The next step is to simplify the last term in (29). To this end we combine Jacobi identity and the definition (11) to prove the identity

$$f_{ij}{}^{s}\Gamma^{(\pm)}_{sk} - f_{kj}{}^{s}\Gamma^{(\pm)}_{si} = 2\omega_{ik,j} - f_{ik}{}^{u}\Gamma^{(\mp)}_{uj}. \tag{30}$$

Starting from relation (24) for the inverse metric we can write

$$\big(g^{-1}\big)_{iu} f_{st}{}^{u} = \big(\Gamma^+ B^{-1}\big)_{iv} f_{st}{}^{u}\Gamma^+_{uv},$$

and use (30) to interchange the indices $s \leftrightarrow v$. Several simplifications occur then in relation (29) and one is left with the simple result

$$\Gamma^i_{jk} = \big(f_{is}{}^{k} - \omega^t{}_{s,u}\Gamma^+_{it} G_{ku}\big) G_{sj}. \tag{31}$$

The same procedure, using the second writing of g^{-1} in relation (24), gives another interesting form

$$\Gamma^i_{jk} = \big(-f_{is}{}^{j} + \omega^t{}_{s,u}\Gamma^-_{it} G_{uj}\big) G_{ks}. \tag{32}$$

3.2. Torsion

To get a useful form for the torsion we use relation (31) to compute

$$2T^i_{jk}\Gamma^+_{jr}\Gamma^+_{ks} = f_{ir}{}^k\,\Gamma^+_{ks} - \omega^\gamma{}_{r,\beta}\Gamma^+_{i\gamma}(\Gamma^- G)_{s\beta} - (r \leftrightarrow s).$$

The identity (30) and the easy relation $\Gamma^- G = 2BG - \mathbb{I}$, transform the previous relation into

$$T^i_{jk}\Gamma^-_{rj}\Gamma^-_{sk} = -\omega^\gamma{}_{r,\beta}\Gamma^+_{i\gamma}(BG)_{s\beta} - (r \leftrightarrow s) - \omega_{rs,i}.$$

It is natural to multiply both sides by $(BG)^{-1}_{ur}(BG)^{-1}_{ts}$. Observing that $g^{-1} = (BG)^{-1}\Gamma^-$, we get

$$T^{ijk} = \left(\Gamma^+ B^{-1}\right)_{ks}\omega^\gamma{}_{s,j}\Gamma^+_{i\gamma} - (j \leftrightarrow k) + \left(\Gamma^+ B^{-1}\right)_{js}\left(\Gamma^+ B^{-1}\right)_{kt}\omega_{rs,i}.$$

This result shows that this tensor is much simpler than T_{ijk} since it is a polynomial in the fields ψ. The coefficient of the linear term vanishes from Jacobi's identity and we are left with

$$T^{ijk} = \frac{1}{2} f_{ij,k} - (A\cdot\psi)_{i\alpha}(A\cdot\psi)_{j\beta}\,\omega^{\alpha\beta}_{\ \ i} + \cdots,$$

where the dots indicate circular permutations of the indices i, j, k. We expand the spin connection according to (11) and use the identity (30) to end up with

$$\begin{aligned} 2T^{ijk} = {} & f_{ij,k} - \left(A\cdot\psi B^{-1}\right)_{it}\left(A\cdot\psi B^{-1}\right)_{ju} f_{tu,k} \\ & - f_{ij}^{\ \ s}\left(A\cdot\psi B^{-1}A\cdot\psi\right)_{sk} + \cdots. \end{aligned} \tag{33}$$

Now we can discuss a possibility not yet considered in the literature: the vanishing of the torsion in the dual model. The terms which are independent of ψ require $f_{[ij,k]} = 0$, a first condition which mixes the structure constants and the breaking matrix. Using this relation and the Jacobi identity one can check that the last two terms in (33) are equal. We conclude that the torsion vanishes iff

$$f_{[ij,k]} = 0 \quad \text{and} \quad f_{\alpha s}{}^{(u}\left(B^{-1}\right)_{st} f_{t[k}{}^{v)} f_{ij]}{}^{\alpha} = 0, \quad \forall (u,v)[ijk]. \tag{34}$$

Clearly for a simple algebra, the first constraint never holds, but for solvable algebras both conditions may be satisfied, as will be seen in Section 5 for the Bianchi family.

Let us conclude with an example of Lie algebra, for which the torsion vanishes for any choice of the breaking matrix. Let its generators be $\{X_i,\ i = 1,\ldots,\nu\}$ and take

$$[X_1, X_i] = X_i, \quad i = 2,\ldots,\nu, \qquad [X_i, X_j] = 0, \quad i \neq j \neq 1.$$

3.3. Ricci tensor

The covariant derivatives are defined by

$$D_i v^j = \partial_i v^j + \Gamma^j_{is} v^s = \nabla_i v^j + T^j_{is} v^s, \qquad D_i v_j = \partial_i v_j - \Gamma^s_{ij} v_s = \nabla_i v_j - T^s_{ij} v_s, \tag{35}$$

and the Riemann curvature by

$$[D_k, D_l]v^i = \mathcal{R}^i{}_{s,kl} v^s - 2T^s_{kl} D_s v^i.$$

Its explicit form is given by

$$\mathcal{R}^i{}_{j,kl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ks}\Gamma^s_{lj} - \Gamma^i_{ls}\Gamma^s_{kj}.$$

The Ricci tensor follows from

$$Ric_{ij} = \mathcal{R}^s{}_{i,sj} = \partial_s \Gamma^s_{ji} - \partial_j \Gamma^s_{si} + \Gamma^s_{st}\Gamma^t_{ji} - \Gamma^s_{jt}\Gamma^t_{si}. \tag{36}$$

Using

$$\Gamma^s_{st} = \gamma^s_{st} = \partial_t\big(\ln\sqrt{\det g}\big),$$

we get for it a useful form

$$Ric_{ij} = \partial_s \Gamma^s_{ji} - \Gamma^s_{jt}\Gamma^t_{si} - D_j D_i\big(\ln\sqrt{\det g}\big). \tag{37}$$

In order to compute the first two terms in this relation, we use (31) for the first two connections and (32) for the third one. Apart from trivial cancellations one has to use the identity

$$\omega^s{}_{t,u}\Gamma^+_{as} + \omega^s{}_{u,t}\Gamma^-_{as} = f_{at}{}^s\Gamma^-_{su} - f_{ua}{}^s\Gamma^+_{st}, \tag{38}$$

in order to obtain further strong cancellations of terms, with the final simple result

$$\partial_s \Gamma^s_{ji} - \Gamma^s_{jt}\Gamma^t_{si} = -G_{is}ric_{st}G_{tj} + 2f^s_{st}\omega^t{}_{u,v}G_{iu}G_{vj}. \tag{39}$$

Using (32) and (35), one can check that the last term can be written

$$2f^s_{st}\omega^t{}_{u,v}G_{iu}G_{vj} = D_j V_i, \quad V_i = -2\,G_{it}\,f^s_{st}.$$

Therefore, we end up with

$$Ric_{ij} = -G_{is}ric_{st}G_{tj} + D_j v_i, \quad v_i = V_i - \partial_i \ln\big(\sqrt{\det g}\big). \tag{40}$$

This relation, which displays the relation between the frame geometry of the principal model and the geometry of its dual, will play an essential role in the next section.

Let us conclude with some remarks:

1. This result is different, although related to the ones by Tyurin [28] and Alvarez [1], who expressed the frame geometry of the dual model in terms of the frame geometry of the principal model. The first reference uses supersymmetry while the second uses purely frames. Our approach, using mainly local coordinates computations is valid for any breaking matrix B, while the previous authors have considered only the case $B = \mathbb{I}$. Note also that, in view of the complexity of the dualized vielbein it's a long way from the vielbein components of the Ricci to our relation (40).

2. If we consider a simple algebra $\mathcal{G}$, equipped with its bi-invariant metric (9). Relation (17) shows that the corresponding principal model is Einstein and we will prove that the *dual metric is quasi-Einstein*. To this aim we insert relation (17) into (40), use $f^s_{si} = 0$ to get for the dual theory

$$Ric_{ij} = \frac{\rho}{4}(GBG)_{ij} + D_j v_i.$$

Using relation (32) one can check that

$$D_j\lambda_i = \frac{1}{2}G_{ij} + \frac{1}{2}(GBG)_{ij}, \quad \lambda_i = \big(B^{-1}\big)_{is}\psi^s, \tag{41}$$

from which we deduce

$$Ric_{ij} = -\frac{\rho}{4}G_{ij} + D_j\mathcal{V}_i, \quad \mathcal{V}_i = \partial_i\Big(-\ln\big(\sqrt{\det g}\,\big) + \frac{\rho}{4}\big(B^{-1}\big)_{st}\psi^s\psi^t\Big), \tag{42}$$

which establishes the desired result.

3. One further important point, with respect to string theory, is the dilatonic property of the dualized geometry, i.e., whether the vector V_i is a gradient or not. For the semi-simple groups the dilatonic property does hold since we have $f_{st}^s = 0$.

The failure of this property was first discovered for the dualized Bianchi V metric [19] (see also [13]). In [22,28] it was shown to appear when the isometries are not semi-simple and have traceful structure constants $f_{st}^s \neq 0$, and its interpretation as an anomaly was worked out in [3].

4. One loop divergences of the dualized models

We are now in position to discuss the quantum properties of the dualized models at the one loop level.

Let us first consider the broken principal models with classical action (4). Its one loop counterterm, first computed by Friedan [17], is

$$\frac{1}{4\pi\epsilon}\int d^2x\, ric_{ij}\eta^{\mu\nu}J_\mu^i J_\nu^j, \quad d = 2-\epsilon, \tag{43}$$

where the Ricci components are computed in the vielbein basis.

Renormalizability in the strict field theoretic sense requires that these divergences have to be absorbed by (field independent) deformations of the coupling constants $\hat{\rho}_s$ hidden in the matrix B and possibly a non-linear field renormalization. The renormalizability of the classical theory is ensured by

$$ric_{ij} = \hat{\chi}_s(\rho)\frac{\partial}{\partial\hat{\rho}_s}B_{ij}. \tag{44}$$

The one loop renormalizability is clear for two extreme choices of metrics:

1. The bi-invariant metric, for which relation (17) shows that the principal model is Einstein.

2. The maximally broken metric, for which the matrix B contains $\nu(\nu+1)/2$ independent coupling constants $\hat{\rho}_s$. Since the Ricci is also a symmetric matrix, it can always be absorbed by a deformation of the coupling constants.

For partial breakings of the group G_R, relation (44) may fail to hold and is indeed a constraint which mixes conditions involving the breaking matrix B and the algebra through its structure constants.

In order to compare to the renormalization properties of the dualized theory, let us recall that the most general conditions giving one loop renormalizability are

$$\begin{cases} Ric_{(ij)} = \hat{\chi}_s\dfrac{\partial}{\partial\hat{\rho}_s}g_{ij} + D_{(i}u_{j)}, \\ Ric_{[ij]} = \hat{\chi}_s\dfrac{\partial}{\partial\hat{\rho}_s}h_{ij} + u_s T_{ij}^s + \partial_{[i}U_{j]}, \end{cases} \tag{45}$$

where the $\hat{\rho}_s$ are the coupling constants in the principal model we started from, appearing now in a non-trivial way in the dualized model. The only constraint on the functions $\hat{\chi}_s$ is that they should be field independent.

These relations can be gathered into the single one

$$Ric_{ij} = \hat{\chi}_s \frac{\partial}{\partial \hat{\rho}_s} G_{ij} + D_j u_i + \partial_{[i}(u+U)_{j]}. \tag{46}$$

We are now in position to prove that the one-loop renormalizability of the principal model implies the one-loop renormalizability of its dual. For the reader's convenience we recall relation (40)

$$Ric_{ij} = -G_{is}\, ric_{st}\, G_{tj} + D_j v_i, \qquad v_i = -2G_{it} f_{st}^{s} - \partial_i \ln\bigl(\sqrt{\det g}\bigr),$$

in which we insert (44) to get

$$Ric_{ij} = -\hat{\chi}_l G_{is} \frac{\partial}{\partial \hat{\rho}_l} B_{st} G_{tj} + D_j v_i.$$

The first term is reduced using the identity

$$\frac{\partial}{\partial \hat{\rho}_l} G_{ij}(B,\psi) = -G_{is}(B,\psi)\left(\frac{\partial}{\partial \hat{\rho}_l} B_{st}\right) G_{tj}(B,\psi), \tag{47}$$

to the final form

$$Ric_{ij} = \hat{\chi}_s \frac{\partial}{\partial \rho_s} G_{ij} + D_j v_i. \tag{48}$$

Comparing with relation (46) we conclude to the one-loop renormalizability of the dual model. Furthermore, the vectors u_i and U_i, defined in relation (45), which could be independent, are in fact related up to a gauge transformation by $U_i = -u_i + \partial_i \tau$.

Our next task is to prove that the β functions are the same, so we need a precise definition of the coupling constants. To do this let us switch from the couplings $\{\hat{\rho}_i,\ i = 1,\dots,c\}$ to new couplings (λ, ρ_i) defined by

$$\hat{\rho}_1 = \frac{1}{\lambda}, \qquad \hat{\rho}_{i+1} = \frac{\rho_i}{\lambda}, \quad i = 1,\dots,c-1. \tag{49}$$

We scale similarly the breaking matrix

$$B_{ij}(\hat{\rho}) = \frac{1}{\lambda} S_{ij}(\rho),$$

where, for simplicity, the matrix S can be taken linear in the couplings ρ_s. Then relation (44) becomes

$$\begin{cases} ric_{ij}(B) = ric_{ij}(S) = \left(\chi_\lambda + \sum_s \chi_s \dfrac{\partial}{\partial \rho_s}\right) S_{ij}(\rho), \\ \chi_\lambda = \hat{\chi}_1, \quad \chi_i = \hat{\chi}_i - \rho_i\, \hat{\chi}_1, \quad i = 1,\dots,c-1. \end{cases} \tag{50}$$

The full one loop action is, therefore,

$$\frac{1}{\lambda}\frac{1}{2}\int d^2x \left[\left(1 + \frac{\lambda \chi_\lambda}{2\pi\epsilon}\right) S_{ij}(\rho) + \frac{\lambda}{2\pi\epsilon} \sum_s \chi_s \frac{\partial}{\partial \rho_s} S_{ij}(\rho)\right] J_\mu^i J_\nu^j, \quad \epsilon = 2-d, \tag{51}$$

from which we see that the divergences can be absorbed through coupling constant renormalizations:

$$\lambda_0 = \mu^{\epsilon}\lambda Z_\lambda, \qquad Z_\lambda = 1 - \frac{\lambda\chi_\lambda}{2\pi\epsilon}, \qquad \rho_i^{(0)} = \rho_i Z_i, \qquad Z_i = 1 + \frac{\lambda\chi_i}{2\pi\epsilon}.$$

It follows that the corresponding beta functions are

$$\beta_\lambda = \mu\frac{\partial\lambda}{\partial\mu} = \lambda^2\frac{\partial}{\partial\lambda}Z_\lambda^{(1)} = -\frac{\lambda^2}{2\pi}\chi_\lambda,$$
$$\beta_i = \mu\frac{\partial\rho_i}{\partial\mu} = \lambda\frac{\partial}{\partial\lambda}\big(\rho_i Z_i^{(1)}\big) = \frac{\lambda}{2\pi}\chi_i. \tag{52}$$

For a principal model built with the bi-invariant metric given by (9) one has just the single coupling λ, and

$$\beta_\lambda = \frac{\lambda^2}{2\pi}\frac{\rho}{4}. \tag{53}$$

In order to compute the divergences of the dualized theory in terms of the coupling constants defined in (49) we start from the dual classical action

$$G_{ij}(B,\tilde\psi)\partial_+\tilde\psi^i\partial_-\tilde\psi^j,$$

which we transform according to

$$G(B,\tilde\psi) = \lambda\, G(S,\psi), \quad \psi^i = \lambda\tilde\psi^i \quad \longrightarrow \quad \frac{1}{\lambda}G_{ij}(S,\psi)\,\partial_+\psi^i\,\partial_-\psi^j.$$

The one-loop counterterms follow from the ricci. We start from relation (40) written

$$Ric_{ij} = \lambda^2\big[-G_{is}(S,\psi)\,ric_{st}G_{tj}(S,\psi) + D_j v_i\big].$$

Using (50) we write the first term

$$-\chi_\lambda G_{is}(S,\psi)S_{st}G_{tj}(S,\psi) - \sum_u \chi_u G_{is}(S,\psi)\frac{\partial S_{st}}{\partial\rho_u}G_{tj}.$$

While the second term is reduced using the identity (47), the first term requires more work. One has first to define the vectors

$$w_i = g_{is}\psi^s, \qquad W_i = \psi^s G_{si}, \tag{54}$$

then check the relation

$$D_j w_i + \partial_{[i}W_{j]} = G_{ij} + \frac{1}{2}\psi^s(\partial_j G_{is} + \partial_i G_{sj}) - \Gamma_{ji}^t g_{ts}\psi^s,$$

which upon use of (27) becomes

$$D_j w_i + \partial_{[i}W_{j]} = G_{ij} + \frac{1}{2}\psi^s\partial_s G_{ij}.$$

Eventually relation (28) gives

$$D_j w_i + \partial_{[i}W_{j]} = \frac{1}{2}G_{ij} + \frac{1}{2}(GBG)_{ij}. \tag{55}$$

Scaling appropriately this identity, we have

$$\left[\chi_\lambda G_{ij}(S,\psi) + \sum_u \chi_u \frac{\partial}{\partial\rho_u} G_{ij}(S,\psi) + D_j(v_i - 2\chi_\lambda w_i) - 2\chi_\lambda \partial_{[i} W_{j]}\right] \partial_+\psi^i \partial_-\psi^j.$$

We end up with the renormalized dual theory

$$\begin{cases} \dfrac{1}{\lambda}\displaystyle\int d^2x \left[\left(1 + \frac{\lambda\chi_\lambda}{2\pi\epsilon} + \frac{\lambda}{2\pi\epsilon}\sum_s \chi_s \frac{\partial}{\partial\rho_s}\right) G_{ij} + \frac{\lambda}{2\pi\epsilon}\left(D_j\mathcal{V}_i - 2\chi_\lambda \partial_{[i}W_{j]}\right)\right] \partial_+\psi^i\partial_-\psi^j, \\ \mathcal{V}_i = v_i - 2\chi_\lambda w_i. \end{cases} \tag{56}$$

Comparing this relation with (51) we conclude that, up to the non-linear field redefinition described by the vector $\mathcal{V}_i$ and the gauge transformation described by W_i, the coupling constants renormalization are exactly the same as in the principal model we started from. We have thus established, at the one-loop level, that the principal σ-model renormalizability implies the renormalizability, in the strict field theoretic sense, of its dual and proved that their β functions do coincide.

Remarks

1. What is really new with respect to [28] is that, even working with renormalizability in the strict field theoretic sense, the (possibly strong) breaking of the right isometries G_R does not jeopardize the one-loop renormalizability, and even in this extreme situation the β functions of the principal model and its dual remain the same. This was not obvious since the symmetry breaking is a "hard" breaking, by couplings of power counting dimension two.

2. As already observed in Section 2.3, the isometries of G_L are lost in the dualization process. Hence for the maximal breaking of G_R, no trace seems to remain of the original isometries in the dualized theory. These dual theories constitute a nice example of non-homogeneous metrics with torsion, with no isometries to account for their one-loop renormalizability. Our computation, which puts forward an experimental fact (the one-loop renormalizability) needs some basic theoretical explanation since we know that renormalizability is never accidental but the result of some underlying deeper symmetry.

3. As first observed in [7] for the dualized $SU(2)$ model with symmetry breaking, there appears in the final form of the divergences (56) a gauge transformation W_i. This term is absent for models built on simple Lie groups with their bi-invariant metric B_{ij}. Indeed in this case we have the identities

$$\psi^s G_{si} = G_{is}\psi^s = \left(B^{-1}\right)_{is}\psi^s \Longrightarrow w_i = W_i = \partial_i\left(\frac{1}{2}\left(B^{-1}\right)_{st}\psi^s\psi^t\right) \equiv \lambda_i,$$

which implies $\partial_{[i}W_{j]} = 0$. Then the general identity (55) reduces, for this particular case, to relation (41).

4. The situation at the two-loop level is still unclear since despite negative results in several models [7,23] a more promising and new approach to the problem [24] seems to yield a positive answer.

5. It is well known that the unbroken principal models are integrable (for a review see [29]). On the contrary the broken ones are not believed to be generically integrable, a notable exception being $SU(2)$, whose integrability was shown in [10] for the most general breaking. If this belief is confirmed, our results show that the one-loop quantum equivalence survives to symmetry breaking and therefore the root of this equivalence cannot be integrability.

5. Extension to principal models with torsion

The previous results can be generalized to cover principal models with torsion, with action

$$S=\frac{1}{2}\int d^2x\big(B_{ij}\eta^{\mu\nu}+C_{ij}\epsilon^{\mu\nu}\big)J^i_\mu J^j_\nu, \quad C_{ij}=-C_{ji},$$

where the matrix C has *constant* components. Taking into account the vielbein interpretation of the currents, we define the torsion t_{ijk} as usual by

$$t=\frac{1}{3!}t_{ijk}e^i\wedge e^j\wedge e^k=\frac{1}{2}dC, \quad C=\frac{1}{2}C_{ij}e^i\wedge e^j,$$

which gives

$$t_{ijk}=-\frac{1}{2}\big(f_{ij}{}^sC_{sk}+f_{jk}{}^sC_{si}+f_{ki}{}^sC_{sj}\big).$$

One should first observe that the parallelizing torsion [30] is not of this kind, and second that we have to exclude the case where

$$C_{ij}=f_{ij}{}^s\gamma_s. \tag{57}$$

Indeed, if this relation holds the Bianchi identity (5) gives

$$C_{ij}\epsilon^{\mu\nu}J^i_\mu J^j_\nu=\gamma_s f_{ij}{}^s\epsilon^{\mu\nu}J^i_\mu J^j_\nu=-2\gamma_s\epsilon^{\mu\nu}\partial_\mu J^s_\nu,$$

which is a total divergence. Correspondingly the torsion vanishes as a consequence of the Jacobi identity.

Even if (57) is valid for a semi-simple algebra $\mathcal{G}$, it is not valid for *any* algebra. To see this let us suppose that the center of $\mathcal{G}$, is non-trivial, i.e., there is some generator X_α which commutes with all the other generators. It follows that $f_{\alpha i}^{\ s}\gamma_s\equiv 0$ for all values of i, while $C_{\alpha i}$ can be non-vanishing.

Let us describe briefly how our analysis can be generalized.

The spin connection $\Omega^i{}_j$ now verifies

$$de^i+\Omega^i{}_j\wedge e^j=B^{ij}t_j, \qquad t_i=t_{ist}e^s\wedge e^t.$$

Let us define

$$\Omega^{(\pm)}{}_{ij,k}=\omega_{ij,k}\pm t_{ijk}, \qquad \Omega^{(\pm)\,i}{}_{j,k}=\big(B^{-1}\big)_{is}\Omega^{\pm}{}_{sj,k},$$

then the spin connection one-forms are $\Omega^i{}_j=\Omega^{(-)\,i}{}_{j,s}\,e^s$.

The Ricci tensor has now for components

$$ric_{ij} = \Omega^{(+)\,s}_{\ \ t,s}\,\Omega^{(-)\,t}_{\ \ i,j} - \Omega^{(-)\,s}_{\ \ i,t}\,\Omega^{(+)\,t}_{\ \ j,s}, \qquad \Omega^{(+)\,s}_{\ \ t,s} = \Omega^{(-)\,s}_{\ \ t,s}$$

and is no longer symmetric.

Introducing the notations $\Gamma^{\pm} = B \pm (C + A\cdot\psi)$, we have for the dualized metric $G = (\Gamma^{+})^{-1}$. The connection in the dual theory becomes

$$\Gamma^{i}_{jk} = \left(f_{is}^{\ \ k} - \Omega^{(+)\,t}_{\ \ s,u}\Gamma^{+}_{it}G_{ku}\right)G_{sj} = \left(-f_{is}^{\ \ j} + \Omega^{(-)\,t}_{\ \ s,u}\,\Gamma^{-}_{it}\,G_{uj}\right)G_{ks},$$

from which, after tedious computations, one gets for the Ricci tensor

$$Ric_{ij} = -G_{is}\,ric_{st}\,G_{tj} + D_j v_i, \qquad v_i = -2G_{it}\,f^{s}_{st} - \partial_i \ln\left(\sqrt{\det g}\right), \tag{58}$$

which is strikingly similar to (40).

Let us denote by ρ_s^B the couplings present in the matrix B, by ρ_s^C the couplings present in the matrix C, and ρ_s the couplings present in both matrices. The renormalizability of the principal model with torsion is ensured by

$$\begin{cases} ric_{(ij)} = \left(\chi_s\dfrac{\partial}{\partial\rho_s^B} + \eta_s\dfrac{\partial}{\partial\rho_s}\right)B_{ij}, \\ ric_{[ij]} = \left(\eta_s\dfrac{\partial}{\partial\rho_s} + \xi_s\dfrac{\partial}{\partial\rho_s^C}\right)C_{ij}. \end{cases} \tag{59}$$

Inserting relation (59) into (58) one ends up with

$$Ric_{ij} = \left(\chi_s\frac{\partial}{\partial\rho_s^B} + \eta_s\frac{\partial}{\partial\rho_s} + \xi_s\frac{\partial}{\partial\rho_s^C}\right)G_{ij} + D_j v_i. \tag{60}$$

It follows, by the same arguments as in Section 5, that the dual model is also renormalizable and has the same β functions as the initial principal model with torsion.

6. Dualized Bianchi metrics

Particular dualized models in the Bianchi family have been studied with emphasis either put on the renormalizability properties of the dualized models with symmetry breaking [7,23] or on the dilaton anomaly [13,19]. The aim of this section is to give some detailed analysis of both aspects for the full family.

All the Lie algebras with 3 generators were classified by Bianchi (1897). In a modern presentation [14,15] these algebras are described in terms of the parameter a and the vector $\vec{n} = (n_1, n_2, n_3)$ according to

$$[X_1, X_2] = aX_2 + n_3X_3, \qquad [X_2, X_3] = n_1X_1,$$
$$[X_3, X_1] = n_2X_2 - aX_3, \qquad f^{s}_{st} = -2a\delta_{t1}.$$

The Jacobi identity requires $a \cdot n_1 = 0$.

The algebras of interest appear in the following table

Class A: $a=0$					Class B: $n_1=0$, $a>0$			
Type		n_1	n_2	n_3	Type	a	n_2	n_3
I		0	0	0	V	1	0	0
II		1	0	0	IV	1	0	1
VI_0		0	1	-1	III	1	1	-1
VII_0		0	1	1	VI_a	$a\neq 1$	1	-1
VIII	$su(1,1)$	1	1	-1	VII_a		1	1
IX	$su(2)$	1	1	1				

The adjoint representation is given by

$$T_1=\begin{pmatrix}0&0&0\\0&-a&-n_3\\0&n_2&-a\end{pmatrix},\qquad T_2=\begin{pmatrix}0&a&n_3\\0&0&0\\-n_1&0&0\end{pmatrix},$$
$$T_3=\begin{pmatrix}0&-n_2&a\\n_1&0&0\\0&0&0\end{pmatrix}. \tag{61}$$

The Killing metric $g_{ij}=\mathrm{Tr}(T_iT_j)$ is diagonal with

$$g_{11}=2(a^2-n_2n_3),\qquad g_{22}=-2n_3n_1,\qquad g_{33}=-2n_1n_2.$$

It follows that B VIII and B IX are semi-simple (in fact, simple). Among the remaining non semi-simple algebras, only those in class B have traceful structure constants.

To simplify matters, still keeping the main peculiarities of symmetry breaking, we take the diagonal metric $B_{ij}=r_i\delta_{ij}$. The dual metric tensor is then

$$G=\frac{1}{\Delta_+}\begin{pmatrix}r_2r_3+x^2&r_3z-xy&-r_2y-zx\\-r_3z-xy&r_3r_1+y^2&-r_1x+yz\\r_2y-zx&r_1x+yz&r_1r_2+z^2\end{pmatrix},\qquad \begin{cases}x=n_1\psi^1,\\ y=n_2\psi^2-a\psi^3,\\ z=a\psi^2+n_3\psi^3\end{cases} \tag{62}$$

with

$$\Delta_\pm=r_1r_2r_3+r_1x^2\pm\big(r_2y^2+r_3z^2\big).$$

From (25) we get the torsion

$$\begin{cases}T_{ijk}=t\epsilon_{ijk},\quad t=N/\Delta_+^2,\\ N=\nu\Delta_-+2r_2r_3\big(n_3y^2+n_2z^2-n_1r_1^2\big),\quad \nu=r_1n_1+r_2n_2+r_3n_3.\end{cases} \tag{63}$$

This result shows that for Bianchi V the dualized metric is torsion-free!

Relation (16) gives for the non-vanishing vielbein components of the initial Ricci tensor

$$\begin{cases}ric_{11}=-2a^2+\dfrac{n_1^2r_1^2-(n_2r_2-n_3r_3)^2}{2r_2r_3},\\ ric_{22}=-2a^2\dfrac{r_2}{r_1}+\dfrac{n_2^2r_2^2-(n_3r_3-n_1r_1)^2}{2r_3r_1},\quad ric_{23}=ric_{32}=a\dfrac{(n_2r_2-n_3r_3)}{r_1},\\ ric_{33}=-2a^2\dfrac{r_3}{r_1}+\dfrac{n_3^2r_3^2-(n_1r_1-n_2r_2)^2}{2r_1r_2}.\end{cases} \tag{64}$$

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6.1. Class A dual models and their β functions

We see at a glance from the Ricci that the class A principal models, with 3 independent diagonal couplings are renormalizable at one-loop. From the the previous section this ensures the renormalizability of the dualized model, with the same β functions.

For Bianchi I and II we define

$$r_1 = \frac{1}{\lambda}, \qquad r_2 = \frac{g}{\lambda}, \qquad r_3 = \frac{g'}{\lambda}.$$

Then one gets for the β functions

$$\begin{cases} \text{Bianchi I:} \quad \beta_\lambda = \beta_g = \beta_{g'} = 0, \\ \text{Bianchi II:} \quad \beta_\lambda = -\dfrac{\lambda^2}{4\pi}\dfrac{1}{gg'}, \quad \beta_g = -\dfrac{\lambda}{8\pi}\dfrac{1}{g'}, \quad \beta_{g'} = -\dfrac{\lambda}{8\pi}\dfrac{1}{g}. \end{cases}$$

The result for Bianchi I is obvious, since its metric is flat.

For Bianchi IX (with $\sigma = +1$) and Bianchi VIII (with $\sigma = -1$), we parametrize the couplings according to [1]

$$r_1 = \frac{\sigma}{\lambda}, \qquad r_2 = \frac{\sigma(1+g)}{\lambda}, \qquad r_3 = \frac{1+g'}{\lambda}.$$

With three independent couplings, the $SU(2)_R$ isometries are fully broken. If one takes $g = g'$ the corresponding model has a residual $U(1)_R$ isometry and has been studied in [7], where the quantum equivalence was proved at the one-loop order.

Using (64) and (52) it is a simple matter to compute

$$\begin{cases} \beta_\lambda = -\dfrac{\lambda^2}{4\pi}\dfrac{(1+g-g')(1-g+g')}{(1+g)(1+g')}, \\ \beta_g = \dfrac{\lambda}{2\pi}\dfrac{g(1+g-g')}{(1+g')}, \quad \beta_{g'} = \dfrac{\lambda}{2\pi}\dfrac{g'(1-g+g')}{(1+g)}. \end{cases} \tag{65}$$

For $g = 0$ and $\sigma = 1$ these results agree with [7].

For the remaining models we parametrize the couplings according to

$$r_1 = \sigma\frac{1}{\lambda}, \qquad r_2 = \sigma\frac{1+g}{\lambda}, \qquad r_3 = \frac{1+g'}{\lambda},$$

where $\sigma = +1$ (respectively, $\sigma = -1$) correspond to Bianchi VII_0 (respectively, Bianchi VI_0). We get for the β functions

$$\beta_\lambda = \frac{\lambda^2}{4\pi}\frac{(g-g')^2}{(1+g)(1+g')},$$

$$\beta_g = \frac{\lambda}{2\pi}\frac{(1+g)}{(1+g')}(g-g'), \qquad \beta_{g'} = -\frac{\lambda}{2\pi}\frac{(1+g')}{(1+g)}(g-g'). \tag{66}$$

Let us observe that for $g' = g$ the metric of the principal model is flat, which explains the vanishing of all the β functions.

[1] In the $g = g' = 0$ limit we recover the bi-invariant metrics.

6.2. Class B dual models and their β functions

Let us begin with Bianchi V, which has diagonal ricci, and is therefore renormalizable with three independent couplings

$$r_1 = \frac{1}{\lambda}, \qquad r_2 = \frac{1+g}{\lambda}, \qquad r_3 = \frac{1+g'}{\lambda}.$$

One gets

$$\beta_\lambda = \frac{\lambda^2}{\pi}, \quad \beta_g = \beta_{g'} = 0. \tag{67}$$

For the remaining models in this class the ricci is not diagonal, therefore we conclude to the non-renormalizability of the remaining models with three independent couplings.

However, if we restrict ourselves to two couplings, tuned in such a way to have $ric_{23} = 0$, most of the class B models become renormalizable:

$$\begin{cases} \text{Bianchi III:} & r_1 = \frac{1}{\lambda},\ r_2 = \frac{1+g}{\lambda},\ r_3 = -\frac{1+g}{\lambda}, \\ \text{Bianchi VI}_a(\sigma=-1),\ \text{VII}_a(\sigma=+1): & r_1 = \frac{1}{\lambda},\ r_2 = \frac{1+g}{\lambda},\ r_3 = \sigma\frac{1+g}{\lambda}. \end{cases}$$

Their beta functions are

$$\beta_\lambda = \frac{\lambda^2}{\pi} a^2, \quad \beta_g = 0. \tag{68}$$

For Bianchi IV no choice of diagonal breaking matrix leads to renormalizability.

6.3. Class B dual models and dilaton anomaly

Let us first get a convenient characterization of the absence of the dilaton anomaly. Using relation (28) one has the equivalence

$$V_i = -2G_{it} f_{st}^s = \partial_i \Phi \Longleftrightarrow \partial_i V_j - \partial_j V_i = 0 \Longleftrightarrow G_{su} f_{vu}^v \big(f_{st}^i G_{jt} - f_{st}^j G_{it}\big) = 0.$$

Upon multiplication by $\Gamma_{ai}\,\Gamma_{bj}$ and use of (15), (38) one gets

$$G_{su} f_{vu}^v \big(f_{sb}^t \Gamma_{at} - f_{sa}^t \Gamma_{bt}\big) = 0 \longrightarrow \omega_{ab,s} G_{st} f_{ut}^u = 0.$$

It follows that the equivalence becomes

$$V_i = -2G_{it} f_{st}^s = \partial_i \Phi \Longleftrightarrow \omega_{ab,s} V_s = 0, \quad \forall a, b. \tag{69}$$

Despite the convenient form of the final relation (69), it is fairly difficult to discuss in general. Let us simply observe that the matrices ω_a, with matrix elements defined by $(\omega_a)_{bs} = \omega_{ab,s}$ are singular. So the analysis of (69) depends strongly on the size of the kernel of the ω_a and therefore of the algebra and of the breaking matrix considered.

To discuss this point for the class B of the Bianchi family, we will consider the most general breaking matrix B and we denote its off-diagonal terms by

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$$B_{12} = s_3, \quad B_{23} = s_1, \quad B_{31} = s_2,$$
$$\det B = r_1 r_2 r_3 - r_1 s_1^2 - r_2 s_2^2 - r_3 s_3^2 + 2 s_1 s_2 s_3 \neq 0.$$

Let us notice that this last condition forbids the simultaneous vanishing of s_1, r_2 and r_3.

The matrices ω_a are given generally by

$$(\omega_i)_{jk} = \omega_{ij,k} = -\frac{\nu}{2}\epsilon_{ijk} + \sum_s n_s B_{sk}\epsilon_{sij} + a_i B_{jk} - a_j B_{ik}, \quad a_i = a\delta_{i1}, \quad \nu = \sum_s n_s B_{ss}.$$

For class B we have $\nu = n_2 r_2 + n_3 r_3$. Taking into account the relations

$$G_{11} = \frac{(r_2 r_3 - s_1^2)}{\det \Gamma}, \qquad G_{21} = -\frac{(r_3 s_3 - s_1 s_2 + s_1 y + r_3 z)}{\det \Gamma},$$
$$G_{31} = \frac{(s_3 s_1 - r_2 s_2 + r_2 y + s_1 z)}{\det \Gamma},$$
$$\det \Gamma = \det B + r_2 y^2 + r_3 z^2 + 2 s_1 y z,$$

it is a purely algebraic matter, using (69), to prove that the dilaton anomaly is absent iff

$$\nu \equiv n_2 r_2 + n_3 r_3 = 0 \quad \text{and} \quad \mu \equiv s_1^2 - r_2 r_3 = 0. \tag{70}$$

These constraints show that Bianchi VII$_a$ is always anomalous, but also that an appropriate choice of the couplings can get rid of the anomaly in the other models!

One can summarize the constraints (70) for the class B models and their possibly non-vanishing ricci component:

Model	Constraint ($\epsilon = \pm 1$)	ric_{11}	$\det B \neq 0$
Bianchi III	$r_3 = r_2,\ s_1 = \epsilon r_2$	$2(\epsilon - 1)$	$r_2(s_2 - \epsilon s_3) \neq 0$
Bianchi IV	$r_3 = 0,\ s_1 = 0$	0	$r_2 \cdot s_2 \neq 0$
Bianchi V	$s_1 = \epsilon\sqrt{r_2 r_3}$	0	$\sqrt{\lvert r_2\rvert}\, s_2 - \epsilon\sqrt{\lvert r_3\rvert}\, s_3 \neq 0$
Bianchi VI$_a$ ($a \neq 1$)	$r_3 = r_2,\ s_1 = \epsilon r_2$	$2(a\epsilon - 1)$	$r_2(s_2 - \epsilon s_3) \neq 0$
Bianchi VII$_a$	impossible		

It follows that the models B IV, B V and B III with $\epsilon = +1$ are flat.

We want to show that the restrictions (70) are equivalent to the vanishing of the torsion. To see this we use the constraints (34), which give, when specialized to class B:

$$3 f_{[ij,k]} = \nu, \qquad 3\big(A \cdot \psi\, B^{-1} A \cdot \psi\big)_{s[k}\, f_{ij]}^{\ \ s} = \mu \frac{(a^2 + n_2 n_3)(n_2(\psi^2)^2 + n_3(\psi^3)^2)}{\det B}.$$

In this case it is interesting to compare the vectors $V_i = -2 G_{it} f^s_{st}$ and $g_i = D_i \ln(\sqrt{\det g})$. One can check that the difference $V_i - 2 g_i$ is then covariantly constant, giving for final geometry

$$Ric_{ij} = -G_{is}\, ric_{st}\, G_{tj} + D_j D_i \ln\big(\sqrt{\det g}\big).$$

7. Dualized Bianchi V model at two loops

As observed in the previous sections, dualized models may be *torsionless*: it is therefore important to ascertain which models lead to this phenomenon. To this end we use the constraints (34). Algebraic computations lead to the following conclusions:

1. For the class A models, no choice of the non-singular matrix B leads to vanishing torsion.

2. For the class B models, except Bianchi V, the necessary and sufficient conditions for vanishing torsion are given by the relations (70).

3. Among all the class B models only Bianchi V has a vanishing torsion for an arbitrary breaking matrix B. In this case the torsion potential is an exact 2-form with

$$\begin{cases} H = \frac{1}{2} dA, \qquad \nu_2 = \frac{s_3 s_1 - r_2 s_2}{s_1^2 - r_2 r_3}, \qquad \nu_3 = \frac{s_1 s_2 - r_3 s_3}{s_1^2 - r_2 r_3}, \\ A = \gamma \big(d\psi^1 - \nu_3 d\psi^2 - \nu_2 d\psi^3 \big), \qquad \gamma = \ln\big(\sqrt{\det g}\big). \end{cases}$$

The case where $s_1^2 = r_2 r_3 \neq 0$ is special, with

$$A = \gamma \, d\psi^1 + \frac{1}{r_2} \left(s_3 \ln \left| x^2 - \alpha^2 \right| - \ln \left| \frac{x+\alpha}{x-\alpha} \right| \cdot \psi^2 \right) d\psi^2,$$
$$x = r_2 \psi^3 - s_1 \psi^2, \quad \alpha = s_1 s_3 - r_2 s_2.$$

It follows that the dual model, at least perturbatively, can be analyzed as if it had no WZW coupling! This situation is fairly original: the principal Bianchi V model, which is homogeneous and torsionless, is mapped by T-duality to an inhomogeneous but still torsionless σ-model. It is therefore attractive to check the two-loop equivalence of the models using the firmly established counterterms given by Friedan [17]. Let us consider the simplest Bianchi V dual model, with $B_{ij} = r\delta_{ij}$. Its dualized metric, taken from (62), reads:

$$g = \frac{r}{\Delta} \big(d\psi^1\big)^2 + \frac{1}{r\Delta} \big[r^2 \big(d\psi^2\big)^2 + r^2 \big(d\psi^3\big)^2 + \big(\psi^3 d\psi^2 - \psi^2 d\psi^3\big)^2 \big],$$
$$\Delta = \big(\psi^2\big)^2 + \big(\psi^3\big)^2 + r^2. \tag{71}$$

Following [13] we take for new coordinates

$$\psi^1 = z, \quad \psi^2 + i\psi^3 = \rho e^{i\phi} \quad \Longrightarrow \quad g = \frac{r}{\rho^2 + r^2} \big(dz^2 + d\rho^2\big) + \frac{\rho^2}{r} (d\phi)^2, \tag{72}$$

which bring the metric to a simple diagonal form, with the obvious vielbein

$$g = \sum_{a=1}^{3} e_a^2, \qquad e_1 = \frac{\sqrt{r}}{\sqrt{\rho^2 + r^2}} dz, \quad e_2 = \frac{\sqrt{r}}{\sqrt{\rho^2 + r^2}} d\rho, \quad e_3 = \frac{\rho}{\sqrt{r}} d\phi. \tag{73}$$

One can prove that this metric has two isometries, described by the vector fields $\partial/\partial z$ and $\partial/\partial\phi$.

The geometrical quantities of interest are

$$\begin{cases} \omega_{23} = -\dfrac{1}{\sqrt{r}}\dfrac{\sqrt{\rho^2+r^2}}{\rho}e_3, \qquad \omega_{12} = -\dfrac{1}{\sqrt{r}}\dfrac{\rho}{\sqrt{\rho^2+r^2}}\,e_1, \\ R_{23} = -\dfrac{1}{r}\,e_2\wedge e_3, \qquad R_{31} = \dfrac{1}{r}\,e_3\wedge e_1, \qquad R_{12} = -\dfrac{\sigma}{r}\,e_1\wedge e_2, \qquad \sigma = \dfrac{\rho^2-r^2}{\rho^2+r^2}, \\ Ric_{11} = \dfrac{1-\sigma}{r}, \qquad Ric_{22} = -\dfrac{1+\sigma}{r}, \qquad Ric_{33} = 0, \\ R = Ric_{ss} = -2\dfrac{\sigma}{r}. \end{cases} \tag{74}$$

The one-loop renormalizability relations

$$Ric_{ij} = \chi^{(1)}\frac{\partial}{\partial r}\,g_{ij} + \nabla_{(i}v_{j)},$$

become, using vielbein components

$$\begin{cases} Ric_{ab} = \chi^{(1)}\Big(\big(e^{-1}\big)_b^j\dfrac{\partial}{\partial r}\,e_{aj} + \big(e^{-1}\big)_a^j\dfrac{\partial}{\partial r}\,e_{bj}\Big) + \mathcal{D}_{(a}v_{b)}, \\ \mathcal{D}_a v_b = \hat{\partial}_a\,v_b + \omega_{bs,a}\,v_s, \qquad \hat{\partial}_a = \big(e^{-1}\big)_a^j\partial_i. \end{cases} \tag{75}$$

Relation (75) works with

$$\chi^{(1)} = -2, \qquad v \equiv v_a\,e_a = -\frac{2}{\sqrt{r}}\frac{\rho}{\sqrt{\rho^2+r^2}}\,d\rho. \tag{76}$$

Let us remark that while $\chi^{(1)}$ is uniquely defined, the vector v_i is not unique and we took its simplest form. As it should, the renormalization of the coupling constant r is the same as in the principal model as can be seen from relation (64).

The two-loops counterterms, first computed by Friedan [17], are

$$\frac{1}{16\pi^2\epsilon}\int d^2x\ R_{is,tu}R_{js,tu}\eta^{\mu\nu}J^i_\mu J^j_\nu,$$

where the $R_{is,tu}$ are the vielbein components of the Riemann tensor.

For three dimensional geometries, this counterterm is most easily obtained from the identity

$$(RR)_{ab} \equiv \frac{1}{2}R_{as,tu}R_{bs,tu} = R\,Ric_{ab} - \big(Ric^2\big)_{ab} + \bigg(\mathrm{Tr}\big(Ric^2\big) - \frac{R^2}{2}\bigg)\delta_{ab}, \tag{77}$$

which gives

$$(RR)_{11} = (RR)_{22} = \frac{1}{r^2}(1+\sigma^2), \qquad (RR)_{33} = \frac{2}{r^2}.$$

In order to prove renormalizability we have to solve for $\chi^{(2)}$ and w_a such that

$$(RR)_{ab} = \chi^{(2)}\Big(\big(e^{-1}\big)_b^j\frac{\partial}{\partial r}e_{aj} + \big(e^{-1}\big)_a^j\frac{\partial}{\partial r}e_{bj}\Big) + \mathcal{D}_{(a}w_{b)}. \tag{78}$$

Explicitly, these equations give the differential system

$$\begin{cases} \frac{1}{r^2}\left(1+\sigma^2\right) - \chi^{(2)}\frac{\sigma}{r} = \hat{\partial}_1 w_1 + \omega_{12,1} w_2, & 0 = \hat{\partial}_1 w_2 + \hat{\partial}_2 w_1 - \omega_{12,1} w_1, \\ \frac{1}{r^2}\left(1+\sigma^2\right) - \chi^{(2)}\frac{\sigma}{r} = \hat{\partial}_2 w_2, & 0 = \hat{\partial}_3 w_1 + \hat{\partial}_1 w_3, \\ \frac{2}{r^2} + \chi^{(2)}\frac{1}{r} = \hat{\partial}_3 w_3 - \omega_{23,3} w_2, & 0 = \hat{\partial}_3 w_2 + \hat{\partial}_2 w_3 + \omega_{23,3} w_3. \end{cases} \tag{79}$$

Integrating some relations with respect to the variable ρ we obtain

$$\begin{cases} w_1 = -w_1^{(0)}(\rho)\partial_z W_2(z,\phi) + \frac{W_1(z,\phi)}{\sqrt{\rho^2+r^2}}, & \frac{d}{d\rho}\left(\sqrt{\rho^2+r^2}\, w_1^{(0)}(\rho)\right) = \sqrt{\rho^2+r^2}, \\ w_2 = w_2^{(0)}(\rho) + W_2(z,\phi), & \frac{d}{d\rho} w_2^{(0)}(\rho) = \frac{\sqrt{r}}{\sqrt{\rho^2+r^2}}\left(\frac{1+\sigma^2}{r^2} - \chi^{(2)}\frac{\sigma}{r}\right), \\ w_3 = -\frac{\sqrt{\rho^2+r^2}}{r}\partial_\phi W_2(z,\phi) + \rho W_3(z,\phi). & \end{cases} \tag{80}$$

Inserting these relations into the last left relation of (79) one has

$$\frac{2}{r^2} + \frac{\chi^{(2)}}{r} - \sqrt{r}\,\partial_\phi W_3 = M, \qquad W_2 - \partial_\phi^2 W_2 = N,$$
$$w_2^{(0)}(\rho) + N = \sqrt{r}\frac{\rho}{\sqrt{\rho^2+r^2}} M, \tag{81}$$

where M and N are coordinate independent. Differentiating this last relation with respect to ρ yields a constraint which does not hold, irrespectively of the values taken for M and $\chi^{(2)}$.

The failure of relations (78) means that the two-loops quantum extension chosen for the dual model does not lift the classical equivalence to the quantum level.

In fact we should consider [2] the whole family of metrics

$$g_{ij} \longrightarrow g_{ij} + \gamma_{ij},$$

where γ_{ij} is a one-loop deformation of the classical metric g_{ij} which describes different possible quantum extensions of the same classical dual model. For this modified theory we get an extra contribution at the two-loops level which is

$$\frac{1}{4\pi\epsilon}\int d^2x\left[Ric_{ij}(g+\gamma) - Ric_{ij}(g)\right]\eta^{\mu\nu} J_\mu^i J_\nu^j.$$

Let us examine whether the two-loops renormalizability can be implemented or not. As is well known, one has

$$Ric_{ij}(g+\gamma) - Ric_{ij}(g) = -\frac{1}{2}\Delta_L\gamma_{ij} + \nabla_{(i}\alpha_{j)}, \quad \alpha_i = \nabla^s\gamma_{si} - \frac{1}{2}\nabla_i\gamma^s_{\ s},$$

where Δ_L is Lichnerowicz's laplacian

$$\Delta_L\gamma_{ij} = \nabla^s\nabla_s\gamma_{ij} + 2R_{is,jt}\gamma^{st} - Ric_{is}\gamma^s_{\ j} - Ric_{js}\gamma^s_{\ i}.$$

The connection ∇, the Riemann tensor and the raising or lowering of indices are related to the unperturbed metric g.

[2] We thank G. Bonneau for suggesting to us this idea.

Up to a scaling of γ, the two-loops renormalizability constraints become

$$(RR)_{ij} = \Delta_L \gamma_{ij} + \chi^{(2)} \frac{\partial}{\partial r} g_{ij} + \nabla_{(i} w_{j)}. \tag{82}$$

We will exhibit a solution of these equations for any choice of $\chi^{(2)}$. For this we consider the vielbein components of the deformation

$$\gamma = \frac{1}{r}\big(\gamma_1 e_1^2 + \gamma_2 e_2^2 + \gamma_3 e_3^2\big),$$

and we use the notations

$$\chi^{(2)} = \frac{2}{r}(1-\chi), \quad x = \frac{\rho^2}{\rho^2 + r^2} \in [0, 1[,$$

$$\Phi(x) = \frac{3+\chi}{2(1-x)^2} \int_0^x \frac{\ln(1-u)}{u}\, du.$$

One should notice that for the principal Bianchi V model at two-loops we have $\chi = 0$.

Let us define

$$\begin{cases} \gamma_1(x) = -\dfrac{1+x}{1-x}\gamma_2(0) - \dfrac{4(4+3\chi)x - (5\chi-1)x^2}{8(1-x)^2} \\ \qquad - \dfrac{3+\chi-x}{2(1-x)} \ln(1-x) + \Phi(x), \\ \gamma_2(x) = \gamma_2(0) + \dfrac{4(11+4\chi)x + (11+5\chi)x^2 - 4x^3}{8(1-x)^2} \\ \qquad + \dfrac{8+3\chi+x}{2(1-x)} \ln(1-x) - (1+2x)\Phi(x), \\ \gamma_3(x) = -\gamma_1(x). \end{cases}$$

The vector vielbein components are

$$w_1 = w_3 = 0,$$

$$r^{3/2}\sqrt{x}\, w_2 = 4(1-\chi)x - 2x^2 - (2+\chi)\ln(1-x) - 2x(1-x)\frac{d}{dx}\gamma_2(x).$$

The reader can check that the deformation and the vector given above are indeed solution of (82) for *any value* of χ.

Two main points need to be checked. The first one is the analyticity of the $\gamma_i(x)$ in a neighbourhood of $x = 0$. This follows from the analyticity of $\Phi(x)$ and is explicit on the other terms. The second point is that we are using polar coordinates; in order to secure an analytic dependence with respect to the cartesian coordinates $\psi^1, \psi^2 \approx 0$ we have imposed $\gamma_3(0) = \gamma_2(0)$. The free parameters in this solution are $\gamma_2(0)$ and χ.

As a side remark, let us observe that the deformation obtained above, cannot be written in the form

$$\gamma_{ij} = A \frac{\partial}{\partial r} g_{ij} + D_{(i}\mathcal{W}_{j)},$$

which means that it cannot be interpreted as a *finite* renormalization of the initial metric g_{ij}.

So we can conclude that it is always possible to have a quantum extension of the dualized Bianchi V which does preserve the two-loops renormalizability. Unfortunately nothing, in this process, enforces χ to have the same value as in the principal model we started from. This shows that further constraints are needed to define uniquely the two-loops quantum dual theory.

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C.2 Renormalisability of non-homogeneous T-dualised sigma-models

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Renormalisability of non-homogeneous T-dualised sigma-models

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Abstract

The quantum equivalence between σ-models and their non-Abelian T-dualised partners is examined for a large class of four dimensional non-homogeneous and quasi-Einstein metrics with an isometry group $SU(2) \times U(1)$. We prove that the one-loop renormalisability of the initial torsionless σ-models is equivalent to the one-loop renormalisability of the T-dualised torsionful model. For a subclass of Kähler original metrics, the dual partners are still Kähler (with torsion).

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Keywords: Non-homogeneous sigma models; T-duality; Renormalisation

1. Introduction

The subject of target space duality, or T-duality, in String Theory and in Conformal Field Theory has generated much interest in recent years and extensive reviews covering Abelian, non-Abelian dualities and their applications to string theory and statistical physics are available in the literature [1–3]. The geometrical aspects of this duality can be found in [4]. T-duality provides a method for relating inequivalent string theories. First discovered for the case of σ-models with some Abelian isometry, the concept of T-duality has been recently enlarged to theories with non-Abelian isometries [5–7]. A very important and interesting property of T-duality applied on non-Abelian isometry is that it can map a geometry with such isometries to another which has none. Therefore, non-Abelian T-duality cannot be inverted as in the Abelian case.

By showing that T-duality is a canonical transformation [5,8,9], it was proved that theories in such way related where classically equivalent. Furthermore, this equivalence was still remaining at the one-loop level, in a strict renormalisability sense, in all the many example that have been tested up to now to this duality, with an emphasis put on $SU(2)$ [1,7,10–13]. For example, this one-loop equivalence still remains for principal σ-models whatever strongly broken the right isometries may be [14]. The non-Abelian dualisation of non-homogeneous metrics such as the Schwarzschild black hole or Taub-NUT was performed in [7,12] and in [15]. We propose here the dualisation of the general $SU(2) \times U(1)$ metrics.

Problems arise when one addresses the question of the renormalisability of dualised theories beyond the

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one-loop order. It had been proved that even for the simplest $(SU(2)\times SU(2))/SU(2)$ principal σ-model, the dualised theory is not two-loop renormalisable, in the minimal dimensional scheme [16,17]. However, as shown in [18], a *finite* deformation at the $\hbar$ order of the dualised metric is sufficient for recovering a two-loop renormalisability for this particular model. As it will be shown, the $SU(2)\times U(1)$ σ-models are not in general two-loop renormalisable, even though the one-loop renormalisability remains for their dual partners!

The content of this Letter is the following: in Section 2, we recall the general expression of the $SU(2)\times U(1)$ metrics and set the notations. In Section 3, we make a review of such metrics which give rise to one-loop renormalisable σ-models, as for example the celebrated Taub-NUT and Eguchi–Hanson metrics. In Section 4, we show that only the particular metrics where homogeneity is recovered by some enhancement of the isometries are two-loop renormalisable. In Section 5, we dualise the original theory and show in Section 6 that the one-loop renormalisability survives during the dualisation process. When the original metric is Kähler, we investigate in Section 7 if such a property is still present for the dual partner. Some concluding remarks are offered in Section 8.

2. The $SU(2)\times U(1)$ metric

We consider the four dimensions metrics with cohomogeneity one under a $SU(2)\times U(1)$ isometry. In the more general way, these can write

$$g=\alpha(t)\,dt^2+\beta(t)\big({\sigma_1}^2+{\sigma_2}^2\big)+\gamma(t){\sigma_3}^2,$$

where the σ_i are 1-forms such that

$$d\sigma_i=\varepsilon\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_j,\quad \varepsilon=\pm1.$$

One can always writes ${\sigma_1}^2+{\sigma_2}^2$ and σ_3 under the well-known specific shape

$$\begin{aligned}{\sigma_1}^2+{\sigma_2}^2&=d\theta^2+\sin^2\theta\,d\varphi^2,\\ \sigma_3&=d\psi+\cos\theta d\varphi.\end{aligned}$$

If $\varepsilon=+1$, the triplet of 1-forms $\vec{\sigma}$ is changed under infinitesimal transformations of $su(2)_L\oplus su(2)_R$ as

$$\delta\vec{\sigma}=\vec{\epsilon}_R\wedge\vec{\sigma}.$$

Therefore $\vec{\sigma}$ is a $SU(2)_L$ singlet and a $SU(2)_R$ triplet. If $\beta(t)\neq\gamma(t)$, the $SU(2)_R$ isometries will be broken down to a $U(1)$ and the total isometry group of the metric will then be $SU(2)_L\times U(1)$. Indeed, in order to keep the metric invariant, one then must have $\vec{\epsilon}_R=\{0,0,\mu\}$. If $\varepsilon=-1$, $\vec{\sigma}$ is changed under infinitesimal transformations of $su(2)_L\oplus su(2)_R$ as

$$\delta\vec{\sigma}=\vec{\epsilon}_L\wedge\vec{\sigma},$$

and therefore the isometry group of the metric will be $SU(2)_R\times U(1)$. The choice of ε switches also the autodual components of the Weyl tensor ($W_+\leftrightarrow W_-$). In all cases, when $\beta(t)=\gamma(t)$, the metric has for isometry group $SU(2)_L\times SU(2)_R$ and is conformally flat.

It is then possible to define the σ-model corresponding to these metrics

$$S=\frac{1}{T}\int dx^2\,\eta^{\mu\nu}g_{ij}\,\partial_\mu\phi^i\,\partial_\nu\phi^j,\tag{1}$$

with $\{\phi^0=t,\phi^1=\theta,\phi^2=\varphi,\phi^3=\psi\}$, and address the question of its one-loop and two-loop renormalisability.

In order to derive the Ricci tensor, we define the vierbein $\{e_a\mid a\in\{0,1,2,3\}\}$ as

$$\begin{aligned}e_0&=\sqrt{\alpha(t)}\,dt,\qquad & e_2&=\sqrt{\beta(t)}\,\sigma_2,\\ e_1&=\sqrt{\beta(t)}\,\sigma_1,\qquad & e_3&=\sqrt{\gamma(t)}\,\sigma_3.\end{aligned}$$

In the absence of torsion, the condition for giving one-loop renormalisability is the quasi-Einstein property of the metric

$$Ric_{ab}=\lambda g_{ab}+D_{(a}v_{b)},\tag{2}$$

where the Einstein constant λ will renormalise the coupling while the vector v will renormalise the field.

3. One-loop renormalisation

We will only consider metrics satisfying condition (2) so that the corresponding σ-models are one-loop renormalisable. Of course, as we want to keep the $SU(2)$ symmetry while renormalising, we will only consider here vectors v that depends only on the t coordinate: $v=v(t)$. As the expression of the $SU(2)\times U(1)$ metric (3) we chose does not mix dt,

σ_1, σ_2 and σ_3, both the metric g and the Ricci tensor Ric will be diagonal in the $\{dt, \sigma_1, \sigma_2, \sigma_3\}$ basis and this will hold in the vierbein. As a consequence, $D_{(a}v_{b)}$ must be also diagonal; this is true only for vectors of the form $v = v_0(t)e_0 + \rho\sqrt{\gamma(t)}\, e_3$. The constant ρ is arbitrary as $\sqrt{\gamma(t)}e_3$ is in fact the form dual to the Killing ∂_ψ. We will take $\rho = 0$.

In order to simplify matters, from now on, we will choose the coordinate t so that $\beta(t) = t$. The metric now writes

$$g = \alpha(t)\, dt^2 + t\big({\sigma_1}^2 + {\sigma_2}^2\big) + \gamma(t){\sigma_3}^2. \tag{3}$$

All this being settled, the quasi-Einstein character of the metric (2) can now be expressed as a set of three non-linear differential equations which are

$$\begin{cases} \frac{1}{t^2} + \big(\frac{1}{t} + \frac{\gamma'(t)}{2\gamma(t)}\big)\frac{\alpha'(t)}{\alpha(t)} + \frac{\gamma'(t)^2}{2\gamma(t)^2} - \frac{\gamma''(t)}{\gamma(t)} \\ \quad = 2\lambda\alpha(t) + 2\sqrt{\alpha(t)}\, v_0'(t), \\ 2\big(2 - \frac{\gamma(t)}{t}\big)\alpha(t) + \frac{\alpha'(t)}{\alpha(t)} - \frac{\gamma'(t)}{\gamma(t)} \\ \quad = 4\lambda t\alpha(t) + 2\sqrt{\alpha(t)}\, v_0(t), \\ -\frac{2}{t} + \frac{2}{t^2}\frac{\gamma(t)^2}{\gamma'(t)}\alpha(t) + \frac{\alpha'(t)}{\alpha(t)} + \frac{\gamma'(t)}{\gamma(t)} - \frac{2\gamma''(t)}{\gamma'(t)} \\ \quad = 4\lambda\frac{\gamma(t)}{\gamma'(t)}\alpha(t) + 2\sqrt{\alpha(t)}\, v_0(t). \end{cases} \tag{4}$$

This system is difficult to solve, even though it can still be done for some limited cases as the Einstein one ($v_0 = 0$) and the quasi-Einstein Kähler one. It is possible to eliminate $\alpha(t)$ and $v_0(t)$ in the system (4), leading to a single, deeply non-linear, differential equation of the fourth order in $\gamma(t)$. The general $SU(2) \times U(1)$ quasi-Einstein metric should therefore depend on four parameters.

In order to convince the reader of the large class of models that will be dualised, we will now give a short review of the $SU(2) \times U(1)$ Einstein and quasi-Einstein Kähler metrics.

3.1. Einstein metrics

The metric g will be Einstein if $Ric = \lambda g$. It is possible to integrate the differential system (4) imposing $v_0 = 0$ and one gets

$$\alpha(t) = \frac{1}{1 + At}\frac{1}{\gamma(t)},$$

$$\gamma(t) = \frac{4t}{(1 + \sqrt{1 + At})^2} - \frac{4\lambda t^2}{3}\frac{3 + \sqrt{1 + At}}{(1 + \sqrt{1 + At})^3} + \frac{B}{t}\sqrt{1 + At}, \tag{5}$$

A and B being the integration constants. This family contains many metrics of interest which we recall briefly.

If $A = 0$, we recover the Kähler–Einstein extension of Eguchi–Hanson [20]. If $A \neq 0$ then g identifies with the large class of Einstein metrics derived by Carter [21]. By making the change of coordinates

$$t \to t^2 - n^2, \quad \text{with } A = \frac{1}{n^2} \text{ and } B = -8(M - n)n^3,$$

one can have for g a more simple expression

$$\begin{cases} g = \frac{t^2 - n^2}{f(t)}\, dt^2 + \big(t^2 - n^2\big)\big({\sigma_1}^2 + {\sigma_2}^2\big) \\ \qquad + \frac{4n^2}{t^2 - n^2} f(t){\sigma_3}^2, \\ f(t) = t^2 - 2Mt + n^2 - \frac{\lambda}{3}(t - n)^3(t + 3n). \end{cases} \tag{6}$$

Notice that as A and B are real constants, M and n can be both reals or pure imaginaries. Defining $2n\, d\psi = d\Psi$ and taking the limit $n \to 0$ gives the Schwarzschild metric with cosmological constant

$$g = \frac{1}{1 - \frac{2M}{t} - \frac{\lambda}{3}t^2}\, dt^2 + t^2\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big) + \left(1 - \frac{2M}{t} - \frac{\lambda}{3}t^2\right) d\Psi^2. \tag{7}$$

Other limits of (6) lead to the Page metric on $P_2(\mathbb{C})\#\overline{P_2(\mathbb{C})}$ and to the Taub-NUT metric.

3.2. Quasi-Einstein Kähler metrics

These are the only $SU(2) \times U(1)$ quasi-Einstein metrics known up to now [22]. We suppose here that there is a choice of holomorphic coordinates on which the isometries $SU(2) \times U(1)$ act linearly. It happens that this hypothesis implies the integrability of the complex structure. A necessary condition of the Kähler property is the closing of the Kähler form

$$d(e_0 \wedge e_3 + \varepsilon e_1 \wedge e_2) = d\big(\sqrt{\alpha(t)\gamma(t)}\, dt \wedge \sigma_3 + \beta(t)\, d\sigma_3\big) = 0.$$

It is clear that this relation will hold iff $\beta'(t)^2 = \alpha(t)\gamma(t)$, i.e., $\alpha(t) = \frac{1}{\gamma(t)}$. It is then possible to solve system (4) and one gets for the metric and for the vector v:

$$g = \frac{1}{\gamma(t)}dt^2 + t\big(\sigma_1{}^2 + \sigma_2{}^2\big) + \gamma(t)\sigma_3{}^2,$$
$$v = -C\sqrt{\gamma(t)}\,e_0 = -C\,dt, \tag{8}$$

with

$$\gamma(t) = \frac{De^{Ct}}{t} + t + \frac{2}{C^2 t}\left(1 - \frac{2\lambda}{C}\right)\left(e^{Ct} - 1 - Ct - \frac{1}{2}C^2t^2\right),$$

where C and D are the integration constants.

In the limit $C \to 0$, we have $v = 0$ and thus we are back to the Kähler–Einstein metrics, i.e., the Kähler–Einstein extension of Eguchi–Hanson (the correspondence between the parameters is then $D = B$).[1]

4. Two-loop renormalisation

The two-loop divergences, first computed by Friedan [19], are

$$Div_{ij}^2 = -\frac{\hbar^2 T}{8\pi^2\epsilon}R_{is,tu}R_j{}^{s,tu}, \qquad d = 2-\epsilon.$$

In order to reabsorb these divergences, the counter-terms may come from the renormalisation of the coupling T and the fields $\vec{\phi}$, but also from the renormalisation of the parameters that were let in the metric at one-loop. For example, if one starts with the Einstein metric (6), one should allow for counter-terms renormalising the parameters M, n. In general, if we define such parameters as ρ_c, the theory will be renormalisable at two loops iff one can find some vector $\tilde{v} = \tilde{v}(t)$ and some constants $\tilde{\lambda}$ and χ_c such that

$$\frac{1}{2}R_{is,tu}R_j{}^{s,tu} = \tilde{\lambda}g_{ij} + \chi_c\partial_{\rho_c}g_{ij} + D_{(i}\tilde{v}_{j)}. \tag{9}$$

We will show that, except for the few particular cases where the metric is homogeneous,[2] the $SU(2) \times U(1)$ Einstein and Kähler metrics do not give in a direct way two-loop renormalisable σ-models.

[1] This shows that the four parameters of the general solution of (4) cannot be A, B, C and D as these are not independent.

[2] It was proven in [23] that homogeneous metrics are always renormalisable to all loop order.

4.1. Einstein metrics

In the vierbein basis, one can compute the two-loop divergences for the metric given in (6) and find

$$\frac{1}{2}R_{am,np}R_{bm,np} = 3\left(\frac{(M-n)^2}{(n-t)^6} + \frac{\big(M+n+\frac{8n^3\lambda}{3}\big)^2}{(n+t)^6} + \frac{\lambda^2}{9}\right)\delta_{ab}.$$

Quite surprisingly, the two-loop divergences are conformal to the original metric.

Relation (9) in the vierbein basis becomes

$$\frac{1}{2}R_{am,np}R_{bm,np} = \frac{1}{2}\tilde{\lambda}\delta_{ab} + E_a{}^j(\chi_M\partial_M + \chi_n\partial_n)E_{bj} + \frac{1}{2}D_a\tilde{v}_b + (a \leftrightarrow b),$$

where E_{ai} is defined by $e_a = E_{ai}d\phi^i$. As for the one-loop renormalisation conditions (4), this last relation gives us three equations. These can easily be reduced to two by eliminating $\tilde{v}$. The remaining equations will only depend on the variable t and on the constants $\tilde{\lambda}$, χ_n and χ_M. As these must vanish irrespectively of the values taken by t, one can show that they will be verified in only two particular cases where M and n are fixed such that

$$M^2 = n^2 = -\frac{3}{4\lambda} \quad \text{or} \quad M = n = 0.$$

In both cases, (9) will be satisfied with $\tilde{\lambda} = \frac{\lambda^2}{3}$ and $\chi_M = \chi_n = \tilde{v} = 0$, but it is not surprising as these choice for M and n are the one which enlarge the $SU(2) \times U(1)$ isometries to $SO(5)$, making the metric homogeneous (de Sitter metric).

4.2. Kähler metrics

Proceeding as for the Einstein metrics, one can compute the two-loop divergence using the metric (8). Once again, the parameters C and D must have special values for the action to be two-loop renormalisable. Indeed, one must have ($C = 2\lambda$, $D = 0$) or ($C \to 0$, $D = 0$). In the first case, we recover flat space. In the second case, we get the Fubiny–Study metric on $P_2(\mathbb{C})$ and its non-compact partner which are also two-loop renormalisable with $\tilde{\lambda} = \frac{2}{3}\lambda^2$ and $\tilde{v} = 0$.

The Einstein and Kähler metrics with no more isometries than $SU(2)\times U(1)$ are therefore not renormalisable in the minimal scheme at two loops. This could of course be cured by adding some infinite deformation of the metric itself as in D. Friedan's approach to σ models quantisation, but it is the author belief that a finite deformation keeping the isometries, as explained in [18], would be sufficient.[3]

5. The dual metric

We dualise the initial metric (3) over the $SU(2)$ isometries, keeping aside the $U(1)$. Practically, it consists in dualising the three-dimensional metric [15]

$$g_3 = t\big({\sigma_1}^2+{\sigma_2}^2\big)+\gamma(t){\sigma_3}^2,$$

leaving the term $\alpha(t)\,dt^2$ unchanged. If we define the new fields of the dual metric λ^i, $i\in\{1,2,3\}$, the dual theory of g_3 will writes, in light-cone coordinates

$$\widehat{S}_3=\frac{1}{T}\int dx^2\,\widehat{G}_{3ij}\partial_+\lambda^i\partial_-\lambda^j,$$

where

$$\widehat{G}_{3ij}=\begin{pmatrix} t & \lambda_3 & -\lambda_2\\ -\lambda_3 & t & \lambda_1\\ \lambda_2 & -\lambda_1 & \gamma(t)\end{pmatrix}^{-1}_{ij}.$$

After the following change in coordinates

$$\lambda_1=y\sin(z),\qquad \lambda_2=y\cos(z),\qquad \lambda_3=r,$$

one has for the total dual metric $\hat{g}=\alpha(t)\,dt^2+\widehat{G}_{3(ij)}\,d\lambda^i\,d\lambda^j$

$$\hat{g}=\alpha(t)\,dt^2+\frac{r^2+t^2}{\Delta}\left(dr+\frac{ry}{r^2+t^2}dy\right)^2 + \frac{t}{r^2+t^2}\,dy^2+\frac{ty^2\gamma(t)}{\Delta}\,dz^2, \tag{10}$$

where

$$\Delta=y^2t+\big(r^2+t^2\big)\gamma(t).$$

[3] Here, one should start with the *general metric*, solution of (4), if no new parameters is a required condition for the renormalisation process.

The torsion is defined by $T=\frac{1}{2}dH$ where $H=\frac{1}{2}\widehat{G}_{3[ij]}\,d\lambda^i\wedge d\lambda^j$ is the torsion potential 2-form

$$H=d(z\,dr)+\frac{(r^2+t^2)\gamma(t)}{\Delta}\,dr\wedge dz + \frac{ry\gamma(t)}{\Delta}\,dy\wedge dz. \tag{11}$$

We define $\hat{g}_{ij}$ as the tensor associated to the metric (10) and $\hat{h}_{ij}$ as the torsion potential. Let $\widehat{G}_{ij}=\hat{g}_{ij}+\hat{h}_{ij}$ and $\widehat{Ric}$ be the new Ricci tensor which is not symmetric anymore because of the presence of torsion in the dualised model. Eventually, the dualised action of our $SU(2)\times U(1)$ theory is, in light-cone coordinates

$$\widehat{S}=\frac{1}{T}\int dx^2\,\widehat{G}_{ij}\partial_+\hat{\phi}^i\partial_-\hat{\phi}^j, \tag{12}$$

where the coordinates are $\{\hat{\phi}^0=t,\hat{\phi}^1=r,\hat{\phi}^2=y,\hat{\phi}^3=z\}$. It could be useful to notice that

$$\det\hat{g}=\frac{t^2y^2}{\Delta^2}\alpha(t)\gamma(t).$$

It was proved in [12] that the dualised Eguchi–Hanson model is conformally flat. We have checked that, in the class studied here, this is the *only* case where the Weyl tensor vanishes.

5.1. The SO(3) dual of Schwarzschild

Among all the $SU(2)\times U(1)$ metrics, the Schwarzschild one has an interesting peculiarity as its dual can be obtained in two ways. Indeed, in the original metric (7), due to the split of ${\sigma_3}^2$, the $SU(2)$ isometries appear only in the $({\sigma_1}^2+{\sigma_2}^3)$ term. One can therefore first dualise the "sub-metric" corresponding to this last term and then add the dt^2 and $d\Psi^2$ terms in order to obtain the dualised Schwarzschild metric. Doing this, only two Lagrange multipliers λ^i will appear during the dualisation procedure [15]. But it is still possible to obtain it by first dualising the metric (6) and *then* taking the appropriate limit ($n\to 0$). As $\gamma(t)\to 0$, one has first to make the change of coordinates $dz=\frac{d\psi}{2n}$ before taking the limit. Doing this, one gets for $\hat{g}$:

$$\hat{g}=\frac{1}{1-\frac{2M}{t}-\frac{\lambda}{3}t^2}\,dt^2+\frac{r^2+t^4}{t^2y^2}\left(dr+\frac{ry}{r^2+t^4}dy\right)^2 + \frac{t^2}{r^2+t^4}\,dy^2+\left(1-\frac{2M}{t}-\frac{\lambda}{3}t^2\right)d\Psi^2.$$

Finally, by making the coordinate change $y = \sqrt{s^2 - r^2}$, we get

$$\hat{g} = \frac{1}{1 - \frac{2M}{t} - \frac{\lambda}{3}t^2} dt^2 + \left(1 - \frac{2M}{t} - \frac{\lambda}{3}t^2\right) d\Psi^2 + \frac{1}{t^2(s^2 - r^2)}\left(t^4\, dr^2 + s^2\, ds^2\right). \tag{13}$$

In the special case $\lambda = 0$, we recover the $SO(3)$ dual of Schwarzschild which was one of the first examples for non-Abelian duality [7]. While making $n \to 0$, the torsion potential 2-form H (11) writes as $d(\frac{\Psi dr}{2n}) + O(n)$, and therefore, as H is only defined up to a total derivative, the torsion vanishes, which is consistent with the result found in [7].

We will now address the question of the one loop renormalisability of the dual theory $\widehat{S}$.

6. One-loop renormalisation of the dual metric

We want to prove that the one-loop renormalisation property does survive to the dualisation process. In other words, if the torsionless action (1) is quasi-Einstein, then so is the action (12). In the presence of torsion, this now means that one can find some constant $\hat{\lambda}$ and some vectors $\hat{v}$ and $\hat{w}$ such that

$$\widehat{Ric}_{ij} = \hat{\lambda}\widehat{G}_{ij} + D_j\hat{v}_i + \partial_{[i}\hat{w}_{j]}. \tag{14}$$

This equality gives a system of equations much more complicated than (4), but what is important is that now $\alpha(t)$ and $\gamma(t)$ are not considered as unknown functions. Furthermore, as we suppose the original metric to be quasi-Einstein, the system (4) is assumed to be verified and one can easily derive from it, in an algebraic way, the three functions A, B and C such that

$$\begin{cases} \alpha'(t) = A\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big), \\ \gamma'(t) = B\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big), \\ \gamma''(t) = C\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big). \end{cases} \tag{15}$$

The procedure is the following: we choose some ansatz for $\hat{\lambda}$, $\hat{v}$ and $\hat{w}$ and express relation (14). Then, in this last expression, we replace each occurrence of $\alpha'(t)$, $\gamma'(t)$ and $\gamma''(t)$ by its expression in (15) and check if (14) holds.

We have checked that (14) is verified taking

$$\begin{cases} \hat{\lambda} = \lambda, \\ \hat{v}_i = -2\lambda\hat{g}_{ij}X^j + D_i \log\Delta + v_i, \\ \hat{w}_j = -2\lambda X^j \widehat{G}_{ji}, \end{cases} \tag{16}$$

where X is defined by $X = r\partial_r + y\partial_y$.

Conversely, let us now suppose that $\hat{\lambda}$, $\hat{v}$ and $\hat{w}$ are defined by (16) where λ and v are supposed to be arbitrary. It is possible to show that if (14) holds, then the original metric is quasi-Einstein with $Ric_{ij} = \lambda g_{ij} + D_{(i}v_{j)}$. In order to demonstrate this, we first define the three functions $f_A(t)$, $f_B(t)$ and $f_C(t)$ such that

$$\begin{cases} \alpha'(t) = A\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big) + f_A(t), \\ \gamma'(t) = B\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big) + f_B(t), \\ \gamma''(t) = C\big(t, \alpha(t), \gamma(t), v_0(t), v_0'(t)\big) + f_C(t). \end{cases} \tag{17}$$

Assuming that (14) holds, and after having replaced each occurrence of $\alpha'(t)$, $\gamma'(t)$ and $\gamma''(t)$ by its value in (17), we get some equation system where the unknowns are the functions $f_X(t)$. As this last system must hold irrespectively of the values taken by r and y which are free variables, one can then prove that $f_A(t) = f_B(t) = f_C(t) = 0$. This shows that (15) holds and therefore the quasi-Einstein property of the original metric.

We have proven, for arbitrary functions $\alpha(t)$ and $\gamma(t)$, the equivalence

$$Ric_{ij} = \lambda g_{ij} + D_{(i}v_{j)} \iff \widehat{\mathrm{Ric}}_{ij} = \hat{\lambda}\widehat{G}_{ij} + D_j\hat{v}_i + \partial_{[i}\hat{w}_{j]}, \tag{18}$$

where λ, $\hat{\lambda}$, v and $\hat{v}$ are related by (16).

6.1. Remarks

- The cosmological constant does not change through the dualisation process as it was already proved for T-dualised homogeneous metrics [14]. That means that the coupling will renormalise in exactly the same way that in the initial theory: the one-loop Callan–Symanzik β function is the same for the initial and dualised $SU(2) \times U(1)$ theories.
- As one could expect, the coordinate t which was a spectator coordinate during the dualisation process plays a special role: $\hat{w}_t = 0$ and, up to the $D_t \log\Delta$ term, $\hat{v}_t$ and v_t are equal.

- The $SU(2)$ symmetries where lost during the dualisation process, so at the end, there is just a $U(1)$ symmetry left and therefore the Killing ∂_z is unique. Indeed, $\hat{v}$ and $\hat{w}$ are defined up to this Killing vector, which dual 1-form is $K = \frac{y^2 t\gamma(t)}{\Delta}\,dz$. One then has $D_{(i}K_{j)} = 0$ and $D_{[j}K_{i]} + \partial_{[i}K^s\widehat{G}_{sj]} = 0$.
- One can address the question of the unicity of $\hat{\lambda}$, $\hat{v}$ and $\hat{w}$ which satisfy (14). There will be multiple solutions if one can find some Λ, V and W such that

  $$\Lambda\widehat{G}_{ij} + D_j V_i + \partial_{[i}W_{j]} = 0.$$

  On the one hand, $\hat{w}$ alone is obviously defined up to a gradient while $\hat{v}$ and $\hat{w}$ together are defined up to the Killing vector K; on the other hand, equivalence (18) shows that if multiple solutions exist for $\hat{\lambda}$ and $\hat{v}$ in the dualised metric, then such ambiguity will appear for the original metric. We have checked that, in our case of $SU(2) \times U(1)$ metrics, only flat metric leads to such possibilities.[4] Therefore, except for this trivial original metric and up to the already noticed freedom in $\hat{v}$ and $\hat{w}$, (16) is the unique solution of (14).
- The $SO(3)$ dual of the Schwarzschild metric (13) gives us a nice example of a torsionless quasi-Einstein metric with a $U(1)$ as minimal isometry.

7. Conservation of the Kähler property

Bakas and Sfetsos described, for SUSY applications, how the complex structures were changed when hyper-Kähler metrics were T-dualised [24]. We propose here to show that when one starts with the original metric (8), the dual partner is still Kähler.

If we define

$$\hat{\sigma}_i = -\widehat{G}_{si}\,d\hat{\phi}^s,$$

it is possible to write the dual metric of (8) under the specific shape

$$\hat{g} = \frac{1}{\gamma(t)}\,dt^2 + t\left(\hat{\sigma}_1^2 + \hat{\sigma}_2^2\right) + \gamma(t)\hat{\sigma}_3^2.$$

[4] For flat space ($\beta(t) = \gamma(t) = 1/\alpha(t) = t$), we have $\lambda g_{ij} + D_{(i}v_{j)} = 0$ with $v = -2\lambda\,dt, \forall\lambda \in \mathbb{R}$.

One can then check that the 2-form

$$\hat{\rho} = dt \wedge \hat{\sigma}_3 + t\hat{\sigma}_1 \wedge \hat{\sigma}_2 = \frac{1}{2}\hat{\mathcal{J}}_{ij}\,d\hat{\phi}^i \wedge d\hat{\phi}^j$$

is a Kähler form *with torsion* for the dual metric. Indeed, for the almost complex structure $\hat{\mathcal{J}}$, we have

$$\begin{cases} \hat{\mathcal{J}}_{is}\hat{\mathcal{J}}^{sj} = -\delta_i{}^j, \\ \hat{\mathcal{J}}_{(ij)} = 0, \\ D_i\hat{\mathcal{J}}_{jk} = 0, \end{cases}$$

where D is the covariant derivative with torsion. One should notice here that, in the presence of torsion, the closing condition on the Kähler form is replaced by

$$d\hat{\rho} = (\star dH) \wedge \hat{\rho}.$$

The torsion potential 2-form H is given by the Eq. (11).

8. Concluding remarks

We have considered all of the four-dimensional non-homogeneous metrics with an isometry group $SU(2) \times U(1)$. We have shown that the dual partners are quasi-Einstein (with torsion) iff the original metrics are quasi-Einstein (without torsion). Let us emphasize that this was possible despite the fact that the explicit form of these metrics are not all known yet.

In [17], it was proven that, in the minimal-dimensional scheme, the dualised $SU(2)$ principal σ-model is not two-loop renormalisable although this property holds for its original model. Here, the one-loop renormalisability remains although the starting models are not in general two-loop renormalisable. This is another suggestion that the renormalisability beyond one loop for the original and dualised models are not linked. Indeed, it is our ansatz that for the dualised models investigated here, one could still define a proper theory up to two loops. This could be achieved by adding some *finite* deformation to the dualised metric, as it was done in [18] for the $SU(2)$ principal σ-model, irrespectively of the two-loop renormalisability of the original theory.

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C.3 Dualised σ-models at the two-loop order

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Dualised σ-models at the two-loop order

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Abstract

We adress ourselves the question of the quantum equivalence of non-abelian dualised σ-models on the simple example of the T-dualised $SU(2)\,\sigma$-model. This theory is classically canonically equivalent to the standard chiral $SU(2)\,\sigma$-model. It is known that the equivalence also holds at the first order in perturbations with the same β functions. However, this model has been claimed to be non-renormalisable at the two-loop order. The aim of the present work is the proof that it is — at least up to this order — still possible to define a correct quantum theory. Its target space metric being only modified in a finite manner, all divergences are reabsorbed into coupling and fields (infinite) renormalisations.

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1. Introduction

The subject of classical versus quantum equivalence of T-dualised σ-models has been strongly studied in recent years, and extensive reviews covering abelian, non-abelian dualities and their applications to string theory and statistical physics are available [1–3]. More recent developments on the geometrical aspects of duality can be found in [4].

The interpretation of T-duality as a canonical transformation, for constant backgrounds, was first given by [5,6]. Its more general formulation [7] was applied to the non-abelian case in [8,9].

After the settling of the classical equivalence, the most interesting problem was its study at the quantum level. This was done mostly for dualisations of Lie groups, with emphasis put on $SU(2)$. For this model the one-loop equivalence was established in [10,11]. This one-loop quantum equivalence was recently settled for the general class of models built on $G_L \times G_R/G_D$, with an arbitrary breaking of G_R [12]. An interesting intermediary result

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is an expression for the Ricci tensor of the dualised geometry (with torsion) exhibiting its dependence with respect to the geometrical quantities of the original model. In the same work, the two-loop renormalisability problem was tackled and the need for extra (non-minimal) one-loop order finite counter-terms was emphasized. Some years ago, it was noted that in the minimal dimensional scheme, two-loop renormalisability does not hold for the $SU(2)$ T-dualised model [13].

The aim of the present work is a more precise analysis of this two-loop (in)equivalence for the non-abelian T-duality, still on the simple example of the original $SU(2)$ T-dualised model.

The main remark is that, part of the isometries being somehow lost, the T-dualised models are not — as they should be if one wants to give an all-order analysis — defined by a sufficient system of Ward identities. For example, in our simple case there is, *a priori* only a linear $SU(2)$ [or $O(3)$] invariance, and any $O(3)$ invariant action is allowed (let us remind the reader that in higher-loop corrections to a classical action, all the terms which are not prohibited by some reason such as power counting, isometries or conservation laws . . . , would appear). To our present knowledge, the extra constraints coming from the origin of the model (dualisation of an $(SU(2)_L \times SU(2)_R)/SU(2)_D$ chiral model) are not understood.[1] As it is highly probable that they are linked with the space–time dimension, it is not surprising that a minimal dimensional renormalisation scheme fails: as is well known, when the regularization method does not respect all the properties that define the theory, extra finite counter-terms are needed [15].

The content of this article is the following: in Section 2 we recall the expression of the classical action of the dualised theory and set the notations. In Section 3, we start from the corresponding *a priori* quantum bare action and obtain through $\hbar$ expansion the possible counter-terms that may be added to the classical action in order to reabsorb the divergences. Then in Section 4 we give the 2-loop divergences and in Section 5 we discuss how they match with the candidates in Section 3. Our result is that coupling constant and field renormalisations (infinite and finite ones) are not sufficient to ensure the two-loop existence of the T-dualised theory but the metric itself has to be deformed (in a *finite way*). Some concluding remarks are offered in Section 6.

2. The classical action

At the classical level and in light-cone co-ordinates, the dual action can be writen [10,12]:

$$S = \frac{1}{\lambda} \int G_{ij} \partial_+ \phi^i \partial_- \phi^j ,$$

[1] In [14] the quantisation of a U(1)-invariant non-linear σ model, the so-called Complex sine-Gordon model, was performed by imposing as extra constraints its classical property of factorisation and non-production; there it was shown that *definite extra finite* one-loop counter-terms are needed to enforce this property to one-loop order and then they also restore the two-loop renormalisability.

where $g_{ij} = G_{(ij)}$ is the target space metric and $h_{ij} = G_{[ij]}$ is the torsion potential. The torsion T_{ijk} is defined by $T_{ijk} = \frac{3}{2}\partial_{[i}h_{jk]}$. The connections with torsion Γ^i_{jk} and without torsion γ^i_{jk} respectively write:

$$\Gamma^i_{jk} = \frac{1}{2}g^{is}(\partial_j G_{ks} + \partial_k G_{sj} - \partial_s G_{kj}) = \gamma^i_{jk} + T^i_{jk},$$
$$\gamma^i_{jk} = \frac{1}{2}g^{is}(\partial_j g_{sk} + \partial_k g_{sj} - \partial_s g_{jk}),$$

and the corresponding covariant derivatives are:

$$D_i k_j = \partial_i k_j - \Gamma^s_{ij} k_s = \nabla_i k_j - T^s_{ij} v_s,$$
$$D_i k^j = \partial_i k^j + \Gamma^j_{is} k^s = \nabla_i k^j + T^j_{is} v^s.$$

The Riemann tensor without torsion will be noted $R_{ij,kl}$ whereas we will denote the one with torsion as $\bar{R}_{ij,kl}$.

The expression of the dualised target space metric G_{ij} as a function of the original one is well known and in [12] the various geometrical quantities (Ricci tensor, ...) were also related. In the special case considered here, where the original model is the $SU(2) \times SU(2)/SU(2)$ non-linear σ model, the metric writes:

$$G_{ij}[\vec{\phi}] = \frac{1}{1 + \vec{\phi}^2}\left[\delta_{ij} + \phi^i\phi^j + \epsilon_{ijk}\phi^k\right], \tag{1}$$

where $\vec{\phi}$ is a $SU(2)$ (real) vector representation and the ϕ^i, $i = 1, 2, 3$, are the co-ordinates on the dualised manifold. Then $\vec{\phi}^2$ is a $SO(3)$ invariant and the symmetry is linearly realised. Torsion breaks parity, but the model is invariant under the simultaneous change $\phi \to -\phi$ and $\epsilon_{ijk} \to -\epsilon_{ijk}$. Let us emphasize that no other local symmetry exists for that model.

3. The two-loop order bare action

In order to analyse the two-loop renormalisability of the dualised $SU(2)$ σ-model, we first examine all the possible ways to reabsorb the divergences through local counter-terms. As usual, we allow for finite and infinite renormalisations of both fields and coupling. But, as we shall see later on, this appears as insufficient to reabsorb the various divergences. Thus, we also allow for a finite deformation of the classical metric and torsion potential $g_{ij} + h_{ij} = G_{ij}$ to describe its quantum extension: of course, this *à la Friedan* [16] extension of the notion of renormalisability involves *a priori* an infinite number of new parameters. Let us emphasize that we shall consider *only finite deformations*.

Even if by doing so we obviously introduce too many parameters, we first let them all independent in order to show the announced need for such intrinsic metric deformation.

Let us first write the bare action:

$$S^o = \frac{1}{\lambda^o}\int G^o_{ij}\partial_+\phi^{oi}\partial_-\phi^{oj}, \tag{2}$$

where:

$$\begin{cases} \dfrac{1}{\lambda^o} = \dfrac{1}{\lambda}\left[1 + \dfrac{\hbar\lambda}{2\pi}\left(\dfrac{\Lambda_1}{\varepsilon} + b\right) + \left(\dfrac{\hbar\lambda}{2\pi}\right)^2\left(\dfrac{c}{\varepsilon^2} + \dfrac{\Lambda_2}{\varepsilon} + d\right) + \cdots\right], \\ \vec{\phi}^o = \vec{\phi} + \dfrac{\hbar\lambda}{2\pi}\left(\dfrac{\vec{v}_1(\vec{\phi})}{\varepsilon} + \vec{w}_1(\vec{\phi})\right) + \left(\dfrac{\hbar\lambda}{2\pi}\right)^2\left(\dfrac{\vec{v}_2(\vec{\phi})}{\varepsilon^2} + \dfrac{\vec{w}_2(\vec{\phi})}{\varepsilon} + \vec{x}(\vec{\phi})\right) + \cdots, \\ G^o_{ij} = G_{ij} + \dfrac{\hbar\lambda}{2\pi}\widetilde{G}_{ij} + \left(\frac{\hbar\lambda}{2\pi}\right)^2\widehat{G}_{ij} + \cdots. \end{cases} \tag{3}$$

To express (2) we shall need the Lie derivative $\underset{\vec{k}}{\mathcal{L}}$ and a "second order" Lie derivative $\underset{\vec{k}}{\mathcal{L}}^{(2)}$. Indeed, for any tensor S_{ij} defined on a manifold with co-ordinates ϕ^j, in a change of co-ordinates:

$$S^0_{ij}(\vec{\phi}^0)\partial_+\phi^{oi}\partial_-\phi^{oj} = S_{ij}(\vec{\phi})\partial_+\phi^i\partial_-\phi^j,$$

and if $\vec{\phi}^0 = \vec{\phi} + \eta\vec{k}$ (note that $\vec{k}$ is not a vector field on the manifold):

$$S_{ij}(\vec{\phi}) = S^0_{ij}(\vec{\phi}) - \eta\,\underset{\vec{k}}{\mathcal{L}}\big(S^0_{ij}(\vec{\phi})\big) + \frac{1}{2}\eta^2\,\underset{\vec{k}}{\mathcal{L}}^{(2)}\big(S^0_{ij}(\vec{\phi})\big) + \mathcal{O}\big(\eta^3\big). \tag{4}$$

We remind the reader that

$$\underset{\vec{k}}{\mathcal{L}}(S_{ij}) = k^s\nabla_s S_{ij} + S_{sj}\nabla_i k^s + S_{is}\nabla_j k^s. \tag{5}$$

One can show that

$$\underset{\vec{k}}{\mathcal{L}}^{(2)}(S_{ij}) = \underset{\vec{k}}{\mathcal{L}}\big(\underset{\vec{k}}{\mathcal{L}}(S_{ij})\big) - \underset{k^s\partial_s\vec{k}}{\mathcal{L}}(S_{ij}). \tag{6}$$

With $\nabla_i g_{jk} = 0$, we rewrite Eqs. (5), (6) for $S_{ij} \equiv G_{ij}$ as:

$$\begin{cases} \underset{\vec{k}}{\mathcal{L}}(G_{ij}) = 2D_j k_i + \partial_{[i}\zeta_{j]}, \qquad \zeta_i = 2k^l G_{li}, \\ \underset{\vec{k}}{\mathcal{L}}^{(2)}(G_{ij}) = 2k^s k^u \overline{R}_{si,ju} + 2D_i k^s D_j k_s - 4T_{ius}k^u D_j k^s \\ \qquad\qquad + \underset{(k^s k^u \bar{\gamma}_{su})}{\mathcal{L}}(G_{ij}) + \partial_{[i}\hat{\zeta}_{j]}. \end{cases} \tag{7}$$

$\hat{\zeta}_i$ is some quantity whose computation is useless as, in the same manner as ζ_i, it gives a vanishing contribution to the action or, the torsion potential being always defined up to a gauge transformation, such term can always be put into h_{ij} (moreover, in our particular situation, the $O(3)$ symmetry implies that such $\partial_{[i}\zeta_{j]}$ terms vanish). Then, we shall not write them anymore.

Then, expending (2) with the help of (3), (4), one gets the possible counter-terms at lowest orders:

- 0 order in $\dfrac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}G_{ij}\partial_+\phi^i\partial_-\phi^j;$$

- first order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}\left[\left(\frac{\Lambda_1}{\varepsilon}+b\right)G_{ij}+\underset{\frac{\vec{v}_1}{\varepsilon}+\vec{w}_1}{\mathcal{L}}(G_{ij})+\widetilde{G}_{ij}\right]\partial_+\phi^i\partial_-\phi^j; \tag{8}$$

- at second order in $\frac{\hbar\lambda}{2\pi}$:

$$\frac{1}{\lambda}\left[\frac{1}{\varepsilon^2}(\cdots)+\left(\frac{\Lambda_1}{\varepsilon}\widetilde{G}_{ij}+\underset{\frac{\vec{v}_1}{\varepsilon}}{\mathcal{L}}(\widetilde{G}_{ij})+\frac{\Lambda_1}{\varepsilon}\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+b\underset{\frac{\vec{v}_1}{\varepsilon}}{\mathcal{L}}(G_{ij})\right.\right.$$
$$\left.\left.+\frac{\Lambda_2}{\varepsilon}G_{ij}+\underset{\frac{\vec{w}_2}{\varepsilon}}{\mathcal{L}}(G_{ij})+\frac{1}{2}\underset{\frac{\vec{v}_1}{\varepsilon}+\vec{w}_1}{\mathcal{L}^{(2)}}(G_{ij})\Big|_{\frac{1}{\varepsilon}}\right)+(\cdots)\right]\partial_+\phi^i\partial_-\phi^j, \tag{9}$$

where $Q|_{\frac{1}{\varepsilon}}$ means that we only take the term in $\frac{1}{\varepsilon}$ in the expression Q.

As we don't consider the 3-loop order, in expression (9) we only need the coefficient of $\frac{1}{\varepsilon}$ (the double poles $\frac{\hbar^2}{\varepsilon^2}$ are not new quantities as they are directly related to first order simple poles and it has already been proved that the dualised $SU(2)$ σ-model is one-loop renormalisable [10]).

Using the following identity between Lie derivatives:

$$\underset{\vec{X}}{\mathcal{L}}\underset{\vec{Y}}{\mathcal{L}}-\underset{\vec{Y}}{\mathcal{L}}\underset{\vec{X}}{\mathcal{L}}=\underset{\vec{Z}}{\mathcal{L}} \quad \text{with} \quad Z^i=X^j\partial_jY^i-Y^j\partial_jX^i,$$

the term with the "second order" Lie derivative may be re-expressed:

$$\varepsilon\left[\underset{\frac{\vec{v}_1}{\varepsilon}+\vec{w}_1}{\mathcal{L}^{(2)}}(G_{ij})\Big|_{\frac{1}{\varepsilon}}\right]=\underset{\vec{v}_1}{\mathcal{L}}\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+\underset{\vec{w}_1}{\mathcal{L}}\underset{\vec{v}_1}{\mathcal{L}}(G_{ij})-\underset{(v_1^k\partial_k\vec{w}_1+w_1^k\partial_k\vec{v}_1)}{\mathcal{L}}(G_{ij})$$
$$=2\left[\underset{\vec{v}_1}{\mathcal{L}}\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})-\underset{v_1^k\partial_k\vec{w}_1}{\mathcal{L}}(G_{ij})\right].$$

So, the $\mathcal{O}(\hbar)$ term (8) may be rewritten as:

$$\frac{1}{\lambda}\left[\frac{1}{\varepsilon}\big(\Lambda_1G_{ij}+\underset{\vec{v}_1}{\mathcal{L}}(G_{ij})\big)+\big(\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+bG_{ij}+\widetilde{G}_{ij}\big)\right]\partial_+\phi^i\partial_-\phi^j,$$

and the $\mathcal{O}(\hbar)^2$ term (9) as:

$$\frac{1}{\lambda\varepsilon}\big[\Lambda_1\big(\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+bG_{ij}+\widetilde{G}_{ij}\big)+(\Lambda_2-b\Lambda_1)G_{ij}$$
$$+\underset{\vec{v}_1}{\mathcal{L}}\big(\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+bG_{ij}+\widetilde{G}_{ij}\big)+\underset{(\vec{w}_2-v_1^k\partial_k\vec{w}_1)}{\mathcal{L}}(G_{ij})\big].$$

As a consequence, as expected, any term $\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+bG_{ij}$ may be reabsorbed into the finite deformation $\widetilde{G}_{ij}$ (and *vice-versa*) to the expense of a change in the $\mathcal{O}(\hbar)^2$ parameters:

$$\widetilde{G}_{ij}+\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})+bG_{ij}\to\bar{G}_{ij}\Rightarrow\big\{\Lambda_2\to\Lambda_2-b\Lambda_1,\ \vec{w}_2\to\vec{w}_2-v_1^k\partial_k\vec{w}_1\big\}. \tag{10}$$

Finally, for the term in $\frac{1}{\varepsilon}(\frac{\hbar\lambda}{2\pi})^2$ in the bare action, one has the following expression:

$$\frac{1}{\lambda}\big(\Lambda_1\widetilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\widetilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{W}_2}{\mathcal{L}}(G_{ij}) + \mathcal{H}_{ij}(v_1, w_1)\big), \tag{11}$$

where

$$\begin{cases} \vec{W}_2 = \vec{w}_2 + b\,\vec{v}_1 + \Lambda_1\vec{w}_1 + v_1^s w_1^u \vec{\gamma}_{su}, \\ \mathcal{H}_{ij}(v_1, w_1) = v_1^s w_1^u \overline{R}_{is,uj} + D_i v_1^s D_j w_{1s} - 2T_{ius} v_1^u D_j w_1^s + (\vec{v}_1 \leftrightarrow \vec{w}_1). \end{cases} \tag{12}$$

4. The two-loop order divergences

We use the expression of the covariant divergences given by Hull and Townsend [17],[2] in the background field method and in the *minimal* dimensional scheme, up to the two-loop order:

$$\begin{cases} Div^1_{ij} = -\dfrac{\hbar}{2\pi\varepsilon} Ric_{ij}, \\ Div^2_{ij} = -\frac{\hbar^2\lambda}{8\pi^2\varepsilon}\big(\overline{R}_i^{\,klm}\big(\overline{R}_{klmj} - \frac{1}{2}\overline{R}_{lmkj}\big) + 2T^k_{mn}T^{lmn}\overline{R}_{kijl}\big). \end{cases} \tag{13}$$

In order to ensure the renormalizability of the theory, these divergences should match with the candidate counter-terms given by (8) and (11):

$$\begin{cases} CT^1_{ij} = \dfrac{\hbar}{2\pi\varepsilon}\big(\Lambda_1 G_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(G_{ij})\big), \\ CT^2_{ij} = \dfrac{\hbar^2\lambda}{4\pi^2\varepsilon}\big(\Lambda_1\widetilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\widetilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{W}_2}{\mathcal{L}}(G_{ij}) + \mathcal{H}_{ij}(v_1, w_1)\big). \end{cases} \tag{14}$$

It has been previously proven [12] that the dualised metric is quasi-Einstein as soon as the original metric is Einstein. In our special case, we get:

$$Ric_{ij} = \Lambda G_{ij} + 2D_j v_i, \quad \Lambda = \Lambda_1 = \frac{1}{2}, \quad \vec{v} = \vec{v}_1 = \frac{1}{2}\left(\frac{1-\phi^2}{1+\phi^2}\right)\vec{\phi}. \tag{15}$$

The addition to the effective action of a $\hbar$ finite deformation of the metric and of some finite renormalisations for the coupling and fields (non-minimal scheme) modifies the $\hbar^2$ divergences. The additional term is easily obtained as

$$\begin{aligned} &-\frac{\hbar}{2\pi\varepsilon}\left\{Ric_{ij}\left(G_{kl} + \frac{\hbar\lambda}{2\pi}\big(\underset{\vec{w}_1}{\mathcal{L}}(G_{kl}) + bG_{kl} + \widetilde{G}_{kl}\big)\right) - Ric_{ij}(G_{kl})\right\} \\ &\quad\equiv -\frac{\hbar^2\lambda}{4\pi^2\varepsilon}\Delta_{ij} + \mathcal{O}\big(\hbar^3\big). \end{aligned}$$

Here also, only the combination $\widetilde{G}_{ij} + \underset{\vec{w}_1}{\mathcal{L}}(G_{ij}) + bG_{ij}$ appears. Then, we could decide to reabsorb bG_{ij} and $\underset{\vec{w}_1}{\mathcal{L}}(G_{ij})$ into $\widetilde{G}_{ij}$, but, as announced at the beginning of Section 3, in order to see if they would be sufficient by themselves, we keep them apart in a first step.

[2] We checked for our example that the two other calculations in [18,19] give the same result.

Finally, the dualised $SU2$ σ-model will be renormalisable at two loops if and only if we can find $\{\widetilde{G}_{ij}[\vec{\phi}],\ b,\ \vec{w}_1[\vec{\phi}];\ \Lambda_2,\ \vec{W}_2[\vec{\phi}]\}$ such that:

$$Div^2_{ij} - \frac{\hbar^2\lambda}{4\pi^2\varepsilon}\Delta_{ij} + \frac{\hbar^2\lambda}{4\pi^2\varepsilon}\left(\Lambda_1\widetilde{G}_{ij} + \underset{\vec{v}_1}{\mathcal{L}}(\widetilde{G}_{ij}) + \Lambda_2 G_{ij} + \underset{\vec{W}_2}{\mathcal{L}}(G_{ij}) + \mathcal{H}_{ij}(v_1, w_1)\right) = 0. \tag{16}$$

5. Results

According to the linearly realised symmetry of the T-dualised $SU(2)$ σ-model, the finite deformation of the metric $\widetilde{G}_{ij}$ and the vectors $\vec{w}_1(\phi)$ and $\vec{W}_2(\phi)$ respectively write:

$$\widetilde{G}_{ij} = \alpha(\tau)\delta_{ij} + \beta(\tau)\phi^i\phi^j + \epsilon_{ijk}\gamma(\tau)\phi^k, \qquad \vec{w}_1 = w_1(\tau)\vec{\phi}, \qquad \vec{W}_2 = W_2(\tau)\vec{\phi},$$

where $\tau = \vec{\phi}^2$. Moreover, the symmetry also implies that terms of the form $\partial_{[i}k_{j]}$ or of the form $k^s K^u \gamma^t_{su}$ are equal to zero. It is then possible to re-express (16) as a set of three linear differential equations :

$$\begin{aligned} &W_2(\tau) + \frac{(1+\tau)\Lambda_2}{2} + \frac{45+68\tau-18\tau^2-12\tau^3-3\tau^4}{16(1+\tau)^3} - \frac{1-\tau}{(1+\tau)^2}w_1(\tau) \\ &\quad= \frac{3+10\tau+5\tau^2+2\tau^3}{4(1+\tau)}\alpha(\tau) + \frac{4+5\tau+6\tau^2+\tau^3}{4(1+\tau)}\beta(\tau) \\ &\qquad - \frac{3(1+\tau)(3+\tau)}{2}\gamma(\tau) - \frac{4+11\tau+5\tau^2-\tau^3}{2}\alpha'(\tau) + \frac{\tau}{2}\beta'(\tau) \\ &\qquad - \tau(1+\tau)(3+\tau)\gamma'(\tau) - \tau(1+\tau)^2\alpha''(\tau), \end{aligned} \tag{17}$$

$$\begin{aligned} &3\Lambda_2 - \frac{3(-5+60\tau+10\tau^2+12\tau^3+3\tau^4)}{8(1+\tau)^4} \\ &\quad= \frac{(7+10\tau)}{2}\alpha(\tau) + \frac{(12+5\tau)}{2}\beta(\tau) - 3(11+5\tau)\gamma(\tau) \\ &\qquad + (-17-22\tau+9\tau^2)\alpha'(\tau) + (5+4\tau+\tau^2)\beta'(\tau) - 2(5+2\tau)(3+5\tau)\gamma'(\tau) \\ &\qquad + 2(-5-19\tau-12\tau^2+\tau^3)\alpha''(\tau) + 2\tau\beta''(\tau) - 4\tau(1+\tau)(3+\tau)\gamma''(\tau) \\ &\qquad - 4\tau(1+\tau)^2\alpha^{(3)}(\tau), \end{aligned} \tag{18}$$

$$\begin{aligned} &\Lambda_2 + \frac{3(1-\tau)(13+6\tau+\tau^2)}{8(1+\tau)^3} \\ &\quad= \frac{(5+2\tau)}{2}\alpha(\tau) + \frac{(6+\tau)}{2}\beta(\tau) - (17+3\tau)\gamma(\tau) \\ &\qquad + (-7+\tau)(3+\tau)\alpha'(\tau) - 2(-5+6\tau+\tau^2)\gamma'(\tau) \\ &\qquad - 2\tau(3+\tau)\alpha''(\tau) + 4\tau\gamma''(\tau). \end{aligned} \tag{19}$$

The need for a true deformation $\widetilde{G}_{ij}$ immediately appears: setting both $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ to zero, Eqs. (18) and (19) cannot be satisfied, even if [3] we allowed for some finite renormalisations of the coupling (b) and field ($\vec{w}_1(\vec{\phi})$), both hidden into the vector $\vec{W}_2(\vec{\phi})$ (see Eq. (12)). Then, as first proven in [13], we have checked that:

In a purely dimensional scheme (even with non minimal subtractions), the dualised $SU(2)$ σ model is not renormalisable at the two-loop order.

So, from the discussion in the previous sections, and without restricting the generality of our analysis, one can take b and $\vec{w}_1(\vec{\phi})$ as vanishing quantities.

Remarks.

- As Λ_2 is not a function, but a constant, differentiating Eqs. (18) and (19) will relate $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$. Then, as soon as $\widetilde{G}_{ij}$, the finite one loop renormalisation, has been definitely set, Eq. (17) will give the infinite two-loop renormalisations $\vec{W}_2(\vec{\phi})$ and Λ_2.
- From the previous discussions, we know that $\widetilde{G}_{ij}$ will be fixed up to some $\check{b}\, G_{ij} + \mathcal{L}_{\vec{\check{W}}(\vec{\phi})}(G_{ij})$; it is then natural to use this freedom, for example to reabsorb $\alpha(\tau)$, and to redefine $\widetilde{G}_{ij}$ such that:

$$\widetilde{G}_{ij} = \check{b}\, G_{ij} + \mathcal{L}_{\vec{\check{W}}}(\vec{\phi})(G_{ij}) + \check{G}_{ij}, \qquad \vec{\check{W}}(\vec{\phi}) = \check{W}(\tau)\vec{\phi}$$

with

$$\check{W}(\tau) = \frac{(1+\tau)^2}{2}\alpha(\tau) - \frac{\check{b}(1+\tau)}{2} \Rightarrow \check{G}_{ij} = \check{\beta}(\tau)\phi^i\phi^j + \epsilon_{ijk}\check{\gamma}(\tau)\phi^k$$

with

$$\check{\beta} = \beta - \frac{\check{b}}{1+\tau} - \frac{2(2+\tau)\check{W}}{(1+\tau)^2} - 4\check{W}' \quad \text{and} \quad \check{\gamma} = \gamma - \frac{\check{b}}{1+\tau} - \frac{(3+\tau)\check{W}}{(1+\tau)^2}. \tag{20}$$

We know that, when expressed as functions of $\check{\beta}(\tau)$ and $\check{\gamma}(\tau)$, Eqs. (17), (18), (19) remain unchanged, up to the substitutions discussed in Section 3 (Eq. (10)):

$$\begin{aligned} \Lambda_2 \to \check{\Lambda}_2 &= \Lambda_2 + \frac{\check{b}}{2}, \\ W_2(\vec{\phi}) \to \check{W}_2(\tau) &= W_2(\tau) + \check{b}\frac{1-\tau}{2(1+\tau)} + \frac{1}{2}\check{W}(\tau) \\ &\quad + \frac{1-\tau}{2(1+\tau)}\big[\check{W}(\tau) + 2\tau\check{W}'(\tau)\big]. \end{aligned} \tag{21}$$

[3] One notices also that the parameters b and $\vec{w}_1$ do not appear in (18) and (19). So, the *existence* of some solution to this set of differential equations is independent of the finite renormalisations of both coupling and fields, as is usual in perturbation theory. This freedom corresponds to a change of renormalisation scheme. This absence is only true if we take the very vector $\vec{v}_1$ (15) that reabsorbs the divergences at the one-loop order: otherwise, $\vec{w}_1(\vec{\phi})$ would appear in (18) and (19). This is a check of a correct renormalisation at the one-loop order.

Eqs. (18) and (19) give $\check{\beta}(\tau)$ as a function of $\check{\gamma}(\tau)$ which itself satisfies a non-homogeneous linear fourth order differential equation:

$$\check{\beta}(\tau)-\frac{\check{b}+2\Lambda_2}{6+\tau}=\frac{3(1-\tau)(13+6\tau+\tau^2)}{4(6+\tau)(1+\tau)^3}+\frac{2(17+3\tau)}{6+\tau}\check{\gamma}(\tau)$$
$$-\frac{4(5-6\tau-\tau^2)}{6+\tau}\check{\gamma}'(\tau)-\frac{8\tau}{6+\tau}\check{\gamma}''(\tau), \tag{22a}$$

$$\check{\gamma}^{(4)}(\tau)+\frac{(6-\tau)(7+\tau)}{\tau(6+\tau)}\check{\gamma}^{(3)}(\tau)+\frac{1260-276\tau-91\tau^2+3\tau^3+\tau^4}{4\tau^2(6+\tau)^2}\check{\gamma}''(\tau)$$
$$+\frac{-120+254\tau+57\tau^2+3\tau^3}{8\tau^2(6+\tau)^2}\check{\gamma}'(\tau)-\frac{138+25\tau+\tau^2}{8\tau^2(6+\tau)^2}\left[\check{\gamma}(\tau)-\frac{\check{b}+2\Lambda_2}{2}\right]$$
$$=-\frac{3(6402-8681\tau-5856\tau^2-22\tau^3+390\tau^4+39\tau^5)}{64\tau^2(1+\tau)^5(6+\tau)^2}. \tag{22b}$$

Note that under the change

$$\Gamma(\tau)=\left[\check{\gamma}(\tau)-\frac{\check{b}+2\Lambda_2}{2}\right],\qquad B(\tau)=\left[\check{\beta}(\tau)-3(\check{b}+2\Lambda_2)\right], \tag{23}$$

the parameter $\check{b}$ and the constant Λ_2 disappear from the set (22). Then, Λ_2 being an unknown constant, the general solution of the differential equation (22b) will be

$$\check{\gamma}(\tau)=\Gamma(\tau)+c,\quad \text{where } c \text{ is an arbitrary constant,}$$

and the two-loop coupling constant renormalisation Λ_2 will be:

$$\Lambda_2=c-\frac{\check{b}}{2}.$$

The model will be renormalisable up to two loops iff equation (22b), where $\check{\gamma}(\tau)$ has been replaced by $\Gamma(\tau)$ according to (23), has a solution which is analytic near $\tau=0$. In order to reach such a conclusion, we use the method of Frobenius for linear differential equations [20]. $\tau=0$ is a regular singularity (notice that we are only interested in $\tau\geqslant 0$). The indicial equation of the linear differential equation (22b) around the singular point 0 has four different solutions: $\nu=-\frac{3}{2},\ -\frac{1}{2},\ 0,\ 1$. For each one, we can find convergent series $\tau^\nu\sum_{n=0}^{\infty}c_n\tau^n$ that are independent solutions of the homogeneous equation associated to (22b). We give here the first terms of such series (it happens that for $\nu=-\frac{3}{2}$ we have an exact solution):

$$\check{\gamma}_{-\frac{3}{2}}(\tau)=\frac{1}{\tau^{\frac{3}{2}}}+\frac{1}{20\sqrt{\tau}}-\frac{\sqrt{\tau}}{20},\qquad \check{\gamma}_{-\frac{1}{2}}(\tau)=\frac{1}{\sqrt{\tau}}\left(1-\frac{11}{6}\tau+\frac{35}{108}\tau^2+\cdots\right),$$
$$\check{\gamma}_0(\tau)=1+\frac{23}{840}\tau^2+\cdots,\qquad \check{\gamma}_1(\tau)=\tau\left(1+\frac{1}{42}\tau-\frac{1}{324}\tau^2+\cdots\right). \tag{24}$$

Then, we use the method of variation of parameters to find $\lambda_{-\frac{3}{2}}(\tau)$, $\lambda_{-\frac{1}{2}}(\tau)$, $\lambda_0(\tau)$ and $\lambda_1(\tau)$ such that

$$\Gamma(\tau)=\lambda_{-\frac{3}{2}}(\tau)\check{\gamma}_{-\frac{3}{2}}(\tau)+\lambda_{-\frac{1}{2}}(\tau)\check{\gamma}_{-\frac{1}{2}}(\tau)+\lambda_0(\tau)\check{\gamma}_0(\tau)+\lambda_1(\tau)\check{\gamma}_1(\tau)$$

is the general solution of the inhomogeneous equation (22b) where $\check{\gamma}(\tau)$ has been replaced by $\Gamma(\tau)$ according to (23).

The first terms in the expansion of these functions are:

$$\begin{aligned}
\lambda_{-\frac{3}{2}}(\tau) &= \lambda^o_{-\frac{3}{2}} + \tau^{\frac{7}{2}}\left(\frac{1067}{1680} - \frac{13691}{3780}\tau + \cdots\right),\\
\lambda_{-\frac{1}{2}}(\tau) &= \lambda^o_{-\frac{1}{2}} + \tau^{\frac{5}{2}}\left(-\frac{1067}{240} + \frac{2543509}{100800}\tau + \cdots\right),\\
\lambda_0(\tau) &= \lambda^o_0 + \frac{1067}{192}\tau^2 - \frac{9805}{288}\tau^3 + \cdots,\\
\lambda_1(\tau) &= \lambda^o_1 - \frac{1067}{480}\tau + \frac{27887}{5760}\tau^2 + \cdots.
\end{aligned} \tag{25}$$

The analyticity requirement near $\tau = 0$ enforces the choice $\lambda^o_{-\frac{3}{2}} = \lambda^o_{-\frac{1}{2}} = 0$; $\check{\gamma}(\tau)$ is then expressed as a convergent series in τ, and the same will be true for $\check{\beta}(\tau)$. The final expression for the deformation $\widetilde{G}_{ij}$ depends on 3 constants [$c,\ \lambda^o_0$ and λ^o_1] and an arbitrary function [$\check{W}(\tau)$] and is given by the three functions:

$$\begin{aligned}
\alpha(\tau) &= \frac{\check{b}}{1+\tau} + \frac{2\check{W}}{(1+\tau)^2},\\
\beta(\tau) &= 6c + \frac{\check{b}}{1+\tau} + \frac{2(2+\tau)\check{W}}{(1+\tau)^2} + 4\check{W}'\\
&\quad + \frac{3(1-\tau)(13+6\tau+\tau^2)}{4(6+\tau)(1+\tau)^3} + \frac{2(17+3\tau)}{6+\tau}\Gamma(\tau) - \frac{4(5-6\tau-\tau^2)}{6+\tau}\Gamma'(\tau)\\
&\quad - \frac{8\tau}{6+\tau}\Gamma''(\tau),\\
\gamma(\tau) &= c + \frac{\check{b}}{1+\tau} + \frac{(3+\tau)\check{W}}{(1+\tau)^2} + \Gamma(\tau).
\end{aligned} \tag{26}$$

We now use the up to now free parameter $\check{b}$ to reabsorb the parameter c. Let us define

$$\bar{b} = \check{b} - 2c, \qquad \overline{W}(\tau) = \check{W}(\tau) + c(1+\tau),$$

we get

$$\widetilde{G}_{ij} = \overline{G}_{ij} + \bar{b}G_{ij} + \mathcal{L}_{\vec{\overline{W}}}\, G_{ij}$$

with $\vec{\overline{W}} = \overline{W}(\tau)\vec{\phi}$ and

$$\overline{G}_{ij} = \widetilde{G}_{ij}\big|_{\text{Eq. (26) for } c=\check{b}=\check{W}(\tau)\equiv 0}.$$

The dualised $SU(2)\,\sigma$-model is therefore renormalisable at the two-loop order if and only if we add a finite $\hbar$ deformation of the classical metric, depending on two new parameters λ^o_0 and λ^o_1.

6. Concluding remarks

We have been able to exhibit some set of counter-terms that ensures the two-loop renormalisability of the T-dualised chiral non-linear σ model. The one-loop effective metric is defined *up to two constants* (λ_0^o and λ_1^o), and some finite arbitrary field and coupling renormalisations. As is well known (e.g., in [21]), the two-loop Callan–Symanzik β function (related to Λ_2) [4] depends on these finite counterterms.

We emphasize that, contrarily to D. Friedan's approach to σ models quantisation, where the classical metric receives *infinite* perturbative deformations, our candidate for the deformation of the classical metric is a *finite one*, depending on *only two parameters* (plus the usual infinite, and finite, renormalisations of the fields and of the coupling constant): our ansatz is that a proper understanding of the dualisation process will precisely offer the extra constraints that uniquely define the quantum extension of the classical theory, order by order in perturbation theory, in the same spirit as Ward identities determine what otherwise would appear as new parameters (see also footnote 1).

Note added in proof

For completeness, let us mention that for abelian T-duality similar works were achieved in Refs. [22,23].

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$$\Lambda_2 = \frac{\bar{b}}{2}, \qquad \vec{W}_2 \text{ obtained through (21).}$$

Notice that the normalisation condition $\bar{b} = 0$ (no $\hbar$ extra finite coupling constant renormalisation) enforces $\Lambda_2 = 0$.

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C.4 Quaternionic extension of the double Taub-NUT metric

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Quaternionic extension of the double Taub-NUT metric

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Abstract

Starting from the generic harmonic superspace action of the quaternion-Kähler sigma models and using the quotient approach we present, in an explicit form, a quaternion-Kähler extension of the double Taub-NUT metric. It possesses $U(1) \times U(1)$ isometry and supplies a new example of non-homogeneous Einstein metric with self-dual Weyl tensor. © 2001 Published by Elsevier Science B.V.

1. Introduction

In view of the distinguished role of hyper-Kähler (HK) and quaternion-Kähler (QK) manifolds in string theory (see, e.g., [1–3]), it is important to know the explicit form of the corresponding metrics. One of the approaches to this problem proceeds from the generic actions of bosonic nonlinear sigma models with the HK or QK targets.

A generic action for the bosonic QK sigma models was constructed in [4], based upon the well-known one-to-one correspondence [5] between the QK manifolds and local $N = 2$, $d = 4$ supersymmetry. This relationship was made manifest in [6,7], where the most general off-shell action for the hypermultiplet $N = 2$ sigma models coupled to $N = 2$ supergravity was constructed in the framework of $N = 2$ harmonic superspace (HSS) [8]. The generic QK sigma model bosonic action was derived in [4] by discarding the fermionic fields and part of the bosonic ones in the general HSS sigma model action. The action of physical bosons parametrizing the target QK manifold arises, like in the HK case [9], after elimination of infinite sets of auxiliary fields present in the off-shell hypermultiplet superfields. This amounts to solving some differential equations on the internal sphere S^2 of the $SU(2)$ harmonic variables. It is a difficult problem in general to solve such equations. As was shown in [4], in the case of metrics with isometries the computations can be greatly simplified by using the HSS version of the QK quotient construction [10,11]. An attractive feature of the HSS quotient is that the isometries of the corresponding metric come out as manifest internal symmetries of the HSS sigma model action.

In [4], using these techniques, we explicitly constructed QK extensions of the Taub-NUT and Eguchi–Hanson (EH) HK metrics [12]. In this note we apply the HSS quotient approach to construct a QK extension of the

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4-dimensional "double Taub-NUT" HK metric. The latter was derived from the HSS approach in [13] by directly solving the corresponding harmonic differential equations. It turns out that the HSS quotient allows one to reproduce the same answer much easier, and it remarkably works in the QK case as well. We gauge two commuting $U(1)$ symmetries of the system of three "free" hypermultiplets and, after solving two algebraic constraints and fully fixing gauges, are left with a 4-dimensional QK metric having two $U(1)$ isometries and going onto the double Taub-NUT in the HK limit. It is a new explicit example of non-homogeneous QK metrics. Based on the results of Przanowski [14], Tod [15] and Flaherty [16], this metric gives also a new explicit solution of the coupled Einstein–Maxwell system with self-dual Weyl tensor.

2. The gauged HSS action of the QK double Taub-NUT

Details of the general construction can be found in [4]. Here we apply the HSS quotient approach to explicitly construct a sigma model giving rise to a QK generalization of the "double Taub-NUT" HK metric. The latter belongs to the class of two-center ALF metrics with the $U(1) \times U(1)$ isometry (one $U(1)$ is triholomorphic) and was treated in the HSS approach in [13].

We start with the action of three hypermultiplet superfields,

$$Q^{+a}_A(\zeta), \qquad g^{+r}(\zeta), \qquad a = 1, 2, \quad r = 1, 2, \quad A = 1, 2, \tag{1}$$

possessing no any self-interaction. So, by reasoning of [4,17], this action corresponds to the "flat" QK manifold $\mathbb{H}H^3 \sim Sp(1,3)/Sp(1) \times Sp(3)$. In (1), the indices a and r are the doublet indices of two Pauli–Gürsey-type $SU(2)$ s realized on Q^{+a}_A and g^{+r}, the index A is an extra $SO(2)$ index. These superfields are given on the harmonic analytic $N = 2$ superspace

$$(\zeta) = \left(x^m, \theta^{+\mu}, \bar\theta^{+\dot\mu}, u^{+i}, u^{-k}\right), \tag{2}$$

the coordinates $u^{+i}, u^{-k},\ u^{+i}u^-_i = 1,\ i,k = 1,2$, being the $SU(2)/U(1)$ harmonic variables, and they satisfy the pseudo-reality conditions

$$\text{(a)} \quad Q^{+a}_A \equiv \widetilde{\left(Q^+_{aA}\right)} = \epsilon^{ab} Q^+_{bA}, \qquad \text{(b)} \quad g^{+r} \equiv \widetilde{\left(g^+_r\right)} = \epsilon^{rs} g^+_s, \tag{3}$$

where $\epsilon^{ab}\epsilon_{bc} = \delta^a_c,\ \epsilon^{12} = -1$. The generalized conjugation $\sim$ is the product of the ordinary complex conjugation and a Weyl reflection of the sphere $S^2 \sim SU(2)/U(1)$ parametrized by $u^{\pm i}$. In the QK sigma model action below we shall need only the bosonic components in the θ-expansion of the above superfields:

$$\begin{aligned}
Q^{+a}_A(\zeta) &= F^{+a}_A(x,u) + i\left(\theta^+\sigma^m\bar\theta^+\right)B^{-a}_{mA}(x,u) + \left(\theta^+\right)^2\left(\bar\theta^+\right)^2 G^{(-3a)}_A(x,u), \\
g^{+r}(\zeta) &= g^{+r}_0(x,u) + i\left(\theta^+\sigma^m\bar\theta^+\right)g^{-r}_m(x,u) + \left(\theta^+\right)^2\left(\bar\theta^+\right)^2 g^{(-3r)}(x,u)
\end{aligned} \tag{4}$$

(possible terms $\sim (\theta^+)^2$ and $\sim (\bar\theta^+)^2$ can be shown not to contribute to the final action). The component fields still have a general harmonic expansion off shell. The physical bosonic components $F^{ai}_A(x), g^{ri}(x)$ are defined as the first components in the harmonic expansions of $F^{+a}_A(x,u)$ and $g^{+r}_0(x,u)$

$$\begin{aligned}
&F^{+a}_A(x,u) = F^{ai}_A(x)u^+_i + \cdots, \qquad g^{+r}_0(x,u) = g^{ri}(x)u^+_i + \cdots, \\
&\overline{(F^{ai}_A(x))} = \epsilon_{ab}\epsilon_{ik}F^{bk}_A(x), \qquad \overline{(g^{ri}(x))} = \epsilon_{rs}\epsilon_{ik}g^{sk}(x).
\end{aligned} \tag{5}$$

The selection of two commuting $U(1)$ symmetries to be gauged and the form of the final gauge-invariant HSS action are uniquely determined by the natural requirement that the resulting action have two different limits corresponding to the earlier considered HSS quotient actions of the QK extensions of Taub-NUT and EH metrics [4]. The full action S_{dTN} has the following form

$$S_{\text{dTN}} = \frac{1}{2}\int d\zeta^{(-4)}\mathcal{L}^{+4}_{\text{dTN}} - \frac{1}{2\kappa^2}\int d^4x\big[D(x) + V^{m(ij)}(x)V_{m(ij)}(x)\big], \tag{6}$$

$$\begin{aligned}\mathcal{L}^{+4}_{\text{dTN}} = &-q^+_a\mathcal{D}^{++}q^{+a}\\ &+\kappa^2\big(u^-\cdot q^+\big)^2\big[Q^+_{aA}\mathcal{D}^{++}Q^{+a}_A + g^+_r\mathcal{D}^{++}g^{+r} + W^{++}\big(Q^{+a}_A Q^+_{aB}\epsilon_{AB} - \kappa^2 c^{(ij)}g^+_i g^+_j + c^{(ij)}v^+_i v^+_j\big)\\ &\qquad + V^{++}\big(2\big(v^+\cdot g^+\big) - a^{(ab)}Q^+_{aA}Q^+_{bA}\big)\big].\end{aligned} \tag{7}$$

Here, $d\zeta^{(-4)} = d^4x\,d^2\theta^+\,d^2\bar\theta^+\,du$ is the measure of integration over (2), $(a\cdot b)\equiv a_i b^i$, the covariant harmonic derivative $\mathcal{D}^{++}$ is defined by

$$\mathcal{D}^{++} = D^{++} + \big(\theta^+\big)^2\big(\bar\theta^+\big)^2\big\{D(x)\,\partial^{--} + 6V^{m\,(ij)}(x)u^-_i u^-_j\,\partial_m\big\}, \tag{8}$$

with $D^{++} = \partial^{++} - 2i\theta^+\sigma^m\bar\theta^+\,\partial_m$, $\partial^{\pm\pm} = u^{\pm i}/\partial u^{\mp i}$, the non-propagating fields D, $V^{m(ij)}$ are inherited from the $N=2$ supergravity Weyl multiplet, κ^2 is the Einstein constant (or, from the geometric point of view, the parameter of contraction to the HK case) and the new harmonic v^{+i} is defined by

$$v^{+a} = \frac{q^{+a}}{(u^-\cdot q^+)} = u^{+a} - \frac{(u^+\cdot q^+)}{(u^-\cdot q^+)}u^{-a}.$$

The superfield $q^{+a} = f^{+a}(x,u) + \cdots = f^{ai}(x)u^+_i + \cdots$ is an extra compensating hypermultiplet, with the θ expansion and reality properties entirely analogous to (3), (4). Like in [4], we fully fix the local $SU(2)_c$ symmetry of (6) (which is present in any QK sigma model action) by the gauge condition

$$f^i_a(x) = \delta^i_a\omega(x). \tag{9}$$

The objects defined so far are necessary ingredients of the generic QK sigma model action. The specificity of the given case is revealed in the particular form of $\mathcal{L}^{+4}$ in (7). It includes two analytic gauge abelian superfields $V^{++}(\zeta)$ and $W^{++}(\zeta)$ and two sets of $SU(2)$ breaking parameters $c^{(ij)}$ and $a^{(ab)}$ satisfying the pseudo-reality condition

$$\overline{c^{(ij)}} = \epsilon_{ik}\epsilon_{jl}c^{(kl)} \tag{10}$$

(and the same for $a^{(ab)}$). The Lagrangian (7) can be checked to be invariant under the following two commuting gauge $U(1)$ transformations, with the parameters $\varepsilon(\zeta)$ and $\varphi(\zeta)$:

$$\begin{aligned}&\delta Q^{+a}_A = \varepsilon\big[\epsilon_{AB}Q^{+a}_B - \kappa^2 c^{+-}Q^{+a}_A\big], \qquad \delta g^{+r} = \varepsilon\kappa^2\big[c^{(rn)}g^+_n - c^{+-}g^{+r}\big],\\ &\delta q^{+a} = \varepsilon\kappa^2 c^{(ab)}q^+_b, \qquad \delta W^{++} = \mathcal{D}^{++}\varepsilon \quad \big(c^{+-}\equiv c^{(ik)}v^+_i u^-_k\big),\end{aligned} \tag{11}$$

$$\begin{aligned}&\delta Q^{+a}_A = \varphi\big[a^{(ab)}Q^+_{bA} - \kappa^2\big(u^-\cdot g^+\big)Q^{+a}_A\big], \qquad \delta g^{+r} = \varphi\big[v^{+r} - \kappa^2\big(u^-\cdot g^+\big)g^{+r}\big],\\ &\delta q^{+a} = \varphi\kappa^2\big(u^-\cdot q^+\big)g^{+a}, \qquad \delta V^{++} = \mathcal{D}^{++}\varphi.\end{aligned} \tag{12}$$

This gauge freedom will be fully fixed at the end. The only surviving global symmetries are two commuting $U(1)$. One of them comes from the Pauli–Gürsey $SU(2)$ acting on Q^{+a}_A and broken by the constant triplet $a^{(bc)}$. Another $U(1)$ is the result of breaking of the $SU(2)$ which uniformly rotates the doublet indices of harmonics and those of q^{+a} and g^{+r}. It does not commute with supersymmetry and forms the diagonal subgroup in the product of three independent $SU(2)$s realized on these quantities in the "free" case; this product gets broken down to the diagonal $SU(2)$ and further to $U(1)$ due to the presence of explicit harmonics and constants $c^{(ik)}$ in the interaction terms in (7). These two $U(1)$ symmetries will be isometries of the final QK metric, the first one becoming triholomorphic in the HK limit. The fields $D(x)$ and $V^{(ik)}_m(x)$ are inert under any isometry (modulo some rotations in the indices i, j), and so are $\mathcal{D}^{++}$ and the D, V part of (6).

It can be shown that the action (6), (7) is a generalization of both the HSS quotient actions describing the QK extensions of the EH and Taub-NUT sigma models: putting $g^{+r} = a^{(ab)} = 0$ yields the EH action as it was given in [4,17], putting $Q^{+a}_{A=2}(Q^{+a}_{A=1}) = c^{(ik)} = 0$ yields the Taub-NUT action [4]. Also, fixing the gauge with respect to the λ transformations by the condition $(u^- \cdot g^+) = 0$, varying with respect to the non-propagating superfield V^{++} and eliminating altogether $(v^+ \cdot g^+)$ by the resulting algebraic constraint, we arrive at the form of the action which in the HK limit $\kappa^2 \to 0$ exactly coincides with the HSS action describing the "double Taub-NUT" manifold [13,18]. Thus (6), (7) is the natural QK generalization of the action of [13,18] and therefore the relevant metric is expected to be a QK generalization of the double Taub-NUT HK metric.

3. Towards the target metric

We are going to profit from the opportunity to choose a WZ gauge for W^{++} and V^{++}, in which harmonic differential equations for $f^{+a}(x,u)$, $F^{+b}_A(x,u)$ and $g^{+r}(x,u)$ are drastically simplified.

In this gauge W^{++} and V^{++} have the following short expansion

$$W^{++} = i\theta^+\sigma^m\bar\theta^+ W_m(x) + \left(\theta^+\right)^2\left(\bar\theta^+\right)^2 P^{(ik)}(x)u^-_i u^-_k,$$
$$V^{++} = i\theta^+\sigma^m\bar\theta^+ V_m(x) + \left(\theta^+\right)^2\left(\bar\theta^+\right)^2 T^{(ik)}(x)u^-_i u^-_k \tag{13}$$

(once again, possible terms proportional to $(\theta^+)^2$ and $(\bar\theta^+)^2$ can be omitted). The hypermultiplet superfields have the same expansions as in (4). At the intermediate step it is convenient to redefine these superfields as follows

$$\left(Q^{+a}_A, g^{+r}\right) = \kappa(u^- \cdot q^+)\left(\widehat{Q}^{+a}_A, \hat g^{+r}\right). \tag{14}$$

Due to the structure of the WZ-gauge superfields (13), the highest components in the θ expansions of the redefined HM superfields appear only in the kinetic part of (7). This results in the linear harmonic equations for $f^{+a}(x,u)$, $\widehat{F}^{+b}_A(x,u)$, $\hat g^{+r}(x,u)$:

$$\partial^{++} f^{+a} = 0 \Rightarrow f^{+a} = u^{+a}\omega(x), \qquad \partial^{++}\widehat{F}^{+a}_a = 0 \Rightarrow \widehat{F}^{+a}_A = \widehat{F}^{ai}_A(x)u^+_i,$$
$$\partial^{++}\hat g^{+r} = 0 \Rightarrow \hat g^{+r} = \hat g^{ri}(x)u^+_i, \tag{15}$$

where we have simultaneously fixed the gauge (9).

Next steps are technical and quite similar to those explained in detail in [4] on the examples of the QK extensions of the Taub-NUT and EH metrics. One substitutes the solution (15) back into the action (with the θ and u integrals performed), varies with respect to the rest of non-propagating fields and also substitutes the resulting relations back into the action. At the final stages it proves appropriate to redefine the basic fields once again

$$\widehat{F}^{ai}_A = \frac{1}{\kappa\omega}F^{ai}_A, \qquad \hat g^{ri} = \frac{2}{\kappa\omega}g^{ri} \tag{16}$$

and to fully fix the residual gauge freedom of the WZ gauge for the φ transformations (with the singlet gauge parameter $\varphi(x)$), so as to gauge away the singlet part of $g^{ri}(x)$:

$$g^{ri}(x) = g^{(ri)}(x) \tag{17}$$

(the residual $SO(2)$ gauge freedom, with the parameter $\varepsilon(x)$, will be kept for the moment). In particular, in terms of the thus defined fields we have the following expressions for the fields ω and $V^{(ij)}_m$ which are obtained by varying the full action (6) with respect to D and $V^{(ij)}_m$:

$$\kappa\omega = \frac{1}{\sqrt{1 - \frac{\lambda}{2}g^2 - 2\lambda F^2}}, \qquad V^{(ij)}_m = -16\lambda^2\omega^2\left[F^{a(i}_A\partial_m F^{j)}_{aA} + \frac{1}{4}g^{r(i}\partial_m g^{j)}_r\right], \tag{18}$$

where

$$F^2 \equiv F^{ai}_A F_{ai\,A}, \qquad g^2 \equiv g^{ri} g_{ri}, \qquad \lambda \equiv \frac{\kappa^2}{4}. \tag{19}$$

The final form of the sigma model Lagrangian in terms of the fields $F^{ai}_A(x)$ and $g^{(rk)}(x)$ is as follows (we replaced altogether '∂_m' by 'd', thus passing to the distance in the target QK space instead of its x-space pullback)

$$\frac{1}{\mathcal{D}^2}\left\{\mathcal{D}\left(X + Z + \frac{Y}{4}\right) + \lambda\left(g^2\frac{Y}{8} + 2T\right)\right\} \tag{20}$$

with

$$\mathcal{D} = 1 - \frac{\lambda}{2} g^2 - 2\lambda F^2, \qquad X = \frac{1}{2} dF_{ai\,A}\, dF^{ai}_A, \qquad Y = \frac{1}{2} dg_{ij}\, dg^{ij},$$

$$Z = \frac{1}{4\alpha\beta - \gamma^2}\left\{\gamma(J\cdot K) - \alpha(J\cdot J) - \beta(K\cdot K)\right\},$$

$$T = F^{\;i}_{a\,B}\, dF^{aj}_B\left(F_{ai\,A}\, dF^{\;a}_{j\,A} + \frac{1}{2} g_{ir}\, dg^{r}{}_{j}\right). \tag{21}$$

Here

$$J = \frac{1}{2} a^{ab} F^{\;i}_{a\,A}\, dF_{bi\,A}, \qquad K = -\frac{1}{2}\epsilon_{AB} F^{ai}_A\, dF_{ai\,B} - \frac{\lambda}{2} c_{ij}\, g^{i}{}_{s}\, dg^{sj}, \tag{22}$$

and

$$\alpha = \frac{1}{2}\left(\frac{F^2}{4} - \lambda\hat{c}^2 + \frac{\lambda^2}{2}\hat{c}^2 g^2\right), \qquad \beta = \frac{1}{4}\left(1 + \frac{\hat{a}^2}{4}F^2 - \frac{\lambda}{2}g^2\right),$$

$$\gamma = \frac{1}{4} a^{ab} F^{\;i}_{a\,A} F_{bi\,B}\,\epsilon_{AB} - \lambda(c\cdot g), \tag{23}$$

where

$$\hat{c}^2 \equiv c^{ik} c_{ik}, \qquad \hat{a}^2 = a^{ab} a_{ab}. \tag{24}$$

On top of this, there are two algebraic constraints on the involved fields

$$F^{a(i}_A\, F^{\;j)}_{a\,B}\,\epsilon_{AB} - \lambda\, g^{(li)}\, g^{(rj)}\, c_{(lr)} + c^{(ij)} = 0, \tag{25}$$

$$g^{ij} = a^{ab} F^{\;i}_{a\,B} F^{\;j}_{b\,B}, \tag{26}$$

which come out by varying the action with respect to the auxiliary fields $P^{(ik)}(x)$ and $T^{(ik)}(x)$ in the WZ gauge (13). Keeping in mind these 6 constraints and one residual gauge ($SO(2)$) invariance, we are left with just four independent bosonic target coordinates as compared with 11 such coordinates in (20). The problem is now to explicitly solve (25), (26). But before turning to this issue, let us notice that the sought metric includes three parameters. These are the Einstein constant, related to λ, and two breaking parameters: the triplet $c^{(ij)}$, which breaks the $SU(2)_{\mathrm{SUSY}}$ to $U(1)$, and the triplet $a^{(ab)}$, which breaks the Pauli–Gürsey $SU(2)$ to $U(1)$. The final isometry group is therefore $U(1)\times U(1)$. For convenience we choose the following frame with respect to the broken $SU(2)$ groups

$$c^{12} = ic, \qquad c^{11} = c^{22} = 0, \qquad a^{12} = ia, \qquad a^{11} = a^{22} = 0,$$

with real parameters a and c, and we shift $\lambda \to \lambda/a^2$. Hereafter we shall use this frame, in which, in particular, the squares (24) become

$$\hat{c}^2 = 2c^2, \qquad \hat{a}^2 = 2a^2.$$

4. Solving the constraints

We need to find true coordinates to compute the metric. This step is non-trivial, due to the fact that (25) becomes quartic after substitution of (26). Instead of solving this quartic equation, it proves more fruitful to take as independent coordinates just the components of the triplet $g^{(ri)}$

$$g^{12} = g^{21} \equiv iah, \qquad \overline{h} = h, \qquad g^{11} \equiv g, \qquad g^{22} = \overline{g},$$

and one angular variable from F_A^{ai}. Then, relabelling the components of the latter fields as follows

$$\begin{cases} F_{A=1}^{a=1\ i=2} = \dfrac{1}{2}(\mathcal{F}+\mathcal{K}), & F_{A=1}^{a=1\ i=1} = \dfrac{1}{2}(\mathcal{P}+\mathcal{V}), \\ F_{A=2}^{a=1\ i=2} = \dfrac{1}{2i}(\mathcal{F}-\mathcal{K}), & F_{A=2}^{a=1\ i=1} = \dfrac{1}{2i}(\mathcal{P}-\mathcal{V}), \\ F_{A}^{a=2\ i=1} = -\overline{F_{A}^{a=1\ i=2}}, & F_{A}^{a=2\ i=2} = \overline{F_{A}^{a=1\ i=1}}, \end{cases}$$

we substitute this into (25), (26), and find the following general solution (it amounts to solving a quadratic equation and we choose the solution which is regular in the limit $g = \bar{g} = h = 0$)

$$\mathcal{P} = -iM\, e^{i(\phi+\alpha/\rho_- + \mu\rho_+)}, \qquad \mathcal{F} = R\, e^{i(\phi+\mu\rho_-)},$$
$$\mathcal{K} = iS\, e^{i(\phi-\alpha/\rho_- - \mu\rho_+)}, \qquad \mathcal{V} = L\, e^{i(\phi-\mu\rho_-)}, \quad \rho_\pm = 1 \pm 4\frac{\lambda c}{a^2} \tag{27}$$

and

$$g = at\, e^{i(\alpha/\rho_- + 8\lambda c/a^2 \mu)}. \tag{28}$$

The various functions involved are

$$L = \sqrt{\frac{1}{2}\Big(\sqrt{\Delta_-} + B_-\Big)}, \qquad R = \sqrt{\frac{1}{2}\Big(\sqrt{\Delta_+} + B_+\Big)},$$
$$M = \sqrt{\frac{1}{2}\Big(\sqrt{\Delta_+} - B_+\Big)}, \qquad S = \sqrt{\frac{1}{2}\Big(\sqrt{\Delta_-} - B_-\Big)},$$

with

$$\begin{cases} A_\pm = 1 \pm 2\lambda c\, h, & B_\pm = c\big(1+\lambda r^2\big) \pm h\, A_\mp, \\ \Delta_\pm = B_\pm^2 + t^2 A_\mp^2, & r^2 = h^2 + t^2, \quad g\bar{g} = a^2 t^2. \end{cases}$$

The true coordinates are (ϕ, α, h, t). An extra angle μ parametrizes the local $SO(2)$ transformations (they act as shifts of μ by the parameter $\varepsilon(x)$). In view of the gauge invariance of (20), the final form of the metric should not depend on μ and we can choose the latter at will. For instance, we can change the precise dependence of phases in (27), (28) on ϕ and α. In what follows we shall stick just to the above parametrization.

5. The resulting metric

To get the full metric is fairly involved and Mathematica was intensively used! The final result is

$$g = \frac{1}{4\mathcal{D}^2}\frac{\mathcal{P}}{\mathcal{A}}\left(d\phi + \frac{\mathcal{Q}}{4\mathcal{P}}\,d\alpha\right)^2 + \frac{\mathcal{A}}{\mathcal{D}^2}\left(dh^2 + dt^2 + \frac{t^2}{\mathcal{P}}\big(1+\lambda r^2\big)^2 d\alpha^2\right). \tag{29}$$

It depends on 4 functions

$$\mathcal{D}, \quad \mathcal{A}, \quad \mathcal{P}, \quad \mathcal{Q},$$

given by

$$\mathcal{A}=\frac{a^2}{4}+\frac{1}{8}\big(1-4\lambda c^2\big)\big(1-\lambda r^2\big)\left(\frac{1}{\sqrt{\Delta_+}}+\frac{1}{\sqrt{\Delta_-}}\right)-\lambda ch\left(\frac{1}{\sqrt{\Delta_+}}-\frac{1}{\sqrt{\Delta_-}}\right)+\frac{\lambda c^2}{a^2}\,\frac{4\lambda t^2-(1+\lambda r^2)^2}{\sqrt{\Delta_+}\sqrt{\Delta_-}},$$

$$\mathcal{P}=\big(1+\lambda r^2\big)^2\left(1-\frac{2\lambda c}{a^2}\left(\frac{h+c(1-\lambda r^2)}{\sqrt{\Delta_+}}-\frac{h-c(1-\lambda r^2)}{\sqrt{\Delta_-}}\right)\right)^2$$
$$+\frac{4\lambda^2c^2t^2}{a^4}\left(\frac{1-\lambda r^2-4\lambda ch}{\sqrt{\Delta_+}}-\frac{1-\lambda r^2+4\lambda ch}{\sqrt{\Delta_-}}\right)^2,$$

$$\mathcal{Q}=-\big(1+\lambda r^2\big)^2\left(\frac{h+c(1-\lambda r^2)}{\sqrt{\Delta_+}}+\frac{h-c(1-\lambda r^2)}{\sqrt{\Delta_-}}\right)$$
$$+4\lambda ct^2\left(\frac{1-\lambda r^2-4\lambda ch}{\sqrt{\Delta_+}}-\frac{1-\lambda r^2+4\lambda ch}{\sqrt{\Delta_-}}\right).$$

The overall conformal factor is

$$\mathcal{D}=1-\lambda r^2-2\frac{\lambda}{a^2}\left(\sqrt{\Delta_+}+\sqrt{\Delta_-}\right).$$

To simplify matter we first rescale $c\to c/2$. The relations

$$\Delta_\pm=(1+\lambda c^2)t^2+(h\pm c/2(1-\lambda r^2))^2$$

suggest the following change of coordinates

$$T=\frac{2t}{1-\lambda r^2},\qquad H=\frac{2h}{1-\lambda r^2},\qquad \rho=\sqrt{T^2+H^2}, \tag{30}$$

which has the virtue of reducing the quartic non-linearities according to

$$\Delta_\pm=\frac{(1-\lambda r^2)^2}{4}\,\delta_\pm,\qquad \delta_\pm=\big(1+\lambda c^2\big)T^2+(H\pm c)^2.$$

Further, to get rid of the square roots we use spheroidal coordinates (s,x) defined by

$$\sqrt{1+\lambda c^2}\,T=\sqrt{(s^2-c^2)(1-x^2)},\qquad H=sx,\quad s\geqslant c,\quad x\in[-1,+1].$$

For convenience reasons we scale the angles ϕ and α according to

$$\frac{\phi}{\sqrt{1+\lambda c^2}}\Rightarrow\phi,\qquad \frac{\alpha}{\sqrt{1+\lambda c^2}}\Rightarrow\alpha,$$

and to have a smooth limit for $a\to 0$ we come back to the original λ, $\lambda\to\lambda a^2$.

Putting these changes together, we get the final form of the metric

$$\big(4d^2\big)g=\big(1+\lambda a^2s^2\big)\frac{P}{A}\left(d\phi+\frac{Q}{4P}\,d\alpha\right)^2+\frac{A}{P}\big(s^2-c^2\big)\big(1-x^2\big)\big(1+\lambda a^2c^2x^2\big)(d\alpha)^2$$
$$+A\left(\frac{ds^2}{(s^2-c^2)(1+\lambda a^2s^2)}+\frac{dx^2}{(1-x^2)(1+\lambda a^2c^2x^2)}\right), \tag{31}$$

with

$$\begin{cases} d=1-2\lambda s, \qquad Q=-2\big(1+\lambda a^2c^2\big)\big(s^2-c^2\big)x,\\ 4A=\big(2+a^2s\big)\big(s-2\lambda c^2\big)-a^2c^2d^2x^2,\\ P=c^2\big(1-x^2\big)\big(1+\lambda a^2c^2x^2\big)\,d^2+\big(s^2-c^2\big)\big[1+\lambda a^2c^2x^2-4\lambda^2c^2\big(1-x^2\big)\big].\end{cases}$$

The isometry group $U(1)\times U(1)$ acts as translations of ϕ and α.

6. Geometric structure of the metric

We know that this metric is QK by construction, but in view of the many steps involved, it is a good self-consistency check to verify that it is Einstein with self-dual Weyl tensor. The details will be presented in [19], let us describe the main result. We take for the vierbein

$$e_0 = a(s,x)\left(d\phi + \frac{Q}{4P}\,d\alpha\right), \qquad e_3 = b(s,x)\,d\alpha, \qquad e_1 = \mu\,ds, \qquad e_2 = \nu\,dx,$$

with

$$\begin{cases} a(s,x) = \dfrac{1}{2d}\sqrt{1+\lambda s^2}\sqrt{\dfrac{P}{A}}, & \mu = \dfrac{1}{2d}\sqrt{\dfrac{A}{\mathcal{A}}}, \qquad \mathcal{A} = \left(s^2 - c^2\right)\left(1+\lambda s^2\right), \\ b(s,x) = \dfrac{1}{2d}\sqrt{(s^2-c^2)\mathcal{B}}\sqrt{\dfrac{A}{P}}, & \nu = \dfrac{1}{2d}\sqrt{\dfrac{A}{\mathcal{B}}}, \qquad \mathcal{B} = \left(1-x^2\right)\left(1+\lambda c^2 x^2\right). \end{cases}$$

The spin connection being defined as usual by

$$de_a + \omega_{ab}\wedge e_b = 0, \quad a,b = 0,1,2,3,$$

one has to compute the anti-self-dual spin connection and curvature

$$\omega_i^- = \omega_{0i} - \frac{1}{2}\epsilon_{ijk}\,\omega_{jk}, \qquad R_i^- \equiv R_{0i} - \frac{1}{2}\epsilon_{ijk}\,R_{jk} = d\omega_i^- + \epsilon_{ijk}\omega_j^-\wedge\omega_k^-, \quad i,j,k = 1,2,3.$$

One gets the crucial relation

$$R_i^- = -16\lambda\left(e_0\wedge e_i - \frac{1}{2}\epsilon_{ijk}\,e_j\wedge e_k\right), \tag{32}$$

which shows at the same time that the metric is Einstein, with

$$\mathrm{Ric} = \Lambda g, \qquad \frac{\Lambda}{3} = -16\lambda = -4\kappa^2,$$

and that the Weyl tensor is self-dual, i.e. $W_i^- = 0$.

Let us now consider a few limiting cases.

The quaternionic Taub-NUT limit

Let us show that in the limit $c \to 0$ we get the quaternionic Taub-NUT. We first write the metric (29) in the form

$$g(c\to 0) = \frac{1}{4\mathcal{D}^2}\left\{\frac{(1+\lambda r^2)^2}{\mathcal{A}_0}\left(d\psi + \frac{h}{r}\,d\alpha\right)^2 + \mathcal{A}_0\gamma_0\right\},$$

with

$$\begin{cases} \psi = -2\phi, \qquad \mathcal{A}_0 = a^2 + \dfrac{1}{r} - \lambda r, \qquad \mathcal{D} = 1 - \lambda r^2 - 4\dfrac{\lambda r}{a^2}, \\ \gamma_0 = dh^2 + dt^2 + t^2\,d\alpha^2, \qquad r^2 = h^2 + t^2. \end{cases}$$

Switching to the spherical coordinates r, θ, α for which

$$t = r\sin\theta, \qquad h = r\cos\theta$$

allows one to get the final form

$$g(c\to 0) = \frac{(1+\lambda r^2)^2}{4\mathcal{D}^2}\left\{\frac{1}{\mathcal{A}_0}\sigma_3^2 + \frac{\mathcal{A}_0}{(1+\lambda r^2)^2}\left(dr^2 + r^2\left(\sigma_1^2+\sigma_2^2\right)\right)\right\}, \tag{33}$$

with

$$\sigma_3 = d\psi + \cos\theta\, d\alpha, \qquad \sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2\theta\, d\alpha^2.$$

The derivation of the quaternionic Taub-NUT metric from harmonic superspace was given in [20]. It contains 2 parameters $\tilde{\lambda}$, R, and in the limit $R \to 0$ it reduces to Taub-NUT. One can see that, upon the identifications

$$s = r, \qquad a^2 = 4\tilde{\lambda}^2, \qquad \lambda = -R\tilde{\lambda}^2,$$

the metric $2g(c \to 0)$ is nothing but the quaternionic Taub-NUT.

The quaternionic Eguchi–Hanson limit

This metric was derived using harmonic superspace in [4], and can be written as

$$4C^2\, g = \frac{(\tilde{s}^2 - \tilde{c}^2)}{\tilde{s}B}\,\tilde{\sigma}_3^2 + \tilde{s}B\left(\frac{d\tilde{s}^2}{\tilde{s}^2 - \tilde{c}^2} + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right), \tag{34}$$

where

$$\tilde{s}B = \tilde{s} - \kappa^2\tilde{c}^2, \qquad C = 1 - \kappa^2\tilde{s}, \qquad \tilde{\sigma}_3 = d\phi + \cos\theta\, d\psi, \qquad \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 = d\theta^2 + \sin^2\theta\, d\psi^2.$$

The writing (34) is adapted to the Killing ∂_ϕ; if we switch to the Killing ∂_ψ we can write the metric as

$$\frac{D}{4\tilde{s}BC^2}\big(d\psi + \mathfrak{B}\big)^2 + \frac{\tilde{s}B}{4C^2}\left(\frac{d\tilde{s}^2}{\tilde{s}^2 - \tilde{c}^2} + d\theta^2 + \frac{\tilde{s}^2 - \tilde{c}^2}{D}\sin^2\theta\, d\phi^2\right),$$

with

$$D = \big(\tilde{s}^2 - \tilde{c}^2\big)\cos^2\theta + (\tilde{s}B)^2\sin^2\theta, \qquad \mathfrak{B} = \frac{\tilde{s}^2 - \tilde{c}^2}{D}\cos\theta\, d\phi.$$

If we now take, in the metric (31), the limit $a \to 0$ it becomes proportional to the metric (34) upon the following identifications

$$s = 2\tilde{s}, \qquad c = 2\tilde{c}, \qquad \lambda = \frac{\kappa^2}{4}, \qquad \phi \to \frac{\psi}{2}, \qquad \alpha \to -\phi, \qquad x \to \cos\theta.$$

The hyper-Kähler limit

Relation (32) makes it clear that in the limit $\lambda \to 0$ we recover a Riemann self-dual geometry, which is therefore hyper-Kähler. At the level of the metric, it is most convenient to discuss it using the coordinates (30). Indeed, we obtain the multicentre structure [21–23]

$$\frac{1}{4}\left[\frac{1}{V}\,(d\phi + \mathcal{A})^2 + V\,\gamma_0\right],$$

with the flat 3-metric

$$\gamma_0 = dH^2 + dT^2 + T^2\, d\alpha^2.$$

The potential V and the connection $\mathcal{A}$ are, respectively,

$$V = \frac{1}{4}\left\{a^2 + \frac{1}{\sqrt{\delta_+}} + \frac{1}{\sqrt{\delta_-}}\right\}, \qquad \mathcal{A} = -\frac{1}{4}\left\{\frac{H+c}{\sqrt{\delta_+}} + \frac{H-c}{\sqrt{\delta_-}}\right\}d\alpha, \quad \delta_\pm = T^2 + (H \pm c)^2. \tag{35}$$

The potential shows two centres and $V(\infty) = a^2/4$. An easy computation gives

$$dV = -\underset{\gamma_0}{\star}\, d\mathcal{A},$$

which is the fundamental relation of the multicentre metrics. For $a \neq 0$ we have the double Taub-NUT metric, while for $a = 0$ we are back to the EH metric.

Comparison with other known QK metrics

The QK metric considered here is Einstein with self-dual Weyl tensor. From a general result due to Przanowski [14] and Tod [15], this class of metrics is conformally related to a subclass of Kähler scalar-flat ones. From a result of Flaherty [16], any Kähler scalar-flat metric is a solution of the coupled Einstein–Maxwell equations, with the restriction that the Weyl tensor be self-dual. The explicit solutions of the coupled Einstein–Maxwell equations known so far fall in two classes: the Perjés–Israel–Wilson metrics [24,25] and the Plebanski–Demianski [26] metrics. In general they are not Weyl-self-dual.

For the first class we have checked (details will be given in [19]), that the Weyl-self-dual metrics are conformal to the multicentre metrics. For the metrics in the second class, imposing Weyl self-duality indeed gives rise to a QK metric. In the HK limit, with the same coordinates as in (35), we have found its potential to be

$$V = \frac{1}{\sqrt{\delta_+}} + \frac{m}{\sqrt{\delta_-}}.$$

For $m = 0$ we recover flat space while for $m \neq 1$ it describes a deformation of Eguchi–Hanson with two unequal masses. Thus our metric is also outside the Plebanski–Demianski ansatz, since their HK limits are different. We conclude that it supplies a novel explicit example of the Einstein metrics with the self-dual Weyl tensor and, simultaneously, of the solution of the coupled Einstein–Maxwell system.

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C.5 $U(1)\times U(1)$ quaternionic metrics from harmonic superspace

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$U(1) \times U(1)$ quaternionic metrics from harmonic superspace

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Abstract

We construct, using harmonic superspace and the quaternionic quotient approach, a quaternionic-Kähler extension of the most general two centres hyper-Kähler metric. It possesses $U(1) \times U(1)$ isometry, contains as special cases the quaternionic-Kähler extensions of the Taub-NUT and Eguchi–Hanson metrics and exhibits an extra one-parameter freedom which disappears in the hyper-Kähler limit. Some emphasis is put on the relation between this class of quaternionic-Kähler metrics and self-dual Weyl solutions of the coupled Einstein–Maxwell equations. The relation between our explicit results and the recent general ansatz of Calderbank and Pedersen for quaternionic-Kähler metrics with $U(1) \times U(1)$ isometries is traced in detail. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, there was a surge of interest in the explicit construction of metrics for various classes of the hyper-Kähler (HK) and quaternionic-Kähler (QK) manifolds, caused by the important role these manifolds play in string theory (see, e.g., [1–5]). At present, there exist a few approaches to tackling this difficult problem [6–25]. One of them proceeds from the generic actions of bosonic non-linear sigma models with the HK and QK target manifolds [8–13,19–23].

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Such generic actions, respectively for the HK and QK sigma models, were constructed in [8,12,13] and [19–21,23] within the harmonic superspace (HSS) method [26,27], based on the renowned one-to-one correspondence [28,29] between the HK and QK manifolds on the one hand, and global and local $N = 2$, $d = 4$ supersymmetries on the other. It was proved in [28,29] that the most general self-coupling of $N = 2$ matter supermultiplets (hypermultiplets) in the rigid or local $N = 2$ supersymmetry, necessarily implies, respectively, the HK or QK target geometry for the hypermultiplet physical bosonic fields. Conversely, any HK or QK bosonic sigma model can be lifted to a rigidly or locally $N = 2$ supersymmetric non-linear sigma model. Most general off-shell actions for such $N = 2$ sigma models were constructed in [13,19] in the framework of $N = 2$ harmonic superspace (HSS) [26] as the only one to offer such an opportunity. As was proved in [13, 21] starting from the general definition of HK or QK geometries as the properly constrained Riemannian ones, the corresponding analytic superfield Lagrangians of interaction have a nice geometric interpretation as the HK or QK potentials. These are the fundamental objects of the HK and QK geometries (like the Kähler potential in Kähler geometry). They encode the entire information about the local properties of the relevant bosonic metric, in particular, about its isometries. Then, based on the one-to-one correspondence mentioned above, the generic HK and QK sigma model bosonic actions can be obtained simply by discarding the fermionic fields in the general harmonic superspace sigma model actions. For the QK case such a generic bosonic action was constructed in [23]. The actions of physical bosons containing the explicit HK or QK metric associated with the given harmonic potential appear in general as the result of elimination of infinite sets of auxiliary fields contained in the off-shell hypermultiplet harmonic analytic superfields. This procedure amounts to solving some differential equations on the internal sphere S^2 parametrized by the $SU(2)$ harmonic variables. It is a difficult problem in general to solve such equations. However, as was shown in [9,23], in the cases with isometries the computations can be radically simplified by using the harmonic superspace version of the HK [6,7] or QK [16–18] quotient constructions. One of the attractive features of the HSS quotient is that it allows one, at all steps of computation, to keep manifest the corresponding isometries of the metric which come out as internal symmetries of the HSS sigma model Lagrangian with a transparent origin. It is especially interesting and tempting to apply this method for the explicit calculation of new inhomogeneous QK metrics. Indeed, whereas a lot of the HK metrics of this sort was explicitly constructed (both in 4- and higher-dimensional cases, see, e.g., [14,30–33]), not too many analogous QK metrics are known to date.

In [23], using the HSS quotient techniques, we constructed QK extensions of the well-known [32] Taub-NUT and Eguchi–Hanson 4-dimensional HK metrics and discussed some their distinguished geometric features. In one or another (though rather implicit) form these QK metrics already appeared in the literature (see, e.g., [17,22,34]) and our detailed treatment of them was a preparatory step to reveal capacities of the HSS approach for working out more interesting and less known examples.

In [11], the double Taub-NUT HK metric was derived from the HSS approach by directly solving the corresponding harmonic differential equations. It turns out that the HSS quotient approach allows one to reproduce the same answer much easier, and it nicely works as well in the QK case, where solving similar harmonic equations would bear a much more involved problem. In [35] we constructed a QK extension of the double Taub-NUT metric using the HSS quotient approach.

The present paper is intended, on the one hand, to give the detailed proof of some statements made in the letter [35] and to perform a further comparison with the available ansatzes for QK metrics. On the other hand, we demonstrate here that the HSS quotient approach suggests a further extension of the class of explicit QK metrics presented in [35]. All of them possess $U(1) \times U(1)$ isometry and are characterized by two additional free parameters. In the HK limit they go over into a generalization of the standard double Taub-NUT metric with two unequal "masses", one of the new parameters being just the ratio of these "masses". Another parameter does not show up in the HK limit, but it proves essential at the non-vanishing contraction parameter (Einstein constant). Thus we observe the existence of a one-parameter class of non-equivalent QK metrics having the same HK limit.

In Section 2 we remind the basic facts about the HSS action of generic QK sigma model, as it was derived in [23]. In Section 3 we construct the HSS quotient for the considered case of the QK double-Taub-NUT sigma model: proceed from a sum of the HSS "free" actions of three Q^+ hypermultiplets (having the hyperbolic $\mathbb{H}H^3$ manifold as the target space) and then gauge two common commuting one-parameter symmetries of these actions by two non-propagating $N = 2$ vector multiplets. The freedom in embedding these two symmetries in the variety of symmetries of the "free" action is characterized by two arbitrary constants which specify the most general QK extension of the double Taub-NUT metric.[1] The intermediate steps leading to the final 4-dimensional metrics are described in Section 4. The metric is read off after fixing the appropriate gauges and solving two sets of algebraic constraints appearing as the equations of motion for the auxiliary fields of the gauge multiplets. In Section 5 we bring the metrics into the final form. Using the Przanowski–Tod ansatz [34,36], we make an independent check that the metrics are indeed self-dual Einstein. Several limiting cases are also discussed. In Section 6 we examine our metrics in the context of the literature related to self-dual Einstein geometries [37–43], including Flaherty's equivalence to the (self-dual Weyl) solutions of the coupled Einstein–Maxwell equations [40].

Just after publication of our letter [35] reporting the construction of a QK extension of the double Taub-NUT metric in the HSS approach, Calderbank and Pedersen [43] have obtained the exact linearization of any four-dimensional QK metric with two commuting Killing vectors. After a short review of their results in Section 6.5, we give the precise relation between their coordinates and ours.

2. The generic HSS action of QK sigma models

In [23] the generic action of QK sigma models with $4n$-dimensional target manifold of physical bosons was obtained as a pure bosonic part of the general off-shell HSS action

[1] The QK metric presented in [35] corresponds to the minimal case, when both extra parameters are equal to zero.

of n self-interacting matter hypermultiplets coupled to the so-called principal version of $N=2$ Einstein supergravity [19]. The gauge multiplet of the latter, in the language of $N=2$ conformal SG, consists of the $N=2$ Weyl multiplet ($24+24$ off-shell components), the compensating vector multiplet ($8+8$ off-shell components) and the compensating hypermultiplet ($\infty+\infty$ off-shell components). It is the only version which admits the most general hypermultiplet matter self-couplings and thus, in accord with the theorem of [29], the most general QK metric in the sector of physical bosons. The matter and compensating hypermultiplets are described by the superfields $Q^+_r(\zeta)$ and $q^+_a(\zeta)$, $r=1,\dots,2n$, $a=1,2$, given on the harmonic analytic $N=2$ superspace

$$(\zeta)=\big(x^m,\theta^{+\mu},\bar\theta^{+\dot\mu},u^{+i},u^{-k}\big), \tag{2.1}$$

where the coordinates u^{+i},u^{-k}, $u^{+i}u^-_i=1$, $i,k=1,2$, are the $SU(2)/U(1)$ harmonic variables. These superfields obey the pseudo-reality conditions

$$\text{(a)}\quad Q^{+r}\equiv\widetilde{\big(Q^+_r\big)}=\Omega^{rs}Q^+_s,\qquad \text{(b)}\quad q^{+a}\equiv\widetilde{\big(q^+_a\big)}=\epsilon^{ab}q^+_b, \tag{2.2}$$

where Ω^{rs} and ϵ^{ab} ($\epsilon^{12}=-\epsilon_{12}=-1$) are the skew-symmetric constant $Sp(n)$ and $Sp(1)\sim SU(2)$ tensors. The generalized conjugation is the product of the ordinary complex conjugation and a Weyl reflection of the sphere $S^2\sim SU(2)/U(1)$ parametrized by $u^{\pm i}$. The superspace (2.1) is real with respect to this generalized conjugation which acts in the following way on the superspace coordinates:

$$\widetilde{x^m}=x^m,\qquad \widetilde{\theta^{+\mu}}=\bar\theta^{+\dot\mu},\qquad \widetilde{\bar\theta^{+\dot\mu}}=-\theta^{+\mu},\qquad \widetilde{u^\pm_i}=u^{\pm i},\qquad \widetilde{u^{\pm i}}=-u^\pm_i.$$

In the QH sigma model action to be given below we shall need to know only the bosonic components in the θ-expansion of the above superfields:

$$\begin{aligned}
q^{+a}(\zeta)&=f^{+a}(x,u)+i\big(\theta^+\sigma^m\bar\theta^+\big)A^{-a}_m(x,u)+\big(\theta^+\big)^2\big(\bar\theta^+\big)^2g^{-3a}(x,u),\\
Q^{+r}(\zeta)&=F^{+r}(x,u)+i\big(\theta^+\sigma^m\bar\theta^+\big)B^{-r}_m(x,u)+\big(\theta^+\big)^2\big(\bar\theta^+\big)^2G^{-3r}(x,u)
\end{aligned} \tag{2.3}$$

(possible terms $\sim(\theta^+)^2$ or $\sim(\bar\theta^+)^2$ can be shown to fully drop out from the final action and so can be discarded from the very beginning). The component fields still have general harmonic expansions off shell. The physical bosonic components $F^{ri}(x)$, $f^{ai}(x)$ are defined as the lowest components in the harmonic expansions of $F^{+r}(x,u)$, $f^{+a}(x,u)$

$$\begin{aligned}
&F^{+r}(x,u)=F^{ri}(x)u^+_i+\cdots, &&f^{+a}(x,u)=f^{ai}(x)u^+_i+\cdots,\\
&\overline{\big(F^{ri}(x)\big)}=\Omega_{rs}\epsilon_{ik}F^{sk}(x), &&\overline{\big(f^{ai}(x)\big)}=\epsilon_{ab}\epsilon_{ik}f^{bk}(x).
\end{aligned} \tag{2.4}$$

Further details can be found in [23] and [26].

The bosonic QK sigma model action derived in [23] consists of the two parts

$$\begin{aligned}
S_{\mathrm{QK}}=\frac{1}{2}\int d\zeta^{(-4)}\Big\{&-q^+_a\mathcal{D}^{++}q^{+a}\\
&+\frac{\kappa^2}{\gamma^2}\big(u^-_iq^{+i}\big)^2\big[Q^+_r\mathcal{D}^{++}Q^{+r}+L^{+4}\big(Q^+,v^+,u^-\big)\big]\Big\}\\
-\frac{1}{2\kappa^2}\int d^4x\big[D(x)&+\mathcal{V}^{m\,ij}(x)\mathcal{V}_{m\,ij}(x)\big]\equiv S_{q,Q}+S_{\mathrm{SG}}.
\end{aligned} \tag{2.5}$$

Here, $d\zeta^{(-4)} = d^4x\, d^2\theta^+\, d^2\bar{\theta}^+\, du$ is the measure of integration over (2.1), the covariant harmonic derivative $\mathcal{D}^{++}$ is defined by

$$\mathcal{D}^{++} = D^{++} + (\theta^+)^2(\bar{\theta}^+)^2\{D(x)\partial^{--} + 6\mathcal{V}^{m(ij)}(x)u_i^- u_j^- \partial_m\}, \tag{2.6}$$

with $D^{++} = \partial^{++} - 2i\theta^+\sigma^m\bar{\theta}^+\partial_m$, $\partial^{\pm\pm} = u^{\pm i}/\partial u^{\mp i}$, the non-propagating fields D, $\mathcal{V}_m^{ij} = \mathcal{V}_m^{ji}$ are inherited from the $N = 2$ Weyl multiplet, κ^2 ($[\kappa] = -1$) is the Einstein constant (or, from the geometric standpoint, the parameter of contraction to the HK case), γ ($[\gamma] = -1$) is the sigma model constant (chosen equal to 1 from now on), and the "target" harmonic variable v^{+a} is defined by

$$v^{+a} = \frac{q^{+a}}{u_i^- q^{+i}} = u^{+a} - \frac{u_i^+ q^{+i}}{u_i^- q^{+i}} u^{-a}, \qquad v^{+a}u_a^- = 1. \tag{2.7}$$

The function $L^{+4}(Q^+, v^+, u^-)$ is the analytic QK potential, the object which encodes the full information about the relevant QK metric.

The action (2.5) possesses a local $SU(2)$ invariance, the remnant of the $N = 2$ supergravity gauge group, with $\mathcal{V}_m^{ij}(x)$ as the gauge field. The precise form of the $SU(2)_{\text{loc}}$ transformations leaving the $S_{q,Q}$ part of (2.5) invariant can be inferred from the realization of the group of $N = 2$ conformal SG as restricted diffeomorphisms of the analytic superspace (2.1) [44]. This can be achieved by fixing a WZ gauge for the Weyl multiplet and neglecting all its field components besides $D(x)$, $\mathcal{V}_m^{ik}(x)$, $e_m^a(x) \to \delta_m^a$ and all the residual gauge invariance parameters besides the $SU(2)_{\text{loc}}$ one $\lambda^{ik}(x) = \lambda^{ki}(x)$. These transformations read[2]

$$\begin{aligned}
&\delta u_i^+ = \Lambda^{++}u_i^-, \qquad \delta u_i^- = 0,\\
&\Lambda^{++} = \lambda^{++} + 2i\theta^+\sigma^m\bar{\theta}^+\partial_m\lambda^{+-}\\
&\qquad - (\theta^+)^2(\bar{\theta}^+)^2\big(\Box\lambda^{--} + 4\mathcal{V}^{--m}\partial_m\lambda^{--} - 2\mathcal{V}^{+-m}\partial_m\lambda^{--} - \lambda^{--}D\big),\\
&\delta\theta^{+\mu} = \lambda^{+-}\theta^{+\mu} - i(\theta^+)^2(\sigma^m\bar{\theta}^+)^\mu\partial_m\lambda^{--} \equiv \lambda^{+\mu}(\zeta),\\
&\delta\bar{\theta}^{+\dot{\mu}} = \widetilde{(\delta\theta^{+\mu})} = \bar{\lambda}^{+\dot{\mu}}(\zeta),\\
&\delta x^m = -2i\theta^+\sigma^m\bar{\theta}^+\lambda^{--} + 6(\theta^+)^2(\bar{\theta}^+)^2\mathcal{V}^{--m}\lambda^{--} \equiv \lambda^m(\zeta),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&\delta\mathcal{D}^{++} = -\Lambda^{++}D^0,\\
&D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+\mu}\frac{\partial}{\partial\theta^{+\mu}} + \bar{\theta}^{+\dot{\mu}}\frac{\partial}{\partial\bar{\theta}^{+\dot{\mu}}},
\end{aligned} \tag{2.9}$$

$$\delta q^{+a}(\zeta) \simeq q^{+a\prime}(\zeta') - q^{+a}(\zeta) = -\frac{1}{2}\Lambda(\zeta)q^{+a}(\zeta), \tag{2.10}$$

$$\delta Q^{+r}(\zeta) \simeq Q^{+r\prime}(\zeta') - Q^{+r}(\zeta) = 0, \tag{2.11}$$

$$\Lambda(\zeta) = \partial_m\lambda^m + \partial^{--}\Lambda^{++} - \partial_{+\mu}\lambda^{+\mu} - \partial_{+\dot{\mu}}\bar{\lambda}^{+\dot{\mu}}. \tag{2.12}$$

[2] They were not explicitly given in [23] and earlier papers on the subject.

Here

$$\lambda^{\pm\pm} = \lambda^{ik}(x)u_i^{\pm}u_k^{\pm}, \qquad \lambda^{+-} = \lambda^{ik}(x)u_i^{+}u_k^{-},$$
$$\mathcal{V}_m^{--} = \mathcal{V}_m^{ik}(x)u_i^{-}u_k^{-}, \qquad \mathcal{V}_m^{+-} = \mathcal{V}_m^{ik}(x)u_i^{+}u_k^{-}.$$

To these transformations one should add the transformation laws of the fields $D(x)$ and $\mathcal{V}_m^{ik}(x)$

$$\delta^* D(x) = 2\partial_m \lambda^{ik}(x)\mathcal{V}_{ik}^m(x), \qquad \delta^* \mathcal{V}_m^{ik}(x) = -\partial_m \lambda^{ik}(x) + 2\lambda_j^{(i}(x)\mathcal{V}_m^{k)j}(x), \quad (2.13)$$

which uniquely follow from the transformation law (2.9).[3] It is easy to see that the $S_{\rm SG}$ part of (2.5) is invariant under (2.13), implying the $SU(2)_{\rm loc}$ invariance of the full action (2.5). Note that the QK potential $L^{+4}(Q^+, v^+, u^-)$ in (2.5) is $SU(2)_{\rm loc}$ invariant because its arguments Q^{+r}, v^{+a} and u^{-i} behave as scalars under the above transformations. The transformations (2.8)–(2.12) entail the following simple $SU(2)_{\rm loc}$ transformation rules for the lowest components $f^{+a}(x,u)$, $F^{+r}(x,u)$ in the θ-expansion (2.3)

$$\delta^* f^{+a} = \lambda^{+-} f^{+a} - \lambda^{++}\partial^{--} f^{+a}, \qquad \delta^* F^{+r} = -\lambda^{++}\partial^{--}F^{+r}. \quad (2.14)$$

The procedure of obtaining the QK metric from the action (2.5) goes through a few steps. First one integrates over θs in $S_{q,Q}$, then varies with respect to the non-propagating fields $g^{-3a}(x,u)$, $G^{-3r}(x,u)$, $A_m^{-a}(x,u)$, $B_m^{-r}(x,u)$, $D(x)$ and $\mathcal{V}_m^{ij}(x)$, solve the resulting non-dynamical equations and substitute the solution back into (2.5), thus expressing everything in terms of the physical components $f^{ai}(x)$ and $F^{ri}(x)$. Varying with respect to $D(x)$ and $\mathcal{V}_m^{ik}(x)$ yields the important constraint relating f^{+a} and F^{+r}:

$$\int du\, \big[f^{+a}\partial^{--} f_a^+ - \kappa^2 \big(u^- f^+\big)^2 F^{+r}\partial^{--}F_r^+\big] = \frac{1}{\kappa^2} \quad (2.15)$$

and the general expression for $\mathcal{V}_m^{ik}$ in terms of the hypermultiplet fields

$$\mathcal{V}_m^{ik}(x) = 3\kappa^2 \int du\, u^{-i}u^{-k}\big[f^{+a}\partial_m f_a^+ - \kappa^2\big(u^- f^+\big)^2 F^{+r}\partial_m F_r^+\big]. \quad (2.16)$$

As the next step, one fixes a gauge with respect to the $SU(2)_{\rm loc}$ transformations defined above. Most convenient is the gauge leaving only the singlet part in $f^{ai}(x)$

$$f_a^i(x) = \delta_a^i\, \omega(x) \quad (2.17)$$

(in what follows, we shall permanently use just this gauge). Finally, using the constraint (2.15), one expresses ω in terms of $F^{ri}(x)$, substitutes this expression into the action and reads off the QK metric on the $4n$-dimensional target space parametrized by $F^{ri}(x)$.

An essential assumption is that ω is a constant in the flat (hyper-Kähler) limit which is achieved by putting altogether

$$|\kappa|\omega = 1, \quad (2.18)$$

[3] Though looking rather involved, the transformations (2.8)–(2.13) can be straightforwardly checked to be closed, with the Lie bracket parameter $\lambda_{br}^{ik} = \lambda_2^{il}\lambda_{1l}^{\ \ k} - \lambda_1^{il}\lambda_{2l}^{\ \ k}$.

and then setting

$$\kappa = 0. \tag{2.19}$$

Note that in order to approach the HK limit in (2.5) in the unambiguous way, one should firstly eliminate the non-propagating field $\mathcal{V}_m^{ik}(x)$ by its algebraic equation of motion and also perform varying with respect to the auxiliary field $D(x)$. Taking into account that the composite field $\mathcal{V}_m^{ik}(x) \sim O(\kappa^2)$ and $q^{+a} \to u^{+a}|\kappa|^{-1}$ in the HK limit, one observes that any dependence on q^{+a}, D and $\mathcal{V}_m^{ik}$ disappears in this limit, and (2.5) goes into the HSS action of generic HK sigma model of n hypermultiplets Q^{+r} $(r = 1, \dots, 2n)$ [12,13]. The constraint (2.15) becomes just the identity $1 = 1$. Another possibility is to remove the fields $D(x), \mathcal{V}_m^{ik}(x)$ from (2.5) by equating them to zero. In this case one reproduces the HSS action of the most general conformally-invariant HK sigma model with $n + 1$ hypermultiplets [23,27,45] (the former compensator $q^{+a}(\zeta)$ enters it on equal footing with other hypermultiplets). One can reverse the argument, i.e., start from such HK sigma model action and reproduce the QK sigma model one (2.5) by coupling the HK action to the non-propagating fields $D(x)$ and $\mathcal{V}_m^{ik}(x)$ in order to restore the local $SU(2)$ symmetry and to be able to remove the remaining (non-gauge) bosonic degree of freedom in f^{+a} by the constraint (2.15). This is the content of the so-called "$N = 2$ superconformal quotient" approach to the construction of $4n$-dimensional QK manifolds from the $4(n+1)$-dimensional HK ones [15,24,25,46,47]. In what follows we shall not need to resort to such an interpretation and shall proceed from the general QK sigma model action (2.5).

3. QK extensions of the "double Taub-NUT" sigma model from HSS quotient

As already mentioned, on the road to the explicit QK metrics one needs to solve the differential equations on S^2 for $f^{+a}(x, u)$, $F^{+r}(x, u)$ which follow by varying the QK sigma model action with respect to the non-propagating fields $g^{-3a}(x, u)$ and $G^{-3r}(x, u)$. No regular methods of solving such non-linear equation are known so far, and this can (and does) bear some troubles in general. However, in a number of interesting examples there is a way around this difficulty, the HSS quotient method (it should not be confused with the "superconformal quotient" mentioned in the end of the previous section). It can be applied both in the HK [9] and QK [20,23] cases. In it, one proceeds from a system of several "free" hypermultiplets (with $L^{+4} = 0$ in (2.5), which corresponds to a $\mathbb{H}H^n \sim Sp(1, n)/Sp(1) \times Sp(n)$ sigma model) and gauges some symmetries of this system in the analytic superspace by non-propagating $N = 2$ vector multiplets represented by the gauge superfields $V^{++}(\zeta)$ (once again, only bosonic components of these superfields are of relevance). In one of possible gauges these superfields can be fully integrated out, producing a non-trivial QK (or HK) potential L^{+4} with the necessity to solve non-linear harmonic equations. But in another gauge (Wess–Zumino gauge) the harmonic equations remarkably become *linear* and can be easily solved. All the non-linearity in this gauge proves to be concentrated in non-linear algebraic constraints on the hypermultiplet physical fields. These constraints are enforced by the auxiliary fields of vector multiplets as Lagrange multipliers. They are much easier to solve as compared to the differential equations on S^2. This allows one to get the explicit form of the QK (or HK) metric at cost of a comparatively little effort.

In [23], we exemplified the HSS quotient approach by QK extensions of the Taub-NUT and Eguchi–Hanson (EH) metrics. Here we elaborate on a more interesting and non-trivial case of the QK non-linear sigma model generalizing the HK model with the "double Taub-NUT" target manifold. The HSS action of the latter model was proposed in [9], and the relevant HK metric was directly computed in [11] (it belongs to the class of two-center ALF metrics, with the triholomorphic $U(1) \times U(1)$ isometry). Here we construct, using the HSS QK quotient method, the QK sigma model action going into that of [9,11] in the HK limit. We find an interesting degeneracy suggested by the QK quotient: there is a one-parameter family of the QK metrics, all having $U(1) \times U(1)$ isometry and reproducing the double Taub-NUT metric in the HK limit. More general QK action contains one more parameter which survives in the HK limit and corresponds to a generalization of the double Taub-NUT metric by non-equal "masses" in its two-centre potential.

3.1. Minimal QK double-Taub-NUT HSS action

The actions we wish to construct have as their "parent action" the QK action including three hypermultiplet superfields of the type Q^{+r} with the vanishing L^{+4}. So it corresponds to the "flat" QK manifold $\mathbb{H}H^3 \sim Sp(1,3)/Sp(1) \times Sp(3)$. For our specific purposes we relabel this superfield triade as

$$Q_A^{+a}, \quad g^{+r}, \quad a = 1,2;\ r = 1,2;\ A = 1,2. \tag{3.1}$$

The indices a and r are the doublet indices of two (initially independent) Pauli–Gürsey type $SU(2)$ groups realized on Q^+ and g^+, the index A is an extra $SO(2)$ index. Each of these three superfields satisfies the pseudo-reality condition (2.2).

We wish to end up with a 4-dimensional quaternionic metric. So, following the general strategy of the quotient method, we need to gauge *two commuting* one-parameter ($U(1)$) symmetries of this action. In this case the total number of algebraic constraints and residual gauge invariances in the WZ gauge is expected to be just 8, which is needed for reducing the original 12-dimensional physical bosons target space to the 4-dimensional one. These $U(1)$ symmetries should be commuting, otherwise their gauging would entail gauging the symmetries appearing in their commutator. This would result in further constraints trivializing the theory.

The selection of two commuting symmetries to be gauged and the form of the final gauge-invariant HSS action are to a great extent specified by the natural requirement that the resulting action has two different limits corresponding to the earlier considered HSS quotient actions of the QK extensions of Taub-NUT and Eguchi–Hanson metrics [23]. The simplest gauged action S_{dTN} which meets this demand is

$$S_{\mathrm{dTN}} = \frac{1}{2} \int d\zeta^{(-4)}\, \mathcal{L}^{+4}_{\mathrm{dTN}} - \frac{1}{2\kappa^2} \int d^4x \left[D(x) + \mathcal{V}^{m\,ij}(x)\mathcal{V}_{m\,ij}(x)\right], \tag{3.2}$$

where

$$\mathcal{L}^{+4}_{\mathrm{dTN}} = -q_a^+ \mathcal{D}^{++} q^{+a}$$

$$+\kappa^2(u^- \cdot q^+)^2\big[Q^+_{rA}\mathcal{D}^{++}Q^{+r}_A + g^+_r\mathcal{D}^{++}g^{+r} \\ + W^{++}\big(Q^{+a}_A Q^+_{aB}\epsilon_{AB} - \kappa^2 c^{(ij)}g^+_i g^+_j + c^{(ij)}v^+_i v^+_j\big) \\ + V^{++}\big(2(v^+ \cdot g^+) - a^{(rf)}Q^+_{rA}Q^+_{fA}\big)\big] \qquad (3.3)$$

and the second term ($S_{\rm SG}$) is common for all QK sigma model actions. In (3.3), $V^{++}(\zeta)$ and $W^{++}(\zeta)$ are two analytic gauge abelian superfields, $c^{(ij)}$ and $a^{(rm)}$ are two sets of independent $SU(2)$ breaking parameters satisfying the pseudo-reality conditions

$$\overline{(c^{(ij)})} = \epsilon_{ik}\epsilon_{jl}c^{(kl)}, \qquad \overline{(a^{(rm)})} = \epsilon_{rn}\epsilon_{ms}a^{(ms)}. \qquad (3.4)$$

The Lagrangian can be checked to be invariant under the following two commuting gauge $U(1)$ transformations, with the parameters $\varepsilon(\zeta)$ and $\varphi(\zeta)$:[4]

$$\delta_\varepsilon Q^{+r}_A = \varepsilon\big[\epsilon_{AB}Q^{+r}_B - \kappa^2 c^{ij}v^+_i u^-_j Q^{+r}_A\big], \\ \delta_\varepsilon g^{+r} = \varepsilon\kappa^2\big[c^{(rn)}g^+_n - c^{ij}v^+_i u^-_j g^{+r}\big], \\ \delta_\varepsilon q^{+a} = \varepsilon\kappa^2 c^{(ab)}q^+_b, \qquad \delta_\varepsilon W^{++} = \mathcal{D}^{++}\varepsilon, \qquad (3.5)$$

$$\delta_\varphi Q^{+r}_A = \varphi a^{(rb)}Q^+_{bA} - \varphi\kappa^2\big(u^- \cdot g^+\big)Q^{+r}_A, \\ \delta_\varphi g^{+r} = \varphi\big[v^{+r} - \kappa^2(u^- \cdot g^+)g^{+r}\big], \\ \delta_\varphi q^{+a} = \varphi\kappa^2\big(u^- \cdot q^+\big)g^{+a}, \qquad \delta_\varphi V^{++} = \mathcal{D}^{++}\varphi. \qquad (3.6)$$

This gauge freedom will be fully fixed at the end. The only surviving global symmetries of the action will be two commuting $U(1)$. One of them comes from the Pauli–Gürsey $SU(2)$ acting on Q^{+a}_A and broken by the constant triplet $a^{(bc)}$. Another $U(1)$ is the result of breaking of the $SU(2)$ which uniformly rotates the doublet indices of harmonics and those of q^{+a} and g^{+r}. It does not commute with supersymmetry (in the full $N=2$ supersymmetric version of (3.3)) and forms the diagonal subgroup in the product of three independent $SU(2)$s realized on these quantities in the "free" case; this product gets broken down to the diagonal $SU(2)$, and further to $U(1)$, due to the presence of explicit harmonics and constants $c^{(ik)}$ in the interaction terms in (3.3). These two $U(1)$ symmetries are going to be isometries of the final QK metric, the first one becoming triholomorphic in the HK limit. The fields $D(x)$ and $\mathcal{V}^{(ik)}_m(x)$ are inert under any isometry (modulo some rotations in the indices i, j after fixing the gauge (2.17)), and so are $\mathcal{D}^{++}$ and the $S_{\rm SG}$ part of (3.2). The harmonics v^{+a}, as follows from their definition (2.7), undergo some appropriate transformations induced by those of q^{+a} in (3.5) and (3.6). Note that the presence of the g-field term in the supercurrent (Killing potential) to which W^{++} couples in (3.3), in

[4] To avoid a possible confusion, let us recall that the original general QK sigma model action (2.5) contains a dimensionful sigma model constant γ, $[\gamma] = -1$, which we have put equal to 1 for convenience. Actually, it is present in an implicit form in the appropriate places of Eq. (3.3) and subsequent formulae, thus removing an apparent discrepancy in the dimensions of various involved quantities. From now on, we assign the following dimensions to the basic involved objects and the gauge transformation parameters (in mass units): $[q] = [Q] = 1$, $[W^{++}] = 0$, $[V^{++}] = 1$, $[c] = 2$, $[a] = -1$, $[\varepsilon] = 0$, $[\varphi] = 1$. With this choice, γ nowhere reappears on its own right.

parallel with the v_i^+ term (becoming the Fayet–Iliopoulos term in the HK limit), is required for ensuring the invariance of this supercurrent under the φ gauge transformations. This in turn implies the non-trivial transformation property of g^{+r} under the ε gauge group in (3.5). In the HK limit the g-field term drops out and g^{+r} becomes inert under the ε transformations.

By fixing the appropriate broken $SU(2)$ symmetries in (3.3), we can leave only one real component in each of the $SU(2)$ breaking vectors a^{rf} and c^{ik}. Thus the relevant QK metric is characterized by three real parameters: two $SU(2)$ breaking ones and the Einstein constant κ^2. The $SU(2)$ breaking parameters survive in the HK limit.

The QK EH and Taub-NUT sigma model limits

It is easy to see that the action (3.2), (3.3) is indeed a generalization of the HSS quotient actions describing QK extensions of the EH and Taub-NUT sigma models.

Putting $g^{+r} = a^{(rm)} = 0$ yields the QK EH action as it was given in [20,23]:

$$\begin{aligned} \mathcal{L}^{+4}_{\text{dTN}} \quad \Rightarrow \quad \mathcal{L}^{+4}_{\text{EH}} = {} & -q^+_a \mathcal{D}^{++} q^{+a} \\ & + \kappa^2 \big(u^- \cdot q^+\big)^2 \big[Q^+_{rA} \mathcal{D}^{++} Q^{+r}_A \\ & \qquad + W^{++} \big(Q^{+a}_A Q^+_{aB} \epsilon_{AB} + c^{(ij)} v^+_i v^+_j \big) \big]. \end{aligned} \tag{3.7}$$

Putting $Q^{+a}_2 (Q^{+a}_1) = c^{(ik)} = 0$ yields the QK Taub-NUT action [23]

$$\begin{aligned} \mathcal{L}^{+4}_{\text{dTN}} \quad \Rightarrow \quad \mathcal{L}^{+4}_{\text{TN}} = {} & -q^+_a \mathcal{D}^{++} q^{+a} \\ & + \kappa^2 \big(u^- \cdot q^+\big)^2 \big[g^+_r \mathcal{D}^{++} g^{+r} \\ & \qquad + V^{++} \big(2(v^+ \cdot g^+) - a^{(rf)} Q^+_{r1} Q^+_{f1} \big) \big]. \end{aligned} \tag{3.8}$$

The HSS action with g^{+r} eliminated

Representing g^{+r} as

$$g^{+r} = \big(u^- \cdot g^+\big) v^{+r} - \big(v^+ \cdot g^+\big) u^{-r},$$

fixing the gauge with respect to the φ transformations by the condition

$$\big(u^- \cdot g^+\big) = 0,$$

varying with respect to the non-propagating superfield V^{++} and eliminating altogether $(v^+ \cdot g^+)$ by the resulting algebraic equation,

$$\big(v^+ \cdot g^+\big) \equiv L^{++} = \frac{1}{2} a^{rf} Q^+_{rA} Q^+_{fA},$$

we arrive at the following equivalent form of (3.3), with only two matter hypermultiplets Q^{+a}_A being involved

$$\begin{aligned} \mathcal{L}^{+4}_{\text{dTN}} = {} & -q^+_a \mathcal{D}^{++} q^{+a} \\ & + \kappa^2 \big(u^- \cdot q^+\big)^2 \big[Q^+_{rA} \mathcal{D}^{++} Q^{+r}_A + L^{++} L^{++} + W^{++} \big(Q^{+a}_A Q^+_{aB} \epsilon_{AB} \\ & \qquad - \kappa^2 c^{(ij)} u^-_i u^-_j L^{++} L^{++} + c^{(ij)} v^+_i v^+_j \big) \big]. \end{aligned} \tag{3.9}$$

In the HK limit $\kappa^2 \to 0$ ($q^{+a} \to |\kappa|^{-1}u^{+a}$, $|\kappa|(u^- \cdot q^+) \to 1$) the corresponding action goes into the HSS action describing the double Taub-NUT manifold [9,11].[5] Thus (3.2), (3.3) is the natural QK generalization of the action of [9,11] and therefore the relevant metric is expected to be a QK generalization of the double Taub-NUT HK metric. We shall calculated it and its some generalizations in the next sections by choosing another, Wess–Zumino gauge in the relevant gauged QK sigma model actions.

3.2. Generalizations

In order to better understand the symmetry structure of the action (3.3) and to construct its generalizations, let us make the field redefinition

$$\widehat{Q}^{+a}_A = |\kappa|\big(u^- \cdot q^+\big)Q^{+a}_A, \qquad \hat{g}^{+r} = |\kappa|\big(u^- \cdot q^+\big)g^{+r}_A. \tag{3.10}$$

In terms of the redefined superfields, Eqs. (3.3), (3.5) and (3.6) are simplified to

$$\begin{aligned}\mathcal{L}^{+4}_{\mathrm{dTN}} = {}& -q^+_a\mathcal{D}^{++}q^{+a} + \widehat{Q}^+_{rA}\mathcal{D}^{++}\widehat{Q}^{+r}_A + \hat{g}^+_r\mathcal{D}^{++}\hat{g}^{+r} \\ &+ W^{++}\big[\widehat{Q}^{+a}_A\widehat{Q}^+_{aB}\epsilon_{AB} - \kappa^2 c^{(ij)}\big(\hat{g}^+_i\hat{g}^+_j - q^+_i q^+_j\big)\big] \\ &+ V^{++}\big[2|\kappa|\big(q^+\cdot\hat{g}^+\big) - a^{(rf)}\widehat{Q}^+_{rA}\widehat{Q}^+_{fA}\big],\end{aligned} \tag{3.11}$$

$$\delta_\varepsilon\widehat{Q}^{+r}_A = \varepsilon\,\epsilon_{AB}\widehat{Q}^{+r}_B, \qquad \delta_\varepsilon\hat{g}^{+r} = \varepsilon\,\kappa^2c^{(rn)}\hat{g}^+_n, \qquad \delta_\varepsilon q^{+a} = \varepsilon\kappa^2c^{(ab)}q^+_b, \tag{3.12}$$

$$\delta_\varphi\widehat{Q}^{+r}_A = \varphi a^{(rb)}\widehat{Q}^+_{bA}, \qquad \delta_\varphi\hat{g}^{+r} = \varphi|\kappa|q^{+r}, \qquad \delta_\varphi q^{+a} = \varphi\,|\kappa|\hat{g}^{+a} \tag{3.13}$$

(the gauge superfields W^{++}, V^{++} have the same transformation laws as before).

This form of gauge transformations clearly shows that the corresponding rigid transformations are linear combinations of *four* independent mutually commuting one-parameter symmetries which are enjoyed by the free part of the Lagrangian (3.11): (a) $SO(2)$ symmetry realized on the capital index of $\widehat{Q}^{+r}_A$; (b) a diagonal $U(1)$ subgroup in the product of two commuting $SU(2)_{\mathrm{PG}}$ groups realized on q^{+a} and $\hat{g}^{+r}$, with c^{ik} as the $U(1)$ generator; (c) $U(1)$ subgroup of the $SU(2)_{\mathrm{PG}}$ group acting on $\widehat{Q}^{+r}_A$, with a^{rs} as the $U(1)$ generator; (d) a hyperbolic rotation of q^{+a} and $\hat{g}^{+r}$,

$$\delta\hat{g}^{+r} = \varphi|\kappa|q^{+r}, \qquad \delta q^{+a} = \varphi\,|\kappa|\hat{g}^{+a}. \tag{3.14}$$

Note that the bilinear form invariant under (3.14) is just $c^{(ij)}(\hat{g}^+_i\hat{g}^+_j - q^+_i q^+_j)$. This explains the presence of this expression in the ε-Killing potential (first square brackets in (3.11)): the q^+ term which is needed for making one of two basic constraints of the theory meaningful and solvable (see below) should be accompanied by the proper $\hat{g}^+$ term in order to comply with the symmetry (3.14). One is led to ε-gauge the diagonal $U(1)$ subgroup in the product of two independent $SU(2)_{\mathrm{PG}}$ groups realized on q^{+a} and $\hat{g}^{+r}$ just in order to gain this expression in the relevant Killing potential. In the HK limit $|\kappa|q^{+a} \to u^{+a}$, $\kappa \to 0$ the symmetry (3.14) becomes gauging of the familiar shift symmetry of the free hypermultiplet

[5] For the precise correspondence one should choose $a^{12} = ia$, $a^{11} = a^{22} = 0$ by appropriately fixing the frame with respect to the broken Pauli–Gürsey $SU(2)$ symmetry of Q^{+r}.

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action:

$$\delta \hat{g}^{+r} = \varphi u^{+r}, \qquad \delta u^{+a} = 0. \tag{3.15}$$

Thus we come to the conclusion that our original Lagrangian (3.3) is the simplest and natural choice yielding the double Taub-NUT HK action in the $\kappa \to 0$ limit, but it is by no means the unique one. Indeed, one could gauge two most general independent combinations of the four commuting $U(1)$ symmetries just mentioned. The corresponding generalization of (3.11) which still has a smooth $\kappa \to 0$ limit is as follows

$$\begin{aligned}\mathcal{L}^{+4\prime}_{\rm dTN} = &-q^+_a \mathcal{D}^{++} q^{+a} + \widehat{Q}^+_{rA} \mathcal{D}^{++} \widehat{Q}^{+r}_A + \hat{g}^+_r \mathcal{D}^{++} \hat{g}^{+r} \\ &+ W^{++}\big[\widehat{Q}^{+a}_A \widehat{Q}^+_{aB} \epsilon_{AB} - \kappa^2 c^{(ij)}\big(\hat{g}^+_i \hat{g}^+_j - q^+_i q^+_j\big) - \beta_0\, a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA}\big] \\ &+ V^{++}\big[2|\kappa|\big(q^+ \cdot \hat{g}^+\big) - a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} - \alpha_0\, \kappa^2 c^{(ij)}\big(\hat{g}^+_i \hat{g}^+_j - q^+_i q^+_j\big)\big],\end{aligned} \tag{3.16}$$

with α_0 and β_0 ($[\alpha_0] = -1$, $[\beta_0] = 1$) being two new real independent parameters. It is straightforward to find the precise modification of the gauge transformation rules (3.12), (3.13):

$$\begin{aligned}&\tilde{\delta}_\varepsilon \widehat{Q}^{+r}_A = \delta_\varepsilon \widehat{Q}^{+r}_A + \varepsilon\, \beta_0\, a^{(rb)} \widehat{Q}^+_{bA}, \qquad \tilde{\delta}_\varphi \hat{g}^{+r} = \delta_\varphi \hat{g}^{+r} + \varphi\, \alpha_0\, \kappa^2 c^{(rn)} \hat{g}^+_n, \\ &\tilde{\delta}_\varphi q^{+a} = \delta_\varphi q^{+a} + \varphi\, \alpha_0\, \kappa^2 c^{(ab)} q^+_b\end{aligned} \tag{3.17}$$

(the rest of transformations remains unchanged).

Limits and truncations

In the HK limit the generalized Lagrangian is reduced to

$$\begin{aligned}\mathcal{L}^{+4\prime}_{\rm dTN}(\kappa \to 0) = &\,\widehat{Q}^+_{rA} D^{++} \widehat{Q}^{+r}_A + \hat{g}^+_r D^{++} \hat{g}^{+r} \\ &+ W^{++}\big[\widehat{Q}^{+a}_A \widehat{Q}^+_{aB} \epsilon_{AB} - \beta_0\, a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + c^{(ij)} u^+_i u^+_j\big] \\ &+ V^{++}\big[2\big(u^+ \cdot \hat{g}^+\big) - a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + \alpha_0\, c^{(ij)} u^+_i u^+_j\big].\end{aligned} \tag{3.18}$$

It is easy to see that the α_0 term in the second bracket in (3.18) can be removed by the redefinition

$$\hat{g}^{+r} \quad \Rightarrow \quad \hat{g}^{+r} - \frac{1}{2}\alpha_0\, c^{ri} u^+_i, \tag{3.19}$$

which does not affect the kinetic term of $\hat{g}^{+r}$. At the same time, no such a redefinition is possible in the QK Lagrangian (3.16), so α_0 is the essentially new parameter of the corresponding QK metric. This α_0-freedom disappears in the HK limit.

Thus the associated class of QK metrics includes two extra free parameters α_0 and β_0 besides the $SU(2)$ breaking parameters and Einstein constant which characterize the minimal case treated before. But only one of them, β_0, is retained in the HK limit. Here we encounter a new (to the best of our knowledge) phenomenon of violation of the one-to-one correspondence between the HK manifolds and their QK counterparts.

It remains to understand the meaning of the parameter β_0. At $\beta_0 = 0$, we have the α_0-modified QK double Taub-NUT action. To see what happens at non-zero β_0, it is instructive

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to take a modified EH limit in (3.18). Let us redefine

$$a^{ik} = \frac{1}{\beta_0}\tilde{a}^{ik}$$

and then put $\hat{g}^{+r} = 0$, $\beta_0 \to \infty$ with keeping $\tilde{a}^{ik}$ finite and non-vanishing. Then (3.18) goes into

$$\mathcal{L}^{+4\prime}_{\mathrm{EH}}(\kappa \to 0) = \widehat{Q}^{+}_{rA} D^{++} \widehat{Q}^{+r}_{A} + W^{++}\big[\widehat{Q}^{+a}_{A}\widehat{Q}^{+}_{aB}\epsilon_{AB} - \tilde{a}^{(rf)}\widehat{Q}^{+}_{rA}\widehat{Q}^{+}_{fA} + c^{(ij)}u^{+}_{i}u^{+}_{j}\big]. \tag{3.20}$$

It is shown in Section 5.5 that this HSS action produces a generalization of the standard two-centre Eguchi–Hanson metric by bringing in two unequal "masses" $1-a$ and $1+a$ in the numerators of poles in the relevant two-centre potential, with $a = \sqrt{\frac{1}{2}\tilde{a}^{ik}\tilde{a}_{ik}}$ (c^{ik} specifies the centres like in the standard EH case [9]). Then it is clear that the action (3.18) describes a similar non-equal masses modification of the double Taub-NUT metric as a non-trivial "hybrid" of the Taub-NUT and unequal masses EH metrics, with β_0 measuring the ratio of the masses.

The general Lagrangian (3.16) has still two commuting rigid $U(1)$ symmetries which constitute the $U(1) \times U(1)$ isometry of the related QK metric. As distinct from the QK Taub-NUT and EH truncations (3.8) and (3.7) of (3.2), in which the isometries are enhanced to $U(2)$ [20,23], the same truncations made in the Lagrangian (3.16) lead to generalized QK Taub-NUT and EH metrics having only $U(1) \times U(1)$ isometries. In the QK Taub-NUT truncation which is performed by putting $\widehat{Q}^{+a}_{2} = \beta_0 = 0$, $c^{ik} = 0$, $\alpha_0 c^{ik} \equiv \tilde{c}^{ik} \neq 0$ in (3.16),

$$\mathcal{L}^{+4\prime}_{\mathrm{dTN}} \quad \Rightarrow \quad \mathcal{L}^{+4\prime}_{\mathrm{TN}} = -q^{+}_{a}\mathcal{D}^{++}q^{+a} + \widehat{Q}^{+}_{r1}\mathcal{D}^{++}\widehat{Q}^{+r}_{1} + \hat{g}^{+}_{r}\mathcal{D}^{++}\hat{g}^{+r} + V^{++}\big[2|\kappa|\big(q^{+}\cdot\hat{g}^{+}\big) - a^{(rf)}\widehat{Q}^{+}_{r1}\widehat{Q}^{+}_{f1} - \kappa^{2}\tilde{c}^{(ij)}\big(\hat{g}^{+}_{i}\hat{g}^{+}_{j} - q^{+}_{i}q^{+}_{j}\big)\big], \tag{3.21}$$

this isometry is again enhanced to $U(2)$ after taking the HK limit, because any dependence on the breaking parameter $\tilde{c}^{ik}$ disappears in this limit (after the redefinition like (3.19)). At the same time, in the QK EH truncation ($\hat{g}^{+r} = \alpha_0 = a^{rf} = 0$, $\beta_0\, a^{rf} \equiv \tilde{a}^{rf} \neq 0$ in (3.16)) the $U(1) \times U(1)$ isometry is retained in the HK limit, as clearly seen from the form of the limiting HK Lagrangian (3.20) (parameters $\tilde{a}^{ik}$ break $SU(2)_{\mathrm{PG}}$ and c^{ik} break the $SU(2)$ which rotates harmonics).

Alternative HSS quotient

Finally, we wish to point out that the QK sigma model actions we considered up to now give rise to the QK metrics which are one or another generalization of the HK double Taub-NUT metric. This is closely related to the property that one of the symmetries of the free QK action of (q^{+a}, $\widehat{Q}^{+a}_{A}$, $\hat{g}^{+r}$) which we gauge always includes as a part the hyperbolic $\hat{g}^{+r}$, q^{+a} rotation (3.14) becoming a pure shift (3.15) of g^{+r} in the HK limit. This ensures the existence of the QK Taub-NUT truncation for the considered class of QK metrics. An essentially different class of QK metrics can be constructed by gauging two independent combinations of those mutually commuting $U(1)$ symmetries of the free action which are realized as the homogeneous phase transformations of the involved superfields. The most general gauged QK sigma model of this kind is specified by the following superfield Lagrangian

$$\begin{aligned}\mathcal{L}^{+4}_{\text{dEH}} = {}& -q^+_a \mathcal{D}^{++} q^{+a} + \widehat{Q}^+_{rA} \mathcal{D}^{++} \widehat{Q}^{+r}_A + \hat{g}^+_r \mathcal{D}^{++} \hat{g}^{+r} \\ & + W^{++}\big[\widehat{Q}^{+a}_A \widehat{Q}^+_{aB} \epsilon_{AB} + \gamma_0 d^{(ik)} \hat{g}^+_i \hat{g}^+_k + \beta_0 a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + c^{(ik)} \kappa^2 q^+_i q^+_j\big] \\ & + V^{++}\big[d^{(ik)} \hat{g}^+_i \hat{g}^+_k - a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + \alpha_0 \kappa^2 c^{(ij)} q^+_i q^+_j\big], \end{aligned} \tag{3.22}$$

where the involved constants are different from those in (3.16), despite being denoted by the same letters. To see to which kind of the 4-dimensional HK sigma model the QK Lagrangian (3.22) corresponds, let us examine its HK limit

$$\begin{aligned}\mathcal{L}^{+4}_{\text{dEH}}(\kappa \to 0) = {}& \widehat{Q}^+_{rA} D^{++} \widehat{Q}^{+r}_A + \hat{g}^+_r D^{++} \hat{g}^{+r} \\ & + W^{++}\big[\widehat{Q}^{+a}_A \widehat{Q}^+_{aB} \epsilon_{AB} + \gamma d^{(ik)} \hat{g}^+_i \hat{g}^+_k + \beta a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} \\ & \quad + c^{(ik)} u^+_i u^+_j\big] \\ & + V^{++}\big[d^{(ik)} \hat{g}^+_i \hat{g}^+_k - a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + \alpha c^{(ij)} u^+_i u^+_j\big]. \end{aligned} \tag{3.23}$$

Under the truncation $\hat{g}^{+r} = 0$, $\alpha_0 = 0$, $\beta_0 a^{rf} \equiv \tilde{a}^{rf} \neq 0$, $a^{rf} = 0$ it goes into the Lagrangian (3.20) which corresponds to the EH model with unequal masses, while under the truncation $Q^{+a}_2 = 0$, $Q^{+a}_1 \equiv Q^{+a}$, $\gamma_0 = \beta_0 = 0$, $\alpha_0 c^{ik} \equiv \tilde{c}^{ik} \neq 0$, $c^{ik} = 0$ it is reduced to the following expression

$$\begin{aligned}\mathcal{L}^{+4\,\prime\prime}_{\text{EH}} = {}& \widehat{Q}^+_r D^{++} \widehat{Q}^{+r} + \hat{g}^+_r D^{++} \hat{g}^{+r} \\ & + V^{++}\big[d^{(ik)} \hat{g}^+_i \hat{g}^+_k - a^{(rf)} \widehat{Q}^+_{rA} \widehat{Q}^+_{fA} + \tilde{c}^{(ij)} u^+_i u^+_j\big]. \end{aligned} \tag{3.24}$$

This HSS Lagrangian can be shown to yield again a EH sigma model with unequal masses. The parameters of this model are different from those pertinent to the first truncation. Thus (3.23) defines a "hybrid" of two different EH sigma models, and the associated QK sigma model (3.22) could be called the "QK double EH sigma model".[6]

As the final remark, we note that in the system of three hypermultiplets in the HK case one can define mutually-commuting independent shifting symmetries of the form (3.15) separately for each hypermultiplet. Accordingly, one can use them to define different HSS quotient actions (actually, all such actions, with at least two independent shifting symmetries (3.15) gauged, yield the Taub-NUT sigma model, while those where all three such symmetries are gauged yield a trivial free 4-dimensional HK sigma model). No such an option exists in the QK case: any other hyperbolic rotation like (3.14) (in the planes $(\widehat{Q}^{+a}_1, q^{+i})$ or $(\widehat{Q}^{+a}_2, q^{+i})$) does not commute with (3.14) and the third one of the same kind. For this reason, we are allowed to use only one such hyperbolic symmetry in the gauged combinations of independent $U(1)$ symmetries in the course of constructing the relevant HSS quotient actions. Of course, this is related to the fact that the full symmetry

[6] We expect that the related QK metrics fall into the class of QK metrics described by the Plebanski–Demianski ansatz [39]; this is not the case for the QK double Taub-NUT metrics, see Section 6.4.

of the "flat" QK action of $(q^{+a}, \widehat{Q}_A^{+a}, \hat{g}^{+r})$ is $Sp(1,3)$, while the analogous symmetry of the relevant limiting HK action is a contraction of $Sp(1,3)$, with a bigger number of the mutually commuting abelian subgroups.

4. From the HSS actions to QK metrics

4.1. Preparatory steps

As already mentioned, the basic advantage of the HSS quotient as compared to the approach based on solving non-linear harmonic equations is the opportunity to choose the WZ gauge for W^{++} and V^{++} by using the ε and φ gauge freedom (see (3.5), (3.6)). In this gauge the harmonic differential equations for the lowest components $f^{+a}(x,u)$, $\widehat{F}_A^{+r}(x,u)$ and $\hat{g}^{+r}(x,u)$ of the superfields $q^{+a}(\zeta)$, $\widehat{Q}_A^{+r}(\zeta)$ and $\hat{g}^{+r}(\zeta)$ become linear and can be straightforwardly solved.

In the WZ gauge the gauge superfields has the following short expansion

$$W^{++}(\zeta) = i\theta^+\sigma^m\bar{\theta}^+ W_m(x) + (\theta^+)^2(\bar{\theta}^+)^2 P^{(ik)}(x)u_i^- u_k^-,$$
$$V^{++}(\zeta) = i\theta^+\sigma^m\bar{\theta}^+ V_m(x) + (\theta^+)^2(\bar{\theta}^+)^2 T^{(ik)}(x)u_i^- u_k^- \tag{4.1}$$

(like in (2.3), we omitted possible terms proportional to the monomials $(\theta^+)^2$ and $(\bar{\theta}^+)^2$ because the equations of motion for the corresponding fields are irrelevant to our problem of computing the final target QK metrics). At the intermediate steps it is convenient to deal with the hypermultiplet superfields $\widehat{Q}_A^{+a}$, $\hat{g}^{+r}$ related to the original superfields by (3.10). They have the same θ expansions (2.3), with "hat" above all the component fields. Due to the structure of the WZ-gauge (4.1), the highest components in the θ expansions of the superfields $\widehat{Q}_A^{+a}$, $\hat{g}^{+r}$ and q^{+a} ($\widehat{G}_A^{-3a}(x,u)$, $\hat{g}^{-3r}(x,u)$ and $f^{-3a}(x,u)$) appear only in the kinetic part of (3.3). This results in the linear harmonic equations for $f^{+a}(x,u)$, $\widehat{F}_A^{+r}(x,u)$ and $\hat{g}^{+r}(x,u)$:

$$\partial^{++} f^{+a} = 0 \quad \Rightarrow \quad f^{+a} = f^{ai}(x)u_i^+,$$
$$\partial^{++} \widehat{F}^{+r} = 0 \quad \Rightarrow \quad \widehat{F}_A^{+b} = \widehat{F}_A^{bi}(x)u_i^+,$$
$$\partial^{++} \hat{g}^{+r} = 0 \quad \Rightarrow \quad \hat{g}^{+r} = \hat{g}^{ri}(x)u_i^+. \tag{4.2}$$

It is easy to check that these equations are covariant under the $SU(2)_{\text{loc}}$ transformations (2.14) which act on f^{+a}, $\widehat{F}_A^{+r} = |\kappa|(u^-\cdot f^+)F_A^{+r}$ and $\hat{g}^{+r} = |\kappa|(u^-\cdot f^+)g^{+r}$ as follows:

$$\delta^* f^{+a} = \lambda^{+-} f^{+a} - \lambda^{++}\partial^{--} f^{+a}, \qquad \delta^*\widehat{F}_A^{+r} = \lambda^{+-}\widehat{F}_A^{+r} - \lambda^{++}\partial^{--}\widehat{F}_A^{+r},$$
$$\delta^*\hat{g}^{+r} = \lambda^{+-}\hat{g}^{+r} - \lambda^{++}\partial^{--}\hat{g}^{+r} \tag{4.3}$$

(in checking this, one must use the properties $\partial^{++}\lambda^{++} = 0$, $\partial^{++}\lambda^{+-} = \lambda^{++}$, $[\partial^{++}, \partial^{--}] = \partial^0 \equiv u^{+i}\partial/\partial u^{+i} - u^{-i}\partial/\partial u^{-i}$ and $\partial^0(f^{+a}, \widehat{F}^{+r}, \hat{g}^{+r}) = (f^{+a}, \widehat{F}^{+r}, \hat{g}^{+r})$). These transformations entail the following ones for the bosonic fields of physical dimension

$$\delta f^{ai}(x) = \lambda^i{}_k(x) f^{ak}(x), \qquad \delta\widehat{F}_A^{ri}(x) = \lambda^i{}_k(x)\widehat{F}_A^{rk}(x),$$
$$\delta\hat{g}^{ri}(x) = \lambda^i{}_k(x)\hat{g}^{rk}(x). \tag{4.4}$$

This step is common for all QK sigma model actions considered in the previous section. The next common step is to vary with respect to the SG fields $D(x)$ and $\mathcal{V}_m^{ik}(x)$ in order to obtain the appropriate particular forms of the constraint (2.15) and the expression (2.16). Bearing in mind the harmonic "shortness" (4.2), we find

$$\frac{\kappa^2}{2} f^2 = 1 + \frac{\kappa^2}{2}\big(\widehat{F}^2 + \hat{g}^2\big), \tag{4.5}$$

$$\mathcal{V}_m^{ik} = \kappa^2\big(f^{a(i}\partial_m f_a^{j)} - \widehat{F}_A^{r(i}\partial_m \widehat{F}_{rA}^{j)} - \hat{g}^{r(i}\partial_m \hat{g}_r^{j)}\big), \tag{4.6}$$

where

$$f^2 = f^{ai} f_{ai}, \qquad \widehat{F}^2 = \widehat{F}_A^{ri}\widehat{F}_{ri\,A}, \qquad \hat{g}^{ri}\hat{g}_{ri}.$$

Taking into account the constraint (4.5), it is easy to check that the $SU(2)_{\rm loc}$ transformation laws (4.3) imply just the transformation law (2.13) for the composite gauge field (4.6) .

One more common step is enforcing the gauge (2.17)

$$f_a^i(x) = \delta_a^i\,\omega(x) \quad\Rightarrow\quad f^{ai} f_{ai} \quad\Rightarrow\quad 2\omega^2. \tag{4.7}$$

For what follows it will be useful to give how the residual gauge symmetries of the WZ gauge (4.1) with the parameters $\varepsilon(x) = \varepsilon(\zeta)|$ and $\varphi(x) = \varphi(\zeta)|$ are realized in the gauge (4.7) (in the general case of gauge symmetries (3.17), (3.12), (3.13))

$$\tilde{\delta}_\varepsilon \widehat{F}_A^{ri} = \varepsilon\epsilon_{AB}\widehat{F}_B^{ri} + \varepsilon\,\beta_0 a^{rs}\widehat{F}_{s\,A}^{\ i} - \lambda_\varepsilon^{ik}\widehat{F}_{k\,A}^{r}, \qquad \tilde{\delta}_\varepsilon \hat{g}^{ri} = \varepsilon\,\kappa^2 c^{rn}\hat{g}_n^{\ i} - \lambda_\varepsilon^{ik}\hat{g}_{\ k}^{r}, \tag{4.8}$$

$$\begin{aligned}
&\tilde{\delta}_\varphi \widehat{F}_A^{ri} = \varphi\, a^{rs}\widehat{F}_{s\,A}^{\ i} - \lambda_\varphi^{ik}\widehat{F}_{k\,A}^{r},\\
&\tilde{\delta}_\varphi \hat{g}^{ri} = \varphi\,|\kappa|\epsilon^{ri}\omega + \varphi\,\alpha_0\,\kappa^2 c^{rs}\hat{g}_s^{\ i} - \lambda_\varphi^{ik}\hat{g}_{\ k}^{r},\\
&\tilde{\delta}_\varphi \omega = \frac{1}{2}\varphi\,|\kappa|\big(\epsilon_{ia}\hat{g}^{ai}\big),
\end{aligned} \tag{4.9}$$

where $\lambda_\varepsilon^{ik}, \lambda_\varphi^{ik}$ are the parameters of two different induced $SU(2)_{\rm loc}$ transformations needed to preserve the gauge (4.7)

$$\lambda_\varepsilon^{ik} = -\varepsilon\,\kappa^2 c^{ik}, \qquad \lambda_\varphi^{ri} = -\varphi\left(\frac{|\kappa|}{\omega}\,\hat{g}^{(ri)} + \alpha_0\,\kappa^2 c^{ri}\right). \tag{4.10}$$

From now on, we fully fix the residual $\varphi(x)$ gauge symmetry by gauging away the singlet part of $g^{ri}(x)$:

$$\epsilon_{ir} g^{ri}(x) = 0 \quad\Rightarrow\quad g^{ri}(x) = g^{(ri)}(x). \tag{4.11}$$

The residual $SO(2)$ gauge freedom, with the parameter $\varepsilon(x)$, will be kept for the moment.

We shall explain further steps on the example of the simplest QK double Taub-NUT action (3.2), (3.3) and then indicate the modifications which should be made in the resulting physical bosons action in order to encompass the general case (3.16).

These steps are technical (though sometimes amounting to rather lengthy computations) and quite similar to those expounded in [23] on the examples of the QK extensions of the Taub-NUT and EH metrics. So here we shall describe them rather schematically.

Firstly one substitutes the solution (4.2) back into the action (3.2), (3.3) (with the θ-integration performed) and varies with respect to the remaining non-propagating (vector) fields of the hypermultiplet superfields ($A_m^{-a}(x,u)$, $\widehat{B}_{Am}^{-a}(x,u)$ and $\hat{b}_m^{-r}(x,u)$ in the θ-expansions of q^{+a}, $\widehat{Q}_A^{+a}$ and $\hat{g}^{+r}$, respectively). Then one substitutes the resulting expressions for these fields into the action (together with those for $\omega(x)$ and $\mathcal{V}_m^{ik}(x)$, Eqs. (4.5), (2.16)) and performs the u-integration. At this stage it is convenient to redefine the remaining fields as follows[7]

$$F_A^{ai} = \frac{1}{\kappa\omega}\widehat{F}_A^{ai}, \qquad g^{ri} = \frac{2}{\kappa\omega}\hat{g}^{ri}. \tag{4.12}$$

In terms of the redefined fields and with taking account of the gauges (4.7), (4.11), the composite fields ω and $\mathcal{V}_m^{ij}$ are given by the following expressions:

$$\kappa\,\omega = \frac{1}{\sqrt{1-\frac{\lambda}{2}g^2 - 2\lambda F^2}}, \qquad \mathcal{V}_m^{(ij)} = -16\lambda^2\omega^2\left[F_A^{a(i}\partial_m F_{aA}^{j)} + \frac{1}{4}g^{r(i}\partial_m g_r^{j)}\right], \tag{4.13}$$

where

$$F^2 \equiv F_A^{ai}F_{aiA}, \qquad g^2 \equiv g^{ri}g_{ri}, \qquad \lambda \equiv \frac{\kappa^2}{4}. \tag{4.14}$$

After substituting everything back into the action we get the following intermediate expression for the x-space Lagrangian density $\mathcal{L}_{\mathrm{dTN}}(x)$:

$$\mathcal{L}_{\mathrm{dTN}}(x) = \mathcal{L}_0(x) + \mathcal{L}_{\mathrm{vec}}(x), \tag{4.15}$$

where

$$\mathcal{L}_0(x) = \frac{1}{\mathcal{D}^2}\left\{\mathcal{D}\left(X + \frac{Y}{4}\right) + \lambda\left(g^2\cdot\frac{Y}{8} + 2T\right)\right\} \tag{4.16}$$

with

$$\mathcal{D} = 1 - \frac{\lambda}{2}g^2 - 2\lambda F^2, \qquad X = \frac{1}{2}\partial^m F_{ai\,A}\,\partial_m F_A^{ai}, \qquad Y = \frac{1}{2}\partial^m g_{ij}\,\partial_m g^{ij},$$
$$T = F_{a\,B}^{\,i}\,\partial^m F_B^{aj}\left(F_{ai\,A}\,\partial_m F_{j\,A}^{a} + \frac{1}{2}g_{ir}\,\partial_m g_{\ j}^{r}\right), \tag{4.17}$$

and

$$\mathcal{L}_{\mathrm{vec}}(x) = \frac{1}{\mathcal{D}}\left[\alpha\left(W^m W_m\right) + \beta\left(V^m V_m\right) + \gamma\left(W^m V_m\right) + W^m K_m + V^m J_m\right], \tag{4.18}$$

with

$$J_m = \frac{1}{2}a^{ab}\,F_{a\,A}^{\,i}\,\partial_m F_{bi\,A}, \qquad K_m = -\frac{1}{2}\epsilon_{AB}\,F_A^{ai}\,\partial_m F_{ai\,B} - \frac{\lambda}{2}c_{ij}\,g_{\ s}^{i}\,\partial_m g^{sj},$$
$$\alpha = \frac{1}{2}\left(\frac{F^2}{4} - \lambda\,\hat{c}^2 + \frac{\lambda^2}{2}\hat{c}^2 g^2\right), \qquad \beta = \frac{1}{4}\left(1 + \frac{\hat{a}^2}{4}F^2 - \frac{\lambda}{2}g^2\right),$$
$$\gamma = \frac{1}{4}a^{ab}\,F_{a\,A}^{\,i}\,F_{bi\,B}\,\epsilon_{AB} - \lambda(c\cdot g), \tag{4.19}$$

[7] This relation was misprinted in [35].

$$\hat{c}^2 \equiv c^{ik} c_{ik}, \qquad \hat{a}^2 = a^{ab} a_{ab}. \tag{4.20}$$

After integrating out the non-propagating gauge fields $W^m(x)$ and $V^m(x)$, the part $\mathcal{L}_{\text{vec}}$ acquires the typical non-linear sigma model form

$$\mathcal{L}_{\text{vec}} \quad \Rightarrow \quad \frac{1}{\mathcal{D}} Z, \quad Z = \frac{1}{4\alpha\beta - \gamma^2} \big\{ \gamma\,(J \cdot K) - \alpha\,(J \cdot J) - \beta\,(K \cdot K) \big\}. \tag{4.21}$$

The resulting sigma model action should be supplemented by two algebraic constraints on the involved fields

$$F_A^{a(i} F_{a\,B}^{j)} \epsilon_{AB} - \lambda\, g^{(li)} g^{(rj)} c_{(lr)} + c^{(ij)} = 0, \tag{4.22}$$

$$g^{ij} - a^{ab} F_{a\,B}^{i} F_{b\,B}^{j} = 0, \tag{4.23}$$

which follow from varying the action with respect to the auxiliary fields $P^{(ik)}(x)$ and $T^{(ik)}(x)$ in the WZ gauge (4.1). Keeping in mind these 6 constraints and one residual gauge ($SO(2)$) invariance, one is left just with four independent bosonic target coordinates as compared with eleven such coordinates explicitly present in (4.16), (4.21). The problem now is to solve Eqs. (4.22), (4.23), and thus to obtain the final sigma model action with 4-dimensional QK target manifold. This will be the subject of our further presentation.

Here, as the convenient starting point for the geometrical treatment in Section 5, it is worth to give how the full distance looks before solving the constraints (4.22), (4.23)

$$\mathfrak{g} = \frac{1}{\mathcal{D}^2} \left\{ \mathcal{D}\left(X' + Z' + \frac{Y'}{4} \right) + \lambda \left(g^2 \cdot \frac{Y'}{8} + 2\,T' \right) \right\}. \tag{4.24}$$

The quantities with "prime" are obtained from those defined above by replacing altogether "∂_m" by "d", thus passing to the distance in the target space. For instance,

$$X' = \frac{1}{2}\, dF_{ai\,A}\, dF_A^{ai}, \qquad Y' = \frac{1}{2}\, dg_{ij}\, dg^{ij}. \tag{4.25}$$

Note that this metric includes three free parameters. These are the Einstein constant related to λ ($\lambda \equiv \kappa^2/4$), and two $SU(2)$ breaking parameters: the triplet $c^{(ij)}$, which breaks the $SU(2)_{\text{SUSY}}$ to $U(1)$, and the triplet $a^{(ab)}$, which breaks the Pauli–Gürsey $SU(2)$ to $U(1)$. The final isometry group is therefore $U(1) \times U(1)$. Constraints (4.22), (4.23) are manifestly covariant under these isometries. For convenience, from now on we choose the following frame with respect to the broken $SU(2)$ groups

$$c^{12} = ic, \qquad c^{11} = c^{22} = 0, \qquad a^{12} = ia, \qquad a^{11} = a^{22} = 0, \tag{4.26}$$

with real parameters a and c. In this frame, the squares (4.20) become

$$\hat{c}^2 = 2c^2, \qquad \hat{a}^2 = 2a^2.$$

Let us now discuss which modifications the distance (4.24) undergoes if one starts from the general QK double Taub-NUT action corresponding to the Lagrangian (3.16). Since the difference between (3.2) and (3.16) is solely in the structure of supercurrents (Killing potentials) to which gauge superfields W^{++} and V^{++} couple, the only modifications entailed by passing to (3.16) are the appropriate changes in the Z'-part of (4.24) and in the constraints (4.22), (4.23). Namely, one should make the following replacements in Z':

$$\begin{aligned}
\alpha \quad &\Rightarrow \quad \hat{\alpha} = \alpha + \frac{1}{16}\beta_0^2 \hat{a}^2 F^2 + \frac{1}{4}\beta_0 a^{rf} F^i_{rA} F_{fiB}\epsilon_{AB}, \\
\beta \quad &\Rightarrow \quad \hat{\beta} = \beta - \lambda\alpha_0 (g\cdot c) - \frac{1}{2}\alpha_0^2 \hat{c}^2 \left(1 - \frac{\lambda}{2} g^2\right), \\
\gamma \quad &\Rightarrow \quad \hat{\gamma} = \gamma + \frac{1}{8}\beta_0 \hat{a}^2 F^2 - \lambda\alpha_0 \hat{c}^2 \left(1 - \frac{\lambda}{2} g^2\right), \\
K_m \quad &\Rightarrow \quad \widehat{K}_m = K_m + \frac{1}{2}\beta_0 a^{rf} F^i_{rA}\, dF_{fiA}, \\
J_m \quad &\Rightarrow \quad \widehat{J}_m = J_m - \frac{1}{2}\lambda\alpha_0 c^{lr} g_{lk}\, dg^k_r
\end{aligned} \tag{4.27}$$

and pass to the following modification of the constraints (4.22), (4.23):

$$F^{a(i}_A F^{j)}_{aB}\epsilon_{AB} - \lambda g^{(li)} g^{(rj)} c_{(lr)} - \beta_0 a^{ab} F^i_{aB} F^j_{bB} + c^{(ij)} = 0, \tag{4.28}$$

$$g^{ij} - a^{ab} F^i_{aB} F^j_{bB} + \alpha_0\left[c^{ij} - \lambda g^{(li)} g^{(rj)} c_{(lr)}\right] = 0. \tag{4.29}$$

4.2. Solving the constraints

In order to find the final form of the QK target metric corresponding to the HSS Lagrangian (3.3) or its generalization (3.16), we should solve the constraints (4.22), (4.23) or their generalization (4.28), (4.29). It is a non-trivial step to find the true coordinates to solve these constraints. Indeed, a direct substitution of g^{ij} from (4.23) into (4.22) gives a quartic constraint for F^{ai}_A which is very difficult to solve as compared to the HK case [9,11] where the analogous constraint is merely quadratic. In the general case (4.28), (4.29) the situation is even worse.

In view of these difficulties, it proves more fruitful to take as independent coordinates just the components of the triplet $g^{(ri)}$,

$$g^{12} = g^{21} \equiv iah, \qquad \bar{h} = h, \qquad g^{11} \equiv g, \qquad g^{22} = \bar{g},$$

and one angular variable from F^{ai}_A. Then the above 6 constraints and one residual gauge invariance (the $\varepsilon(x)$ one) allow us to eliminate the remaining 7 components of F^{ai}_A in terms of 4 independent coordinates thus defined. Following the same strategy as in the previous subsection, we shall first explain how to solve Eqs. (4.22) and (4.23) in this way and then indicate the modifications giving rise to the solution of the general two-parameter set of constraints (4.28), (4.29). We relabel the components of F^{ai}_A as follows

$$\begin{aligned}
F^{a=1\ i=2}_{A=1} &= \frac{1}{2}(\mathcal{F} + \mathcal{K}), &\qquad F^{a=1\ i=1}_{A=1} &= \frac{1}{2}(\mathcal{P} + \mathcal{V}), \\
F^{a=1\ i=2}_{A=2} &= \frac{1}{2i}(\mathcal{F} - \mathcal{K}), &\qquad F^{a=1\ i=1}_{A=2} &= \frac{1}{2i}(\mathcal{P} - \mathcal{V}), \\
F^{a=2\ i=1}_{A} &= -\overline{F^{a=1\ i=2}_{A}}, &\qquad F^{a=2\ i=2}_{A} &= \overline{F^{a=1\ i=1}_{A}},
\end{aligned}$$

and substitute this into (4.22), (4.23). After some simple algebra, the constraints can be equivalently rewritten in the following form

$$\begin{aligned} &\text{(a)} \quad \mathcal{P}\overline{\mathcal{F}} = -\frac{i}{2a}A_-, \quad \text{(and c.c.);} \\ &\text{(b)} \quad \mathcal{V}\overline{\mathcal{K}} = -\frac{i}{2a}A_+, \quad \text{(and c.c.);} \end{aligned} \tag{4.30}$$

$$\begin{aligned} &\text{(c)} \quad \mathcal{F}\overline{\mathcal{F}} - \mathcal{P}\overline{\mathcal{P}} = B_+; \\ &\text{(d)} \quad \mathcal{V}\overline{\mathcal{V}} - \mathcal{K}\overline{\mathcal{K}} = B_-. \end{aligned} \tag{4.31}$$

Here

$$A_\pm = 1 \pm 2\lambda a^2 ch, \qquad B_\pm = c\big(1 + \lambda a^2 r^2\big) \pm h A_\mp,$$
$$r^2 = h^2 + t^2, \qquad g\bar{g} = a^2 t^2.$$

Next, one expresses $\overline{\mathcal{P}}$ and $\overline{\mathcal{K}}$ from (4.30) and substitutes them into (4.31), which gives two quadratic equations for $\mathcal{F}\overline{\mathcal{F}} \equiv X$ and $\mathcal{V}\overline{\mathcal{V}} \equiv Y$,

$$X^2 - XB_+ - \frac{1}{4}t^2 A_-^2 = 0, \qquad Y^2 - YB_- - \frac{1}{4}t^2 A_+^2 = 0. \tag{4.32}$$

Solving these equations, selecting the solution which is regular in the limit $g = \bar{g} = h = 0$ and properly fixing the phases of $\mathcal{F}, \mathcal{P}, \mathcal{V}$ and $\mathcal{K}$ in terms of the phase of g with taking account of the residual $\varepsilon(x)$ gauge freedom, we find the general solution of (4.22), (4.23) in the following concise form

$$\begin{aligned} &\mathcal{P} = -iMe^{i(\phi+\alpha/\rho_- - \mu\rho_+)}, \qquad \mathcal{F} = Re^{i(\phi-\mu\rho_-)}, \\ &\mathcal{K} = iSe^{i(\phi-\alpha/\rho_- + \mu\rho_+)}, \qquad \mathcal{V} = Le^{i(\phi+\mu\rho_-)}, \\ &\rho_\pm = 1 \pm 4\lambda c \end{aligned} \tag{4.33}$$

and

$$g = ate^{i(\alpha/\rho_- - 8\mu\lambda c)}. \tag{4.34}$$

The various functions involved are

$$L = \sqrt{\frac{1}{2}(\sqrt{\Delta_-} + B_-)}, \qquad R = \sqrt{\frac{1}{2}(\sqrt{\Delta_+} + B_+)},$$
$$M = \sqrt{\frac{1}{2}(\sqrt{\Delta_+} - B_+)}, \qquad S = \sqrt{\frac{1}{2}(\sqrt{\Delta_-} - B_-)},$$

where

$$\Delta_\pm = B_\pm^2 + t^2 A_\mp^2.$$

The true coordinates are (ϕ, α, h, t). An extra angle μ parametrizes the residual local $SO(2)$ transformations which act as shifts of μ by the parameter $\varepsilon(x)$, $\mu \to \mu + \varepsilon$. To see this, one must rewrite the ε-transformation law of F_A^{ri} following from that of $\widehat{F}_A^{ri}$, Eq. (4.8) (at $\beta_0 = \alpha_0 = 0$),

$$\delta_\varepsilon F_A^{ri} = \varepsilon \epsilon_{AB} F_B^{ri} + \varepsilon \kappa^2 c^{ik} F_{kA}^r,$$

in terms of the newly defined variables and in the $SU(2)$ frame (4.26)

$$\delta_\varepsilon \mathcal{F} = -i\varepsilon\rho_- \mathcal{F}, \qquad \delta_\varepsilon \mathcal{V} = i\varepsilon\rho_- \mathcal{V}, \qquad \delta_\varepsilon \mathcal{P} = -i\varepsilon\rho_+ \mathcal{P}, \qquad \delta_\varepsilon \mathcal{K} = i\varepsilon\rho_+ \mathcal{K},$$
$$\delta_\varepsilon h = 0, \qquad \delta_\varepsilon g = -8i\varepsilon\lambda c g. \tag{4.35}$$

As a consequence of gauge invariance of (4.16), the final form of the metric should not depend on μ and we can choose the latter at will. For instance, we can change the precise dependence of phases in (4.33), (4.34) on ϕ and α. In what follows we shall stick just to the above parametrization. Explicitly keeping μ at the intermediate steps of calculations is a good self-consistency check: this gauge parameter should fully drop out from the correct final expression for the metric.

Finally, let us indicate the modifications which should be made in the above solution to adapt it to the general set of constraints (4.28), (4.29). It is convenient to represent the latter in the following equivalent form

$$F_A^{a(i} F_{aB}^{j)} \epsilon_{AB} - \beta_0 g^{(ij)} + (1 - \alpha_0\beta_0)\left[c^{(ij)} - \lambda g^{(li)} g^{(rj)} c_{(lr)}\right] = 0, \tag{4.36}$$
$$g^{ij} - a^{ab} F_{aB}^i F_{bB}^j + \alpha_0\left[c^{ij} - \lambda g^{(li)} g^{(rj)} c_{(lr)}\right] = 0. \tag{4.37}$$

Then, following the same line as in the case of $\beta_0 = \alpha_0 = 0$, one gets the general solution in the form

$$\mathcal{P} = -i\widetilde{M} e^{i(\phi + \alpha/\rho_- - \mu\rho_+)} e^{-i\mu\beta_0 a}, \quad \mathcal{F} = \widetilde{R} e^{i(\phi - \mu\rho_-)} e^{-i\mu\beta_0 a},$$
$$\mathcal{K} = i\widetilde{S} e^{i(\phi - \alpha/\rho_- + \mu\rho_+)} e^{-i\mu\beta_0 a}, \qquad \mathcal{V} = \widetilde{L} e^{i(\phi + \mu\rho_-)} e^{-i\mu\beta_0 a}, \tag{4.38}$$

where the functions with "tilde" are related to those defined earlier by the following replacements

$$A_\pm \quad \Rightarrow \quad \widetilde{A}_\pm = (1 \pm a\beta_0)(1 - 2\alpha_0\lambda a c h) \pm 2\lambda a^2 c h,$$
$$B_\pm \quad \Rightarrow \quad \widetilde{B}_\pm = \left[1 \pm \frac{\alpha_0}{a}(1 \mp a\beta_0)\right] B_\pm - a\left[\beta_0 + \frac{\alpha_0}{a^2}(1 \mp a\beta_0)\right] h. \tag{4.39}$$

The appearance of an additional phase factor in (4.34) is due to the fact that in the general case the ε transformations (4.35) acquire the common extra piece proportional to β_0:

$$\delta_\varepsilon \mathcal{F} \quad \Rightarrow \quad \delta_\varepsilon \mathcal{F} - i\varepsilon\beta_0 a \mathcal{F},$$

etc. The QK Taub-NUT and QK EH truncations of the general solution correspond to imposing the following conditions:

$$\text{QK Taub-NUT:} \quad \beta_0 = 0, \quad c = 0, \quad \alpha_0 c \equiv \tilde{\alpha}_0 \neq 0, \tag{4.40}$$
$$\text{QK EH:} \quad \alpha_0 = 0, \quad \beta_0 a \equiv \tilde{\beta}_0 \neq 0, \quad a \Rightarrow 0. \tag{4.41}$$

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Respectively, in these two limits we have

QK Taub-NUT:

$$\widetilde{A}_{\pm} = (1 - 2\lambda a\tilde{\alpha}_0 h) \equiv \widetilde{A}, \qquad \widetilde{B}_{\pm} = \pm\left[h + \frac{\tilde{\alpha}_0}{a}(1 + \lambda a^2 r^2)\right] \equiv \pm\widetilde{B},$$
$$\widetilde{\Delta}_{+} = \widetilde{\Delta}_{-} = \widetilde{B}^2 + t^2\widetilde{A}^2 \quad \Rightarrow \quad \widetilde{L} = \widetilde{M}, \quad \widetilde{R} = \widetilde{S}, \tag{4.42}$$

QK EH:

$$\widetilde{A}_{\pm} = (1 \pm \tilde{\beta}_0), \qquad \widetilde{B}_{\pm} = c \pm (1 \mp \tilde{\beta}_0)h,$$
$$\widetilde{\Delta}_{\pm} = \left[c \pm (1 \mp \tilde{\beta}_0)\right]^2 + (1 \mp \tilde{\beta}_0)^2 t^2. \tag{4.43}$$

Note that in the Taub-NUT case we can obviously choose, up to a gauge freedom,

$$\mathcal{P} = \mathcal{V}, \qquad \mathcal{F} = \mathcal{K} \quad \Rightarrow \quad e^{2i\mu} = -ie^{i\alpha},$$

which, according to the above definition of $\mathcal{P}, \mathcal{V}, \mathcal{F}, \mathcal{V}$, corresponds just to the truncation $Q^{+a}_{A=2} = 0$ at the level of the general HSS Lagrangian (3.16). Also note that for taking the QK EH limit in the original form of constraints (4.36), (4.37) in the unambiguous way, one should firstly rescale $g^{ik} \to ag^{(ik)}$. The corresponding limiting QK metrics can be obtained by taking the limits (4.40), (4.41) in the QK metric associated with the general choice of $\alpha_0 \neq 0$, $\beta_0 \neq 0$.

5. The structure of general metric

5.1. First set of coordinates

To obtain the metric, we substitute the explicit form (4.38) of the coordinates into the distance (4.24) and compute it. The algebraic manipulations to be done in order to cast the resulting expression in a readable form are rather involved, and Mathematica was intensively used while doing this job. To simplify matters, we make the change of coordinates

$$T = \frac{2t}{1 - a^2\lambda r^2}, \qquad H = \frac{2h}{1 - a^2\lambda r^2}, \qquad \rho = \sqrt{T^2 + H^2} \tag{5.1}$$

and use the notations

$$\beta = \frac{a\beta_0}{1 - 4c\lambda}, \qquad c_{\pm} = \left(\frac{1}{1 \mp a\beta_0} \pm \frac{\alpha_0}{a}\right)c,$$

$$\delta_{\pm} = \frac{4\Delta_{\pm}}{\left(1 - a^2\lambda r^2\right)^2} = \left(1 + 4a^2\lambda c_{\pm}^2\right)T^2 + \left(H \pm 2c_{\pm}\right)^2.$$

The final result for the metric $\mathfrak{g}$ can be presented in terms of 4 functions D, A, P, Q

$$4D^2\mathfrak{g} = \frac{P}{A}\left(d\phi + \frac{Q}{4\,P}\,d\alpha\right)^2 + A\left(\mathfrak{g}_0 + \frac{1 + a^2\lambda\rho^2}{P}T^2\,d\alpha^2\right) \tag{5.2}$$

where

$$\mathfrak{g}_0 = \frac{dH^2 + dT^2 + a^2\lambda(T\,dH - H\,dT)^2}{1 + a^2\lambda\rho^2} \tag{5.3}$$

is the metric on the two-sphere ($a^2\lambda < 0$), on flat space ($a^2\lambda = 0$), or on the hyperbolic plane ($a^2\lambda > 0$).

The various involved functions are as follows

$$\begin{aligned}
D &= \frac{\mathcal{D}}{1 - a^2\lambda r^2} = 1 - \lambda\big((1 + a\beta_0)\sqrt{\delta_-} + (1 - a\beta_0)\sqrt{\delta_+}\big),\\
A &= \frac{a^2}{4} + \frac{1}{4}\left((1 + a\beta_0)\frac{1 - 4a^2\lambda c_-^2}{\sqrt{\delta_-}} + (1 - a\beta_0 r)\frac{1 - 4a^2\lambda c_+^2}{\sqrt{\delta_+}}\right)\\
&\quad + a^2\lambda H\left(\frac{(1 + a\beta_0)c_-}{\sqrt{\delta_-}} - \frac{(1 - a\beta_0)c_+}{\sqrt{\delta_+}}\right) - \frac{4c^2\lambda}{1 - a^2{\beta_0}^2}\frac{1 + a^2\lambda H^2}{\sqrt{\delta_-}\sqrt{\delta_+}},\\
P &= \big(1 + a^2\lambda\rho^2\big)\left(1 - 2c\lambda\frac{H + 2c_+}{\sqrt{\delta_+}} + 2c\lambda\frac{H - 2c_-}{\sqrt{\delta_-}}\right)^2\\
&\quad + 4c^2\lambda^2 T^2\left(-a\alpha_0 - \frac{1 + 2a^2\lambda c_- H}{\sqrt{\delta_-}} + \frac{1 - 2a^2\lambda c_+ H}{\sqrt{\delta_+}}\right)^2,\\
Q &= -\big(1 + a^2\lambda\rho^2\big)\left(2\beta(1 + 2c\lambda) + \big(1 + \beta(1 + 4c\lambda)\big)\frac{H - 2c_-}{\sqrt{\delta_-}}\right.\\
&\qquad \left. + \big(1 - \beta(1 + 4c\lambda)\big)\frac{H + 2c_+}{\sqrt{\delta_+}} - 4c\beta\lambda\frac{(H - 2c_-)(H + 2c_+)}{\sqrt{\delta_-}\sqrt{\delta_+}}\right)\\
&\quad - 2ca\lambda T^2\left(a^2\alpha_0 - 2\big(1 - 2c{\alpha_0}^2\big)\beta a\lambda\right.\\
&\qquad + \big(a + \alpha_0 + \alpha_0\beta(1 + 4c\lambda)\big)\frac{1 + 2c_- a^2\lambda H}{\sqrt{\delta_-}}\\
&\qquad - \big(a - \alpha_0 + \alpha_0\beta(1 + 4c\lambda)\big)\frac{1 - 2c_+ a^2\lambda H}{\sqrt{\delta_+}}\\
&\qquad \left. - \frac{2\beta}{a}\frac{(1 + 2c_- a^2\lambda H)(1 - 2c_+ a^2\lambda H)}{\sqrt{\delta_-}\sqrt{\delta_+}}\right).
\end{aligned} \tag{5.4}$$

The isometry group $U(1) \times U(1)$ acts by translations on ϕ and α.

5.2. Second set of coordinates

In order to verify that $\mathfrak{g}$ is self-dual Einstein (see Section 5.4), it is more convenient to use coordinates s and x defined by

$$T = s\sqrt{1 - x^2}, \qquad H = sx. \tag{5.5}$$

We then get for the metric the expression

$$4D^2\mathfrak{g} = \frac{P}{A}\left(d\phi + \frac{Q}{4\,P}\,d\alpha\right)^2$$

$$+A\left(\frac{ds^2}{1+a^2\lambda s^2}+\frac{s^2\,dx^2}{1-x^2}+\frac{s^2(1+a^2\lambda s^2)(1-x^2)}{P}\,d\alpha^2\right). \tag{5.6}$$

The functions A, P, Q and D are still the same as in (5.4), up to the substitution (5.5), and the functions $\delta_\pm$ can be written as

$$\delta_\pm=\frac{1}{a^2\lambda}\big[\big(1+4a^2\lambda c_\pm^2\big)\big(1+a^2\lambda s^2\big)-\big(1\mp 2a^2\lambda c_\pm sx\big)^2\big].$$

5.3. Third set of coordinates ($\alpha_0=\beta_0=0$ case)

In the limit $\alpha_0\to 0$ and $\beta_0\to 0$, the metric $\mathfrak{g}$ reduces to the quaternionic extension of the double Taub-NUT metric given in [35]. For this particular case one can get rid of the square roots by switching to the spheroidal coordinates (u,θ),

$$T=\frac{\sqrt{u^2-4c^2}}{\sqrt{1+4a^2\lambda c^2}}\sin\theta,\qquad H=u\cos\theta. \tag{5.7}$$

In these coordinates:

$$\sqrt{\delta_\pm}=u\pm 2c\cos\theta.$$

It is convenient to scale the angles ϕ and α according to

$$\hat\phi=\frac{\phi}{1+4a^2\lambda c^2},\qquad \hat\alpha=\frac{\alpha}{1+4a^2\lambda c^2}.$$

Then the metric at $\alpha_0=\beta_0=0$ becomes

$$\begin{aligned}4D^2\mathfrak{g}&=\big(1+a^2\lambda u^2\big)\frac{\widehat{P}}{\widehat{A}}\left(d\hat\phi+\frac{\widehat{Q}}{4\widehat{P}}\,d\hat\alpha\right)^2\\&\quad+\widehat{A}\left(\widehat{\mathfrak{g}}_0+\frac{(u^2-4c^2)(1+4a^2\lambda c^2\cos^2\theta)}{\widehat{P}}\sin^2\theta\,d\hat\alpha^2\right),\end{aligned} \tag{5.8}$$

where

$$\widehat{\mathfrak{g}}_0=\big(u^2-4c^2\cos^2\theta\big)\mathfrak{g}_0=\frac{du^2}{(u^2-4c^2)(1+a^2\lambda u^2)}+\frac{d\theta^2}{1+4a^2\lambda c^2\cos^2\theta}$$

and

$$\begin{aligned}4\widehat{A}&=4\big(u^2-4c^2\cos^2\theta\big)A=\big(2+a^2u\big)\big(u-8c^2\lambda\big)-4a^2c^2D^2\cos^2\theta,\\ \widehat{P}&=\frac{(1+4a^2\lambda c^2)(u^2-4c^2\cos^2\theta)}{1+a^2\lambda u^2}P\\&=4c^2\sin^2\theta\big(1+4a^2\lambda c^2\cos^2\theta\big)D^2\\&\quad+\big(u^2-4c^2\big)\big(1+4a^2\lambda c^2\cos^2\theta-16\lambda^2c^2\sin^2\theta\big),\\ \widehat{Q}&=\frac{(1+4a^2\lambda c^2)(u^2-4c^2\cos^2\theta)}{1+a^2\lambda u^2}Q=-2\big(u^2-4c^2\big)\big(1+4a^2\lambda c^2\big)\cos\theta,\\ D&=1-2\lambda u.\end{aligned} \tag{5.9}$$

5.4. Einstein and self-dual Weyl properties of the metric

A four-dimensional QK metric is nothing but an Einstein metric with self-dual Weyl tensor. This property should be inherent to the metric $\mathfrak{g}$ given by (5.2), since we started from the generic HSS action for QK sigma models. However, checking these properties explicitly is a good test of the correctness of our computations.

We first consider the particular case $\alpha_0 = \beta_0 = 0$ because the use of the spheroidal-like coordinates (5.7) greatly simplifies the metric as can be seen from relations (5.8) and (5.9). Despite these simplifications, intensive use of Mathematica was needed to compute the spin connection, the anti-self-dual curvature R_i^- and to check the crucial relation (see Appendix A for the notation):

$$R_i^- = -16\lambda\, \Xi_i^-.$$

It simultaneously establishes that the metric is indeed self-dual Einstein, with

$$Ric(g) = \Lambda\, g, \qquad \frac{\Lambda}{3} = -16\lambda, \qquad W_i^- = 0.$$

For non-vanishing α_0 or β_0, such a check is no longer feasible because of the strong increase in complexity of various functions appearing in the metric. Moreover, in this case we failed to find any proper generalization of the spheroidal-like coordinates (5.7) which would allow us to get rid of the square roots $\sqrt{\delta_+}$ and $\sqrt{\delta_-}$.

In order to by-pass these difficulties we have used an approach due to Przanowski [36] and Tod [34], which reduces the verification of the self-dual Einstein property to simpler checks. We shall begin with a description of their construction.

One starts from an Einstein metric g (more precisely, $Ric(g) = \Lambda g$). Furthermore it will be supposed that this metric has (at least) one Killing vector with the associated 1-form $K = K_\mu\, dx^\mu$. Differentiating K gives

$$dK = dK_i^+ \Xi_i^+ + dK_i^- \Xi_i^-, \qquad \Xi_i^\pm = e_0 \wedge e_i \pm \frac{1}{2}\epsilon_{ijk} e_j \wedge e_k,$$

for some vierbein of the metric g. We can extract, from dK, an integrable complex structure $\mathcal{I}$ and a coordinate w according to

$$\mathcal{I} = \frac{dK_i^-}{\sqrt{\sum_i (dK_i^-)^2}}\, \Xi_i^-, \qquad w = -\frac{\Lambda}{3\sqrt{\sum_i (dK_i^-)^2}}. \tag{5.10}$$

Using these elements one can formulate

Proposition 1 ([34,36]). *There exist real coordinates w, ν and μ such that any Einstein metric g with self-dual Weyl tensor and a Killing vector ∂_ϕ can be written as*

$$g = \frac{1}{w^2}\left[\frac{1}{\mathcal{W}}(d\phi + \Theta)^2 + \mathcal{W}\big(e^\nu\big(d\nu^2 + d\mu^2\big) + dw^2\big)\right]. \tag{5.11}$$

This metric will be self-dual Einstein iff

(a) $\quad -2\dfrac{\Lambda}{3}\mathcal{W} = 2 - w\partial_w \nu,$

$$\text{(b)} \quad \big(\partial_\nu^2 + \partial_\mu^2\big)\nu + \partial_w^2\big(e^\nu\big) = 0,$$
$$\text{(c)} \quad -d\Theta = \partial_\nu \mathcal{W} d\mu \wedge dw + \partial_\mu \mathcal{W} dw \wedge d\nu + \partial_w\big(\mathcal{W} e^\nu\big) d\nu \wedge d\mu. \tag{5.12}$$

The following remarks are in order:

1. The relation (5.12b) is the celebrated continuous Toda equation.
2. Except for this Toda equation, the checks of the self-dual Einstein property are reduced to solving first-order partial differential equations.
3. Relation (5.11) shows that any self-dual Einstein metric with at least one Killing is conformal to a subclass of Kähler scalar-flat metrics (see Section 6.1 for the proof).

Let us now use this approach to analyze our metric (5.6) in the (s, x) coordinates and to check whether it obeys the conditions (5.12).

We take for vierbein

$$e_0 = \frac{1}{\sqrt{W}}\big(d\phi + \Theta\big), \qquad e_1 = \frac{\sqrt{A}}{2D} \frac{ds}{\sqrt{1 + a^2\lambda s^2}},$$
$$e_2 = \frac{\sqrt{A}}{2D} \frac{s\,dx}{\sqrt{1 - x^2}}, \qquad e_3 = \frac{\sqrt{W}}{4D^2} s\sqrt{1 + a^2\lambda s^2}\sqrt{1 - x^2}\,d\alpha,$$
$$W = \frac{4D^2 A}{P},$$

and consider the Killing ∂_ϕ, with the 1-form

$$K = \frac{1}{W}(d\phi + \Theta) = \frac{1}{\sqrt{W}} e_0.$$

The computation of $\sum_i (dK_i^-)^2$ eventually leads to the identification

$$w = -\frac{\Lambda}{3} \frac{D}{4\lambda\sqrt{\delta(\hat{c})}}, \tag{5.13}$$

where

$$\delta(\hat{c}) = \frac{1}{a^2\lambda}\Big[\big(1 + 4a^2\lambda\hat{c}^2\big)\big(1 + a^2\lambda s^2\big) - \big(1 - 2a^2\lambda\hat{c}sx\big)^2\Big],$$
$$2\hat{c} = \big(1 - a\beta_0\big)c_+ - \big(1 + a\beta_0\big)c_- = 2c\frac{\alpha_0}{a}. \tag{5.14}$$

Then, comparing the metric $\mathfrak{g}$ in the form (5.6) with (5.11), we express the quantities $\mathcal{W}$, μ and e^ν entering (5.11) in terms of ours

$$\mathcal{W} = \frac{W}{w^2}, \qquad \mu = \alpha, \qquad e^\nu = \frac{s^2\big(1 + a^2\lambda s^2\big)\big(1 - x^2\big)w^4}{16D^4}. \tag{5.15}$$

Simultaneously, we obtain the expressions for the partial derivatives of ν

$$\partial_x \nu = -\frac{4D^2\partial_s w}{(1 - x^2)w^2}, \qquad \partial_s \nu = \frac{4D^2\partial_x w}{s^2(1 + a^2\lambda s^2)w^2}. \tag{5.16}$$

Two expressions for the mixed derivative $\partial_s \partial_x \nu$ coincide as a consequence of the relation:

$$s^2\big(1+a^2\lambda s^2\big)\partial_s^2 D + \big(1-x^2\big)\partial_x^2 D = 0. \tag{5.17}$$

Checking the relation (5.12a) suggests the identification

$$\frac{\Lambda}{3} = -16\lambda. \tag{5.18}$$

Then the remaining equations (b) and (c) in (5.12) have been explicitly checked using Mathematica, and shown to be valid. This proves that our general metric (5.6) is self-dual Einstein.

5.5. Limiting cases

The hyper-Kähler limit Using the coordinates H and T (defined in (5.1)), in the limit $\lambda \to 0$, the metric (5.2) can be written as the multicentre structure

$$4\mathfrak{g}(\lambda \to 0) = \frac{1}{V}(d\Phi + \mathcal{A})^2 + V\mathfrak{g}_0(\lambda \to 0),$$

with the flat 3-metric and the angle Φ defined by

$$\mathfrak{g}_0(\lambda \to 0) = dH^2 + dT^2 + T^2\,d\alpha^2, \qquad \Phi = \phi - \frac{a\beta_0}{2}\alpha.$$

The potential V and the connection $\mathcal{A}$ are, respectively,

$$V = \frac{1}{4}\left(a^2 + \frac{1+a\beta_0}{\sqrt{\delta_-}} + \frac{1-a\beta_0}{\sqrt{\delta_+}}\right), \tag{5.19}$$

$$\mathcal{A} = -\frac{1}{4}\left((1+a\beta_0)\frac{H-2c_-}{\sqrt{\delta_-}} + (1-a\beta_0)\frac{H+2c_+}{\sqrt{\delta_+}}\right)d\alpha, \tag{5.20}$$

with

$$\delta_\pm = (H \pm 2c_\pm)^2 + T^2, \quad c_\pm = \frac{c}{1 \mp a\beta_0}.$$

Since α_0 is an irrelevant parameter in the limit $\lambda \to 0$ (it can be removed from the metric by a shift of H), we put it equal to zero from the very beginning.

The potential shows two centres at $T = 0$, $H = \mp 2c_\pm$ with different masses $1 \mp a\beta_0/4$ and $V(\infty) = a^2/4$. An easy computation gives the fundamental multicentre relation

$$dV = -\underset{\mathfrak{g}_0(\lambda\to 0)}{\star} d\mathcal{A}.$$

For $a \neq 0$, $\beta_0 = 0$, we have the double Taub-NUT metric; for $a \neq 0$, $c = 0$ and $a \neq 0$, $a\beta_0 = \pm 1$, ($c_\pm$ finite), we have the Taub-NUT metric; for $a = 0$, we have the Eguchi–Hanson metric.

The quaternionic Taub-NUT limit

In order to show that in the limit $c \to 0$ we recover the quaternionic Taub-NUT metric, we switch to new coordinates $(\hat{s}, \hat{\theta})$ defined by

$$H = \frac{2}{a}\frac{\hat{s}\cos\hat{\theta}}{1-\lambda\hat{s}^2}, \qquad T = \frac{2}{a}\frac{\hat{s}\sin\hat{\theta}}{1-\lambda\hat{s}^2}.$$

The metric $\mathfrak{g}$ coincides, up to a constant factor $\frac{1}{2a}$, with the metric given by relation (5.4) in [22]:

$$2a\mathfrak{g}(c \to 0) = \frac{1}{2}\left(\frac{\widehat{B}}{\hat{s}\widehat{C}^2}\,d\hat{s}^2 + \frac{\hat{s}\widehat{B}}{\widehat{C}^2}\left(\sigma_1^2+\sigma_2^2\right) + \frac{\hat{s}\widehat{A}^2}{\widehat{B}\widehat{C}^2}\sigma_3^2\right),$$

where

$$\widehat{A} = 1 - \widehat{R}\hat{\lambda}^2\hat{s}^2, \qquad \widehat{B} = 1 + \hat{\lambda}^2\hat{s}(4+\widehat{R}\hat{s}), \qquad \widehat{C} = 1 + \widehat{R}\hat{s} + \widehat{R}\hat{\lambda}^2\hat{s}^2,$$

and

$$\begin{aligned} &\sigma_1^2+\sigma_2^2 = d\hat{\theta}^2 + \sin^2\hat{\theta}\,d\alpha^2,\\ &\sigma_3 = (-2d\phi + a\beta_0\,d\alpha) + \cos\hat{\theta}\,d\alpha,\\ &\widehat{R} = -4\frac{\lambda}{a}, \qquad \hat{\lambda}^2 = \frac{a}{4}. \end{aligned} \tag{5.21}$$

Various limits of the quaternionic Taub-NUT metric can be found in [22]. Let us just remark here that in the limit $\widehat{R} \to 0$ we once again recover the standard Taub-NUT metric.

The quaternionic Eguchi–Hanson limit

In the limit $a \to 0$ with $a\beta_0 = \tilde{\beta}_0 \neq 0$, it is more convenient to study the metric in coordinates in which the square roots disappear. Thus, we define the coordinates $\tilde{s}$ and $\tilde{\theta}$ by

$$T = \frac{2}{1-\tilde{\beta}_0^2}\sqrt{\tilde{s}^2-c^2}\,\sin\tilde{\theta}, \qquad H = \frac{2}{1-\tilde{\beta}_0^2}\tilde{s}\cos\tilde{\theta} + c_- - c_+,$$

so that

$$\sqrt{\delta_\pm} = \frac{2}{1-\tilde{\beta}_0^2}(\tilde{s} \pm c\cos\tilde{\theta}).$$

The metric can now be expressed as

$$4(1-\tilde{\beta}_0^2)\widetilde{C}^2\mathfrak{g}(a \to 0, \tilde{\beta}_0) = \frac{\tilde{s}^2-c^2}{\tilde{s}\widetilde{B}}\mathcal{G}^2 + \tilde{s}\widetilde{B}\left(\frac{d\tilde{s}^2}{\tilde{s}^2-c^2} + d\tilde{\theta}^2 + \sin^2\tilde{\theta}\mathcal{H}^2\right),$$

with

$$\widetilde{C} = 1 - \frac{\kappa^2}{1-\tilde{\beta}_0^2}(\tilde{s} - c\tilde{\beta}_0\cos\tilde{\theta}), \qquad \mathcal{G} = -(1+\beta\cos\tilde{\theta})\,d\alpha + 2\cos\tilde{\theta}\,d\phi,$$

$$\tilde{s}\widetilde{B} = \tilde{s} - \kappa^2c^2 + c\tilde{\beta}_0\cos\tilde{\theta}, \qquad \mathcal{H} = \frac{1}{\tilde{s}\widetilde{B}}\left[-(\tilde{s}-c)\beta\,d\alpha + 2\left(\tilde{s}-\kappa^2c^2\right)d\phi\right],$$

where

$$\kappa^2 = 4\lambda, \qquad \beta = \frac{\tilde{\beta}_0}{1 - 4c\lambda}.$$

One can see that in the limit $a \to 0$, the parameter α_0 fully drops out from the metric. If we now take the limit $\tilde{\beta}_0 = a\beta_0 \to 0$, we reproduce the quaternionic Eguchi–Hanson metric derived in [23] (see Eq. (4.7) of this reference):

$$4\widetilde{C}^2\mathfrak{g}(a \to 0) = \frac{\tilde{s}^2 - c^2}{\tilde{s}\widetilde{B}}\tilde{\sigma}_3^2 + \tilde{s}\widetilde{B}\left(\frac{d\tilde{s}^2}{\tilde{s}^2 - c^2} + \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2\right),$$

with

$$\widetilde{C} = 1 - \kappa^2\tilde{s}, \qquad \tilde{\sigma}_3 = (-d\alpha) + \cos\tilde{\theta}(2\,d\phi),$$
$$\tilde{s}\widetilde{B} = \tilde{s} - \kappa^2 c^2, \qquad \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 = d\tilde{\theta} + \sin^2\tilde{\theta}(2\,d\phi)^2.$$

In conclusion, let us point out that, whereas the parameters a, c and β_0 have a counterpart in the HK limit, this is not the case for the parameter α_0. This distinguished parameter is specific just for the QK metrics.

6. Connection with the literature

Metrics with self-dual Weyl tensor may appear as:

1. Kähler scalar-flat metrics;
2. self-dual Einstein metrics (considered in this work);
3. metrics in the system of coupled Einstein–Maxwell fields.

In order to exhibit the relationships between these classes and to find out how our metrics correlate with them, let us begin with the description, due to LeBrun, of the Kähler scalar-flat metrics with one Killing vector.

6.1. Kähler scalar-flat metrics in LeBrun setting

These metrics, with self-dual Weyl tensor, have received attention in [41]. There, it was proved that any such metric, with at least one Killing vector $K = \partial_t$, can be written as

$$g = \frac{1}{\mathcal{W}}(dt + \widetilde{\Theta})^2 + \mathcal{W}\big[dw^2 + e^v\big(dv^2 + d\mu^2\big)\big] = \sum_{A=0}^{3} e_A^2, \tag{6.1}$$

where the functions v and $\mathcal{W}$ must be solutions of the following equations

$$\big(\partial_\nu^2 + \partial_\mu^2\big)v + \partial_w^2\big(e^v\big) = 0, \qquad \big(\partial_\nu^2 + \partial_\mu^2\big)\mathcal{W} + \partial_w^2\big(\mathcal{W}e^v\big) = 0. \tag{6.2}$$

The connection one-form $\widetilde{\Theta}$ is then obtained from

$$d\widetilde{\Theta} = \partial_\nu(\mathcal{W})\,d\mu \wedge dw + \partial_\mu(\mathcal{W})\,dw \wedge d\nu + \partial_w(\mathcal{W}e^v)\,d\nu \wedge d\mu. \tag{6.3}$$

The vierbein, defined in relation (6.1), is taken to be

$$e_0 = \frac{dt + \widetilde{\Theta}}{\sqrt{\mathcal{W}}}, \qquad e_1 = \sqrt{\mathcal{W}e^v}\, dv, \qquad e_2 = \sqrt{\mathcal{W}e^v}\, d\mu, \qquad e_3 = \sqrt{\mathcal{W}}\, dw.$$

In terms of the self-dual two-forms $\Xi_i^{\pm} = e_0 \wedge e_i \pm \frac{1}{2}\epsilon_{ijk} e_j \wedge e_k$ the Kähler form is anti-self-dual,

$$\Omega = dw \wedge (dt + \widetilde{\Theta}) + \mathcal{W}e^v dv \wedge d\mu = -\Xi_3^-, \tag{6.4}$$

while the Ricci form is self-dual,

$$2\hat{\rho} = \frac{1}{\sqrt{e^v}}\partial_v\left(\frac{\partial_w v}{\mathcal{W}}\right) \cdot \Xi_1^+ + \frac{1}{\sqrt{e^v}}\partial_\mu\left(\frac{\partial_w v}{\mathcal{W}}\right) \cdot \Xi_2^+ + \partial_w\left(\frac{\partial_w v}{\mathcal{W}}\right) \cdot \Xi_3^+. \tag{6.5}$$

Now, comparing (5.11) and (6.1), we observe that any self-dual Einstein metric with at least one Killing, in particular the metric (5.2), is conformally related to a subclass of Kähler scalar-flat metrics, with the identifications:

$$\widetilde{\Theta} = -\Theta, \qquad dt = -d\phi, \qquad d\mu = d\alpha, \qquad g = w^2 \mathfrak{g}.$$

In [41], a large class of explicit solutions of (6.2) was obtained. Taking

$$q = \sqrt{2w}, \qquad e^v = q^2, \qquad V = q^2 W, \qquad \gamma = \frac{dv^2 + d\mu^2 + dq^2}{q^2},$$

where γ is the hyperbolic 3-space, these metrics have the form

$$q^2\left[\frac{1}{V}(dt + \Theta)^2 + V\gamma\right], \tag{6.6}$$

where V is some real harmonic function on γ.

LeBrun obtained the potential V as a sum of monopoles in this hyperbolic space. In the limit where the hyperbolic space becomes flat, one recovers the multicentre metrics. However, the possibility that these metrics could be conformally Einstein has been ruled out by Pedersen and Tod in [42]. Therefore the metrics (6.6) bear no relation to our metric (5.2).

6.2. Flaherty's equivalence

Let us now examine Flaherty's equivalence relating Kähler scalar-flat metrics and self-dual metrics solving the coupled Einstein–Maxwell field equations.

In [40] Flaherty has proved:

Proposition 2. *The following two classes of metrics are equivalent*:

1. *Any Kähler scalar-flat metric.*
2. *Any metric which is a solution of the coupled Einstein–Maxwell equations*

$$Ric_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho} g^{\rho\sigma} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right),$$
$$dF^- = 0, \qquad dF^+ = 0, \tag{6.7}$$

with self-dual Weyl tensor ($W^- = 0$).

In this equivalence the self-dual parts of the Maxwell field strength are given by

$$F^- \propto \Omega, \qquad F^+ \propto \hat{\rho},$$

where Ω denotes the Kähler form and $\hat{\rho}$ the Ricci form of the Kähler metric.

In the euclidean case, this equivalence can be easily checked for metrics with at least one Killing vector, using the LeBrun framework. One can check Eqs. (6.7) and find the self-dual parts of the field strength two-forms:

$$F^- = -\frac{m}{2}\Omega, \qquad F^+ = \frac{2}{m}\hat{\rho}, \tag{6.8}$$

where m is an arbitrary real parameter.

This equivalence and the property that any self-dual Einstein metric with one Killing is conformal to some Kähler scalar-flat metric suggest that the Weyl-self-dual metrics which solve the Einstein–Maxwell system may hide, up to some conformal factor, a self-dual Einstein metric. Let us now examine two known classes of the metrics giving solution of the Einstein–Maxwell system (in general, they are not Weyl-self-dual) in order to see whether the metric (5.2) is conformally related to any of them. We shall find that the answer is negative in both cases. This means that (5.2) determines a new explicit solution of the Einstein–Maxwell system, with the conformal factor w given in (5.13).

6.3. The metrics of Perjès–Israel–Wilson

These metrics are solutions of the Einstein–Maxwell field equations. They were derived independently, for the minkowskian signature, by Perjès [37] and Israel and Wilson [38]. Their continuation to the euclidean signature was given by Yuille [48] and Whitt [49] who discussed their global properties and their possible applications in the path integral approach to quantum gravity.

These metrics have at least one Killing vector ∂_t. Their local form is given by

$$g = \frac{1}{V}(dt + \mathcal{A})^2 + V\gamma_0, \qquad V = U\widetilde{U}, \qquad \gamma_0 = d\vec{x}\cdot d\vec{x}. \tag{6.9}$$

The real functions U and $\widetilde{U}$ must be harmonic

$$\Delta U = \Delta\widetilde{U} = 0, \tag{6.10}$$

and the connection one-form $\mathcal{A}$ is constrained by

$$\underset{\gamma_0}{\star}\, d\mathcal{A} = \widetilde{U}\, dU - U\, d\widetilde{U}. \tag{6.11}$$

The star and laplacian are taken with respect to the three-dimensional flat space with cartesian coordinates $\vec{x}$. Clearly, when U or $\widetilde{U}$ are constant we come back to the multicentre metrics.

In order to check the previous assertions, let us define the vierbein e_A by

$$e_0 = \frac{1}{\sqrt{V}}(dt + \mathcal{A}), \qquad e_i = \sqrt{V}\, dx_i, \quad i = 1, 2, 3.$$

It is an easy task to compute the matrices A, B and C giving the curvature (see Appendix A for the definitions and notation). One finds, upon using the relations (6.10), (6.11), the simple expressions

$$A_{ij} = \frac{1}{V}\left[\frac{\partial^2_{ij}U}{U} - 3\frac{\partial_i U \partial_j U}{U^2} + \delta_{ij}\frac{(\partial_l U)^2}{U^2}\right],$$
$$B_{ij} = -\frac{1}{V^2}\partial_i U \partial_j \widetilde{U},$$
$$C_{ij} = \frac{1}{V}\left[\frac{\partial^2_{ij}\widetilde{U}}{\widetilde{U}} - 3\frac{\partial_i \widetilde{U} \partial_j \widetilde{U}}{\widetilde{U}^2} + \delta_{ij}\frac{(\partial_l \widetilde{U})^2}{\widetilde{U}^2}\right], \tag{6.12}$$

where the derivatives are taken with respect to the cartesian coordinates $\vec{x}$. The scalar curvature $R = 4(\mathrm{Tr}\, A)$ vanishes as it should.

The first equation in (6.7) gives for the field strength

$$F \equiv F^- + F^+ = \partial_i\big[U^{-1}\big] \cdot \Xi_i^- - \partial_i\big[\widetilde{U}^{-1}\big] \cdot \Xi_i^+.$$

Using (6.10), (6.11) one can check that these field strengths indeed obey the Maxwell equations:

$$dF^+ = dF^- = 0.$$

Let us prove the following:

Proposition 3. *The Perjès–Israel–Wilson metrics are self-dual Weyl only in the two cases*:

1. *When $\widetilde{U}$ is a constant*: *they are homothetic to the multicentre metrics.*
2. *When $\widetilde{U} = m/|\vec{x} - \vec{x}_0|$*: *they are conformal to the multicentre metrics.*

Proof. Let us impose, for instance, the condition that the Weyl tensor is self-dual (i.e., $W^- = 0$). Using (6.12) and (6.10), the corresponding constraints can be written as

$$\partial_i \partial_j\left(\frac{1}{\widetilde{U}^2}\right) - \frac{1}{3}\delta_{ij}\Delta\left(\frac{1}{\widetilde{U}^2}\right) = 0. \tag{6.13}$$

Acting on the left-hand side by ∂_i gives

$$\partial_j \Delta\left(\frac{1}{\widetilde{U}^2}\right) = 0 \quad \Longrightarrow \quad \Delta\left(\frac{1}{\widetilde{U}^2}\right) = \mathrm{const} \equiv 6B.$$

Then one can integrate relation (6.13) to

$$\left(\frac{1}{\widetilde{U}^2}\right) = A + \vec{f} \cdot \vec{x} + Br^2, \qquad r^2 = \vec{x} \cdot \vec{x},$$

where A and $\vec{f}$ are integration constants. The requirement that $\widetilde{U}$ is harmonic (Eq. (6.10)) amounts to the relation

$$\vec{f} \cdot \vec{f} = 4AB.$$

If B vanishes, the harmonic function $\widetilde{U}$ is evidently reduced to a constant which can be scaled to 1. Then the relations (6.10), (6.11) imply that the metric is homothetic to some multicentre one.

If B does not vanish, we can write $\widetilde{U} = m/|\vec{x} - \vec{x}_0|$ which can be simplified to $1/r$ by rescaling and translation of $\vec{x}$. The metric (6.9) becomes

$$g = r^2 \hat{g},$$

with

$$\hat{g} = \left[\frac{1}{\widehat{V}}(d\tau + \mathcal{A})^2 + \widehat{V}\hat{\gamma}\right], \qquad \hat{\gamma} = \frac{1}{r^4}\hat{\gamma}_0, \qquad \widehat{V} = rU.$$

Using spherical coordinates we have

$$\hat{\gamma}_0 = dr^2 + r^2\, d\Omega^2 \quad \Longrightarrow \quad \hat{\gamma} = d\rho^2 + \rho^2\, d\Omega^2, \quad \rho = 1/r,$$

thus establishing that $\hat{\gamma}$ is flat. Then relation (6.11) becomes

$$-\underset{\hat{\gamma}}{\star} d\mathcal{A} = d\widehat{V},$$

showing that $\hat{g}$ is some multicentre. This completes the proof. □

Proposition 3 tells us that the metrics of Perjès–Israel–Wilson, when they have self-dual Weyl tensor, are never conformal to Einstein metrics (with non-vanishing cosmological constant). This implies that our metric (5.2) can never be transformed to the Perjès–Israel–Wilson form.

6.4. The Plebanski–Demianski metric

In [39] Plebanski and Demianski have derived a minkowskian solution of the coupled Einstein–Maxwell field equations. Its euclidean version, obtained by complex changes of coordinates and parameters, can be written in the form

$$g_{\mathrm{PD}} = \sum_{A=0}^{3} e_A^2,$$

with the vierbein

$$e_0 = \frac{1}{1+pq}\sqrt{\frac{p^2 - q^2}{X(p)}}\, dp, \qquad e_1 = \frac{1}{1+pq}\sqrt{\frac{p^2 - q^2}{Y(q)}}\, dq,$$

$$e_2 = \frac{1}{1+pq}\sqrt{\frac{X(p)}{p^2 - q^2}}\big(d\tau + q^2\, d\sigma\big), \qquad e_3 = \frac{1}{1+pq}\sqrt{\frac{Y(q)}{p^2 - q^2}}\big(d\tau + p^2\, d\sigma\big),$$

where

$$X(p) = \left(g_0^2 - \gamma + \frac{\lambda}{6}\right) - 2lp + \epsilon p^2 - 2mp^3 - \left(e^2 + \gamma + \frac{\lambda}{6}\right)p^4,$$

$$Y(q) = \left(e^2 + \gamma - \frac{\lambda}{6}\right) - 2mq - \epsilon q^2 - 2lq^3 - \left(g_0^2 - \gamma - \frac{\lambda}{6}\right)q^4.$$

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It displays 6 real parameters besides the cosmological constant λ and possesses $U(1)\times U(1)$ isometry realized by shifts of τ and σ.

The meaning of the parameters e, g_0, l and m follows from:

Proposition 4. *The Plebanski–Demianski metrics are*

- *Einstein for* $e = g_0 = 0$.
- *Einstein with self-dual Weyl tensor* $(W^- = 0)$ *for* $e = g_0 = 0$ *and* $l = m$.
- *Einstein with anti-self-dual Weyl tensor* $(W^+ = 0)$ *for* $e = g_0 = 0$ *and* $l = -m$.

Proof. The proposition follows from the computation of the curvature matrices A, B and C defined in Appendix A.

We are going to show that our metric (5.2) lies outside the above ansatz. To this end, we shall work with an anti-self-dual Weyl tensor $(W^- = 0)$ and analyze the $\lambda \to 0$ limit of g_{PD}. We switch to the triholomorphic Killing vector ∂_ϕ by making the change of coordinates

$$d\phi = d\tau, \qquad d\alpha = d\sigma + d\tau.$$

It leads to the limiting metric

$$g_{\mathrm{PD}}(\lambda \to 0) = \frac{1}{V}(d\phi + \mathcal{A})^2 + V\gamma_0, \tag{6.14}$$

with the potential

$$V = \frac{(1+pq)^2(p^2-q^2)}{\mathcal{D}}, \quad \mathcal{D} = \left(1-q^2\right)^2 X(p) + \left(1-p^2\right)^2 Y(q), \tag{6.15}$$

the gauge field one-form

$$\mathcal{A} = \frac{q^2(1-q^2)X(p) + p^2(1-p^2)Y(q)}{\mathcal{D}}\, d\alpha, \tag{6.16}$$

and the three-dimensional metric

$$\gamma_0 = \frac{\mathcal{D}}{(1+pq)^4}\left(\frac{dp^2}{X(p)} + \frac{dq^2}{Y(q)}\right) + \frac{X(p)Y(q)}{(1+pq)^4}\, d\alpha^2. \tag{6.17}$$

One can explicitly check the relation

$$\underset{\gamma_0}{\star}\, d\mathcal{A} = \pm dV.$$

To prove that (6.14) is indeed multicentre, we define cartesian coordinates $\vec{x}$ by

$$\begin{aligned}
x &= A \sin\left[\sqrt{m^2 + \gamma(2\gamma+\epsilon)}\,\alpha\right],\\
y &= A \cos\left[\sqrt{m^2 + \gamma(2\gamma+\epsilon)}\,\alpha\right],\\
z &= B,
\end{aligned} \tag{6.18}$$

with

$$A=\frac{1}{(1+pq)^2}\sqrt{\frac{X(p)Y(q)}{m^2+\gamma(2\gamma+\epsilon)}},$$
$$B=-\frac{m(p-q)(1-pq)+\gamma(p^2+q^2)+\epsilon pq}{\sqrt{m^2+\gamma(2\gamma+\epsilon)}(1+pq)^2}.$$

One can check that these coordinates make manifest the flatness of the metric (6.17)

$$\gamma_0=(dx)^2+(dy)^2+(dz)^2.$$

For comparing (6.14) with the HK limit of our metric we need to express the potential V in terms of the coordinates (6.18).

For $m=0$, as observed in the original paper [39], the metric (6.14) is flat: this is a special case which needs a separate analysis. We have

$$V=\frac{1}{2\sqrt{\gamma(\epsilon-2\gamma)}}\frac{1}{\sqrt{x^2+y^2+Z^2}},\qquad Z=z+\frac{\epsilon}{2\sqrt{\gamma(2\gamma+\epsilon)}},$$

provided that the expressions within square roots are positive. This potential corresponds to a mass at the origin, and is known to yield a flat four-dimensional metric [32].

For $m\neq 0$, we define new parameters by

$$\cosh\phi=\frac{\epsilon-2\gamma}{4m},\quad \phi\geqslant 0,\qquad c=\frac{\sqrt{(\epsilon-2\gamma)^2-16m^2}}{4\sqrt{m^2+\gamma(2\gamma+\epsilon)}},$$

and

$$Z=z+\frac{2\gamma+\epsilon}{4\sqrt{m^2+\gamma(2\gamma+\epsilon)}},\qquad d_\pm=x^2+y^2+(Z\pm c)^2.$$

In this notation, the potential (6.15) becomes

$$V=\mu\left(\frac{\eta}{\sqrt{d_-}}+\frac{1/\eta}{\sqrt{d_+}}\right),\tag{6.19}$$

with

$$\eta^2=\frac{e^{-\phi}}{\sqrt{c^2+1}+c},\qquad \mu=\frac{\sqrt{c}}{4m(\sinh\phi)^{3/2}}.$$

The HK limit of the Plebanski–Demianski metric therefore gives an ALE generalization of the Eguchi–Hanson metric (recovered for $\eta=1$) with two different masses. It is reduced to the flat metric, up to rescaling, in the limits $\eta\to 0$ and $\eta\to\infty$.

The potential (6.19) is a particular case $a=0$, $a\beta_0\neq 0$ of our limiting HK potential (5.19):

$$V=\frac{1}{4}\left(a^2+\frac{1+a\beta_0}{\sqrt{\delta_-}}+\frac{1-a\beta_0}{\sqrt{\delta_+}}\right).$$

The conclusion is that our general metric (5.2) cannot be embedded into the Plebanski–Demianski class of self-dual Einstein metrics because their HK limits are different. □

Summarizing the discussion in Sections 6.3 and 6.4, we observe that our metric (5.2) cannot be reduced to either known class of metrics solving the Einstein–Maxwell equations. Hence, by Flaherty's equivalence, it provides (up to conformal factor (5.13)) a new family of explicit solutions of this system. For the minimal case $\alpha_0 = \beta_0 = 0$ this fact was pointed out in [35].

6.5. The linearization by Calderbank and Pedersen

Quite recently, while we were typing this article, Calderbank and Pedersen [43] have exhibited a class of self-dual Einstein metrics with two commuting (and hypersurface generating) Killing vectors. To describe their metrics, two main ingredients are needed:

1. A function $F(\rho, \eta)$ which is a solution of the linear differential equation

$$\rho^2(F_{\rho\rho} + F_{\eta\eta}) = \frac{3}{4}F. \tag{6.20}$$

   It is an eigenfunction of the laplacian in the hyperbolic plane $\mathcal{H}^2$ with metric

$$g_0(\mathcal{H}^2) = \frac{d\rho^2 + d\eta^2}{\rho^2}, \quad \rho > 0. \tag{6.21}$$

2. The set of one-forms

$$\alpha = \sqrt{\rho}\, d\alpha, \quad \beta = \frac{d\phi + \eta\, d\alpha}{\sqrt{\rho}}.$$

In terms of these, the full metric is

$$g = \frac{F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)}{4F^2} g_0(\mathcal{H}^2) + \frac{[(F - 2\rho F_\rho)\alpha - 2\rho F_\eta \beta)^2 + (2\rho F_\eta \alpha - (F + 2\rho F_\rho)\beta]^2}{F^2[F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)]}. \tag{6.22}$$

The main result of [43] is a theorem which states that these metrics with two commuting Killings are self-dual Einstein with non-vanishing scalar curvature and that *any* such metric has locally the structure given by the expression (6.22).

In order to get a deeper insight into the construction of [43], it is convenient to pass to a function G according to $F = G/\sqrt{\rho}$. The metric g becomes

$$G^2 g = \frac{1}{\mathcal{W}}(d\phi + \Theta)^2 + \mathcal{W}\gamma, \qquad \Theta = \left(\frac{GG_\eta}{G_\rho^2 + G_\eta^2} - \eta\right) d\alpha, \tag{6.23}$$

with

$$\mathcal{W} = \frac{1}{\rho}\frac{GG_\rho}{G_\rho^2 + G_\eta^2} - 1, \qquad \gamma = \rho^2\, d\alpha^2 + \left(G_\rho^2 + G_\eta^2\right)\left(d\rho^2 + d\eta^2\right).$$

Following Tod, we can now compute the anti-self-dual part of dK, where K is the 1-form associated with the Killing ∂_ϕ. After some algebra, using (6.20), we obtain

$$K = \frac{1}{G^2\mathcal{W}}(d\phi + \Theta), \qquad dK^- = -\frac{1}{G\sqrt{G_\rho^2 + G_\eta^2}}\big(G_\rho\, \Xi_1^- + G_\eta\, \Xi_2^-\big),$$

from which we conclude that in fact G is proportional to Tod's coordinate w, defined in (5.10). Taking $G = w$, relation (6.20) becomes

$$w_{\rho\rho} + w_{\eta\eta} = \frac{1}{\rho} w_\rho. \tag{6.24}$$

Using relations (6.4) and (6.5), and switching from Tod's coordinates (w, ν) to the (ρ, η) coordinates, we can obtain the Kähler form Ω and the Ricci form $\hat{\rho}$ in this setting:

$$\begin{aligned} \Omega &= -dw \wedge d\phi + \big(\eta w_\rho - \rho w_\eta\big)\, d\rho \wedge d\alpha + (\rho w_\rho + \eta w_\eta - w)\, d\eta \wedge d\alpha, \\ \hat{\rho} &= -d\left[\frac{1}{w\mathcal{W}}(d\phi + \Theta) + \frac{1}{w}(d\phi - \eta\, d\alpha)\right]. \end{aligned} \tag{6.25}$$

The Kähler form Ω is closed as a consequence of (6.24). One can check that Ω and $\hat{\rho}$ possess opposite self-dualities. In view of Flaherty's equivalence, the metrics described by the Calderbank–Pedersen ansatz are conformally related to a subclass of metrics solving the coupled Einstein–Maxwell equations. Then the two-forms (6.25) specify the field strengths of the corresponding Maxwell field (6.8).

We are now in a position to establish the precise connection between our coordinates s and x and the coordinates ρ and η in the hyperbolic plane $\mathcal{H}^2$. To this end, we have to identify the pieces which are independent of the choice of basis for the Killing vectors, i.e., the pieces involving γ. One gets the correspondence:

$$\begin{aligned} \rho &= \frac{4s}{\delta(\hat{c})}\sqrt{1 - x^2}\sqrt{1 + a^2 s^2 \lambda}, \\ \eta &= \frac{2}{\hat{c}}\left(\frac{s(s + 2\hat{c}x)}{\delta(\hat{c})} - 1\right), \end{aligned} \tag{6.26}$$

where $\delta(\hat{c})$ was defined in (5.14) and $\hat{c} = \alpha_0 c/a$. Let us notice that the coordinate η is defined up to an additive constant that can always be reabsorbed through a redefinition of the Killing ∂_ϕ. The check of Eq. (5.12a) gives $\Lambda = 3 \Leftrightarrow \lambda = -1/16$. One can then invert relations (6.26):

$$\begin{aligned} x &= \frac{|2 - c\alpha_0|\sqrt{\big(\eta - \frac{2a}{2 - c\alpha_0}\big)^2 + \rho^2} - |2 + c\alpha_0|\sqrt{\big(\eta + \frac{2a}{2 + c\alpha_0}\big)^2 + \rho^2}}{2\sqrt{(c\alpha_0\eta + 2a)^2 + c^2\alpha_0^2\rho^2}}, \\ s &= 8\frac{\sqrt{\big(\frac{c\alpha_0}{a}\eta + 2\big)^2 + \big(\frac{c\alpha_0}{a}\big)^2\rho^2}}{|2 - c\alpha_0|\sqrt{\big(\eta - \frac{2a}{2 - c\alpha_0}\big)^2 + \rho^2} + |2 + c\alpha_0|\sqrt{\big(\eta + \frac{2a}{2 + c\alpha_0}\big)^2 + \rho^2}}. \end{aligned}$$

As discussed in Section 5.5, in the analysis of the QK Eguchi–Hanson limit, for $a \to 0$ the parameter α_0 becomes irrelevant since it disappears from the metric. The above coordinate s is well defined in the limit $a \to 0$ only if we first put $\alpha_0 = 0$.

Having the explicit expressions for s, x, it is then possible to compute $w(\rho, \eta)$ which was given in (5.13) as a function of s, x:

$$w = \frac{1}{4}|2 - c\alpha_0|\sqrt{\left(\eta - \frac{2a}{2 - c\alpha_0}\right)^2 + \rho^2} + \frac{1}{4}|2 + c\alpha_0|\sqrt{\left(\eta + \frac{2a}{2 + c\alpha_0}\right)^2 + \rho^2} + \frac{|c|}{8}\sqrt{\left(\eta - \frac{2}{c}(1 - a\beta_0)\right)^2 + \rho^2} + \frac{|c|}{8}\sqrt{\left(\eta + \frac{2}{c}(1 + a\beta_0)\right)^2 + \rho^2}.$$

It is easy to check that $w(\rho, \eta)$ satisfies Eq. (6.24).

Let us finally give $w(\rho, \eta)$ in the two interesting cases $a \to 0$ (QK-EH) and $c \to 0$ (QK-TN):

$$w_{\text{QK-EH}}(\rho, \eta) = \sqrt{\eta^2 + \rho^2} + \frac{|c|}{8}\sqrt{\left(\eta - \frac{2}{c}\right)^2 + \rho^2} + \frac{|c|}{8}\sqrt{\left(\eta + \frac{2}{c}\right)^2 + \rho^2},$$

$$w_{\text{QK-TN}}(\rho, \eta) = \frac{1}{2} + \frac{1}{2}\sqrt{(\eta - a)^2 + \rho^2} + \frac{1}{2}\sqrt{(\eta + a)^2 + \rho^2}.$$

Using these relations we can, e.g., compute the forms Ω and $\hat{\rho}$ (6.25) for our metrics and, via the correspondence (6.8), to find the relevant Maxwell field strengths.

7. Conclusions

In this paper, proceeding from the general HSS formulation of QK sigma models, we have constructed a wide class of $U(1) \times U(1)$ 4-dimensional QK metrics extending most general two-centre HK metrics. These QK metrics supply, via Flaherty's equivalence [40], a new family of explicit solutions of the coupled Einstein–Maxwell equations. We have given the precise embedding of our metrics in the framework of general $U(1) \times U(1)$ ansatz of Calderbank and Pedersen [43].

The HSS approach gives QK metrics in the form which admits a transparent interpretation of the involved parameters as the symmetry breaking ones and possesses a clear hyper-Kähler limit, with the Einstein constant as a contraction parameter. However, despite these attractive features, it does not immediately provide the natural coordinates best suited to display the final linearization of the self-dual Einstein equations along the line of Ref. [43]. It would be interesting to explore what the choice of such coordinates means in the language of the original hypermultiplet superfields parametrizing the general HSS action of QK sigma models. One more obvious direction of further study could consist in applying our HSS methods for explicit construction of higher-dimensional QK metrics generalizing, e.g., the HK metrics constructed in [14].

One of possible physical applications of the QK metrics presented here is to utilize them in the context of gauged five-dimensional supergravities. The latter seemingly provide an appropriate framework for supersymmetric extensions of the famous Randall–Sundrum scenario (for a recent review, see [50]). The presence of matter hypermultiplets seems necessary for the existence of such (smooth) extensions (see, e.g., [51]). To analyse various possibilities, it is important to know the structure of the scalar potential which is obtained by gauging isometries of the QK manifold parametrized by the hypermultiplets. Until now, in the actual computations (e.g., in [51,52]) there was mainly used the so-called universal hypermultiplet [1] corresponding to the homogeneous QK manifold $SU(2,1)/U(2)$. It would be tempting to study models with non-homogeneous QK manifolds possessing isometries and, in particular, with those considered here. It is straightforward to gauge the $U(1) \times U(1)$ isometries of our HSS actions following the general recipe of Ref. [20] (in order to generate scalar potentials, the relevant gauge supermultiplets should be propagating, in contrast to the non-propagating ones employed in the HSS quotient). The $SU(2,1)/U(2)$ QK manifold is a special case [23] of the QK Eguchi–Hanson limit of our $U(1) \times U(1)$ class of QK manifolds, so the scalar potentials associated with our metrics and inheriting all free parameters of the latter may offer new possibilities as compared to the case of universal hypermultiplet.

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Appendix A. Definitions and notation

For a given metric g, the vierbein $e_a,\ a = 0,1,2,3$, is such that

$$g = \sum_a e_a^2.$$

The spin connection ω_{ab} is defined by

$$de_a + \omega_{ab} \wedge e_b = 0, \qquad \omega_{ab} = -\omega_{ba},$$

with self-dual components

$$\omega_i^{\pm} = \omega_{0i} \pm \frac{1}{2}\epsilon_{ijk}\omega_{jk},$$

and similarly for the curvature

$$R_{ab} = d\omega_{ab} + \omega_{as} \wedge \omega_{sb} = \frac{1}{2} R_{ab,st} e_s \wedge e_t \quad \rightarrow \quad R_i^{\pm} = R_{0i} \pm \frac{1}{2}\epsilon_{ijk} R_{jk}.$$

We take for the Ricci tensor and scalar curvature

$$Ric_{ab} = R_{as,bs}, \qquad R = Ric_{ss}.$$

It is useful to define the two-forms of definite self-duality by

$$\Xi_i^{\pm} = e_0 \wedge e_i \pm \frac{1}{2}\epsilon_{ijk} e_j \wedge e_k.$$

Using this basis, the curvature and Ricci tensor are encoded in the three matrices A, B and C such that

$$R_i^+ = A_{ij}\,\Xi_j^+ + B_{ij}\,\Xi_j^-, \qquad R_i^- = B^t_{ij}\,\Xi_j^+ + C_{ij}\,\Xi_j^-,$$

where the matrices A and C are symmetric.

The Ricci components in the vierbein basis are

$$Ric_{00} = \mathrm{Tr}(A+B), \qquad Ric_{0i} = -\frac{1}{2}\epsilon_{ijk}\big(B_{jk} - B^t_{jk}\big),$$
$$Ric_{ij} = \mathrm{Tr}(A-B)\delta_{ij} + B_{ij} + B^t_{ij},$$

and the scalar curvature is

$$R = 4(\mathrm{Tr}\,A) = 4(\mathrm{Tr}\,C).$$

The Einstein condition $Ric_{ab} = \Lambda\delta_{ab}$ is seen to be equivalent to the vanishing of the matrix B and we have $\mathrm{Tr}\,C = \mathrm{Tr}\,A = \Lambda$.

One further defines the Weyl tensor

$$W_{ab,cd} = R_{ab,cd} + \frac{R}{6}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})$$
$$- \frac{1}{2}(\delta_{ac}Ric_{bd} - \delta_{ad}Ric_{bc} + \delta_{bd}Ric_{ac} - \delta_{bc}Ric_{ad}).$$

The corresponding two-forms

$$W_{ab} = \frac{1}{2}W_{ab,cd}\,e_c \wedge e_d,$$

and their self-dual parts are given by

$$W_i^+ \equiv W_{0i} + \frac{1}{2}\epsilon_{ijk}W_{jk} = W^+_{ij}\,\Xi_j^+, \qquad W^+_{ij} = A_{ij} - \frac{1}{3}(\mathrm{Tr}\,A)\delta_{ij},$$
$$W_i^- \equiv W_{0i} - \frac{1}{2}\epsilon_{ijk}W_{jk} = W^-_{ij}\,\Xi_j^-, \qquad W^-_{ij} = C_{ij} - \frac{1}{3}(\mathrm{Tr}\,C)\delta_{ij}.$$

We conclude that for an Einstein space with self-dual Weyl tensor (i.e., $W_i^- = 0$) we should have

$$C_{ij} = \frac{\Lambda}{3}\delta_{ij} \quad\Longleftrightarrow\quad R_i^- = \frac{\Lambda}{3}\Xi_i^-.$$

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Résumé :

Dans la première partie de la thèse, nous étudions le problème de l'équivalence quantique de modèles sigma reliés entre eux par la T-dualité non-abelienne.

Nous prouvons que la renormalisabilité à une boucle de divers modèles initiaux implique la renormalisabilité à une boucle de leur partenaire dualisé, et qu'ils partagent les mêmes fonctions β. Ceci est fait pour tous les modèles sigma principaux $(G_L \times G_R)/G_D$, quelle que soit la brisure de G_R, ainsi que pour la large classe de métriques à quatre dimensions, inhomogènes et d'isométrie $SU(2) \times U(1)$.

Pour l'exemple simple du modèle sigma T-dualisé $SU(2)$, dont la non-renormalisabilité à deux boucles a été démontrée dans le schéma dimensionel minimal, nous prouvons qu'il est encore possible, à cet ordre, de définir une théorie quantique correcte en modifiant, à l'ordre $\hbar$, la métrique de l'espace cible de façon finie.

Dans la seconde partie, nous construisons de façon explicite, grâce au superespace harmonique et à l'approche du quotient quaternionique, une extension quaternion-Kähler de la métrique hyper-Kähler à deux centres la plus générale. Elle possède le groupe d'isométrie $U(1) \times U(1)$ et contient comme cas particuliers les extensions quaternion-Kähler des métriques de Taub-NUT et d'Eguchi-Hanson. Elle fait aussi apparaître un paramètre supplémentaire qui disparaît dans la limite hyper-Kähler.

Summary :

In the first part of the thesis, we adress ourselves the question of the quantum equivalence of non abelian T-dualised σ-models.

We prove that the one-loop renormalisability of initial σ-models does imply the one-loop renormalisability of their dualised partner, and that they share the same β functions. This is done for any principal σ-models defined on a group manifold $(G_L \times G_R)/G_D$ with arbitrary breaking of G_R, and for the large class of four dimensional non-homogeneous metrics with an isometry group $SU(2) \times U(1)$.

For the simple example of the T-dualised $SU(2)$ σ-model, which has been claimed to be non-renormalisable at the two-loop order, we prove that it is - at least up to this order - still possible to define a correct quantum theory by modifying, at the $\hbar$ order, its target space metric in a finite manner.

In the second part, we construct, using harmonic superspace and the quaternionic quotient approach, an explicit quaternionic-Kähler extension of the most general two centres hyper-Kähler metric. It possesses $U(1) \times U(1)$ isometry and contains as special cases the quaternionic-Kähler extensions of the Taub-NUT and Eguchi-Hanson metrics. It exhibits an extra one-parameter freedom which disappears in the hyper-Kähler limit.