Interprétation probabiliste de l'équation de Landau.

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Introduction

Cette thèse porte sur une approche probabiliste d'une équation aux dérivées partielles issue de la physique, appelée l'équation de Landau. Cette équation a été jusqu'à maintenant étudiée avec des méthodes d'analyse. On va ici donner les résultats obtenus grâce aux outils probabilistes et expliquer ce que peut apporter cette approche par rapport à ce qui a déja été fait. Cette introduction a pour but de rappeler les expressions des équations nécessaires à notre étude, d'expliquer en détail l'interprétation probabiliste de ces équations et de donner les principaux résultats contenus dans la thèse.

1 L'équation de Landau

L'équation de Landau, aussi appelée équation de Fokker-Planck-Landau, est une équation aux dérivées partielles non linéaire qui décrit le mouvement des particules dans un plasma. Cette équation a été obtenue par Lev Davidovitch Landau en 1936 à partir de l'équation de Boltzmann. Pour un certain type de gaz, les gaz de Coulomb, l'équation de Boltzmann n'est en effet plus adaptée. Dans ces gaz, il y a énormément de collisions rasantes de particules. Les vitesses des particules évoluent de façon continues au cours du temps. L'équation de Boltzmann ne permet pas de décrire ce phénomène. L. D. Landau a alors obtenu une nouvelle équation, l'équation de Landau, comme limite asymptotique d'équations de Boltzmann lorsque les collisions rasantes deviennent prépondérantes (le passage de l'équation de Boltzmann vers l'équation de Landau est approfondi dans la Section 4).

L'équation de Landau s'écrit en dimension $d \geq 2$ dans le cas spatialement homogène de la façon suivante

$$\frac{\partial f}{\partial t} = Q_L\left(f, f\right) \tag{0.1}$$

où $f\left(t, v\right)$ est la densité de particules qui ont la même vitesse $v \in \mathbb{R}^d$ au même instant $t \in \mathbb{R}^+$ et Q_L est l'opérateur quadratique suivant

$$Q_L\left(f,f\right)\left(v,t\right) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}\left(v - v_*\right)\left(f\left(t, v_*\right)\frac{\partial f}{\partial v_j}\left(t,v\right) - f(t,v)\frac{\partial f}{\partial v_{*j}}\left(t, v_*\right)\right)\right\} \tag{0.2}$$

avec $a = (a_{ij})_{1\leq i,j\leq d}$ une matrice symétrique positive qui dépend de la nature des collisions.

L'équation (0.1) est en effet considérée dans le cas spatialement homogène car la densité f est supposée indépendante de la position des particules. Le cas non-spatialement homogène est à ce jour un problème ouvert. On peut cependant mentionner l'article de Villani et Desvillettes, [15], sur l'équation de Landau 'linéarisée' et l'article de Lucquin-Desreux et Mancini, [31], qui donne une méthode numérique pour une équation de Boltzmann 'linéarisée' non spatialement homogène.

Lorsque les particules entrent en collision suivant une force en $1/r^s$, où r est la distance entre les particules et $s \geq 2$, la matrice a est de la forme suivante

$$a(z) = \Lambda |z|^{\gamma+2} \Pi(z)$$

où Λ est une constante positive, $\gamma = (s-(2d-1))/(s-1)$, et $\Pi = (\Pi_{ij})_{1\leq i,j\leq d}$ est la projection orthogonale sur $(z)^{\perp}$, c'est à dire $\Pi_{ij}(z) = \delta_{ij} - z_i z_j / |z|^2$.
On introduit la même classification que pour l'équation de Boltzmann:

- si $\gamma > 0$, on parle de potentiels durs,
- si $\gamma = 0$, il s'agit d'un gaz de Maxwell,
- si $\gamma < 0$, on parle de potentiels mous,
- si $\gamma = -3$, il s'agit d'un gaz de Coulomb.

Villani et Desvillettes ont étudié l'équation de Landau dans le cas des potentiels durs, plus exactement lorsque $\gamma \in]0,1]$, dans deux articles [13, 14]. Ils ont étudié l'existence et la régularité de solutions et leur comportement en temps grand. Villani a fait par ailleurs une étude approfondie des gaz de Maxwell dans [40]. Pour les potentiels mous, $\gamma \in]-2,0]$, les articles de Goudon, [23], et de Villani, [41], donnent l'existence d'une solution faible dans L^1 en utilisant le passage entre l'équation de Boltzmann et l'équation de Landau lorsque les collisions rasantes deviennent prépondérantes. Dans ces deux articles, la condition initiale est une fonction ayant un moment d'ordre 2 et une entropie finis. Le cas le plus intéressant d'un point de vue physique est le cas d'un gaz de Coulomb, mais c'est aussi le cas le plus difficile d'un point de vue mathématique. On peut mentionner l'article de Villani [41] qui aborde le cas coulombien. Tous ces travaux utilisent des outils analytiques, cette thèse a pour but d'étudier l'équation de Landau par une approche probabiliste.

Dans cette thèse, on considère une matrice de collision a de la forme suivante:

$$a(z) = \Lambda h(|z|) |z|^{\gamma+2} \Pi(z) \tag{0.3}$$

pour des potentiels 'modérément mous': $\gamma \in]-1,0]$, où h est une fonction positive, bornée et localement lipschitzienne sur $\mathbb{R}^+$ et Λ est une constante strictement positive.

Tanaka, [38], a introduit les bases de l'approche probabiliste de l'équation de Boltzmann pour un gaz de Maxwell sous des hypothèses restrictives. Ensuite Horowitz, Karandikar, [26], Desvillettes, Graham, Méléard, Fournier, [12], [20], ont étendu les résultats à des situations plus physiques. Nous reviendrons sur certains de ces articles de façon plus précise par la suite. Ces travaux ont servi de base à l'introduction d'une approche probabiliste de l'équation de Landau, permettant non seulement d'avoir des conditions initiales mesures, de travailler dans l'espace de Hilbert L^2 (grâce au choix de γ), mais aussi d'avoir une meilleure compréhension de certains phénomènes comme le passage de l'équation de Boltzmann à l'équation de Landau.

L'équation (0.1) s'écrit formellement sous forme faible de la façon suivante : pour toute fonction test ϕ

$$\frac{d}{dt}\int_{\mathbb{R}^d}\phi(v)\,f(v,t)\,dv=\int_{\mathbb{R}^d\times\mathbb{R}^d}L^{\phi}(v,v_*)\,f(v,t)dvf(v_*,t)\,dv_* \tag{0.4}$$

où l'opérateur L^{ϕ} est défini par

$$L^{\phi}(v,v_*)=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(v-v_*)\frac{\partial^2\phi(v)}{\partial v_i\partial v_j}+\sum_{i=1}^{d}b_i(v-v_*)\frac{\partial\phi(v)}{\partial v_i} \tag{0.5}$$

avec $b:\ \mathbb{R}^d\to\ \mathbb{R}^d$ donné par

$$b_i(z)=\sum_{j=1}^{d}\frac{\partial\,a_{ij}(z)}{\partial z_j}=-(d-1)\,h(|z|)\,|z|^{\gamma}z_i. \tag{0.6}$$

Remarque 1.1 *L'équation de Landau (0.4) conserve la masse, la quantité de mouvement et l'énergie cinétique. (Il suffit de considérer des fonctions tests de la forme* $\phi(v)=1,v_i,|v|^2$ *.)*

Définition 1.2 *Une fonction f, définie sur $\mathbb{R}^d\times[0,+\infty[$, est dite* **solution fonction** *de l'équation de Landau de condition initiale f_0 si $f(.,0)=f_0(.)$ et si l'équation (0.4) est satisfaite pour tout $t\geq 0$ et pour toute fonction test $\phi\in\mathcal{C}_b^2(\mathbb{R}^d)$, où $\mathcal{C}_b^2(\mathbb{R}^d)$ représente l'ensemble des fonctions à valeurs réelles, bornées, de classe $\mathcal{C}^2$ et à dérivées bornées.*

On peut supposer sans restriction que $\int f(v,0)\,dv=1$. La conservation de la masse implique alors $\int f(v,t)\,dv=1$ pour tout temps $t\geq 0$. Donc au lieu de chercher une fonction f, il est assez naturel de chercher un flot $(P_t)_{t\geq 0}$ de mesures de probabilité sur $\mathbb{R}^d$ satisfaisant une équation similaire à (0.4) en remplacant la mesure $f(v,t)dv$ par $P_t(dv)$.

Définition 1.3 *Une* **solution mesure** *de l'équation de Landau de condition initiale P_0 est un flot de mesures de probabilité $(P_t)_{t\geq 0}$ tel que pour tout $t\geq 0$ et $\phi\in\mathcal{C}_b^2(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d}\phi(v)\,P_t(dv)\ =\ \int_{\mathbb{R}^d}\phi(v)\,P_0(dv)+\int_{\mathbb{R}^d\times\mathbb{R}^d}L^{\phi}(v,v_*)\,P_s(dv)\,P_s(dv_*)\,ds. \tag{0.7}$$

Nous allons étudier, dans la première partie de cette thèse, l'existence et l'unicité d'une solution mesure pour l'équation de Landau, puis, chercher à en déduire l'existence d'une solution fonction. Enfin, lorsqu'il y a existence de solutions fonctions, nous étudierons leur régularité. Par ailleurs, comme l'équation de Landau a été obtenue comme limite asymptotique d'équations de Boltzmann, la deuxième partie de la thèse sera consacrée à la convergence des solutions des équations de Boltzmann lorsque les collisions rasantes deviennent prédominantes.

2 Approche probabiliste de l'équation de Landau

Cette partie est développée dans le Chapitre 1.

La première étape de cette thèse est de montrer l'existence d'une famille $(P_t)_{t\geq 0}$ de mesures de probabilité qui soit solution de (0.7). Il est équivalent de chercher une famille $(Z_t)_{t\geq 0}$ de variables aléatoires sur $\mathbb{R}^d$ telle que la loi de Z_t soit P_t et telle que l'égalité suivante soit satisfaite pour tout $\phi \in \mathcal{C}_b^2\left(\mathbb{R}^d\right)$, et tout $t \geq 0$:

$$E\left[\phi\left(Z_t\right)\right] = E\left[\phi\left(Z_0\right)\right] + \int_0^t E\left[\int_{\mathbb{R}^d} L^\phi\left(Z_s, v_*\right) P_s\left(dv_*\right)\right] ds$$

où L^ϕ est l'opérateur de diffusion (0.5).

D'un point de vue probabiliste, on va en fait chercher à satisfaire une propriété plus forte. L'équation (0.7) est considérée comme l'équation d'évolution des marginales d'un processus de Markov dont la loi est définie par le problème de martingales suivant:

Définition 2.1 *Soit P_0 une mesure de probabilité ayant un moment d'ordre 2 fini. Une mesure de probabilité P sur $\mathcal{C}\left([0,+\infty[\,,\mathbb{R}^d\right)$ satisfait le problème de martingales (LPM) si pour toute fonction $\phi \in \mathcal{C}_b^2\left(\mathbb{R}^d\right)$*

$$M_t^\phi = \phi\left(X_t\right) - \phi\left(X_0\right) - \int_0^t \int_{\mathbb{R}^d} L^\phi\left(X_s, v_*\right) P_s\left(dv_*\right) ds \tag{LPM}$$

est une martingale sous la loi P, où $P_t = P \circ X_t^{-1}$ et X est le processus canonique sur $\mathcal{C}\left([0,+\infty[\,,\mathbb{R}^d\right)$ (espace des fonctions continues sur $[0,+\infty[$ à valeurs dans $\mathbb{R}^d$).

Ce problème de martingales est non linéaire car la loi intervient dans le troisième terme de M^ϕ. En prennant l'espérance de M^ϕ, on remarque que si la mesure P est solution du problème de martingales (LPM), alors la famille des marginales temporelles $(P_t)_{t\geq 0}$ de P est une solution mesure de l'équation de Landau.

Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace de probabilité.

Afin de résoudre le problème (LPM), nous introduisons une équation différentielle stochastique non linéaire dont le terme de diffusion est un bruit blanc (selon la définition de Walsh, [42]). En effet, le crochet de la martingale locale M^ϕ, pour la fonction test $\phi(x) = x$, est donnée par

$$\left\langle M_i^x, M_j^x \right\rangle_t = \int_0^t \int_{\mathbb{R}^d} a_{ij}\left(X_s - v_*\right) P_s\left(dv_*\right) ds.$$

Comme la matrice a est symétrique, il existe une matrice σ sur $\mathbb{R}^d \times \mathbb{R}^{d'}$ telle que

$$a = \sigma.\sigma^* \tag{0.8}$$

où σ^* est la matrice transposée de σ. D'après El Karoui et Méléard, [17] Théorème III-10, il existe sur une extension de $(\Omega, \mathcal{F}, \mathbb{P})$ d' bruits blancs $\tilde{W}_j$ indépendants de mesure de covariance $P_t\left(dv_*\right) dt$ sur $\mathbb{R}^d \times [0,+\infty[$ tels que

$$M_{i,t}^x = \sum_{j=1}^{d'} \int_0^t \int_{\mathbb{R}^d} \sigma_{ij}(X_s - v_*) \tilde{W}_j(dv_*, ds).$$

Notons $\tilde{W}^{d'} = \left(\tilde{W}_1, ..., \tilde{W}_{d'}\right)$ le bruit blanc de dimension d' ainsi défini.
Soit X_0 un vecteur aléatoire indépendant de $\tilde{W}^{d'}$ de loi P_0. On introduit alors l'équation différentielle stochastique non linéaire suivante, pour tout $t \geq 0$,

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} \sigma\left(X_s - v_*\right).\tilde{W}^{d'}\left(dv_*, ds\right) + \int_0^t \int_{\mathbb{R}^d} b\left(X_s - v_*\right) P_s\left(dv_*\right) ds \qquad (0.9)$$

telle que la loi de X soit P.

Réciproquement, la formule d'Itô permet de montrer que la loi P, d'un processus X solution de (0.9), est une solution du problème de martingales (LPM), et par conséquent le flot $(P_t)_{t\geq 0}$ associé est une solution mesure de l'équation de Landau.

Cependant, l'écriture (0.9) n'est pas tout à fait satisfaisante car la loi de X intervient dans le bruit blanc. Le théorème de représentation de Skorohod (voir [35]) permet de simplifier l'équation.

Considérons un deuxième espace de probabilité, l'espace $([0,1], \mathcal{B}([0,1]), d\alpha)$ où $d\alpha$ est la mesure de Lebesgue sur $[0,1]$. Pour éviter toute confusion dans la suite, E et $\mathcal{L}$ représenteront respectivement l'espérance et la loi d'une variable définie sur $(\Omega, \mathcal{F}, \mathbb{P})$ et E_α et $\mathcal{L}_\alpha$ l'espérance et la loi d'une variable définie sur $([0,1], \mathcal{B}([0,1]), d\alpha)$.
Soit $W^{d'} = (W_1, ..., W_{d'})$ le bruit blanc de dimension d' où les W_i sont des bruits blancs espace-temps indépendants de mesure covariance $d\alpha dt$ sur $[0,1] \times \mathbb{R}^+$.
Au lieu de chercher un processus X solution de (0.9), on va chercher un couple de processus vérifiant la définition suivante

Définition 2.2 *Soit X_0 un vecteur aléatoire de $\mathbb{R}^d$ tel que $E\left[|X_0|^2\right] < +\infty$. Un couple de processus (X, Y) défini sur $(\Omega, \mathcal{F}, \mathbb{P}) \times ([0,1], \mathcal{B}([0,1]), d\alpha)$ est solution de $(LEDS)$ si X est adapté par rapport à la filtration naturelle de $W^{d'}$ et si pour tout $t \geq 0$*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}\left(d\alpha, ds\right) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds$$

avec $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$.

Dans le cas où les coefficients σ et b sont des fonctions lipschitziennes et où la condition initiale X_0 admet un moment d'ordre 2 fini, l'existence d'une solution forte à l'équation $(LEDS)$ et l'unicité en loi de la solution sont obtenus grâce à une méthode de Picard améliorée (du fait de la non linéarité de l'équation). Ce résultat implique l'existence d'une solution mesure pour l'équation de Landau pour une mesure de probabilité initiale P_0 admettant un moment d'ordre 2 fini. De plus, l'unicité de la solution mesure de l'équation de Landau est montrée en utilisant les résultats d'Horowitz et Karandikar ([26] Théorème 5.2) sur les problèmes de martingales.

Dans le cas plus général que nous considèrons, $\gamma \in \left]-1, 0\right]$, les coefficients σ et b sont à croissance linéaire. Les solutions de l'équation de Landau sont alors approchées par des solutions d'une équation aux coefficients 'régularisés' où les coefficients σ et b sont remplacés par des fonctions lipschitziennes bien choisies. L'existence d'une solution au problème de martingale (LPM), et donc l'existence d'une solution mesure à l'équation de Landau, sont ainsi obtenues par approximations, lorsque la mesure de probabilité initiale admet un moment d'ordre 2 fini lorque $\gamma \in]-1,0[$ et d'ordre 3 lorsque $\gamma = 0$. Ce résultat est montré en utilisant des critères de tension et par conséquent, comme pour les méthodes analytiques, cette approche ne permet pas d'obtenir l'unicité de la solution pour les potentiels 'modérément mous'.

On remarque que pour obtenir l'existence d'une solution mesure, seule la régularité des coefficients $a = \sigma.\sigma^*$ et b de l'équation de Landau est utilisée, et non leur expression exacte.

3 Existence d'une solution fonction à l'équation de Landau

Cette partie est développée dans le Chapitre 2.

Connaissant l'existence d'une solution mesure, la seconde étape de cette étude consiste à en déduire l'existence d'une solution fonction. On constate que si $(P_t)_{t\geq 0}$ est une solution mesure et si pour tout $t > 0$ la mesure P_t admet une densité $f(t,.)$ par rapport à la mesure de Lebesgue, c'est à dire $P_t(dv) = f(t,v)\,dv$, alors la fonction f est solution fonction de l'équation de Landau (0.4).

Cette partie se limitera au cas où il y a existence forte d'une solution à l'équation $(LEDS)$, c'est à dire lorsque les fonctions σ et b sont lipschitziennes, afin de pouvoir utiliser un outil probabiliste très efficace: le calcul de Malliavin. Le calcul de Malliavin permet de construire un calcul différentiel lié à l'alea et d'obtenir une formule d'intégration par parties pour les variables aléatoires afin de montrer l'existence et la régularité de densité par rapport à la mesure de Lebesgue de la loi d'une variable aléatoire (voir le livre de Nualart [33]).

On va ici l'utiliser dans le cas particulier de processus définis par une équation différentielle stochastique dont le terme de diffusion est un bruit blanc.

Soit (X,Y) la solution de l'équation stochastique non linéaire $(LEDS)$. L'article de Bally et Pardoux [3] nous a servi de base pour montrer que la loi P_t de X_t est absolument continue par rapport à la mesure de Lebesgue sur $\mathbb{R}^d$ pour tout $t > 0$, lorsque la mesure initiale P_0 admet un moment d'ordre 2 et n'est pas une mesure de Dirac. De plus, le calcul de Malliavin permet aussi d'obtenir la régularité $\mathcal{C}^\infty$ de la densité lorsque la condition initiale admet en plus des moments de tous ordres et lorsque les coefficients σ et b sont suffisamment réguliers.

L'hypothèse 'P_0 n'est pas une masse de Dirac' est naturelle. En effet, si la condition initiale est une masse de Dirac, alors toutes les vitesses initiales sont les mêmes. Par conséquent, il n'y a pas de choc de particules, le système n'évolue pas en fonction du temps. La seule solution de l'équation de Landau est la masse de Dirac initiale P_0, qui bien évidemment n'admet pas de densité par rapport à la mesure de Lebesgue. Par ailleurs, les hypothèses sur la condition initiale sont peu restrictives car elles permettent d'avoir une condition initiale dégénérée comme par exemple une somme de deux mesures de Dirac.

Dans cette partie, la conservation de la quantité de mouvement et de l'énergie cinétique (voir Remarque 1.1), et les expressions exactes des coefficients de l'équation de

Landau sont essentiels dans la preuve du théorème d'existence de densité.

Les méthodes utilisant le calcul de Malliavin nécessitent en général une hypothèse de non-dégénérescence sur les coefficients de l'équation différentielle stochastique pour obtenir l'existence de densité. Cependant, l'équation, que l'on considère, a une matrice de collision a dégénérée. C'est la non linéarité de l'équation $(LEDS)$ qui permet de conclure.

Après la recherche de solutions et de leur régularité, la suite de cette thèse est consacrée à la construction de l'équation de Landau. L. D. Landau a obtenu cette équation comme limite asymptotique d'équations de Boltzmann lorsque les collisions rasantes deviennent prépondérantes. Nous allons donner maintenant une interprétation probabiliste de cette convergence.

4 Liens entre l'équation de Boltzmann et l'équation de Landau en dimension 3

Cette partie est développée dans le Chapitre 3.

4.1 L'équation de Boltzmann

L'équation de Boltzmann, que nous allons considérer, est définie dans le cas spatialement homogène par

$$\frac{\partial f}{\partial t} = Q_B(f, f) \tag{0.10}$$

où f est la densité, dans un gaz suffisamment dilué, de particules qui ont la même vitesse au même instant et Q_B est l'opérateur de collision quadratique suivant

$$Q_B(f,f)(t,v) = \int_{v_* \in \mathbb{R}^d} \int_{\sigma \in S^{d-1}} \bigg(f(t,v')f(t,v'_*) - f(t,v)f(t,v_*) \bigg) B(|v-v_*|, \theta) d\sigma dv_*$$

où v, v_* sont les vitesses avant collision et v', v'_* les vitesses après collision, et θ est appelé angle de déviation. On a les relations suivantes

$$\begin{aligned} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad ; \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2} \\ \cos\theta &= \left\langle \frac{v-v_*}{|v-v_*|}, \sigma \right\rangle. \end{aligned}$$

La fonction positive B est appelée section efficace.

Cette partie se limitera à l'étude en dimension 3, qui est le cas le plus intéressant pour les physiciens. Lorsque les particules entrent en intéraction selon une force en $1/r^s$, où r est la distance entre les particules et $s \geq 2$, la section efficace $B(z,\theta)$ admet une singularité lorsque θ tend vers 0. Elle satisfait cependant la condition $\int_0^\pi |\theta|^2 B(z,\theta) d\theta <$

∞ pour tout $z \in \mathbb{R}^3$. L'explosion en 0 s'explique d'un point de vue physique comme l'accumulation de collisions rasantes.

L'équation de Boltzmann dans le cas général est très difficile à étudier. De nombreux travaux se placent sous l'hypothèse de troncature $\int_0^\pi B(z,\theta)d\theta < \infty$, mais cette hypothèse n'a aucun sens physique. Puis, le cas des molécules Maxwelliennes, c'est à dire lorsque la section efficace $B(z,\theta) = \beta(\theta)$ ne dépend que de θ, a été étudié sans hypothèse de troncature. Tanaka, [38], a étudié ce cas lorsque$\int_0^\pi \theta\beta(\theta)d\theta < \infty$, puis Horowitz, Karandikar [26], Desvillettes [11], et Fournier, Méléard [18], [21], l'ont étudié sous l'hypothèse physique $\int_0^\pi \theta^2\beta(\theta)d\theta < +\infty$. Goudon [23], Villani et Desvillettes, [41, 13], ont étudié le cas de molécules non Maxwelliennes en utilisant des outils analytiques. Fournier-Méléard [20] ont obtenu des résultats via une approche probabiliste en dimension 2 et lorsque la section efficace, vue comme fonction de la vitesse, est bornée.

Concernant la convergence, via une approche probabiliste, des solutions des équations de Boltzmann vers une solution de l'équation de Landau suivant l'asymptotique des collisions rasantes, pour des potentiels 'modérément mous', il faut tout d'abord étendre les résultats de l'approche probabiliste de l'équation de Boltzmann à ces potentiels. Dans un travail écrit en commun avec Sylvie Méléard, on a donc étudié l'équation de Boltzmann en dimension 3 pour des sections efficaces de la forme suivante:

$$B(z,\theta) = \psi(z)\beta(\theta), \tag{0.11}$$

avec $\psi(z) = h(|z|)|z|^\gamma$, $\gamma \in]-1,0]$, h une fonction positive bornée localement lipschitzienne et $\beta :]0,\pi] \to \mathbb{R}^+$ telle que $\int_0^\pi \theta^2\beta(\theta)d\theta < \infty$.

L'amplitude des sauts lors du choc des particules est définie par

$$A(v,v_*,\theta,\varphi) = v' - v \tag{0.12}$$

où φ et θ sont les coordonnées sphériques de $\sigma \in S^2$ dans le repère où l'axe polaire est $v - v_*$. Cette amplitude satisfait l'inégalité suivante

$$|A(v,v_*,\theta,\varphi)| \leq |\theta|\,|v - v_*|. \tag{0.13}$$

La principale difficulté de notre étude réside dans le fait que la fonction A n'est pas une fonction lipschitzienne en les variables v et v_*. Comme l'a montré Tanaka dans [38] et Fournier, [21], elle vérifie cependant une propriété de fonction "presque" lipschitzienne (A est lipschitzienne à une rotation près) ce qui suffit pour obtenir l'existence d'une solution au sens probabiliste.

L'équation (0.10) est étudiée sous sa forme faible. Comme l'intégrale $\int_0^\pi \theta\beta(\theta)d\theta$ peut être infinie, on regarde l'équation compensée ainsi définie

$$\frac{d}{dt}\int_{\mathbb{R}^3} \phi(v)f(t,v)dv = \int_{\mathbb{R}^3\times\mathbb{R}^3} K_\beta^\phi(v,v_*)\, f(t,v)dvf(t,v_*)dv_* \tag{0.14}$$

où le noyau K_β^ϕ est donné par

$$K_\beta^\phi(v,v_*) = -B\psi(v-v_*)(v-v_*).\nabla\phi(v) \tag{0.15}$$
$$+\int_0^{2\pi}\int_0^{\pi}\Big(\phi(v+A(v,v_*,\theta,\varphi))-\phi(v)-A(v,v_*,\theta,\varphi).\nabla\phi(v)\Big)\psi(v-v_*)\beta(\theta)d\theta d\varphi$$

et $B=\pi\int_0^\pi(1-\cos\theta)\beta(\theta)d\theta$.

De même que pour l'équation de Landau, la notion de solution fonction et la notion probabiliste de solution mesure (grâce à la conservation de la masse de (0.14)) sont introduites pour l'équation de Boltzmann.

Un des résultats obtenu dans cette partie est l'existence d'une solution mesure de l'équation de Boltzmann lorsque la section efficace satisfait les hypothèses (0.11). L'expression (0.15) du noyau K_β^ϕ permet d'observer un saut de discontinuité de la vitesse v à la vitesse $v+A$ lors du choc. On associe alors à l'équation de Boltzmann un problème de martingales et une équation différentielle stochastique non linéaire dont le terme stochastique est une mesure de Poisson (voir [20]). Les solutions du problème de martingales sont des probabilités sur l'espace de Skorohod $\mathbb{D}([0,+\infty[\,,\mathbb{R}^3)$, espace des fonctions continues à droite et limitées à gauche.
Les mêmes techniques, que pour la résolution de l'équation de Landau, permettent de montrer l'existence d'une solution mesure à l'équation de Boltzmann en dimension 3 lorsque la condition initiale est une mesure de probabilité admettant un moment d'ordre 2 fini si $\gamma\in\,]-1,0[$ et d'ordre 4 fini si $\gamma=0$. Les hypothèses sur la condition initiale permettent des mesures dégénérées comme par exemple des mesures de Dirac. Ce résultat généralise les travaux de Tanaka [38] et de Horowitz et Karandikar [26] aux potentiels mous. Par contre, comme pour les méthodes analytiques, cette approche ne donne pas de résultat d'unicité de la solution.

4.2 L'asymptotique des collisions rasantes

Cette section a pour but d'étudier le lien entre les équations de Boltzmann et de Landau dans le cas de potentiels 'modérément mous', $\gamma\in]-1,0]$. On considère par conséquent des sections efficaces $\beta^\varepsilon(\theta)$ dépendant d'un paramètre de collision ε de manière à privilégier les collisions rasantes.

Arsen'ev et Buryak, [1], ont montré la convergence de l'équation de Boltzmann vers l'équation de Landau sous des hypothèses très restrictives sur la section efficace et sur la condition initiale. Degond et Lucquin, [9], ont étudié la convergence des opérateurs dans le cas plus physique des potentiels de Colomb ($\gamma=-3$) en utilisant l'approximation suivante:

$$\beta^\varepsilon(\theta)=\frac{1}{|\log\varepsilon|}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\mathbb{I}_{\theta\geq\varepsilon}$$

Desvillettes, [10], a fait un travail similaire en utilisant la normalisation suivante

$$\beta^\varepsilon(\theta)=\frac{1}{\varepsilon^3}\beta\left(\frac{\theta}{\varepsilon}\right)$$

mais pour des potentiels excluant le cas coulombien. La normalisation introduite par Villani, [41], qui inclut celle de Degond-Lucquin et celle de Desvillettes, sera utilisée dans cette section. On considère donc des sections efficaces $\beta^\varepsilon : [0,\pi] \to \mathbb{R}^+$ telles que

$$\forall \theta_0 > 0 \ \beta^\varepsilon(\theta) \underset{\varepsilon\to 0}{\longrightarrow} 0 \text{ uniformément sur } \theta \geq \theta_0 \tag{0.16}$$

$$\Lambda^\varepsilon = \pi \int_0^\pi \sin^2(\theta/2)\beta^\varepsilon(\theta)\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} \Lambda > 0 \tag{0.17}$$

La constante Λ qui apparait dans (0.17) est la même que celle de la définition de la matrice de collision a de l'équation de Landau (voir (0.3)). Les limites (0.16) et (0.17) impliquent les convergences suivantes:

$$\int_0^\pi \beta^\varepsilon(\theta)\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} +\infty, \tag{0.18}$$

$$\text{et pour } k \geq 3, \int_0^\pi \sin^k(\theta/2)\beta^\varepsilon(\theta)\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{0.19}$$

On constate que lorsque ε tend vers 0, on modélise les collisions rasantes. En effet, plus ε est petit, plus la fonction β^ε se concentre sur les petites valeurs de l'angle de déviation θ. Par conséquent, plus ε est petit, plus l'amplitude des sauts A est petite (voir (0.13)), par contre la masse $\int_0^\pi \beta^\varepsilon(\theta)\, d\theta$ grandit (voir (0.18)). D'un point de vue probabiliste, ce comportement est interprété de la façon suivante: plus il y a des collisions rasantes, plus il y a de chocs de particules avec une amplitude de plus en plus petite. Il semble donc naturel de passer d'un processus de sauts, le processus de Boltzmann, à un processus de diffusion, le processus de Landau.

4.3 Convergence des processus

Soit P^ε une mesure de probabilité sur $\mathbb{D}([0,+\infty[\,,\mathbb{R}^3)$, solution de l'équation de Boltzmann pour $\gamma \in\,]-1,0]$ où β est remplacé par β^ε, défini par (0.16) et (0.17), dans la section efficace (définie par (0.11)).

Lorsque ε tend vers 0, la suite $(P^\varepsilon)_{\varepsilon>0}$ est tendue pour une mesure initiale admettant un moment d'ordre 4. De plus, un critère de C-tension permet d'observer que ses points d'adhérence ne chargent que l'espace des fonctions continues $\mathcal{C}([0,+\infty[\,,\mathbb{R}^3)$. Enfin, une étude basée sur les problèmes de martingales identifie les points d'adhérence, qui sont en fait solutions du problème de martingales (LPM) lié à l'équation de Landau.

Dans le cas de molécules Maxwelliennes ($\gamma = 0$, h =constante), il suffit que la condition initiale admette un moment d'ordre 2 et grâce à l'unicité de la solution de l'équation de Landau, il y a en fait convergence de la suite $(P^\varepsilon)_{\varepsilon>0}$.

L'intérêt de cette approche est la compréhension de la convergence au niveau microscopique des processus. Les processus de Boltzmann sautent de plus en plus avec une amplitude de plus en plus petite lorsque ε décroît et convergent finalement vers un processus de diffusion (continu). De plus, ce résultat n'impose pas de condition forte sur la condition initiale qui peut être dégénérée comme par exemple une mesure de Dirac.

4.4 Simulation de la convergence

Méléard et Fournier, [21], ont construit une approximation particulaire des solutions de Boltzmann en dimension 3 et lorsque $\gamma = 0$. Leur algorithme va servir de base pour simuler la convergence des équations Boltzmann vers l'équation de Landau lorsque les collisions rasantes deviennent prépondérantes.

A notre connaissance, il n'y a pas de résolution numérique par des méthodes déterministes de l'équation de Landau vue comme limite asymptotique d'équations de Boltzmann, sauf dans [34] où une méthode spectrale donne une manière concrète d'étudier cette limite (mais aucun résultat numérique n'est donné). Par ailleurs, les méthodes particulaires déterministes ne sont pas adaptées à l'étude en dimension 3 de l'équation de Landau. Quelques algorithmes de Monte-Carlo pour l'équation de Landau existent mais sans preuve de convergence, voir Takizuka and Abe [39] et Wang, Okamoto, Nakajima, Murakami [43]. Ils utilisent l'aspect diffusif de l'équation et non l'asymptotique des collisions rasantes.

On considère ici des sections efficaces tronquées (afin qu'elles soient simulables) qui dépendent du paramètre ε des collisions rasantes. Une méthode de Monte-Carlo, en remplacant la non linéarité par la mesure empirique du système, permet de définir un système de particules en interaction.

Le système que l'on considère conserve la quantité de mouvement et l'énergie cinétique. A ε fixé, Fournier et Méléard, [21], ont déja établi que lorsque le paramètre de troncature et le nombre de particules tendent vers l'infini, la mesure empirique du système converge en loi vers un processus de Boltzmann. Cette section établit une convergence uniforme en ε. Par conséquent, lorsque le paramètre ε tend vers 0, on simule le passage des équations de Boltzmann à celle de Landau. Dans le Chapitre 3, des exemples de simulations sont donnés lorsque $\gamma \in]-1, 0]$. Par ailleurs, notre modèle est implémentable lorsque $\gamma = -3$ (cas coulombien). Quelques simulations sont faites dans ce cas en prennant les mêmes conditions initiales que Cordier, Degond et Lemou, [7], afin de comparer les résultats.

Le code de notre algorithme de simulation est donné en Appendice de cette thèse.

5 Asymptotique des collisions rasantes en dimension 2, convergence des solutions fonctions

Cette partie est développée dans le Chapitre 4.

La partie précédente a montré le lien entre les solutions mesures des équations de Boltzmann et de Landau lorsque les collisions rasantes deviennent prépondérantes pour des potentiels 'modérément mous'. Il est alors naturel d'étudier la convergence des solutions fonctions dans le cas où elles existent. Fournier, [18], a obtenu l'existence d'une solution fonction pour l'équation de Boltzmann pour un gaz de Maxwell ($\gamma = 0$ et h =constante dans l'expression (0.11) de la section efficace), en dimension 2 (à cause du manque de régularité des coefficients dans $\mathbb{R}^3$, voir [20] Lemma 2.6). Il y a de plus existence d'une solution fonction pour l'équation de Landau dans ce cas (voir Partie 3). Par conséquent, cette partie se placera sous ces conditions. Lorsque le gaz est maxwellien et grâce à l'unicité de la solution de Landau (voir Partie 2), une analyse fine du calcul de Malliavin permet de montrer la convergence ponctuelle des solutions fonctions des équations de Boltzmann vers la solution fonction de l'équation de Landau. Aucun résultat similaire, sur la convergence ponctuelle des solutions fonctions, n'a été obtenu par des méthodes analytiques.

Considérer les collisions rasantes signifie que l'on privilégie les petits angles de déviation. On va ici considérer une approximation des collisions rasantes particulière, introduite par Desvillettes ([10]): pour tout $\varepsilon > 0$, on définit β^ε sur $[-\varepsilon\pi, \varepsilon\pi] \setminus \{0\}$ par

$$\beta^\varepsilon(\theta) = \frac{1}{\varepsilon^3}\beta\left(\frac{\theta}{\varepsilon}\right) \tag{0.20}$$

où β est une fonction positive sur $[-\pi, \pi]\setminus\{0\} \to \mathbb{R}^+$ telle que $\int_{-\pi}^{\pi} \theta^2\beta(\theta)d\theta < \infty$. Cette approximation vérifie les hypothèses (0.16) et (0.17) satisfaites par l'approximation utilisée dans la Partie 4.

En dimension 2, dans le cas d'un gaz de Maxwell, l'équation de Boltzmann s'écrit:

$$\frac{\partial f}{\partial t} = Q_{B^\varepsilon}(f, f) \tag{0.21}$$

où l'opérateur de collision Q_{B^ε} est donné par

$$Q_{B^\varepsilon}(f,f)(t,v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta=-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\beta^\varepsilon(\theta)\, d\theta dv_*$$

La relation entre les vitesses avant et après collision en dimension 2 est la suivante:

$$v' = v + A(\theta)(v - v_*)\ ;\ v'_* = v - A(\theta)(v - v_*)$$

avec

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}.$$

L'existence d'une solution fonction à partir d'une solution mesure de l'équation de Boltzmann a été établie par Fournier, [18], en utilisant le calcul de Malliavin pour des processus de sauts. La solution fonction est donnée par l'inverse de la transformée de Fourier de la mesure solution. Donc si on note respectivement P_t^ε et P_t les solutions mesures des équations de Boltzmann et de Landau, et $\hat{P}_t^\varepsilon$, $\hat{P}_t$ leur transformée de fourier, alors les solutions fonction f^ε et f sont données par:

$$f^\varepsilon(t,v) = \int_{\mathbb{R}^2} e^{-i<x,v>}\hat{P}_t^\varepsilon(x)dx \quad \text{et} \quad f(t,v) = \int_{\mathbb{R}^2} e^{-i<x,v>}\hat{P}_t(x)dx.$$

Vu que les distributions P^ε convergent vers P lorsque ε tend vers 0, il y a convergence ponctuelle des transformées de Fourier. Il suffit par conséquent de montrer que les fonctions $\hat{P}^\varepsilon$ sont bornées uniformément en ε par une fonction intégrable pour obtenir la convergence de f^ε vers f.

Lorsque la mesure initiale P_0 n'est pas une mesure de Dirac et admet des moments de tous ordres, on montre que la suite $(f^\varepsilon)_{\varepsilon>0}$ des solution fonctions de l'équation de Boltzmann converge ponctuellement sur $\mathbb{R}^2$ lorsque ε tend vers 0 et sa limite f est solution fonction de l'équation de Landau. De plus, f est de classe $\mathcal{C}^\infty$ et il y a convergence des dérivées. La preuve de ce résultat est basée sur une étude approfondie des preuves de Fournier, [18], en fixant avec attention les divers paramètres intervenant.

Cette partie établit une convergence forte des solutions de l'équation de Boltzmann vers la solution de l'équation de Landau lorsque les collisions deviennent rasantes pour un gaz de Maxwell. Goudon, [23], et Villani, [41], ont montré une convergence faible dans L^1, mais dans le cas plus général des potentiels mous et en dimension 3. Il semble que leur méthode ne peut pas donner de résultat plus fort.

En utilisant l'algorithme de Monte Carlo de la Section 4.4, des simulations de la solution de l'équation de Landau et de l'évolution de son entropie en fonction du temps sont réalisées en dimension 2 dans le Chapitre 4 lorsque la condition initiale est très dégénérée (la somme de deux mesures de Dirac).

6 Conclusion

Au regard des résultats obtenus sur l'équation de Landau par des outils probabilistes, développés dans cette thèse, un certain nombre de points mériterait d'être approfondi.

Tout d'abord en ce qui concerne la Section 3, il serait intéressant de mieux comprendre quel rôle joue la non linéarité de l'équation dans l'obtention d'existence de densité. La matrice de collision a est dégénérée, cependant est-ce sa forme explicite qui permet de conclure ou est-ce dû à des propriétés géométriques de celle-ci ? En résumé, ce résultat d'existence de densité est peut-être généralisable à d'autres équations non linéaires ayant des coefficients dégénérés .

Par ailleurs, le lien entre les équations de Boltzmann et de Landau nous a permis de construire un système de particules simulable pour approcher les solutions de l'équation de Landau. Il existe déja quelques algorithmes de Monte-Carlo mais sans preuve de convergence (voir Takizuka and Abe [39] et Wang, Okamoto, Nakajima, Murakami [43]) qui utilisent l'aspect diffusif de l'équation et non l'asymptotique des collisions rasantes. Il serait intéressant de montrer la convergence d'un tel algorithme et de le comparer à celui établi dans cette thèse.

Enfin, cette thèse se place sous l'hypothèse que l'équation est spatialement homogène. Dans le cas non spatialement homogène, un terme de transport apparait:

$$\frac{\partial f}{\partial t} = Q_L(f, f) + \nabla_x f.v$$

où $f(t, v, x)$ est la densité de particules qui ont la même vitesse v à la même position x et au même instant t et Q_L est l'opérateur de collision défini par (0.2).
Avec C. Buet, S. Cordier et S. Méléard, nous réfléchissons à une approche numérique de cette équation. Notre but est d'utiliser l'algorithme construit dans cette thèse et qui suit l'asymptotique des collisions rasantes, afin de simuler les solutions de l'équation de Landau dans le cas non spatialement homogène. On utilise une méthode de 'splitting'

qui consiste à regarder successivement le terme de collision, $Q_L(f, f)$ et le terme de transport, $\nabla_x f.v$.

Les chapitres qui suivent sont indépendants et les notations utilisées sont définies à l'intérieur de chacun. Le premier chapitre est accepté pour publication dans la revue Annals of Applied Probability, le second est accepté dans la revue Stochastic Processes and their Applications et le troisième dans la revue Journal of statistical Physics. Le dernier chapitre est soumis pour publication.
Une bibliographie globale est placée à la fin de cette thèse.

Partie I

Étude de l'équation de Landau par des outils probabilistes

Chapitre 1

Solving Landau equation for some soft potentials through a probabilistic approach

Abstract: This article deals with a way to solve the spatially homogeneous Landau equation using probabilistic tools. Thanks to the study of a non-linear stochastic differential equation driven by a space-time white noise, we state the existence of a measure solution of the Landau equation with a probability measure initial data, for a generalization of the Maxwellian molecules case. Then, by approximation of the Landau coefficients, the first result helps us to state the existence of a measure solution for some soft potentials ($\gamma \in (-1, 0)$). This second part is based on the use of nonlinear stochastic differential equations and some martingale problems.

Ce travail a été accepté pour publication dans la revue *Annals of Applied Probability.*

1 Introduction

The Landau equation is obtained as a limit of the Boltzmann equation, when all the collisions become grazing. In the spatially homogeneous case, it writes:

$$\frac{\partial f}{\partial t}(v,t) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}(v - v_*)\left[f(v_*,t)\frac{\partial f}{\partial v_j}(v,t) - f(v,t)\frac{\partial f}{\partial v_{*j}}(v_*,t)\right]\right\} \tag{1.1}$$

where $f(v,t) \geq 0$ is the density of particles having the velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$, and $(a_{ij}(z))_{1 \leq i,j \leq d}$ is a nonnegative symmetric matrix depending on the interaction between the particles.

This equation is also called the Fokker-Planck-Landau equation. Arsen'ev and Buryak, [1], have shown that the solutions of the Boltzmann equation converge toward the solutions of the Landau equation when grazing collisions prevail. On that topic, one can read the paper of Villani, [41], which gives a lot of references.
If we assume, for example, that any two particles at distance r interact with a force proportional to $\frac{1}{r^s}$, the matrix a has the following expression, up to a multiplicative constant,

$$a_{ij}(z) = |z|^{\gamma+2} \Pi_{ij}(z)$$

where

- $|z|$ is the euclidean norm of z in $\mathbb{R}^d$,
- $\Pi(z)$ is the orthogonal projection on $z^\perp$ $(z \neq 0)$, i.e. $\Pi_{ij}(z) = \delta_{ij} - (z_i z_j)/|z|^2$,
- $\gamma = (s-(2d-1))/(s-1)$.

The Landau equation has a physical sense when $d = 3$. However, we will prove some results in more general cases $(d \geq 1)$. Moreover, in this paper, we will consider a matrix a of the form:

$$a_{ij}(z) = |z|^{\gamma+2} \Pi_{ij}(z)\, h\left(|z|^2\right)$$

where h is a bounded nonnegative locally Lipschitz continuous function. We define

$$b_i(z) = \sum_{j=1}^{d} \partial_j a_{ij}(z).$$

So by integration by parts, for any test function φ, we can write a weak formulation of the Landau equation, at least formally,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \varphi(v) f(v,t)\, dv &= \frac{1}{4}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, a_{ij}(v-v_*)\left(\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*)\right) \\ &+ \frac{1}{2}\sum_{i=1}^{d} \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, b_i(v-v_*)\left(\partial_i\varphi(v) - \partial_i\varphi(v_*)\right) \quad (1.2) \end{aligned}$$

where $\partial_i\varphi = \partial\varphi/\partial v_i$ and $\partial_{ij}\varphi = \partial^2\varphi/\partial v_i \partial v_j$.

The properties of the equation depend heavily on γ :

- $\gamma > 0$, one speaks of hard potentials,
- $\gamma = 0$ corresponds to the case of Maxwellian molecules,
- $\gamma < 0$, one speaks of soft potentials,

- $\gamma = -3$ corresponds to the Coulomb interaction.

Villani studies carefully the Landau equation for Maxwellian molecules in [40]. Desvillettes and Villani, [13], prove in the existence of solution, in a weak sense, for hard potentials under some conditions on the initial data. Little is known about soft potentials, we can mention the work of Villani, [41], and the one of Goudon, [23]. Those two independent articles prove the existence of a weak function solution of the Landau equation when $\gamma \in (-2,0)$ and when the initial data is a nonnegative function with finite mass, energy and entropy, using the convergence of the solutions of the Boltzmann equation toward the solutions of the Landau equation.
Our paper deals with an original probabilistic way to solve the spatially homogeneous Landau equation for $\gamma \in (-1,0]$. Thanks to this method, we can assume weaker conditions on the initial data than in the previous articles. We restrict our study to the case $\gamma \in (-1,0]$ to have enough regularity on the Landau coefficients to obtain the existence of solutions (the coefficients have linear growth, see (1.13) and (1.14)).

Remark 1.1 *Choosing $\varphi(v) = 1, v_i$, or $|v|^2/2$, we can easily check that the mass, the momentum and the kinetic energy are conserved.*

So, if we suppose that $\int_{\mathbb{R}^d} f(v,0)\,dv = 1$, we can define the probability flow $(P_t)_{t\geq 0}$ by $P_t(dv) = f(v,t)\,dv$.
Since the functions $z \longmapsto a_{ij}(z)$ and $z \longmapsto b_i(z)$ are respectively even and odd for any i,j, we obtain a new expression of the Landau weak formulation, which will be the base of our study,

$$\begin{aligned} \frac{d}{dt}\int \varphi(v)\,P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\,a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v) \qquad (1.3)\\ &+\sum_{i=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\,b_i(v-v_*)\right)\partial_i\varphi(v). \end{aligned}$$

Definition 1.2 *Let P_0 be a probability measure on $\mathbb{R}^d$ with a finite two-order moment (i.e. $\int_{\mathbb{R}^d}|v|^2 P_0(dv) < \infty$). A measure solution of the Landau equation (1.3) with initial data P_0 is a probability flow $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$ satisfying (0.1) for any function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$, where $\mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$ is the space of bounded functions of class $\mathcal{C}^2$ on $\mathbb{R}^d$ with bounded derivatives.*

Remark 1.3 *With an abuse of notation, we will still say that a probability measure P on $\mathcal{C}([0,T],\mathbb{R}^d)$ is a measure solution of the Landau equation when its time-marginals flow is a measure solution in the sense of Definition 1.2.*

There are two ways to solve the equation (1.3) in a probabilistic sense. The first consists in finding a probability measure P which satisfies a nonlinear martingale problem. Funaki, [22], solves this martingale problem when the matrix a is a nondegenerate matrix. But, the collision matrix a of the Landau equation is degenerated. The second way consists in associating with the Landau equation (1.3) a nonlinear stochastic differential equation driven by a space-time white noise. Those two methods are in relation. Indeed, a solution of the differential equation is a solution of the martingale problem and a solution of the martingale problem is a weak solution of the differential equation (see [17]).

The benefit of the second method is that one can develop a Malliavin calculus to state the existence of a density and then to show the existence of a weak function-solution of the Landau equation (1.2), when the coefficients are smooth enough. If, for any $t>0$, P_t has a density with respect the Lebesgue measure on $\mathbb{R}^d$, i.e. there exists a nonnegative function $f(.,t)$ such that $P_t(dv)=f(v,t)\,dv$, then f is a weak function-solution of the Landau equation (1.2). This question is studied in Chapter 2 of this thesis.

In this paper, we are firstly interested in solving the Landau equation with regular coefficients (for example, $\gamma=0$ and $h=$ constant). In this case, we solve a nonlinear differential stochastic equation driven by a white noise to find a measure solution of the Landau equation.

Secondly, using the results obtained in the first part, we study the Landau equation with $\gamma\in(-1,0]$ and h some bounded continuous function. We approximate the coefficients by some coefficients having the same regularity as in the first part. Then, thanks to the study of martingale problems and of nonlinear stochastic differential equations, we state the existence of a measure solution of the Landau equation with $\gamma\in(-1,0]$. Moreover, we obtain a weak solution for the associated nonlinear stochastic differential equation.

Notations

- $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ is the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and for $k\in\mathbb{N}$, $\mathcal{C}_b^k\left([0,T],\mathbb{R}^d\right)$ is the space of bounded functions of class $\mathcal{C}^k$ with all its derivatives bounded up to order k.

- $\mathcal{M}_{d,d'}(\mathbb{R})$ is the set of $d\times d'$ matrix on $\mathbb{R}$.

- If $(P^n)_{n\geq 0}$ and P are probability measures, we denote by $P^n\Longrightarrow P$ the convergence in distribution of the sequence (P^n) toward P.

- K is an arbitrary notation for a constant (K can change from line to line).

We consider, as it was mentioned above, a matrix a which has the following form:

$$a_{ij}(z)=|z|^{\gamma+2}\,h\left(|z|^2\right)\left(\delta_{ij}-z_iz_j/|z|^2\right) \tag{1.4}$$

with h some bounded nonnegative locally Lipschitz continuous function on $\mathbb{R}_+$ and $\gamma\in(-1,0]$. Then, the vector b has the following expression

$$b_i(z)=\sum_{j=1}^{d}\partial_j a_{ij}(z)=-(d-1)\,h\left(|z|^2\right)|z|^{\gamma}\,z_i. \tag{1.5}$$

For example, in dimension 2, a and b are given by

$$\begin{aligned} a(z) &= |z|^\gamma h\left(|z|^2\right)\begin{bmatrix} {z_2}^2 & -z_1z_2 \\ -z_1z_2 & {z_1}^2 \end{bmatrix}, \\ b(z) &= -|z|^\gamma h\left(|z|^2\right)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \end{aligned}$$

and in dimension 3, they are given by

$$\begin{aligned} a(z) &= |z|^\gamma h\left(|z|^2\right)\begin{bmatrix} {z_2}^2+{z_3}^2 & -z_1z_2 & -z_1z_3 \\ -z_1z_2 & {z_1}^2+{z_3}^2 & -z_2z_3 \\ -z_1z_3 & -z_2z_3 & {z_1}^2+{z_2}^2 \end{bmatrix}, \\ b(z) &= -2|z|^\gamma h\left(|z|^2\right)\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \end{aligned}$$

As a is a symmetric nonnegative matrix, there exists a matrix σ in $\mathcal{M}_{d,d'}(\mathbb{R})$ such that

$$a = \sigma.\sigma^* \tag{1.6}$$

where σ^* is the adjoint matrix of σ and d' is an integer≥ 1. There is not uniqueness of σ, one can take for example

$$\sigma(z) = \frac{1}{|z|^{\frac{\gamma}{2}+1}\sqrt{h\left(|z|^2\right)}}a(z) \tag{1.7}$$

($\Pi(z)$ *is a projection, then* $a(z).a(z) = |z|^{\gamma+2}h\left(|z|^2\right)a(z)$), or in dimension two

$$\sigma(z) = |z|^{\frac{\gamma}{2}}\sqrt{h\left(|z|^2\right)}\begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \tag{1.8}$$

and in dimension three

$$\sigma(z) = |z|^{\frac{\gamma}{2}}\sqrt{h\left(|z|^2\right)}\begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix} \tag{1.9}$$

If we denote by c a constant >0 such that $\forall z \in \mathbb{R}^d$ $h\left(|z|^2\right) \leq c$, one can notice that

$$\begin{aligned} |a(z)| &\leq c|z|^{\gamma+2} \\ |b(z)| &\leq (d-1)c|z|^{\gamma+1} \end{aligned}$$

and, in the previous examples, $|\sigma(z)| \leq \sqrt{c}|z|^{\frac{\gamma}{2}+1}$.

2 The Landau equation with regular coefficients

In this section, we deal with the case of Lipschitz continuous coefficients b and σ.

2.1 A nonlinear stochastic differential equation associated with the Landau equation

We associate with the Landau equation a nonlinear stochastic differential equation driven by a space-time white noise which gives a probabilistic interpretation of the Landau equation (1.3). We highlight the nonlinearity using two probability spaces.
Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and $([0,1], \mathcal{B}([0,1]), d\alpha)$ be an auxiliary probability space, where $d\alpha$ is the Lebesgue measure on $[0,1]$.
The Skorohod Theorem (see [35]) links up those two spaces : it states that for any probability measure P on the polish space $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$, with the topology of the uniform convergence, there exists a random variable $Y : ([0,1], \mathcal{B}([0,1]), d\alpha) \to \mathcal{C}\left([0,T], \mathbb{R}^d\right)$ which has the distribution P.
For the clarity of the exposition, we will denote by E the expectation and $\mathcal{L}$ the distribution of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and E_α, $\mathcal{L}_\alpha$ for a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$.
For $k \geq 2$, we define $\mathcal{P}_k$ the space of continuous adapted processes $X = (X_t)_{t\geq 0}$ from $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ to $\mathbb{R}^d$, such that $\forall T > 0$ $E\left[\sup_{0\leq t\leq T} |X_t|^k\right] < \infty$, and $\mathcal{P}_{k,\alpha}$ the space of continuous processes $Y = (Y_t)_{t\geq 0}$ from $([0,1], \mathcal{B}([0,1]), d\alpha)$ to $\mathbb{R}^d$, such that $\forall T > 0$ $E_\alpha\left[\sup_{0\leq t\leq T} |Y_t|^k\right] < \infty$.
Let $T > 0$ be arbitrary fixed. ρ_T denotes the Vaserstein metric on the space of probability measure on $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$ defined by

$$\rho_T^2(P,Q) = \inf\left\{E\left(\sup_{0\leq t\leq T} |A_t - B_t|^2\right) : \begin{array}{c} A \text{ and } B \text{ processes on } \mathcal{C}\left([0,T], \mathbb{R}^d\right) \\ \text{with distribution } P \text{ and } Q \text{ respectively} \end{array}\right\}$$

We define the d'-dimensional process $W^{d'}$ by

$$W^{d'} = \begin{pmatrix} W_1 \\ \vdots \\ W_{d'} \end{pmatrix}$$

where the W_i are independent space-time white noises with covariance measure $d\alpha dt$ on $[0,1] \times [0,\infty)$ (according to the definition of Walsh [42]).
Let X_0 be a random vector on $\mathbb{R}^d$ independent of $W^{d'}$ with a finite moment of order 2.
Let σ and b be the functions defined by (1.6) and (1.5) respectively.
We consider the following nonlinear stochastic differential equation

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \qquad (NSDE(\sigma, b))$$

with $\mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)$ $\forall t \geq 0$.

Proposition 2.1 *Assume that the coefficients b and σ are Lipschitz continuous. Let P_0 be a probability measure with a finite moment of order 2. Let X_0, Y_0 be random variables such that $\mathcal{L}(X_0) = \mathcal{L}_\alpha(Y_0) = P_0$. If we assume that there exists a solution (X,Y) of $(NSDE(\sigma,b))$, in $\mathcal{P}_2 \times \mathcal{P}_{2,\alpha}$, with initial data (X_0,Y_0), such that $\forall t \geq 0$ $\mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)$. Then the common flow $(P_t)_{t\geq 0}$ is a measure solution of the Landau equation with initial data P_0.*

Proof. Let $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$.
Using Itô's formula, we obtain

$$\begin{aligned}\varphi(X_t) &= \varphi(X_0) + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_{ij}\varphi(X_s)\,a_{ij}(X_s-y)\,P_s(dy)\,ds \\ &+\sum_{i=1}^{d}\int_0^t\int_{\mathbb{R}^d}\partial_i\varphi(X_s)\,b_i(X_s-y)\,P_s(dy)\,ds \\ &+\sum_{i=1}^{d}\sum_{k=1}^{d'}\int_0^t\int_0^1\sigma_{i,k}(X_s-Y_s(\alpha))\,\partial_i\varphi(X_s)\,W_k(d\alpha,ds).\end{aligned}$$

According to [42] Theorem 2.5, $\forall i,k$ $\int_0^t\int_0^1\sigma_{i,k}(X_s-Y_s(\alpha))\,\partial_i\varphi(X_s)\,W_k(d\alpha,ds)$ is a martingale. So the expectation of $\varphi(X_t)$ satisfies

$$\begin{aligned}E[\varphi(X_t)] &= E[\varphi(X_0)] + \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t E\left[\partial_{ij}\varphi(X_s)\left(\int_{\mathbb{R}^d}a_{ij}(X_s-y)\,P_s(dy)\right)\right]ds \\ &+\sum_{i=1}^{d}\int_0^t E\left[\partial_i\varphi(X_s)\left(\int_{\mathbb{R}^d}b_i(X_s-y)\,P_s(dy)\right)\right]ds.\end{aligned}$$

Since $\mathcal{L}(X_t) = P_t$ $\forall t \geq 0$, the proposition is proved. ■
Consequently, it is enough to solve the nonlinear stochastic differential equation to find a measure solution of the Landau equation.

2.2 Solving a nonlinear stochastic differential equation driven by a white noise

We use the same notations as in part 2.1.

Definition 2.2 *Let η and f be two continuous functions. Let $W^{d'}$ be a process on $\mathbb{R}^{d'}$ having independent white noises components on $[0,1]\times[0,+\infty)$ with covariance measure $d\alpha dt$ and X_0 be a random variable with finite moment of order 2. We consider Y_0 a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that $\mathcal{L}_\alpha(Y_0) = \mathcal{L}(X_0)$. We will say that a couple (X,Y) is solution of the nonlinear stochastic differential $(NSDE(\eta,f))$ if for any $t \geq 0$*

$$X_t = X_0 + \int_0^t\int_0^1\eta(X_s-Y_s(\alpha)).W^{d'}(d\alpha,ds) + \int_0^t\int_0^1 f(X_s-Y_s(\alpha))\,d\alpha ds$$

and $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$.

We state the existence of a solution of $(NSDE\,(\eta, f))$ under some conditions on the regularity of the functions η and f:
Assumption (H): η and f are globally Lipschitz continuous functions from $\mathbb{R}^d$ respectively to $\mathcal{M}_{d,d'}(\mathbb{R})$ and to $\mathbb{R}^d$, where d and d' are integers ≥ 1.

To simplify the expressions, we consider in this part $\mathbf{d} = \mathbf{d'}= \mathbf{1}$. Nevertheless the same arguments can be applied when the dimensions are higher.
The following method, based on a stochastic calculus for a white noise, is a variation of the method built by Desvillettes, Graham and Méléard, [12], in the different case of Poisson measure.

Definition 2.3 *Let W be a space-time white noise with covariance measure $d\alpha dt$ on $[0,1] \times [0, +\infty)$, X_0 an independent random variable with finite 2-order moment, Z a $\mathcal{P}_2$-process and Y a $\mathcal{P}_{2,\alpha}$-process. The following equation*

$$X_t = X_0 + \int_0^t \int_0^1 \eta\left(Z_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(Z_s - Y_s(\alpha)\right) d\alpha ds \qquad (1.10)$$

defines an application Φ by $Z, Y, X_0, W \longmapsto X = \Phi(Z, Y, X_0, W)$.

We first state a technical lemma:

Lemma 2.4 *If X_0 and W are such as in Definition 2.3. For $i = 1, 2$, we consider the processes $Z^i \in \mathcal{P}_2$ and $Y^i \in \mathcal{P}_{2,\alpha}$. We define $X^i = \Phi(Z^i, Y^i, X_0, W)$, $i = 1, 2$. Then $X^i \in \mathcal{P}_2$. Moreover, for any $T > 0$, there exists a constant $K > 0$ such that*

$$E\left[\sup_{0\leq t\leq T} \left|X_t^1 - X_t^2\right|^2\right] \leq K \left\{ \int_0^T E\left[\left|Z_s^1 - Z_s^2\right|^2\right] ds + \int_0^T E_\alpha\left[\left|Y_s^1 - Y_s^2\right|^2\right] ds \right\}.$$

Proof. It is clear that the processes $(X_t^i)_{t\geq 0}$ are continuous.
Let $T > 0$. Using the Burkholder-Davis-Gundy and the Hölder inequalities, we obtain that

$$\begin{aligned} E\left[\sup_{0\leq t\leq T} \left|X_t^1 - X_t^2\right|^2\right] \;\leq\; & 2\left\{ CE\left(\int_0^T \int_0^1 \left[\eta\left(Z_s^1 - Y_s^1(\alpha)\right) - \eta\left(Z_s^2 - Y_s^2(\alpha)\right)\right]^2 d\alpha ds \right)\right. \\ & \left. + TE\left(\int_0^T \int_0^1 \left[f\left(Z_s^1 - Y_s^1(\alpha)\right) - f\left(Z_s^2 - Y_s^2(\alpha)\right)\right]^2 d\alpha ds \right) \right\}. \end{aligned}$$

Since η and f are Lipschitz continuous, if we denote by K_η and K_f their lipschitz constant respectively, we have

$$E\left[\sup_{0\leq t\leq T} \left|X_t^1 - X_t^2\right|^2\right] \leq 4\left(CK_\eta^2 + TK_f^2\right) \left\{ \int_0^T E\left(\left|Z_s^1 - Z_s^2\right|^2\right) ds + \int_0^T E_\alpha\left(\left|Y_s^1 - Y_s^2\right|^2\right) ds \right\}.$$

The lemma is proved. ■
We give now a solving method for a standard stochastic differential equation.

Theorem 2.5 *Assume that W is a space-time white noise with covariance measure $d\alpha dt$ on $[0,1]\times[0,\infty)$, X_0 is an independent random variable with finite 2-order moment and Y a $\mathcal{P}_{2,\alpha}$-process. If η and f satisfy the Assumption (H), the equation*

$$X_t = X_0 + \int_0^t \int_0^1 \eta\left(X_s - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s - Y_s(\alpha)\right) d\alpha ds \qquad (1.11)$$

has a unique strong solution X belonging to $\mathcal{P}_2$.

Proof. We prove the existence of a solution of (1.11) which belongs to $\mathcal{P}_2$ using a standard method of approximation of the solution by the following Picard sequence, for any $t \geq 0$

$$\begin{aligned} X_t^0 &= X_0 \\ X_t^{n+1} &= X_0 + \int_0^t \int_0^1 \eta\left(X_s^n - Y_s(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s^n - Y_s(\alpha)\right) d\alpha ds. \end{aligned}$$

(The proof is easy and can be adapted from the proof of theorem 2.7.)
Moreover, using Gronwall's Lemma, we state the strong uniqueness on $[0,T]$ for any $T>0$. ■

Remark 2.6 *Let X be a solution of the stochastic differential equation (1.11). We write $X = \Phi(X, Y, X_0, W)$.*
The strong uniqueness of X implies as usual the uniqueness in law. Moreover, the distribution of X depends on the process Y only through the distribution family $\left(\mathcal{L}_\alpha(Y_t)\right)_{t\geq 0}$. Consequently if Y and Y' are two α-processes such that $\left(\mathcal{L}_\alpha(Y_t)\right)_{t\geq 0} = \left(\mathcal{L}_\alpha(Y_t')\right)_{t\geq 0}$, the solutions of the stochastic differential equations $X = \Phi(X, Y, X_0, W)$ and $X' = \Phi(X', Y', X_0, W)$ have the same distribution: $\mathcal{L}(X) = \mathcal{L}(X')$.

Proof. We define, for any $t \geq 0$, the flow $P_t = \mathcal{L}_\alpha(Y_t)$ and a martingale measure W^P on $[0,1]\times[0,+\infty)$ such that $\forall A \in \mathcal{B}([0,1])$, $\forall t \geq 0$,

$$W_t^P(A) = \int_0^t \int_0^1 \mathbb{I}_A\left(Y_s(\alpha)\right) W(d\alpha, ds).$$

We notice that

$$\begin{aligned} \mathcal{L}\left(W_t^P(A)\right) &= \mathcal{N}\left(0, \int_0^t \int_0^1 \mathbb{I}_A\left(Y_s(\alpha)\right) d\alpha ds\right) \\ &= \mathcal{N}\left(0, \int_0^t \int_{\mathbb{R}} \mathbb{I}_A(v) P_s(dv) ds\right) \end{aligned}$$

where $\mathcal{N}(\lambda, k)$ is the Normal distribution with expectation λ and variance k. Moreover, if $A \cap B = \emptyset$, we have $W_t^P(A\cup B) = W_t^P(A) + W_t^P(B)$.
So W^P is a white noise with covariance measure $P_s(dv)\,ds$ (according to the definition of Walsh, [42]). Then, we can rewrite (1.11) in the following way

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} \eta\left(X_s - y\right) W^P(dy, ds) + \int_0^t \int_{\mathbb{R}} f\left(X_s - y\right) P_s(dy)\, ds.$$

A white noise is entirely defined by its covariance measure, and the one of W^P is $\nu(dy, ds) = P_s(dy)\,ds$. Consequently, the distribution of X depends only on the distribution of Y through its flow $(\mathcal{L}_\alpha(Y_t))_{t\geq 0}$. ■

We now study the nonlinear stochastic differential equation $(NSDE(\eta, f))$.

Theorem 2.7 *Assume that $W^{d'}$ is a process on $\mathbb{R}^{d'}$ having independent white noises components on $[0,1] \times [0,+\infty)$ with covariance measure $d\alpha dt$, and assume that X_0 is an independent random vector on $\mathbb{R}^d$ with finite moment of order 2. Then, under the Assumption (H), there exists a couple (X,Y) solution of the nonlinear equation $(NSDE(\eta, f))$. Moreover, $(X,Y) \in \mathcal{P}_2 \times \mathcal{P}_{2,\alpha}$.*
We notice that the distribution of X depends only on the distribution $P_0 = \mathcal{L}(X_0)$ and not on the specific choice of the white noise and of X_0.

Proof. We prove this theorem in dimension $d = d' = 1$. The proof is almost the same in higher dimension if we work with each component, but the expressions are more complex. We now use a generalization of the Picard iteration method. We construct two recursive sequences:

- Let X^0 such that $\forall s \geq 0$ $X_s^0 = X_0$ and Y^0 such that $\forall s \geq 0$ $Y_s^0 = Y_0$, where Y_0 is a random variable on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that $\mathcal{L}_\alpha(Y_0) = \mathcal{L}(X_0)$ (obtained by Skorohod's Theorem).

- We define

$$X_t^{n+1} = X_0 + \int_0^t \int_0^1 \eta\left(X_s^n - Y_s^n(\alpha)\right) W(d\alpha, ds) + \int_0^t \int_0^1 f\left(X_s^n - Y_s^n(\alpha)\right) d\alpha ds.$$

  On the probability space $([0,1], \mathcal{B}([0,1]), d\alpha)$, we construct a continuous process Y^{n+1} such that $\mathcal{L}_\alpha(Y^{n+1} \mid Y^0, .., Y^n) = \mathcal{L}(X^{n+1} \mid X^0, .., X^n)$.

  In particular, we have for any $n \geq 0$ $\mathcal{L}_\alpha(Y^0, .., Y^n) = \mathcal{L}(X^0, .., X^n)$.

Let us define $g_n(t) = E\left[\sup_{0\leq s\leq t}\left(X_s^{n+1} - X_s^n\right)^2\right]$.
Lemma 2.4 implies

$$\begin{aligned}
g_n(t) &\leq K\left\{\int_0^t E\left[\left|X_s^n - X_s^{n-1}\right|^2\right] ds + \int_0^t E_\alpha\left[\left|Y_s^n - Y_s^{n-1}\right|^2\right] ds\right\} \\
&= 2K\int_0^t E\left[\left|X_s^n - X_s^{n-1}\right|^2\right] ds \leq 2K\int_0^t g_{n-1}(s)\,ds \\
&\ \ \vdots \\
&\leq (2K)^n \int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} g_0(t_n)\,dt_n.
\end{aligned}$$

For a fixed $T > 0$, it is easy to state that g_0 is bounded on $[0,T]$.
If we define $C = \sup_{0\leq t\leq T} g_0(t)$, we have $g_n(t) \leq C(2K)^n T^n/n!$.

Then, for any $T > 0$, the sequence $(X^n)_{n\geq 0}$ converges for the norm $\|U\| = \left\| \sup_{0\leq s\leq T} U_s \right\|_{\mathbb{L}^2}$ and, using Borel-Cantelli's Lemma, (X^n) converges almost surely uniformly on $[0,T]$ toward a continuous process X. Consequently, $(Y^n)_{n\geq 0}$ converges also in $\mathbb{L}^2$ and a.s.. We denote by Y its limit. Since $\mathcal{L}_\alpha(Y^0,..,Y^n) = \mathcal{L}(X^0,..,X^n)\ \forall n$, we have $\mathcal{L}_\alpha(Y) = \mathcal{L}(X)$. In particular, for any $T > 0$,

$$\sup_{n\geq 0} E\left(\sup_{0\leq t\leq T} |X_t^n|^2\right) = \sup_{n\geq 0} E_\alpha\left(\sup_{0\leq t\leq T} |Y_t^n|^2\right) < \infty.$$

Using dominated convergence theorem, we easily check that (X,Y) is effectively a solution of the nonlinear stochastic differential equation

$$X_t \overset{a.s.}{=} X_0 + \int_0^t \int_0^1 \eta(X_s - Y_s(\alpha))\, W(d\alpha, ds) + \int_0^t \int_0^1 f(X_s - Y_s(\alpha))\, d\alpha ds$$

Moreover, thanks to the strong uniqueness proved in Theorem 2.5 and consequently to the uniqueness in law, the distribution of X depends only on $P_0 = \mathcal{L}(X_0)$. ■

Theorem 2.8 *Under Assumption (H), uniqueness in law holds for a solution of $((NSDE(\eta, f))$.*

Proof. Assume that (U,V) is a solution on $\mathcal{C}([0,T], \mathbb{R}^d)$ with initial data X_0 of

$$U = \Phi(U, V, X_0, W) \quad \text{with } \mathcal{L}_\alpha(V) = \mathcal{L}(U) = Q.$$

Assume that (X,Y) is the solution given by Theorem 2.7 of

$$X = \Phi(X, Y, X_0, W) \ \text{ with } \mathcal{L}_\alpha(Y) = \mathcal{L}(X) = P.$$

We want to state that $P = Q$.
Let $T > 0$.
Let $\tau \in \left]0, T\right]$, let ρ_τ be the Vaserstein metric on the space of probability measures on $\mathcal{C}([0,\tau], \mathbb{R}^d)$ defined by

$$\rho_\tau(P,Q)^2 = \inf\left\{ E_\alpha\left(\sup_{0\leq t\leq \tau} |A_t - B_t|^2\right) : \mathcal{L}_\alpha(A) = P, \mathcal{L}_\alpha(B) = Q \right\}.$$

We prove that there exists at least one $\tau > 0$ such that $\rho_\tau(P,Q) = 0$.
Let $\varepsilon > 0$, there exists A^ε and B^ε, two $\mathcal{P}_{2,\alpha}$-processes, such that $\mathcal{L}_\alpha(A^\varepsilon) = P, \mathcal{L}_\alpha(B^\varepsilon) = Q$ and

$$\rho_\tau(P,Q)^2 \leq E_\alpha\left(\sup_{0\leq t\leq \tau} |A_t^\varepsilon - B_t^\varepsilon|^2\right) \leq \rho_\tau(P,Q)^2 + \varepsilon.$$

Let X^ε be the solution of $X^\varepsilon = \Phi(X^\varepsilon, A^\varepsilon, X_0, W)$ given by Theorem 2.5. Since $\mathcal{L}_\alpha(A^\varepsilon) = \mathcal{L}_\alpha(Y) = P$ and following Remark 2.6, we have $\mathcal{L}(X^\varepsilon) = \mathcal{L}(X)$.
If U^ε is the solution of $U^\varepsilon = \Phi(U^\varepsilon, B^\varepsilon, X_0, W)$ obtained in Theorem 2.5, we have also $\mathcal{L}(U^\varepsilon) = \mathcal{L}(U)$.

Thanks to the bound given in the proof of Lemma 2.4, we have the following inequality

$$\begin{aligned} E\left[\sup_{0\le t\le\tau}\left|X_t^\varepsilon-U_t^\varepsilon\right|^2\right] &\le 4\left(CK_\eta^2+\tau K_f^2\right)\left\{\int_0^\tau E\left[\left|X_s^\varepsilon-U_s^\varepsilon\right|^2\right]ds+\int_0^\tau E_\alpha\left[\left|A_s^\varepsilon-B_s^\varepsilon\right|^2\right]ds\right\} \\ &\le 4\left(CK_\eta^2+\tau K_f^2\right)\left\{\int_0^\tau E\left[\sup_{0\le u\le s}\left|X_u^\varepsilon-U_u^\varepsilon\right|^2\right]ds+\tau\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right)\right\}\end{aligned}$$

and by Gronwall's Lemma, we obtain

$$E\left[\sup_{0\le t\le\tau}\left|X_t^\varepsilon-U_t^\varepsilon\right|^2\right]\le 4\tau\left(CK_\eta^2+\tau K_f^2\right)\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right).$$

Thus, for any $\varepsilon>0$,

$$\rho_\tau\left(P,Q\right)^2\le 4\tau\left(CK_\eta^2+\tau K_f^2\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right)\left(\rho_\tau\left(P,Q\right)^2+\varepsilon\right).$$

If we choose $\tau>0$ such that $4\tau\left(CK_\eta^2+\tau K_f^2\right)\exp\left(4\tau\left(CK_\eta^2+\tau K_f^2\right)\right)<1$, then $\rho_\tau\left(P,Q\right)=0$.
We have uniqueness in law on $[0,\tau]$, but we would like to obtain uniqueness in law on $[0,T]$. We will extend the property by induction.
For $n\ge1$, we define $X^n=\left(X_{n\tau+t}\right)_{t\ge0}$, and we define similarly Y^n,U^n,V^n ....
Let us assume that we have uniqueness in law on $[0,n\tau]$. Then, in particular, $\mathcal{L}\left(X_{n\tau}\right)=\mathcal{L}\left(U_{n\tau}\right)$. We consider the process $\tilde{U}$ solution of

$$\tilde{U}_{t+n\tau}=X_{n\tau}+\int_{n\tau}^{t+n\tau}\int_0^1\eta\left(\tilde{U}_s-V_s\left(\alpha\right)\right)W\left(d\alpha,ds\right)+\int_{n\tau}^{t+n\tau}\int_0^1 f\left(\tilde{U}_s-V_s\left(\alpha\right)\right)d\alpha ds \tag{1.12}$$

with initial data $X_{n\tau}$.
We can rewrite (1.12) in the following way

$$\tilde{U}_t^n=X_{n\tau}+\int_0^t\int_0^1\eta\left(\tilde{U}_s^n-V_s^n\left(\alpha\right)\right)\tilde{W}\left(d\alpha,ds\right)+\int_0^t\int_0^1 f\left(\tilde{U}_s^n-V_s^n\left(\alpha\right)\right)d\alpha ds$$

where $\tilde{W}$ is a white noise with covariance $d\alpha dt$ on $[0,1]\times[0,\infty)$ defined by $\forall A\in\mathcal{B}\left([0,1]\right)$

$$\begin{aligned}\tilde{W}\left(A\times[0,t]\right) &= W(A\times[0,n\tau+t])-W\left(A\times[0,n\tau]\right) \\ &= W\left(A\times[n\tau,n\tau+t]\right)\end{aligned}$$

(if $A\in\mathcal{B}\left([0,1]\right)$ is fixed, $\left(W\left(A\times[0,t]\right)\right)_{t\ge0}$ is an independent increment process).
According to the uniqueness in law, obtained in Remark 2.6, $\mathcal{L}\left(\tilde{U}^n\right)=\mathcal{L}\left(U^n\right)$ on $[0,\tau]$ and thus $\mathcal{L}\left(\tilde{U}^n\right)=\mathcal{L}_\alpha\left(V^n\right)$ on $[0,\tau]$.

Consequently, thanks to the first part of this proof, we have $\mathcal{L}\left(\tilde{U}^n\right)=\mathcal{L}\left(X^n\right)$ on $[0,\tau]$. We deduce from the recurrent hypothesis that the flows $\left(\mathcal{L}_\alpha\left(V_t\right)\right)_{0\le t\le\tau+n\tau}$ and $\left(\mathcal{L}_\alpha\left(Y_t\right)\right)_{0\le t\le\tau+n\tau}$ are the same. According to Remark 2.6, we have $\mathcal{L}\left(X\right)=\mathcal{L}\left(U\right)$ on $[0,(n+1)\tau]$. Hence, by induction, we conclude $\mathcal{L}\left(X\right)=\mathcal{L}\left(U\right)$ on $[0,T]$ for any $T>0$.
■

2.3 Existence of a measure solution of the Landau equation with regular coefficients

In the previous part, we have proved the existence and uniqueness in law of a solution of the nonlinear stochastic differential equation $(NSDE(\eta, f))$ when η and f satisfy Assumption (H).

According to Proposition 2.1, we have finally stated the following theorem

Theorem 2.9 *Assume that P_0 is a probability measure with a finite moment of order 2. There is a measure solution $(P_t)_{t\geq 0}$ with initial data P_0 to the Landau equation*

$$\begin{aligned}\frac{d}{dt}\int \varphi(v)\, P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, a_{ij}(v-v_*)\right)\partial_{ij}\varphi(v) \\ &+\sum_{i=1}^{d}\int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, b_i(v-v_*)\right)\partial_i\varphi(v)\end{aligned}$$

where $(a_{ij})_{0\leq i,j\leq d}$ is a matrix of the form $a=\sigma.\sigma^*$, with σ and b satisfying Assumption (H).

Remark 2.10 *If we assume that $\gamma = 0$, choosing for σ the expression (1.8) in dimension 2, or (1.9) in dimension 3, we can notice that if h is a bounded nonnegative function of class C^1 such that there exists a constant $K>0$ with $h'(x)\leq \frac{K}{x^2}$ when $x\to+\infty$, σ and b satisfy Assumption (H). In particular, if h is a constant function (the Maxwellian case), σ and b satisfy Assumption (H). We can generalize those properties in dimension $d\geq 3$.*

When the initial data is a probability measure with a finite moment of order 2, we have thus proved the existence of a measure solution of the Landau equation (1.3) under some conditions on the function h. Nevertheless, with this approach, we cannot state the uniqueness of a measure solution.

3 Study of the Landau equation for some soft potentials

We use the same notations as in Section 2.

The case $\gamma\in(-1,0]$ with h some bounded locally Lipschitz continuous nonnegative function is more difficult than the previous case, because the continuous coefficients b and σ are no more Lipschitz continuous on $\mathbb{R}^d$. We will use the results obtained in the Section 2 approaching the coefficients σ and b by two sequences (σ^n) and (b^n) of Lipschitz continuous functions. Then, for any $n\geq 0$, we build a sequence of random couples (X^n, Y^n) solution of the nonlinear differential equation:

$$X_t^n = X_0 + \int_0^t\int_0^1 \sigma^n\left(X_s^n - Y_s^n(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t\int_0^1 b^n\left(X_s^n - Y_s^n(\alpha)\right)d\alpha ds \qquad (NSDE(\sigma^n, b^n))$$

Our aim is to show that the sequence (X^n) converges, in a certain sense, toward a process X, and, if we denote by P the distribution of X, to state that P satisfies a nonlinear martingale problem. We will see that this last property has two mains consequences: the existence of a measure solution of the Landau equation when $\gamma \in (-1,0]$ and h some bounded locally Lipschitz continuous function, and the existence of a weak solution of the nonlinear stochastic differential equation:

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \tag{$NSDE(\sigma,b)$}$$

where σ and b are defined by (1.6), (1.4) and (1.5).
For the last stage (theorem 3.7), we use results obtained by El Karoui and Méléard in [17], and thereby we need a symmetric condition on σ *(consequently, $d' = d$)*. So we choose in this section the expression (1.7) given in the introduction, i.e. $\sigma_{ij}(z) = |z|^{\frac{\gamma}{2}+1}\sqrt{h\left(|z|^2\right)}\left(\delta_{ij} - \frac{z_i z_j}{|z|^2}\right)$.
As $\gamma \in (-1,0]$, we can notice that σ and b have linear growth: if we denote by $c = \sup_{z\in\mathbb{R}^d} h\left(|z|^2\right)$, we have (differentiating the case the case $|z| \geq 1$ from the case $|z| < 1$)

$$\begin{aligned} |b(z)| &\leq c|z|^{\gamma+1} \leq c(d-1)(|z|+1), && (1.13)\\ |\sigma(z)| &\leq \sqrt{c}|z|^{\frac{\gamma}{2}+1} \leq \sqrt{c}(|z|+1). && (1.14) \end{aligned}$$

Those inequalities will be very helpful below.

We give first a technical lemma.

Lemma 3.1 *We assume that $X_0 \in \mathbb{L}^k$ for $k \geq 2$. If $Z \in \mathcal{P}^k$ and $Y \in \mathcal{P}^k_\alpha$, then the process X defined by*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(Z_s - Y_s(\alpha)\right).W^d(d\alpha, ds) + \int_0^t \int_0^1 b\left(Z_s - Y_s(\alpha)\right) d\alpha ds$$

belongs to $\mathcal{P}^k$.

Proof. The i^{th} component of X is given by

$$X_{i,t} = X_{i,0} + \sum_{j=1}^d \int_0^t \int_0^1 \sigma_{i,j}\left(Z_s - Y_s(\alpha)\right) W_j(d\alpha, ds) + \int_0^t \int_0^1 b_i\left(Z_s - Y_s(\alpha)\right) d\alpha ds.$$

For some $T > 0$, by the Burkholder-Davis-Gundy and the Hölder inequalities, we have

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}|X_{i,t}|^k\right] \leq\ & 3^{k-1}\left\{E\left[|X_{i,0}|^k\right] + C_k d^{\frac{k-2}{2}} T^{\frac{k-2}{2}} \sum_{j=1}^d \int_0^T \int_0^1 E\left(|\sigma_{i,j}\left(Z_s - Y_s(\alpha)\right)|^k\right) d\alpha ds\right. \\ & \left. + T^{k-1} \int_0^T \int_0^1 E\left(|b_i\left(Z_s - Y_s(\alpha)\right)|^k\right) d\alpha ds\right\}. \end{aligned}$$

Since $\gamma \in (-1, 0]$, using (1.13) and (1.14), we obtain

$$\begin{aligned} E\left[\sup_{0\le t\le T}|X_{i,t}|^k\right] &\le 3^{k-1}\left\{E\left[|X_{i,0}|^k\right] + C_k d^{\frac{k}{2}} T^{\frac{k-2}{2}} c^{\frac{k}{2}} \int_0^T\int_0^1 E\left((|Z_s - Y_s(\alpha)| + 1)^k\right) d\alpha ds\right. \\ &\left. + T^{k-1}c^k (d-1)^k \int_0^T\int_0^1 E\left((|Z_s - Y_s(\alpha)| + 1)^k\right) d\alpha ds\right\}. \end{aligned}$$

So, there exists $K > 0$ such that

$$E\left[\sup_{0\le t\le T}|X_t|^k\right] \le K\left\{E\left[|X_0|^k\right] + \int_0^T E\left(|Z_s|^k\right) ds + \int_0^T E_\alpha\left(|Y_s|^k\right) ds\right\}.$$

The lemma is proved. ■

3.1 Approximation of the solution

Construction of the approximation

Let χ be an even smooth function: $\chi(z) = \begin{cases} 1 \text{ if } |z| \ge 2 \\ 0 \text{ if } |z| \le 1 \end{cases}$ such that for any $z \in \mathbb{R}^d$, $0 \le \chi(z) \le 1$.
We define

$$\begin{aligned} a^n(z) &= \chi(nz)\, a(z), \\ b^n(z) &= \chi(nz)\, b(z), \\ \sigma^n(z) &= \sqrt{\chi(nz)}\sigma(z). \end{aligned}$$

Then, σ^n and b^n satisfy the Assumption (H) of Section 2. Moreover, we can notice that

$$|a^n| \le |a|, \quad |b^n| \le |b| \text{ and } |\sigma^n| \le |\sigma|.$$

We consider the following approximation of the Landau equation: for any $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$

$$\begin{aligned} \frac{d}{dt}\int \varphi(v) P_t^n(dv) &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}^d} P_t^n(dv)\left(\int_{\mathbb{R}^d} P_t^n(dv_*)\, a_{ij}^n(v - v_*)\right)\partial_{ij}\varphi(v) \\ &+ \sum_{i=1}^d \int_{\mathbb{R}^d} P_t^n(dv)\left(\int_{\mathbb{R}^d} P_t^n(dv_*)\, b_i^n(v - v_*)\right)\partial_i\varphi(v). \end{aligned} \tag{1.15}$$

(We have chosen χ even to keep the conservation of the mass, of the momentum and of the energy in the approximation of the Landau equation.)
For any arbitrary $T > 0$, we define as follows the martingale problem (MP^n) associated with this equation: let X be the canonical process on $\mathcal{C}([0,T], \mathbb{R}^d)$ (i.e., for $w \in \mathcal{C}([0,T], \mathbb{R}^d)$ $X_t(\omega) = w(t)$), and let us define the second order differential operator

$$L^n(Q)\varphi(x) = \frac{1}{2}\sum_{i,j=1}^d \int a_{ij}^n(x-y)\, Q(dy)\, \partial_{ij}^2\varphi(x) + \sum_{i=1}^d \int b_i^n(x-y)\, Q(dy)\, \partial_i\varphi(x)$$

where Q is a probability measure and $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$.
We will say that a probability measure Q on $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$ is a solution of the nonlinear martingale problem (MP^n) if

$$M_t^n = \varphi\left(X_t\right) - \varphi\left(X_0\right) - \int_0^t L^n\left(Q_s\right)\varphi\left(X_s\right) ds$$

is a Q-martingale for any $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$, where $Q_s = Q \circ X_s^{-1}$. Taking the expectation of M_t^n, we notice that a solution of the martingale problem is a measure solution of (1.15).
If we assume that $X_0 \in \mathbb{L}^k$, adapting the proofs of Section 2.2, we show the existence of a solution $(X^n, Y^n) \in \mathcal{P}^k \times \mathcal{P}_\alpha^k$, unique in law, of the nonlinear stochastic differential equation

$$X_t^n = X_0 + \int_0^t \int_0^1 \sigma^n\left(X_s^n - Y_s^n(\alpha)\right).W^d(d\alpha, ds) + \int_0^t \int_0^1 b^n\left(X_s^n - Y_s^n(\alpha)\right) d\alpha ds.$$

Moreover, if we denote by $P^n = \mathcal{L}\left(X^n\right) = \mathcal{L}_\alpha\left(Y^n\right)$ the common distribution, P^n satisfies the martingale problem (MP^n), for any n.

Tightness of the sequence (P^n)

Proposition 3.2 *Assume that X_0 is a square integrable random vector of $\mathbb{R}^d$, the sequence of probability distributions (P^n) built in Section 3.1 is tight.*

According to Aldous's criterium and Rebolledo's criterium (see [29]), it is enough to prove the following lemma to state the proposition.

Lemma 3.3 *If $X_0 \in \mathbb{L}^k$, with $k \geq 2$, there is a constant $C > 0$ such that for any $T > 0$ $\sup_{n\geq 0} E\left[\sup_{0\leq t\leq T}\left|X_t^n\right|^k\right] < C$.*

Proof. (Lemma 3.3)
We notice that $\left|\sigma^n(z)\right| \leq \sqrt{c}\,|z|^{\frac{\gamma}{2}+1}$ and $\left|b^n(z)\right| \leq c\,(d-1)\,|z|^{\gamma+1}$ where $c = \sup_{z\in\mathbb{R}^d} h\left(|z|^2\right)$.
Since $\mathcal{L}\left(X^n\right) = \mathcal{L}_\alpha\left(Y^n\right)$, according to the proof of Lemma 3.1, we have

$$\begin{aligned} E\left[\sup_{0\leq u\leq t}\left|X_u^n\right|^k\right] &\leq 3^{k-1}\left\{E\left[\left|X_0\right|^k\right] + K\int_0^t E\left(\left|X_s^n\right|^{k\left(\frac{\gamma}{2}+1\right)} + \left|X_s^n\right|^{k(\gamma+1)}\right) ds\right\} \\ &\leq K_1 + K_2\int_0^t E\left(\sup_{0\leq u\leq s}\left|X_u^n\right|^k\right) ds \end{aligned}$$

with K_1 and K_2 independent of n.
Using Gronwall's Lemma, we have $E\left[\sup_{0\leq u\leq T}\left|X_u^n\right|^k\right] \leq K_1 e^{K_2 T}$. The lemma is proved.
■
Proof. (Proposition 3.2)

We denote by $M^n + A^n$ the Doob-Meyer decomposition of X^n, i.e.

$$\begin{aligned} M_t^n &= \int_0^t \int_0^1 \sigma^n \left(X_s^n - Y_s^n(\alpha)\right) .W^d(d\alpha, ds) \\ A_t^n &= X_0 + \int_0^t \int_0^1 b^n \left(X_s^n - Y_s^n(\alpha)\right) d\alpha ds \end{aligned}$$

then for any $T > 0$ there exists a constant $K > 0$ such that for any $\eta > 0$, $\delta > 0$, $t \in [0, T]$ and $n \geq 0$

$$\sup_{\theta \leq \delta} \mathbb{P}\left(\left|A_{t+\theta}^n - A_t^n\right| > \eta\right) \leq KE\left[\sup_{0 \leq s \leq T} |X_s^n|^2\right] \frac{\delta^2}{\eta^2}$$

and

$$\sup_{\theta \leq \delta} \mathbb{P}\left(\left|\langle M^n \rangle_{t+\theta} - \langle M^n \rangle_t\right| > \eta\right) \leq KE\left[\sup_{0 \leq s \leq T} |X_s^n|^2\right] \frac{\delta}{\eta}$$

where $\langle M \rangle$ is the bracket of M. According to Lemma 3.3, the two sequences (A^n) and (M^n) satisfy the hypothesis of Aldous's criterium. Then, for any $T > 0$, according to Rebolledo's criterium, the sequence (P^n), where P^n is the distribution of (X^n), is tight in the space of probability measures on $\mathcal{C}\left([0, T], \mathbb{R}^d\right)$. ■
Consequently, there is a subsequence of (P^n) which converges toward a probability distribution P. Let us now identify this distribution.

3.2 The nonlinear martingale problem associated with the probability measure P

For a probability measure Q, we define the elliptic operator

$$L(Q)\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{d} \int a_{ij}(x-y) Q(dy) \partial_{ij}^2 \varphi(x) + \sum_{i=1}^{d} \int b_i(x-y) Q(dy) \partial_i \varphi(x)$$

where $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$.
For any arbitrary $T > 0$, we define the nonlinear martingale problem (MP) : a probability measure Q on $\mathcal{C}\left([0, T], \mathbb{R}^d\right)$ is a solution of (MP) if

$$M_t = \varphi(X_t) - \varphi(X_0) - \int_0^t L(Q_s) \varphi(X_s) ds$$

is a Q-martingale, where $Q_s = Q \circ X_s^{-1}$.

Theorem 3.4 *Let P_0 be a probability measure with a finite 2-order moment if $\gamma \in (-1, 0)$ and with a finite 3-order moment if $\gamma = 0$. Let P^n be a solution of (MP^n) with initial data P_0 for any $n \geq 0$ and P be a cluster point of the sequence (P^n). Then P satisfies the martingale problem (MP).*

Remark 3.5 *Assume that P_0 has a finite k-order moment. Let P be a cluster point of (P^n). Thanks to Lemma 3.3, there exists a constant $C > 0$ such that*

$$E_P\left[\sup_{0\leq t\leq T}|X_t|^k\right] < C$$

where E_P is the expectation under the distribution P.

Proof. (Remark 3.5)
Up to a subsequence, (P^n) converges toward the distribution P. According to the Skorohod Theorem (see [35]), there exists a sequence of random processes $(Y^n)_{n\geq 0}$ and a process Y defined on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that

$$\begin{aligned} \mathcal{L}_\alpha(Y^n) &= P^n \ \ \forall n \geq 0 \\ \mathcal{L}_\alpha(Y) &= P \end{aligned}$$

and $Y^n \underset{n\to\infty}{\longrightarrow} Y$ a.s.. Using Fatou's Lemma and Lemma 3.3, we notice that

$$E_P\left[\sup_{0\leq t\leq T}|X_t|^k\right] \leq \liminf_{n\to\infty} E_{P^n}\left[\sup_{0\leq t\leq T}|X_t|^k\right] \leq C.$$

■

Proof. (Theorem 3.4)
Let M be the process defined by $M_t = \varphi(X_t) - \varphi(X_0) - \int_0^t L(P_s)\varphi(X_s)\,ds$.
To prove that P satisfies the martingale problem (MP), we have to state that M is a P-martingale.
Let (g_i) be a sequence of continuous bounded functions.
M is a P-martingale if and only if for any $0 \leq s \leq t$, $k \geq 1$ and $0 \leq s_1 \leq .. \leq s_k \leq s$, M satisfies

$$E_P\left[(M_t - M_s)\, g_1(X_{s_1})\ldots g_k(X_{s_k})\right] = 0.$$

We choose $0 \leq s \leq t$, $k \geq 1$ and $0 \leq s_1 \leq .. \leq s_k \leq s$.
We know that, for any n, P^n is a solution of the martingale problem (MP^n). We will still denote by (P^n) a subsequence of (P^n) which converges toward $P : P^n \Longrightarrow P$.
As M^n is a P^n-martingale, we have $E_{P^n}\left[(M_t^n - M_s^n)\, g_1(X_{s_1})\ldots g_k(X_{s_k})\right] = 0$.
Let us prove in the following that

$$E_{P^n}\left[(M_t^n - M_s^n)\, g_1(X_{s_1})\ldots g_k(X_{s_k})\right] \underset{n\to\infty}{\longrightarrow} E_P\left[(M_t - M_s)\, g_1(X_{s_1})\ldots g_k(X_{s_k})\right]. \quad (1.16)$$

Since φ, g_1,..., g_k are bounded continuous functions and $x \to x_t$ is a continuous function, $x \to \varphi(x_t)\, g_1(x_t)\ldots g_k(x_t)$ is a bounded continuous function, and then $\forall t \geq 0$

$$E_{P^n}\left[\varphi(X_t)\, g_1(X_t)\ldots g_k(X_t)\right] \underset{n\to\infty}{\longrightarrow} E_P\left[\varphi(X_t)\, g_1(X_t)\ldots g_k(X_t)\right]. \quad (1.17)$$

Knowing convergence (1.17), we just have to check the following convergence

$$\begin{aligned} &E_{P^n}\left[\left(\int_s^t L^n(P_u^n)\,\varphi(X_u)\,du\right) g_1(X_{s_1})\ldots g_k(X_{s_k})\right] \\ &\qquad \underset{n\to\infty}{\longrightarrow} E_P\left[\left(\int_s^t L(P_u)\,\varphi(X_u)\,du\right) g_1(X_{s_1})\ldots g_k(X_{s_k})\right]. \end{aligned} \quad (1.18)$$

We can write

$$\begin{aligned}
&E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&\qquad -E_P\left[\left(\int_s^t L\left(P_u\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&= E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&\qquad -E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&+E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&\qquad -E_P\left[\left(\int_s^t L\left(P_u\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right].
\end{aligned} \tag{1.19}$$

We will use a product-space to simplify those expressions. As $P^n \Longrightarrow P$, we notice that $P^n \otimes P^n \Longrightarrow P \otimes P$ when n goes to $+\infty$. If we denote by (X, Y) the canonical process on $\mathcal{C}\left([0,T],\mathbb{R}^d\right) \times \mathcal{C}\left([0,T],\mathbb{R}^d\right)$, we notice that

$$\begin{aligned}
&E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&= \frac{1}{2}\sum_{i,j=1}^d \int_s^t E_{P^n\otimes P^n}\left[a_{ij}^n\left(X_u - Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]du \\
&+ \sum_{i=1}^d \int_s^t E_{P^n\otimes P^n}\left[b_{ij}^n\left(X_u - Y_u\right)\partial_{i}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]du.
\end{aligned}$$

We make the same transformation for the others expectations of the second term of (1.19), and we divide in two parts the convergence study of (1.18).
As $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d,\mathbb{R}\right)$, and $g_1, ..., g_k$ are bounded functions, there exists a finite constant $m > 0$ such that $m = \sup\left(\|\partial\varphi\|_\infty, \|\varphi\|_\infty, \|g_i\|_\infty, i = 1,..,k\right)$.
Part I: We state that

$$\begin{aligned}
&\Bigg|E_{P^n}\left[\left(\int_s^t L^n\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right] \\
&-E_{P^n}\left[\left(\int_s^t L\left(P_u^n\right)\varphi\left(X_u\right)du\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\Bigg| \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

We first study the convergence of the term with the coefficients a_{ij}^n and a_{ij}.

$$\begin{aligned}
E_1 &= \Bigg|E_{P^n\otimes P^n}\left[\int_s^t a_{ij}^n\left(X_u - Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right] \\
&\quad -E_{P^n\otimes P^n}\left[\int_s^t a_{ij}\left(X_u - Y_u\right)\partial_{ij}\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\Bigg| \\
&\le m^{k+1}E_{P^n\otimes P^n}\left[\int_s^t \left|a_{ij}^n\left(X_u - Y_u\right) - a_{ij}\left(X_u - Y_u\right)\right|du\right].
\end{aligned}$$

Since $\left|a_{ij}^{n}(z)\right| \leq |a_{ij}(z)| \leq c|z|^{\gamma+2}$ and $a_{ij}^{n}(z) = a_{ij}(z)$ on $|z| \geq \frac{2}{n}$,

$$E_1 \leq 2m^{k+1}c\int_s^t E_{P^n\otimes P^n}\left[|X_u - Y_u|^{\gamma+2}\,\mathbb{I}_{|X_u-Y_u|\leq\frac{2}{n}}\right]du.$$

As $\gamma + 2 > 0$, there finally exists a constant $K > 0$ such that

$$\left|E_{P^n\otimes P^n}\left[\int_s^t a_{ij}^{n}(X_u - Y_u)\,\partial_{ij}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right.$$
$$\left.-E_{P^n\otimes P^n}\left[\int_s^t a_{ij}(X_u - Y_u)\,\partial_{ij}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right| \leq \frac{K}{n^{\gamma+2}}.$$

We can use the same arguments for the term with the coefficients b_i^n and b_i. Hence, we obtain

$$\left|E_{P^n\otimes P^n}\left[\int_s^t b_i^{n}(X_u - Y_u)\,\partial_{i}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right.$$
$$\left.-E_{P^n\otimes P^n}\left[\int_s^t b_i(X_u - Y_u)\,\partial_{i}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right| \leq \frac{K}{n^{\gamma+1}}.$$

Consequently, since $\gamma \in (-1, 0]$, we have proved,

$$\left|E_{P^n}\left[\left(\int_s^t L^n(P_u^n)\,\varphi(X_u)\,du\right)g_1(X_{s_1})\ldots g_k(X_{s_k})\right]\right.$$
$$\left.-E_{P^n}\left[\left(\int_s^t L(P_u^n)\,\varphi(X_u)\,du\right)g_1\left(X_{s_1}^n\right)\ldots g_k\left(X_{s_k}^n\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Part II: We now state that,

$$\left|E_{P^n}\left[\left(\int_s^t L(P_u^n)\,\varphi(X_u)\,du\right)g_1(X_{s_1})\ldots g_k(X_{s_k})\right]\right.$$
$$\left.-E_{P}\left[\left(\int_s^t L(P_u)\,\varphi(X_u)\,du\right)g_1(X_{s_1})\ldots g_k(X_{s_k})\right]\right| \underset{n\to\infty}{\longrightarrow} 0. \qquad (1.20)$$

As in the part I, we first study the term with the coefficients a_{ij}, i.e. the convergence

$$\left|E_{P^n\otimes P^n}\left[\int_s^t a_{ij}(X_u - Y_u)\,\partial_{ij}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right.$$
$$\left.-E_{P\otimes P}\left[\int_s^t a_{ij}(X_u - Y_u)\,\partial_{ij}\varphi(X_u)\,g_1(X_{s_1})\ldots g_k(X_{s_k})\,du\right]\right| \underset{n\to\infty}{\overset{?}{\longrightarrow}} 0. \qquad (1.21)$$

Let $f : \mathcal{C}\left([0,t], \mathbb{R}^d\right) \times \mathcal{C}\left([0,t], \mathbb{R}^d\right) \longrightarrow \mathbb{R}$ be the continuous function defined by

$$(x, y) \longmapsto f(x, y) = \int_s^t a_{ij}(x_u - y_u)\,\partial_{ij}\varphi(x_u)\,du.g_1(x_{s_1})\ldots g_k(x_{s_k}).$$

So, we can rewrite (1.21):

$$\left|E_{P^n\otimes P^n}\left[f\left(X,Y\right)\right]-E_{P\otimes P}\left[f\left(X,Y\right)\right]\right|\underset{n\to\infty}{\overset{?}{\longrightarrow}}0.$$

The function f is not a bounded function, hence we cannot just use the convergence in distribution to conclude. But we have the following estimate on the function f

$$\begin{aligned}\left|f\left(x,y\right)\right| &\leq m^{k+1}\int_s^t\left|a_{ij}\left(x_u-y_u\right)\right|du\\ &\leq m^{k+1}c\int_s^t\left|x_u-y_u\right|^{\gamma+2}du\\ &\leq K\left(\sup_{0\leq u\leq t}\left|x_u\right|^{\gamma+2}+\sup_{0\leq u\leq t}\left|y_u\right|^{\gamma+2}\right)\end{aligned}$$

Thus, using Lemma 3.3 with $k=2$ when $\gamma\in(-1,0)$ and with $k=3$ when $\gamma=0$, we easily prove that

$$\lim_{C\to+\infty}\sup_{n\geq0}E_{P^n\otimes P^n}\left[\left|f(X,Y)\right|\mathbb{I}_{\left|f(X,Y)\right|>C}\right]=0$$

Consequently, we have

$$\left|E_{P^n\otimes P^n}\left[f\left(X,Y\right)\right]-E_{P\otimes P}\left[f\left(X,Y\right)\right]\right|\underset{n\to\infty}{\longrightarrow}0.$$

To finish the proof of (1.20), we still have to check

$$\begin{aligned}&\Bigg|E_{P^n\otimes P^n}\left[\int_s^t b_i\left(X_u-Y_u\right)\partial_i\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\\ &\quad -E_{P\otimes P}\left[\int_s^t b_i\left(X_u-Y_u\right)\partial_i\varphi\left(X_u\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)du\right]\Bigg|\underset{n\to\infty}{\overset{?}{\longrightarrow}}0.\end{aligned}$$

Using the continuous function $\tilde{f}:\mathcal{C}\left([0,t],\mathbb{R}^d\right)\times\mathcal{C}\left([0,t],\mathbb{R}^d\right)\longrightarrow\mathbb{R}$ defined by

$$(x,y)\longmapsto\tilde{f}\left(x,y\right)=\int_s^t b_i\left(x_u-y_u\right)\partial_i\varphi\left(x_u\right)du.g_1\left(x_{s_1}\right)...g_k\left(x_{s_k}\right).$$

we state as above that

$$\left|E_{P^n\otimes P^n}\left[\tilde{f}\left(X,Y\right)\right]-E_{P\otimes P}\left[\tilde{f}\left(X,Y\right)\right]\right|\underset{n\to\infty}{\longrightarrow}0.$$

<ins>Conclusion:</ins> according to Parts I and II, we have proved the convergence (1.18). Then, thanks to (1.17), we have

$$E_{P^n}\left[\left(M_t^n-M_s^n\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right]\underset{n\to\infty}{\longrightarrow}E_P\left[\left(M_t-M_s\right)g_1\left(X_{s_1}\right)...g_k\left(X_{s_k}\right)\right].$$

Hence, since $E_{P^n}\left[\left(M_t^n - M_s^n\right) g_1\left(X_{s_1}\right) ... g_k\left(X_{s_k}\right)\right] = 0$, we have, for any $0 \leq s < t, 0 \leq s_1 \leq ... \leq s_k \leq s$,

$$E_P\left[\left(M_t - M_s\right) g_1\left(X_{s_1}\right) ... g_k\left(X_{s_k}\right)\right] = 0.$$

So, P satisfies the martingale problem (MP). ■
There are two main consequences of this theorem. The first one concerns the existence of a solution to the Landau equation when $\gamma \in (-1, 0]$:

Theorem 3.6 *Let a and b be defined by (1.4) and (1.5) respectively. Let P_0 be a probability measure with a finite moment of order 2 when $\gamma \in (-1, 0)$ and of order 3 when $\gamma = 0$. There exists a measure solution $(P_t)_{t \geq 0}$ with initial data P_0 of the Landau equation*

$$\begin{aligned} \frac{d}{dt} \int \varphi(v) P_t(dv) &= \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} P_t(dv) \left(\int_{\mathbb{R}^d} P_t(dv_*) a_{ij}(v - v_*) \right) \partial_{ij} \varphi(v) \\ &+ \sum_{i=1}^{d} \int_{\mathbb{R}^d} P_t(dv) \left(\int_{\mathbb{R}^d} P_t(dv_*) b_i(v - v_*) \right) \partial_i \varphi(v) \end{aligned}$$

when $\gamma \in (-1, 0]$ and h is a bounded continuous nonnegative function.

The second one states that the distribution P can be also interpreted as the distribution of a weak solution of a nonlinear stochastic differential equation:

Theorem 3.7 *Let the matrix a and the vector b be defined by (1.4) and (1.5) respectively. Let X_0 be a random variable with a finite moment of order 2 if $\gamma \in (-1, 0)$ and of order 3 if $\gamma = 0$. Then, there exists a weak solution X of the nonlinear stochastic differential equation:*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right) . W^d(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds \qquad (NSDE(\sigma, b))$$

where σ is a symmetric matrix such that $\sigma^* \sigma = a$.

Proof. Let P be a cluster point of (P^n). Let X be a process with distribution P.
We firstly state the following lemma:

Lemma 3.8 *The process $M_t = X_t - X_0 - \int_0^t b(X_s, P_s) ds$ is a continuous local P-martingale and its bracket is given by $\langle M_i, M_j \rangle_t = \int_0^t a_{ij}(X_s, P_s) ds$ where $b(X_s, P_s) = \int b(X_s - y) P_s(dy)$ and $a_{ij}(X_s, P_s) = \int a_{ij}(X_s - y) P_s(dy)$.*

Proof. We denote by $a \wedge b = \min(a, b)$ and $B_R = \left\{x \in \mathbb{R}^d : |x| \leq R\right\}$.
Using the functions $\varphi_i \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$ such that $\varphi_i(x) = x_i$ on B_R, for $i = 1, .., d$, it is easy to check that M is a continuous local P-martingale.

Using the functions $\varphi_{ij} \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$ such that $\varphi_{ij}(x) = x_i x_j$ on B_R, we state that the processes

$$\begin{aligned} N_{ij,t} &= X_{i,t}X_{j,t} - X_{i,0}X_{j,0} - \int_0^t a_{ij}\left(X_s, P_s\right) ds \\ &\quad - \int_0^t b_i\left(X_s, P_s\right) X_{j,s} ds - \int_0^t b_j\left(X_s, P_s\right) X_{i,s} ds \end{aligned}$$

are continuous local P-martingales, $i, j \in \{1, .., d\}$.
Moreover, $M_{i,t}M_{j,t} - \int_0^t a_{ij}\left(X_s, P_s\right) ds = N_{ij,t} - X_{i,0}M_{j,t} - X_{j,0}M_{i,t}$. Then, $\langle M_i, M_j \rangle_t = \int_0^t a_{ij}\left(X_s, P_s\right) ds$. ■
According to [17] Theorem III-10 *(σ is a symmetric matrix)*, we conclude that there are, on an extension of the probability space, d continuous orthogonal martingale measures $\left(W_k^P\right)_{k=1..d}$ with intensity $P_s(dy)\, ds$ on $\mathbb{R}^d \times [0, \infty)$ such that, for any $k = 1...d$

$$M_{i,t} = \sum_{k=1}^d \int_0^t \int_{\mathbb{R}^d} \sigma_{ik}\left(X_s - y\right) W_k^P\left(ds, dy\right).$$

As the measure $P_s(dy)\, ds$ is deterministic, using [17] Theorem III-3, we deduce that the W_k^P are white noises and

$$X_{i,t} = X_0 + \sum_{k=1}^d \int_0^t \int \sigma_{ik}\left(X_s - y\right) W_k^P\left(ds, dy\right) + \int_0^t \int b_i\left(X_s - y\right) P_s(dy)\, ds. \quad (1.22)$$

We can easily rewrite the equation (1.22) under the expression $(NSDE(\sigma, b))$ (see the proof of Remark 2.6). Consequently, we have proved the theorem. ■

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Chapitre 2

Existence and regularity of a weak function-solution for some Landau equations with a stochastic approach

Abstract : Using the Malliavin Calculus, this paper proves the existence of a weak function-sochlution of class $\mathcal{C}^\infty$ with bounded derivatives of the Landau equation for a generalization of Maxwellian molecules when the initial data is a probability measure.

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1 Introduction

The Landau equation, also called the Fokker-Planck-Landau equation, is obtained as limit of the Boltzmann equation when all the collisions become grazing. Its expression, in the spatially homogeneous case, is:

$$\frac{\partial f}{\partial t}(v,t) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} dv_* a_{ij}(v-v_*)\left[f(v_*,t)\frac{\partial f}{\partial v_j}(v,t) - f(v,t)\frac{\partial f}{\partial v_{*j}}(v_*,t)\right]\right\} \tag{2.1}$$

where $f(v,t) \geq 0$ is the density of particles with velocity $v \in \mathbb{R}^d$ at time $t \in \mathbb{R}^+$, and $(a_{ij}(z))_{1\leq i,j\leq d}$ is a nonnegative symmetric matrix depending on the interaction between the particles.

In this paper, we study the Landau equation for a generalization of Maxwell gas. We consider a matrix a of the form

$$a_{ij}(z) = h\left(|z|^2\right)\left(|z|^2\delta_{ij} - z_i z_j\right) \tag{2.2}$$

where h is a positive continuous function on $\mathbb{R}_+$ such that there exist $m, M > 0$ with $\forall z \in \mathbb{R}^d$

$$m \leq h\left(|z|^2\right) \leq M. \tag{2.3}$$

When h is a constant, we recognize the coefficient of the Landau equation for Maxwellian molecules.
We define the vector b by

$$b_i(z) = \sum_{j=1}^{d} \partial_j a_{ij}(z) = -(d-1)\, h\left(|z|^2\right) z_i. \tag{2.4}$$

Then, by integration by parts, we can give a weak formulation of the equation (2.1), and consequently we define the notion of weak function-solution:

Definition 1.1 *Let $f(.,0)$ be a nonnegative function on $\mathbb{R}^d$ with finite mass and energy. A nonnegative function f on $\mathbb{R}^d \times \mathbb{R}^+$ is a* <u>*weak function-solution*</u> *of the Landau equation with initial data $f(.,0)$, if f satisfies the following equation for any test function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$*

$$\begin{aligned}
&\frac{d}{dt}\int \varphi(v)\, f(v,t)\, dv = \\
&\qquad \frac{1}{4}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, a_{ij}(v-v_*)\left(\partial_{ij}\varphi(v) + \partial_{ij}\varphi(v_*)\right) \\
&\qquad +\frac{1}{2}\sum_{i=1}^{d} \int_{\mathbb{R}^d\times\mathbb{R}^d} dv dv_* f(v,t) f(v_*,t)\, b_i(v-v_*)\left(\partial_i\varphi(v) - \partial_i\varphi(v_*)\right)
\end{aligned} \tag{2.5}$$

where $\partial_i\varphi = \frac{\partial\varphi}{\partial v_i}$ and $\partial_{ij}\varphi = \frac{\partial^2\varphi}{\partial v_i \partial v_j}$.

The equation (2.5) conserves mass, momentum and energy. Thus, if there exists a weak function-solution f of (2.5) with an initial data satisfying $\int_{\mathbb{R}^d} f(v,0)\, dv = 1$, the measure P_t on $\mathbb{R}^d$ given by $P_t(dv) = f(v,t)\, dv$ is a probability measure, for any $t \geq 0$. Thus, we define a probabilistic notion of solutions of the Landau equation :

Definition 1.2 *Let P_0 be a probability measure on $\mathbb{R}^d$ with a finite two-order moment (i.e. $\int_{\mathbb{R}^d} |v|^2 P_0(dv) < \infty$). A* <u>*measure-solution*</u> *of the Landau equation (2.6) with initial data P_0 is a probability flow $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$ satisfying*

$$\begin{aligned}
\frac{d}{dt}\int \varphi(v)\, P_t(dv) &= \frac{1}{2}\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, a_{ij}(v-v_*)\right) \partial_{ij}\varphi(v) \\
&\quad + \sum_{i=1}^{d} \int_{\mathbb{R}^d} P_t(dv)\left(\int_{\mathbb{R}^d} P_t(dv_*)\, b_i(v-v_*)\right) \partial_i\varphi(v)
\end{aligned} \tag{2.6}$$

for any function $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$.

This approach allows us to have weaker conditions on the initial data, i.e. we can assume that the initial data is a probability measure and not necessarily a density of probability.

Remark 1.3 *With an abuse of notation, we will still say that a probability measure P on $\mathcal{C}\left([0,T],\mathbb{R}^d\right)$ is a measure-solution of the Landau equation when its time-marginals flow is a measure-solution in the sense of Definition 1.2.*

We have already stated the existence of a probability measure-solution of the Landau equation (2.6) in Chapter 1 of this thesis. We are here interested in proving with a stochastic approach the existence of a bounded weak function-solution of (2.5) of class $\mathcal{C}^\infty$ with bounded derivatives.
Let us briefly recall the main results of Chapter 1. We will associate with the Landau equation (2.6) a nonlinear stochastic differential equation driven by a space-time white noise. We highlight the nonlinearity using two probability spaces: $(\Omega,\mathcal{F},\mathbb{P})$ and $([0,1],\mathcal{B}([0,1]),d\alpha)$, where $d\alpha$ is the Lebesgue measure on $[0,1]$. In order to avoid any confusion, we will denote by E the expectation and $\mathcal{L}$ the distribution of a random variable on $(\Omega,\mathcal{F},\mathbb{P})$ and E_α, $\mathcal{L}_\alpha$ for a random variable on $([0,1],\mathcal{B}([0,1]),d\alpha)$.
Since a is a nonnegative symmetric matrix, there exists a matrix σ of order $d\times d'$ such that

$$a=\sigma.\sigma^* \tag{2.7}$$

where σ^* is the adjoint matrix of σ.
We define a d'- dimensional space-time white noise on $[0,1]\times[0,\infty)$, by

$$W^{d'}=\begin{pmatrix} W_1 \\ \vdots \\ W_{d'} \end{pmatrix} \tag{2.8}$$

where the W_i are independent space-time white noises with covariance measure $d\alpha dt$ on $[0,1]\times[0,\infty)$ (according to the definition of Walsh, [42]). We consider its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_t=\sigma\{W^{d'}([0,s]\times A):0\leq s\leq t, A\in\mathcal{B}([0,1])\}$.
For $k\geq 2$, we define $\mathcal{P}_k$ the space of continuous adapted processes $X=(X_t)_{t\geq 0}$ from $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)$ to $\mathbb{R}^d$, such that $E\left[\sup_{0\leq t\leq T}|X_t|^k\right]<\infty$ for any $T>0$, and $\mathcal{P}_{k,\alpha}$ the space of continuous processes $Y=(Y_t)_{t\geq 0}$ from $([0,1],\mathcal{B}([0,1]),d\alpha)$ to $\mathbb{R}^d$, such that $E_\alpha\left[\sup_{0\leq t\leq T}|Y_t|^k\right]<\infty$ for any $T>0$.
Let X_0 be a random vector on $\mathbb{R}^d$, independent of $W^{d'}$, with a finite moment of order 2.
We consider the following nonlinear stochastic differential equation:

Definition 1.4 *Let X_0 and $W^{d'}$ be defined as below. A couple of processes (X,Y) on $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)\times([0,1],\mathcal{B}([0,1]),d\alpha)$ is solution of the nonlinear stochastic differential equation (NSDE(σ,b)) if for any $t\geq 0$*

$$X_t=X_0+\int_0^t\int_0^1\sigma\left(X_s-Y_s(\alpha)\right).W^{d'}(d\alpha,ds)+\int_0^t\int_0^1 b\left(X_s-Y_s(\alpha)\right)d\alpha ds$$

and $\mathcal{L}(X)=\mathcal{L}_\alpha(Y)$.

We notice, using Ito's Formula, that the distribution of a solution of $(NSDE\,(\sigma, b))$ is a weak measure-solution of the Landau equation (2.6) with initial data $P_0 = \mathcal{L}\,(X_0)$.
In Chapter 1 Theorem 2.7, we have proved the following theorem for $k = 2$, but, adapting the proofs, it is still true for any $k \geq 2$.

Theorem 1.5 *Assume that $W^{d'}$ is a d'-dimensional space-time white noise and assume that X_0 is an independent random vector on $\mathbb{R}^d$ with finite moment of order k. If the functions σ and b, defined by (2.7), (2.2) and (2.4), are Lipschitz continuous, there exists a couple (X, Y), unique in law, solution of the nonlinear equation $(NSDE\,(\sigma, b))$ with $(X, Y) \in \mathcal{P}_k \times \mathcal{P}_{k,\alpha}$.*

Corollary 1.6 *Assume that P_0 is a probability measure with a finite moment of order 2. There exists a measure-solution $(P_t)_{t\geq 0}$ with initial data P_0 to the Landau equation (2.6) when σ and b are Lipschitz continuous functions.*

Corollary 1.7 *Assume that P_0 is a probability measure with a finite moment of order 2. There is uniqueness of the measure-solution $(P_t)_{t\geq 0}$ with initial data P_0 to the Landau equation (2.6) when σ and b are Lipschitz continuous functions.*

Proof. (Proof of Corollary 1.7)
• We just have to prove the uniqueness of the solution $(Q_t^\mu)_{t\geq 0}$ of the linear Landau equation, i.e.

$$\begin{aligned} \frac{d}{dt} \int \varphi\,(v)\, Q_t^\mu\,(dv) &= \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} Q_t^\mu\,(dv) \left(\int_{\mathbb{R}^d} \mu_t\,(dv_*)\, a_{ij}\,(v - v_*) \right) \partial_{ij} \varphi\,(v) \\ &\quad + \sum_{i=1}^{d} \int_{\mathbb{R}^d} Q_t^\mu\,(dv) \left(\int_{\mathbb{R}^d} \mu_t\,(dv_*)\, b_i\,(v - v_*) \right) \partial_i \varphi\,(v) \qquad (2.9) \end{aligned}$$

where $(\mu_t)_{t\geq 0}$ is a probability measure family on $\mathbb{R}^d$.
Indeed, if we assume that there is uniqueness of the solution of the linear Landau equation.
We consider P^1 and P^2 two solutions of the *(nonlinear)* Landau equation. Let Y^1 and Y^2 be processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ such that $\mathcal{L}_\alpha(Y^1) = P^1$ and $\mathcal{L}_\alpha(Y^2) = P^2$.
Let X^1 be the solution of the standard stochatic differential equation (see Chapter 1. Theorem 2.5):

$$X_t^1 = X_0 + \int_0^t \int_0^1 \sigma\left(X_s^1 - Y_s^1\,(\alpha)\right).W^{d'}\,(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s^1 - Y_s^1\,(\alpha)\right) d\alpha ds$$

We define similarly a process X^2 replacing Y^1 with Y^2.
Using Itô's Formula, $\mathcal{L}(X^1) = Q^{P^1}$ is a solution of the linear Landau equation (2.9) with $\mu_t = P_t^1$. Moreover, as P^1 is a solution of the Landau equation, it is also a solution of the linear Landau equation with $\mu_t = P_t^1$. Thanks to the uniqueness of the solution of the linear Landau equation, we have $P^1 = Q^{P^1}$ and similarly $P^2 = Q^{P^2} = \mathcal{L}(X^2)$.
Consequently $\mathcal{L}(X^1) = \mathcal{L}_\alpha(Y^1)$ and $\mathcal{L}(X^2) = \mathcal{L}_\alpha(Y^2)$, thus (X_1, Y_1) and (X_2, Y_2) are two solutions of the nonlinear stochastic differential equation $(NSDE\,(\sigma, b))$. Thanks

to the uniqueness result of Theorem 1.5, we have $\mathcal{L}(X^1) = \mathcal{L}(X^2)$, which implies that $P^1 = P^2$.

• The linear Landau equation (2.9) satisfies the assumptions of [4] Theorem 5.2, then there is uniqueness of the measure-solution of (2.9). ■

Remark 1.8 *We notice that we can choose*

$$\sigma(z) = \sqrt{h\left(|z|^2\right)} \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \tag{2.10}$$

in dimension two, and

$$\sigma(z) = \sqrt{h\left(|z|^2\right)} \begin{bmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{bmatrix} \tag{2.11}$$

in dimension three. Then, if h is a bounded function of class $\mathcal{C}^1$ with $h'(x) = O\left(\frac{1}{x^2}\right)$ when $x \to +\infty$, σ and b are Lipschitz continuous functions of class $\mathcal{C}^1$ on $\mathbb{R}^d$, for $d = 2, 3$. This property can be generalized in higher dimension.

The aim of this article is to find a weak function-solution of the Landau equation when the initial data is a probability measure. To state the existence of a weak function-solution of (2.5) from a measure-solution, it is enough to show that the measure-solution is absolutely continuous with respect the Lebesgue measure. Indeed, if $(P_t)_{t\geq 0}$ is a measure-solution of (2.6) with initial data P_0 and if there exists a nonnegative function f_t on $\mathbb{R}^d$ such that $P_t(dv) = f_t(v)\,dv$ for any $t > 0$, then the function f defined by $f(v,t) = f_t(v)$ for any $v \in \mathbb{R}^d$, $t > 0$, is a weak function-solution of (2.5) with initial data P_0.
The idea consists in using the relation between the Landau equation and the nonlinear differential equation $(NSDE(\sigma, b))$. In fact, we develop a Malliavin Calculus for the value X_t, $t > 0$, of the solution X of $(NSDE(\sigma, b))$ obtained in Theorem 1.5, inspired by the methods used by Bally and Pardoux, [3], and by Nualart, [33].
The Maxwellian case (i.e., when the function h is a constant) is studied in detail with an analytic approach by Villani in [40]. When the initial data is a nonnegative function f_0 with finite mass and energy, Villani has proved the existence and the uniqueness of a bounded solution of (2.1) of class $\mathcal{C}^\infty$.
We prove here the existence of a bounded weak function-solution of the Landau equation (2.5) of class $\mathcal{C}^\infty$ with bounded derivatives when the initial data is a probability measure with finite moments for some bounded functions h.

1.1 About the Malliavin calculus for a white noise

Let us describe the main steps of the Malliavin calculus for a white noise. We use in the following the same notation as Nualart [33].
Let $W^{d'}$ be a d'-dimensional space-time white noise.

Let $\mathcal{S}$ be the class of random variables F having the following form

$$F = f\left(W^{d'}(g_1), .., W^{d'}(g_n)\right),$$

where f is a $\mathcal{C}^\infty$ real valued function on $\mathbb{R}^n$ with partial derivatives having polynomial growth, $g = (g_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq d'}}$ is a matrix with components in $\mathbb{L}^2([0,1]\times[0,\infty), d\alpha ds)$, and

$$W^{d'}(g_i) = \sum_{j=1}^{d'} \int_0^\infty \int_0^1 g_{ij}(s,\alpha) W_j(d\alpha, ds).$$

Assuming that $(r,z) \in [0,\infty)\times[0,1]$, we define the first order Malliavin derivative $D^l_{(r,z)}F$ of F in relation to the l^{th} white noise W_l at point (r,z), with $l \in \{1,...,d'\}$, by

$$D^l_{(r,z)}F = \sum_{i=1}^n \partial_i f\left(W^{d'}(g_1), .., W^{d'}(g_n)\right) g_{il}(r,z).$$

For $k \geq 1$, set $\lambda_k = ((r_1,z_1),...,(r_k,z_k))$ with $r_m \in [0,\infty)$ and $z_m \in [0,1]$, $m = 1,..,k$. We define by iteration the derivatives of order k. Let $(l_1,..,l_k)$ be a k-uplet of $\{1,..,d'\}$, we denote by

$$D^{l_1,..,l_k}_{\lambda_k}F = D^{l_k}_{(r_k,z_k)} D^{l_{k-1}}_{(r_{k-1},z_{k-1})} .. D^{l_1}_{(r_1,z_1)}F.$$

We denote by $\mathbb{D}^{k,p}$ the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{m=1}^k \sum_{l_1,..,l_m=1}^{d'} E\left(\left\|D^{l_1,..,l_m}F\right\|^p_{\mathbb{L}^2(\Lambda_m)}\right)\right]^{\frac{1}{p}},$$

where

$$\left\|D^{l_1,..,l_m}F\right\|^2_{\mathbb{L}^2(\Lambda_m)} = \int_{\Lambda_m} \left|D^{l_1,..,l_m}_{\lambda_m}F\right|^2 d\lambda_m$$

with $\Lambda_m = ([0,\infty)\times[0,1])^m$ and $\lambda_m = ((r_1,z_1),...,(r_m,z_m)) \in \Lambda_m$.

We also denote by $\mathbb{D}^\infty$ the subspace of the infinitely differentiable variables:

$$\mathbb{D}^\infty = \bigcap_{p\geq 1} \bigcap_{k\geq 1} \mathbb{D}^{k,p}.$$

When F is a random vector in $\mathbb{R}^d$, we differentiate component by component and we denote by DF the matrix $(DF)_{i,l} = D^l F_i$, $1 \leq i \leq d$,$1 \leq l \leq d'$. The Malliavin matrix is defined by

$$I = \int_0^T \int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^* dzdr.$$

In this paper, under suitable assumptions on σ and b and integrability conditions on the initial data X_0, we show that for any $t > 0$ the value X_t of X obtained in Theorem 1.5 satisfies the conditions of one of those two following theorems.

Theorem (a) (see [33] Theorem 2.1.2)
Let $F = (F_1, .., F_d)$ be a random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions :
(i) F_i belongs to the space $\mathbb{D}^{1,p}$, $p > 1$, for any $i = 1, .., d$.
(ii) The Malliavin matrix $I = \int_0^\infty \int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^* dzdr$ is invertible a.s..
Then the distribution of F is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^d$.

Theorem (b) (see [33] Corollary 2.1.2)
Let $F = (F_1, .., F_d)$ be a random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions :
(i) F_i belongs to $\mathbb{D}^\infty$, for any $i = 1, .., d$.
(ii) The Malliavin matrix $I = \int_0^\infty \int_0^1 D_{(r,z)}F.\left(D_{(r,z)}F\right)^* dzdr$ satisfies

$$(\det I)^{-1} \in \bigcap_{p>1} \mathbb{L}^p(\Omega).$$

Then F has an infinitely differentiable density.

1.2 Notations

- $\mathcal{C}\left([0,T], \mathbb{R}^d\right)$ is the space of continuous functions from $[0,T]$ to $\mathbb{R}^d$, and for $k \in \mathbb{N}$, $\mathcal{C}_b^k\left([0,T], \mathbb{R}^d\right)$ is the space of functions of class $\mathcal{C}^k$ with all its derivatives bounded up to order k.

- $\mathcal{M}_{d,d'}(\mathbb{R})$ is the set of $d \times d'$ matrix on $\mathbb{R}$.

- For $k \geq 2$, a random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ belongs to $\mathbb{L}^k$ if Z has a finite moment of order k, i.e. $E\left[|Z|^k\right] < \infty$.

- K is an arbitrary notation for a positive constant (K can change from line to line).

2 Computation of the derivatives of X

2.1 The first derivative

Assumption (H^1): σ and b are Lipschitz continuous functions of class $\mathcal{C}^1$ from $\mathbb{R}^d$ to $\mathcal{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$ respectively.
We denote by K_σ and K_b their Lipschitz constants.

Theorem 2.1 *We assume that X_0 has a finite 2-order moment. Let (X, Y) be the solution of the nonlinear stochastic differential equation $(NSDE(\sigma, b))$ obtained in theorem 1.5. (Y will play a parameter role in the following.)*

Under Assumption (H^1), $\forall t \in [0,T]$ $\forall i = 1,..,d$, $X_{i,t} \in \mathbb{D}^{1,2}$. *The* i^{th} *component of its derivative in relation to the* l^{th} *white noise at point* $(r,z) \in [0,\infty) \times [0,1]$ *is given by*

$$\begin{aligned} D^l_{(r,z)} X_{i,t} &= \sigma_{i,l}\left(X_r - Y_r(z)\right) \\ &+ \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)} X_{m,s} W_k(d\alpha, ds) \\ &+ \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)} X_{m,s} d\alpha ds \end{aligned}$$

if $t \geq r$ *and* $D^l_{(r,z)} X_{i,t} = 0$ *if* $t < r$.

Proof. Since X_0 is independant of the white noise, we notice that we can extend the definition of the sets $\mathbb{D}^{k,p}$ and $\mathbb{D}^\infty$ to the case we are considering.
We consider the Picard sequence of $\mathcal{P}_2$-processes defined by

$$\begin{aligned} X_t^0 &= X_0, \\ X_t^{n+1} &= X_0 + \int_0^t \int_0^1 \sigma\left(X_s^n - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) \\ &\quad + \int_0^t \int_0^1 b\left(X_s^n - Y_s(\alpha)\right) d\alpha ds. \end{aligned} \tag{2.12}$$

Then, the i^{th} component writes

$$X_{i,t}^{n+1} = X_{i,0} + \int_0^t \int_0^1 \sum_{k=1}^{d'} \sigma_{i,k}\left(X_s^n - Y_s(\alpha)\right) W_k(d\alpha, ds) + \int_0^t \int_0^1 b_i\left(X_s^n - Y_s(\alpha)\right) d\alpha ds.$$

According to Chapter 1 Theorem 2.5, the sequence (X^n) satisfies

$$\sup_n E\left[\sup_{0\leq r\leq T} |X_r^n|^2\right] < \infty \tag{2.13}$$

and converges for the norm $\|U\| = \left\|\sup_{0\leq t\leq T} |U_t|\right\|_{\mathbb{L}^2}$ to X.
Let $T > 0$ be arbitrary fixed. Let $t \in [0,T]$ and $(r,z) \in [0,T] \times [0,1]$ be fixed.
We show firstly by recurrence that for any $n \geq 0$ X_t^n is differentiable at point (r,z) in the Malliavin sense.
Recurrence Hypothesis:
(i) $X_{i,t}^n \in \mathbb{D}^{1,2}$ $\forall t \in [0,T]$ $\forall i = 1,...,d$.
(ii) $\sup_{t\in[0,T]} \sum_{l=1}^{d'} E\left(\int_0^\infty \int_0^1 \left|D^l_{(r,z)} X_t^n\right|^2 dzdr\right) < \infty$ where $\left|D^l_{(r,z)} X_t^n\right|^2 = \sum_{i=1}^{d}\left(D^l_{(r,z)} X_{i,t}^n\right)^2$.
For $n = 0$, Recurrence Hypothesis is satisfied.
We assume that it is true at rank n. According to [33] Proposition 1.2.2, since σ and b are functions of class $\mathcal{C}_b^1$, we notice that $\forall i = 1,..,d$ $\forall k = 1,..,d'$, $\sigma_{i,k}\left(X_t^n - Y_t(\alpha)\right) \in \mathbb{D}^{1,2}$ and $b_i\left(X_t^n - Y_t(\alpha)\right) \in \mathbb{D}^{1,2}$.

As for the Brownian Motion, we can show that derivative and integral commute (see [33]), then $X_{i,t}^{n+1} \in \mathbb{D}^{1,2}$ $\forall t \in [0,T]$ $\forall i = 1,...,d$. Moreover, its derivative at point $(r,z) \in [0,T] \times [0,1]$ in relation to the l^{th} white noise W_l is given by

$$\begin{aligned} D_{(r,z)}^{l} X_{i,t}^{n+1} &= \sigma_{i,l}\left(X_r^n - Y_r(z)\right) \\ &+ \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^{l} X_{m,s}^{n} W_k(d\alpha, ds) \\ &+ \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i\left(X_s^n - Y_s(\alpha)\right) D_{(r,z)}^{l} X_{m,s}^{n} d\alpha ds \end{aligned}$$

if $r \leq t$, and $D_{(r,z)}^{l} X_{i,t}^{n+1} = 0$ else.
We still have to check that

$$\sup_{t\in[0,T]} \sum_{l=1}^{d'} E\left[\int_0^{\infty} \int_0^1 \left|D_{(r,z)}^{l} X_t^{n+1}\right|^2 dzdr\right] < \infty.$$

We define

$$S_n(t) = \sum_{l=1}^{d'} E\left[\int_0^{\infty} \int_0^1 \left|D_{(r,z)}^{l} X_t^{n}\right|^2 dzdr\right] = \sum_{l=1}^{d'} E\left[\int_0^{t} \int_0^1 \left|D_{(r,z)}^{l} X_t^{n}\right|^2 dzdr\right].$$

According to Recurrence Hypothesis, $\sup_{t\in[0,T]} S_n(t) < \infty$. Let us study $\sup_{t\in[0,T]} S_{n+1}(t)$.
Let $l \in \{1,..,d'\}$ be arbitrary fixed. We divide in three parts the expectation $E\left[\int_0^t \int_0^1 \left|D_{(r,z)}^{l} X_{i,t}^{n+1}\right|^2 dzdr\right]$.
We define

$$\begin{aligned} E_1 &= E\left[\int_0^t \int_0^1 \left|\sigma_{i,l}\left(X_r^n - Y_r(z)\right)\right|^2 dzdr\right] \\ &\leq 4K_\sigma^2 E\left[\int_0^t \int_0^1 \left|X_r^n\right|^2 + \left|Y_r(z)\right|^2 dzdr\right] + 2T\left|\sigma(0)\right|^2 \\ &\quad \text{since } \sigma \text{ is Lipschitz continuous} \\ &\leq 4K_\sigma^2 T\left(\sup_n E\left[\sup_{0\leq r\leq T} \left|X_r^n\right|^2\right] + E_\alpha\left[\sup_{0\leq r\leq T} \left|Y_r\right|^2\right]\right) + 2T\left|\sigma(0)\right|^2. \end{aligned}$$

According to (2.13), we have $\sup_{0\leq t\leq T} E_1 < \infty$.

We define

$$\begin{aligned} E_2 &= E\left[\int_0^t\int_0^1\left|\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s^n-Y_s(\alpha)\right)D^l_{(r,z)}X^n_{m,s}W_k(d\alpha,ds)\right|^2 dzdr\right] \\ &= \int_0^t\int_0^1 E\left[\int_r^t\int_0^1\sum_{k=1}^{d'}\left(\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s^n-Y_s(\alpha)\right)D^l_{(r,z)}X^n_{m,s}\right)^2 d\alpha ds\right]dzdr \\ &\quad \text{since } W_k \text{ are independent white noises} \\ &\leq d\sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1\left(\partial_m\sigma_{i,k}\left(X_s^n-Y_s(\alpha)\right)D^l_{(r,z)}X^n_{m,s}\right)^2 d\alpha ds\right]dzdr \\ &\quad \text{using Hölder's Inequality.} \end{aligned}$$

Since the partial derivatives of σ are bounded by K_σ,

$$\begin{aligned} E_2 &\leq dK_\sigma^2\sum_{k=1}^{d'}\sum_{m=1}^{d}\int_0^t\int_0^1 E\left[\int_r^t\int_0^1\left[D^l_{(r,z)}X^n_{m,s}\right]^2 d\alpha ds\right]dzdr \\ &= d'dK_\sigma^2\int_0^t E\left[\int_0^s\int_0^1\left|D^l_{(r,z)}X^n_s\right|^2 dzdr\right]ds \\ &\quad \text{using Fubini's Theorem.} \end{aligned}$$

We consider now

$$E_3 = E\left[\int_0^t\int_0^1\left|\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i\left(X_s^n-Y_s(\alpha)\right)D^l_{(r,z)}X^n_{m,s}d\alpha ds\right|^2 dzdr\right].$$

Using the same method as for integral E_2, we finally prove that

$$S_{n+1}(t) \leq C_0 + C_1\int_0^t S_n(s)ds \leq C_0 + C_1T\sup_{0\leq t\leq T}S_n(t) < +\infty, \tag{2.14}$$

with

$$\begin{aligned} C_0 &= 12dd'K_\sigma^2T\left(\sup_n E\left[\sup_{0\leq r\leq T}\left|X_r^n\right|^2\right] + E_\alpha\left[\sup_{0\leq r\leq T}\left|Y_r\right|^2\right]\right) + 6dd'T\left|\sigma(0)\right|^2 \\ C_1 &= 6d^2\max\left(d'K_\sigma^2, K_b^2T\right). \end{aligned}$$

Thus, Recurrence Hypothesis is satisfied for any $n \geq 0$.
Since $S_0 = 0$, we notice that we have in fact a stronger result than property (ii). The estimate (2.14) implies that $\sup_{n\geq 0}\sup_{t\in[0,T]}S_n(t) \leq C_0e^{C_1T}$.

Finally, we have proved

$$\forall n \geq 0\ \forall t\in[0,T]\quad \forall i=1,...,d\ \ X^n_{i,t}\in\mathbb{D}^{1,2} \tag{2.15}$$

$$\sup_{n\geq 0}\sup_{t\in[0,T]}\sum_{l=1}^{d'}E\left(\int_0^t\int_0^1\left|D^l_{(r,z)}X^n_t\right|^2 dzdr\right) < \infty. \tag{2.16}$$

Since the sequence (X^n) converges uniformly on $[0,T]$ in $\mathbb{L}^2$ to X and thanks to (2.15) and (2.16), we deduce that X is differentiable (see [33] lemma 1.2.3). Moreover, the sequence of derivatives (DX^n) converges to DX for the weak topology on $\mathbb{L}^2\left([0,T]\times[0,1]\times\Omega\right)$. Thus, the theorem is proved. ■

2.2 The upper order derivatives

We state that X belongs to $\mathbb{D}^\infty$ under a stronger assumption on σ and b.
Assumption (H^∞): σ and b are functions of class $\mathcal{C}^\infty$ with bounded derivatives of any orders ≥ 1 from $\mathbb{R}^d$ to $\mathcal{M}_{d,d'}(\mathbb{R})$ and $\mathbb{R}^d$ respectively.

Notations: Let $k\geq 1$. We define $\lambda_k=\left((r_1,z_1),...,(r_k,z_k)\right)$ and

$$\hat{\lambda}_m=\left((r_1,z_1),...,(r_{m-1},z_{m-1}),(r_{m+1},z_{m+1}),...,(r_k,z_k)\right),$$

with $r_m\in[0,t]$ and $z_m\in[0,1]$ for $m=1,..,k$.
Let us now define $l(E)=l_{\varepsilon_1},...,l_{\varepsilon_\eta}$ and $\lambda(E)=\left((r_{\varepsilon_1},z_{\varepsilon_1}),...,(r_{\varepsilon_\eta},z_{\varepsilon_\eta})\right)$ for any subset $E=\{\varepsilon_1,...,\varepsilon_\eta\}$ of $\{1,...,k\}$. We consider

$$\Sigma^i_{j,(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)=\sum\sum_{k_1,..,k_\nu=1}^{d}\partial_{k_1,..,k_\nu}\sigma_{i,j}\left(X_s-Y_s(\alpha)\right)D^{l(E_1)}_{\lambda(E_1)}X_{k_1,s}..D^{l(E_\nu)}_{\lambda(E_\nu)}X_{k_\nu,s}$$

$$\beta^i_{(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)=\sum\sum_{k_1,..,k_\nu=1}^{d}\partial_{k_1,..,k_\nu}b_i\left(X_s-Y_s(\alpha)\right)D^{l(E_1)}_{\lambda(E_1)}X_{k_1,s}..D^{l(E_\nu)}_{\lambda(E_\nu)}X_{k_\nu,s},$$

where the first sum is taken on all partitions $E_1\cup...\cup E_\nu=\{1,...,k\}$.
We define at last,

$$\Sigma^i_j\left((s,\alpha)\right)=\sigma_{ij}\left(X_s-Y_s(\alpha)\right).$$

We denote by $r_1\vee...\vee r_k=\sup\{r_1,...,r_k\}$.

Theorem 2.2 *Assume that $X_0\in\mathbb{L}^p$, for any $p\geq 1$. Under Assumption (H^∞), $\forall t\geq 0$ $X_t\in\mathbb{D}^\infty$. Moreover, the i^{th} component of one of its derivative of order k at point $\lambda_k=\left((r_1,z_1),...,(r_k,z_k)\right)$ is given by the following equation*

$$\begin{aligned} D^{l_1,..,l_k}_{\lambda_k}X_{i,t} &= \sum_{m=1}^{k}\Sigma^i_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}\left((r_m,z_m),\hat{\lambda}_m\right) \\ &+ \int_{r_1\vee..\vee r_k}^{t}\int_0^1\sum_{j=1}^{d'}\Sigma^i_{j,(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)W_j(d\alpha,ds) \\ &+ \int_{r_1\vee..\vee r_k}^{t}\int_0^1\beta^i_{(l_1,..,l_k)}\left((s,\alpha),\lambda_k\right)d\alpha ds \end{aligned} \tag{2.17}$$

if $t\geq r_1\vee...\vee r_k$ and $D^{l_1,...,l_k}_{\lambda_k}X_{i,t}=0$ if $t<r_1\vee..\vee r_k$.

Remark 2.3 *In expression (2.17) of the k^{th} derivative, the terms in the first sum with $r_m < r_1 \vee ... \vee r_k$ are equal to 0.*

Proof. We use again the Picard sequence $(X^n)_{n\geq 0}$ defined by (2.12). For any $p \geq 2$, $n \geq 0$, $X^n \in \mathbb{L}^p$ and (X^n) converges uniformly to X in $\mathbb{L}^p$. As σ and b satisfy Assumption (H^∞), using the same method as in the previous paragraph, we prove that $X_t^n \in \mathbb{D}^{1,p}$ $\forall p \geq 1$ for any $t \geq 0$. By recurrence, we prove that $\forall t \geq 0$, $\forall n \geq 0$ $X_t^n \in \mathbb{D}^\infty$.

Let us fix $T > 0$.

Recurrence Hypothesis (h_n):

(i) $X_{i,t}^n \in \mathbb{D}^\infty$, $\forall t \in [0,T]$, $\forall i = 1, ..., d$.

(ii) $\sup_{t\in[0,T]} \sum_{l_1,..,l_k=1}^{d'} E\left(\int_{([0,t]\times[0,1])^k} \left|D_{\lambda_k}^{l_1,..,l_k} X_{i,t}^n\right|^p d\lambda_k\right) < \infty \ \ \forall p \geq 1, \forall k \geq 1$.

(iii) the derivatives of order k have the following expression:

$$\begin{aligned} D_{\lambda_k}^{l_1,..,l_k} X_{i,t}^{n+1} &= \sum_{m=1}^{k} \Sigma_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}^{n,i} \left((r_m, z_m), \hat{\lambda}_m\right) \\ &+ \int_{r_1\vee..\vee r_k}^{t} \int_0^1 \sum_{j=1}^{d'} \Sigma_{j,(l_1,..,l_k)}^{n,i} ((s,\alpha), \lambda_k) W_j(d\alpha, ds) \\ &+ \int_{r_1\vee..\vee r_k}^{t} \int_0^1 \beta_{(l_1,..,l_k)}^{n,i} ((s,\alpha), \lambda_k) \, d\alpha ds \end{aligned} \tag{2.18}$$

if $t \geq r_1 \vee .. \vee r_k$ and $D_{\lambda_k}^{l_1,..,l_k} X_{i,t}^n = 0$ else, where Σ^n and $\beta^n$are defined as Σ and β replacing X with X^n.

Hypothesis (h_0) is satisfied.

Let us assume that Hypothesis (h_n) is true, and let us study (h_{n+1}). According to Assumption (H^∞) and adapting the computation of the first derivative, it is easy to state that the two first properties are satisfied. We just check the expression of the k^{th} derivative by recurrence on k.

For $k = 1$, we have

$$\begin{aligned} D_{(r,z)}^{l} X_{i,t}^{n+1} &= \Sigma_l^{n,i}((r,z)) + \int_r^t \int_0^1 \sum_{j=1}^{d'} \Sigma_{j,(l)}^{n,i} ((s,\alpha), (r,z)) W_j(d\alpha, ds) \\ &+ \int_r^t \int_0^1 \beta_{(l)}^{n,i} ((s,\alpha), (r,z)) \, d\alpha ds, \end{aligned}$$

then the expression (2.18) is satisfied.

We assume that the expression (2.18) of the k^{th} derivative is true, and we now compute the derivative of order $k+1$

$$
\begin{aligned}
& D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(D^{l_1,..,l_k}_{\lambda_k}X^{n+1}_{i,t}\right) = \\
& D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\sum_{m=1}^{k}\Sigma^{n,i}_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k)}\left(\left(r_m,z_m\right),\hat{\lambda}_m\right)\right) \\
+ \; & \Sigma^{n,i}_{l_{k+1},(l_1,..,l_k)}\left(\left(r_{k+1},z_{k+1}\right),\lambda_k\right) \\
+ \; & \int_{r_1\vee..\vee r_k\vee r_{k+1}}^{t}\int_0^1\sum_{j=1}^{d'}D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\Sigma^{n,i}_{j,(l_1,..,l_k)}\left(\left(s,\alpha\right),\lambda_k\right)\right)W_j\left(d\alpha,ds\right) \\
+ \; & \int_{r_1\vee..\vee r_k\vee r_{k+1}}^{t}\int_0^1 D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(\beta^{n,i}_{(l_1,..,l_k)}\left(\left(s,\alpha\right),\lambda_k\right)\right)d\alpha ds.
\end{aligned}
$$

Using some elementary computations, we obtain

$$
\begin{aligned}
D^{l_{k+1}}_{(r_{k+1},z_{k+1})}\left(D^{l_1,..,l_k}_{\lambda_k}X^{n+1}_{i,t}\right) &= \sum_{m=1}^{k}\Sigma^{n,i}_{l_m,(l_1,..l_{m-1},l_{m+1},..,l_k,l_{k+1})}\left(\left(r_m,z_m\right),\hat{\lambda}_m\right) \\
&+\Sigma^{n,i}_{l_{k+1},(l_1,..,l_k)}\left(\left(r_{k+1},z_{k+1}\right),\hat{\lambda}_{k+1}\right) \\
&+\int_{r_1\vee..\vee r_{k+1}}^{t}\int_0^1\sum_{j=1}^{d'}\Sigma^{n,i}_{j,(l_1,..,l_k,l_{k+1})}\left(\left(s,\alpha\right),\lambda_{k+1}\right)W_j\left(d\alpha,ds\right) \\
&+\int_{r_1\vee..\vee r_{k+1}}^{t}\int_0^1\beta^{n,i}_{(l_1,..,l_k,l_{k+1})}\left(\left(s,\alpha\right),\lambda_k\right)d\alpha ds.
\end{aligned}
$$

So by recurrence, the property (iii) of (h_{n+1}) is proved and consequently for any $n \geq 0$ Recurrence Hypothesis (h_n) is satisfied.
Moreover, as in the computation of the first derivative, we have a stronger property than (ii) in (h_n):

Lemma 2.4 *If we denote by*

$$
\begin{aligned}
S_{n,k}\left(t\right) &= \sum_{l_1,..,l_k=1}^{d'}E\left(\int_{([0,t]\times[0,1])^k}\left|D^{l_1,..,l_k}_{\lambda_k}X^n_t\right|^p d\lambda_k\right) \\
M_k &= \sup_{0\leq q\leq k}\sup_{n\geq 0}\sup_{t\in[0,T]}S_{n,q}\left(t\right),
\end{aligned}
$$

then for any $k \geq 1$, $M_k < \infty$.

Proof. The proof is similar to the proof of Theorem 2.1. ■

As (X^n) converges to X in $\mathbb{L}^p$ uniformly on $[0,T]$ for any $T > 0$, the process X satisfies the conditions of [33] lemma 1.5.4). Then, the theorem is proved. ■

3 Existence of a weak function-solution of the Landau equation

Under some suitable conditions on the function h, the Landau coefficients satisfy Assumption (H^1) (see Remark 1.8). Consequently, if X_0 belongs to $\mathbb{L}^2$, the process X solution of $(NSDE(\sigma, b))$ is differentiable in the Malliavin sense. Let us now study the Malliavin matrix $I_t = \int_0^T \int_0^1 D_{(r,z)} X_t. \left(D_{(r,z)} X_t\right)^* dzdr$ for any $t > 0$ to state the following theorem.

Theorem 3.1 *Assume that X_0 is a $\mathbb{R}^d$-valued random vector with a finite 2-order moment. Let σ and b be the coefficients of the Landau equation defined respectively by (2.7), (2.2) and (2.4). We assume that σ and b are Lipschitz continuous of class $\mathcal{C}^1$. If the distribution of X_0 is not a Dirac mass and if we denote by (X, Y) the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds. \tag{$NSDE(\sigma, b)$}$$

then, for any $t > 0$ the distribution P_t of X_t is absolutely continuous with respect the Lebesgue measure.

Corollary 3.2 *Let P_0 be a probability measure such that $\int |x|^2 P_0(dx) < \infty$. Let σ and b be the coefficients of the Landau equation defined respectively by (2.7), (2.2) and (2.4). We assume that σ and b are Lipschitz continuous of class $\mathcal{C}^1$. If P_0 is not a Dirac measure, there exists a unique weak function-solution of the Landau equation with initial data P_0.*

Proof. (Corollary 3.2)
Let X_0 be a random vector with distribution P_0 and X be a solution of $(NSDE(\sigma, b))$ with initial data X_0. If we denote by f_t the density of the distribution of X_t, then, using Itô's Formula, the function f, defined by $f(x,t) = f_t(x)$ for $t > 0$, is a weak function solution of the Landau equation (2.5) with initial data P_0.
The uniqueness is given by Corollary 1.7. ■

Remark 3.3 *Without any restriction, we can assume that* $\mathbf{E}[\mathbf{X_0}] = \mathbf{0}$ *to simplify the computations.*

Proof. (Remark 3.3)
By conservation of momentum, if we define for any $t \geq 0$, $X'_t = X_t - E[X_0]$, the expectation of X' is equal to 0 and X' satisfies the following equation

$$X'_t = X'_0 + \int_0^t \int_0^1 \sigma\left(X'_s - Y'_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X'_s - Y'_s(\alpha)\right) d\alpha ds,$$

with $Y'_s(\alpha) = Y_s(\alpha) - E[X_0]$.
As $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$, we also have $\mathcal{L}(X') = \mathcal{L}_\alpha(Y')$.

If we prove that the distribution of X_t' has a density f_t' with respect the Lebesgue measure, then X_t has a density given by $f_t(z) = f_t'(z - E[X_0])$. ■

Proof. (Theorem 3.1)
We recall the expression of the first Malliavin derivative of X at point $(r,z) \in [0,\infty) \times [0,1]$:

$$\begin{aligned} D^l_{(r,z)} X_{i,t} &= \sigma_{i,l}(X_r - Y_r(z)) \\ &+ \int_r^t \int_0^1 \sum_{k=1}^{d'} \sum_{m=1}^{d} \partial_m \sigma_{i,k}(X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s} W_k(d\alpha, ds) \\ &+ \int_r^t \int_0^1 \sum_{m=1}^{d} \partial_m b_i (X_s - Y_s(\alpha)) D^l_{(r,z)} X_{m,s} d\alpha ds \end{aligned}$$

if $t \geq r$ and $D^l_{(r,z)} X_{i,t} = 0$ else.
We fix $(r,z) \in [0,\infty) \times [0,1]$ and we define

$$\begin{aligned} S_k(.) &= (\partial_m \sigma_{i,k}(.))_{1 \leq i,m \leq d} \\ B(.) &= (\partial_m b_i(.))_{1 \leq i,m \leq d} \end{aligned}$$

Thus we give a matricial expression of the derivative of X

$$\begin{aligned} D_{(r,z)} X_t &\underset{\text{if } t \geq r}{=} \sigma(X_r - Y_r(z)) + \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k(X_s - Y_s(\alpha)) . D_{(r,z)} X_s W_k(d\alpha, ds) \\ &+ \int_r^t \int_0^1 B(X_s - Y_s(\alpha)) . D_{(r,z)} X_s d\alpha ds \\ D_{(r,z)} X_t &\underset{\text{if } t<r}{=} 0. \end{aligned}$$

Let us define the semimartingale Z^r, for any $t \geq r$,

$$Z_t^r = \int_r^t \int_0^1 \sum_{k=1}^{d'} S_k(X_s - Y_s(\alpha)) W_k(d\alpha, ds) + \int_r^t \int_0^1 B(X_s - Y_s(\alpha)) d\alpha ds.$$

As S and B are bounded, $(Z_t^r)_{t \geq r}$ is a continuous semimartingale and the first derivative satisfies the equation

$$D_{(r,z)} X_t = \sigma(X_r - Y_r(z)) + \int_r^t dZ_s^r . D_{(r,z)} X_s. \qquad (2.19)$$

Using the results of Jacod in [27], there is a unique solution of (2.19) defined almost surely, for any $t \geq r$ by

$$D_{(r,z)} X_t = \mathcal{E}(Z)_t^r . \sigma(X_r - Y_r(z)),$$

with $\mathcal{E}(Z)_t^r$ invertible for any $t \geq r$.

We fix now $t > 0$.
We want to apply Theorem (a), thus study if the Malliavin matrix I_t is invertible a.s..

$$\begin{aligned} I_t &= \int_0^\infty \int_0^1 D_{(r,z)} X_t. \left(D_{(r,z)} X_t\right)^* drdz \\ &= \int_0^t \int_0^1 D_{(r,z)} X_t. \left(D_{(r,z)} X_t\right)^* drdz \\ &= \int_0^t \int_0^1 \mathcal{E}(Z)_t^r .\sigma(X_r - Y_r(z)) .\sigma^*(X_r - Y_r(z)) . (\mathcal{E}(Z)_t^r)^* drdz \\ &= \int_0^t \mathcal{E}(Z)_t^r . \left(\int_0^1 a(X_r - Y_r(z))\, dz\right) . (\mathcal{E}(Z)_t^r)^* dr. \end{aligned}$$

I_t is a nonnegative symmetric matrix, then I_t is invertible if and only if $V^*.I_t.V > 0$ for any $V \in \mathbb{R}^d \setminus \{0\}$.
We define $\Gamma_r = \int_0^1 a(X_r - Y_r(z))\, dz$.

We prove the theorem by contradiction.
Assumption : let us suppose that I_t is not "invertible a.s.".
Then, there exists a subset $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) > 0$, such that $\forall \omega \in \Omega_1$ $I_t(\omega)$ is not invertible.
Let Ω_2 be such that $\mathbb{P}(\Omega_2) = 1$ and $\forall \omega \in \Omega_2$ $\forall r \leq t$, $\mathcal{E}(Z)_t^r(\omega)$ is invertible.
We define $\Omega_0 = \Omega_1 \cap \Omega_2$, and we notice that $\mathbb{P}(\Omega_0) > 0$.
Let us fix $\omega_0 \in \Omega_0$.

- As $I_t(\omega_0)$ is not invertible, there exists a vector $V = V(\omega_0, t) \in \mathbb{R}^d \setminus \{0\}$ such that

$$\begin{aligned} V^*.I_{\omega_0}(t).V &= \int_0^t V^*.\mathcal{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathcal{E}(Z)_t^r(\omega_0))^*.V dr \\ &= 0. \end{aligned}$$

As for any $r \leq t$, $\mathcal{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathcal{E}(Z)_t^r(\omega_0))^*$ is a nonnegative symmetric matrix, we notice that

$$V^*.\mathcal{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathcal{E}(Z)_t^r(\omega_0))^*.V \geq 0.$$

Then, on a subset J_{ω_0} of full measure in $[0,t]$,$V^*.\mathcal{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathcal{E}(Z)_t^r(\omega_0))^*.V = 0$, which implies that $\forall r \in J_{\omega_0}$, $\mathcal{E}(Z)_t^r(\omega_0).\Gamma_r(\omega_0).(\mathcal{E}(Z)_t^r(\omega_0))^*$ is a non invertible matrix. However, since $\Omega_0 \subset \Omega_2$, $\mathcal{E}(Z)_t^r(\omega_0)$ is invertible for any $r \leq t$, and consequently $\Gamma_r(\omega_0)$ is not invertible for $r \in J_{\omega_0}$.

- Let us now study if the situation "$\Gamma_r(\omega_0)$ non invertible for almost all r" is possible.

Using Lebesgue's Theorem, we notice that the mapping $r \to \Gamma_r(\omega_0)$ is continuous. Consequently, $\Gamma_r(\omega_0)$ non invertible for almost all r implies that $\Gamma_r(\omega_0)$ is non invertible for any $r \in [0,t]$.
Let $V = (V_i)_{1 \leq i \leq d}$ be a vector in $\mathbb{R}^d \setminus \{0\}$.

Using the lower bound (2.3) of h, since $E[X_t] = 0$ and $\mathcal{L}(X_t) = \mathcal{L}_\alpha(Y_t)$ $\forall t \geq 0$, we obtain the following estimate

$$\begin{aligned}
&V^*.\Gamma_r(\omega_0).V \\
&= \int_0^1 h\left(|X_r(\omega_0) - Y_r(z)|^2\right)\Bigg[|V|^2|X_r(\omega_0) - Y_r(z)|^2 \\
&\qquad\qquad - \left(\sum_{i=1}^d V_i(X_{i,r}(\omega_0) - Y_{i,r}(z))\right)^2\Bigg]dz \\
&\geq m\left[|V|^2|X_r(\omega_0)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega_0)\right)^2 + E\left(|X_r|^2|V|^2 - < X_r, V >^2\right)\right].
\end{aligned}$$

Using Cauchy-Schwarz's inequality, we notice that

$$|V|^2|X_r(\omega_0)|^2 - \left(\sum_{i=1}^d V_i X_{i,r}(\omega_0)\right)^2 \geq 0$$

and consequently,

$$V^*.\Gamma_r(\omega_0).V \geq m\,E\left(|X_r|^2|V|^2 - < X_r, V >^2\right) \tag{2.20}$$

$\Gamma_r(\omega_0)$ non invertible means that for any $r \in [0,t]$ there exists $V_r = V_r(\omega_0) \in \mathbb{R}^d \setminus \{0\}$ such that $V_r^*.\Gamma_r(\omega_0).V_r = 0$. We can assume that $|V_r| = 1$ without restriction. Nevertheless, using expression (2.20), this implies that there is equality in Cauchy-Schwarz, i.e. $\forall\, r \in [0,t]$ there exists a random variable U_r such that for any $\omega \in \Omega$,

$$X_r(\omega) = U_r(\omega)\,V_r. \tag{2.21}$$

Using the conservation of the momentum, of the kinetic energy and $|V_r| = 1$, we notice that $E[U_r] = E[X_0] = 0$ and $E[U_r{}^2] = E[|X_0|^2]$. Moreover, since the distribution of X_0 is not a Dirac measure, $E[U_r^2] \neq 0$ for any $r \in [0,t]$.

• The distribution of a solution of $(NSDE(\sigma, b))$ is a measure-solution of the Landau equation (2.6). Then, we will now study if the distribution of a process defined by (2.21) can be a solution of the Landau equation.
We denote by Q the distribution of the process U and by Q_t the distribution on $\mathbb{R}$ of U_t. Using (2.21), the equation (2.6) writes

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} \varphi(x\,V_t)\,Q_t(dx) &= \frac{1}{2}\sum_{i,j=1}^d \int_{\mathbb{R}\times\mathbb{R}} a_{ij}((x-y)\;V_t)\,\partial_{ij}\varphi(x\,V_t)\,Q_t(dx)\,Q_t(dy) \\
&\quad + \sum_{i=1}^d \int_{\mathbb{R}\times\mathbb{R}} b_i((x-y)\;V_t)\,\partial_i\varphi(x\,V_t)\,Q_t(dx)\,Q_t(dy)
\end{aligned}$$

for any test function $\varphi \in \mathcal{C}_b^2\left(\mathbb{R}^d, \mathbb{R}\right)$.

As $|V_t| = 1$, for $i,j \in \{1,..,d\}$

$$\begin{aligned} a_{ij}\left((x-y)\;V_t\right) &= (x-y)^2\, h\left((x-y)^2\right)\left(\delta_{ij} - V_{i,t}V_{j,t}\right), \\ b_i\left((x-y)\;V_t\right) &= -\left(d-1\right)(x-y)\, h\left((x-y)^2\right) V_{i,t}. \end{aligned}$$

Then,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \varphi\left(x\,V_t\right) Q_t\left(dx\right) \\ &= \frac{1}{2}\sum_{i,j=1}^{d}\left(\delta_{ij} - V_{i,t}V_{j,t}\right)\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2\, h\left((x-y)^2\right)\partial_{ij}\varphi\left(x\,V_t\right) Q_t\left(dx\right) Q_t\left(dy\right) \\ &- \left(d-1\right)\sum_{i=1}^{d} V_{i,t}\int_{\mathbb{R}\times\mathbb{R}}(x-y)\, h\left((x-y)^2\right)\partial_i\varphi\left(x\,V_t\right) Q_t\left(dx\right) Q_t\left(dy\right). \end{aligned}$$

We now explicit the equation satisfied by the 2-order moments of X: let $k,l \in \mathbb{N}$, $k \neq l$. Using $\varphi\left(v\right) = v_k^2$ or $\varphi\left(v\right) = v_k v_l$, $v \in \mathbb{R}^d$, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} V_{k,t}^{\,2}\, x^2 Q_t\left(dx\right) &= E\left[|X_0|^2\right]\frac{d}{dt}V_{k,t}^{\,2} \\ &= \left(1 - d\,V_{k,t}^{\,2}\right)\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2\, h\left((x-y)^2\right) Q_t\left(dx\right) Q_t\left(dy\right), \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} V_{k,t}V_{l,t}x^2 Q_t\left(dx\right) &= E\left[|X_0|^2\right]\frac{d}{dt}V_{k,t}V_{l,t} \\ &= -d\,V_{k,t}V_{l,t}\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2\, h\left((x-y)^2\right) Q_t\left(dx\right) Q_t\left(dy\right). \end{aligned}$$

Let us define $f\left(t\right) = \displaystyle\int_{\mathbb{R}\times\mathbb{R}}(x-y)^2\, h\left((x-y)^2\right) Q_t\left(dx\right) Q_t\left(dy\right)$. As h satisfies (2.3) and $E\left[|U_t|^2\right] \neq 0$, for any $t \geq 0$, we notice that $f\left(t\right) > 0$.
Let us now compute $\frac{d}{dt}\left(V_{k,t}^{\,2}V_{l,t}^{\,2}\right)$ using two different ways :

$$\begin{aligned} E\left[|X_0|^2\right]\frac{d}{dt}\left(V_{k,t}^{\,2}V_{l,t}^{\,2}\right) &= V_{k,t}^{\,2}E\left[|X_0|^2\right]\frac{d}{dt}V_{l,t}^{\,2} + V_{l,t}^{\,2}E\left[|X_0|^2\right]\frac{d}{dt}V_{k,t}^{\,2} \\ &= V_{k,t}^{\,2}f\left(t\right) + V_{l,t}^{\,2}f\left(t\right) - 2d\,V_{k,t}^{\,2}V_{l,t}^{\,2}f\left(t\right), \\ E\left[|X_0|^2\right]\frac{d}{dt}\left(V_{k,t}^{\,2}V_{l,t}^{\,2}\right) &= 2V_{k,t}V_{l,t}E\left[|X_0|^2\right]\frac{d}{dt}\left(V_{k,t}V_{l,t}\right) \\ &= -2d\,V_{k,t}^{\,2}V_{l,t}^{\,2}f\left(t\right). \end{aligned}$$

Then $V_t = 0$ which is impossible.
Finally, I_t is invertible a.s. for any $t > 0$ and according to Theorem (a), the theorem is proved. ■

Remark 3.4 *We notice that the matrix* $\Gamma_r = \displaystyle\int_0^1 a\left(X_r - Y_r\left(z\right)\right)dz$ *is invertible a.s., whereas* $\det\left(a\left(X_r - Y_r\left(z\right)\right)\right) = 0$ *for any* r,z. *In fact, thanks to the nonlinearity of equation* $\left(NSDE\left(\sigma,b\right)\right)$*, we can conclude that the Malliavin matrix has an inverse a.s..*

Remark 3.5 *A consequence of Theorem 3.1 is*

$$E\left(|X_t|^2|V|^2 - \langle X_t, V\rangle^2\right) > 0$$

for any $V \in \mathbb{R}^d \setminus \{0\}$, for any $t > 0$, when the random vector X_0 is not a constant.

4 Regularity of the weak function-solution

Theorem 4.1 *Let X_0 be a random vector such that $E\left[|X_0|^p\right] < \infty$ for any $p \geq 1$. Let σ and b be the coefficients of the Landau equation respectively defined by (2.7), (2.2) and (2.4). We assume that σ and b are Lipschitz continuous and infinitely differentiable with bounded derivatives. If the distribution of X_0 is not a Dirac mass and if we denote by X the solution of the nonlinear stochastic differential equation*

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\left(X_s - Y_s(\alpha)\right).W^{d'}(d\alpha, ds) + \int_0^t \int_0^1 b\left(X_s - Y_s(\alpha)\right) d\alpha ds. \tag{$NSDE(\sigma,b)$}$$

then for any $t > 0$ the distribution of X_t has a bounded density of class $\mathcal{C}^\infty$ with bounded derivatives with respect the Lebesgue measure on $\mathbb{R}^d$.

Corollary 4.2 *Let P_0 be a probability measure such that $\int |x|^p P_0(dx) < \infty$ for any $p \geq 1$. Let σ and b be the coefficients of the Landau equation defined respectively by (2.7), (2.2) and (2.4). We assume that σ and b are Lipschitz continuous and infinitely differentiable with bounded derivatives. If P_0 is not a Dirac measure, there exists a unique weak function solution of the Landau equation with initial data P_0 which is moreover of class $\mathcal{C}^\infty$, bounded on $\mathbb{R}^d$ and with bounded derivatives.*

Remark 4.3 *Using the expressions (2.10) or (2.11), we notice that, if h is a bounded function of class such that $h^{(l)}(x) = O\left(\frac{1}{x^{l+1}}\right)$ when $x \to +\infty$ for any $l \geq 1$, σ and b are Lipschitz continuous functions of class $\mathcal{C}^\infty$ with bounded derivatives.*

Proof. As in the previous part, we assume that $E[X_0] = 0$ to simplify the computations. As σ and b satisfy Assumption (H^∞), the process X is infinitely differentiable in the Malliavin sense. We need to study the moments of the inverse of the determinant of the Malliavin matrix I_t at time t, for any $t > 0$, to apply Theorem (b). The expression of the determinant is complex, nevertheless we can notice that in dimension d,

$$(\det I_t)^{1/d} \geq \inf_{|V|=1} \langle I_t V, V\rangle,$$

where $\langle .,. \rangle$ is the euclidean scalar product in $\mathbb{R}^d$.
Moreover, see [2] lemma 3.4, the property (ii) of theorem (b) is satisfied as soon as for any $k \in \mathbb{N}$

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} \mathbb{P}\left((\det I_t)^{1/d} < c\varepsilon\right) = 0, \tag{2.22}$$

where c is a positive constant which will be computed later.

Let $t>0$ be fixed.
As $(\det I_t)^{\frac{1}{d}} \geq \inf_{|V|=1}\langle I_t V, V\rangle$, we want to find a lower bound for $\inf_{|V|=1}\langle I_t V, V\rangle$.
Let ε be such that $0<\varepsilon<\frac{t}{2}$. We consider $V=(V_i)_{1\leq i\leq d}\in\mathbb{R}^d$ such that $|V|=1$.

$$\begin{aligned}\langle I_t V, V\rangle &= \int_0^t\int_0^1\sum_{l=1}^{d'}\left(\sum_{i=1}^{d} D^l_{(r,z)}X_{i,t}V_i\right)^2 dzdr\\ &\geq \int_{t-\varepsilon}^t\int_0^1\sum_{l=1}^{d'}\left(\sum_{i=1}^{d} D^l_{(r,z)}X_{i,t}V_i\right)^2 dzdr\\ &\geq \frac{2}{3}I_1 - 2I_2\end{aligned}$$

with

$$\begin{aligned}I_1 &= \int_{t-\varepsilon}^t\int_0^1\sum_{l=1}^{d'}\left(\sum_{i=1}^{d}\sigma_{i,l}\left(X_r - Y_r(z)\right)V_i\right)^2 dzdr,\\ I_2 &= \int_{t-\varepsilon}^t\int_0^1\sum_{l=1}^{d'}\left[\sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{k=1}^{d'}\sum_{m=1}^{d}\partial_m\sigma_{i,k}\left(X_s - Y_s(\alpha)\right)D^l_{(r,z)}X_{m,s}W_k(d\alpha,ds)\right.\\ &\left.+\sum_{i=1}^{d}V_i\int_r^t\int_0^1\sum_{m=1}^{d}\partial_m b_i\left(X_s - Y_s(\alpha)\right)D^l_{(r,z)}X_{m,s}d\alpha ds\right]^2 dzdr.\end{aligned}$$

Then

$$\inf_{|V|=1}\langle I_t V, V\rangle \geq \frac{2}{3}\inf_{|V|=1} I_1 - 2\sup_{|V|=1} I_2.$$

We want to minimize the first integral:

$$\begin{aligned}I_1 &= \int_{t-\varepsilon}^t\int_0^1\sum_{l=1}^{d'}\left(\sum_{i=1}^{d}\sigma_{i,l}\left(X_r - Y_r(z)\right)V_i\right)^2 dzdr\\ &= \int_{t-\varepsilon}^t\int_0^1\sum_{i,j=1}^{d}a_{i,j}\left(X_r - Y_r(z)\right)V_iV_j dzdr\\ &= \int_{t-\varepsilon}^t V^*.\Gamma_r.V\,dr\end{aligned}$$

Using the results of Section 3, we obtain

$$I_1 \geq m\int_{t-\varepsilon}^t E\left(|X_r|^2|V|^2 - \langle X_r, V\rangle^2\right)dr.$$

We define the function $f(V,r) = E\left(|X_r|^2|V|^2 - \langle X_r, V\rangle^2\right)$. We notice that f is a positive continuous function (see Remark 3.5) on the compact subset $D = \left\{V \in \mathbb{R}^d : |V| = 1\right\} \times \left\{r : \frac{t}{2} \le r \le t\right\}$, then f reaches its minimum. So, if we denote by

$$\tilde{c} = \inf\left\{f(V,r) : |V| = 1 \text{ and } \frac{t}{2} \le r \le t\right\},$$

we notice that $\tilde{c}$ is independent of $\omega \in \Omega$, $\tilde{c} > 0$ and

$$I_1 \ge m.\tilde{c}.\varepsilon.$$

Let us now study $E\left[\sup_{|V|=1} I_2^p\right]$ for $p \ge 1$.

$$\begin{aligned} I_2 &= \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l=1}^{d'}\left[\sum_{i=1}^{d} V_i \int_r^t\int_0^1 \sum_{k=1}^{d'}\sum_{m=1}^{d} \partial_m\sigma_{i,k}\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s} W_k(d\alpha, ds)\right.\\ &\quad \left. + \sum_{i=1}^{d} V_i \int_r^t\int_0^1 \sum_{m=1}^{d} \partial_m b_i\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s}\,d\alpha ds\right]^2 dzdr. \end{aligned}$$

Using Burkholder-Davis-Gundy's and Hölder's inequalities, and the fact that $|V| = 1$, we notice that

$$\begin{aligned} &E\left[\sup_{|V|=1} I_2^p\right]\\ &\le K\varepsilon^{p-1}\left\{\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,m} E\left[\left|\sum_{k=1}^{d'}\int_r^t\int_0^1 \partial_m\sigma_{i,k}\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s} W_k(d\alpha, ds)\right|^{2p}\right] dzdr\right.\\ &\qquad \left. + \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,m} E\left[\left|\int_r^t \partial_m b_i\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s}\,d\alpha ds\right|^{2p}\right] dzdr\right\}\\ &\le K\varepsilon^{p-1}\left\{\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,k,m} E\left[\left|\int_r^t\int_0^1 \left(\partial_m\sigma_{i,k}\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s}\right)^2 d\alpha ds\right|^{p}\right] dzdr\right.\\ &\qquad \left. + \int_{t-\varepsilon}^{t}\int_0^1 \sum_{l,i,m} E\left[\left|\int_r^t \partial_m b_i\left(X_s - Y_s(\alpha)\right) D^l_{(r,z)}X_{m,s}\,d\alpha ds\right|^{2p}\right] dzdr\right\}. \end{aligned}$$

As the derivatives of σ and b are bounded, using Hölder's inequality, we obtain

$$\begin{aligned} E\left[\sup_{|V|=1} I_2^p\right] &\le K\varepsilon^{2p-2}\left\{\int_{t-\varepsilon}^{t}\int_0^1 \sum_{l} E\left[\int_r^t \left|D^l_{(r,z)}X_s\right|^{2p} ds\right] dzdr\right\}\\ &\quad \text{using Fubini's Theorem}\\ &\le K\varepsilon^{2p-2}\int_{t-\varepsilon}^{t} E\left[\int_{t-\varepsilon}^{s}\int_0^1 \left|D_{(r,z)}X_s\right|^{2p} dzdr\right] ds. \end{aligned}$$

Then, for any $p \ge 2$ there exists a constant $K = K(p, d, d', t)$ such that

$$E\left[\sup_{|V|=1} I_2^p\right] \le K\varepsilon^{2p-1}\sup_{0\le s\le t} E\left[\int_0^s\int_0^1 \left|D_{(r,z)}X_s\right|^{2p} dzdr\right].$$

Let us now check that $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}\left((\det I_t)^{\frac{1}{d}}<c\varepsilon\right)=0$, where $k\in\mathbb{N}$ is fixed and $c=\frac{1}{3}m\tilde{c}$ with $\tilde{c}$ the constant built in the study of the first integral I_1.
Let $p\in\mathbb{N}$ such that $p>k+1$.

$$\begin{aligned}
\mathbb{P}\left((\det I_t)^{\frac{1}{d}}<c\varepsilon\right) &\leq \mathbb{P}\left(\inf_{|V|=1}\langle I_tV,V\rangle<c\varepsilon\right)\\
&\leq \mathbb{P}\left(\frac{2}{3}\inf_{|V|=1}I_1-2\sup_{|V|=1}I_2<c\varepsilon\right)\\
&\leq \mathbb{P}\left(\sup_{|V|=1}I_2>\frac{c\varepsilon}{2}\right)\\
&\quad\text{using Tchebychev's Inequality}\\
&\leq \left(\frac{2}{c}\right)^p\varepsilon^{-p}E\left[\sup_{|V|=1}I_2^p\right]\\
&\leq K\varepsilon^{p-1}.
\end{aligned}$$

Thus, $\lim_{\varepsilon\to 0}\varepsilon^{-k}\mathbb{P}\left((\det I_t)^{\frac{1}{d}}<c\varepsilon\right)=0$. We can apply Theorem (b), then there exists a density $f_t(v)$ to the distribution $P_t(dv)$ for any $t>0$.
Using [33] Lemma 2.1.5, the density $f_t(v)$ is given by

$$f_t(v)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-i\langle x,v\rangle}\hat{\nu}_t(x)\,dx,$$

where $\hat{\nu}_t(x)=\int_{\mathbb{R}^d}e^{i\langle x,y\rangle}P_t(dy)$ is the Fourier transform of P_t. Thanks to Theorem (b), we have the following estimates

$$\left|x_1^{\alpha_1}...x_d^{\alpha_d}\hat{\nu}_t(x)\right|\leq C_{\alpha,t}$$

for any $\alpha=\{\alpha_1,..,\alpha_d\}\in\mathbb{N}$ with $C_{\alpha,t}$ depending on the moments of $(\det I_t)^{-1}$ and on the moments of the derivatives of X. Then f_t is a bounded function on $\mathbb{R}^d$ by

$$f_t(v)\leq C_{\{2,..,2\},t}\int_{\mathbb{R}^d}\left(1\wedge_{i=1}^d\frac{1}{x_i^2}\right)dx$$

and its derivatives have similar estimates. ■

Partie II

Lien entre les équations de Boltzmann et de Landau

Chapitre 3

Convergence from Boltzmann to Landau processes with soft potential and particle approximations

Abstract: Our aim in this paper is to show how a probabilistic interpretation of the Boltzmann and Landau equations gives a microscopic understanding of these equations. We firstly associate stochastic jump processes with the Boltzmann equations we consider. Then we renormalize these equations following asymptotics which make prevail the grazing collisions, and prove the convergence of the associated Boltzmann jump processes to a diffusion process related to the Landau equation. The convergence is pathwise and also implies a convergence at the level of the partial differential equations. The best feature of this approach is the microscopic understanding of the transition between the Boltzmann and the Landau equations, by an accumulation of very small jumps. We deduce from this interpretation an approximation result for a solution of the Landau equation via colliding stochastic particle systems. This result leads to a Monte-Carlo algorithm for the simulation of solutions by a conservative particle method which enables to observe the transition from Boltzmann to Landau equations. Numerical results are given.

Ce travail a été écrit en collaboration avec Sylvie Méléard[1] et a été accepté pour publication dans la revue *Journal of Statistical Physics*.

1 Introduction.

Our aim in this paper is to show how a probabilistic interpretation of the Boltzmann and Landau equations gives a microscopic understanding of these equations.

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In the first part of the paper, we consider spatially homogeneous soft potential Boltzmann equations without angular cutoff for a large class of initial data, and relate them to jump processes solutions of Poisson-driven stochastic differential equations. These results extend results due to Tanaka in the Maxwellian case and for L^1-hypotheses on the cross-section [38] and generalized by Horowitz-Karandikar [26] in the L^2-case and by Fournier-Méléard [20] for non Maxwell molecules in dimension 2.

This probabilistic representation has been proved usefull either to obtain existence of measure solutions of the Boltzmann equation for a large class of measure initial data or to prove, at least in dimension two, the existence of positive smooth solutions to the Boltzmann equation, improving thus the analytical results (Graham-Méléard [25], Fournier [18]). It also allows to get numerical Monte-Carlo methods for Boltzmann equations without cutoff (Desvillettes-Graham-Méléard [12], Fournier-Méléard [21]).

In this paper we show more specifically that the microscopic stochastic representation of the Boltzmann equations leads to a natural and intuitive understanding of the transition to Landau equations, when grazing collisions prevail.

The Fokker-Planck-Landau equation, or Landau equation, is derived from the Boltzmann (see [30]) and is usually considered as an approximation of the homogeneous Boltzmann equation in the limit of grazing collisions. Many authors have been interested in proving rigourously this convergence, in different cases of scaterring cross-section and initial data. Firstly Arsen'ev and Buryak [1] proved the convergence of solutions of the Bolzmann equation towards solutions of the Landau equation under very restrictive assumptions. Then, Desvillettes [10] gave a mathematical framework for more physical situations, but excluding the main case of Coulomb potential studied by Degond and Lucquin, [9], for which the Boltzmann equation is not realistic (see [41]) and the Landau equation appears naturally. More recently, Goudon [23] and Villani [41] proved the convergence of Boltzmann equations towards the Landau equation, with bounded entropy and energy function as initial data. They use analytical techniques as convergence theorems or spectral analysis, showing a L^1-convergence for a bounded entropy and energy initial condition. However, these results could be relaxed without the entropy assumption in a weak-$*$ convergence.

In the grazing collision asymptotics, the cross-section in the Boltzmann operator is renormalized by a small parameter depending on the nature of the collisions. In this paper, we consider asymptotics which include those of Degond-Lucquin [9] and Desvillettes [10]. We show how the accumulation of grazing collisions can be interpreted at the level of the jump processes as an accumulation of small jumps. Then we prove the convergence in law, in the Skorohod space, of sequences of renormalized Boltzmann processes to a diffusion process, called Landau process, which describes the microscopic random behaviour of the Fokker-Planck-Landau equation. We immediately deduce a convergence result at the level of the partial differential equations for general initial data. Unhappily, the probabilistic tools oblige us to use a L^2-framework, which necessitates the consideration of potentials $\gamma \in (-1, 0]$. In particular, our theorical approach do not recover the interesting coulombian case, even if the Monte-Carlo algorithm also makes sense in this case.

As in the analytical framework, uniqueness is an open problem for all the equations we consider. All the convergence results we prove are obtained by a compactness method which only gives converging subsequences.

The pathwise interpretation of the equations (in the probabilistic framework) provides a natural approximation by interacting colliding particle systems of the Fokker-Planck-Landau equations. The collision rate and the amplitude of jumps of the particles are related to the size of the system. We prove the convergence of its empirical measures to a weak solution of the Landau equation, when the size of the system growths. We deduce from this theorical result a simple simulation algorithm, based upon particles conserving momentum and kinetic energy.
We finally discuss about numerical results. The main interest of our approach is to observe in the simulations the transition from the renormalized Boltzmann equations to the Landau equation (see Section 6 Figure 1).

The paper is organized as follows : in Section 2, we explain the pathwise interpretation of the Boltzmann equation with soft potential, and solve the nonlinear Poisson-driven stochastic differential equation. In Section 3, we study the convergence in law of the renormalized Boltzmann processes to a Landau process and deduce the convergence of solutions of the Boltzmann equations to the ones of the Landau equation when the grazing collisions prevail. In Section 4, we study the approximating particle systems. We describe the pathwise Monte-Carlo algorithm in Section 5. Numerical results are discussed in Section 6.

Notations

- $\mathbb{D}_T$ will denote the Skorohod space $\mathbb{D}([0,T],\mathbb{R}^3)$ of cadlag functions from $[0,T]$ into $\mathbb{R}^3$. The space $\mathbb{D}_T$ endowed with the Skorohod topology is a Polish space.
- $\mathcal{C}_T$ is the space $C([0,T],\mathbb{R}^3)$ of continuous functions from $[0,T]$ into $\mathbb{R}^3$ and $C_b^2(\mathbb{R}^3)$ is the space of real bounded functions of class $\mathcal{C}^2$ with bounded derivatives.
- $\mathcal{P}(\mathbb{R}^3)$ is the set of probability measures on $\mathbb{R}^3$ and $\mathcal{P}_2(\mathbb{R}^3)$ the subset of probability measures with a finite second order moment. Similarly, $\mathcal{P}(\mathbb{D}_T)$ denotes the space of probability measures on $\mathbb{D}_T$ and $\mathcal{P}_2(\mathbb{D}_T)$ is the subset of probability measures with a finite second order moment: $q \in \mathcal{P}_2(\mathbb{D}_T)$ if $\int_{x\in\mathbb{D}_T} \sup_{t\in[0,T]} |x(t)|^2 q(dx) < \infty$.
- Let A and B be two matrices with same dimensions. The symbol $A : B$ denotes the real $\sum_{i,j} A_{ij}B_{ij}$ and A^t is the transpose matrix of the matrix A.
- K will denote a real positive constant of which the value may change from line to line.
- We define the natural number **p** by $p = 2$ when $\gamma \in (-1,0)$ and $p = 4$ when $\gamma = 0$.

2 The Boltzmann Process

2.1 The equation

The Boltzmann equation we consider describes the evolution of the density $f(t,v)$ of particles with velocity $v \in \mathbb{R}^3$ at time t in a rarefied homogeneous gas:

$$\frac{\partial f}{\partial t} = Q_B(f,f), \tag{3.1}$$

where Q_B is a quadratic collision kernel preserving momentum and kinetic energy,

$$Q_B(f,f)(t,v) \;=\; \int_{v_*\in I\!R^3}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\Big(f(t,v')f(t,v'_*)-f(t,v)f(t,v_*)\Big)B(|v-v_*|,\theta)d\theta d\varphi dv_* \tag{3.2}$$

with $v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma$ and $v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma$, the unit vector σ having colatitude θ and longitude φ in the spherical coordinates in which $v-v_*$ is the polar axis. The nonnegative function B is called the cross-section.
We are interested in cases for which the molecules in the gas interact according to an inverse power law in $1/r^s$ with $s\geq 2$. The physical cross-sections $B(z,\theta)$ tend to infinity when θ goes to zero, but satisfy $\int_0^\pi |\theta|^2 B(z,\theta)d\theta<\infty$ for each z. Physically, this explosion near 0 comes from the accumulation of grazing collisions.

In this general (spatially homogeneous) setting, the Boltzmann equation is difficult to study. A large literature deals with the non physical equation with angular cutoff, namely under the assumption $\int_0^\pi B(z,\theta)d\theta<\infty$. More recently, the case of Maxwell molecules, for which the cross-section $B(z,\theta)=\beta(\theta)$ only depends on θ, has been studied without the cutoff assumption. In the Maxwell context, Tanaka [38] was considering the case where $\int_0^\pi \theta\beta(\theta)d\theta<\infty$, and Horowitz, Karandikar [26], Desvillettes [11], and Fournier, Méléard [18], [21], have worked under the physical assumption $\int_0^\pi \theta^2\beta(\theta)d\theta<+\infty$. In the non Maxwell case, by analytical methods, Goudon [23] and Villani [41] obtain existence results. With a probabilistic approach, Fournier-Méléard [20] obtain such results in dimension 2 and for cross-sections bounded as velocity function. We generalize here this approach in dimension 3 and for unbounded (as velocity field) soft potential cross-sections of the form

$$B(z,\theta)=\psi(z)\beta(\theta), \tag{3.3}$$

with

$$\psi(z)=h(|z|)|z|^\gamma,$$

$\gamma\in(-1,0]$ and h a bounded nonnegative locally Lipschitz continuous function and β from $(0,\pi]\to I\!R+$ such that $\int_0^\pi \theta^2\beta(\theta)d\theta<\infty$.

Remark 2.1 *The probabilistic tools oblige us to work with moderately soft potentials, $\gamma\in(-1,0]$. In this case, we have the following usefull estimates: for each $\gamma\in(-1,0]$, for each $z\in I\!R^3$,*

$$|z|^{2+\gamma}\leq |z|^2+1\;;\;\;|z|^{2+2\gamma}\leq|z|^2+1. \tag{3.4}$$

We define the jump amplitude

$$a(v,v_*,\theta,\varphi)=v'-v=\frac{\cos\theta-1}{2}(v-v_*)+\frac{\sin\theta}{2}\Gamma(v-v_*,\varphi). \tag{3.5}$$

where for $x\in I\!R^3$, $\varphi\in[0,2\pi)$,

$$\Gamma(x,\varphi)=\cos\varphi I(x)+\sin\varphi J(x) \tag{3.6}$$

and $\frac{1}{|x|}(x, I(x), J(x))$ is an orthonormal basis of $I\!R^3$. One can choose, for example,

$$I(x) = \begin{cases} \frac{|x|}{\sqrt{x_1^2+x_2^2}}(-x_2, x_1, 0) & \text{if} \quad x_1^2+x_2^2>0 \\ (x_3, 0, 0) & \text{if } x_1^2+x_2^2=0 \end{cases} \quad ; \; J(x) \quad = \quad \frac{x}{|x|} \wedge I(x).$$

The main difficulty is that a is not a Lipschitz continuous function on the variables v and v_*. It just satisfies an "almost"-Lipschitz property of a (Lipschitz up to a rotation), as proved in [38] or in its "fine" version in [21]. However, this property will be suffisant to obtain existence results.

Lemma 2.2 *There exists a measurable function $\varphi_0 : I\!R^3 \times I\!R^3 \mapsto [0, 2\pi[$, such that for all v, v_*, w, w_* in $I\!R^3$, $\theta \in [0,\pi]$, $\varphi \in [0, 2\pi]$,*

$$|a(v, v_*, \theta, \varphi) - a(w, w_*, \theta, \varphi + \varphi_0(v - v_*, w - w_*))| \leq 3\theta\left(|v - w| + |v_* - w_*|\right), \quad (3.7)$$
$$|a(v, v_*, \theta, \varphi)| \leq 2\,|\sin(\theta/2)|\,|v - v_*|. \quad (3.8)$$

Equation (3.1) has to be understood in a weak sense, i.e. f is a solution of the equation if for any test function ϕ, $\frac{\partial}{\partial t} < f, \phi >=< Q_B(f,f), \phi >$ where $< .,. >$ denotes the duality bracket between L^1 and L^∞ functions. By a standard integration by parts, we define a solution f as satisfying for each $\phi \in C_b^2(I\!R^3)$

$$\frac{\partial}{\partial t}\int_{I\!R^3} f(t,v)\phi(v)dv = \int_{I\!R^3\times I\!R^3}\int_0^{2\pi}\int_0^{\pi}(\phi(v')-\phi(v))B(v-v_*,\theta)d\theta d\varphi f(t,v)dv f(t,v_*)dv_*.$$

Since $\int_0^\pi \theta\beta(\theta)d\theta$ may be infinite, the RHS term may explode. Thus we have to compensate it, and taking into account the conservation of the mass, we obtain finally the following definition of probability measure solutions of (3.1).

Definition 2.3 *We say that a probability measure family $(P_t)_{t\geq 0}$ is a measure-solution of the Boltzmann equation (3.1) if for each $\phi \in C_b^2(I\!R^3)$*

$$\langle \phi, P_t\rangle = \langle \phi, P_0\rangle + \int_0^t \langle K^{\phi}_{\beta,\gamma}(v, v_*), P_s(dv)\, P_s(dv_*)\rangle\, ds, \quad (3.9)$$

where $K^{\phi}_{\beta,\gamma}$ is defined in the compensated form

$$K^{\phi}_{\beta,\gamma}(v, v_*) = -b\psi(v - v_*)(v - v_*).\nabla\phi(v) \quad (3.10)$$
$$+ \int_0^{2\pi}\int_0^{\pi}\Big(\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v) - a(v, v_*, \theta, \varphi).\nabla\phi(v)\Big)\psi(v - v_*)\beta(\theta)d\theta d\varphi$$

and where

$$b = \pi\int_0^{\pi}(1 - \cos\theta)\beta(\theta)d\theta. \quad (3.11)$$

2.2 The probabilistic approach

We consider (3.9) as the evolution equation for the marginals of a Markov process which law is defined by a martingale problem.

Definition 2.4 *Let β be a cross section such that $\int_0^\pi \theta^2\beta(\theta)d\theta < +\infty$ and Q_0 in $\mathcal{P}_2(I\!R^3)$. We say that $Q \in \mathcal{P}(I\!D(I\!R_+, I\!R^3))$ solves the nonlinear martingale problem (BMP) starting at Q_0 if under Q, the canonical process V satisfies for any $\phi \in C_b^2(I\!R^3)$*

$$\phi(V_t) - \phi(V_0) - \int_0^t \int_{I\!R^3} K^\phi_{\beta,\gamma}(V_s, v_*)Q_s(dv_*)ds \tag{3.12}$$

is a square-integrable martingale and the law of V_0 is Q_0. Here, the nonlinearity appears through Q_s which denotes the marginal of Q at time s.

Remark 2.5 *Taking expectations in (3.12), we remark that if Q is a solution of (BMP), then its time-marginal family $(Q_t)_{t\geq 0}$ is a measure-solution of the Boltzmann equation, in the sense of Definition 2.3.*

Our first aim is to prove the existence of a solution to the martingale problem (3.12) and then to obtain the existence of a measure-solution to the Boltzmann equation.

Theorem 2.6 *Assume that Q_0 is a probability measure on $I\!R^3$ with a finite p-order moment and that $B(z,\theta) = \psi(z)\beta(\theta)$ is a cross-section satisfying Hypothesis (3.3), with $p = 2$ when $\gamma \in (-1, 0)$ and $p = 4$ when $\gamma = 0$. Then*
1) The nonlinear martingale problem (BMP) with initial data Q_0 has a solution $Q \in \mathcal{P}_2(I\!D_T)$.
2) Moreover, $E_Q(\sup_{t\leq T}|X_t|^p) < +\infty$, where X is the canonical process on $I\!D_T$.

Remark 2.7 *There is no assumption on Q_0, except the existence of a p-order moment. This allows us in particular to consider degenerate initial data, as Dirac measures. The point 1) in Theorem 2.6 exhibits in particular a measure-solution to the Boltzmann equation (3.1) for each initial data $Q_0 \in \mathcal{P}_4(I\!R^3)$.*
Our method gives no hope to obtain a uniqueness result.

We will prove this theorem using stochastic calculus tools. We generalize here the results of Tanaka and Horowitz-Karandikar [26] to soft potential cases, introducing a specific nonlinear stochastic differential equation giving a pathwise version of the probabilistic interpretation.
We are looking for a stochastic process belonging to $I\!D(I\!R_+, I\!R^3)$ and with a law Q solution of (3.12). It can be given as solution of the nonlinear stochastic differential equation

$$\begin{aligned} V_t = V_0 - b\int_0^t \int_{I\!R^3} \psi(V_s - z)(V_s - z)Q_s(dz)ds \\ + \int_0^t \int_{I\!R^3} \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, z, \theta, \varphi)\mathbf{1}_{\{x\leq\psi(V_{s-}-z)\}}\tilde{N}^*(dx, d\theta, d\varphi, dz, ds)\end{aligned}$$

where $\tilde{N}^*$ is the compensated martingale of an inhomogeneous Poisson-point measure on $I\!R_+ \times [0,\pi] \times [0,2\pi] \times I\!R^3 \times I\!R_+$ with intensity $dx\beta(\theta)d\theta d\varphi Q_t(dz)dt$. The nonlinearity appears through Q_s, which is the law of V_s for each s.
We consider a compensated form of the Poisson-point measure following Definition 2.3. We can easily remark using Itô's formula that the law Q of a solution V of this stochastic differential equation is a solution of (3.12) and $(Q_t)_{t\geq 0}$ is a solution of the Boltzmann equation. That gives a pathwise mean-field interacting representation of the Boltzmann process: the process jumps following a Poisson-point measure which picks independent colliding particules having the same law as the process itself. The jump takes place if $x \leq \psi(V_{s-} - z)$ and the amplitude of the jump is equal to a.

Technically, to obtain a more intrinsic representation, we use the Skorohod representation and describe the behaviour of the colliding particules on an auxiliary probability space. So we now consider two probability spaces : the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. In order to avoid any confusion, the processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α, and the laws $\mathcal{L}_\alpha$.

Definition 2.8 *We say that (V, W, N, V_0) is a solution of (SDE) if*
(i) (V_t) is an adapted cadlag $I\!D_T$-valued process such that $E(\sup_{t\in[0,T]} |V_t|^2) < +\infty$,
(ii) (W_t) is a α-process such that $E_\alpha(\sup_{t\in[0,T]} |W_t|^2) < +\infty$,
(iii) $N(\omega, dt, d\alpha, dx, d\theta, d\varphi)$ is a $\{\mathcal{F}_t\}$-Poisson point measure on $[0,T] \times [0,1] \times I\!R_+ \times [0,\pi] \times [0,2\pi]$ with intensity $m(dt, d\alpha, dx, d\theta, d\phi) = dtd\alpha dx\beta(\theta)d\theta d\varphi$ and $\tilde{N}$ is its compensated martingale,
(iv) V_0 is a square integrable variable independent of N,
(v) $\mathcal{L}(V) = \mathcal{L}_\alpha(W)$,
(vi)

$$V_t = V_0 - b\int_0^t \int_0^1 \psi(V_s - W_s(\alpha))(V_s - W_s(\alpha))d\alpha ds$$
$$+ \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi)\mathbf{1}_{\{x\leq\psi(V_{s-}-W_{s-}(\alpha))\}}\tilde{N}(ds, d\alpha, dx, d\theta, d\varphi)$$

Remark 2.9 *Of course, as before, the law of V is then a solution of (BMP) with initial law $Q_0 = \mathcal{L}(V_0)$.*

Let us now prove in many steps Theorem 2.6. We obtain the existence of weak solutions of the martingale problem (BMP) under Hypothesis (3.3), as limits in law of solutions of regularized equations.
The first step generalizes the result of Fournier-Méléard [20] obtained in dimension 2. The specific difficulty in dimension 3 is the lack of Lipschitz continuity of a described in lemma 2.2. We will prove

Proposition 2.10 *Assume that $B(z,\theta) = \hat{\psi}(z)\beta(\theta)$ with $\hat{\psi}$ a nonnegative bounded and locally Lipschitz continuous function, and β integrating θ (hence, no compensation is* *needed). Assume that V_0 is a p-order moment random variable. Then the nonlinear stochastic differential equation (SDE) which can be rewritten in this case*

$$V_t = V_0 + \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}, W_{s-}(\alpha), \theta, \varphi) \mathbf{1}_{\{x \leq \hat{\psi}(V_{s-} - W_{s-}(\alpha))\}} N(ds, d\alpha, dx, d\theta, d\varphi) \tag{3.13}$$

has a weak solution, and moreover, for every $T > 0$,

$$E(\sup_{t \leq T} |V_t|^p) < +\infty. \tag{3.14}$$

Proof. The proof mixes arguments from [21] (adapted from Tanaka) to control the lack of Lipschitz regularity of a and from [20] Theorem 3.4. in the non Maxwell case.
Let us assume that the function $\hat{\psi}$ is bounded by M. Let us define

$$\hat{a}(v, w, \theta, \varphi, x) = a(v, w, \theta, \varphi) \mathbf{1}_{\{x \leq \hat{\psi}(v-w)\}}$$

and its cutoff versions

$$\hat{a}_n(v, w, \theta, \varphi, x) = \hat{a}(v \wedge n \vee (-n), w \wedge n \vee (-n), \theta, \varphi, x).$$

We remark that

$$\int |\hat{a}_n(v, w, \theta, \varphi, x)| dx \leq M\theta |v - w| \tag{3.15}$$

$$\int |\hat{a}_n(v, w, \theta, \varphi, x) - \hat{a}_n(v', w', \theta, \varphi + \varphi_0(v - w, v' - w'), x)| dx \leq K_n(|v - v'| + |w - w'|) \tag{3.16}$$

Thanks to these properties, we are able to construct, by a sophisticated Picard iteration mixing results of [21] and [20], a solution of

$$V_t^n = V_0 + \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} \hat{a}_n(V_{s-}^n, W_{s-}^n(\alpha), \theta, \varphi, x) N(ds, d\alpha, dx, d\theta, d\varphi) \tag{3.17}$$

satisfying moreover that

$$\sup_n E(\sup_{s \leq t} |V_s^n|^p) < +\infty. \tag{3.18}$$

This Picard iteration takes into account the specific property (3.16). The trick is to observe that the image measure of a Poisson point measure with intensity $ds dx d\alpha \beta(\theta) d\theta d\varphi$ by the rotation $\varphi \mapsto \varphi + \varphi_0$ is still a Poisson point measure with the same intensity measure. That is technical and we refer to [38] or [21] for more details.

Property (3.18) implies that the laws Q^n of V^n are uniformly tight on the path space .

Let us now prove that each limit point Q of this sequence is solution of the nonlinear martingale problem associated with (3.13), i.e. that for (X_t) the canonical process on $\mathbb{D}_T$ and for $\phi \in C_b^1(\mathbb{R}^3)$, $t > 0$,

$$H_t^\phi = \phi(X_t) - \phi(X_0) - \int_0^t \int_0^M \int_0^\pi \int_0^{2\pi} \Big(\phi(X_u + \hat{a}(X_u, w, \theta, \varphi, x)) - \phi(X_u)\Big) Q_u(dw)\beta(\theta) d\theta d\varphi dx du$$

is a Q-martingale, knowing that the similar quantity $H_t^{n,\phi}$, with $\hat{a}_n$ instead of $\hat{a}$ and Q^n instead of Q, is a Q^n-martingale. The only new difficulty in dimension 3 consists in proving that the function, for $s \leq t$,

$$K(X,Y) = \int_s^t \int_0^M \int_0^\pi \int_0^{2\pi} \Big(\phi(X_u + \hat{a}(X_u, Y_u, \theta, \varphi, x)) - \phi(X_u)\Big) \beta(\theta) d\theta d\varphi dx du$$

is continuous on $\mathbb{D}_T \times \mathbb{D}_T$, although a is not Lipschitz continuous. Using the translation invariance of the Lebesgue's measure $d\varphi$ and the periodicity of $\hat{a}$ in the variable φ , we write

$$|K(X,Y) - K(X',Y')| \leq M \int_s^t \int_0^\pi \int_0^{2\pi} \Big(\Big|\phi(X_u) - \phi(X'_u)\Big|$$
$$+ \Big|\phi(X_u + \hat{a}(X_u, Y_u, \theta, \varphi)) - \phi(X'_u + \hat{a}(X'_u, Y'_u, \theta, \varphi + \varphi_0(X_u - Y_u, X'_u - Y'_u)))\Big|\Big) \beta(\theta) d\theta d\varphi du$$

and thanks to Lemma 2.2, we see that the RHS term tends to 0 when the uniform distance between (X,Y) and (X',Y') tends to 0.
A standard proof allows us to conclude that Q is solution of the nonlinear martingale problem (BMP) associated with (3.13). Moreover, using a representation theorem, we can exhibit an enlarged probability space, on which the canonical process is solution of (3.13) (a similar argument is more detailed in the end of the proof of Theorem 2.6). The property (3.14) follows easily from (3.18). ■
Let us now prove Theorem 2.6.

Proof. In order to apply Proposition 2.10, we consider some cutoff of the cross-section in both variables.

We introduce the following **approximating model:**
Let $l, k \in \mathbb{N}^*$ and define

$$\beta_l(\theta) = \beta(\theta) \mathbf{1}_{|\theta| \geq \frac{1}{l}} \ ; \ \psi_k(z) = h(|z|)(|z|^\gamma \wedge k), \forall z \in \mathbb{R}^3.$$

Each function ψ_k is locally Lipschitz continuous and is bounded by kH, where H is a bound of the function h. Thanks to Proposition 2.10 and for each (k,l), there exists a weak solution to the nonlinear stochastic differential equation (SDE_{kl}):

$$V_t^{k,l} = V_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}^{k,l}, W_{s-}^{k,l}(\alpha), \theta, \varphi) \times \mathbf{1}_{\{x \leq \psi_k(V_{s-}^{k,l} - W_{s-}^{k,l}(\alpha))\}} N_{k,l}(ds, d\alpha, dx, d\theta, d\varphi) \qquad (3.19)$$

where $N_{k,l}$ is a point Poisson measure with intensity $dsd\alpha dx\beta_l(\theta)d\theta d\varphi$ on $[0,T]\times[0,1]\times[0,kH]\times[0,\pi]\times[0,2\pi]$. So the associated nonlinear martingale problem $(BMP_{k,l})$ has a solution $P^{k,l}$. The aim is now to prove that the sequence $(P^{k,l})$ of probability measures on the path space $I\!D_T$ is uniformly tight and that each limit point is solution of the initial nonlinear martingale problem (BMP).
Since the limit case has sense only in a compensated form, we write each equation (3.19) in its compensated form:

$$\begin{aligned} V_t^{k,l} &= V_0 - b_l\int_0^t\int_0^1 \psi_k(V_s^{k,l}-W_s^{k,l}(\alpha))(V_s^{k,l}-W_s^{k,l}(\alpha))d\alpha ds \\ &+\int_0^t\int_0^1\int_{I\!R_+}\int_0^\pi\int_0^{2\pi} a(V_{s-}^{k,l},W_{s-}^{k,l}(\alpha),\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(V_{s-}^{k,l}-W_{s-}^{k,l}(\alpha))\}}\tilde N_{k,l}(ds,d\alpha,dx,d\theta,d\varphi)\end{aligned}$$

where

$$b_l=\pi\int_0^\pi(1-\cos\theta)\beta_l(\theta)d\theta.$$

Lemma 2.11

$$\sup_{k,l}E(\sup_{t\le T}|V_t^{k,l}|^p)<+\infty. \tag{3.20}$$

Proof. (of the Lemma) Thanks to (3.4), we obtain easily that

$$\begin{aligned} E(\sup_{s\le t}|V_s^{k,l}|^p) &\le K\left(E(|V_0|^p)+\int_0^t\int_0^1 E(\sup_{u\le s}|V_u^{k,l}-W_u^{k,l}(\alpha)|^p+1)d\alpha ds\right)\\ &\le K\left(1+\int_0^t E(\sup_{u\le s}|V_u^{k,l}|^p)ds\right) \end{aligned}\tag{3.21}$$

and the constant number K does not depend on k and l. By Proposition 2.10, $E(\sup_{s\le T}|V_s^{k,l}|^p)$ is finite for each k,l and the proof is obtained by Gronwall's lemma. ∎
It is thus classical to show that the Aldous criterion is satisfied.
Hence the sequence $(P^{k,l})$ is tight.
Let us now identify each limit point of $(P^{k,l})$. Let Q be a limit value of this sequence. We consider the compensated martingale problems. Let $(X_t)_t$ be the canonical process on $I\!D_T$ and for $\phi\in C_b^2(I\!R^3)$, $t>0$, we set

$$\begin{aligned} H_t^\phi &= \phi(X_t)-\phi(X_0)+b\int_0^t\int_{w\in I\!R^2}\nabla\phi(X_u).(X_u-w)\psi(X_u-w)Q_u(dw)du\\ &-\int_0^t\int_0^\pi\int_0^{2\pi}\int_{I\!R^3}(\phi(X_u+a(X_u,w,\theta,\varphi,x))-\phi(X_u)\\ &\qquad -a(X_u,w,\theta,\varphi,x)(X_u-w).\nabla\phi(X_u))\,\psi(X_u-w)Q_u(dw)\beta(\theta)d\theta d\varphi du\end{aligned}$$

and $H_t^{k,l,\phi}$ denotes a similar quantity with ψ_k instead of ψ, β_l instead of β, b_l instead of b and $P_u^{k,l}$ instead of Q_u. The probability measure Q will be a solution of the nonlinear martingale problem (BMP) with initial law Q_0 if it satisfies for each $0\le s_1<...<s_p<s<t\le T$, each $G\in C_b((I\!R^3)^p)$,

$$<(H_t^\phi-H_s^\phi)G(X_{s_1},...,X_{s_p}),Q>=0. \tag{3.22}$$

Since $P^{k,l}$ is a solution of $(BMP_{k,l})$, we already know that

$$< (H_t^{k,l,\phi} - H_s^{k,l,\phi}) G(X_{s_1}, ..., X_{s_p}), P^{k,l} >= 0.$$

Since the sequence $(P^{k,l})$ satisfies the Aldous criterion, the law Q is the law of a quasi-càg process (cf. [28] p. 321). Then the mapping $F : x \mapsto (\phi(x_t) - \phi(x_s))G(x_{s_1}, ..., x_{s_p})$ is Q-a.e. continuous and bounded from $I\!D_T$ to $I\!R$. Thus $< F, P^{k,l} >$ tends to $< F, Q >$ as k, l tend to infinity.
Now, let us successively prove that

$$\begin{aligned} T_1 &= < \Big(\int_s^t \int_0^\pi \int_0^{2\pi} \int_{I\!R^3} (\phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi).\nabla\phi(X_u)) \\ &\qquad (\psi(X_u - w) - \psi_k(X_u - w))\, P_u^{k,l}(dw)\beta_l(\theta)d\theta d\varphi du \Big) G(X_{s_1}, ..., X_{s_p}), P^{k,l} >, \end{aligned}$$

$$\begin{aligned} T_2 &= < \Big(\int_s^t \int_0^\pi \int_0^{2\pi} \int_{I\!R^3} (\phi(X_u + a(X_u, w, \theta, \varphi)) - \phi(X_u) - a(X_u, w, \theta, \varphi).\nabla\phi(X_u)) \\ &\qquad \psi(X_u - w)(\beta_l(\theta) - \beta(\theta)) P_u^{k,l}(dw) d\theta d\varphi du \Big) G(X_{s_1}, ..., X_{s_p}), P^{k,l} >, \\ T_3 &= < G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{I\!R^3} K_{\beta,\gamma}^{\phi}(X_u, Y_u), P^{k,l}(dX) \otimes P^{k,l}(dY) > \\ &\quad - < G(X_{s_1}, ..., X_{s_p}) \int_s^t \int_{I\!R^3} K_{\beta,\gamma}^{\phi}(X_u, Y_u), Q(dX) \otimes Q(dY) >, \end{aligned}$$

and the term T_4 similar to T_1 corresponding to the drift term, tend to 0 as k, l tend to infinity.

Term T_1:

$$\begin{aligned} |T_1| &\leq K \int_0^\pi \theta^2 \beta_l(\theta) d\theta < \int_s^t \int_{I\!R^3} |X_u - w|^2 (|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k) \\ &\qquad P_u^{k,l}(dw)du, P^{k,l} > \\ &\leq K < \int_s^t \int_{I\!R^3} |X_u - w|^{2+\gamma} \mathbf{1}_{\{|X_u - w|^\gamma \geq k\}} P_u^{k,l}(dw)du, P^{k,l} > \\ &\leq K < \int_s^t \int_{I\!R^3} |X_u - w|^{2+\gamma} \mathbf{1}_{\{|X_u - w| \leq (k)^{\frac{1}{\gamma}}\}} P_u^{k,l}(dw)du, P^{k,l} > \\ &\leq K(k)^{\frac{2+\gamma}{\gamma}} \end{aligned}$$

and T_1 tends to zero when k tends to infinity, uniformly in l since $\int_0^\pi \theta^2 \beta_l(\theta) d\theta \leq \int_0^\pi \theta^2 \beta(\theta) d\theta < +\infty$, and since $\frac{2+\gamma}{\gamma} < 0$.

Term T_4: By a similar study with the drift term, we obtain

$$\begin{aligned} |T_4| &\leq K < \int_s^t \int_{I\!R^3} |X_u - w|(|X_u - w|^\gamma - (|X_u - w|^\gamma) \wedge k) P_u^{k,l}(dw)du, P^{k,l} > \\ &\leq K(k)^{\frac{1+\gamma}{\gamma}} \end{aligned}$$

and T_4 tends to zero when k tends to infinity, since $\gamma \in (-1, 0]$.
Term T_2:

$$\begin{aligned} |T_2| &\leq K \int_0^\pi \theta^2 |\beta_l(\theta) - \beta(\theta)| d\theta < \int_s^t \int_{\mathbb{R}^3} (|X_u - w|^{2+\gamma}) P_u^{k,l}(dw) du, P^{k,l} > \\ &\leq K \left(\sup_{k,l} E_{P^{k,l}} (\sup_{u \leq T} |X_u|^2) + 1 \right) \int_{-\pi}^{\pi} \theta^2 |\beta_l(\theta) - \beta(\theta)| d\theta \end{aligned}$$

which tends to 0 as l tends to infinity, uniformly in k thanks to Lemma 2.11.

Term T_3: Let us define the function $F(x, y)$ on $\mathbb{D}_T \times \mathbb{D}_T$ by $F(x, y) = \int_s^t K^\phi_{\beta,\gamma}(x_u, y_u) du$. The function F is $Q \otimes Q$-a.e. continuous by a similar argument as in the proof of Proposition 2.10 and is not bounded.

$$\begin{aligned} |F(x, y)| &\leq K \int_0^\pi \theta^2 \beta(\theta) d\theta \left(\sup_{s \leq u \leq t} (|x_u - y_u|^{2+\gamma} + |x_u - y_u|^{1+\gamma}) \right) \\ &\leq K \left(\sup_{u \leq T} |x_u|^{2+\gamma} + \sup_{u \leq T} |y_u|^{2+\gamma} \right). \end{aligned}$$

Now, the measure $P^{k,l} \otimes P^{k,l}$ converges obviously to $Q \otimes Q$. We remark that

$$|F(x, y)| \mathbf{1}_{\{|F(x,y)| \geq C\}} \leq$$
$$K \left(\sup_{u \leq T} |x(u)|^{2+\gamma} + \sup_{u \leq T} |y(u)|^{2+\gamma} \right) \left(\mathbf{1}_{\{\sup_{u \leq T} |x(u)|^{2+\gamma} \geq C/2K\}} + \mathbf{1}_{\{\sup_{u \leq T} |y(u)|^{2+\gamma} \geq C/2K\}} \right)$$

Thus, thanks to Lemma 2.11 with $p = 2$ when $\gamma \in (-1, 0)$ and with $p = 4$ when $\gamma = 0$ and thanks to Estimates (3.4), we have

$$\lim_{C \to +\infty} \sup_{k,l} < |F(x, y)| \mathbf{1}_{\{|F(x,y)| \geq C\}}, P^{k,l} \otimes P^{k,l} >= 0$$

Consequently, $|T_3|$ converges to 0 when k, l tends to $+\infty$.
We have thus proved that each limit law of the sequence $(P^{k,l})$ is solution of the martingale problem (BMP). Since such limits exist thanks to the Aldous criterion, we deduce obviously from this approach the existence of at least one solution to (BMP).
Let us now show that each solution Q of (BMP) is a weak solution of (SDE).
The canonical process X is a semimartingale under Q. Then a comparison between the Itô formula and the martingale problem proves that X is a pure jump process and that its Lévy measure is the image measure of the measure m on $[0, T] \times [0, 1] \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi]$ by the mapping $(s, \alpha, x, \theta, \varphi) \mapsto a(X_{s-}, W_{s-}(\alpha), \theta, \varphi) \mathbf{1}_{\{x \leq \psi(X_{s-} - W_{s-}(\alpha))\}}$. Then by a representation theorem for point measures [16], there exist on an enlarged probability space a square integrable variable V_0 and a Poisson-point measure N with intensity m such that (X, W, N, V_0) is a solution of (SDE). ■

3 Convergence of renormalized Boltzmann Processes towards a Landau Process

3.1 A probabilistic interpretation of the Landau equation

The Landau equation, also called the Fokker-Planck-Landau equation, describes the collisions of particles in a plasma and is obtained as limit of Boltzmann equations when the collisions become grazing. In the spatially homogeneous case, it writes in $I\!R^3$:

$$\frac{\partial f}{\partial t} = Q_L(f,f) \tag{3.23}$$

with

$$Q_L(f,f)(t,v) = \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial}{\partial v_i}\left\{\int_{I\!R^3} dv_* A_{ij}(v-v_*)\left[f(t,v_*)\frac{\partial f}{\partial v_j}(t,v) - f(t,v)\frac{\partial f}{\partial v_{*j}}(t,v_*)\right]\right\}$$

where $f(t,v) \geq 0$ is the density of particles having velocity $v \in I\!R^3$ at time $t \in I\!R^+$, and $(A_{ij}(z))_{1\leq i,j\leq 3}$ is a nonnegative symmetric matrix depending on the interaction between the particles, of the form

$$\begin{aligned} A(z) &= \Lambda |z|^{\gamma+2}\,\Pi(z)\,h(|z|) \\ &= \Lambda |z|^{\gamma} h(|z|)\begin{bmatrix} z_2^2+z_3^2 & -z_1z_2 & -z_1z_3 \\ -z_1z_2 & z_1^2+z_3^2 & -z_2z_3 \\ -z_1z_3 & -z_2z_3 & z_1^2+z_2^2 \end{bmatrix} \end{aligned} \tag{3.24}$$

where $\Pi(z)$ is the orthogonal projection on $(z)^{\perp}$, Λ is a positive constant and h is a nonnegative locally Lipschitz continuous bounded function.
By integration by parts, see [41], a weak formulation of the equation (3.23) writes, at least formally, for any test function $\phi \in C_b^2(I\!R^3)$,

$$\begin{aligned} \frac{d}{dt}\int \phi(v) f(t,v)\,dv &= \frac{1}{2}\sum_{i,j=1}^{3}\int_{I\!R^3\times I\!R^3} dv dv_* f(t,v) f(t,v_*) A_{ij}(v-v_*)\,\partial_{ij}^2\phi(v) \\ &\quad + \sum_{i=1}^{3}\int_{I\!R^3\times I\!R^3} dv dv_* f(t,v) f(t,v_*)\, b_i(v-v_*)\,\partial_i\phi(v) \end{aligned} \tag{3.25}$$

where $b_i(z) = \sum_{j=1}^{3}\partial_j A_{ij}(z) = -2\Lambda h(|z|)\,|z|^{\gamma} z_i$.

As for the Boltzmann equation, the equation (3.25) conserves the mass, thus we give a definition of probability-measure solutions of the Landau equation :

Definition 3.1 *Let P_0 belong to $\mathcal{P}_2(I\!R^3)$. A probability measure solution of the Landau equation (3.26) with initial data P_0 is a probability measure family $(P_t)_{t\geq 0}$ on $I\!R^3$ satisfying*

$$\langle \phi, P_t\rangle = \langle \phi, P_0\rangle + \int_o^t \langle L^{\phi}(v,v_*), P_s(dv)\,P_s(dv_*)\rangle ds \tag{3.26}$$

for any function $\phi \in C_b^2(\mathbb{R}^3)$ *where* L^ϕ *is the Landau kernel defined on* $\mathbb{R}^3 \times \mathbb{R}^3$ *by :*

$$\begin{aligned} L^\phi(v, v_*) &= \frac{1}{2}\sum_{i,j=1}^{3} \partial_{ij}^2 \phi(v) A_{ij}(v - v_*) + \sum_{i=1}^{3} \partial_i \phi(v) b_i(v - v_*) \\ &= \frac{1}{2} J_\phi(v) : A(v - v_*) + b(v - v_*) . \nabla \phi(v) \end{aligned}$$

with $J_\phi = \left(\partial_{ij}^2 \phi\right)_{1 \le i,j \le 3}$.

We now consider the martingale problem associated with the Landau equation and defined as follows.

Definition 3.2 *Let* P_0 *belong to* $\mathcal{P}_2(\mathbb{R}^3)$.
Let $(Y_s)_{s \ge 0}$ *be the canonical process on* $\mathcal{C}_T$. *A probability measure* $P \in \mathcal{P}(\mathcal{C}_T)$ *is a solution of the martingale problem* (LMP) *with initial data* P_0 *if the law of* Y_0 *is* P_0 *and if for any* $\phi \in C^2(\mathbb{R}^3)$,

$$\phi(Y_t) - \phi(Y_0) - \int_0^t \int_{\mathbb{R}^3} L^\phi(Y_s, v_*) P_s(dv_*) ds$$

is a P*-martingale, where* $P_s = P \circ Y_s^{-1}$.

Remark 3.3 *The time-marginal family of a solution of the martingale problem* (LMP) *is a measure-solution of the Fokker-Planck-Landau equation.*

We already built, in Chapter 1 of this thesis, by a direct probabilistic approach, a Landau process solution of a nonlinear stochastic differential equation driven by a white noise and deduced the existence of a measure-solution of the Landau equation for any dimension ≥ 2 and for $\gamma \in (-1, 0]$. We obtain here a new proof of the existence of a solution to the Landau process (and then of a solution to the Landau equation) as limit of Boltzmann processes.

3.2 Asymptotic of Boltzmann processes towards a Landau process

We are now interested in stating the convergence in law of Boltzmann processes with 'moderately soft potentials', obtained in Section 2, towards a Landau process when the collisions become grazing. With this aim in view, we consider cross-sections β^ε depending on the grazing collision parameter ε, as in Villani [41]. The function β^ε from $[0, \pi]$ to $\mathbb{R}^+$ satisfies

$$\forall \theta_0 > 0 \; \beta^\varepsilon(\theta) \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ uniformly on } \theta \ge \theta_0 \tag{3.27}$$

$$\Lambda^\varepsilon = \pi \int_0^\pi \sin^2(\theta/2) \beta^\varepsilon(\theta) d\theta \underset{\varepsilon \to 0}{\longrightarrow} \Lambda > 0 \tag{3.28}$$

Let us remark that these hypotheses contain the case $\beta^\varepsilon(\theta) = \frac{1}{|\log\varepsilon|}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\mathbb{I}_{\theta\geq\varepsilon}$ introduced by Degond-Lucquin [9] for a Coulomb potential ($\gamma = -3$) and $\beta^\varepsilon(\theta) = \frac{1}{\varepsilon^3}\beta\left(\frac{\theta}{\varepsilon}\right)$ introduced by Desvillettes [10] for non Coulomb potentials.

Let us notice that

Lemma 3.4 *1)*

$$\int_0^\pi \beta^\varepsilon(\theta)\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} +\infty,$$

2) For $k \geq 3$,

$$\int_0^\pi \sin^k(\theta/2)\beta^\varepsilon(\theta)\, d\theta \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

The proof is left to the reader.

For each $\varepsilon > 0$, for $\gamma \in (-1,0]$, we define the Boltzmann kernel $K^\phi_{\beta^\varepsilon,\gamma}$ on $\mathbb{R}^3\times\mathbb{R}^3$, as in (3.10), by

$$K^\phi_{\beta^\varepsilon,\gamma}(v,v_*) = -b^\varepsilon\psi(v-v_*)(v-v_*).\nabla\phi(v) \\ \int_0^{2\pi}\int_0^\pi \Big(\phi(v+a(v,v_*,\theta,\varphi)) - \phi(v) - a(v,v_*,\theta,\varphi).\nabla\phi(v)\Big)\psi(v-v_*)\beta^\varepsilon(\theta)d\theta d\varphi \tag{3.29}$$

with $b^\varepsilon = \pi\int_0^\pi (1-\cos\theta)\,\beta^\varepsilon(\theta)\, d\theta$.

We notice that the Boltzmann kernels converge towards the Landau kernel when $\varepsilon \to 0$, for any $v, v_* \in \mathbb{R}^3$ and $\phi \in C^2_b\left(\mathbb{R}^3\right)$ (for more details, see the convergence of the term E_1 in Section 3.4).
We denote by $(B^\varepsilon MP)$ the martingale problem associated with the Boltzmann equation defined as in Definition 2.4 replacing $K^\phi_{\beta,\gamma}$ with $K^\phi_{\beta^\varepsilon,\gamma}$. In the previous section, we have proved the existence of a solution Q^ε of $(B^\varepsilon MP)$. We are now interested in the asymptotic behaviour of the sequence $(Q^\varepsilon)_{\varepsilon>0}$ when ε tends to 0.
We state the following main theorem.

Theorem 3.5 *Consider a bounded locally Lipschitz continuous nonnegative function h, $\gamma \in (-1,0]$, β^ε satisfying (3.27) and (3.28) and Q_0 a finite fourth-order moment probability measure. Let $Q^\varepsilon \in \mathcal{P}(\mathbb{D}_T)$ be a solution of the nonlinear martingale problem $(B^\varepsilon MP)$ with kernel $K_{\beta^\varepsilon,\gamma}$ defined by (3.29) and initial data Q_0.*
Then the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight when ε tends to 0, and any of its subsequences converges towards a solution $P \in \mathcal{P}(\mathcal{C}_T)$ of the nonlinear martingale problem (LMP), associated with the Landau equation (3.26) having diffusion matrix defined by (3.24), with initial law Q_0.

Remark 3.6 *When $\gamma = 0$ and under some regularity assumptions on h, we have proved in Chapter 2 Corollary 1.7 the uniqueness of a solution P to the martingale problem (LMP). Then, in this case, the sequence $(Q^\varepsilon)_{\varepsilon>0}$ converges towards this unique solution P.*

Let us notice that Villani [41] and Goudon [23] proved the existence of weak function solutions of the Landau equation for soft potentials using the convergence of the solutions of the Boltzmann equation towards the solutions of the Landau equation. The interest of our approach is the understanding of this convergence at the microscopic level of processes. When ε decreases, the Boltzmann processes jump more and more often with smaller jumps, and then converge finally to a (continuous) diffusion process. Moreover, our convergence result is true for general (even degenerate, as Dirac measures) initial data and leads naturally to particle approximations.

3.3 C-Tightness of the sequence $(Q^\varepsilon)_{\varepsilon>0}$

We assume that Q_0 has a finite fourth-order moment.
Let Q^ε be a solution of the martingale problem $(B^\varepsilon MP)$ obtained in Theorem 2.6 and X the canonical process on $I\!D_T$. Thanks to the point 2) of Theorem 2.6, for any $\varepsilon > 0$, the probability Q^ε satisfies $E_{Q^\varepsilon}(\sup_{0\leq t\leq T}|X_t|^4) \leq K^\varepsilon$ with K^ε a positive constant depending on ε only through $\int_{-\pi}^{\pi}\sin^4(\theta/2)\beta^\varepsilon(\theta)\,d\theta$, $\int_{-\pi}^{\pi}\sin^2(\theta/2)\beta^\varepsilon(\theta)\,d\theta$ and b^ε according to Lemma 2.2. Using Lemma 3.4 and (3.28) we notice that the sequence $(K^\varepsilon)_{\varepsilon>0}$ converges as ε tends to 0. Then there exists $K > 0$ such that

$$\sup_{\varepsilon>0} E_{Q^\varepsilon}(\sup_{0\leq t\leq T}|X_t|^4) \leq K \tag{3.30}$$

Thanks to the Aldous criterion, we deduce, with similar arguments as in Section 2, that the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{P}(I\!D_T)$, and then each limit point P of $(Q^\varepsilon)_{\varepsilon>0}$ belongs to $\mathcal{P}(I\!D_T)$.
We now prove that the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is moreover C-tight, in the sense of Jacod-Shiryaev [28] p. 315, and then P will belong to $\mathcal{P}(\mathcal{C}_T)$.
As the sequence $(Q^\varepsilon)_{\varepsilon>0}$ is tight and according to [28] Proposition 3.26 (iii), we just have to prove that for any $\eta > 0$, for $\Delta X_t = X_t - X_{t^-}$,

$$\lim_{\varepsilon\to 0} Q^\varepsilon(\sup_{t\leq T}|\Delta X_t| > \eta) = 0.$$

We use the stochastic differential equation (SDE) introduced in Section 2.2. Let V^ε be a process with distribution Q^ε such that

$$\begin{aligned} V_t^\varepsilon &= V_0 - b^\varepsilon \int_0^t \int_0^1 \psi(V_s^\varepsilon - W_s^\varepsilon(\alpha))(V_s^\varepsilon - W_s^\varepsilon(\alpha))d\alpha ds \\ &+ \int_0^t \int_0^1 \int_{I\!R_+} \int_0^\pi \int_0^{2\pi} a(V_{s-}^\varepsilon, W_{s-}^\varepsilon(\alpha), \theta, \varphi)\mathbf{1}_{\{x\leq\psi(V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha))\}}\tilde{N}^\varepsilon(ds, d\alpha, dx, d\theta, d\varphi) \end{aligned}$$

with $\mathcal{L}_\alpha(W^\varepsilon) = \mathcal{L}(V^\varepsilon) = Q^\varepsilon$ and $\tilde{N}^\varepsilon(ds, d\alpha, dx, d\theta, d\varphi)$ is the compensated martingale of a Poisson measure with intensity $m^\varepsilon(dt, d\alpha, dx, d\theta, d\varphi) = dt d\alpha dx \beta^\varepsilon(\theta)\,d\theta d\varphi$.
Then, by Tchebychev and Burkholder-Davis-Gundy inequalities for jump semimartin-

gales and Lemma 2.2,

$$\begin{aligned}
&Q^{\varepsilon}(\sup_{t\leq T}|\Delta X_t|>\eta)\leq \frac{1}{\eta^4}E(\sup_{t\leq T}|\Delta V_t^{\varepsilon}|^4)\leq \frac{1}{\eta^4}E(\sum_{t\leq T}|\Delta V_t^{\varepsilon}|^4)\\
&\leq \frac{1}{\eta^4}E\left(\int_0^T\int_0^1\int_0^{\pi}\int_0^{2\pi}\left|a(V_{s-}^{\varepsilon},W_{s-}^{\varepsilon}(\alpha),\theta,\varphi)\right|^4\psi(V_{s-}^{\varepsilon}-W_{s-}^{\varepsilon}(\alpha))\beta^{\varepsilon}(\theta)\,d\theta d\varphi d\alpha ds\right)\\
&\leq \frac{K}{\eta^4}\left(\int_0^T\int_0^1 E(\left|V_{u-}^{\varepsilon}-W_{u-}^{\varepsilon}(\alpha)\right|^{\gamma+4})d\alpha du\right)\int_0^{\pi}\left|\sin(\theta/2)\right|^4\beta^{\varepsilon}(\theta)\,d\theta
\end{aligned}$$

with K independent of ε. Thanks to estimates (3.4) and (3.30), we obtain

$$Q^{\varepsilon}(\sup_{t\leq T}|\Delta X_t|>\eta)\leq \frac{KT}{\eta^4}\int_0^{\pi}|\sin(\theta/2)|^4\beta^{\varepsilon}(\theta)\,d\theta$$

As $\int_0^{\pi}\left|\sin(\theta/2)\right|^4\beta^{\varepsilon}(\theta)\,d\theta$ tends to 0 as ε tends to 0, the sequence $(Q^{\varepsilon})_{\varepsilon>0}$ is C-tight.

3.4 Identification of the limit point values P

Let P be a limit value of the sequence (Q^{ε}). Then P is the limit of a subsequence (Q^{ε}) that we will still denote by (Q^{ε}) for simplicity. We wish to prove that P is a solution of the martingale problem (LMP). Let $\phi\in C_b^2\left(I\!R^3\right)$. We define the two following processes on $I\!D_T$

$$M_t^{\varepsilon} = \phi(X_t)-\phi(X_0)-\int_0^t\int_{I\!R^3}K_{\beta^{\varepsilon},\gamma}^{\phi}(X_s,v_*)\,Q_s^{\varepsilon}(dv_*)\,ds \tag{3.31}$$

$$M_t = \phi(X_t)-\phi(X_0)-\int_0^t\int_{I\!R^3}L^{\phi}(X_s,v_*)\,P_s(dv_*)\,ds \tag{3.32}$$

The probability measure P will be a solution of the nonlinear martingale problem (LMP) with initial law Q_0 if it satisfies, for any $0\leq s_1<...<s_p<s<t\leq T$ and $G\in C_b((I\!R^3)^p)$,

$$<(M_t-M_s)G(X_{s_1},...,X_{s_p}),P>=0$$

However, Q^{ε} is a solution of $(B^{\varepsilon}MP)$, then, for any $0\leq s_1<...<s_p<s<t\leq T$ and $G\in C_b((I\!R^3)^p)$,

$$<(M_t^{\varepsilon}-M_s^{\varepsilon})G(X_{s_1},...,X_{s_p}),Q^{\varepsilon}>=0$$

Thus, we want to state the following convergence

$$E_{Q^{\varepsilon}}\left((M_t^{\varepsilon}-M_s^{\varepsilon})\,G(X_{s_1},...,X_{s_p})\right)\xrightarrow[\varepsilon\to 0]{?}E_P\left((M_t-M_s)\,G(X_{s_1},...,X_{s_p})\right)$$

1. Since (Q^{ε}) is C-tight, the distribution P charges only the set $\mathcal{C}_T$, then the mapping $F:x\mapsto(\phi(x_t)-\phi(x_s))G(x_{s_1},...,x_{s_p})$ is P-continuous and bounded from $I\!D_T$ to $I\!R$. Thus $<F,Q^{\varepsilon}>$ tends to $<F,P>$ as ε tends to zero.

2. We now study the convergence of $E_{Q^\varepsilon}\left(\left\{\int_s^t \int_{I\!R^3} K^{\phi}_{\beta^\varepsilon,\gamma}\left(X_u, v_*\right) Q^\varepsilon_u\left(dv_*\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right)$ to $E_P\left(\left\{\int_s^t \int_{I\!R^3} L^{\phi}\left(X_u, v_*\right) P_u\left(dv_*\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right)$.

If we denote by (X,Y) the canonical process on $I\!D_T \times I\!D_T$, we can write

$$E_{Q^\varepsilon}\left(\left\{\int_s^t < K^{\phi}_{\beta^\varepsilon,\gamma}\left(X_u, v_*\right), Q^\varepsilon_u\left(dv_*\right) > du\right\} G(X_{s_1}, ..., X_{s_p})\right) - E_P\left(\left\{\int_s^t < L^{\phi}\left(X_u, v_*\right), P_u\left(dv_*\right) > du\right\} G(X_{s_1}, ..., X_{s_p})\right) = E_1 + E_2$$

with

$$\begin{aligned} E_1 &= E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\left\{\int_s^t \left(K^{\phi}_{\beta^\varepsilon,\gamma}\left(X_u, Y_u\right) - L^{\phi}\left(X_u, Y_u\right)\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right) \\ E_2 &= E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\left\{\int_s^t L^{\phi}\left(X_u, Y_u\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right) \\ &\quad - E_{P\otimes P}\left(\left\{\int_s^t L^{\phi}\left(X_u, Y_u\right) du\right\} G(X_{s_1}, ..., X_{s_p})\right) \end{aligned}$$

(a) Study of E_1:

$$|E_1| \le K E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t \left|K^{\phi}_{\beta^\varepsilon,\gamma}\left(X_u, Y_u\right) - L^{\phi}\left(X_u, Y_u\right)\right| du\right) \tag{3.33}$$

The Taylor development of ϕ writes

$$\phi\left(v+u\right) = \phi\left(v\right) + u.\nabla\phi\left(v\right) + \frac{1}{2}u^t.J_\phi\left(v\right).u + O\left(|u|^3\right)$$

We notice that $u^t.J_\phi\left(v\right).u = J_\phi\left(v\right) : u.u^t$. Then we divide the expectation of the right term in (3.33) in three parts :

$$E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t \left|K^{\phi}_{\beta^\varepsilon,\gamma}\left(X_u, Y_u\right) - L^{\phi}\left(X_u, Y_u\right)\right| du\right) \le E_{11} + E_{12} + E_{13}$$

with

$$\begin{aligned} E_{11} &= K(|-2\Lambda + b^\varepsilon|) E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t |\psi(X_u - Y_u)(X_u - Y_u).\nabla\phi\left(X_u\right)| du\right) \\ E_{12} &= K E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t \left(\psi(X_u - Y_u)\right)\left|J_\phi\left(X_u\right) : \left(\Lambda|X_u - Y_u|^2 \Pi\left(X_u - Y_u\right)\right.\right.\right. \\ &\quad \left.\left.\left. - \int_0^{2\pi}\int_0^{\pi} a(X_u, Y_u, \theta, \varphi).a^t(X_u, Y_u, \theta, \varphi)\beta^\varepsilon(\theta) d\theta d\varphi\right)\right| du\right) \\ E_{13} &= K E_{Q^\varepsilon\otimes Q^\varepsilon}\left(\int_s^t \psi(X_u - Y_u)\left(\int_0^{2\pi}\int_0^{\pi} |a(X_u, Y_u, \theta, \varphi)|^3 \beta^\varepsilon(\theta) d\theta d\varphi\right) du\right) \end{aligned}$$

- Using estimates (3.4) and thanks to (3.30), we get
$$E_{11} \le K\left|-2\Lambda + b^{\varepsilon}\right|$$
As $b^{\varepsilon} \underset{\varepsilon\to 0}{\longrightarrow} 2\Lambda$, E_{11} converges towards 0 as ε tends to 0.
- Let us now study E_{12}. After some computations, we prove that
$$\begin{aligned}&\int_0^{2\pi} a(X_u,Y_u,\theta,\varphi).a^t(X_u,Y_u,\theta,\varphi)d\varphi\\ =\ &\frac{\pi}{4}\left[\Pi\left(X_u-Y_u\right)\sin^2\theta + 2\left(I-\Pi\left(X_u-Y_u\right)\right)(\cos\theta-1)^2\right]\left|X_u-Y_u\right|^2\end{aligned}$$
Then
$$\int_0^{2\pi}\int_0^{\pi} a(X_u,Y_u,\theta,\varphi).a^t(X_u,Y_u,\theta,\varphi)\beta^{\varepsilon}(\theta)d\theta d\varphi \underset{\varepsilon\to 0}{\longrightarrow} \Lambda\Pi\left(X_u-Y_u\right)\left|X_u-Y_u\right|^2$$
Thanks to (3.30), we conclude that E_{12} converges towards 0 as ε tends to 0.
- Using similar arguments and Lemma 2.2, we prove the same convergence for E_{13}.

Finally, we have proved that $E_1 \underset{\varepsilon\to 0}{\longrightarrow} 0$.

(b) Study of E_2

$$\begin{aligned}E_2 \ =\ & E_{Q^{\varepsilon}\otimes Q^{\varepsilon}}\left(\left\{\int_s^t L^{\phi}\left(X_s,Y_s\right)ds\right\}G(X_{s_1},...,X_{s_p})\right)\\ &-E_{P\otimes P}\left(\left\{\int_s^t L^{\phi}\left(X_s,Y_s\right)ds\right\}G(X_{s_1},...,X_{s_p})\right)\end{aligned}$$

The functions $f_{ij}^A : I\!D_T\times I\!D_T\to I\!R$, $(x,y)\longmapsto G(X_{s_1},...,X_{s_p})\int_s^t A_{ij}\left(x_u-y_u\right)\partial_{ij}\phi\left(x_u\right)du$ and $f_i^b : I\!D_T\times I\!D_T\to I\!R$, $(x,y)\longmapsto G(X_{s_1},...,X_{s_p})\int_s^t b_i\left(x_u-y_u\right)\partial_i\phi\left(x_u\right)du$ are continuous functions ($\gamma\in(-1,0]$), but not necessarily bounded. Nevertheless, using similar arguments as in the proof of Theorem 2.6 in Section 2, we obtain $E_2 \underset{\varepsilon\to 0}{\longrightarrow} 0$.

Conclusion For any $(t,s,s_1,...,s_p)\in(I\!R_+)^{p+2}$, with $0\le s_1\le ...\le s_p\le s<t$, we have proved
$$E_{Q^{\varepsilon}}\left(\left(M_t-M_s\right)G(X_{s_1},...,X_{s_p})\right) \underset{n\to\infty}{\longrightarrow} E_P\left(\left(M_t-M_s\right)G(X_{s_1},...,X_{s_p})\right)$$
which implies that
$$E_P\left(\left(M_t-M_s\right)G(X_{s_1},...,X_{s_p})\right)=0$$
So, $(M_t)_{t\ge 0}$ is a P-martingale and P satisfies the martingale problem (LMP).

4 A stochastic particle approximation

Our aim here is the construction of some simulable stochastic colliding particle systems converging in a certain sense to the law of a Landau process. More precisely we consider cutoff cross-sections (to obtain simulable systems) which depend on a grazing collision parameter ε. We define the interacting particle systems by a Monte-Carlo approach, consisting in replacing the nonlinearity with the empirical measure of the system. These particle systems will conserve momentum and kinetic energy. If ε is fixed, it has already been proved that when the parameter of the cutoff and the size of the system tend to infinity, the empirical measures of the system tend to the law of a Boltzmann process (see for example [21]). The novelty here is that this convergence is uniform in the parameter ε. If moreover ε tends to 0, we observe the transition from Boltzmann to Landau equations on the particle system. This result will be exploited in the last section to construct an efficient Monte-Carlo algorithm which allows to see effectively this transition.

We consider the sequence of cutoff cross-sections

$$B_{k,\varepsilon}(z,\theta) = \psi_k(z)\beta^\varepsilon(\theta) \tag{3.34}$$

where $\psi_k(z) = h(|z|)(|z|^\gamma \wedge k)$, h is a locally Lipschitz function bounded by H, $\gamma \in (-1,0]$, ε a parameter tending to 0, β^ε is a $L^1([0,\pi])$-function satisfying (3.27) and (3.28), k is a positive integer.

In order to define the interacting systems, we will "replace" the nonlinearity in (3.13) with the empirical measure of the system. Hence we introduce a family of independent Poisson-point measures $(N^{\varepsilon,ij})_{1\leq i<j\leq n}$ on $[0,\pi]\times[0,2\pi]\times[0,kH]\times[0,T]$ with intensities $\frac{1}{n-1}\beta^\varepsilon(\theta)d\theta d\varphi dx dt$. For $i>j$, we set $N^{\varepsilon,ij} = N^{\varepsilon,ji}$ (we thus choose a binary mean-field interaction, close to the physical interpretation). We define the process $(V^{k\varepsilon,in})_{1\leq i\leq n}$ solution of the following stochastic differential system:

$$\begin{aligned} V_t^{k\varepsilon,in} &= V_0^i + \sum_{j\neq i, j=1}^n \int_0^t \int_0^{kH} \int_0^{2\pi} \int_0^\pi a(V_{s-}^{k\varepsilon,in}, V_{s-}^{k\varepsilon,jn}, \theta, \varphi) \\ &\qquad \times \mathbf{1}_{\{x\leq \psi_k(V_{s-}^{k\varepsilon,in} - V_{s-}^{k\varepsilon,jn})\}} N^{\varepsilon,ij}(d\theta, d\varphi, dx, ds). \end{aligned} \tag{3.35}$$

We construct it easily, working recursively on each interjump interval of the point process $(N^{\varepsilon,ij})_{1\leq i,j\leq n}$. The equations are not compensated since for a ε fixed, $\beta^\varepsilon \in L^1([0,\pi])$. The system conserves momentum and kinetic energy and is a $(\mathbb{R}^3)^n$-valued pure-jump Markov process with the generator defined for $\phi \in C_b((\mathbb{R}^3)^n)$ by

$$\begin{aligned} \frac{1}{n-1} \sum_{1\leq i,j\leq n} \int_0^{2\pi}\int_0^\pi \int_0^{kH} \frac{1}{2}\Big(&\phi(v^n + \mathbf{e_i}.a(v_i,v_j,\theta,\varphi)\mathbf{1}_{\{x\leq\psi_k(v_i-v_j)\}} \\ &+ \mathbf{e_j}.a(v_j,v_i,\theta,\varphi)\mathbf{1}_{\{x\leq\psi_k(v_i-v_j)\}}) - \phi(v^n)\Big) dx \beta^\varepsilon(\theta) d\theta d\varphi. \end{aligned} \tag{3.36}$$

Here $v^n = (v_1,...,v_n)$ denotes the generic point of $(\mathbb{R}^3)^n$ and $\mathbf{e_i} : h \in \mathbb{R}^3 \mapsto \mathbf{e_i}.h = (0,...,0,h,0,...,0) \in (\mathbb{R}^3)^n$ with h at the i-th place.

Let us denote by

$$\mu^{k\varepsilon,n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{V^{k\varepsilon,in}}$$

the empirical measure of this system and by $\pi^{k\varepsilon,n}$ its law, which is a probability measure on $\mathcal{P}(I\!D([0,T], I\!R^3))$.

Theorem 4.1 *Assume that $Q_0 \in \mathcal{P}_4(I\!R^3)$. Let $(V_0^i)_{i\geq 1}$ be independent Q_0-distributed random variables. Then the sequence $(\pi^{k\varepsilon,n})_{k,\varepsilon,n}$ is uniformly tight for the weak convergence and any limit point charges only probability measures which are solutions of (LMP). Thus any limit point (for the convergence in law) of the sequence $(\mu^{k\varepsilon,n})$ is a solution of (LMP).*

Proof. To prove this theorem, we will show
1) the tightness of $(\pi^{k\varepsilon,n})_{k,\varepsilon,n}$ in $\mathcal{P}(\mathcal{P}(I\!D([0,T], I\!R^3)))$,
2) the identification of the limit values of $(\pi^{k\varepsilon,n})_{k,\varepsilon,n}$ as solutions of the nonlinear martingale problem (LMP).
One knows (cf. [36]) that the tightness of $(\pi^{k\varepsilon,n})_{k,\varepsilon,n}$ is equivalent to the tightness of the laws of the semimartingales $V^{k\varepsilon,1n}$ belonging to $\mathcal{P}(I\!D([0,T], I\!R^3))$. This tightness is due to

$$\sup_{k,\varepsilon,n} E(\sup_{t\leq T} |V_t^{k\varepsilon,1n}|^4) < +\infty. \tag{3.37}$$

This moment condition is obtained by a good use of Burkholder-Davis-Gundy and Doob's inequalities for (3.35).
Let us now prove that each limit value of $(\pi^{k\varepsilon,n})$ is a solution of the nonlinear martingale problem (LMP). Consider one of them, de,oted by $\pi^\infty \in \mathcal{P}(\mathcal{P}(I\!D([0,T], I\!R^3)))$. It is the limit point of a subsequence we still denote by $(\pi^{k\varepsilon,n})$.
We define for $\phi \in C_b^1(I\!R^3)$, $0 \leq s_1, ..., s_p \leq s < t$, $G \in C_b((I\!R^3)^p)$, $Q \in \mathcal{P}(I\!D_T)$ and for X the canonical process on $I\!D([0,T], I\!R^3)$

$$F(Q) = \left\langle G(X_{s_1}, ..., X_{s_p})\left(\phi(X_t) - \phi(X_s) - \int_s^t \int_{I\!R^3} L^\phi(X_u, v_*), Q_u(dv_*)du\right), Q\right\rangle. \tag{3.38}$$

Our aim is to prove that $< |F|, \pi^\infty >= 0$.
The mapping F is not continuous since the projections are not continuous for the Skorohod topology. However, for any $Q \in \mathcal{P}(I\!D_T)$, the mapping $X \mapsto X_t$ is Q-almost surely continuous for all t outside of an at most countable set D_Q, and then F is continuous at the point Q if $s, t, s_1, ..., s_p$ are not in D_Q. Here we use the continuity and the boundedness of ϕ, G and also the continuity of $(q, v) \mapsto \int_{I\!R^3} L^\phi(v, w)q(dw)$ on $\mathcal{P}(I\!D([0,T], I\!R^3)) \times I\!R^3$. Thus, if $s, t, s_1, ..., s_p$ are not in D_Q, F is π^∞-a.s. continuous. Then,

$$\left\langle F^2, \pi^\infty \right\rangle = \lim_{k,\varepsilon,n} \left\langle F^2, \pi^{k\varepsilon,n} \right\rangle$$

But $\left\langle |F|, \pi^{k\varepsilon,n} \right\rangle \leq \left\langle |F^{k\varepsilon}|, \pi^{k\varepsilon,n} \right\rangle + \left\langle |F - F^{k\varepsilon}|, \pi^{k\varepsilon,n} \right\rangle$ where

$$F^{k\varepsilon}(Q) = \left\langle G(X_{s_1}, ..., X_{s_p})\left(\phi(X_t) - \phi(X_s) - \int_s^t \int_{I\!R^3} K^\phi_{\beta^\varepsilon,k}(X_u, v_*), Q_u(dv_*)du\right), Q\right\rangle \tag{3.39}$$

in which $K^{\phi}_{\beta^\varepsilon,k}$ is obtained as $K^{\phi}_{\beta^\varepsilon,\gamma}$ but where $|z|^\gamma$ has been replaced by $|z|^\gamma \wedge k$. In this case and since $\int_0^\pi \beta^\varepsilon(\theta)d\theta < +\infty$, $K^{\phi}_{\beta^\varepsilon,k}$ also writes

$$\begin{aligned} K^{\phi}_{\beta^\varepsilon,k}(v,v_*) &= \int_0^{2\pi}\int_0^\pi (\phi(v+a(v,v_*,\theta,\varphi)) - \phi(v))\psi_k(v-v_*)\beta^\varepsilon(\theta)d\theta d\varphi \\ &= \int_0^{2\pi}\int_0^\pi\int_0^{kH} (\phi(v+a(v,v_*,\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(v-v_*)\}}) - \phi(v))dx\beta^\varepsilon(\theta)d\theta d\varphi. \end{aligned}$$

Firstly,

$$\begin{aligned} &\left\langle (F^{k\varepsilon})^2, \pi^{k\varepsilon,n}\right\rangle = E((F^{k\varepsilon}(\mu^{k\varepsilon,n}))^2) \\ &= E\left(\left(\frac{1}{n}\sum_{i=1}^n (M_t^{k\varepsilon,i\phi} - M_s^{k\varepsilon,i\phi})G(V_{s_1}^{k\varepsilon,in},...,V_{s_p}^{k\varepsilon,in})\right)^2\right) \\ &= \frac{1}{n}E\left(\left((M_t^{k\varepsilon,1\phi} - M_s^{k\varepsilon,1\phi})G(V_{s_1}^{k\varepsilon,1n},...,V_{s_p}^{k\varepsilon,1n})\right)^2\right) \qquad (3.40) \\ &\quad +\frac{n-1}{n}E\left((M_t^{\varepsilon,1\phi} - M_s^{\varepsilon,1\phi})(M_t^{\varepsilon,2\phi} - M_s^{\varepsilon,2\phi})G(V_{s_1}^{k\varepsilon,1n},...,V_{s_p}^{k\varepsilon,1n})G(V_{s_1}^{k\varepsilon,2n},...,V_{s_p}^{k\varepsilon,2n})\right) \end{aligned}$$

where $M^{k\varepsilon,i\phi}$ is the martingale defined by

$$\begin{aligned} &M_t^{k\varepsilon,i\phi} = \phi(V_t^{k\varepsilon,in}) - \phi(V_0^i) \\ &-\frac{1}{n-1}\sum_{j=1}^n \int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^\pi \left(\phi(V_s^{k\varepsilon,in} + a(V_s^{k\varepsilon,in},V_s^{k\varepsilon,jn},\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}}) \right. \\ &\qquad\qquad \left. -\phi(V_s^{k\varepsilon,in})\right)\beta^\varepsilon(\theta)d\theta d\varphi dx ds \end{aligned}$$

and with Doob-Meyer process given by

$$\begin{aligned} &< M^{k\varepsilon,i\phi} >_t \\ &= \frac{1}{n-1}\sum_{j=1}^n \int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^\pi \left(\phi(V_s^{k\varepsilon,in} + a(V_s^{k\varepsilon,in},V_s^{k\varepsilon,jn},\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}}) \right. \\ &\qquad\qquad \left. -\phi(V_s^{k\varepsilon,in})\right)^2\beta^\varepsilon(\theta)d\theta d\varphi dx ds \end{aligned}$$

and for $i \neq j$,

$$\begin{aligned} &< M^{k\varepsilon,i\phi}, M^{k\varepsilon,j\phi} >_t \qquad (3.41) \\ &= \frac{1}{n-1}\int_0^t\int_0^{kH}\int_0^{2\pi}\int_0^\pi \left(\phi(V_s^{k\varepsilon,in} + a(V_s^{k\varepsilon,in},V_s^{k\varepsilon,jn},\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}} - \phi(V_s^{k\varepsilon,in})\right) \\ &\left(\phi(V_s^{k\varepsilon,jn} + a(V_s^{k\varepsilon,jn},V_s^{k\varepsilon,in},\theta,\varphi)\mathbf{1}_{\{x\le\psi_k(V_s^{k\varepsilon,in}-V_s^{k\varepsilon,jn})\}} - \phi(V_s^{k\varepsilon,jn})\right)\beta^\varepsilon(\theta)d\theta d\varphi dx ds. \end{aligned}$$

The right terms in (3.40) go to 0 thanks to the expression of the Doob-Meyer process, to the uniform integrability proved in (3.37). Moreover the convergence is uniform on k,ε. Hence

$$\lim_n \langle |F^{k\varepsilon}|, \pi^{k\varepsilon,n}\rangle = 0, \text{ uniformly in } k,\varepsilon.$$

Otherwise, the quantity $\langle |F - F^{k\varepsilon}|, \pi^{k\varepsilon,n}\rangle = E(|F - F^{k\varepsilon}|(\mu^{k\varepsilon,n}))$ can be written in an analogous form to the right term of (3.33) replacing Q^ε by $\mu^{k\varepsilon,n}$. Its study is thus controled in a similar way than the term E_1 in Section 3.3. Then it converges to 0 uniformly in k and n as ε tends to 0.
Finally, we have proved that

$$\langle |F|, \pi^\infty\rangle = 0.$$

Thus, $F(Q)$ is π^∞-a.s. equal to 0, for every $s, t, s_1, ..., s_p$ outside of the countable set D_Q. It is sufficient to assure that π^∞-a.s., Q is a solution of the nonlinear martingale problem (LMP). Let us remark to conclude that each solution Q of the limit martingale problem is in fact a probability measure on $\mathcal{C}_T$. This remark allows us to deduce immediately the following corollary. ■

Corollary 4.2 *Assume $Q_0 \in \mathcal{P}_4(I\!R^3)$ and consider a sequence $\mu^{k_r\varepsilon_r,n_r}$ which converges to Q. Then the probability measure-valued process $(\mu_t^{k_r\varepsilon_r,n_r})_{t\geq 0}$ converges in probability to the flow $(Q_t)_{t\geq 0}$ in the space $I\!D([0,T], \mathcal{P}(I\!R^3))$ endowed with the uniform topology.*

5 The Monte-Carlo algorithm

We deduce from the above study an algorithm associated with the binary mean-field interacting particle system, which enables to observe the transition from the Boltzmann equations to the Landau equation.
At our knowledge, no effective numerical resolution of the Landau equation seen as limit of Boltzmann equations has been obtained by deterministic methods, except in [34] in which a spectral method furnishes a concrete way to study this limit (without numerical resolution). Moreover, according to discussions with numericians, it seems that the deterministic particle methods do not work for the 3D Landau equation. Some numerical Monte-Carlo algorithms for the Landau equation exist, but without convergence proofs, see Takizuka and Abe [39] and Wang, Okamoto, Nakajima, Murakami [43], and they are inspired by the diffusion structure of the Landau equation and do not follow the asymptotics of grazing collisions.

From now on, the quantities h, γ, k and β^ε defining the cross-section B, the initial distribution Q_0, the terminal time $T > 0$ and the size $n \geq 2$ of the particle system are fixed. We denote by $B_{k,\varepsilon}(z,\theta) = \psi_k(z)\beta^\varepsilon(\theta)$ the corresponding cross-section with cutoff. Because of Theorem 4.1 and Corollary 4.2, we simulate a particle system following (3.36), i.e. the whole path $(V_t^n)_{t\in[0,T]} \in I\!D([0,T], (I\!R^3)^n)$.
First of all, we assume that V_0^n is simulated according to the initial distribution $Q_0^{\otimes n}$. Then, we denote by $0 < T_1 < ... < T_k$ the successive jump times until T of a standard Poisson process with parameter $n\pi kH\|\beta^\varepsilon\|_1$.
Before the first collision, the velocities do not change, so that we set $V_s^n = V_0^n$ for all $s < T_1$. Let us describe the first collision. We choose at random a couple (i,j) of particles according a uniform law over $\{(p,m) \in \{1,...,n\}^2;\ m \neq p\}$. We choose x uniformly on the interval $[0, kH]$, we choose the first angle of collision φ uniformly on $[0, 2\pi]$ and we

finally choose the collision angle θ following the law $\frac{\beta^\varepsilon(\theta)}{\|\beta^\varepsilon\|_1}d\theta$. Then we set

$$\begin{aligned} V_{T_1}^{n,i} &= V_0^{n,i} + a(V_0^{n,i}, V_0^{n,j}, \theta, \varphi)\mathbf{1}_{\{x \le \psi_k(V_0^{n,i} - V_0^{n,j})\}} \\ V_{T_1}^{n,j} &= V_0^{n,j} + a(V_0^{n,j}, V_0^{n,i}, \theta, \varphi)\mathbf{1}_{\{x \le \psi_k(V_0^{n,i} - V_0^{n,j})\}} \\ V_{T_1}^{n,p} &= V_0^{n,p} \text{ if } p \neq \{i,j\} \end{aligned}$$

Since nothing happens between T_1 and T_2, we set $V_s^n = V_{T_1}^n$ for all $s \in [T_1, T_2[$.
Iterating this method, we simulate $V_{T_1}^n, V_{T_2}^n, ..., V_{T_k}^n$, i.e. the whole path $(V_t^n)_{t\in[0,T]}$, which was our aim.
Notice that this algorithm is very simple and takes a few lines of program and does not require to discretize time. It furthermore conserves momentum and kinetic energy. Let us remark that at least formally, this algorithm can be adapted in a similar way to the coulombian case, since the soft potential term is cut off for the simulations.

6 Numerical results

We use the previous Monte-Carlo algorithm to estimate the fourth-order moment of a solution of the Landau equation. By this method, one conserves momentum and kinetic energy, and one follows the asymptotics of grazing collisions.
We consider the cross-section $B_{k,\varepsilon}(z,\theta) = \psi_k(z)\beta^\varepsilon(\theta)$ with $\psi_k(z) = |z|^{-\gamma} \wedge k$ and β^ε satisfying Assumptions (3.3), (3.27) and (3.28).
For each ε, k, we denote by $Q^{k,\varepsilon}$ the solution of the martingale problem with cross-section $B_{k,\varepsilon}$ obtained in Theorem 2.6. We know that for each ε, k, $\left(Q^{k,\varepsilon}\right)$ is a cluster point as n tends to infinity of the empirical measure $\mu^{k,\varepsilon,n}$ associated with a simulable particle system. We also know that $\left(Q^{k,\varepsilon}\right)_{\varepsilon>0,k\ge0}$ is tight and that any limit point P is a solution of the martingale problem (LMP) associated with the Landau equation.
At last, we define :

$$\begin{aligned} m_\gamma^{k,\varepsilon,n}(t) &= \int_{\mathbb{R}^3} |v|^4 \mu_t^{k,\varepsilon,n}(dv) \quad ; \quad m_\gamma^{k,\varepsilon}(t) = \int_{\mathbb{R}^3} |v|^4 Q_t^{k,\varepsilon}(dv) \\ \text{and } m_\gamma(t) &= \int_{\mathbb{R}^3} |v|^4 P_t(dv). \end{aligned}$$

We mention that there is no explicit computation of the fourth-order moment m_t for the Landau equation in our context.

6.1 The 'moderately soft' potential case, $\gamma \in (-1, 0]$

We fix $\gamma = -0.8$ and we consider the following asymptotics

$$\beta^\varepsilon(\theta) = \frac{1}{2\pi\varepsilon^3 \sin\left(\frac{\theta}{2\varepsilon}\right)^2}\mathbf{1}_{\varepsilon \le \left|\frac{\theta}{\varepsilon}\right| \le \pi}$$

These functions satisfy Assumptions (3.3) for any $\varepsilon > 0$ and (3.27), (3.28) when ε tends to zero. We notice that $\|\beta^\varepsilon\|_1 = \frac{1}{\pi\varepsilon^2}\tan^{-1}(\varepsilon/2)$ and $\Lambda^\varepsilon = \pi\int\beta^\varepsilon(\theta)\sin^2\left(\frac{\theta}{2}\right)d\theta$ converges towards $\Lambda = \pi\ln 2$ as ε tends to 0.

We also consider the initial distribution on $I\!R^3$, $Q_0\,(dv) = \mathbf{1}_{[-1/2;1/2]^3}\,(v)\,dv$.

We first estimate $m_{-0.8}(t)$ at time $t = \frac{1}{2\pi}$. We consider $n = 50000$ particles.
First of all, when we consider the mean over 100 simulations of $m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$, we notice that it converges very fastly in k. Hence the error due to the *spatial* cutoff is small :

k	1	4	6	10	50
$m_{-0.8}^{k,0.1,50000}(\frac{1}{2\pi})$	0.09742	0.09873	0.09881	0.09878	0.09875

So we fix $k = 6$ in all what follows.

We now study the convergence of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ as ε tends to zero. Taking each time the mean over 100 simulations, we observe in Figure 1 the convergence of the fourth-order moments for the Boltzmann equation to the one for the Landau equation when ε becomes small.

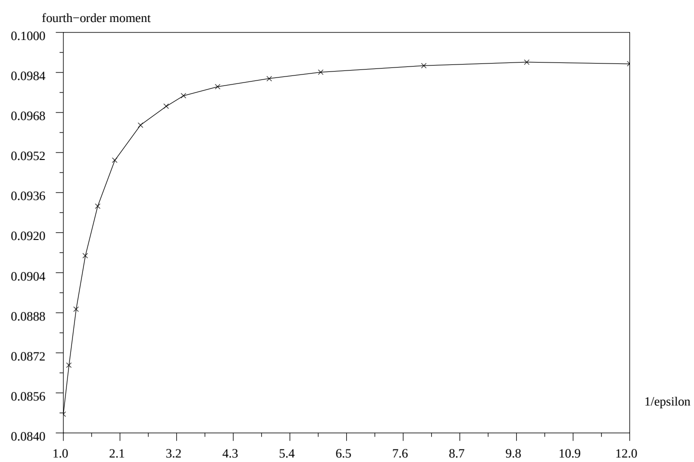

Figure 1. Evolution in $1/\varepsilon$ of $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$.

One can notice that $m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi})$ tends to 0.0988, with a speed of convergence in $\left|m_{-0.8}^{6,\varepsilon,50000}(\frac{1}{2\pi}) - 0.0988\right| \simeq 0.015 * \varepsilon^2$, when ε tends to zero. Hence, the choice $\varepsilon = 0.1$ seems reasonable to describe the Landau behaviour.
Our algorithm describes precisely the convergence of the Boltzmann equation to the Landau equation. But we take into account all small jumps, then the duration of computation is not optimal. For example, when $\varepsilon = 0.1$ and $k = 6$, there is arround 25.10^6 shocks of particles on the time interval $[0, \frac{1}{2\pi}]$.

Let us now study the speed of convergence of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ to $m_{-0.8}^{6,0.1}(\frac{1}{2\pi})$, when n tends to infinity. We obtain the Figure 2.

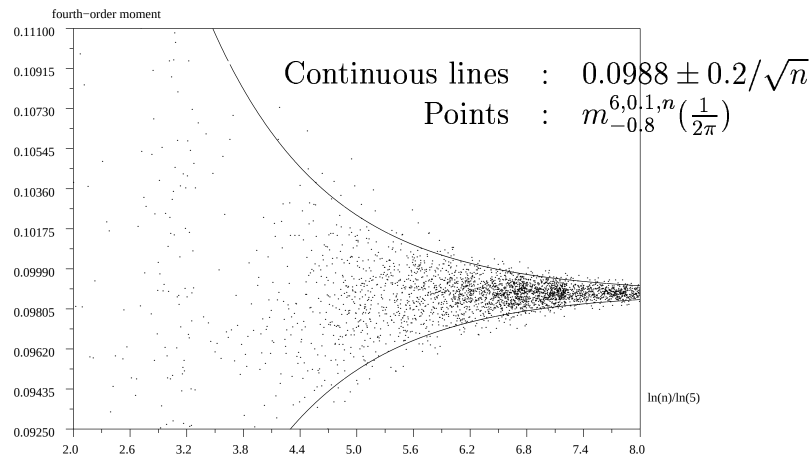

Figure 2. Evolution of $m_{-0.8}^{6,0.1,n}(\frac{1}{2\pi})$ as $n \to +\infty$.

The speed of convergence is in $1/\sqrt{n}$. It seems that a central limit theorem holds. (A proof of a similar central limit theorem has been obtained by Fournier-Méléard [19] from 2D Boltzmann equations without cutoff and for Maxwell molecules).

At last, we observe the evolution in time of the fourth-order moment. (Our method conserves the energy, then the two-order moment is constant in time). We fix again $k = 6$ and $\varepsilon = 0.1$ and we observe the moments of order 4 for some values of $t \in [0, 1]$:

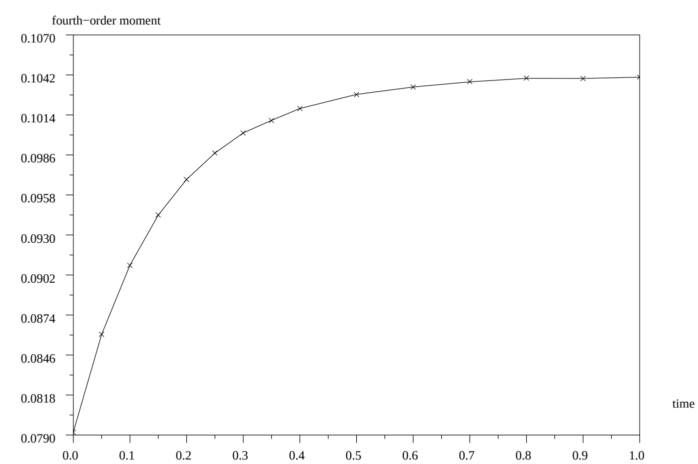

Figure 3. Evolution in time of $m_{-0.8}^{6,0.1,50000}(t)$.

6.2 The coulombian case

Our theorical results are satisfied for a potential $\gamma \in (-1, 0]$. But our numerical approach works in the interesting case of Coulomb molecules, $\gamma = -3$.
We now consider our algorithm with $\gamma = -3$ and with the same initial condition as in Buet, Cordier, Degond and Lemou [7]. We consider $n = 50000$ particles and each

value is obtained taking the mean over 100 simulations. We take as initial condition the measure Q_0 with the following density with respect to the Lebesgue measure:

$$f(0,v) = \frac{1}{2}\left(M_{\mathcal{N},v_{01},v_{th}} + M_{\mathcal{N},v_{02},v_{th}}\right)$$

where $M_{\mathcal{N},u,v_{th}}$ is the Maxwellian function on $I\!R^3$

$$M_{\mathcal{N},u,v_{th}}(v) = \frac{\mathcal{N}}{(2\pi v_{th}^2)^{3/2}} exp\left(-\frac{|v-u|^2}{2v_{th}^2}\right)$$

with $\mathcal{N} = 5$, $v_{th} = 0.45$, $v_{01} = (2,3,3)$ and $v_{02} = (4,3,3)$.
Moreover we take the cross-sections defined in [9]:

$$\beta^\varepsilon(\theta) = \frac{1}{|\log \varepsilon|}\frac{\cos(\theta/2)}{\sin^3(\theta/2)}\mathbb{I}_{\theta\geq\varepsilon}$$

In this situation, Λ^ε converges towards $\Lambda = \frac{1}{2}$ as ε tends to 0.
Since the initial distribution is not a probability measure (its mass is equal to 5), we adapt the results obtained by Méléard in [32], and we consider the algorithm with the empirical measure $\mu^{k,\varepsilon,n} = \frac{5}{n}\sum_{i=1}^n \delta_{V^{k\varepsilon,in}}$ and the jump times of a standard Poisson process with parameter $\frac{5n\pi k\|\beta^\varepsilon\|_1}{2}$.

We first estimate the fourth-order moment $m_{-3}(t)$ at time $t = 0.06$.
As for the previous simulations, the algorithm converges very fastly in k. Then we fix again $k = 6$.
We observe that the convergence in ε of the fourth-order moment of the Boltzmann equation to the one of the Landau equation is very fast:

ε	0.9	0.6	0.2	0.1	0.08
$m_{-3}^{6,\varepsilon,50000}(0.06)$	4389.5	4389.1	4389.9	4388.9	4388.5

The choice of $\varepsilon = 0.2$ seems to be reasonable to describe the Landau moment.
At last, we fix $k = 6$ and $\varepsilon = 0.2$ and we observe the evolution in time of the fourth-order moment. We find the same evolution as the one described in [7].

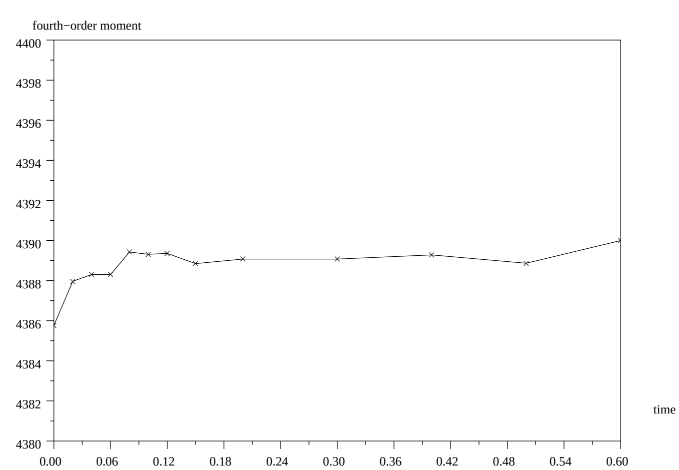

Figure 4. Evolution in time of $m_{-3}^{6,0.2,50000}(t)$.

Acknowledgements We thank Jacques Portes for many helpful discussions on the numerical part of this paper.

Chapitre 4

Pointwise convergence of Boltzmann solutions for grazing collisions in a Maxwell gas via a probabilitistic interpretation

Abstract: Using probabilistic tools, this work states a pointwise convergence of function solutions of the 2-dimensional Boltzmann equation to the function solution of the Landau equation for Maxwellian molecules when the collisions become grazing. To this aim, we use the results of Fournier [18] on the Malliavin calculus for the Boltzmann equation. Then, using the particle system introduced by Guérin and Méléard (see Chapter 3 Section 4), some simulations of the solution of the Landau equation will be given. This result is original and has not been obtained for the moment by analytical methods.

1 Introduction.

The Boltzmann equation describes the behaviour of particles in a rarefied gas. More precisely, in dimension 2, it describes the density $f(t, v, x)$ of particles having the velocity $v \in \mathbb{R}^2$ at time $t \geq 0$ and at point $x \in \mathbb{R}^2$. We consider in this work the spatially homogenous case, which means that the density does not depend on the position x of particles. In 1936, Landau [30] derived from the Boltzmann equation a new equation called the Fokker-Planck-Landau equation, usually considered as an approximation of the homogeneous Boltzmann equation in the limit of grazing collisions. These equations take the form

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{4.1}$$

where Q is a quadratic operator depending on the nature of the collisions. In this paper, we consider the case of a Maxwell gas in dimension 2. Then the Boltzmann equation writes

$$\frac{\partial f}{\partial t} = Q_B(f,f) \tag{BE}$$

with a collision operator Q_B given by

$$Q_B(f,f)(t,v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta=-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\beta(\theta)\, d\theta dv_*$$

where v, v_* are the pre-collisional velocities and v', v'_* the post-collisional velocities and where the cross-section β is an even positive function from $[-\pi,\pi]\backslash\{0\}$ to $\mathbb{R}^+$ such that $\int_{-\pi}^{\pi} \theta^2\beta(\theta)d\theta < \infty$.
The relation between the post-collisional velocities and the pre-collisional velocities in dimension 2 is the following

$$v' = v + A(\theta)(v - v_*) \ ; \ v'_* = v - A(\theta)(v - v_*)$$

with

$$A(\theta) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}.$$

We are interested in cases for which the molecules in the gas interact according to an inverse power law in $\frac{1}{d^s}$ with $s \geq 2$, where d is the distance between particles. Consequently, the function β has a singularity in 0 of the form $\beta(\theta) \underset{\theta\to 0}{\sim} C\theta^{-\frac{s+1}{s-1}}$, with C a positive constant. We assume that
Assumption (A): β is an even positive function on $[-\pi,\pi]\backslash\{0\}$ of the form $\beta = \beta_0 + \beta_1$ such that

1) β_1 is an even and positive function on $[-\pi,\pi]$,

2) there exist $k_0 > 0$, $\theta_0 \in (0,\pi)$ and $r \in (1,3)$ such that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r}\mathbb{I}_{[-\theta_0,\theta_0]}(\theta)$.

The second equation we consider is the Landau equation:

$$\frac{\partial f}{\partial t} = Q_L(f,f) \tag{LE}$$

with the collision operator Q_L defined by

$$Q_L(f,f) = \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^2} dv_* a_{ij}(v - v_*)\left[f(t,v_*)\frac{\partial f}{\partial v_j}(t,v) - f(t,v)\frac{\partial f}{\partial v_{*j}}(t,v_*)\right]\right\}$$

with $a = (a_{ij})_{1\leq i,j\leq 2}$ a nonnegative symmetric matrix of the form in the Maxwell case

$$a(z) = \Lambda\,|z|^2\,\Pi(z) \tag{4.2}$$

where $\Pi(z)$ is the orthogonal projection on $(z)^{\perp}$ and Λ is a positive constant which will be precise below.

Many authors have been interested in proving rigorously the convergence of Boltzmann to Landau, in different cases of scattering cross-section and initial data. Firstly Arsen'ev and Buryak [1] proved the convergence of solutions of the Boltzmann equation towards solutions of the Landau equation under very restrictive assumptions. Desvillettes [10] gave a mathematical framework for more physical situations, but excluding the case of Coulomb potential which has been studied by Degond and Lucquin [9]. Degond and Lucquin stated an asymptotic development of the Boltzmann kernel when the collisions become grazing. Then, Goudon [23] and Villani [41] proved in two independent works the existence of a solution of the Landau equation for soft potentials using the asymptotic of grazing collisions, with a bounded entropy and energy function as initial data. More recently, Guérin and Méléard (see Chapter 3) proved the convergence of solutions of the Boltzmann equation to a solution of the Landau equation for 'moderately soft' potentials with a probabilistic representation when the initial data is a probability measure with a finite fourth-order moment. All those works prove an L^1-weak convergence of the solutions.

The aim of this work is to prove a pointwise convergence of function-solutions of the Boltzmann equation to the function-solution of the Landau equation on $\mathbb{R}^2$ for a Maxwell gas, which is unknown by analytical methods. Fournier [18] and Guérin (see Chapter 2) proved respectively from probability measure solutions the existence of weak function solutions of the Boltzmann equation and of the Landau equation when the initial data is not a Dirac measure. To this aim, they used an efficient probabilistic tool: the Malliavin calculus for processes with jumps in [18] and the Malliavin calculus for white noises in this thesis Chapter 2. From the result of Guérin and Méléard (see Chapter 3) on the convergence of the probability measure solutions following the asymptotic of grazing collisions, it seems to be natural to study the convergence of function solutions.

In the asymptotics of grazing collisions, we only consider collisions with an infinitesimal angle of deviation. To this aim, we renormalize the cross-section β of the Boltzmann equation to concentrate on such collisions. We use the approximation introduced by Desvillettes [10]: for any $\varepsilon > 0$, let β^{ε} be the function defined on $[-\varepsilon\pi, \varepsilon\pi] \backslash \{0\}$ by

$$\beta^{\varepsilon}(\theta) = \frac{1}{\varepsilon^3}\beta\left(\frac{\theta}{\varepsilon}\right) \tag{4.3}$$

We notice that the mass of the function β^{ε} concentrates on the values of θ near of 0 when ε tends to 0, i.e. when the collisions become grazing, in the following sense:

$$\text{for any } \theta_0 > 0,\ \beta^{\varepsilon}(\theta) \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ uniformly on } \theta \geq \theta_0 \tag{4.4}$$

$$\text{and } \int_{-\varepsilon\pi}^{\varepsilon\pi} \sin\left(\frac{\theta}{2}\right)^2 \beta^{\varepsilon}(\theta)\, d\theta \underset{\varepsilon \to 0}{\longrightarrow} \Lambda \tag{4.5}$$

where $\Lambda = \frac{1}{2}\int_0^{\pi} \theta^2 \beta(\theta) d\theta > 0$ is the constant appearing in the expression (4.2) of the matrix a. This asymptotic (4.3) is a particular case of the one introduced by Villani in [41], and used by Guérin and Méléard (see Chapter 3). We prove here the following theorem:

Theorem 1.1 *Let β be an even function on $[-\pi,\pi]\backslash\{0\}$ satisfying Assumption (A). Assume that the initial data P_0 is a probability measure with finite moments of all orders and P_0 is not a Dirac mass.*

We define $\beta^\varepsilon(\theta) = \varepsilon^{-3}\beta(\theta/\varepsilon)$ and we denote by f^ε the function-solution of the Boltzmann equation (BE) associated with the cross-section β^ε (obtained by Fournier in [18]). The function f^ε is of class $\mathcal{C}^\infty$ on $\mathbb{R}^2$ ([18] Theorem 3.2).

Then the sequence $(f^\varepsilon(t,.))_{\varepsilon>0}$ is pointwise convergent on $\mathbb{R}^2$ as ε tends to 0 for any $t>0$ and the limiting function f is the function-solution of the Landau equation. Moreover, $f(t,.)$ is of class $\mathcal{C}^\infty$ and there is pointwise convergence of derivatives of all orders.

This theorem states a strong convergence result of solutions of the Boltzmann equation to the solution of Landau equation for a Maxwell gas when the collisions become grazing. Goudon [23] and Villani [41] proved a L^1-weak convergence, but in the more general case of soft potentials and in dimension 3. It seems that their methods can not give a stronger result.

Theorem 1.1 gives a new proof of the existence of regular function-solution for the Landau equation via a probabilistic approach.

We have to restrict our study to the dimension 2 because of the nonregularity of the Boltzmann coefficients in $\mathbb{R}^3$ (see [21] Lemma 2.6). Fournier [18] built the functions f^ε using the Fourier transform of the probability measure solutions. Consequently, thanks to the convergence of Chapter 3, it suffices to prove that the Fourier transforms of the Boltzmann measure-solutions are uniformly bounded by integrable functions on $\mathbb{R}^2$ when the collisions become grazing to obtain the convergence of the function-solutions. The proof is based upon a careful study of the results of Fournier [18].

In the last part of this paper, we use the Monte-Carlo algorithm following the asymptotic of grazing collisions developed by Guérin and Méléard, in Chapter 3 Section 4. We firstly simulate the convergence of solutions of the Boltzmann equation to the solution of the Landau equation for a degenerate initial distribution, and then we observe the behaviour in time of the solution of the Landau equation and of its entropy.

The paper is organized as follows. In Section 2, we give some definition connected with the probabilistic approach of the Boltzmann and Landau equations. Then, we give the main ideas of the proof of Theorem 1.1. We detail the proof in Section 4. At last, some simulations of the solution of the Landau equation are given.

Notations

- $\mathbb{D}_T$ will denote the Skorohod space $\mathbb{D}([0,T],\mathbb{R}^2)$ of cadlag functions from $[0,T]$ into $\mathbb{R}^2$.
- $C_b^2(\mathbb{R}^2)$ is the space of real bounded functions of class $\mathcal{C}^2$ with bounded derivatives.
- $\mathcal{M}_2(\mathbb{R})$ is the set of matrices of order 2×2. The matrix A^* is the adjoint of the matrix A and the matrix I denote the identity matrix in $\mathcal{M}_2(\mathbb{R})$.
- The bracket $\langle .,.\rangle$ denote the scalar product in $\mathbb{R}^2$.

2 Some Definitions

Let β be defined by Assumption (A) and β^ε be defined by (4.3). We define the Boltzmann equation (BE^ε) associated with the cross-section β^ε:

$$\frac{\partial f}{\partial t} = Q_{B^\varepsilon}(f,f) \tag{BE^ε}$$

with

$$Q_{B^\varepsilon}(f,f)(t,v) = \int_{v_*\in\mathbb{R}^2}\int_{\theta=-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\beta^\varepsilon(\theta)\,d\theta dv_*.$$

The collision operators of the Boltzmann and the Landau equations preserve momentum and kinetic energy. Equations of the form (4.1) have to be understood in a weak sense, i.e. f is a solution of the equation if for test functions ϕ,

$$\frac{d}{dt}\int_{\mathbb{R}^2}\phi(v)f(t,v)dv = \int_{\mathbb{R}^2}\phi(v)Q(f,f)(t,v)dv$$

As detailed for example in [18], a standard integration by parts and a compensation due to the bad integrability behaviour of β^ε yield to the definition of a function-solution of the Boltzmann equation:

Definition 2.1 *Let $\varepsilon > 0$ be fixed. A function-solution of (BE^ε) is a function f^ε satisfying for any $\phi \in C_b^2(\mathbb{R}^2)$ the equation*

$$\frac{d}{dt}\int_{\mathbb{R}^2} f^\varepsilon(t,v)\phi(v)dv = \int_{\mathbb{R}^2\times\mathbb{R}^2} K^\phi_{\beta^\varepsilon}(v,v_*)\, f^\varepsilon(t,v)dv f^\varepsilon(t,v_*)dv_* \tag{4.6}$$

where $K^\phi_{\beta^\varepsilon}$ is defined by

$$\begin{aligned} K^\phi_{\beta^\varepsilon}(v,v_*) &= -b^\varepsilon\nabla\phi(v)\,.\,(v-v_*) \\ &\quad + \int_{-\varepsilon\pi}^{\varepsilon\pi}\Big(\phi(v+A(\theta)(v-v_*)) - \phi(v) - A(\theta)(v-v_*)\,.\nabla\phi(v)\Big)\beta^\varepsilon(\theta)\,d\theta \end{aligned} \tag{4.7}$$

with $b^\varepsilon = \frac{1}{2}\int_{-\varepsilon\pi}^{\varepsilon\pi}(1-\cos\theta)\beta^\varepsilon(\theta)d\theta$.

Using the conservation of the mass in (4.6), we introduce a definition of probability measure solutions of (BE^ε):

Definition 2.2 *Let $\varepsilon > 0$ be fixed. Let P_0 be a probability measure with a finite 2-order moment. A measure family $(P_t^\varepsilon)_{t\geq 0}$ is a measure-solution of (BE^ε) if it satisfies for any $\phi \in \mathcal{C}_b^2(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^2}\phi(v)P_t^\varepsilon(dv) = \int_{\mathbb{R}^2}\phi(v)P_0(dv) + \int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2} K^\phi_{\beta^\varepsilon}(v,v_*)\,P_s^\varepsilon(dv)\,P_s^\varepsilon(dv_*)\,ds \tag{4.8}$$

In the same way, we give the following definition of a function-solution for the Landau equation:

Definition 2.3 *A function f is a function-solution of (LE) if f satisfies for each $\phi \in C_b^2(\mathbb{R}^2)$*

$$\frac{d}{dt}\int_{\mathbb{R}^2} f(t,v)\phi(v)\,dv = \int_{\mathbb{R}^2\times\mathbb{R}^2} L^{\phi}(v,v_*)\,f(t,v)dvf(t,v_*)\,dv \tag{4.9}$$

where L^{ϕ} is the Landau kernel defined on $\mathbb{R}^2\times\mathbb{R}^2$ by :

$$L^{\phi}(v,v_*) = \frac{1}{2}\sum_{i,j=1}^{2}\partial_{ij}^2\phi(v)\ a_{ij}(v-v_*) + \sum_{i=1}^{2}\partial_i\phi(v)\,b_i(v-v_*)$$

with $b_i(z) = \sum_{j=1}^{2}\partial_j a_{ij}(z) = -\Lambda z_i.$

We also state a definition of measure-solutions of (LE) as in Definition 2.2.

Remark 2.4 *We notice that the Boltzmann kernel $K_{\beta^{\varepsilon}}^{\phi}$ is pointwise convergent on $\mathbb{R}^2\times\mathbb{R}^2$ to the Landau kernel L^{ϕ} when ε tends to 0 for any $\phi\in C_b^2(\mathbb{R}^2)$ (See for example [23] or [41]).*

3 The convergence of the function-solutions

We give in this section the main idea of the proof of Theorem 1.1.

In all the following, P_0 is assumed to be a probability measure with a finite two-order moment and β a positive even function on $[-\pi,\pi]\backslash\{0\}$ satisfying Assumption (A).
In the probabilistic study of the Boltzmann equation, we consider in fact (4.8) as the evolution equation of the flow of a jump process. The distribution of this process will be solution of the following nonlinear martingale problem:

Definition 3.1 *Let $\varepsilon>0$ be fixed. We say that a probability measure P^{ε} on $\mathbb{D}_T$ solves the nonlinear martingale problem (MP^{ε}) starting at P_0 if for X the canonical process under P^{ε}, the law of X_0 is P_0 and for any $\phi\in C_b^2(\mathbb{R}^2)$,*

$$\phi(X_t)-\phi(X_0)-\int_0^t\int_{\mathbb{R}^2}K_{\beta^{\varepsilon}}^{\phi}(X_s,v_*)\,P_s^{\varepsilon}(dv_*)\,ds \tag{4.10}$$

is a square-integrable martingale, where P_s^{ε} is the marginal of P^{ε} at time s.

Remark 3.2 *Taking expectation in (4.10), we notice that if P^{ε} is a solution of (MP^{ε}),then $(P_t^{\varepsilon})_{t\geq 0}$ is a measure-solution of (BE^{ε}).*

Fournier proved in [18] the existence of a solution P^ε of (MP^ε) for any $\varepsilon > 0$. Moreover, Guérin and Méléard in Chapter 2 stated the tightness of the sequence $(P^\varepsilon)_{\varepsilon>0}$ when the collisions becomes grazing ($\varepsilon \to 0$) in the more general case of soft potentials and in dimension 3 (using the same arguments, the convergence theorem is still true in dimension 2). In the particular case of Maxwellian molecules, there is convergence of the sequence $(P^\varepsilon)_{\varepsilon>0}$ to the measure-solution of the Landau equation (LE) thanks to the uniqueness of this solution (see Chapter 2 Corollary 1.7). We will use those results under the following form:

Theorem 3.3 *Let $\beta^\varepsilon = \varepsilon^{-3}\beta(\theta/\varepsilon)$. For any $\varepsilon > 0$, there exists a solution P^ε of the martingale problem (MP^ε). Moreover, the sequence $(P^\varepsilon)_{\varepsilon>0}$ converges as ε goes to 0 to a distribution P which time-marginal family is the measure-solution of the Landau equation.*

Let us notice that to obtain a function-solution from a measure-solution $(P_t^\varepsilon)_{t\geq 0}$, it suffices to prove that $\forall t > 0$ P_t^ε admits a density $f^\varepsilon(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Then the function f^ε satisfies Definition 2.1. Fournier [18] stated the following theorem using the Malliavin calculus for processes with jumps:

Theorem 3.4 *Let $\varepsilon \in (0,1)$ be fixed. Assume that P_0 is not a Dirac measure.*
1) The Boltzmann equation (BE^ε) admits a function-solution f^ε with initial data P_0.
2) If P_0 belongs to L^p for any $p \geq 1$, then for any $t > 0$, for any couple $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exists a constant $C_{t,\alpha}^\varepsilon$ such that the following inequality holds for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2)$ with compact support

$$\left| \int_{\mathbb{R}^2} \partial_\alpha \varphi(v)\, P_t^\varepsilon(dv) \right| \leq C_{t,\alpha}^\varepsilon \|\varphi\|_\infty \tag{4.11}$$

where ∂_α denotes the partial derivative $\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x_1 \partial^{\alpha_2}x_2}$. Consequently, the function-solution f^ε is infinitely differentiable on $\mathbb{R}^2$ and is given by:

$$f^\varepsilon(t,v) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{P}_t^\varepsilon(x)\, e^{-i<v,x>} dx$$

where $\hat{P}_t^\varepsilon$ is the Fourier transform of P_t^ε.

We want to state the convergence of the function-solutions f^ε of the Boltzmann equation (BE^ε) when the grazing collisions prevail.
Thanks to the convergence of measure-solutions $(P_t^\varepsilon)_{t\geq 0}$ of the Boltzmann equation to the measure-solution $(P_t)_{t\geq 0}$ of the Landau equation (see Theorem 3.3), the sequence $\left(\hat{P}_t^\varepsilon\right)_{\varepsilon>0}$ is pointwise convergent on $\mathbb{R}^2$ to the Fourier transform $\hat{P}_t$ of P_t, for any $t \geq 0$. Approximating the functions $\varphi(v) = e^{i<v,x>}$ with $x = (x_1, x_2) \in \mathbb{R}^2$, by compact support functions of class $\mathcal{C}^\infty$, we obtain from inequality (4.11) that $\forall x \in \mathbb{R}^2$ and $\forall \alpha_1, \alpha_2 \geq 2$

$$\left|\hat{P}_t^\varepsilon(x)\right| \leq \inf\left\{1, \frac{C_{t,(\alpha_1,\alpha_2)}^\varepsilon}{x_1^{\alpha_1} x_2^{\alpha_2}}\right\}$$

Thus if we prove that the constants $C^{\varepsilon}_{t,\alpha}$ are uniformly bounded in ε by a constant $C_{t,\alpha}$ for any $\alpha \in \mathbb{N}^2$, using the Lebesgue theorem, we easily deduce that the function-solutions $f^{\varepsilon}(t,v)$ (and its derivatives of any orders) of the Boltzmann equation converge as ε goes to 0 to the function-solution $f(t,v) = \int_{\mathbb{R}^2} \hat{P}_t(x)\, e^{i<v,x>} dx$ (respectively, its derivatives) of the Landau equation for any $v \in \mathbb{R}^2$ and $t > 0$. Consequently the theorem will be proved.

4 The proof of Theorem 1.1

We assume from now without restriction that $\varepsilon \in (0, 1/2]$.
To state that the constants $C^{\varepsilon}_{t,\alpha}$ appearing in (4.11) are uniformly bounded in ε, we have to study the proof of Theorem 3.4. Fournier, [18], proved the existence of function-solutions by the mean of a nonlinear differential equation giving a pathwise version of the probabilistic interpretation.

4.1 The Pathwise approach

Let $\varepsilon > 0$ be fixed, P_0 be a probability measure with a finite 2-order moment and β satisfy Assumption (A).
Let us consider two probability spaces to highlight the nonlinearity of the equation : the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and the second one is $([0,1], \mathcal{B}([0,1]), d\alpha)$. The processes on $([0,1], \mathcal{B}([0,1]), d\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α and the laws by $\mathcal{L}_\alpha$.
On $(\Omega, \mathcal{F}, P)$ we consider a Poisson measure $N^{\varepsilon}(d\theta, d\alpha, dt)$ on $[-\pi, \pi] \times [0,1] \times [0,T]$ with intensity measure $\nu^{\varepsilon}(d\theta, d\alpha, dt) = \beta^{\varepsilon}(\theta)\, d\theta d\alpha dt$ and with compensated measure $\tilde{N}^{\varepsilon}(d\theta, d\alpha, dt)$.

Theorem 4.1 *(see [18] Theorem 2.8) Let V_0 be a random variable with distribution P_0. There exists a couple of processes $(V^{\varepsilon}, W^{\varepsilon})$ on $\Omega \times [0,1]$ satisfying the nonlinear stochastic differential equation (SDE^{ε}):*

$$V^{\varepsilon}_t = V_0 + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta)(V^{\varepsilon}_{s-} - W^{\varepsilon}_{s-}(\alpha))\tilde{N}^{\varepsilon}(ds, d\alpha, d\theta)$$
$$-b^{\varepsilon} \int_0^t \int_0^1 \left(V^{\varepsilon}_{s-} - W^{\varepsilon}_{s-}(\alpha)\right) d\alpha ds$$

with $\mathcal{L}(V^{\varepsilon}) = \mathcal{L}_\alpha(W^{\varepsilon}) = P^{\varepsilon}$.

Moreover $E[\sup_{0\le t\le T} |V^{\varepsilon}_t|^2] = E_\alpha[\sup_{0\le t\le T} |W^{\varepsilon}_t|^2] < \infty$. There is uniqueness in law of P^{ε}.

Corollary 4.2 *Thanks to Itô's formula, the measure P^{ε} is also a solution of the martingale problem (MP^{ε}). Consequently, $(P^{\varepsilon}_t)_{t\ge 0}$ is a measure-solution of the Boltzmann equation for Maxwellian molecules.*

Let us denote from now by $(V^{\varepsilon}, W^{\varepsilon})$ the couple solution of (SDE^{ε}) and P^{ε} the distribution of the process V^{ε} obtained in Theorem 4.1. Let us give a technical lemma (see Chapter 3 Section 3.3):

Lemma 4.3 *Assume that X_0 is a random vector in $\mathbb{R}^2$ belonging to L^p for any $p \geq 1$. Then for any $T > 0$, $p \geq 1$, there exists a constant K_p independent of ε such that*

$$E[\sup_{0\leq t\leq T} |V_t^\varepsilon|^p] = E_\alpha[\sup_{0\leq t\leq T} |W_t^\varepsilon|^p] \leq K_p \tag{4.12}$$

Using the Malliavin calculus for a stochastic differential equation driven by a Poisson process, Fournier, [18], proved that each time-marginal P_t^ε satisfies (4.11) for any $t > 0$ and the coefficients $C_{t,\alpha}^\varepsilon$ depend on the Malliavin derivatives of V^ε. Consequently, to control C_α^ε we have to estimate the Malliavin's derivatives.

4.2 Some recalls on the Malliavin calculus

The Malliavin calculus in the case of a stochastic differential equation driven by a Poisson process, also called the stochastic calculus of variations, has been adapted to the case of the Boltzmann equation by Graham and Méléard, [25], and Fournier, [18], from the arguments of Bichteler, Gravereaux and Jacod in [5] and [6].
Let us consider a fixed time interval $[0,T]$, $T > 0$. Let $\varepsilon \in \left(0, \frac{1}{2}\right]$ be fixed.
Let us explain the main idea of this framework. We build a perturbation replacing θ with $\theta + < \lambda, v^\varepsilon >$ in order to obtain a new family of random measures N_λ^ε (for $\lambda \in \Lambda$, Λ being a neighborhood of 0 in $\mathbb{R}^2$ and v^ε a well-chosen predictable function from $\Omega \times [0,T] \times [-\varepsilon\theta_0, \varepsilon\theta_0] \times [0,1]$ to $\mathbb{R}^2$). Then, we build a family of probability measures $P_\lambda^\varepsilon = G_{\lambda,T}^\varepsilon . P^\varepsilon$ on Ω such that $\mathcal{L}\left((V_0, N_\lambda^\varepsilon)\,|P_\lambda^\varepsilon\right) = \mathcal{L}\left((V_0, N^\varepsilon)\,|P^\varepsilon\right)$. By this way, we obtain a perturbed process V_λ^ε satisfying $\mathcal{L}\left(V_{\lambda,t}^\varepsilon | P_\lambda^\varepsilon\right) = \mathcal{L}\left(V_t^\varepsilon | P^\varepsilon\right)$, and thus $E\left[\varphi\left(V_{\lambda,t}^\varepsilon\right) G_{\lambda,t}^\varepsilon\right] = E\left[\varphi\left(V_t^\varepsilon\right)\right]$, for any Borel bounded function φ on $\mathbb{R}^2$. Differentiating this equality at $\lambda = 0$, using an L^2-differentiate of $V_{\lambda,t}^\varepsilon$ and $G_{\lambda,t}^\varepsilon$, we finally obtain an equality of the form

$$E\left[\varphi'\left(V_t^\varepsilon\right).DV_t^\varepsilon\right] = -E\left[\varphi\left(V_t^\varepsilon\right) DG_t^\varepsilon\right]$$

which is the first step to satisfy inequality (4.11) of Theorem 3.4.
Consequently, the constant $C_{t,\alpha}^\varepsilon$ appearing in (4.11) depends on the moments of the derivatives of V_t^ε, of $det^{-1}(DV_t^\varepsilon)$ and of the derivatives of DG_t^ε. Under some assumptions on the initial data P_0, Fournier, [18], obtained estimates of those moments. Consequently, we still have to state that those moments are uniformly bounded in ε to prove Theorem 1.1. The derivatives of V_t^ε and DG_t^ε depend strongly on the random function v^ε introduced in the perturbation. The function v^ε used by Fournier in [18] does not allow to obtain uniform bounds of the moments in $\varepsilon \in (0, 1/2]$ (see Remark 4.5). So, we consider another perturbation which we describe now.

4.3 The perturbation and the Malliavin derivatives

Let δ^ε be a nonnegative even function on $[-\varepsilon\theta_0, \varepsilon\theta_0]$ defined by

$$\delta^\varepsilon(\theta) = c\,\varepsilon^{1-r}\,|\theta|^{r+1}\left(1 - \frac{|\theta|}{\varepsilon\theta_0}\right) \tag{4.13}$$

with c a constant independent of ε such that $c \leq \left[\theta_0^r \left(\theta_0 + r + 2 + r2^{r-1}\right)\right]^{-1}$. We notice that

$$\delta^\varepsilon (\theta) + \left| (\delta^\varepsilon)' (\theta) \right| < 1.$$

Let g^ε be a $\mathbb{R}^2$-valued predictable function such that for any $\omega, t, \alpha, \varepsilon$, the map $\theta \to g^\varepsilon (\omega, t, \theta, \alpha)$ is of class $\mathcal{C}^1$ with $\|g^\varepsilon\|_\infty + \|g^{\varepsilon\prime}\|_\infty \leq 1$ where $g^{\varepsilon\prime}$ is the derivative of g^ε with respect to θ.
We then define the random function v^ε on $\Omega \times [0, T] \times [-\varepsilon\theta_0, \varepsilon\theta_0] \times [0, 1]$ by

$$v^\varepsilon (\omega, t, \theta, \alpha) = g^\varepsilon (\omega, t, \theta, \alpha) \, \delta^\varepsilon (\theta) \tag{4.14}$$

We denote $v^{\varepsilon\prime}$ the derivative of v^ε with respect to θ.
Let $\Lambda \subset B(0, 1)$ be a neighborhood of 0 in $\mathbb{R}^2$. For $\lambda \in \Lambda$, we consider the following perturbation

$$\gamma^{\varepsilon,\lambda} (\omega, t, \theta, \alpha) = \theta + \langle \lambda, v^\varepsilon (\omega, t, \theta, \alpha) \rangle .$$

We notice that the map $\theta \longmapsto \gamma^{\varepsilon,\lambda} (\omega, t, \theta, \alpha)$ is an increasing bijection from $[-\varepsilon\theta_0, \varepsilon\theta_0]$ into itself (for any $\varepsilon \leq \frac{1}{2}$ and $|\theta| \leq \varepsilon\theta_0$, $|v^{\varepsilon\prime} (\theta)| < 1$ thanks to the choice of c).
Recalling that $\beta = \beta_1 + \beta_0$, the Poisson measure N split into $N_0 + N_1$, where N_0 and N_1 are independent Poisson measures on $[0, T] \times [0, 1] \times [-\pi, \pi]$ with intensities $\nu_0(d\theta, d\alpha, ds) = \beta_0(\theta) d\theta d\alpha ds$ and $\nu_1(d\theta, d\alpha, ds) = \beta_1(\theta) d\theta d\alpha ds$ respectively. We denote by $\tilde{N}_0$ and $\tilde{N}_1$ the associated compensated measures.
For $\lambda \in \Lambda$, we define $N_0^{\varepsilon,\lambda} = \gamma^{\varepsilon,\lambda} (N_0^\varepsilon)$ the image measure of N_0^ε by the map $\gamma^{\varepsilon,\lambda}$: if $A \subset [0, T] \times [0, 1] \times [-\varepsilon\theta_0, \varepsilon\theta_0]$ is a Borel set,

$$N_0^{\varepsilon,\lambda} (\omega, A) = \int_0^T \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} \mathbb{I}_A \left(s, \gamma^{\varepsilon,\lambda} (\omega, s, \theta, \alpha), \alpha\right) N_0^\varepsilon (\omega, d\theta, d\alpha, ds) .$$

We consider the shift $S^{\varepsilon,\lambda}$ defined by

$$V_0 \circ S^{\varepsilon,\lambda} (\omega) = V_0 (\omega), \; N_0^\varepsilon \circ S^{\varepsilon,\lambda} (\omega) = N_0^{\varepsilon,\lambda} (\omega), \text{ and } N_1^\varepsilon \circ S^{\varepsilon,\lambda} (\omega) = N_1^\varepsilon (\omega) .$$

Proposition 4.4 *Let $G^{\varepsilon,\lambda}$ be the Doléans-Dade martingale:*

$$G_t^{\varepsilon,\lambda} = 1 + \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} G_{s^-}^{\varepsilon,\lambda} \left(Y^{\varepsilon,\lambda} (s, \theta, \alpha) - 1\right) \tilde{N}_0^\varepsilon (d\theta, d\alpha, ds)$$

where $Y^{\varepsilon,\lambda}$ is the following predictable real valued function on $\Omega \times [0, T] \times [-\varepsilon\theta_0, \varepsilon\theta_0] \times [0, 1]$

$$Y^{\varepsilon,\lambda} (\omega, s, \theta, \alpha) = \left(1 + \langle \lambda, v^{\varepsilon\prime} (\omega, t, \theta, \alpha) \rangle\right) \frac{\beta_0^\varepsilon \left(\gamma^{\varepsilon,\lambda} (\omega, t, \theta, \alpha)\right)}{\beta_0^\varepsilon (\theta)}.$$

Then $G_t^{\varepsilon,\lambda}$ is positive for any $t \in [0, T]$.

Proof. Let us notice that

$$\left|Y^{\varepsilon,\lambda}(s,\theta,\alpha)-1\right| \leq |\lambda|\, d^{\varepsilon}(\theta)$$

with $d^{\varepsilon}(\theta) = \delta^{\varepsilon}(\theta) + |\delta^{\varepsilon\prime}(\theta)| + r2^{r-1}\frac{\delta^{\varepsilon}(\theta)}{|\theta|}$. According to Appendix Lemma 6.2, $d^{\varepsilon} \in \bigcap_{p\geq 2} L^p\left(\beta_0^{\varepsilon}(\theta)\, d\theta\right)$ with moments uniformly bounded in ε. Consequently, $G^{\varepsilon,\lambda}$ is well defined and if

$$M_t^{\varepsilon,\lambda} = 1 + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} \left(Y^{\varepsilon,\lambda}(s,\theta,\alpha)-1\right) \tilde{N}_0^{\varepsilon}(d\theta, d\alpha, ds)$$

then (see Jacod, Shiryaev, [28] p. 59),

$$G_t^{\varepsilon,\lambda} = e^{M_t^{\varepsilon,\lambda}} \prod_{s\leq t}\left(1+\Delta M_s^{\varepsilon,\lambda}\right) e^{-\Delta M_s^{\varepsilon,\lambda}}.$$

Moreover, since $\varepsilon \leq 1/2$, for $|\theta| \leq \varepsilon\theta_0$

$$\begin{aligned}\left|Y^{\varepsilon,\lambda}(s,\theta,\alpha)-1\right| &\leq d^{\varepsilon}(\theta) \leq \frac{1}{2} c\theta_0^r \left[\theta_0 + r + 2 + r2^{r-1}\right] \\ &\leq 1/2\end{aligned}$$

thanks to the choice of c (see (4.13)). Thus, the jumps of $M^{\varepsilon,\lambda}$ are greater than $-1/2$ which implies that $G_t^{\varepsilon,\lambda}$ is positive. ■

Let $P^{\varepsilon,\lambda}$ be the probability measure defined by $P^{\varepsilon,\lambda} = G_T^{\varepsilon,\lambda}.P^{\varepsilon}$. Using the Girsanov Theorem for random measures, we notice that $P^{\varepsilon,\lambda} \circ \left(S^{\varepsilon,\lambda}\right)^{-1} = P^{\varepsilon}$ (for more details see [18] Proposition 3.7). We consider now the perturbed process $V^{\varepsilon,\lambda} = V^{\varepsilon} \circ S^{\varepsilon,\lambda}$. Using the results of Fournier, [18], and Lemma 6.2, we notice that $V^{\varepsilon,\lambda}$ and $G^{\varepsilon,\lambda}$ belong to L^p for any $p \geq 1$ with bounded moments in ε, and they are differentiable at $\lambda = 0$. Following the proofs in [18] Section 3.2, we can give the expressions of their derivatives:

- the derivative of $G^{\varepsilon,\lambda}$ at $\lambda = 0$ is the following random vector in $\mathbb{R}^2$

$$DG_t^{\varepsilon} = \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} \left(v^{\varepsilon\prime}(s,\theta,\alpha) - r\frac{v^{\varepsilon}(s,\theta,\alpha)}{\theta}\right) \tilde{N}_0^{\varepsilon}(d\theta, d\alpha, ds)$$

- the derivative of V_t^{ε} is a 2×2 matrix which satisfies the equation ([18] Theorem 3.12)

$$\begin{aligned} DV_t^{\varepsilon} &= -\frac{b^{\varepsilon}}{2}\int_0^t DV_s^{\varepsilon} ds + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta) DV_{s^-}^{\varepsilon}\, \tilde{N}^{\varepsilon}(d\theta, d\alpha, ds) \\ &+ \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} A'(\theta)(V_{s^-} - W_{s^-}(\alpha))(v^{\varepsilon}(s,\theta,\alpha))^*\, N_0^{\varepsilon}(d\theta, d\alpha, ds) \end{aligned} \tag{4.15}$$

which can be also written ([18] Proposition 3.15)

$$DV_t^{\varepsilon} = M_t^{\varepsilon}.H_t^{\varepsilon} \tag{4.16}$$

where M^{ε} is the following invertible Doléans-Dade martingale

$$M_t^{\varepsilon} = I - \frac{b^{\varepsilon}}{2}\int_0^t M_s^{\varepsilon} ds + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta)\, M_{s^-}^{\varepsilon}\, \tilde{N}^{\varepsilon}(d\theta, d\alpha, ds) \tag{4.17}$$

and

$$H_t^\varepsilon = \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} \left(M_{s^-}^\varepsilon\right)^{-1} \left(I + A\left(\theta\right)\right)^{-1} A'\left(\theta\right) \left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon\left(\alpha\right)\right) \left(v^\varepsilon\left(s,\theta,\alpha\right)\right)^* N_0^\varepsilon\left(d\theta, d\alpha, ds\right) \tag{4.18}$$

We want to state that the moments of the derivatives of V_t^ε, of $det^{-1}(DV_t^\varepsilon)$ and of the derivatives of DG_t^ε are uniformly bounded in ε. We will just give here a detailed proof of the term $det^{-1}(DV_t^\varepsilon)$. We easily obtain the bounds for the two other terms studying the construction of DG_t^ε and of DV_t^ε ([18] Section 3.2), using the definition (4.14) of v^ε and the bounds given in Lemma 6.2.

Remark 4.5 *The derivatives of V_t^ε and of DG_t^ε depend strongly on v^ε. The choice of v^ε is important. The moments of DG_t^ε are uniformly bounded in ε, if there exists a positive constant K_1 independent of ε such that*

$$\int_0^{\varepsilon\theta_0} \left(\delta^\varepsilon(\theta) + |\delta^{\varepsilon\prime}(\theta)| + r\frac{\delta^\varepsilon(\theta)}{\theta}\right)^2 \beta_0^\varepsilon(\theta)d\theta \leq K_1$$

The moments of DV_t^ε are uniformly bounded in ε, if there exists a positive constant K_2 independent of ε such that

$$\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\beta_0^\varepsilon(\theta)d\theta \leq K_2$$

Nevertheless, the integral $\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\beta_0^\varepsilon(\theta)d\theta$ must not tend to 0 as ε goes to 0. If not, the variable DV_t^ε converges to 0 in L^2 as ε tends to 0 (see Expression (4.15) of DV_t^ε) , and we have no hope to obtain uniform bounds for the term $det^{-1}(DV_t^\varepsilon)$.

In the sequel, we will consider more precisely the perturbation v^ε defined by

$$v^\varepsilon\left(t,\theta,\alpha\right) = \bar{g}\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon\left(\alpha\right), M_{s^-}^\varepsilon, \theta\right)\delta^\varepsilon\left(\theta\right)$$

with for any $x \in \mathbb{R}^2$, $y \in \mathcal{M}_2\left(\mathbb{R}\right)$

$$\begin{aligned} \bar{g}\left(x,y,\theta\right) &= \left(A'\left(\theta\right)x\right)^* \left(\left(I + A\left(\theta\right)\right)^{-1}\right)^* \left(y^{-1}\right)^* \zeta\left(x,y,\theta\right) \\ \zeta\left(x,y,\theta\right) &= h\left(A'\left(\theta\right)x\right)k\left(I + A\left(\theta\right)\right)k\left(y\right) \end{aligned}$$

where δ^ε is defined by (4.13) and the functions h and k satisfy the following assumptions:

- h is the function from $\mathbb{R}^2$ to $(0,1]$ defined by $h(x) = \left(1 + |x|^2\right)^{-1}$,

- k is a function from $\mathcal{M}_2\left(\mathbb{R}\right)$ to $[0,1]$ such that $k\left(y\right) = 0$ if and only if $\det y = 0$ and such that the map

$$y \longmapsto \begin{cases} \left(y^{-1}\right)^* k\left(y\right) & \text{if } \det y \neq 0 \\ 0 & \text{if } \det y = 0 \end{cases} \tag{4.19}$$

is of class $\mathcal{C}_b^\infty$ from $\mathcal{M}_2\left(\mathbb{R}\right)$ to itself.

Consequently, the process H^ε introduced in (4.16) writes

$$\begin{aligned} H_t^\varepsilon &= \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} (M_{s^-}^\varepsilon)^{-1} \Gamma\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon(\alpha), \theta\right) \left[(M_{s^-}^\varepsilon)^{-1}\right]^* \\ &\qquad \times \zeta\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon(\alpha), M_{s^-}^\varepsilon, \theta\right) \delta^\varepsilon(\theta) N_0^\varepsilon(d\theta, d\alpha, ds) \end{aligned}$$

with for any $x \in \mathbb{R}^2$,

$$\Gamma(x, \theta) = (I + A(\theta))^{-1} (A'(\theta) x) (A'(\theta) x)^* \left((I + A(\theta))^{-1}\right)^*.$$

4.4 Study of $det^{-1}(DV_t^\varepsilon)$

Since the derivative of V_t^ε can be written as $DV_t^\varepsilon = M_t^\varepsilon . H_t^\varepsilon$ for any $t \geq 0$, we study independently the term M_t^ε and the term H_t^ε.

Theorem 4.6 *Assume (A) and $P_0 \in \cap_{p<\infty} L^p$. For every $t \geq 0$, $(\det M_t^\varepsilon)^{-1}$ admits moments of all orders uniformly in ε.*

Proof. By [18] Theorem 3.20, M_t^ε is invertible and its inverse $(M_t^\varepsilon)^{-1}$ satisfies the equation

$$\begin{aligned} (M_t^\varepsilon)^{-1} &= I - \frac{b^\varepsilon}{2} \int_0^t (M_s^\varepsilon)^{-1} ds - \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} (M_{s^-}^\varepsilon)^{-1} (I + A(\theta))^{-1} A(\theta) \tilde{N}^\varepsilon(d\theta, d\alpha, ds) \\ &\quad + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} (M_{s^-}^\varepsilon)^{-1} A(\theta) (I + A(\theta))^{-1} A(\theta) \beta^\varepsilon(\theta) d\theta d\alpha ds \end{aligned} \tag{4.20}$$

with

$$(I + A(\theta))^{-1} A(\theta) = \frac{\sin\theta}{\cos\theta + 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$A(\theta)(I + A(\theta))^{-1} A(\theta) = \frac{1}{2} \frac{\sin\theta}{\cos\theta + 1} \begin{pmatrix} -\sin\theta & 1 - \cos\theta \\ \cos\theta - 1 & -\sin\theta \end{pmatrix}$$

Since $\int_0^\pi \theta^2 \beta(\theta) d\theta < \infty$, the sequence $(b^\varepsilon)_{\varepsilon > 0}$ is bounded.
We notice that,

$$\int_{-\varepsilon\pi}^{\varepsilon\pi} \left(\frac{|\sin\theta|}{\cos\theta + 1}\right)^p \beta^\varepsilon(\theta) d\theta = \varepsilon^{-2} \int_{-\pi}^{\pi} \left(\frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta + 1}\right)^p \beta(\theta) d\theta$$

and for any $\varepsilon \in (0, \frac{1}{2}]$, the function

$$\theta \longmapsto \frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta + 1} \beta(\theta)$$

is continuous on $[-\pi, \pi] \setminus \{0\}$ and for ε small enough,

$$\frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta + 1} \leq \varepsilon\theta$$

Consequently, the sequence $\left(\int\left(\frac{|\sin\theta|}{\cos\theta+1}\right)^p\beta^\varepsilon(\theta)\,d\theta\right)_{\varepsilon\in(0,1/2]}$ is bounded for any $p\geq 2$.
Using the same arguments, we notice that the integrals

$$\int_{-\varepsilon\pi}^{\varepsilon\pi}\left(\frac{\sin^2\theta+|\sin\theta\,(1-\cos\theta)|}{\cos\theta+1}\right)^p\beta^\varepsilon(\theta)\,d\theta=\varepsilon^{-2}\int_{-\pi}^{\pi}\left(\frac{\sin^2\varepsilon\theta+|\sin\varepsilon\theta\,(1-\cos\varepsilon\theta)|}{\cos\varepsilon\theta+1}\right)^p\beta(\theta)\,d\theta$$

are uniformly bounded in ε, $\varepsilon\in(0,\frac{1}{2}]$, for any $p\geq 1$.
Then, using usual estimates, Gronwall's Lemma in (4.20), we easily deduce that for any $p\geq 1$, there exists a constant K_p (independent of ε) such that $\forall\varepsilon\in(0,\frac{1}{2}]$,

$$E\left[(M_t^\varepsilon)^{-p}\right]\leq K_p.$$

Thus, $(\det M_t^\varepsilon)^{-1}$ is uniformly bounded in ε in L^p for any $t\geq 0$. ■

Theorem 4.7 *Assume that (A) is satisfied and $P_0\in L^p$ for any $p\geq 1$. For every $t\geq 0$, $(\det H_t^\varepsilon)^{-1}$ admits moments of all orders uniformly in ε.*

Lemma 4.8 *The map $(\varepsilon,t,Y)\longmapsto\mathcal{L}(\langle V_t^\varepsilon,Y\rangle)$ is weakly continuous on $\left[0,\frac{1}{2}\right]\times[0,T]\times\{Y\in\mathbb{R}^2:|Y|=1\}$ where $P_t^0=\mathcal{L}(V_t^0)$ is the measure-solution of the Landau equation at time t.*

Proof. Let (ε_n,t_n,Y_n) be a sequence such that $(\varepsilon_n,t_n,Y_n)\underset{n\to\infty}{\to}(\varepsilon,t,Y)$ in $[0,\frac{1}{2}]\times[0,T]\times\{Y\in\mathbb{R}^2:|Y|=1\}$.
Let $\psi\in\mathcal{C}_b^2(\mathbb{R})$ and we define ψ_Y on $\mathbb{R}^2$ of class $\mathcal{C}_b^2$ by $v\longmapsto\psi_Y(v)=\psi(\langle v,Y\rangle)$. We consider the sequence

$$d_n=E\left[\psi_Y(V_t^\varepsilon)-\psi_{Y_n}\left(V_{t_n}^{\varepsilon_n}\right)\right].$$

We want to state that $d_n\longrightarrow 0$ as n goes to $+\infty$.
Let (Z^1,Z^2) be the canonical process on $\mathbb{D}_T\times\mathbb{D}_T$. Let us define $P_{t_n}^{\varepsilon_n}=\mathcal{L}\left(V_{t_n}^{\varepsilon_n}\right)$.
<u>If $\varepsilon>0$</u>: Since the family of time marginal $\left(P_{t_n}^{\varepsilon_n}\right)_{n\geq 0}$ of the probability measure P^{ε_n} is a solution of (4.8), we notice that :

$$\begin{aligned}d_n &= E\left[\psi_Y(V_0)-\psi_{Y_n}(V_0)\right]+\int_{t_n}^{t}E_{P^\varepsilon\otimes P^\varepsilon}\left[K_{\beta^\varepsilon}^{\psi_Y}\left(Z_s^1,Z_s^2\right)\right]ds\\&\quad+\int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K_{\beta^\varepsilon}^{\psi_Y}\left(Z_s^1,Z_s^2\right)\right]-E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_{Y_n}}\left(Z_s^1,Z_s^2\right)\right]\right)ds\\&= A_n+B_n+C_n\end{aligned}$$

Since ψ is globally Lipschitz, obviously A_n tends to 0 as n goes to $+\infty$.
We rewrite the term C_n under the form:

$$\begin{aligned}C_n &= \int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K_{\beta^\varepsilon}^{\psi_Y}\left(Z_s^1,Z_s^2\right)\right]-E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_{Y_n}}\left(Z_s^1,Z_s^2\right)\right]\right)ds\\&= \int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K_{\beta^\varepsilon}^{\psi_Y}\left(Z_s^1,Z_s^2\right)\right]-E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K_{\beta^\varepsilon}^{\psi_Y}\left(Z_s^1,Z_s^2\right)\right]\right)ds\\&\quad+\int_0^{t_n}E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K_{\beta^\varepsilon}^{\psi_Y-\psi_{Y_n}}\left(Z_s^1,Z_s^2\right)\right]ds+\int_0^{t_n}E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K_{\beta^\varepsilon-\beta^{\varepsilon_n}}^{\psi_{Y_n}}\left(Z_s^1,Z_s^2\right)\right]ds\end{aligned}$$

We easily prove the convergence of the law $P^{\varepsilon_n} \otimes P^{\varepsilon_n}$ to $P^{\varepsilon} \otimes P^{\varepsilon}$ when n goes to $+\infty$. For any $\phi \in \mathcal{C}_b^2(\mathbb{R})$, $\varepsilon > 0$ fixed, the function $(v, v_*) \longmapsto K_{\beta^\varepsilon}^{\phi}(v, v_*)$ is continuous and a simple computation shows that for any $v, v^* \in \mathbb{R}^2$

$$\left|K_{\beta^\varepsilon}^{\phi}(v, v_*)\right| \leq C \left\|\phi''\right\|_\infty \left(\int |\theta|^2 \beta^\varepsilon(\theta)\, d\theta\right) |v - v_*|^2 + |b^\varepsilon| \left\|\phi'\right\|_\infty |v - v_*| \qquad (4.21)$$

Using the bounds (4.12) of the moment of V^ε, we deduce that B_n and C_n converge to 0 as n goes to $+\infty$. So $d_n \to 0$ when n tends to $+\infty$.
Thus the function $(\varepsilon, t, Y) \to \mathcal{L}(\langle V_t^\varepsilon, Y\rangle)$ is weakly continuous on $(0, \frac{1}{2}] \times [0, T] \times \{Y \in \mathbb{R}^2 : |Y| = 1\}$.
If $\varepsilon = 0$: As $\left(P_{t_n}^{\varepsilon_n}\right)_{n \geq 0}$ and $(P_t^0)_{t \geq 0}$ are measure-solutions of the Boltzmann equation and of the Landau equation respectively, we rewrite d_n:

$$\begin{aligned} d_n &= E\left[\psi_Y(V_0) - \psi_{Y_n}(V_0)\right] + \int_{t_n}^{t} E_{P^0 \otimes P^0}\left[L^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right] ds \\ &\quad + \int_0^{t_n} \left(E_{P^0 \otimes P^0}\left[L^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right] - E_{P^{\varepsilon_n} \otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_{Y_n}}\left(Z_s^1, Z_s^2\right)\right]\right) ds \\ &= A_n' + B_n' + C_n' \end{aligned}$$

As in the previous case, we divide the term C_n' into three parts

$$\begin{aligned} C_n' &= \int_0^{t_n} \left(E_{P^0 \otimes P^0}\left[L^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right] - E_{P^{\varepsilon_n} \otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_{Y_n}}\left(Z_s^1, Z_s^2\right)\right]\right) ds \\ &= \int_0^{t_n} E_{P^0 \otimes P^0}\left[L^{\psi_Y}\left(Z_s^1, Z_s^2\right) - K_{\beta^{\varepsilon_n}}^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right] ds + \int_0^{t_n} E_{P^{\varepsilon_n} \otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_Y - \psi_{Y_n}}\left(Z_s^1, Z_s^2\right)\right] ds \\ &\quad + \int_0^{t_n} \left(E_{P^0 \otimes P^0}\left[K_{\beta^{\varepsilon_n}}^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right] - E_{P^{\varepsilon_n} \otimes P^{\varepsilon_n}}\left[K_{\beta^{\varepsilon_n}}^{\psi_Y}\left(Z_s^1, Z_s^2\right)\right]\right) ds \end{aligned}$$

We notice that for any $\phi \in \mathcal{C}_b^2(\mathbb{R})$, $v, v_* \in \mathbb{R}^2$

$$L^\phi(v, v_*) \leq C\left(\left\|\phi''\right\|_\infty |v - v_*|^2 + \left\|\phi'\right\|_\infty |v - v_*|\right)$$

Using the same arguments as above, the convergence of the Boltzmann kernel to the Landau kernel and the convergence of measure-solutions of the Boltzmann equation to the measure solution of the Landau equation, we obtain the convergence of d_n to 0 as $n \to +\infty$. Consequently, $\mathcal{L}\left(\langle V_{t_n}^{\varepsilon_n}, Y_n\rangle\right) \underset{n\to\infty}{\to} \mathcal{L}(\langle V_t^0, Y\rangle)$.
Finally, the map $(\varepsilon, t, Y) \longmapsto \mathcal{L}(\langle V_t^\varepsilon, Y\rangle)$ is weakly continuous on $[0, \frac{1}{2}] \times [0, T] \times \{Y \in \mathbb{R}^2 : |Y| = 1\}$. ∎
We now state a technical lemma of nondegeneracy of the law of V_t^ε:

Lemma 4.9 *Assume that (A) is satisfied, $V_0 \in \cap_{p<\infty} L^p$ and $E[V_0] = 0$. Let $t_0 > 0$ be fixed. There exists $\eta > 0$, $q > 0$ and $\xi > 0$ (depending on t_0) such that for any $\varepsilon \in [0, \frac{1}{2}]$, for any $t \in [t_0, T]$ and for any $X, Y \in \mathbb{R}^2$ with $|Y| = 1$,*

$$P\left(\langle V_t^\varepsilon - X, Y\rangle^2 > \eta, |V_t^\varepsilon|^2 < \xi\right) > q$$

where $\mathcal{L}(V_t^0)$ is the solution of the Landau equation at time t.

Proof. Fournier ([18] Lemma 3.22) proved this Lemma for any fixed $\varepsilon \geq 0$. So we study step by step his proof to state that η, q and ξ do not depend of ε.
Let us notice that it is enough to show that there exists $\eta > 0$, $q > 0$ such that for any $t \in [t_0, T]$, for any $\varepsilon \geq 0$ and for any $X, Y \in \mathbb{R}^2$ with $|Y| = 1$,

$$P\left(\langle V_t^\varepsilon - X, Y\rangle^2 > \eta\right) > 2q$$

Indeed, since $\sup_{\varepsilon\geq 0} E\left[\sup_{0\leq t\leq T} |V_t^\varepsilon|^2\right] \leq K$, using Bienayme-Tchebichev's inequality, there exists $\xi > 0$ such that $P\left(|V_t^\varepsilon|^2 < \xi\right) > 1 - q$ and ξ does not depend of ε.
Step1: Let $t \geq t_0$, $\varepsilon \geq 0$ and $|Y| = 1$ be fixed. The distribution of V_t^ε admits a density with respect to the Lebesgue measure, hence the distribution of $\langle V_t^\varepsilon, Y\rangle$ has a density on $\mathbb{R}$. Using the conservation of the momentum, we notice that $E\left(\langle V_t^\varepsilon, Y\rangle\right) = E\left(\langle V_0, Y\rangle\right) = 0$.
Consequently, there exists $\eta\left(t, \varepsilon, Y\right) > 0$ and $q\left(t, \varepsilon, Y\right) > 0$ such that

$$P\left(\langle V_t^\varepsilon, Y\rangle > \sqrt{\eta\left(\varepsilon, t, Y\right)}\right) > 2q\left(\varepsilon, t, Y\right) \text{ and } P\left(\langle V_t^\varepsilon, Y\rangle < -\sqrt{\eta\left(\varepsilon, t, Y\right)}\right) > 2q\left(\varepsilon, t, Y\right)$$

Step2: Using Lemma 4.8 and Portemanteau's Theorem, for any $t \in [t_0, T]$, for any $\varepsilon \in \left[0, \frac{1}{2}\right]$ and $Y \in \mathbb{R}^2$ with $|Y| = 1$, there is a neighborhood $\mathcal{V}\left(\varepsilon, t, Y\right)$ of $\left(\varepsilon, t, Y\right)$ such that for any $\left(\varepsilon', t', Y'\right) \in \mathcal{V}\left(\varepsilon, t, Y\right)$

$$P\left(\langle V_{t'}^{\varepsilon'}, Y'\rangle > \sqrt{\eta\left(\varepsilon, t, Y\right)}\right) > 2q\left(\varepsilon, t, Y\right)$$

We consider a finite covering $\cup_{i=1}^N \mathcal{V}\left(\varepsilon_i, t_i, Y_i\right)$ of the compact set $[0, \frac{1}{2}] \times [t_0, T] \times \{Y \in \mathbb{R}^2 : |Y| = 1\}$.
If we define $\eta = \inf_{i\leq N} \eta\left(\varepsilon_i, t_i, Y_i\right)$ and $q = \inf_{i\leq N} q\left(\varepsilon_i, t_i, Y_i\right)$, we notice that

$$P\left(\langle V_t^\varepsilon, Y\rangle > \sqrt{\eta}\right) > 2q$$

for any $(\varepsilon, t, Y) \in [0, \frac{1}{2}] \times [t_0, T] \times \{Y \in \mathbb{R}^2 : |Y| = 1\}$.
In the same way, $P\left(\langle V_t^\varepsilon, Y\rangle < -\sqrt{\eta}\right) > 2q$ for any $t \in [t_0, T]$ and $Y \in \mathbb{R}^2$ with $|Y| = 1$.
Step3: Let $X \in \mathbb{R}^2$, $t \in [t_0, T]$, $\varepsilon \geq 0$ and $|Y| = 1$ be fixed. If $\langle X, Y\rangle \leq 0$,

$$P\left(\langle V_t^\varepsilon - X, Y\rangle^2 > \eta\right) \geq P\left(\langle V_t^\varepsilon, Y\rangle > \sqrt{\eta}\right) > 2q$$

and if $\langle X, Y\rangle > 0$,

$$P\left(\langle V_t^\varepsilon - X, Y\rangle^2 > \eta\right) \geq P\left(\langle V_t^\varepsilon, Y\rangle < -\sqrt{\eta}\right) > 2q$$

The lemma is proved. ■

Proof. (Theorem 4.7)
We fix $t_0 > 0$, and we prove the theorem for every $t \geq t_0$ which suffices.
We choose k such that $k\left(y\right) = 1$ as soon as $|\det y| \geq d_0$ with $d_0 = \inf_{|\theta|\leq\theta_0} \left|\det\left(I + A\left(\theta\right)\right)\right| > 0$.

First of all, we prove that $(\det(F^\varepsilon H_t^\varepsilon))^{-1}$ belongs to L^p uniformly in ε for any $p \geq 1$ where F^ε is the random variable defined by

$$F^\varepsilon = \sup_{s\in[0,T]} \left\{ \left(1 + \frac{1}{4}\left(|V_s^\varepsilon|^2 + \xi\right)\right) \times \left(k\left(M_{s-}^\varepsilon\right) \left\| \left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^* \right\|_{op}^2\right)^{-1} \right\}$$

with $\left\| \left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^* \right\|_{op}^2$ the operator norm of $\left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^*$ and ξ defined by Lemma 4.9. To this aim, using Lemma 6.1, we estimate the quantity for $p \geq 2$

$$\begin{aligned} \mathbb{E} &= E\left[\int_{X\in\mathbb{R}^2} |X|^p \exp\left(-X^* F^\varepsilon H_t^\varepsilon X\right) dX\right] \\ &= \int_{\rho=0}^{\infty} \int_{|Y|=1} \rho^p E\left[\exp\left(-\rho^2 F^\varepsilon \times Y^* H_t^\varepsilon Y\right)\right] dY\, d\rho \end{aligned}$$

Thanks to Lemma 4.9, we can state (see the proof of [18] Theorem 3.24) that for $\rho > 0$, $t \geq t_0$ and $Y \in \mathbb{R}^2$ with $|Y| = 1$,

$$E\left[\exp\left(-\rho^2 F^\varepsilon \times Y^* H_t^\varepsilon Y\right)\right] \leq \exp\left(-q\left(t - t_0\right) \int_0^{\varepsilon\theta_0} \left(1 - e^{-\eta\rho^2\delta^\varepsilon(\theta)}\right) \beta_0^\varepsilon(\theta)\, d\theta\right)$$

with η independent of ε issue from Lemma 4.9. Thus, there exists a constant $K > 0$ (independent of ε) such that for any $p \geq 1$, $t > t_0$ and $\varepsilon > 0$

$$\mathbb{E} \leq K \int_0^\infty \rho^p \exp\left(-q\left(t - t_0\right) \int_0^{\varepsilon\theta_0} \left(1 - e^{-\rho^2\eta\delta^\varepsilon(\theta)}\right) \beta_0^\varepsilon(\theta)\, d\theta\right) d\rho$$

Moreover, using Appendix Lemma 6.3, we can write

$$\mathbb{E} \leq \frac{K}{\sqrt{\eta}} \left[\int_0^{\sqrt{k^\varepsilon}} \rho^p \exp\left(-K_1\rho^2\right) d\rho + \int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) d\rho\right]$$

where $K_1 = qC_1\left(t - t_0\right)$, $K_2 = qC_2\left(t - t_0\right)$ are positive constants independent of ε (with C_1 and C_2 constants defined in Lemma 6.3), and $k^\varepsilon = 2^{(r+2)}\theta_0^{-(r+1)}\varepsilon^{-2}/c$.
Let us study the first term.
We notice that $k^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} +\infty$, thus we can write for ε small enough

$$\begin{aligned} \int_0^{\sqrt{k^\varepsilon}} \rho^p \exp\left(-K_1\rho^2\right) d\rho &\leq 1 + \int_1^{\sqrt{k^\varepsilon}} \rho^{2p+1} \exp\left(-K_1\rho^2\right) d\rho \\ &\leq 1 + \int_1^{k^\varepsilon} \rho^p \exp\left(-K_1\rho\right) d\rho \\ &\leq C_{K_1,p}\left(1 + \left(k^\varepsilon\right)^p \exp\left(-K_1\, k^\varepsilon\right)\right) \end{aligned}$$

with $C_{K_1,p}$ a positive constant independent of ε. Consequently, this integral is uniformly bounded in ε, $\varepsilon \in \left(0, \frac{1}{2}\right]$.

Let us now study the second term

$$\int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) d\rho.$$

We notice that $K_2 \varepsilon^{-\frac{4}{r+1}} \longrightarrow +\infty$ and $k^\varepsilon \longrightarrow +\infty$ when ε tends to 0. Let us recall that $r \in (1,3)$, then for any $q \geq 1$, for any $\varepsilon > 0$, $\rho^q \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \longrightarrow 0$ as ρ goes to $+\infty$. Consequently, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, for any $\rho > \sqrt{k^\varepsilon}$,

$$\rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \leq \rho^{-2}$$

and

$$\begin{aligned} \int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) d\rho &\leq \int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^{-2} d\rho \\ &\leq (k^\varepsilon)^{-1/2} \end{aligned}$$

which implies that

$$\int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) d\rho \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

We then deduce that for any $p \geq 1$ there exists K_p independent of ε such that

$$E\left[\int_{X \in \mathbb{R}^2} |X|^p \exp\left(-X^* F^\varepsilon H_t^\varepsilon X\right) dX\right] \leq K_p$$

We conclude that for any $t > t_0$, $(\det F^\varepsilon H_t^\varepsilon)^{-1} = \left((F^\varepsilon)^2 \det H_t^\varepsilon\right)^{-1}$ belongs to L^p uniformly in ε for any $p \geq 1$.
Moreover, it is possible to choose k such that $F^\varepsilon \leq F_1^\varepsilon \times F_2^\varepsilon$ with

$$F_1^\varepsilon = \sup_{[0,T]} \left(1 + \frac{1}{4} |V_s^\varepsilon|^2 + \frac{\xi}{4}\right) \text{ and } F_2^\varepsilon = \sup_{[0,T]} \left(k\left(M_s^\varepsilon\right) \left\| \left(\left(M_s^\varepsilon\right)^{-1}\right)^* \right\|_{op}^2\right)^{-1}.$$

The random variable F_1^ε has moments of all orders independent of ε thanks to (4.12). From the definition (4.17) of M^ε, we easily prove that the moment of $\sup_{t \in [0,T]} |M_t^\varepsilon|$ are uniformly bounded in ε. So we obtain that F_2^ε has the same property thanks to Theorem 4.6 and the following estimate (see the proof of [18] Theorem 3.24) ,

$$F_2^\varepsilon \leq \sup_{[0,T]} \left(1 + |M_s^\varepsilon|^8\right) \times \sup_{[0,T]} \left|\left(M_s^\varepsilon\right)^{-1}\right|^2.$$

Thus, for any $p \geq 2$, there exists $C_p > 0$ such that for any $\varepsilon \in (0, \frac{1}{2}]$,

$$\begin{aligned} E\left[|\det H_t^\varepsilon|^{-p}\right] &= E\left[|F^\varepsilon|^{2p} \times |\det\left(F^\varepsilon H_t^\varepsilon\right)|^{-p}\right] \\ &\leq E\left[|F^\varepsilon|^{4p}\right]^{\frac{1}{2}} E\left[|\det\left(F^\varepsilon H_t^\varepsilon\right)|^{-2p}\right]^{\frac{1}{2}} \\ &\leq C_p < \infty. \end{aligned}$$

The theorem 4.7 is proved. ■

Consequently, according to Theorem 4.7 and Theorem 4.6, for any $p \geq 1$ there exists a constant C_p such that for any $\varepsilon > 0$

$$E[|det(DV_t^\varepsilon)|^{-p}] \leq C_p.$$

Then, Theorem 1.1 on the convergence of the function-solutions is proved.

5 Some numerical results

Guérin and Méléard (see Chapter 3 Section 4) built a Monte-Carlo algorithm of simulation by a conservative particle method following the asymptotic of grazing collisions. In this section, we will use this algorithm to simulate the convergence of the function-solutions of the Boltzmann equation to the function-solution of the Landau equation.
Let us consider an initial measure

$$P_0 = \frac{1}{2}\left(\delta_{(-1,1)} + \delta_{(1,-1)}\right)$$

and the following approximation β^ε of the grazing collisions

$$\beta^\varepsilon(\theta) = \frac{1}{2\pi\varepsilon^3 \sin(\frac{\theta}{2\varepsilon})^2} \mathbb{I}_{\varepsilon \leq |\frac{\theta}{\varepsilon}| \leq \pi}.$$

Let $\varepsilon > 0$ be fixed. We define $(V^{\varepsilon,1n}, ..., V^{\varepsilon,nn})$ the n-particles system in $(\mathbb{R}^2)^n$ introduced by Guérin and Méléard (Chapter 3 Section 4) which is a $(\mathbb{R}^2)^n$-valued pure-jump Markov process with generator defined for $\phi \in C_b((\mathbb{R}^2)^n)$ by

$$\frac{1}{n-1} \sum_{1\leq i,j \leq n} \int_{-\pi}^{\pi} \frac{1}{2}\Big(\phi\left(v^n + \mathbf{e_i}.A(\theta)(v_i - v_j) + \mathbf{e_j}.A(\theta)(v_j - v_i)\right) - \phi(v^n)\Big)\beta^\varepsilon(\theta)d\theta.$$

Here $v^n = (v_1, ..., v_n)$ denotes the generic point of $(\mathbb{R}^2)^n$ and $\mathbf{e_i} : h \in \mathbb{R}^2 \mapsto \mathbf{e_i}.h = (0, ..., 0, h, 0, ..., 0) \in (\mathbb{R}^2)^n$ with h at the i-th place.
They have proved in Chapter 3 Theorem 4.1 that the empirical measure $\mu^\varepsilon = \frac{1}{n}\sum_{i=1}^n \delta_{V^{\varepsilon,in}}$ on $\mathcal{P}(\mathbb{D}_T)$ associated with the system converges to the measure-solution P of the Landau equation when n tends to $+\infty$ and ε tends to 0. Then, for any $\phi \in \mathcal{C}_b(\mathbb{D}_T)$,

$$\frac{1}{n}\sum_{i=1}^n \phi\left(V^{\varepsilon,in}\right) \underset{\substack{n\to+\infty \\ \varepsilon\to 0}}{\longrightarrow} \int_{\mathbb{R}^2} \phi(v)\, P(dv) \tag{4.22}$$

Let us explain how we simulate the function-solution from the particle system.
Let $t > 0$ be fixed. Thanks to the convergence of the empirical measure μ^ε, the function $g_{h_1,h_2}^{\varepsilon,n}$ on $\mathbb{R}^2$ defined by

$$x = (x_1, x_2) \longmapsto g_{h_1,h_2}^{\varepsilon,n}(x) = \frac{1}{nh_1h_2}\sum_{i=1}^n \mathbb{I}_{x_1 < V_{t,1}^{\varepsilon,in} \leq x_1+h_1} . \mathbb{I}_{x_2 < V_{t,2}^{\varepsilon,in} \leq x_2+h_2}$$

converges to $F_{h_1,h_2}(x) = \frac{1}{h_1 h_2} \int\int_{(x_1,x_1+h_1]\times(x_2,x_2+h_2]} P_t(dv)$ as $n \to +\infty$ and $\varepsilon \to 0$ for any step $h_1, h_2 > 0$. Moreover, the function $F_{h_1,h_2}(x)$ is pointwise convergent to the density $f(t,x)$ of the probability measure P_t on $\mathbb{R}^2$ when $h_1, h_2 \to 0$. Thus, the function $g^{\varepsilon,n}_{h_1,h_2}$ is an estimator of the function-solution f of the Landau equation.

For the simulations, we consider 500000 particles and we choose the step $h_1 = h_2 = 0.1$.

We first observe the behaviour of the entropy of the solution f of the Landau equation which is defined by

$$H(t) = \int_{\mathbb{R}^2} f(t,v)\ln(f(t,v))dv$$

Replacing the density f with its estimator $g^{\varepsilon,n}_{h_1,h_2}$in the expression of H, we simulate the entropy and we observe in Figure 1. its evolution in ε when $t = 0.005$.

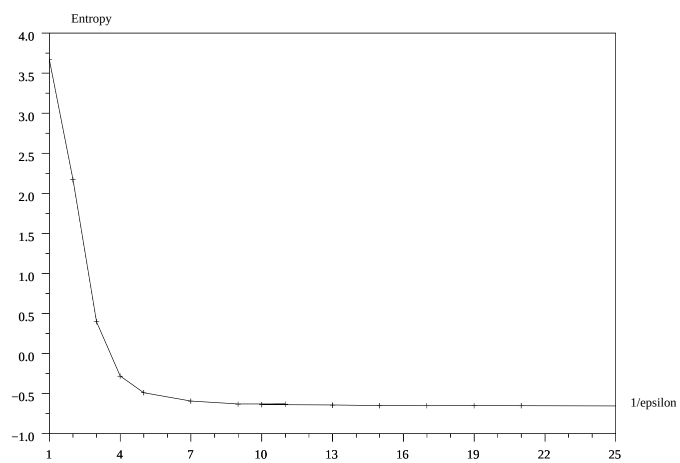

Figure 1. Evolution in $1/\varepsilon$ of the entropy.

The choice of $\varepsilon = 0.1$ seems to be reasonable to describe the Landau behaviour.

From now, we fix $\varepsilon = 0.1$ and we observe in Figure 2. the decay in time of the entropy of the solution of the Landau equation. We note that the entropy converge to -2.833 when t goes to infinity.

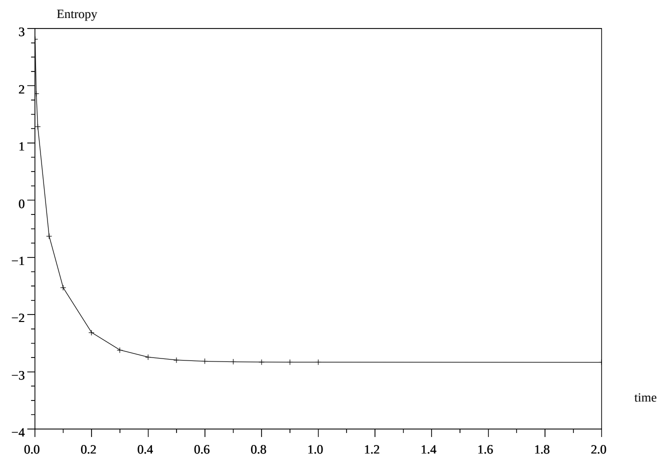

Figure 2. Evolution in time of the entropy.

Moreover, Figure 3. gives the behaviour in time of the function-solution f of the Landau equation ($\varepsilon = 0.1$). We observe a result proved by Villani [40]: the convergence of f to a Maxwellian function. As the 2-order moments of f are given by the following expression (see [40] Section 2.)

$$\int_{\mathbb{R}^2} v_i v_j f(t,v)dv = (1-e^{-8t})\delta_{ij}\int_{\mathbb{R}^2}\frac{|v|^2}{2}P_0(dv) + e^{-8t}\int_{\mathbb{R}^2} v_i v_j P_0(dv),$$

the function f converges to the following Maxwellian function when the time goes to infinity

$$M(v) = \frac{1}{2\pi}exp\left(\frac{-|v|^2}{2}\right).$$

Let us notice that the entropy associated to M is equal to $-1-\ln(2\pi) \approx 2.838$ which is approximately the limit value obtain in Figure 2.

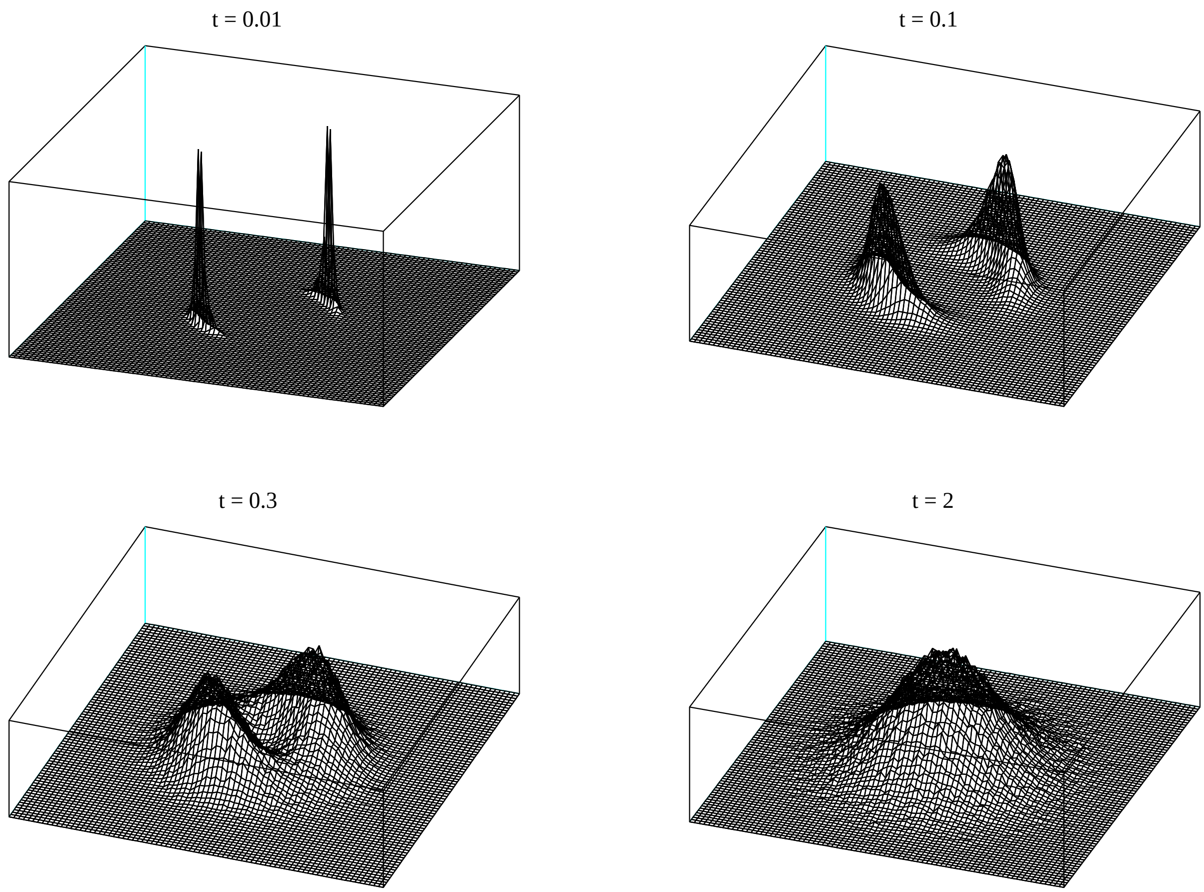

Figure 3. Evolution in time of the function-solution of the Landau equation.

6 Appendix

We first mention a useful lemma proved in [5], page 92.

Lemma 6.1 *For any $p \geq 1$, there exists a constant C_p such that for any 2×2 symmetric positive matrix A,*

$$(\det A)^{-p} \leq C_p \int_{X \in \mathbb{R}^2} |X|^{4p-2} e^{-X^* A X} dX.$$

Let us now give some estimates on the function δ^ε introduced in (4.14) and defined on $[-\varepsilon\theta_0, \varepsilon\theta_0]$ by

$$\delta^\varepsilon(\theta) = c\varepsilon^{1-r} |\theta|^{r+1} \left(1 - \frac{|\theta|}{\varepsilon\theta_0}\right)$$

with $c \leq \left[\theta_0^r \left(r2^{r-1} + r + 2 + \theta_0\right)\right]^{-1}$.

Lemma 6.2 *Assume that* $\varepsilon \in \left(0, \frac{1}{2}\right]$.
- $\delta^\varepsilon \in \bigcap_{p\geq 1} L^p\left(\beta_0^\varepsilon(\theta)\, d\theta\right)$ *with moments uniformly bounded in* ε.
- Let $d^\varepsilon(\theta) = \delta^\varepsilon(\theta) + \left|(\delta^\varepsilon)'(\theta)\right| + r2^{r-1}\frac{\delta^\varepsilon(\theta)}{|\theta|}$. *Then* $d^\varepsilon \in \bigcap_{p\geq 2} L^p\left(\beta_0^\varepsilon(\theta)\, d\theta\right)$ *with moments uniformly bounded in* ε.

Proof. Let us recall that $\beta_0^\varepsilon(\theta) = \varepsilon^{-3}\beta_0(\theta/\varepsilon) = k_0\varepsilon^{r-3}|\theta|^{-r}\,\mathbb{I}_{|\theta|\leq\varepsilon\theta_0}$. Thanks to the choice of the constant c, the function δ^ε is bounded by 1. Then, it is enough to estimate its first moment:

$$\begin{aligned}\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\,\beta_0^\varepsilon(\theta)\, d\theta &\leq ck_0\varepsilon^{-2}\int_0^{\varepsilon\theta_0}\theta d\theta \\ &\leq \frac{ck_0\theta_0^2}{2}\end{aligned}$$

Then the first point of the Lemma is proved.
We notice that the function d^ε is also bounded by 1. So we just have to study the integral $\int_0^{\varepsilon\theta_0}\left(d^\varepsilon(\theta)\right)^2\beta_0^\varepsilon(\theta)\, d\theta$. The function d^ε is the sum of three terms. We already know that $\int_0^{\varepsilon\theta_0}\left(\delta^\varepsilon(\theta)\right)^2\beta_0^\varepsilon(\theta)\, d\theta$ is uniformly bounded in ε. We estimate now the two other terms:
- Study of the second term:

$$(\delta^\varepsilon)'(\theta) = \frac{c(r+1)}{\varepsilon^{r-1}}\theta^r\left(1-\frac{\theta}{\varepsilon\theta_0}\right) - \frac{c}{\varepsilon^r\theta_0}\theta^{r+1} \quad \text{if } \theta\in[0,\varepsilon\theta_0].$$

Thus

$$\begin{aligned}\int_0^{\varepsilon\theta_0}\left((\delta^\varepsilon)'(\theta)\right)^2\beta_0^\varepsilon(\theta)\, d\theta &\leq k_0c^2\varepsilon^{-(r+1)}\int_0^{\varepsilon\theta_0}\left((r+1)+\frac{\theta}{\varepsilon\theta_0}\right)^2\theta^r d\theta \\ &\leq 2k_0c^2\theta_0^{r+1}\frac{r^2+4r+4}{r+3}.\end{aligned}$$

- Study of the third term:

$$\begin{aligned}\int_0^{\varepsilon\theta_0}\left(\frac{\delta^\varepsilon(\theta)}{\theta}\right)^2\beta_0^\varepsilon(\theta)\, d\theta &\leq c^2k_0\varepsilon^{-(r+1)}\int_0^{\varepsilon\theta_0}\theta^r d\theta \\ &\leq \frac{c^2\theta_0^{r+1}k_0}{r+1}.\end{aligned}$$

The lemma is proved. ■

Lemma 6.3 *Let* $r\in(1,3)$ *and* $x>0$. *Let* $k^\varepsilon = 2^{r+2}\theta_0^{-(r+1)}\varepsilon^{-2}/c$.
a) For any $x\geq k^\varepsilon$ *there exists a constant* $C_1>0$ *independent of* ε *such that*

$$\int_0^{\varepsilon\theta_0}\left(1-e^{-x\delta^\varepsilon(\theta)}\right)\beta_0^\varepsilon(\theta)\, d\theta \geq C_1\varepsilon^{-\frac{4}{r+1}}x^{\frac{r-1}{r+1}}.$$

b) For any $x\leq k^\varepsilon$ *there exists a constant* $C_2>0$ *independent of* ε *such that*

$$\int_0^{\varepsilon\theta_0}\left(1-e^{-x\delta^\varepsilon(\theta)}\right)\beta_0^\varepsilon(\theta)\, d\theta \geq C_2 x.$$

Proof. Since $\beta_0^\varepsilon(\theta) = k_0 \varepsilon^{r-3} |\theta|^{-r} \mathbb{I}_{|\theta| \leq \varepsilon\theta_0}$, we write

$$\begin{aligned} I(\varepsilon, x) &= \int_0^{\varepsilon\theta_0} \left(1 - e^{-x\delta^\varepsilon(\theta)}\right) \beta_0^\varepsilon(\theta)\, d\theta = k_0 \varepsilon^{r-3} \int_0^{\varepsilon\theta_0} \left(1 - e^{-x\delta^\varepsilon(\theta)}\right) \theta^{-r} d\theta \\ &\geq k_0 \varepsilon^{r-3} \int_0^{\frac{\varepsilon\theta_0}{2}} \left(1 - e^{-x\tilde{\delta}^\varepsilon(\theta)}\right) \theta^{-r} d\theta \end{aligned}$$

with $\tilde{\delta}^\varepsilon(\theta) = \frac{c}{2}\varepsilon^{1-r}\theta^{r+1}$. We notice that $k^\varepsilon = 1/\tilde{\delta}^\varepsilon(\frac{\varepsilon\theta_0}{2})$.
We use in the proof the following inequality:

$$\text{if } x \in [0,1],\ 1 - e^{-x} \geq \frac{x}{2}.$$

a) The function $\tilde{\delta}^\varepsilon$ is increasing and its inverse function is

$$\left(\tilde{\delta}^\varepsilon\right)^{-1}(y) = \left(\frac{2\varepsilon^{r-1}}{c} y\right)^{1/(r+1)} \quad \text{for } y > 0.$$

If $x \geq 1/\tilde{\delta}^\varepsilon(\frac{\varepsilon\theta_0}{2})$, we notice that $\left(\tilde{\delta}^\varepsilon\right)^{-1}(x^{-1}) \leq \frac{\varepsilon\theta_0}{2}$, thus

$$I(\varepsilon, x) \geq k_0 \varepsilon^{r-3} \int_0^{\left(\tilde{\delta}^\varepsilon\right)^{-1}(x^{-1})} \left(1 - e^{-x\tilde{\delta}^\varepsilon(\theta)}\right) \theta^{-r} d\theta.$$

As $\tilde{\delta}^\varepsilon$ is an increasing function, $x\tilde{\delta}^\varepsilon(\theta) \leq 1$ for any $\theta \in \left[0, \left(\tilde{\delta}^\varepsilon\right)^{-1}(x^{-1})\right]$. Thus, we conclude

$$\begin{aligned} I(\varepsilon, x) &\geq \frac{k_0}{2} \varepsilon^{r-3} x \int_0^{\left(\tilde{\delta}^\varepsilon\right)^{-1}(x^{-1})} \tilde{\delta}^\varepsilon(\theta)\, \theta^{-r} d\theta \\ &\geq \frac{k_0 c}{4} \varepsilon^{-2} x \int_0^{\left(\tilde{\delta}^\varepsilon\right)^{-1}(x^{-1})} \theta d\theta \\ &\geq \frac{k_0 c}{8} \left(\frac{2}{c}\right)^{\frac{2}{r+1}} \varepsilon^{-\frac{4}{r+1}} x^{\frac{r-1}{r+1}}. \end{aligned}$$

b) If $x \leq 1/\tilde{\delta}^\varepsilon(\frac{\varepsilon\theta_0}{2})$, then we have clearly $x\tilde{\delta}^\varepsilon(\theta) \leq 1$ and

$$\begin{aligned} I(\varepsilon, x) &\geq \frac{k_0}{2} \varepsilon^{r-3} x \int_0^{\frac{\varepsilon\theta_0}{2}} \tilde{\delta}^\varepsilon(\theta)\, \theta^{-r} d\theta \\ &\geq \frac{k_0 c}{4} \varepsilon^{-2} x \int_0^{\frac{\varepsilon\theta_0}{2}} \theta d\theta \\ &\geq \frac{k_0 c \theta_0^2}{32} x. \end{aligned}$$

■

Appendice: Code de simulation

Voici le code de simulation, voir Chapitre 3 Section 6.1, du moment d'ordre 4 à l'instant $t = 0.1$ d'une solution de l'équation de Landau en dimension 3, via l'asymptotique des collisions rasantes, pour $\gamma = -0.8$ et pour la condition initiale suivante:

$$Q_0(dv) = \mathbb{I}_{[-1/2,1/2]^3}dv$$

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>
#include <time.h>


const int    n = 50000;
const int    NbExp = 100;
const double M =6.0;
const double e = 0.5;
const double t = 0.1;
const double PI = 3.1415926536;

// loi uniforme
double unif()
{
 double res;
 res = (random()+0.0)/RAND_MAX;
 return res;
}

// loi de theta
double LoiAngle (double L10, double e0, double u0)
{
 double res, aux;
 aux = L10*u0;
 res = 2.0*e0*(PI/2.0-atan(aux));
```

```
 return res;
}

// loi normale
double normal(double u0, double v0)
{
 double res;
 res = sqrt(-2.0*log(u0))*cos(2.0*PI*v0);
 return res;
}

// modellisation d'une loi de poisson par le T.L.C.
long poisson(double n0, double lambda0)
{
 double res1;
 long res;
 res1 = lambda0 +sqrt(lambda0)*n0;
 res = (long)floor(res1);
 return res;
}

// fonction psi
double psi(double x, double M0, double M10)
{
 double res;
 res = M0;
 if (x>M10)
  res = pow(x,(-0.8));
 return res;
}


main(void)
{
 long   a, b, c, d, g, h, i, j, k, Nbs;
 double B1, L1, L, L2, M1, ma, lambda;
 double aux, phi, theta, ct, stsp, stcp, x, s, gauss, norm, norm01, psi1;
 double v0, v1, v2, I0, I1, I2, J0, J1, J2;
 double v[3][n];
 double moyenne;
 double vect[NbExp];
 FILE   *f;

 double unif();
 double LoiAngle(double, double, double);
```

```
double normal(double, double);
long   poisson(double, double);
double psi(double, double, double);

srand((unsigned)time(NULL));

L=1.0/e;
L2 = e/2.0; L1 = 1.0/(tan(L2));
B1 = L1/(PI*e*e);
f = fopen("Evolt01.dat","w");

fprintf(f,"\n*********************\n");
fprintf(f,"* Equation de Landau *\n");
fprintf(f,"*********************\n\n");
fprintf(f,"1/2-Norme L1 de beta : %f\n",B1);
fprintf(f,"\nValeur de e : %f\n",e);
fprintf(f,"Valeur de L : %f\n",L);
fprintf(f,"Valeur de M : %f\n",M);
fprintf(f,"Valeur de t : %f\n",t);

M1 = pow(M,-1.25);
lambda = PI*n*B1*M*t;
moyenne = 0.0;
fprintf(f,"\nLambda : %f\n",lambda);


for (a=1;a<=NbExp;a++)
{
 // nombre de sauts
 gauss = normal(unif(),unif());
 Nbs = poisson(gauss,lambda);

 // initialisation des vitesses
 for (b=0;b<n;b++)
 {
  v[0][b] = unif()-0.5;
  v[1][b] = unif()-0.5;
  v[2][b] = unif()-0.5;
 }

  // initialisation du compteur
  k = 0;

  // choc des particules
  for (c=1;c<=Nbs;c++)
```

```
{
 // choix de deux particules differentes
 i = (long)floor(n*unif());
 j = (long)floor(n*unif());
 while (i==j)
  j = (long)floor(n*unif());

 v0 = v[0][i]-v[0][j];
 v1 = v[1][i]-v[1][j];
 v2 = v[2][i]-v[2][j];
 norm = sqrt(v0*v0+v1*v1+v2*v2);

  // test lie a l'indicatrice
  x = M*unif();
  psi1 = psi(norm,M,M1);
  if (x<=psi1)
  {
   k = k+1;

   // choix de theta
   theta = LoiAngle(L1,e,unif());

   // choix de phi uniformement sur [0;2*pi]
   phi = 2.0*PI*unif();

   stcp = sin(theta)*cos(phi)/2.0;
   stsp = sin(theta)*sin(phi)/2.0;
   ct   = (cos(theta)-1.0)/2.0;

   // coordonnees du repere mobile
   norm01 = sqrt(v0*v0+v1*v1);
   if (norm01==0.0)
   {
    I0 = v2; I1 = 0.0; I2 = 0.0;
    J0 = 0.0; J1 = fabs(v2); J2 = 0.0;
   }
   else
   {
    I0 = -norm*v1/(norm01+0.0); I1 = norm*v0/(norm01+0.0); I2 = 0.0;
    J0 = -v0*v2/(norm01+0.0); J1 = -v1*v2/(norm01+0.0); J2 = norm01;
   }

   // nouvelles vitesses
   v[0][i] = v[0][i]+ct*v0+stcp*I0+stsp*J0;
   v[1][i] = v[1][i]+ct*v1+stcp*I1+stsp*J1;
```

```
      v[2][i] = v[2][i]+ct*v2+stcp*I2+stsp*J2;
      v[0][j] = v[0][j]-ct*v0-stcp*I0-stsp*J0;
      v[1][j] = v[1][j]-ct*v1-stcp*I1-stsp*J1;
      v[2][j] = v[2][j]-ct*v2-stcp*I2-stsp*J2;
     }
     // affichage de certains resultats
     if ((c%2000000==0)&&(a%20==0))
     {
      fprintf(f,"\nValeur de a : %d\n",a);
      fprintf(f,"Valeur de c : %d\n",c);
      fprintf(f,"Norm : %lf\n",norm);
      fprintf(f,"Valeur de V[0][1] : %lf\n",v[0][1]);
      fprintf(f,"Valeur de V[0][2] : %lf\n",v[0][2]);
     }
    }

   if (a%20==0)
   {
    fprintf(f,"\nValeur de a : %d\n",a);
    fprintf(f,"Nombre de sauts : %d\n",Nbs);
    fprintf(f,"Nombre de test accepte : %d\n",k);
    fflush(f);
    fflush(fM);
   }

   ma = 0.0;
   // moyenne sur nbre particules des moments d'ordre 4
   for (d=0;d<n;d++)
   {
    aux = v[0][d]*v[0][d]+v[1][d]*v[1][d]+v[2][d]*v[2][d];
    ma = ma+aux*aux;
   }
   ma = ma/(n+0.0);


   // moyenne des moments sur le nbre d'experiences
   moyenne = moyenne+ma;
  }

  moyenne = moyenne/(NbExp+0.0);
  fprintf(f,"\n Moyenne des moments d'ordre 4 %lf\n",moyenne);

  fclose(f);
  fclose(fM);
}
```

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