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Introduction

L'étude des problèmes de perturbations singulières occupe une place importante parmi les problèmes mathématiques étudiés de nos jours et dans le passé. En effet, ce type de problèmes se présente souvent dans la nature; ils sont caractérisés par la présence d'un "petit" paramètre qui, lorsqu'il tend vers zéro, donne lieu généralement à un phénomène physique appelé "couches limites". Dans les études théoriques, ces problèmes apparaissent de deux façons différentes : Soit physiquement, le petit paramètre étant une donnée du problème, par exemple un coefficient de diffusion, ou un coefficient de viscosité, etc.., et dans ce cas on cherche à comparer la solution de ce problème à celle du problème limite qui correspond (au moins formellement) à la valeur nulle de ce paramètre. Soit inversement, les perturbations singulières peuvent apparaître indirectement comme un outil de démonstration; ce sont les méthodes dites de "régularisation" ou de "viscosité artificielle" qui consistent à renforcer la régularité de l'équation en ajoutant un terme plus régulier affecté d'un petit coefficient. Plus précisément, si l'on a à résoudre un problème aux limites pour un opérateur $P(x, \partial/\partial x)$, on construit une famille d'opérateurs $P(x, \partial/\partial x; \varepsilon)$ plus faciles à étudier et tels que $P(x, \partial/\partial x; 0) = P(x, \partial/\partial x)$, puis on essaye de passer à la limite en $\varepsilon \to 0$. C'est le cas par exemple lorsque le problème initial (non régularisé) n'admet pas de solutions.

En l'absence du terme régularisant, qui est en général le terme le plus régulier dans une équation, on aboutit à un problème limite d'un type différent du problème régularisé et dont la solution peut être régulière ou singulière. Dans cette thèse nous

nous intéressons à des problèmes de perturbations singulières dont la solution limite est singulière. Dans la théorie des équations aux dérivées partielles, ceci se produit en particulier lorsque le problème est non linéaire et la solution limite n'est pas régulière ou encore lorsque le domaine est peu régulier. De tels problèmes apparaissent par exemple en mécanique des fluides parfaits en présence de tourbillons (vortex) singuliers à 0, 1 ou 2 dimensions (voir e.g. Chorin & Majda [CM]); c'est le cas aussi pour le problème du bord arrière pour l'écoulement autour d'une aile; en élasticité lorsque des flambements apparaissent, par exemple dans les problèmes de bifurcations, le problème de la charge critique pour une poutre ou une plaque élastoplastique (voir e.g. Do [Do], Lods & Madare [LoM], Mardare [Md], etc.) et dans la navigation à savoir le problème du contrôle automatique de la trajectoire d'un mobile, problèmes dont plusieurs aspects sont traités dans [AVK], [Bar], [Bau], [GlY] et dans la bibliographie de ces articles. Plusieurs études sont faites autour de ce type de problèmes et beaucoup de problèmes intéressants de perturbations singulières se situent dans ce cadre là. C'est aussi l'objet de cette thèse dans laquelle on étudie deux classes de perturbations singulières ayant un caractère commun lié au fait que la solution limite, *i.e.* lorsque le petit paramètre ε est nul, possède une ou plusieurs singularités sur des parties du domaine.

Les deux premiers chapitres (constituant la première partie de la thèse) sont consacrés à l'étude d'un problème de perturbations singulières lié à des équations aux dérivées partielles linéaires dont le second membre est une fonction discontinue (voire une distribution). Notons que plusieurs propriétés et techniques dans la théorie des équations différentielles avec second membre régulier demeurent toujours valables pour les équations différentielles avec second membre singulier. En revanche, le concept de solution pour ce type d'équations doit être précisé.
Généralement, pour étudier un problème de perturbations singulières on suppose que toutes les données du problème (second membre, données sur le bord, données initiales pour les problèmes d'évolution,..) sont suffisamment régulières et alors la présence d'un petit paramètre, que l'on fait tendre vers zéro, donne en général des couches limites

apparaissant près de la frontière du domaine et ceci à cause de la différence produite entre les traces sur le bord de la solution régularisée et de la solution limite. Ainsi, une question naturelle consiste à étudier ce qui se passe si on considère en plus une fonction source discontinue ou singulière pour un problème de perturbations singulières. Pour cela et dans le but d'acquérir une expérience pour ce type de problèmes, on a choisi d'étudier deux problèmes réguliers l'un associé au laplacien et l'autre au bilaplacien. L'analyse mathématique menée dans cette première partie montre que les singularités de la fonction source (second membre de l'équation) donnent naissance à des couches limites dans la solution du problème régularisé, *i.e.* la solution régularisée subit une grande variation et sa norme explose dans H^1 lorsque le petit paramètre tend vers zéro, près du bord ce qui est bien connu, et aussi au voisinage des surfaces de singularités de la fonction source.
Dans le premier chapitre on s'intéresse à une régularisation elliptique du second ordre, tandis que dans le deuxième chapitre, on étudie un problème dont la régularisation est d'ordre 4.

La deuxième classe de perturbations singulières considérée dans cette thèse correspond à l'utilisation d'une méthode de régularisation en ajoutant une viscosité artificielle à l'équation initiale. Plus précisément, on s'intéresse dans la deuxième partie de la thèse, à l'étude d'une régularisation elliptique, proposée par Temam [Te1], [Te2], d'un problème de calculs de variations, le problème des surfaces minimales. Ceci nous amène donc à étudier un problème de perturbations singulières lié aux surfaces minimales. D'habitude, pour les problèmes de perturbations singulières, on utilise une méthode suggérée par Lions [Li] qui consiste à introduire des fonctions appelées "correcteurs" qui, moyennant certaines propriétés de la théorie des couches limites, vérifient des équations plus simples que celles du problème régularisé et qui sont généralement explicitement données. Toutefois, pour le problème des surfaces minimales, la solution n'existe pas nécessairement. Ainsi, on ne s'intéresse pas aux correcteurs puisque l'intérêt porte alors sur la solution du problème initial (limite) plutôt que sur la solution régularisée. Dans ce cadre là, on décrit dans le troisième chapitre une approximation définie explicitement pour la solution du problème des surfaces minimales régularisées de révolution.

Historiquement, la théorie des surfaces minimales a attiré l'attention de plusieurs mathématiciens depuis le $XVIII^{ème}$ siècle, mais une importante avance a été apportée par le mathématicien belge Joseph Plateau ($1801 - 1883$) qui a commencé à croire que ses expériences sur les configurations des films de savons sont les bases d'une nouvelle branche en mathématique qui sera appelée plus tard par Lebesgue "le problème de Plateau" dans son fameux article [L]. Ce problème consiste à trouver une surface d'aire minimale limitée par un contour donné. De nombreuses applications provenant de la physique, la chimie, l'architecture et la biologie font appelle à cette théorie. Les propriétés des films et des bulles de savons ont été intensivement étudiées par Euler ($1707-1783$), Laplace, Lagrange ($1736 - 1813$), Legendre, Riemann, Schwarz, ..etc. Parmi les problèmes dérivant du problème de Plateau, le problème des surfaces minimales qui est lui même un cas particulier des surfaces ayant une courbure moyenne donnée, leur courbure moyenne étant en fait nulle.

Actuellement, le problème des surfaces minimales occupe toujours une place particulière dans la théorie des équations aux dérivées partielles dont une très large bibliographie porte sur ce sujet. Les études modernes du problème sont dues en particulier à De Girogi et l'équipe italienne autour de lui, cf par exemple Bombieri, De Giorgi & Miranda [BDM], Giusti [G], etc. Le problème que nous traitons dans cette partie est aussi lié à un problème considéré par Marcellini & Miller [MM], article dans lequel ils étudient le même problème qu'ici considéré sur un disque.

0.1 1^{ere} partie: Couches limites à l'intérieur du domaine.

0.1.1 Chapitre 1: Problèmes elliptiques associés au laplacien.

sec1.1]

On considère l'équation du second ordre :

$$-\varepsilon \Delta u^\varepsilon + u^\varepsilon = H, \qquad \text{dans} \quad \Omega, \tag{0.1.1}$$

où Ω est un canal de R^3 donné par

$$\Omega = (0, L_1) \times (0, L_2) \times (-h, h),$$

et on pose

$$\Gamma = (0, L_1) \times (0, L_2) \times \{-h, h\}.$$

Içi, H est la fonction d'Heaviside dans la direction orthogonale au canal ($H = H(z)$), et $\Phi = \Phi(x, y, z)$ est une fonction donnée sur Γ aussi régulière que nécessaire. On a choisi de travailler sur un canal dans le but d'éviter les technicités dues à la courbure du domaine. On associe à (0.1.1) la condition aux limites

$$u^\varepsilon = \Phi, \qquad \text{sur} \quad \Gamma, \tag{0.1.2}$$

et la condition de périodicité :

$$u^\varepsilon \text{ est periodique de periodes } L_1 \text{ et } L_2 \text{ dans les directions } x \text{ et } y, \tag{0.1.3}$$

Le problème limite associé à (0.1.1) est simplement obtenu en prenant formellement $\varepsilon = 0$ dans (0.1.1), ce qui donne

$$u^0 = H(z) = \begin{cases} 1 & \text{si } z \in (0, h), \\ \\ 0 & \text{si } z \in (-h, 0). \end{cases} \tag{0.1.4}$$

On voit que u^0 ne vérifie pas (0.1.2) ce qui produit, pour u^ε, des couches limites près de la frontière $z = \pm h$; et en outre u^0 contient une discontinuité en $z = 0$ et celle-ci donne naissance, pour la solution régularisée u^ε, à une couche limite non classique en $z=0$, donc à l'intérieur du domaine. Dans une telle situation, on ne peut pas espérer une convergence dans $H^1(\Omega)$ de u^ε vers u^0, lorsque ε tend vers zéro (cette convergence entraînant la convergence des traces, voir e.g. [Li]). En effet, d'une part on sait que $u^\varepsilon \in H^4(\Omega)$ puisque $H \in L^2(\Omega)$, et d'autre part u^0 n'appartient pas à $H^1(\Omega)$ puisque $\partial u^0/\partial z = \delta \notin L^2(\Omega)$. D'où la nécessité d'introduire un correcteur pour prendre en compte ces singularités et pour décrire cette nouvelle catégorie de couches limites; c'est l'objet de cette partie dont le but essentiel est de trouver le premier terme dans le développement asymptotique de u^ε solution de (0.1.1)–(0.1.3).

Le premier résultat de cette partie est le suivant :

.1]

Théorème 1 *: Pour u^ε solution de (0.1.1)–(0.1.3), il existe un correcteur θ^ε fonction de x, y et $z/\varepsilon^{1/2}$, dont l'expression est donnée explicitement, et il existe une constante C indépendante de ε tels que*

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{L^2(\Omega)} \leq C(\Omega,\Phi)\ \varepsilon, \tag{0.1.5}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^1(\Omega)} \leq C(\Omega,\Phi)\ \varepsilon^{1/2}. \tag{0.1.6}$$

Ensuite, on s'intéresse à des fonctions sources de plus en plus singulières. Par exemple en remplaçant dans l'équation (0.1.1) H par $\delta^{(k)}$, on aboutit à l'étude du problème suivant :

$$-\varepsilon\Delta u_k^\varepsilon + u_k^\varepsilon = \delta^{(k)}, \qquad \text{dans } \Omega, \tag{0.1.7}$$

$$u_k^\varepsilon = \Phi, \qquad \text{sur } \Gamma, \tag{0.1.8}$$

où u_k^ε vérifient les mêmes conditions aux limites en $z=\pm h$ et de périodicité que u^ε.

Le problème limite est simplement

$$u_k^0 = \delta^{(k)}, \qquad \text{dans } \Omega. \tag{0.1.9}$$

Le but est toujours d'expliciter le premier terme dans le développement asymptotique de u_k^ε, et on a un résultat analogue au Théorème 1 :

1.2]

Théorème 2 *: Pour tout $k \in N$, il existe un correcteur θ_k^ε fonction de x, y, $z/\varepsilon^{1/2}$ et k et une constante C dépendant des données du problème, sauf ε, tels que*

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{L^2(\Omega)} \leq C(\Omega, \Phi, k)\ \varepsilon^{5/4}, \tag{0.1.10}$$

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^1(\Omega)} \leq C(\Omega, \Phi, k)\ \varepsilon^{3/4}. \tag{0.1.11}$$

Remarque 1 : Ici aussi le correcteur θ_k^ε est donné explicitement. D'autre part on utilise la lettre C pour designer une constante dont la valeur peut changer d'une place à l'autre.

Remarque 2 : Les résultats annoncés ci-dessus peuvent être étendus aux problèmes analogues dans un canal Ω de R^n, en prenant en compte convenablement la courbure du domaine.

0.1.2 Chapitre 2: Problèmes elliptiques d'ordre 4 (bilaplacien).

Dans ce paragraphe on s'intéresse à l'étude d'un problème de perturbations singulières lié à un opérateur d'ordre 4. Plus précisément, on considère le système suivant :

$$\varepsilon \Delta^2 u^\varepsilon + u^\varepsilon = H, \qquad \text{dans} \quad \Omega, \tag{0.1.12}$$

$$u^\varepsilon = \Phi_0, \qquad \frac{\partial u^\varepsilon}{\partial \nu} = \Phi_1, \qquad \text{sur} \quad \Gamma, \tag{0.1.13}$$

et

$$u^\varepsilon \text{ est periodique de periodes } L_1 \text{ et } L_2 \text{ dans les directions } x \text{ et } y, \tag{0.1.14}$$

où Ω est un canal comme au paragraphe 0.1.1, et Φ_0 et Φ_1 sont deux fonctions aussi régulières que nécessaires.

Le problème limite ($\varepsilon = 0$) est donné par (0.1.4).

L'analyse mathématique montre que la fonction u^ε solution de (0.1.12)–(0.1.14) contient deux couches limites l'une classique se produisant près du bord $z = \pm h$ et l'autre non classique due à la discontinuité de u^0 qui est elle même conséquence du fait que la fonction

source H est discontinue. Par conséquent, on introduit un correcteur qui pourra prendre en compte ces deux singularités. Celui-ci est explicitement obtenu en utilisant des techniques appropriées de la théorie des couches limites.
On arrive à ce résultat

Théorème 3 : *Soit u^ε la solution de (0.1.12)–(0.1.14) et u^0 la solution de (0.1.4). Il existe alors un correcteur θ^ε dépendant uniquement de x, y et $z/\varepsilon^{1/4}$, et une constante C indépendante de ε tels que :*

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{L^2(\Omega)} \leq C\ \varepsilon^{5/8}, \tag{0.1.15}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^1(\Omega)} \leq C\ \varepsilon^{3/8}, \tag{0.1.16}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^2(\Omega)} \leq C\ \varepsilon^{1/8}. \tag{0.1.17}$$

Les estimations (0.1.15)–(0.1.17) constituent en fait une confirmation pour le choix du correcteur proposé à priori de manière intuitive. Ces résultats prouvent que le correcteur proposé θ^ε est un “bon” choix.
En renforçant la singularité de la fonction source, on se ramène à étudier le comportement asymptotique de la solution du problème suivant :

$$\varepsilon \Delta^2 u_k^\varepsilon + u_k^\varepsilon = \delta^{(k)}, \qquad \text{dans} \ \ \Omega, \tag{0.1.18}$$

$$u_k^\varepsilon = \Phi_0, \qquad \frac{\partial u_k^\varepsilon}{\partial \nu} = \Phi_1, \qquad \text{sur} \ \ \Gamma, \tag{0.1.19}$$

et

$$u_k^\varepsilon \text{ est periodique de periodes } L_1 \text{ et } L_2 \text{ dans les directions } x \text{ et } y. \tag{0.1.20}$$

Le problème limite s'écrit alors

$$u_k^0 = \delta^{(k)}, \qquad \text{dans} \ \ \Omega. \tag{0.1.21}$$

Comme précédemment, on a un résultat de convergence qui confirme le choix du correcteur proposé et donne ainsi le premier terme dans le développement asymptotique de u_k^ε :

Théorème 4 : *Pour u_k^ε solution de (0.1.18)-(0.1.20) et u_k^0 solution de (0.1.21), il existe un correcteur θ_k^ε, fonction de $x, y, z/\varepsilon^{1/4}$ et k, et une constante positive C dépendant des données sauf ε tels que :*

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{L^2(\Omega)} \leq C\ \varepsilon^{5/8}, \tag{0.1.22}$$

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^1(\Omega)} \leq C\ \varepsilon^{3/8}, \tag{0.1.23}$$

$$\|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^2(\Omega)} \leq C\ \varepsilon^{1/8}. \tag{0.1.24}$$

Remarque 3 : On observe que, lorsque ε tend vers zéro, la vitesse de convergence dans les estimations obtenues pour une régularisation d'ordre 4 est plus faible que celle correspondant à une régularisation du second ordre.

0.2 2^{eme} partie: Un problème de perturbation singulière lié au problème des surfaces minimales.

0.2.1 Chapitre 3: Concept de solution généralisée.

Position du problème.

Il est bien connu que les surfaces de courbure moyenne donnée sont les points critiques d'une certaine fonctionnelle. Pour des surfaces qui sont graphes d'une fonction $u : \Omega \to R$, ces surfaces sont solutions du problème variationnel suivant :

$$\inf_{u=\Phi\ sur\ \partial\Omega} \int_\Omega \{\sqrt{1+|\nabla u|^2} - Hu\}\ dx, \tag{0.2.1}$$

où H est la courbure moyenne donnée de cette surface, Φ est une fonction donnée sur $\partial\Omega$ et Ω est un ouvert borné de R^n.

Les surfaces minimales sont solutions du problème (0.2.1) avec $H = 0$ et dans ce cas (0.2.1) est équivalent à un problème de Dirichlet formé par l'équation d'Euler-Lagrange associé à (0.2.1) et la condition de Dirichlet donnée sur le bord, soit

$$-\,div(T(\nabla u)) = 0, \quad \text{dans} \quad \Omega, \tag{0.2.2}$$

$$u = \Phi, \quad \text{sur} \quad \Gamma, \tag{0.2.3}$$

où $\Gamma = \partial\Omega$ et où l'on a noté

$$T(v) = \frac{v}{\sqrt{1+v^2}}.$$

En fait, l'existence d'une solution pour le problème (0.2.2)-(0.2.3) ci-dessus n'est pas toujours garantie, tandis que l'unicité est facile à démontrer. Plusieurs études faites montrent que le problème d'existence est fortement lié à la géométrie du domaine Ω et plus précisément à la géométrie de sa frontière $\partial\Omega$. Pour cette raison Bombieri, De Giorgi & Miranda [BDM], par des méthodes purement géométriques, et Temam [Te1], par un argument de dualité, ont défini une solution faible u de (0.2.2)-(0.2.3) appelée "solution généralisée" dont on donnera la définition exacte dans le paragraphe 0.2.1 (u est unique sauf peut être, pour des domaines très particuliers, à une constante additive près). La difficulté essentielle pour l'existence d'une solution pour (0.2.2)-(0.2.3) est que, généralement, une solution de (0.2.2) ne vérifie pas forcement (0.2.3). Ainsi, il est nécessaire de bien comprendre le comportement de u près de Γ. Pour cette solution u, Temam montre dans [Te1] qu'on a, sur Γ, la propriété suivante

$$\frac{\partial u}{\partial \nu} = \pm\infty, \qquad \text{là où} \quad u \neq \Phi. \tag{0.2.4}$$

Cette condition expose clairement le problème d'existence d'une solution classique (régulière) pour (0.2.2)-(0.2.3).

Cependant, en raison d'autres difficultés liées à la nonlinéarité de l'équation analogue à (0.2.2) pour les surfaces à courbure moyenne donnée, celles-ci produisent en particulier des "points verticaux" à l'intérieur du domaine , ce qui rend les solutions de ce problème de moins en moins régulière, voir e.g. Atkinson & Peletier [AP] et Atkinson, Peletier & Serrin [APS].

La solution généralisée.

Dans ce paragraphe, nous donnons la définition d'une solution généralisée en adoptant celle proposée par Temam dans [Te1] et [Te2] :

Définition 0.2.1 *: Une fonction $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ est dite solution généralisée de (0.2.1) ou (0.2.2)-(0.2.3) si elle vérifie :*

(i) u est une solution de (0.2.2), l'équation d'Euler associé à (0.2.1).

(ii) Toute suite minimisante $\{v_m\}_m$ de (0.2.1) converge vers u au sens suivant :

$$\begin{aligned} v_m &\to u, \quad \text{dans} \quad L^1(\Omega)/R, \\ \frac{\partial v_m}{\partial x_i} &\to \frac{\partial u}{\partial x_i}, \quad \text{dans} \quad L^1(\Omega'), \quad \forall\ \Omega' \subset\subset \Omega,\ i = 1,..,n. \end{aligned}$$

Comme signalé précédemment, l'existence de cette solution généralisée est obtenue par Temam en considérant le problème dual de (0.2.1) et en utilisant les techniques de l'analyse convexe ou encore par la méthode de régularisation qui sera l'objet du paragraphe suivant. Bombieri, De Giorgi & Miranda et Giusti [G] ont obtenu cette solution par extension de la fonctionnelle de (0.2.1) aux classes des fonctions à variations bornées.
En résumé, si la condition (0.2.3) n'est pas vérifiée, on définit alors une solution généralisée au sens de la Définition 0.2.1. Cependant, il est naturel de voir dans quels cas la condition aux limites (0.2.3) est satisfaite; c'est l'objet de la proposition suivante

Proposition 0.2.1 *(Temam [Te1]). Supposons qu'il existe une solution $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ de (0.1.2) vérifiant :*

$$\exists x_0 \in \partial\Omega \text{ tel que } \limsup_{\substack{x \to x_0 \\ x \in \Omega}} |\nabla v(x)| < +\infty. \tag{0.2.5}$$

Alors, pour $\Phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, il existe $u \in W^{1,1}(\Omega) \cap W^{1,\infty}(\Omega) \cap L^\infty(\Omega'), \forall \Omega' \subset\subset \Omega$, qui est solution de (0.2.2) et vérifie

$$u = \Phi, \quad p.p., \tag{0.2.6}$$

sur une partie γ de Γ de mesure non nulle, et plus précisément sur

$$\{x/x \in \Gamma;\ \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| < +\infty\}.$$

De plus, Temam montre dans [Te2] que si $\gamma \subset \Gamma$ est telle que $\gamma \neq \emptyset, \gamma$ est de classe C^3 et de courbure moyenne positive et $\Phi \in L^\infty(\Omega)$ est continue sur γ, alors $u = \Phi$ sur γ.

Problème elliptique régularisé des surfaces minimales.

Afin d'obtenir la solution généralisée mentionnée au paragraphe 0.2.1, Temam [Te1] introduit une méthode de régularisation du problème des surfaces minimales. Nous suivons içi cette méthode et nous introduisons alors, pour $\varepsilon > 0$, le problème elliptique régularisé lié aux surfaces minimales comme suit :

$$
\begin{aligned}
-\,\varepsilon\,\Delta u^{\varepsilon} - \operatorname{div}(T(\nabla u^{\varepsilon})) &= 0, \quad \text{in } \Omega, && (0.2.7)\\
u^{\varepsilon} &= \Phi, \quad \text{on } \Gamma. && (0.2.8)
\end{aligned}
$$

Ainsi, l'équation (0.2.7) est uniformément elliptique et la fonctionnelle associée à (0.2.7)-(0.2.8) est strictement convexe. Par conséquent l'existence de $u^{\varepsilon} \in C^{2,\alpha}(\overline{\Omega})$ solution de (0.2.7)-(0.2.8) est obtenu facilement par des méthodes classiques de calcul des variations. Mis à part l'intérêt du problème régularisé du point de vue physique, celui-ci constitue un bon chemin pour construire une solution généralisée pour le problème (0.2.2)-(0.2.3). De plus, en prenant formellement $\varepsilon = 0$, on retrouve le problème exact des surfaces minimales. Ceci met en évidence l'intérêt de bien comprendre le comportement de u^{ε}. C'est le but essentiel de ce troisième chapitre. Dans ce cadre là, Temam [Te1] montre que u^{ε} tend vers la solution généralisée u^0 lorsque $\varepsilon \to 0$. Plus précisément, le résultat suivant est prouvé :

Théorème 5 *: (cf. [Te1]). Soit $\Phi \in W^{1,1}(\Omega) \cap L^{\infty}(\Omega)$. Pour u^{ε} solution de (0.2.7)-(0.2.8), il existe une unique u^0 solution généralisée de (0.2.2)-(0.2.3) telle que :*

$$u^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} u^0 \textit{ fortement dans } L^{\alpha}(\Omega), \quad 1 < \alpha < \frac{n}{n-1}, \tag{0.2.9}$$

$$u^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} u^0 \textit{ fortement dans } H^1(\Omega'), \quad \forall\, \Omega' \subset\subset \Omega. \tag{0.2.10}$$

Finalement, le problème régularisé lié aux surfaces minimales forme un problème de perturbations singulières non classique pour lequel les méthodes classiques de perturbations singulières [Li], [VL] ne sont pas valables. En effet, la solution limite, comme démontré dans plusieurs travaux y compris celui de ce troisième chapitre, a une dérivée infinie sur certaines parties du bord.

0.2.2 Chapitre 4: Les surfaces minimales de révolution.

Dans cette partie sur la quelle se concentre notre effort, nous nous intéressons à l'étude du problème des surfaces minimales en dimension deux dans le cas de la symétrie radiale. Malgré la simplification apportée par la réduction de la dimension de l'espace, ce problème s'est avéré difficile à résoudre à cause de la spécificité de sa nonlinéarité. Nous pensons aussi qu'il peut servir de modèle à des problèmes avec des géométries plus compliquées qui feront l'objet d'études ultérieures.
Dans tout ce qui suit, on considère Ω une couronne de R^2 donnée par :

$$\Omega = \Big\{x \in R^2/0 < R_1 < |x| < R_2\Big\}, \quad (R_1 < R_2).$$

Vu la géométrie du domaine, on cherche des solutions radiales pour le problème variationnel (0.2.1) (dans le cas $H = 0$), qui se transforme ainsi, à une constante près, au problème suivant :

$$\inf_{u=\Phi\ sur\ \partial\Omega} \int_{R_1}^{R_2} \{r\sqrt{1+(u_r)^2}\}\ dr. \tag{0.2.11}$$

L'équation d'Euler-Lagrange associée à (0.2.11) avec la condition de Dirichlet sur le bord s'écrit

$$-\frac{d}{dr}\Big(rT(u_r^0)\Big) = 0, \qquad \text{dans} \quad \Omega, \tag{0.2.12}$$

$$u^0 = \Phi, \qquad \text{sur} \quad \Gamma; \tag{0.2.13}$$

où l'on a noté $\Gamma = \{|x| = R_i, i = 1, 2.\}$, et

$$\Phi(r) = \begin{cases} K_1 & \text{si} \quad r = R_1, \\ & \qquad\qquad (K_1 \neq K_2), \\ K_2 & \text{si} \quad r = R_2, \end{cases} \tag{0.2.14}$$

K_1, K_2 étant deux constantes réelles.

De même, les équations associées au problème régularisé (0.2.7)- (0.2.8) s'écrivent dans ce cas :

$$\frac{d}{dr}\left\{r\Big(\varepsilon u^\varepsilon_r + T(u^\varepsilon_r)\Big)\right\} = 0, \qquad \text{in } \Omega, \tag{0.2.15}$$

$$u^\varepsilon = \Phi, \qquad \text{on } \Gamma. \tag{0.2.16}$$

Les problèmes (0.2.12)-(0.2.13) et (0.2.15)- (0.2.16) étant stables par translation et par symétrie, on peut, sans perte de généralité, supposer que Φ est de la forme :

$$\Phi(r) = \begin{cases} K & \text{si} \quad r = R_1, \\ & \\ 0 & \text{si} \quad r = R_2, \end{cases} \qquad (K > 0). \tag{0.2.17}$$

Avant d'annoncer le résultat principal de ce chapitre, nous donnons d'abord quelques résultats préliminaires.

Proposition 0.2.2 *: Il existe une constante "seuil" $K_* > 0$ qui dépend de R_1 et R_2 telle que :*

(i) Pour tout $0 \leq K \leq K_*$, le problème des surfaces minimales (0.2.12)-(0.2.13) admet une solution (unique).

(ii) Pour tout $K > K_*$, le problème (0.2.12)-(0.2.13) n'admet pas de solution et la solution généralisée u^0 vérifie alors

$$\frac{du^0}{dr} = -\infty, \quad \text{en} \quad r = R_1.$$

Dans la suite de ce travail, on suppose que $K > K_*$. Ceci nous met dans le cadre des problèmes de perturbations singulières pour lesquels la solution limite est singulière. Par conséquent on ne peut espérer une convergence de u^ε vers u^0 ni dans $H^1(\Omega)$ ni même dans $L^\infty(\Omega)$. Cependant, on a le résultat suivant :

Proposition 0.2.3 *Pour tout $\delta > 0$, on a*

$$u^\varepsilon_r \underset{\varepsilon \to 0}{\longrightarrow} u^0_r, \quad \textit{fortement dans} \quad L^\infty(R_1 + \delta, R_2). \tag{0.2.18}$$

Ensuite, on remarque que l'équation (0.2.15) est simplement équivalente à

$$\varepsilon u_r^\varepsilon + T(u_r^\varepsilon) = -\frac{R_\varepsilon}{r}, \quad \text{dans } \Omega, \tag{0.2.19}$$

où $R_\varepsilon > 0$ est une constante en r.

Ainsi, il est clair que l'étude du comportement de u_r^ε est fortement liée à celle de R_ε quand $\varepsilon \to 0$. En fait, on démontre le résultat suivant :

Proposition 0.2.4 *: La constante (en r) R_ε décroît quand $\varepsilon \searrow 0$, et elle vérifie :*

$$\lim_{\varepsilon \to 0} R_\varepsilon = R_1. \tag{0.2.20}$$

Finalement, on a le résultat principal de ce chapitre qui définit explicitement une approximation de u^ε tout en confirmant ce choix par une estimation dans l'espace de Sobolev $H^1(\Omega)$ (qui implique convergence des traces) :

Théorème 6 *Pour u^ε la solution du problème régularisé (0.2.15)-(0.2.16), il existe une fonction v^ε, explicitement donnée, et une constante C qui dépend des données sauf de ε, telles que :*

$$\|u^\varepsilon - v^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1/14}\sqrt{|\ln \varepsilon|}. \tag{0.2.21}$$

Conclusions et Perspectives.

Cette thèse m'a permis d'acquérir une bonne expérience dans le domaine des perturbations singulières et de certains aspects des équations aux dérivées partielles non linéaires. En particulier, je me suis familiarisé avec un nouveau type de couches limites qui sont les couches limites provenant des singularités du second membre d'une équation aux dérivées partielles. Cette étude ouvre, pour moi, un nouvel axe de recherche; par exemple on pourra étendre l'étude aux problèmes suivants :

- Les problèmes d'évolution associés au laplacien et au bilaplacien.
- Des problèmes non linéaires avec second membre singulier.

De même pour le problème des surfaces minimales, on pourra attaquer d'autres difficultés liées à ce sujet :

- Généraliser le résultat obtenu sur d'autres domaines.
- Étudier le problème d'évolution associé sur la couronne et ensuite dans un domaine quelconque.

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Première partie

Interior boundary layers generated by a singularity in the source function

Chapitre 1

Boundary layers generated by singularity in the source function

Abstract

Our aim in this article is to study singularly perturbed problems which display boundary layers in the interior of the domain. These interior boundary layers which supplement the usual boundary layers at the boundary, are generated by discontinuities in the data. Second order linear elliptic one-dimensional and multidimensional problems are considered in this article.

1.1 Introduction

Numerous problems of mathematical physics, in particular in fluid and solid mechanics, contain a small parameter leading to singular perturbations (see e.g. [C], [E], [Fri]). Classically boundary layers may occur at the boundary for small values of the parameter, when the boundary conditions for the perturbed and limit problems do not match; an extensive literature exists on that questions in the mathematical and engineering literatures (see among many other references [BP], [CW], [Lag],[Li], [OSY], [S], [TW], [VL]). Less classical are the boundary layers appearing inside the domain, generally when the problem is nonlinear and the limit solution of the limit problem is not smooth; this occurs for instance with vortex structures in fluid mechanics or when folding occurs in elasticity. As a very simplified model for such outstanding problems, we consider here such singular perturbations problems in the second order linear elliptic case. We start in Section 1.2 by studying one dimensional problems which are explicitly solvable ; these problems are needed for the higher dimensional case which we consider in Section 1.3 and Section 1.4. Higher order elliptic operators and time dependent problems will be considered elsewhere.

1.2 The one-dimensional case

In this section, we restrict ourselves to 1-D problems. The equation and boundary conditions for u^ε read

$$-\varepsilon\ \frac{d^2\,u^\varepsilon}{dz^2} + u^\varepsilon = H, \qquad in \ \ \Omega, \tag{1.2.1}$$

$$u^\varepsilon = 0, \qquad on \ \ \partial\Omega, \tag{1.2.2}$$

where

$$\Omega = (-h, h), \qquad \partial\Omega = \{-h, h\},$$

and H is the Heaviside function

$$H = H(z) = \begin{cases} 1 & \text{if } z \in (0, h), \\ \\ 0 & \text{if } z \in (-h, 0). \end{cases} \tag{1.2.3}$$

By taking, $\varepsilon = 0$ in (1.2.1), we obtain formally the limit problem which reduces to

$$u^0 = H. \tag{1.2.4}$$

Concerning the convergence as $\varepsilon \to 0$ of u^ε (solution of (1.2.1)-(1.2.2)) to u^0 (solution of (1.2.4)), one can observe, firstly, that u^ε and u^0 do not have the same boundary values; this produces a boundary layer at $z = h$ that will not be our primary aim here. Secondly we notice that u^0 contains a discontinuity in the interior of Ω, at $z = 0$, which is not present in u^ε as we will see. Our primary aim is to study the boundary layers generated near $z=0$; in particular we aim to derive the "corrector" which absorbs the main part of the singularity at $z=0$.

First we need to resolve (1.2.1)-(1.2.2) explicitly. A natural way to find u^ε is by insuring its regularity. If we look, at least, for u^ε square integrable, then d^2u^ε/dz^2 is square integrable as well and then :

$$u^\varepsilon \in H^2(\Omega) \hookrightarrow C^1(\overline{\Omega}),$$

by the Sobolev injection theorem. Hence we write

$$u^\varepsilon(z) = u_1^\varepsilon(z)\chi_{(0,h)} + u_2^\varepsilon(z)\chi_{(-h,0)}, \tag{1.2.5}$$

where u_1^ε and u_2^ε must satisfy

$$-\varepsilon\ \frac{d^2\, u_1^\varepsilon}{dz^2} + u_1^\varepsilon = 1, \qquad 0<z<h, \tag{1.2.6}$$

$$-\varepsilon\ \frac{d^2\, u_2^\varepsilon}{dz^2} + u_2^\varepsilon = 0, \qquad -h<z<0, \tag{1.2.7}$$

$$u_1^\varepsilon(h) = 0, \qquad u_2^\varepsilon(-h) = 0, \tag{1.2.8}$$

$$u_1^\varepsilon(0^+) = u_2^\varepsilon(0^-), \qquad \frac{du_1^\varepsilon}{dz}(0^+) = \frac{du_2^\varepsilon}{dz}(0^-). \tag{1.2.9}$$

From (1.2.6) and (1.2.7) we see that

$$\begin{aligned} u_1^\varepsilon(z) &= 1 + A_\varepsilon \exp\big(\frac{z-h}{\sqrt{\varepsilon}}\big) + B_\varepsilon \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big), && (1.2.10)\\ u_2^\varepsilon(z) &= C_\varepsilon \exp\big(\frac{z}{\sqrt{\varepsilon}}\big) + D_\varepsilon \exp\big(-\frac{z+h}{\sqrt{\varepsilon}}\big), && (1.2.11)\end{aligned}$$

then we apply (1.2.8) and (1.2.9) to (1.2.10)-(1.2.11) and end up with a linear system of the form (here $a=c=-1$ and $b=d=0$)

$$(S)\begin{cases} A_\varepsilon + \alpha(\varepsilon)B_\varepsilon & = \ a,\\ \qquad\qquad \alpha(\varepsilon)C_\varepsilon + D_\varepsilon & = \ b,\\ \alpha(\varepsilon)A_\varepsilon + B_\varepsilon - C_\varepsilon - \alpha(\varepsilon)D_\varepsilon & = \ c,\\ \alpha(\varepsilon)A_\varepsilon - B_\varepsilon - C_\varepsilon + \alpha(\varepsilon)D_\varepsilon & = \ d,\end{cases}$$

where we have set

$$\alpha(\varepsilon) = \exp\,(-\frac{h}{\sqrt{\varepsilon}}). \tag{1.2.12}$$

The explicit resolution of (S) yields

$$\begin{aligned} A_\varepsilon &= -1 + \frac{\alpha(\varepsilon)}{2}\frac{1+\alpha(\varepsilon)+(\alpha(\varepsilon))^2}{(1+(\alpha(\varepsilon))^2)(1+\alpha(\varepsilon))}, && (1.2.13)\\ B_\varepsilon &= \frac{-1-\alpha(\varepsilon)-2(\alpha(\varepsilon))^2}{2(1+(\alpha(\varepsilon))^2)(1+\alpha(\varepsilon))}, && (1.2.14)\\ C_\varepsilon &= \frac{1-\alpha(\varepsilon)}{2(1+(\alpha(\varepsilon))^2)(1+\alpha(\varepsilon))}, && (1.2.15)\\ D_\varepsilon &= -\frac{\alpha(\varepsilon)}{2}\frac{1-\alpha(\varepsilon)}{2(1+(\alpha(\varepsilon))^2)(1+\alpha(\varepsilon))}, && (1.2.16) \end{aligned}$$

which can be approximated as follows :

$$\begin{aligned} A_\varepsilon &= \overline{A}_\varepsilon + O(\alpha(\varepsilon)), \quad \overline{A}_\varepsilon = -1, && (1.2.17)\\ B_\varepsilon &= \overline{B}_\varepsilon + O(\alpha(\varepsilon))^2), \quad \overline{B}_\varepsilon = -\frac{1}{2}, && (1.2.18)\\ C_\varepsilon &= \overline{C}_\varepsilon + O(\alpha(\varepsilon)), \quad \overline{C}_\varepsilon = \frac{1}{2}, && (1.2.19)\\ D_\varepsilon &= \overline{D}_\varepsilon + O(\alpha(\varepsilon)), \quad \overline{D}_\varepsilon = 0. && (1.2.20) \end{aligned}$$

Hence, we deduce an approximate value of u^ε denoted by $\overline{u}^\varepsilon$:

$$\overline{u}^\varepsilon(z) = \begin{cases} 1 - \exp\,(\dfrac{z-h}{\sqrt{\varepsilon}}) - \dfrac{1}{2}\exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0 < z < h, \\ \\ \dfrac{1}{2}\exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h < z < 0, \end{cases} \tag{1.2.21}$$

and

$$u^\varepsilon(z) = \overline{u}^\varepsilon(z) + \begin{cases} O(1)\exp\,(\dfrac{z-2h}{\sqrt{\varepsilon}}) + O(1)\exp\,(-\dfrac{z+2h}{\sqrt{\varepsilon}}), & 0 < z < h, \\ \\ O(1)\exp\,(\dfrac{z-h}{\sqrt{\varepsilon}}) + O(1)\exp\,(-\dfrac{z+2h}{\sqrt{\varepsilon}}), & -h < z < 0. \end{cases} \tag{1.2.22}$$

Note that the three functions $\exp\,(\dfrac{z-2h}{\sqrt{\varepsilon}})$, $\exp\,(-\dfrac{z+2h}{\sqrt{\varepsilon}})$ and $\exp\,(\dfrac{z-h}{\sqrt{\varepsilon}})\chi_{(-h,0)}$ verify

$$\|f\|_{H^m(-h,h)} = O\Big(\varepsilon^{1/4-m/2}\alpha(\varepsilon)\Big). \tag{1.2.23}$$

Hence we can write

$$u^\varepsilon(z) = \overline{u}^\varepsilon(z) + e.l.t, \tag{1.2.24}$$

where e.l.t. stands for exponentially lower terms, and we have

$$\|u^\varepsilon - \overline{u}^\varepsilon\|_{L^2(\Omega)} \leq c\ \varepsilon^{1/4}\ \alpha(\varepsilon), \tag{1.2.25}$$

$$\|u^\varepsilon - \overline{u}^\varepsilon\|_{H^1(\Omega)} \leq c\ \varepsilon^{-1/4}\ \alpha(\varepsilon). \tag{1.2.26}$$

On the other hand, (1.2.4) and (1.2.21) imply

$$\|\overline{u}^\varepsilon - u^0\|_{L^2(\Omega)} \leq c\ \varepsilon^{1/4}, \tag{1.2.27}$$

$$\|\overline{u}^\varepsilon - u^0\|_{H^1(\Omega)} \leq c\ \varepsilon^{-1/4}. \tag{1.2.28}$$

By the triangle inequality and (1.2.25) to (1.2.28), we deduce that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \leq c\ \varepsilon^{1/4}, \tag{1.2.29}$$

$$\|u^\varepsilon - u^0\|_{H^1(\Omega)} \leq c\ \varepsilon^{-1/4}. \tag{1.2.30}$$

Remark 1 *:* All the C's which have been (and will be) used represent a constant independent of ε which may be different at each place.

Remark 2 *:* The estimates (1.2.29) and (1.2.30) show that we have convergence of u^ε to u^0 in the strong topology of $L^2(\Omega)$, but this is not verified in $H^1(\Omega)$-norm, because of the singularity of H at $z = 0$ (and of the difference in boundary values at $z = h$). Our object will be to introduce a corrector corresponding to this singularity : this will be done in Section 1.3 in a more general context.

In the next part, we resolve equations similar to (1.2.1)-(1.2.2), but with stronger singularities of the source function. Beside its intrinsic interest, this will be useful subsequently. Hence, one considers the problem

$$-\ \varepsilon\ \frac{d^2\ v^\varepsilon}{dz^2} + v^\varepsilon = \delta^{(k)}, \qquad in\ \ (-h,h), \tag{1.2.31}$$

$$v^{\varepsilon} = 0, \qquad on \quad z = \pm h, \tag{1.2.32}$$

where $k \in N$, and δ stands for the Dirac measure at 0, and $\delta^{(k)}$ for its kth derivative. Here, we have to consider two cases, depending on either k is even or odd :

Case 1 : If $k = 2q$, $q \in N$, then for all $1 \leq j \leq q+1$, we define recursively

$$v_{2j}^{\varepsilon} = v_{2j-2}^{\varepsilon} + \frac{\delta^{(2(q-j))}}{\varepsilon^{j}}, \tag{1.2.33}$$

where $v_0^{\varepsilon} = v^{\varepsilon}$. We introduce furthermore this convention

$$\delta^{(-1)} = H, \qquad and \qquad \delta^{(-2)} = G, \tag{1.2.34}$$

with

$$G(z) = \int_0^z H(t)\,dt. \tag{1.2.35}$$

Next, we add the relation (1.2.33) for $j = 1, .., q+1$ and we find

$$v^{\varepsilon} = v_{2q+2}^{\varepsilon} - \sum_{j=1}^{q+1} \Big(\frac{\delta^{(2(q-j))}}{\varepsilon^{j}}\Big), \tag{1.2.36}$$

and $v_{2q+2}^{\varepsilon}(k = 2q)$ satisfies

$$-\,\varepsilon\ \frac{d^2 v_{2q+2}^{\varepsilon}}{dz^2} + v_{2q+2}^{\varepsilon} = \frac{G}{\varepsilon^{q+1}}, \qquad in \ \ (-h, h), \tag{1.2.37}$$

$$v_{2q+2}^{\varepsilon}(-h) = 0, \qquad v_{2q+2}^{\varepsilon}(h) = \frac{h}{\varepsilon^{q+1}}, \tag{1.2.38}$$

$$v_{2q+2}^{\varepsilon} \quad \text{is continous at} \quad z = 0, \tag{1.2.39}$$

$$\Big[\frac{dv_{2q+2}^{\varepsilon}}{dz}\Big](z = 0) \ = \ 0. \tag{1.2.40}$$

The resolution of (1.2.37) to (1.2.40) leads to a linear system of type (S) with $a = b = c = 0$ and $d = -\frac{\sqrt{\varepsilon}}{\varepsilon^{q+1}}$. Thus, we can obtain the explicit expression of v_{2q+2}^{ε} of which we give an approximation $\overline{v}_{2q+2}^{\varepsilon}$ to within an e.l.t :

$$\overline{v}_{2q+2}^{\varepsilon}(z) = \frac{1}{\varepsilon^{q+1}} \begin{cases} z + \dfrac{\sqrt{\varepsilon}}{2} \exp\big(-\dfrac{z}{\sqrt{\varepsilon}}\big), & 0 < z < h, \\ \\ \dfrac{\sqrt{\varepsilon}}{2} \exp\big(\dfrac{z}{\sqrt{\varepsilon}}\big), & -h < z < 0. \end{cases} \tag{1.2.41}$$

Thanks to (1.2.34) , (1.2.36) and (1.2.41), we deduce that

$$\overline{v}^{\varepsilon}(z) = -\sum_{j=1}^{q}\Big(\frac{\delta^{(2(q-j))}}{\varepsilon^{j}}\Big) + \frac{1}{2\varepsilon^{q+1/2}}\begin{cases} \exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0<z<h, \\ \\ \exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h<z<0. \end{cases} \tag{1.2.42}$$

Case 2 *:* If $k = 2q-1$, $q \in N^{*}$, we define similarly :

$$\begin{cases} v^{\varepsilon}_{2j-1} = & v^{\varepsilon}_{2j-3} + \dfrac{\delta^{(2(q-j)-1)}}{\varepsilon^{j}}, \qquad \forall\ 1 \le j \le q, \\ \\ v^{\varepsilon}_{-1} \ \ = & v^{\varepsilon}. \end{cases} \tag{1.2.43}$$

Therefore, we obtain

$$v^{\varepsilon} = v^{\varepsilon}_{2q-1} - \sum_{j=1}^{q}\Big(\frac{\delta^{(2(q-j)-1)}}{\varepsilon^{j}}\Big), \tag{1.2.44}$$

where $v^{\varepsilon}_{2q-1}(k = 2q-1)$ satisfies the following equations :

$$-\,\varepsilon\ \frac{d^2 v^{\varepsilon}_{2q-1}}{dz^2} + v^{\varepsilon}_{2q-1} = \frac{H}{\varepsilon^{q}}, \quad in\ \ (-h,h), \tag{1.2.45}$$

$$v^{\varepsilon}_{2q-1}(-h) = 0, \qquad v^{\varepsilon}_{2q-1}(h) = \frac{1}{\varepsilon^{q}}, \tag{1.2.46}$$

$$v^{\varepsilon}_{2q-1} \quad \text{is of class } C^1 \text{ at } \quad z = 0. \tag{1.2.47}$$

Solving again (1.2.45) separately on $(-h,0)$ and $(0,h)$ and matching the values at $z=0$, we obtain a linear system of type (S) with $a = b = d = 0$ and $c = -\frac{1}{\varepsilon^{q}}$ which leads us to the approximate solution (up to an e.l.t) :

$$\overline{v}^{\varepsilon}_{2q-1}(z) = \frac{1}{\varepsilon^{q}}\begin{cases} 1 - \dfrac{1}{2}\exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0<z<h, \\ \\ \dfrac{1}{2}\exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h<z<0. \end{cases} \tag{1.2.48}$$

Combining (1.2.34) , (1.2.44) and (1.2.48) we deduce then an approximation of v^{ε} in this case

$$\overline{v}^{\varepsilon}(z) = -\sum_{j=1}^{q-1}\Big(\frac{\delta^{(2(q-j))}}{\varepsilon^{j}}\Big) + \frac{1}{2\varepsilon^{q}}\begin{cases} -\exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0<z<h, \\ \\ \exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h<z<0. \end{cases} \tag{1.2.49}$$

Finally, we gather these results in the next Proposition

Proposition 1.2.1 The solution of (1.2.31)-(1.2.32) has an approximation to within an e.l.t. denoted by $\overline{v}^\varepsilon$ and given by

$$\overline{v}^\varepsilon(z) = -\sum_{j=1}^{k/2}\Big(\frac{\delta^{(k-2j)}}{\varepsilon^j}\Big) + \frac{1}{2\ \varepsilon^{\frac{k+1}{2}}}\begin{cases} \exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0 < z < h, \\ \\ \exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h < z < 0, \end{cases} \tag{1.2.50}$$

if k is an even number and

$$\overline{v}^\varepsilon(z) = -\sum_{j=1}^{\frac{k-1}{2}}\Big(\frac{\delta^{(k-2j)}}{\varepsilon^j}\Big) + \frac{1}{2\ \varepsilon^{\frac{k+1}{2}}}\begin{cases} -\exp\,(-\dfrac{z}{\sqrt{\varepsilon}}), & 0 < z < h, \\ \\ \exp\,(\dfrac{z}{\sqrt{\varepsilon}}), & -h < z < 0, \end{cases} \tag{1.2.51}$$

if k is an odd number .

1.3 The multidimensional case.

In this section, we may consider Ω an open set of R^n, but just for simplicity in the notations, we consider Ω a bounded open set of R^3 as follows

$$\Omega = (0, L_1) \times (0, L_2) \times (-h, h),$$

and we write

$$\Gamma = (0, L_1) \times (0, L_2) \times \{-h, h\}.$$

We will be interested in the solution u^ε of the following problem

$$-\varepsilon\Delta u^\varepsilon + u^\varepsilon = H, \qquad in \ \ \Omega, \tag{1.3.1}$$

$$u^\varepsilon = \Phi, \qquad on \ \ \Gamma, \tag{1.3.2}$$

and

$$u^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{1.3.3}$$

Here, $H = H(z)$ is the same as in (1.2.3) and $\Phi = \Phi(x, y, z)(z = \pm h)$ is a given function as regular as necessary. Furthermore the "inviscid" equation is still given by (1.2.4) .
The first part of this section (Section 1.3.1) will be dedicated to describing the corrector corresponding to this problem and to prove some tangential estimates with the help of standard energy methods . These estimates are useful to derive bounds in $L^2(\Omega)$ and $H^1(\Omega)$ which will be the object of the second part(Section 1.3.2) and we obtain finally the asymptotic behavior of u^ε in these two spaces .

1.3.1 Explicit form of the corrector and tangential derivative estimates

It is obvious that the convergence of u^ε to u^0 in the $H^1(\Omega)$-norm fails to be true . Thus, we propose to introduce a primary form of the zero th order corrector $\overline{\theta}^{0,\varepsilon}$, the form of which is suggested by the resolution of the analogous problem in dimension one (see Section 1.2), $\overline{\theta}^{0,\varepsilon} = \overline{u}^\varepsilon - u^0$:

$$\overline{\theta}^{0,\varepsilon}(z) = \begin{cases} (\Phi(x,y,h) - 1)\exp\left(\dfrac{z-h}{\sqrt{\varepsilon}}\right) - \dfrac{1}{2} exp\left(-\dfrac{z}{\sqrt{\varepsilon}}\right), & 0 < z < h, \\ \\ \Phi(x,y,-h)\exp\left(-\dfrac{z+h}{\sqrt{\varepsilon}}\right) + \dfrac{1}{2}\exp\left(\dfrac{z}{\sqrt{\varepsilon}}\right), & -h < z < 0. \end{cases} \tag{1.3.4}$$

It is important to notice that $\overline{\theta}^{0,\varepsilon}$ is not continuous at $z = 0$. In fact

$$\overline{\theta}^{0,\varepsilon}(z = 0^+) = (\Phi(x,y,h) - 1)\alpha(\varepsilon) - \frac{1}{2}, \tag{1.3.5}$$

$$\overline{\theta}^{0,\varepsilon}(z = 0^-) = \Phi(x,y,-h)\alpha(\varepsilon) + \frac{1}{2}. \tag{1.3.6}$$

Then, $\overline{\theta}^{0,\varepsilon}$ verifies

$$-\varepsilon\frac{\partial^2\overline{\theta}^{0,\varepsilon}}{\partial z^2} + \overline{\theta}^{0,\varepsilon} = \varepsilon(1 + \mu(x,y)\alpha(\varepsilon))\delta' + \sqrt{\varepsilon}\lambda(x,y)\alpha(\varepsilon)\delta, \tag{1.3.7}$$

where we wrote

$$\mu(x,y) = \Phi(x,y,-h) - \Phi(x,y,h) + 1,$$

$$\lambda(x,y) = -\Phi(x,y,-h) - \Phi(x,y,h) + 1.$$

Notice that in the right-hand side of (1.3.7), there are singularities which are generated by the data Φ at the boundary Γ. These are not our aim here since Φ is a given function and we will be interested in singularities generated by the discontinuity of u^0 aiming for methods that apply to more general equations.

Hence, from the expression of $\overline{\theta}^{0,\varepsilon}$ given by (1.3.4), we define a corrector function θ^ε by setting

$$\theta^\varepsilon = \overline{\theta}^{0,\varepsilon} - \varphi^\varepsilon, \tag{1.3.8}$$

where

$$\varphi^\varepsilon(x,y,z) = \Big[\Phi(x,y,h)(\frac{z}{\sqrt{\varepsilon}}+1)\chi_{(0,h)} + \Phi(x,y,-h)(-\frac{z}{\sqrt{\varepsilon}}+1)\chi_{(0,-h)}\Big]\alpha(\varepsilon). \tag{1.3.9}$$

Then θ^ε verifies

$$\begin{aligned}
-\varepsilon\frac{\partial^2\theta^\varepsilon}{\partial z^2} + \theta^\varepsilon &= \varepsilon\delta' + \varepsilon\alpha(\varepsilon)\delta' + \sqrt{\varepsilon}\alpha(\varepsilon)\delta - \varphi^\varepsilon, \quad in \ \ \Omega, && (1.3.10)\\
\theta^\varepsilon(z=h) &= -u^0(h) + \Phi(x,y,h) - \varphi^\varepsilon(x,y,h) - \frac{1}{2}\alpha(\varepsilon), && (1.3.11)\\
\theta^\varepsilon(z=-h) &= -u^0(-h) + \Phi(x,y,-h) - \varphi^\varepsilon(x,y,-h) + \frac{1}{2}\alpha(\varepsilon). && (1.3.12)
\end{aligned}$$

Next, we consider $w^\varepsilon = u^\varepsilon - u^0 - \theta^\varepsilon$ which satisfies the following boundary conditions

$$\begin{aligned}
w^\varepsilon(z=h) &= \varphi^\varepsilon(x,y,h) + \frac{1}{2}\alpha(\varepsilon),\\
w^\varepsilon(z=-h) &= \varphi^\varepsilon(x,y,-h) - \frac{1}{2}\alpha(\varepsilon).
\end{aligned}$$

Since w^ε has a nonzero boundary value on Γ, we define

$$\widetilde{w}^\varepsilon = w^\varepsilon - \psi^\varepsilon, \tag{1.3.13}$$

where

$$\psi^\varepsilon(x,y,z) = \frac{\varphi^\varepsilon(x,y,h) - \varphi^\varepsilon(x,y,-h) + \alpha(\varepsilon)}{2h}z + \frac{\varphi^\varepsilon(x,y,h) + \varphi^\varepsilon(x,y,-h)}{2}, \tag{1.3.14}$$

and now $\widetilde{w}^\varepsilon$ vanishes on $\Gamma(z=\pm h)$.

Thus, the equation and boundary conditions for $\widetilde{w}^\varepsilon$ read

$$\begin{aligned} -\varepsilon\Delta\widetilde{w}^\varepsilon+\widetilde{w}^\varepsilon &= -\psi^\varepsilon+\varepsilon\frac{\partial^2\theta^\varepsilon}{\partial x^2}+\varepsilon\frac{\partial^2\theta^\varepsilon}{\partial y^2}+\varepsilon\frac{\partial^2\psi^\varepsilon}{\partial x^2}+ \\ &\quad +\varepsilon\frac{\partial^2\psi^\varepsilon}{\partial y^2}-\varepsilon\alpha(\varepsilon)\delta'-\sqrt{\varepsilon}\alpha(\varepsilon)\delta, \quad in\ \Omega, && (1.3.15) \\ \widetilde{w}^\varepsilon &= 0, \quad on\ \Gamma, && (1.3.16) \end{aligned}$$

and

$$\widetilde{w}^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \qquad (1.3.17)$$

We have the following result

Theorem 1.3.1 : *There exist constants C_1, C_2 and C_3 depending only on Ω and Φ such that $w^\varepsilon=u^\varepsilon-u^0-\theta^\varepsilon$ satisfies the following estimates*

$$\left\|\frac{\partial w^\varepsilon}{\partial x}\right\|_{L^2(\Omega)} \le C_1\ \varepsilon^{5/4}, \qquad (1.3.18)$$

$$\left\|\frac{\partial w^\varepsilon}{\partial x}\right\|_{H^1(\Omega)} \le C_2\ \varepsilon^{3/4}, \qquad (1.3.19)$$

$$\left\|\frac{\partial w^\varepsilon}{\partial x}\right\|_{H^2(\Omega)} \le C_3\ \varepsilon^{1/4}. \qquad (1.3.20)$$

Proof . The function ψ^ε is "small" in $H^m(\Omega)$ for all $m\in N$, more precisely ψ^ε verifies

$$\begin{aligned} \|\psi^\varepsilon\|_{H^m(\Omega)} &\le c\{\|\varphi^\varepsilon(x,y,h)\|_{H^m(\omega)}+\|\varphi^\varepsilon(x,y,-h)\|_{H^m(\omega)}+\alpha(\varepsilon)\} \\ &\le (\text{Thanks to } (1.3.9)) \\ &\le c\varepsilon^{-1/2}\alpha(\varepsilon) && (1.3.21) \end{aligned}$$

where $\omega=(0,L_1)\times(0,L_2)$, it is then sufficient to prove these estimates for $\widetilde{w}^\varepsilon$.

By differentiating (1.3.15)–(1.3.17) in x, we find

$$\begin{aligned} -\varepsilon\,\Delta\Big(\frac{\partial\widetilde{w}^\varepsilon}{\partial x}\Big)+\frac{\partial\widetilde{w}^\varepsilon}{\partial x} &= -\frac{\partial\psi^\varepsilon}{\partial x}+\varepsilon\frac{\partial^3\theta^\varepsilon}{\partial x^3}+\varepsilon\frac{\partial^3\theta^\varepsilon}{\partial x\partial y^2}+ \\ &\quad +\varepsilon\frac{\partial^3\psi^\varepsilon}{\partial x^3}+\varepsilon\frac{\partial^3\psi^\varepsilon}{\partial x\partial y^2}, \quad in\ \Omega, && (1.3.22) \\ \frac{\partial\widetilde{w}^\varepsilon}{\partial x} &= 0, \quad on\ \Gamma, && (1.3.23) \end{aligned}$$

$$\frac{\partial \widetilde{w}^\varepsilon}{\partial x} \text{ is periodic with periods } L_1 \; and \; L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{1.3.24}$$

Observe that the right-hand side of (1.3.22) is regular (belongs to $L^2(\Omega)$). Thus, we can apply the standard energy estimate methods.

We multiply (1.3.22) by $\partial w^\varepsilon/\partial x$, integrate over Ω and notice that the boundary terms vanish due to (1.3.23) and (1.3.24). We then deduce

$$\begin{aligned}\varepsilon\left\|\nabla\Big(\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\Big)\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right\|^2_{L^2(\Omega)} &= \left(-\frac{\partial \psi^\varepsilon}{\partial x} + \varepsilon\frac{\partial^3\theta^\varepsilon}{\partial x^3} + \varepsilon\frac{\partial^3\theta^\varepsilon}{\partial x\partial y^2}, \frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right)_{L^2(\Omega)} + \\ &+\varepsilon\left(\frac{\partial^3\psi^\varepsilon}{\partial x^3} + \frac{\partial^3\psi^\varepsilon}{\partial x\partial y^2}, \frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right)_{L^2(\Omega)}.\end{aligned} \tag{1.3.25}$$

Thanks to the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}\varepsilon\left\|\nabla\Big(\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\Big)\right\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right\|^2_{L^2(\Omega)} &\le \varepsilon^2\left\|\frac{\partial^3\theta^\varepsilon}{\partial x^3}\right\|^2_{L^2(\Omega)} + \varepsilon^2\left\|\frac{\partial^3\theta^\varepsilon}{\partial x\partial y^2}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial \psi^\varepsilon}{\partial x}\right\|^2_{L^2(\Omega)} + \\ &+\varepsilon^2\left(\left\|\frac{\partial^3\psi^\varepsilon}{\partial x^3}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial^3\psi^\varepsilon}{\partial x\partial y^2}\right\|^2_{L^2(\Omega)}\right),\end{aligned} \tag{1.3.26}$$

Observe that, by the expression (1.3.8) of θ^ε, we have

$$\left\|\frac{\partial^3\theta^\varepsilon}{\partial x^3}\right\|^2_{L^2(\Omega)} \le \frac{1}{2}\left\{\left\|\Phi_{xxx}(z=h)\right\|^2_{L^2(\omega)} + \left\|\Phi_{xxx}(z=-h)\right\|^2_{L^2(\omega)}\right\}\varepsilon^{1/2}, \tag{1.3.27}$$

and

$$\left\|\frac{\partial^3\theta^\varepsilon}{\partial x\partial y^2}\right\|^2_{L^2(\Omega)} \le \frac{1}{2}\left\{\left\|\Phi_{xyy}(z=h)\right\|^2_{L^2(\omega)} + \left\|\Phi_{xyy}(z=-h)\right\|^2_{L^2(\omega)}\right\}\varepsilon^{1/2}. \tag{1.3.28}$$

Combining (1.3.26), (1.3.27) and (1.3.28), we deduce that

$$\left\|\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right\|_{L^2(\Omega)} \le C_1(\Omega,\Phi)\,\varepsilon^{5/4}, \tag{1.3.29}$$

$$\left\|\frac{\partial \widetilde{w}^\varepsilon}{\partial x}\right\|_{H^1(\Omega)} \le C_2(\Omega,\Phi)\,\varepsilon^{3/4}. \tag{1.3.30}$$

From (1.3.22), the triangle inequality, (1.3.27), (1.3.28) and (1.3.29), we obtain (1.3.20). This completes the proof of Theorem 1.3.1.

Remark 3 *:* Notice that in Theorem 1.3.1, $\partial\widetilde{w}^\varepsilon/\partial x$ can be replaced by any combination of x and y-derivatives of $\widetilde{w}^\varepsilon$ of arbitrary order and the proof is similar.

Remark 4 *:* The bounds proved in Theorem 1.3.1 are not sufficient in terms of boundary layers theory since we have only considered the tangential derivatives. Nevertheless, they are necessary for the derivation of the subsequent estimates .

1.3.2 Convergence in $L^2(\Omega)$ and $H^1(\Omega)$

In this part, we are interested in $L^2(\Omega)$ and $H^1(\Omega)$- estimates for w^ε itself; we will show how these estimates can be derived with the help of those obtained for $\partial w^\varepsilon/\partial x$ and $\partial w^\varepsilon/\partial y$. For this purpose, we rewrite (1.3.22) in the form

$$\begin{aligned} -\varepsilon\frac{\partial^2\widetilde{w}^\varepsilon}{\partial z^2}+\widetilde{w}^\varepsilon &= -\psi^\varepsilon+\varepsilon\Big(\frac{\partial^2\theta^\varepsilon}{\partial x^2}+\frac{\partial^2\theta^\varepsilon}{\partial y^2}+\frac{\partial^2\psi^\varepsilon}{\partial x^2}+\frac{\partial^2\psi^\varepsilon}{\partial y^2}+ \\ &\qquad +\frac{\partial^2\widetilde{w}^\varepsilon}{\partial x^2}+\frac{\partial^2\widetilde{w}^\varepsilon}{\partial y^2}\Big)-\varepsilon\alpha(\varepsilon)\delta'-\sqrt{\varepsilon}\alpha(\varepsilon)\delta, \qquad in \ \ \Omega. \end{aligned} \tag{1.3.31}$$

For the sake of simplicity in the notation, we write temporarily w instead of $\widetilde{w}^\varepsilon$ and we set

$$g=\frac{\partial^2\theta^\varepsilon}{\partial x^2}+\frac{\partial^2\theta^\varepsilon}{\partial y^2}+\frac{\partial^2\psi^\varepsilon}{\partial x^2}+\frac{\partial^2\psi^\varepsilon}{\partial y^2}+\frac{\partial^2\widetilde{w}^\varepsilon}{\partial x^2}+\frac{\partial^2\widetilde{w}^\varepsilon}{\partial y^2}-\frac{\psi^\varepsilon}{\varepsilon}. \tag{1.3.32}$$

Now, we see (1.3.31) as a one-dimensional equation in z

$$-\varepsilon\frac{\partial^2 w}{\partial z^2}+w=\varepsilon g-\varepsilon\alpha(\varepsilon)\delta'-\sqrt{\varepsilon}\alpha(\varepsilon)\delta, \tag{1.3.33}$$

with boundary conditions

$$w=0, \quad \text{on} \quad z=\pm h, \tag{1.3.34}$$

and we intend, at this stage, to use the results from Section 1.2.

We propose to look for a solution of (1.3.33)-(1.3.34) of this form :

$$w=w_1+w_2+w_3+\alpha(\varepsilon)H+\frac{1}{\sqrt{\varepsilon}}\alpha(\varepsilon)G, \tag{1.3.35}$$

where w_1 verifies

$$
(P_1)\begin{cases}
-\varepsilon\dfrac{d^2w_1}{dz^2}+w_1=-\alpha(\varepsilon)H, \qquad in \quad (-h,h),\\[2em]
w_1=0, \quad \text{on} \quad z=\pm h,\\[2em]
[w_1](z=0)=0,\\[2em]
\int_0^z w_1(z')\mathrm{d}z' \quad \text{is continuous} \quad \text{at} \quad z=0,
\end{cases}
$$

and w_2 verifies

$$
(P_2)\begin{cases}
-\varepsilon\dfrac{d^2w_2}{dz^2}+w_2=-\dfrac{1}{\sqrt{\varepsilon}}\alpha(\varepsilon)G, \qquad in \quad (-h,h),\\[2em]
w_2=0, \qquad \text{on} \quad z=\pm h,\\[2em]
[\dfrac{dw_2}{dz}](z=0)=0,\\[2em]
w_2 \quad \text{is continuous} \quad \text{at} \quad z=0.
\end{cases}
$$

Finally, w_3 satisfies

$$
\begin{aligned}
-\varepsilon\frac{\partial^2 w_3}{\partial z^2}+w_3 &= \varepsilon g, \qquad in \quad (-h,h) && (1.3.36)\\
w_3 &= 0, \qquad at \quad z=h && (1.3.37)\\
w_3 &= -(1+\frac{h}{\sqrt{\varepsilon}})\alpha(\varepsilon), \qquad at \quad z=-h. && (1.3.38)
\end{aligned}
$$

The resolution of (P_1) and (P_2) is similar to the the resolution of (1.2.1), (1.2.2) in Section 1.2 . Therefore, we deduce an approximation for w_1 and w_2 to within l.o.t (where l.o.t stands for lower order terms)

$$
\overline{w}_1(z)=\begin{cases}
\alpha(\varepsilon)(\exp(\dfrac{z-h}{\sqrt{\varepsilon}})-1)-\frac{1}{2}\alpha(\varepsilon)\exp(-\dfrac{z}{\sqrt{\varepsilon}}), \qquad 0<z<h,\\[2em]
-\frac{1}{2}\alpha(\varepsilon)\exp(\dfrac{z}{\sqrt{\varepsilon}}), \qquad\qquad -h<z<0,
\end{cases} \qquad (1.3.39)
$$

$$\overline{w}_2(z) = \begin{cases} \dfrac{\alpha(\varepsilon)}{\sqrt{\varepsilon}}(h\exp(\dfrac{z-h}{\sqrt{\varepsilon}}) - z) - \dfrac{h}{2}\alpha(\varepsilon)\exp(-\dfrac{z}{\sqrt{\varepsilon}}), & 0 < z < h, \\ \\ -\dfrac{h}{2}\alpha(\varepsilon)\exp(\dfrac{z}{\sqrt{\varepsilon}}), & -h < z < 0, \end{cases} \tag{1.3.40}$$

and

$$w_1 = \overline{w}_1 + l.o.t, \qquad w_2 = \overline{w}_2 + l.o.t, \quad in \quad H^m(\Omega), \forall m \geq 0. \tag{1.3.41}$$

Afterwards, we write the form of the solution of (1.3.36)–(1.3.38) as follows

$$\begin{aligned} w_3(z) &= K\exp(\frac{z-h}{\sqrt{\varepsilon}}) + K'\exp(-\frac{z+h}{\sqrt{\varepsilon}}) + \\ &\quad + \begin{cases} \exp(\dfrac{z}{\sqrt{\varepsilon}})I(z) & \text{if } \ z \in (0,h), \\ \\ \exp(-\dfrac{z}{\sqrt{\varepsilon}})J(z) & \text{if } \ z \in (-h,0), \end{cases} \end{aligned} \tag{1.3.42}$$

where

$$I(z) = -\int_0^z \exp(-\frac{2t}{\sqrt{\varepsilon}})\Big(\int_0^t g(s)\exp(\frac{s}{\sqrt{\varepsilon}})\mathrm{d}s\Big)\mathrm{d}t, \qquad \forall z \in (0,h), \tag{1.3.43}$$

$$J(z) = -\int_0^z \exp(\frac{2t}{\sqrt{\varepsilon}})\Big(\int_0^t g(s)\exp(-\frac{s}{\sqrt{\varepsilon}})\mathrm{d}s\Big)\mathrm{d}t, \qquad \forall z \in (-h,0), \tag{1.3.44}$$

and K,K' are constants with respect to z but they generally depend on x and y. We recall that other terms may depend either on x and y, but this is not mentioned in our writing just for simplicity. It is important to remark that the particular solution of (1.3.36)–(1.3.38) which is given by (1.3.42) is of class C^1 at $z = 0$.

Since w_3 verifies the boundary conditions (1.3.37) and (1.3.38) , we deduce that

$$\begin{aligned} \overline{w}_3(x,y,z) &= -(1+\frac{h}{\sqrt{\varepsilon}})\alpha(\varepsilon)\exp(\frac{z-h}{\sqrt{\varepsilon}}) + J(-h)\exp(\frac{z-2h}{\sqrt{\varepsilon}}) - \\ &\quad -I(h)\exp(\frac{z}{\sqrt{\varepsilon}}) - J(-h)\exp(-\frac{z}{\sqrt{\varepsilon}}) + \\ &\quad +I(h)\exp(-\frac{z+2h}{\sqrt{\varepsilon}}) + \begin{cases} \exp(\dfrac{z}{\sqrt{\varepsilon}})I(z) & \text{if } \ z \in (0,h), \\ \\ \exp(-\dfrac{z}{\sqrt{\varepsilon}})J(z) & \text{if } \ z \in (-h,0), \end{cases} \end{aligned} \tag{1.3.45}$$

and

$$w_3(x,y,z) = \overline{w}_3(x,y,z) + l.o.t. \tag{1.3.46}$$

Finally, we insert all these results in (1.3.35) and we obtain an approximation $\overline{w} = \overline{w}^\varepsilon$ of $w = \widetilde{w}^\varepsilon$:

$$\begin{aligned}\overline{w}(x,y,z) &= -\alpha(\varepsilon)\exp\,(-\frac{z}{\sqrt{\varepsilon}}) + \Big(I(z) - I(h)\Big)\exp\,(\frac{z}{\sqrt{\varepsilon}}), + I(h)\exp\,(-\frac{z+2h}{\sqrt{\varepsilon}}) + \\ &\quad + J(-h)\Big(\exp\,(\frac{z-2h}{\sqrt{\varepsilon}}) - \exp\,(-\frac{z}{\sqrt{\varepsilon}})\Big),\end{aligned} \tag{1.3.47}$$

for all $(x,y,z) \in \omega \times (0,h)$, and

$$\begin{aligned}\overline{w}(x,y,z) &= -\alpha(\varepsilon)\exp\,(\frac{z}{\sqrt{\varepsilon}}) + \Big(J(z) - J(-h)\Big)\exp\,(-\frac{z}{\sqrt{\varepsilon}}) + J(-h)\exp\,(\frac{z-2h}{\sqrt{\varepsilon}}) + \\ &\quad + I(h)\Big(\exp\,(-\frac{z+2h}{\sqrt{\varepsilon}}) - \exp\,(\frac{z}{\sqrt{\varepsilon}})\Big),\end{aligned} \tag{1.3.48}$$

for all $(x,y,z) \in \omega \times (-h,0)$, and we infer from (1.3.41) and (1.3.46) that

$$w^\varepsilon = \overline{w}^\varepsilon + l.o.t., \quad in \quad H^m(\Omega),\ \forall m \geq 0. \tag{1.3.49}$$

Hence, by respecting the definition of *l.o.t.*, we have, for ε small,

$$\|w^\varepsilon\|_{H^m(\Omega)} \leq 2\ \|\overline{w}^\varepsilon\|_{H^m(\Omega)}. \tag{1.3.50}$$

Furthermore, the *l.o.t.* in (1.3.49) is in fact an *e.l.t.* and it verifies

$$\|l.o.t.(\overline{w}^\varepsilon)\|_{H^m(\Omega)} \leq c\ \varepsilon^{1/4-m/2}\alpha(\varepsilon). \tag{1.3.51}$$

We now state and prove the following result :

Theorem 1.3.2 *: For θ^ε given by (1.3.8), there exist two constants C_4 and C_5 independent of ε such that*

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{L^2(\Omega)} \leq C_4(\Omega,\Phi)\ \varepsilon, \tag{1.3.52}$$

$$\|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^1(\Omega)} \leq C_5(\Omega,\Phi)\ \varepsilon^{1/2}. \tag{1.3.53}$$

Proof . Thanks to (1.3.49), it is sufficient to prove these estimates for $\overline{w} = \overline{w}^\varepsilon$.
We have

$$\|\overline{w}\|^2_{L^2(\Omega)} = \|\overline{w}\|^2_{L^2(\omega\times(0,h))} + \|\overline{w}\|^2_{L^2(\omega\times(-h,0))}. \tag{1.3.54}$$

On the other hand, by (1.3.47), we have

$$\begin{aligned}\|\overline{w}\|_{L^2(\omega\times(0,h))} \leq\ & \sqrt{L_1L_2}\alpha(\varepsilon)\Big\|\exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(0,h)} + \\ & +\Big\|\Big(I(z)-I(h)\Big)\exp\big(\frac{z}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(\omega\times(0,h))} + \\ & +\Big\|I(h)\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(\omega\times(0,h))} + \\ & +\Big\|J(-h)\Big(\exp\big(\frac{z-2h}{\sqrt{\varepsilon}}\big)-\exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\Big)\Big\|_{L^2(\omega\times(0,h))}. \end{aligned} \tag{1.3.55}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|I(z)-I(h)| \leq\ & \int_z^h \exp\big(-\frac{2t}{\sqrt{\varepsilon}}\big)\|g\|_{L^2_z(0,h)}\Big(\int_0^t \exp\big(\frac{2s}{\sqrt{\varepsilon}}\big)\Big)^{1/2}\mathrm{d}t, \\ \leq\ & \frac{1}{\sqrt{2}}\|g\|_{L^2_z(0,h)}\ \varepsilon^{3/4}\ \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big), \\ & \forall z\in(0,h),\end{aligned} \tag{1.3.56}$$

which implies

$$\Big\|(I(z)-I(h))\exp\big(\frac{z}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(\omega\times(0,h))} \leq \frac{h}{\sqrt{2}}\ \|g\|_{L^2(\Omega)}\ \varepsilon^{3/4}. \tag{1.3.57}$$

Similarly, we have

$$|I(h)| \leq \frac{1}{\sqrt{2}}\ \|g\|_{L^2_z(0,h)}\ \varepsilon^{3/4}, \quad \forall\ x,y\in\omega. \tag{1.3.58}$$

Hence, we deduce

$$\begin{aligned}\Big\|I(h)\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(\omega\times(0,h))} \leq\ & \|I(h)\|_{L^\infty(\omega)}\ \sqrt{L_1L_2}\ \Big\|\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big)\Big\|_{L^2_z(0,h)}, \\ \leq\ & c\ \|g\|_{L^2(\Omega)}\ \varepsilon\ \alpha(\varepsilon).\end{aligned} \tag{1.3.59}$$

Similar estimates hold for $J(-h)$, and we find

$$\|J(-h)\|_{L^2(\omega)} \leq c\ \|g\|_{L^2(\Omega)}\ \varepsilon^{3/4}, \tag{1.3.60}$$

and hence

$$\Big\|J(-h)\Big(\exp\big(\frac{z-2h}{\sqrt{\varepsilon}}\big)-\exp\big(\frac{z}{\sqrt{\varepsilon}}\big)\Big)\Big\|_{L^2(\omega\times(0,h))} \leq c\ \|g\|_{L^2(\Omega)}\ \varepsilon. \tag{1.3.61}$$

It is obvious that

$$\Big\| \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\Big\|_{L^2(0,h)} \leq \frac{1}{\sqrt{2}}\ \varepsilon^{1/4}. \tag{1.3.62}$$

Now, inserting (1.3.57), (1.3.59), (1.3.61) and (1.3.62) in (1.3.55), we deduce

$$\|\overline{w}\|_{L^2(\omega\times(0,h))} \leq c\ (\varepsilon^{1/4}\alpha(\varepsilon) + \varepsilon^{3/4}\|g\|_{L^2(\Omega)}). \tag{1.3.63}$$

According to (1.3.4) and (1.3.8), θ^ε verifies

$$\Big\|\frac{\partial^2\theta^\varepsilon}{\partial x^2}\Big\|_{L^2(\Omega)} \leq c(\Omega,\Phi)\ \varepsilon^{1/4}, \tag{1.3.64}$$

and

$$\Big\|\frac{\partial^2\theta^\varepsilon}{\partial y^2}\Big\|_{L^2(\Omega)} \leq c(\Omega,\Phi)\ \varepsilon^{1/4}. \tag{1.3.65}$$

This together with (1.3.18), and taking into account Remark 4, and (1.3.21), we find that

$$\|g\|_{L^2(\Omega)} \leq c\ \varepsilon^{1/4}. \tag{1.3.66}$$

Thanks to (1.2.12),(1.3.63) and (1.3.66), we conclude that

$$\|\overline{w}\|_{L^2(\omega\times(0,h))} \leq c\ \varepsilon. \tag{1.3.67}$$

Recall (1.3.48) which is similar to (1.3.47). By the same techniques, we obtain

$$\|\overline{w}\|_{L^2(\omega\times(-h,0))} \leq c\ \varepsilon. \tag{1.3.68}$$

This together with (1.3.67) implies (1.3.52).

Now for the second estimate in Theorem 1.3.2, we differentiate (1.3.47) and (1.3.48) with respect to z and we obtain

$$\begin{aligned} \frac{\partial\overline{w}}{\partial z}(x,y,z) \quad = \quad & \frac{1}{\sqrt{\varepsilon}}\Big\{\alpha(\varepsilon)\exp\big(-\frac{z}{\sqrt{\varepsilon}}\big) + \Big(I(z) - I(h)\Big)\exp\big(\frac{z}{\sqrt{\varepsilon}}\big) \\ & + J(-h)\Big(\exp\big(\frac{z-2h}{\sqrt{\varepsilon}}\big) + \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\Big) - I(h)\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big)\Big\} \\ & - \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\int_0^z g(s)\exp\big(\frac{s}{\sqrt{\varepsilon}}\big)\ \mathrm{d}s, \qquad \forall\ 0 < z < h, \end{aligned} \tag{1.3.69}$$

$$\begin{aligned}\frac{\partial \overline{w}}{\partial z}(x,y,z) \;=\; & \frac{1}{\sqrt{\varepsilon}}\Big\{-\alpha(\varepsilon)\exp\big(\frac{z}{\sqrt{\varepsilon}}\big) - \Big(J(z)-J(-h)\Big)\exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\\ & - I(h)\Big(\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big)+\exp\big(\frac{z}{\sqrt{\varepsilon}}\big)\Big) + J(-h)\exp\big(\frac{z-2h}{\sqrt{\varepsilon}}\big)\Big\}\\ & - \exp\big(\frac{z}{\sqrt{\varepsilon}}\big)\int_0^z g(s)\exp\big(-\frac{s}{\sqrt{\varepsilon}}\big)\,\mathrm{d}s, \qquad \forall\ -h<z<0. \quad (1.3.70)\end{aligned}$$

It is important to notice that w is continuous at $z=0$ and consequently ∇w is regular ($\nabla w \in L^2(\Omega)$). This allows us to work separately on $(-h,0)$ and $(0,h)$ since ∇w does not contain any singularity.

We may observe that $\partial\overline{w}/\partial z$ contains terms similar to those in $\overline{w}$, except its last one which shall be bounded with the help of the Cauchy-Schwarz inequality :

$$\Big|\int_0^z g(s)\exp\big(\frac{s}{\sqrt{\varepsilon}}\big)\,\mathrm{d}s\Big| \le \frac{1}{\sqrt{2}}\|g\|_{L^2_z(0,h)}\ \varepsilon^{1/4}\ \exp\big(\frac{z}{\sqrt{\varepsilon}}\big), \quad \forall z\in(0,h), \qquad (1.3.71)$$

which further implies

$$\begin{aligned}\Big\|\exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\int_0^z g(s)\exp\big(\frac{s}{\sqrt{\varepsilon}}\big)\,\mathrm{d}s\Big\|_{L^2(\omega\times(0,h))} & \le\ c\ \|g\|_{L^2(\Omega)}\ \varepsilon^{1/4},\\ & \le\ (\text{Thanks to (1.3.66)}),\\ & \le\ c\,\varepsilon^{1/2}. \qquad (1.3.72)\end{aligned}$$

We then infer that

$$\Big\|\frac{\partial\overline{w}}{\partial z}\Big\|_{L^2(\omega\times(0,h))} \le c\,\varepsilon^{1/2}. \qquad (1.3.73)$$

In a similar way we prove that

$$\Big\|\frac{\partial\overline{w}}{\partial z}\Big\|_{L^2(\omega\times(-h,0))} \le c\,\varepsilon^{1/2}. \qquad (1.3.74)$$

To conclude, observe that

$$\begin{aligned}\frac{\partial\overline{w}}{\partial x}(x,y,z) \;=\; & \Big(I'(z)-I'(h)\Big)\exp\big(\frac{z}{\sqrt{\varepsilon}}\big) +\\ & + J'(-h)\Big(\exp\big(\frac{z-2h}{\sqrt{\varepsilon}}\big) - \exp\big(-\frac{z}{\sqrt{\varepsilon}}\big)\Big) +\\ & + I'(h)\exp\big(-\frac{z+2h}{\sqrt{\varepsilon}}\big) \quad 0<z<h. \qquad (1.3.75)\end{aligned}$$

Here $I'(z)$ or $J'(z)$ has an expression similar to $I(z)$ or $J(z)$ provided that we replace g by $\partial g/\partial x$.

Thanks to (1.3.18) in Theorem 1.3.1 combined with Remark 4, (1.3.27) and (1.3.28), we deduce, by differentiating (1.3.32) with respect to x, that

$$\left\|\frac{\partial g}{\partial x}\right\|_{L^2(\Omega)} \leq c\,\varepsilon^{1/4}. \tag{1.3.76}$$

Therefore $\partial g/\partial x$ verifies the same estimate as g.

Thus, we have

$$\left\|\frac{\partial \overline{w}}{\partial x}\right\|_{L^2(\omega\times(0,h))} \leq c\,\varepsilon. \tag{1.3.77}$$

Similarly, we can prove the same estimate on $\omega \times (-h, 0)$, *i.e.*

$$\left\|\frac{\partial \overline{w}}{\partial x}\right\|_{L^2(\Omega)} \leq c\,\varepsilon. \tag{1.3.78}$$

It is obvious that $\partial\overline{w}/\partial x$ can be replaced by $\partial\overline{w}/\partial y$ in (1.3.78) .

Finally, we conclude that

$$\|\nabla\overline{w}\|_{L^2(\Omega)} \leq c\,\varepsilon^{1/2}. \tag{1.3.79}$$

This completes the proof of Theorem 1.3.2 .

Remark 5 *:* We may observe that the choice of a particular solution for (1.3.36)–(1.3.38), which was given by (1.3.42), is important to obtain (1.3.68) and (1.3.74).

Remark 6 *:* For this kind of problems, we can not use standard a priori estimates for the $\tilde{w}^\varepsilon$-equation (see (1.3.15)) since the right-hand side of (1.3.15) is not in $L^2(\Omega)$.

1.4 Higher order singularities for the source function.

In this section, we will see how Theorem 1.3.2 can be extended if the source function is even more singular. In this case, we must introduce a new corrector corresponding to the singularity in the limit solution.

We are interested in the following model problem

$$\begin{aligned} -\varepsilon\Delta u^\varepsilon + u^\varepsilon &= \delta^{(k)}, \qquad in \;\; \Omega, & (1.4.1)\\ u^\varepsilon &= \Phi, \qquad on \;\; \Gamma, & (1.4.2)\end{aligned}$$

and

$$u^\varepsilon \textit{ is periodic with periods } L_1 \textit{ and } L_2 \textit{ in the } x \textit{ and } y \textit{ directions.} \tag{1.4.3}$$

Here Φ is a given function, $\delta^{(k)}$ is the k^{th} derivative of the Dirac measure δ (in the z variable), $\delta^{(k)} = \frac{\partial^k}{\partial z^k}(\delta)$ and Ω is a channel in R^n. For the sake of simplicity, we consider here Ω a channel in R^3 (the same as in Section 1.3). Note that the restriction on the dimension of space is not essential since we are treating a linear problem. For the existence of u^ε solution of (1.4.1)– (1.4.3), the reader is refered e.g. to the book of Lions-Magenes [LM].

The limit problem is simply

$$u^0 = \delta^{(k)}. \tag{1.4.4}$$

Our aim now is to define an appropriate corrector corresponding to problem (1.4.1)–(1.4.3) and prove a convergence result which gives the first order in the asymptotic expansion of u^ε in suitable Sobolev spaces. We have the following theorem

Theorem 1.4.3 . *For every $k \in N$, there exists a corrector θ_k^ε function of x, y and $z/\varepsilon^{1/2}$ and C_6 and C_7 two constants independent of ε such that*

$$\|u^\varepsilon - u^0 - \theta_k^\varepsilon\|_{L^2(\Omega)} \leq C_6(\Omega, \Phi)\ \varepsilon^{5/4}, \tag{1.4.5}$$

$$\|u^\varepsilon - u^0 - \theta_k^\varepsilon\|_{H^1(\Omega)} \leq C_7(\Omega, \Phi)\ \varepsilon^{3/4}. \tag{1.4.6}$$

Proof . To derive the explicit corrector, we refer to the resolution of the corresponding one-dimensional problem. Then, as in Section 1.2 , we have to consider two cases depending on either k is even or odd.

Firstly, we consider an even value of k, $k = 2q$, $q \in N$. From the resolution of the analogous problem in dimension one (see Proposition 1.2.1) , we propose the following primary corrector

$$\overline{\theta}_e^{0,\varepsilon}(x,y,z) \quad = \quad -\sum_{j=0}^{q} \frac{\delta^{(2(q-j))}}{\varepsilon^j} + \left\{ \begin{array}{l} \dfrac{1}{2\varepsilon^{q+1/2}} \exp\left(-\dfrac{z}{\sqrt{\varepsilon}}\right)\chi_{(0,h)}, \\ \dfrac{1}{2\varepsilon^{q+1/2}} \exp\left(\dfrac{z}{\sqrt{\varepsilon}}\right)\chi_{(-h,0)}, \end{array} \right.$$

$$+\begin{cases} \Phi(x,y,h)\exp\,(\dfrac{z-h}{\sqrt{\varepsilon}})\chi_{(0,h)}, \\ \Phi(x,y,-h)\exp\,(-\dfrac{z+h}{\sqrt{\varepsilon}})\chi_{(-h,0)}, \end{cases} \tag{1.4.7}$$

which satisfies

$$\begin{aligned} -\varepsilon\frac{\partial^2\overline{\theta}_e^{0,\varepsilon}}{\partial z^2}+\overline{\theta}_e^{0,\varepsilon} &= \varepsilon\delta^{(k+2)}+[\Phi(x,y,-h)-\Phi(x,y,h)]\varepsilon\alpha(\varepsilon)\delta' \\ &\quad -[\Phi(x,y,-h)+\Phi(x,y,h)]\sqrt{\varepsilon}\alpha(\varepsilon)\delta, \qquad (1.4.8)\\ \overline{\theta}_e^{0,\varepsilon}(h) &= \Phi(x,y,h)+\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}, \qquad (1.4.9)\\ \overline{\theta}_e^{0,\varepsilon}(-h) &= \Phi(x,y,-h)+\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}, \qquad (1.4.10) \end{aligned}$$

and

$$\overline{\theta}_e^{0,\varepsilon} \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{1.4.11}$$

With the purpose at absorbing the singularities produced in the right-hand side of (1.4.8), by the data Φ at $z=0$ (these singularities are very "small" since $\alpha(\varepsilon)$ verifies (1.2.12)), we define a corrector θ_e^ε as follows

$$\theta_e^\varepsilon=\overline{\theta}_e^{0,\varepsilon}-\varphi^\varepsilon, \tag{1.4.12}$$

where φ^ε is given by (1.3.9). Then the equation and boundary conditions for θ_e^ε read

$$\begin{aligned} -\varepsilon\frac{\partial^2\theta_e^\varepsilon}{\partial z^2}+\theta_e^\varepsilon &= \varepsilon\delta^{(k+2)}, \qquad in \;\; \Omega, \qquad (1.4.13)\\ \theta_e^\varepsilon(z=h) &= \Phi(x,y,h)+\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}-\varphi^\varepsilon(x,y,h), \qquad (1.4.14)\\ \theta_e^\varepsilon(z=-h) &= \Phi(x,y,-h)+\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}-\varphi^\varepsilon(x,y,-h). \qquad (1.4.15) \end{aligned}$$

Let now $w_e^\varepsilon=u^\varepsilon-u^0-\theta_e^\varepsilon$ which verifies

$$w_e^\varepsilon=\varphi^\varepsilon(x,y,.)-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}, \qquad on \;\; z=\pm h.$$

Hence, we define $\widetilde{w}_e^\varepsilon=w_e^\varepsilon-\psi_e^\varepsilon$, where

$$\psi_e^\varepsilon(x,y,z)=\frac{\varphi^\varepsilon(x,y,h)-\varphi^\varepsilon(x,y,-h)}{2h}z+\frac{\varphi^\varepsilon(x,y,h)+\varphi^\varepsilon(x,y,-h)}{2}-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1/2}}, \tag{1.4.16}$$

and from (1.4.1), (1.4.4) and (1.4.13), we deduce that

$$\begin{aligned} -\varepsilon\Delta\widetilde{w}_e^\varepsilon+\widetilde{w}_e^\varepsilon &= -\psi_e^\varepsilon+\varepsilon\frac{\partial^2\psi_e^\varepsilon}{\partial x^2}+\varepsilon\frac{\partial^2\psi_e^\varepsilon}{\partial y^2}+ \\ &\quad +\varepsilon\frac{\partial^2\theta_e^\varepsilon}{\partial x^2}+\varepsilon\frac{\partial^2\theta_e^\varepsilon}{\partial y^2}, \quad in\ \Omega, \qquad (1.4.17) \\ \widetilde{w}_e^\varepsilon &= 0, \quad on\ \Gamma, \qquad (1.4.18) \end{aligned}$$

and

$$\widetilde{w}_e^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \qquad (1.4.19)$$

Thus, we multiply (1.4.17) by $\widetilde{w}_e^\varepsilon$, integrate over Ω using the Green formula, we see that the boundary terms vanish thanks to the periodicity and boundary conditions on $\widetilde{w}_e^\varepsilon$. From the Cauchy-Schwarz and Young inequalities, we deduce that

$$\begin{aligned} \varepsilon\|\nabla\widetilde{w}_e^\varepsilon\|^2_{L^2(\Omega)}+\frac{1}{2}\|\widetilde{w}_e^\varepsilon\|^2_{L^2(\Omega)} &\leq \|\psi_e^\varepsilon\|_{L^2(\Omega)}+\varepsilon^2\left\|\frac{\partial^2\psi_e^\varepsilon}{\partial x^2}+\frac{\partial^2\psi_e^\varepsilon}{\partial y^2}\right\|^2_{L^2(\Omega)}+ \\ &\quad +\varepsilon^2\left\|\frac{\partial^2\theta_e^\varepsilon}{\partial x^2}+\frac{\partial^2\theta_e^\varepsilon}{\partial y^2}\right\|^2_{L^2(\Omega)}. \qquad (1.4.20) \end{aligned}$$

On the other hand, from the expression (1.3.9) of φ^ε, it is easy to see that

$$\|\varphi^\varepsilon(.,.,\pm h)\|_{H^m(\omega)} \leq c\ \varepsilon^{-1/2}\alpha(\varepsilon). \qquad (1.4.21)$$

This further implies

$$\|\psi_e^\varepsilon\|_{L^2(\Omega)} \leq c\ \varepsilon^{-q-1/2}\alpha(\varepsilon), \qquad (1.4.22)$$

and

$$\left\|\frac{\partial^{m+n}\psi_e^\varepsilon}{\partial x^m\partial y^n}\right\|_{L^2(\Omega)} \leq c\ \varepsilon^{-1/2}\alpha(\varepsilon), \quad \forall (m,n)\in N^*\times N. \qquad (1.4.23)$$

From (1.4.7), (1.4.12) and (1.4.21), we conclude that

$$\left\|\frac{\partial^{m+n}\theta_e^\varepsilon}{\partial x^m\partial y^n}\right\|_{L^2(\Omega)} \leq c\ \varepsilon^{1/4}, \quad \forall (m,n)\in N^*\times N. \qquad (1.4.24)$$

Combining (1.4.20), (1.4.22), (1.4.23) and (1.4.24), we deduce the estimates (1.4.5) and (1.4.6) for this case.

Secondly, we consider an odd value for k, $k = 2q-1$, $q \in N^*$. Similarly, we define explicitly the corresponding primary corrector by setting

$$\begin{aligned} \overline{\theta}_o^{0,\varepsilon}(x,y,z) &= -\sum_{j=0}^{q-1} \frac{\delta^{(2(q-j)-1)}}{\varepsilon^j} + \begin{cases} -\dfrac{1}{2\varepsilon^q} \exp\big(-\dfrac{z}{\sqrt{\varepsilon}}\big)\chi_{(0,h)}, \\ \dfrac{1}{2\varepsilon^q} \exp\big(\dfrac{z}{\sqrt{\varepsilon}}\big)\chi_{(-h,0)}, \end{cases} \\ &+ \begin{cases} \Phi(x,y,h) \exp\big(\dfrac{z-h}{\sqrt{\varepsilon}}\big)\chi_{(0,h)}, \\ \Phi(x,y,-h) \exp\big(-\dfrac{z+h}{\sqrt{\varepsilon}}\big)\chi_{(-h,0)}. \end{cases} \end{aligned} \tag{1.4.25}$$

After some computations, we see that the above function verifies the following equation and boundary conditions :

$$-\varepsilon \frac{\partial^2 \overline{\theta}_o^{0,\varepsilon}}{\partial z^2} + \overline{\theta}_o^{0,\varepsilon} = \varepsilon\delta^{(k+2)} + [\Phi(x,y,-h) - \Phi(x,y,h)]\varepsilon\alpha(\varepsilon)\delta' - [\Phi(x,y,-h) + \Phi(x,y,h)]\sqrt{\varepsilon}\alpha(\varepsilon)\delta, \tag{1.4.26}$$

$$\overline{\theta}_o^{0,\varepsilon}(h) = \Phi(x,y,h) - \frac{\alpha(\varepsilon)}{2\varepsilon^q}, \tag{1.4.27}$$

$$\overline{\theta}_o^{0,\varepsilon}(-h) = \Phi(x,y,-h) + \frac{\alpha(\varepsilon)}{2\varepsilon^q}, \tag{1.4.28}$$

and

$$\overline{\theta}_o^{0,\varepsilon} \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{1.4.29}$$

Similarly, to absorb the singularities δ and δ' in the right-hand side of (1.4.26), we define the corrector θ_o^ε as follows :

$$\theta_o^\varepsilon = \overline{\theta}_o^{0,\varepsilon} - \varphi^\varepsilon, \tag{1.4.30}$$

where φ^ε is still given by (1.3.9).

It is easy to see that the equation and boundary conditions for θ_o^ε read

$$-\varepsilon \frac{\partial^2 \theta_o^\varepsilon}{\partial z^2} + \theta_o^\varepsilon = \varepsilon\delta^{(k+2)}, \qquad in\Omega, \tag{1.4.31}$$

$$\theta_e^\varepsilon(z=h) = \Phi(x,y,h) + \frac{\alpha(\varepsilon)}{2\varepsilon^q} - \varphi^\varepsilon(x,y,h), \tag{1.4.32}$$

$$\theta_e^\varepsilon(z=-h) = \Phi(x,y,-h) + \frac{\alpha(\varepsilon)}{2\varepsilon^q} - \varphi^\varepsilon(x,y,-h). \tag{1.4.33}$$

Finally, we consider successively $w_o^\varepsilon = u^\varepsilon - u^0 - \theta_o^\varepsilon$ and $\widetilde{w}_o^\varepsilon = w_o^\varepsilon - \psi_o^\varepsilon$, where we have set

$$\psi_o^\varepsilon(x,y,z) = \Big[\frac{\varphi^\varepsilon(x,y,h) - \varphi^\varepsilon(x,y,-h)}{2h} + \frac{\alpha(\varepsilon)}{2h\varepsilon^q}\Big]z + \frac{\varphi^\varepsilon(x,y,h) + \varphi^\varepsilon(x,y,-h)}{2}; \quad (1.4.34)$$

$\widetilde{w}_o^\varepsilon$ vanishes at the boundary Γ *i.e.* at $z = \pm h$ and it satisfies an equation similar to that satisfied by $\widetilde{w}_e^\varepsilon$ above, provided that we replace ψ_e^ε and θ_e^ε by ψ_o^ε and θ_o^ε respectively.

Moreover, ψ_o^ε is also small in $H^m(\Omega)$, $\forall m \in N$, and verifies, thanks to (1.4.21) and (1.4.34), the estimates

$$\|\psi_o^\varepsilon\|_{L^2(\Omega)} \le c\,\varepsilon^{-q}\alpha(\varepsilon), \quad (1.4.35)$$

and

$$\Big\|\frac{\partial^{m+n}\psi_o^\varepsilon}{\partial x^m \partial y^n}\Big\|_{L^2(\Omega)} \le c\,\varepsilon^{-1/2}\alpha(\varepsilon), \quad \forall (m,n) \in N^* \times N. \quad (1.4.36)$$

On the other hand θ_o^ε satisfies tangential estimates analogous to those verified by θ_e^ε (see (1.4.24)).

Hence, by applying the standard energy methods, the estimates (1.4.5) and (1.4.6) hold also for w_o^ε.

This completes the proof of Theorem 1.4.3.

Remark 7 *:* In general, we may consider many other singularities; in particular any finite linear combination of all singularities studied in this paper can be treated.

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Chapitre 2

Non classical boundary layers for fourth-order equation with singular limit solution

Abstract

In this article we study non classical singular perturbation problems involving boundary layers in the interior of the domain. As usual, these problems contain a small parameter which produces, when this parameter approaches zero, classical boundary layers located at the boundary. If we moreover consider a singular source function, we produce also boundary layers inside the domain. Our aim in this article is to study this kind of boundary layers. We consider, here, a model with a fourth-order differential operator, and the open set is a channel to avoid the technicalities due to the curvature of the boundary. Other stationary problems and time dependent problems will be considered elsewhere.

2.1 Introduction and main results.

Many physical problems are characterized by the presence of a small parameter such as a small diffusion coefficient or a small viscosity coefficient, in particular in fluid and solid mechanics and generally this small parameter generates singular perturbations. Many articles are devoted to such singular perturbation in the fluid mechanics, solid mechanics, and mathematical literatures (see for instance Eckhaus [E], Friedrichs [Fri], Chang and Howes [CH], Germain [G], Balian and Peube [BP], Lions [Li], Vishik and Lyusternik [VL] among many other articles). When the small parameter approaches zero, boundary layers appear in the domain. Classically, in the theory of singular perturbations, the boundary layers appear only at the boundary of the domain. This is the case in particular for the famous example of Friedrichs (1941) concerning the structure of these boundary layers, in the long article of Vishik and Lyusternik [VL] and in the book of Lions [Li]; different aspects of these problems are also considered by Oleinik [O], Frank [Fra], Lagerström [Lag], Ladyzhenskaya [Lad]. The singular perturbation phenomenon occurs when the boundary conditions required for the perturbed and limit problems do not match. In studying the behavior of these boundary layers, we usually assume that the source functions and all data are as regular as necessary. However, this is not always the case in physically relevant problems and it is natural to ask and study what will happen in the absence of regularity. In an earlier article [H], we studied such a problem associated to second-order elliptic problems and we proved that, when the parameter goes to zero and the source function is discontinuous (singular), a discontinuity is produced in the limit solution, which is supplemented by the disparity of one or more boundary conditions. Since the perturbed solution is regular, we note the appearance of boundary layers where singularities exist for the limit solution.

The purpose of this article is to introduce and study similar problems using the terminology of the boundary layer “corrector” introduced by Lions [Li] and used in [TW1-3] and in many other references. In addition to the properties of this type of boundary layers that had been studied in the second-order elliptic case in [H], we emphasize here on the effect of a fourth-order regularisation on the oder of magnitude in ε in the estimates of the convergence results which are usually devoted to confirm the choice of the corrector.

For simplicity, we will focus on the following fourth-order linear model :

$$\varepsilon \Delta^2 u^\varepsilon + u^\varepsilon = H, \qquad in \ \ \Omega, \tag{2.1.1}$$

$$u^\varepsilon = \Phi_0, \qquad \frac{\partial u^\varepsilon}{\partial \nu} = \Phi_1, \qquad on \ \ \Gamma. \tag{2.1.2}$$

Here Φ_0 and Φ_1 are two given (smooth) functions, Ω is a channel of R^n and H is the Heaviside function in the direction orthogonal to the walls (z-direction).

For the sake of simplicity, we assume that $n = 3$ and that Ω is of the following form

$$\Omega = (0, L_1) \times (0, L_2) \times (-h, h),$$

and we write

$$\Gamma = (0, L_1) \times (0, L_2) \times \{-h, h\}.$$

In the x and y directions the boundary condition is periodicity for u^ε :

$$u^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{2.1.3}$$

Note that the restriction on the dimension of space is not essential since we are treating a linear problem. The choice of dimension three is rather arbitrary.

The corresponding inviscid problem is simply

$$u^0 = H(z) = \begin{cases} 1 & \text{if } z \in (0, h), \\ \\ 0 & \text{if } z \in (-h, 0). \end{cases} \tag{2.1.4}$$

We easily observe that u^0 does not fulfill (2.1.2) (this produces a classical boundary layer at $z = \pm h$) and that it displays a discontinuity at $z = 0$ producing there a non classical boundary layer which will be our main objective.

On the other hand, it is obvious that $u^\varepsilon \in H^4(\Omega)$ since $H \in L^2(\Omega)$. Hence, u^ε can not converge to u^0 in $H^1(\Omega)$ as ε approaches to zero since $\partial u^0/\partial z = \delta \notin L^2(\Omega)$, where δ stands for the Dirac distribution concentrated on the surface $z = 0$, and $\partial u^\varepsilon/\partial z \in H^3(\Omega)$. Usually, in the theory of boundary layers, some derivatives of the regularized solution become very large, as ε becomes small, near the boundary only. In our problem this occurs

also inside the domain Ω, more precisely where the limit solution is discontinuous. In Section 2.2, we will illustrate this by just solving the corresponding problem in dimension one, leading to a result useful in the higher dimensional case.
Then in higher dimension, our main results for problem (2.1.1)– (2.1.3) are the following ones :

Theorem 2.1.4 *: Let u^ε be the solution of (2.1.1)–(2.1.3) and u^0 the solution of (2.1.4). There exists a corrector θ^ε depending only on x, y and $z/\varepsilon^{1/4}$, and a constant C independent of ε such that*

$$\begin{aligned} \|u^\varepsilon - u^0 - \theta^\varepsilon\|_{L^2(\Omega)} &\leq C\ \varepsilon^{5/8}, && (2.1.5)\\ \|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^1(\Omega)} &\leq C\ \varepsilon^{3/8}, && (2.1.6)\\ \|u^\varepsilon - u^0 - \theta^\varepsilon\|_{H^2(\Omega)} &\leq C\ \varepsilon^{1/8}. && (2.1.7)\end{aligned}$$

The above result can be extended to more singular cases, e.g. when we replace, in the right-hand side of (2.1.1), $H \in L^2(\Omega)$ by $\delta^{(k)} \in H^{-1-k}(\Omega)$, where $\delta^{(k)}$ stands for the k^{th} derivative, in the z-variable, of the Dirac distribution δ ($\delta = \mathrm{d}H/\mathrm{d}z$).
Namely, we study u_k^ε solution of the following equation

$$\varepsilon\Delta^2 u_k^\varepsilon + u_k^\varepsilon = \delta^{(k)}, \qquad in\ \ \Omega, \tag{2.1.8}$$

and verifying the same boundary and periodicity conditions as u^ε, namely (2.1.2) and (2.1.3).
Here, the limit problem is

$$u_k^0 = \delta^{(k)}, \qquad in\ \ \Omega, \tag{2.1.9}$$

and we have the following result.

Theorem 2.1.5 *: There exists a corrector θ_k^ε, function of $x, y, z/\varepsilon^{1/4}$ and k, and a constant C depending on the data but not on ε and such that :*

$$\begin{aligned} \|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{L^2(\Omega)} &\leq C\ \varepsilon^{5/8}, && (2.1.10)\\ \|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^1(\Omega)} &\leq C\ \varepsilon^{3/8}, && (2.1.11)\\ \|u_k^\varepsilon - u_k^0 - \theta_k^\varepsilon\|_{H^2(\Omega)} &\leq C\ \varepsilon^{1/8}. && (2.1.12)\end{aligned}$$

Remark 8 Throughout this article, C will denote a generic constant depending on all the data except ε. Its value may be different at each occurrence.

Remark 9 The methods used below apply to many other equations as long as the corresponding one-dimensional problem remains explicitly solvable. In particular, we may replace, in (2.1.1), the term u^ε by $c(x,y)u^\varepsilon$.

Remark 10 Our results can be extended to other source functions with general singularities, in particular a finite linear combination of H and the $\delta^{(k)}$ can be treated.

Remark 11 The existence of u_k^ε solution of (2.1.2), (2.1.3), and (2.1.8), is based on the duality argument, see for instance the book of Lions-Magenes [LM].

The rest of this article is organized as follows : In the next section we study the one-dimensional problem corresponding to (2.1.1) and (2.1.8) and associated with homogeneous boundary conditions; this will be useful subsequently to describe the corrector. In Section 2.3, we prove Theorem 2.1.4 and we derive the first order term in the asymptotic expansion of u^ε with respect to ε. The last section is devoted to the proof of Theorem 2.1.5 which is concerned with the enhanced singularity for the source function.

2.2 The one-dimensional case

It is well known, in the theory of boundary layers, that the most important derivative variations are produced in the direction normal to the boundary. As we will see later on, this is still true for the boundary layer in the interior of the domain. Hence, in view of the geometry of Ω, we will be interested in the one-dimensional problems in the z variable. We start with a singularity belonging to $L^2(\Omega)$ ($L^\infty(\Omega)$ in fact).

2.2.1 A discontinuous source function in $L^2(\Omega)$.

We intend to solve explicitly the following problem

$$\varepsilon\,\frac{d^4 u^\varepsilon}{dz^4} + u^\varepsilon = H, \qquad in \quad \Omega = (-h, h), \tag{2.2.1}$$

$$u^\varepsilon = \frac{d\,u^\varepsilon}{dz} = 0, \qquad at \quad z = \pm h. \tag{2.2.2}$$

It is obvious that if $u^\varepsilon \in L^2(\Omega)$, then $d^4 u^\varepsilon/dz^4 \in L^2(\Omega)$. Hence, by the Sobolev embedding theorem,

$$u^\varepsilon \in H^4(\Omega) \hookrightarrow C^3(\overline{\Omega}). \tag{2.2.3}$$

Thus, we write

$$u^\varepsilon(z) = u_1^\varepsilon(z)\chi_{(0,h)} + u_2^\varepsilon(z)\chi_{(-h,0)}, \tag{2.2.4}$$

and we solve separately (2.2.1)-(2.2.2) on $(-h, 0)$ and $(0, h)$, then we match the solutions u_1^ε and u_2^ε and their derivatives of order $1, 2$ and 3 at $z = 0$ with the help of the regularity conditions implied by (2.2.3).

The solutions u_1^ε and u_2^ε have the following form

$$\begin{aligned} u_1^\varepsilon(z) = 1 &+ \exp\big(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{A_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + B_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\} + \\ &+ \exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{C_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + D_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}, \end{aligned} \tag{2.2.5}$$

for all $z \in (0, h)$, and,

$$\begin{aligned} u_2^\varepsilon(z) &= \exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{E_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + F_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\} + \\ &\quad + \exp\big(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{G_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + H_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}, \end{aligned} \tag{2.2.6}$$

for all $z \in (-h, 0)$.

Hence, writing the boundary and regularity conditions ((2.2.2) and (2.2.3)) for u_1^ε and u_2^ε leads to a linear system of the form

$$(\mathcal{S}_\varepsilon) \qquad \mathcal{A}_\varepsilon\, \mathcal{X}_\varepsilon = \mathcal{B}_\varepsilon,$$

where, $\mathcal{B}_\varepsilon = \mathcal{B} = (-1,0,0,0,-1,0,0,0) \in R^8$ is a vector, $\mathcal{X}_\varepsilon = (A_\varepsilon, B_\varepsilon, ..., H_\varepsilon)$ is the unknown variable, and $\mathcal{A}_\varepsilon \in M_8(R)$ is the real square matrix

$$\mathcal{A}_\varepsilon = \begin{bmatrix} \beta_\varepsilon & \gamma_\varepsilon & \alpha(\varepsilon)\beta_\varepsilon & \alpha(\varepsilon)\gamma_\varepsilon & 0 & 0 & 0 & 0 \\ \beta_\varepsilon - \gamma_\varepsilon & \beta_\varepsilon + \gamma_\varepsilon & -\alpha(\beta_\varepsilon + \gamma_\varepsilon) & \alpha(\beta_\varepsilon - \gamma_\varepsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha(\varepsilon)\beta_\varepsilon & -\alpha(\varepsilon)\gamma_\varepsilon & \beta_\varepsilon & -\gamma_\varepsilon \\ 0 & 0 & 0 & 0 & \alpha(\beta_\varepsilon + \gamma_\varepsilon) & \alpha(\beta_\varepsilon - \gamma_\varepsilon) & \gamma_\varepsilon - \beta_\varepsilon & \beta_\varepsilon + \gamma_\varepsilon \\ \alpha(\varepsilon) & 0 & 1 & 0 & -1 & 0 & -\alpha(\varepsilon) & 0 \\ \alpha(\varepsilon) & \alpha(\varepsilon) & -1 & 1 & -1 & -1 & \alpha(\varepsilon) & -\alpha(\varepsilon) \\ 0 & 2\alpha(\varepsilon) & 0 & -2 & 0 & -2 & 0 & 2\alpha(\varepsilon) \\ -2\alpha(\varepsilon) & 2\alpha(\varepsilon) & 2 & 2 & 2 & -2 & -2\alpha(\varepsilon) & -2\alpha(\varepsilon) \end{bmatrix}.$$

Note that in the matrix $\mathcal{A}_\varepsilon$, the lines 1 and 2 (respectively 3 and 4) express the boundary conditions at $z = h$ (respectively at $z = -h$), the lines 5 to 8 express the matching at $z = 0$ of u_1^ε and u_2^ε and their derivatives up to order three.

We have set

$$\alpha = \alpha(\varepsilon) = \exp\left(-\frac{h}{\sqrt{2}\varepsilon^{1/4}}\right), \tag{2.2.7}$$

and

$$\beta_\varepsilon = \cos\left(\frac{h}{\sqrt{2}\varepsilon^{1/4}}\right), \qquad \gamma_\varepsilon = \sin\left(\frac{h}{\sqrt{2}\varepsilon^{1/4}}\right). \tag{2.2.8}$$

We easily deduce a simple approximate solution of $(\mathcal{S}_\varepsilon)$, denoted by $\overline{\mathcal{X}}_\varepsilon = (\overline{A}_\varepsilon, .., \overline{H}_\varepsilon)$. Thanks to the relation

$$\beta_\varepsilon^3 + \gamma_\varepsilon^3 + \beta_\varepsilon^2\gamma_\varepsilon + \gamma_\varepsilon^2\beta_\varepsilon = \beta_\varepsilon + \gamma_\varepsilon,$$

which is a simple consequence of the fact that β_ε and γ_ε satisfy $\beta_\varepsilon^2 + \gamma_\varepsilon^2 = 1$, we find

$$\begin{aligned} \overline{A}_\varepsilon &= -(\beta_\varepsilon + \gamma_\varepsilon), & \overline{B}_\varepsilon &= \beta_\varepsilon - \gamma_\varepsilon, \\ \overline{C}_\varepsilon &= -\frac{1}{2}, & \overline{D}_\varepsilon &= -\beta_\varepsilon^2(\alpha(\varepsilon))^2, \\ \overline{E}_\varepsilon &= \frac{1}{2}, & \overline{F}_\varepsilon &= (\beta_\varepsilon - \gamma_\varepsilon)\alpha(\varepsilon), \\ \overline{G}_\varepsilon &= -\frac{1 + 2\beta_\varepsilon\gamma_\varepsilon}{2}\alpha(\varepsilon), & \overline{H}_\varepsilon &= -\beta_\varepsilon^2\alpha(\varepsilon). \end{aligned}$$

When $f_\varepsilon = A_\varepsilon, .., $ or H_ε, it verifies

$$f_\varepsilon = \overline{f}_\varepsilon + O(\alpha(\varepsilon)), \tag{2.2.9}$$

so that $\overline{f}_\varepsilon$ is an approximation of f_ε at exponential order.

This leads to an approximation $\overline{u}^\varepsilon$ of u^ε, up to an order subsequently specified :

$$\begin{aligned}\overline{u}^\varepsilon(z) \;&=\; 1 + \exp\big(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\overline{A}_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \overline{B}_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\} + \\ &\quad + \exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\overline{C}_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \overline{D}_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\},\end{aligned} \tag{2.2.10}$$

for all $z \in (0, h)$ and

$$\begin{aligned}\overline{u}^\varepsilon(z) \;&=\; \exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\overline{E}_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \overline{F}_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\} + \\ &\quad + \exp\big(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\overline{G}_\varepsilon \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \overline{H}_\varepsilon \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\},\end{aligned} \tag{2.2.11}$$

for all $z \in (-h, 0)$.

Now, it is easy to see that $\overline{u}^\varepsilon$ verifies the following boundary conditions :

$$\begin{aligned}\overline{u}^\varepsilon(-h) \;&=\; -(\beta_\varepsilon - \gamma_\varepsilon)(\alpha(\varepsilon))^2, \\ \overline{u}^\varepsilon(h) \;&=\; -\frac{1}{2}\beta_\varepsilon\alpha(\varepsilon) - 2\beta_\varepsilon^2\gamma_\varepsilon(\alpha(\varepsilon))^3, \\ \frac{d\overline{u}^\varepsilon}{dz}(-h) \;&=\; \frac{\varepsilon^{-1/4}}{\sqrt{2}}\Big\{\frac{1}{2}(\beta_\varepsilon + \gamma_\varepsilon) - 2(\beta_\varepsilon - \gamma_\varepsilon)\beta_\varepsilon^2\Big\}\alpha(\varepsilon), \\ \frac{d\overline{u}^\varepsilon}{dz}(h) \;&=\; \frac{\varepsilon^{-1/4}}{\sqrt{2}}(\beta_\varepsilon - \gamma_\varepsilon)^2(\alpha(\varepsilon))^2.\end{aligned}$$

We observe that the boundary values of $\overline{u}^\varepsilon$ and $d\overline{u}^\varepsilon/dz$ converge exponentially fast to zero as $\varepsilon \longrightarrow 0$. This shows how $\overline{u}^\varepsilon$ approximates u^ε near the boundary. At $z = 0$ where $\overline{u}^\varepsilon$ and its derivatives of order $1, 2$, and 3 are not continuous, we have :

$$\begin{aligned}\overline{u}^\varepsilon(z = 0^-) \;&=\; \frac{1}{2} - \big(\frac{1}{2} + \beta_\varepsilon\gamma_\varepsilon\big)(\alpha(\varepsilon))^2, \\ \overline{u}^\varepsilon(z = 0^+) \;&=\; \frac{1}{2} - (\beta_\varepsilon + \gamma_\varepsilon)\,\alpha(\varepsilon), \\ \frac{d\overline{u}^\varepsilon}{dz}(z = 0^-) \;&=\; \frac{\varepsilon^{-1/4}}{2\sqrt{2}} + \frac{\beta_\varepsilon - \gamma_\varepsilon}{\sqrt{2}\varepsilon^{1/4}}\,\alpha(\varepsilon) + \frac{1}{\sqrt{2}\varepsilon^{1/4}}\big(\frac{1}{2} + \beta_\varepsilon\gamma_\varepsilon - \beta_\varepsilon^2\big)(\alpha(\varepsilon))^2,\end{aligned}$$

$$\begin{aligned}
\frac{d\overline{u}^\varepsilon}{dz}(z=0^+) &= \frac{\varepsilon^{-1/4}}{2\sqrt{2}} - \frac{\sqrt{2}\gamma_\varepsilon}{\varepsilon^{1/4}}\alpha(\varepsilon) - \frac{\sqrt{2}\beta_\varepsilon^2}{\varepsilon^{1/4}}(\alpha(\varepsilon))^2, \\
\frac{d^2\overline{u}^\varepsilon}{dz^2}(z=0^-) &= \frac{1}{\varepsilon^{1/2}}\{(\beta_\varepsilon - \gamma_\varepsilon)\,\alpha(\varepsilon) + \beta_\varepsilon^2)(\alpha(\varepsilon))^2\}, \\
\frac{d^2\overline{u}^\varepsilon}{dz^2}(z=0^+) &= \frac{1}{\varepsilon^{1/2}}\{(\beta_\varepsilon - \gamma_\varepsilon)\,\alpha(\varepsilon) + 2\beta_\varepsilon^2)(\alpha(\varepsilon))^2\}, \\
\frac{d^3\overline{u}^\varepsilon}{dz^3}(z=0^-) &= -\frac{\varepsilon^{-3/4}}{2\sqrt{2}} + \frac{\beta_\varepsilon - \gamma_\varepsilon}{\sqrt{2}\varepsilon^{3/4}}\,\alpha(\varepsilon) - \frac{1+2\beta_\varepsilon\gamma_\varepsilon + 2\beta_\varepsilon^2}{2\sqrt{2}\varepsilon^{3/4}}\,(\alpha(\varepsilon))^2, \\
\frac{d^3\overline{u}^\varepsilon}{dz^3}(z=0^+) &= -\frac{\varepsilon^{-3/4}}{2\sqrt{2}} + \frac{\sqrt{2}\beta_\varepsilon}{\varepsilon^{3/4}}\,\alpha(\varepsilon) - \frac{2\sqrt{2}\beta_\varepsilon^2}{\varepsilon^{3/4}}\,(\alpha(\varepsilon))^2.
\end{aligned}$$

Consequently, $\overline{u}^\varepsilon$ satisfies the equation

$$\begin{aligned}
\varepsilon\frac{d^4\overline{u}^\varepsilon}{dz^4} + \overline{u}^\varepsilon = {} & H - \varepsilon\Big[(\frac{1}{2} + \beta_\varepsilon\gamma_\varepsilon) - (\beta_\varepsilon + \gamma_\varepsilon)\alpha(\varepsilon)\Big]\,\alpha(\varepsilon)\,\delta^{(3)} - \\
& -\frac{\varepsilon^{3/4}}{\sqrt{2}}\Big[\beta_\varepsilon + \gamma_\varepsilon + (\frac{1}{2} + \beta_\varepsilon\gamma_\varepsilon + \beta_\varepsilon^2)\alpha(\varepsilon)\Big]\alpha(\varepsilon)\delta'' + \varepsilon^{1/2}\beta_\varepsilon^2(\alpha(\varepsilon))^2\delta' + \\
& +\frac{\varepsilon^{1/4}}{\sqrt{2}}\Big[\beta_\varepsilon + \gamma_\varepsilon + (\frac{1}{2} + \beta_\varepsilon\gamma_\varepsilon - \beta_\varepsilon^2)\alpha(\varepsilon)\Big]\alpha(\varepsilon)\delta. \qquad (2.2.12)
\end{aligned}$$

Because of the appearance of $\delta, \delta', \delta''$ and $\delta^{(3)}$ in the right-hand side of (2.2.12), we can not apply the standard energy method to prove the convergence of $u^\varepsilon - \overline{u}^\varepsilon$ to 0 in suitable Sobolev spaces. But, if we replace in (2.2.5) and in (2.2.6), $A_\varepsilon, ..., H_\varepsilon$ by their asymptotic expansions, as $\varepsilon \to 0$, using (2.2.9), we conclude from the expression of $\overline{u}^\varepsilon$ (2.2.10)-(2.2.11) that

$$\begin{aligned}
u^\varepsilon(z) = {} & \overline{u}^\varepsilon(z) + O(\alpha(\varepsilon))\exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}\chi_{(0,h)} + \\
& +O(\alpha(\varepsilon))\exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}\chi_{(-h,0)} + O(\alpha(\varepsilon)) \\
& \Big\{\exp\big(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\big) + \exp\big(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}\Big\{\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) + \sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}, \qquad (2.2.13)
\end{aligned}$$

for all $z \in (-h, h)$.

By simple computations, we see that any of the three functions $g = \exp(-z/\varepsilon^{1/4})$

$\Big\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\Big\}\chi_{(0,h)}$, or $\exp(z/\varepsilon^{1/4})\Big\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\Big\}\chi_{(-h,0)}$ or $\Big\{\exp((z-h)/\varepsilon^{1/4}) + \exp(-(z+h)/\varepsilon^{1/4})\Big\}\Big\{\cos(z/\varepsilon^{1/4}) + \sin(z/\varepsilon^{1/4})\Big\}\chi_{(-h,h)}$ verifies the following estimate

$$\|g\|_{H^m(\Omega)} \leq C\,\varepsilon^{(1-2m)/8}, \quad \forall m \geq 0.$$

Using this bound for $m = 0$ together with the triangle inequality applied to (2.2.13), we obtain

$$\|u^\varepsilon - \overline{u}^\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon^{1/8}\alpha(\varepsilon). \tag{2.2.14}$$

Hence, by differentiating (2.2.13) m times with respect to z and knowing that the $O(\alpha(\varepsilon))$ terms are independent of z, we deduce that $\overline{u}^\varepsilon$ is an approximation of u^ε at exponential order in all the Sobolev spaces $H^m(\Omega)$.

Proposition 2.2.1 : *There exists a constant C independent of ε such that*

$$\|u^\varepsilon - \overline{u}^\varepsilon\|_{H^m(\Omega)} \leq C\,\varepsilon^{(1-2m)/8}\alpha(\varepsilon), \qquad \forall m \in N. \tag{2.2.15}$$

Remark 12 : Notice that the singularities $\delta, \delta', \delta''$ and $\delta^{(3)}$ appearing in the right-hand side of (2.2.12) can be eliminated, as we will see later, by introducing an appropriate corrector function; we may then apply the standard energy estimates and obtain an analogous result to (2.2.15). However, the estimate (2.2.15) will be sufficient to derive the corrector in the multidimensional case subsequently. We have choosen to make explicit the equation (2.2.12) verified by $\overline{u}^\varepsilon$, in order to emphasize this difficulty for the rest of this article.

2.2.2 A singular source function in $H^s(\Omega), s < 0$.

In this section, we are interested in the one-dimensional problem corresponding to (2.1.8). As before, this problem is considered in the normal direction (z) :

$$\varepsilon\,\frac{d^4 v^\varepsilon}{dz^4} + v^\varepsilon = \delta^{(k)}, \qquad in \quad \Omega = (-h, h), \tag{2.2.16}$$

$$v^\varepsilon = \frac{d\,v^\varepsilon}{dz} = 0, \qquad at \quad z = \pm h. \tag{2.2.17}$$

To derive the explicit solution of (2.2.16)-(2.2.17), we have to consider four cases depending on the value of k.

Case 3 : If $k = 4q, q \in N$, then we define recursively $v_0^\varepsilon = v^\varepsilon$ and

$$\begin{aligned} v_4^\varepsilon &= v_0^\varepsilon - \frac{\delta^{(4(q-1))}}{\varepsilon}, \\ v_8^\varepsilon &= v_4^\varepsilon + \frac{\delta^{(4(q-2))}}{\varepsilon^2}, \\ &\vdots \\ v_{4q+4}^\varepsilon &= v_{4q}^\varepsilon + (-1)^{q+1}\frac{\delta^{(-4)}}{\varepsilon^{q+1}}. \end{aligned}$$

By summing these equations, we deduce that

$$v^\varepsilon = -\sum_{j=1}^{q+1}(-1)^j \frac{\delta^{(4(q-j))}}{\varepsilon^j} + v_{4q+4}^\varepsilon, \tag{2.2.18}$$

where we have used this convention

$$\delta^{(-i)} = H_i, \qquad \forall i \in \{1,2,3,4\}, \tag{2.2.19}$$

H_i denoting the integrand of H_{i-1} for $i \in \{2,3,4\}$ (by setting $H_1 = H$), *i.e.*,

$$H_i(z) = \frac{z^{i-1}}{(i-1)!}\chi_{(0,h)}, \qquad \forall i \in \{1,2,3,4\}.$$

Hence, v_{4q+4}^ε verifies the equations

$$\varepsilon \; \frac{d^4\, v_{4q+4}^\varepsilon}{dz^4} + v_{4q+4}^\varepsilon = (-1)^{q+1}\frac{H_4}{\varepsilon^{q+1}}, \qquad in \;\; (-h,h), \tag{2.2.20}$$

$$v_{4q+4}^\varepsilon(z=h) = (-1)^{q+1}\frac{h^3}{6\varepsilon^{q+1}}, \qquad v_{4q+4}^\varepsilon(z=-h) = 0, \tag{2.2.21}$$

$$\frac{d\, v_{4q+4}^\varepsilon}{dz}(h) = (-1)^{q+1}\frac{h^2}{2\varepsilon^{q+1}}, \qquad \frac{d\, v_{4q+4}^\varepsilon}{dz}(-h) = 0, \tag{2.2.22}$$

$$v_{4q+4}^\varepsilon \text{ is of class } C^3 \text{ at } z=0. \tag{2.2.23}$$

Thus, we see that we have reduced the singularity of the right-hand side.

Solving again (2.2.20) separately on $(-h,0)$ and $(0,h)$, matching the solutions at $z = 0$ with the help of (2.2.23) and applying the boundary conditions (2.2.21) and

(2.2.22), we obtain a linear system of type $(\mathcal{S}_\varepsilon)$ with the same matrix $\mathcal{A}_\varepsilon$ and with $\mathcal{B}_\varepsilon = (0,0,\ldots,0,2\sqrt{2}(-1)^q/\varepsilon^{q+1/4}) \in R^8$.

The resolution of this system leads to an approximation $\overline{v}^\varepsilon_{4q+4}$ for v^ε_{4q+4} and consequently (thanks to (2.2.18)) to an approximation for v^ε called $\overline{v}^\varepsilon$ which is given by

$$\begin{aligned}\overline{v}^\varepsilon(z) \;=\; & -\sum_{j=1}^{q}(-1)^j\,\frac{\delta^{(4(q-j))}}{\varepsilon^j} + \frac{\sqrt{2}\,(-1)^q}{4\varepsilon^{q+1/4}}\Big\{\exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\\ & \Big[\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)+\sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big]\chi_{(0,h)} +\\ & +\exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big[\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)-\sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big]\chi_{(-h,0)}\Big\}.\end{aligned} \tag{2.2.24}$$

Note that the approximation $\overline{v}^\varepsilon$ of v^ε is defined in the following sense

$$v^\varepsilon = \overline{v}^\varepsilon + e.s.t., \tag{2.2.25}$$

where $e.s.t.$ stands for exponentially small(er) terms on $(-h,h)$ (exponentially small in all usual norms such as $\mathcal{C}^k, H^m, \ldots$).

Case 4 : If $k = 4q+1, q \in N$, we define recursively

$$v_1^\varepsilon = v^\varepsilon, \qquad \text{and} \quad v^\varepsilon_{4j+1} = v^\varepsilon_{4j-3} + (-1)^j\,\frac{\delta^{(4(q-j)+1)}}{\varepsilon^j}, \quad \forall j \geq 1. \tag{2.2.26}$$

Writing (2.2.26) for $j = 1, 2, \ldots, q+1$, using the convention (2.2.19), and summing these equations, we find

$$v^\varepsilon = -\sum_{j=1}^{q+1}(-1)^j\,\frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + v^\varepsilon_{4q+5}, \tag{2.2.27}$$

where v^ε_{4q+5} satisfies

$$\varepsilon\,\frac{d^4\,v^\varepsilon_{4q+5}}{dz^4} + v^\varepsilon_{4q+5} = (-1)^{q+1}\,\frac{H_3}{\varepsilon^{q+1}}, \qquad in\;\;(-h,h), \tag{2.2.28}$$

$$v^\varepsilon_{4q+5}(z=h) = (-1)^{q+1}\frac{h^2}{2\varepsilon^{q+1}}, \qquad v^\varepsilon_{4q+5}(z=-h) = 0, \tag{2.2.29}$$

$$\frac{d\,v^\varepsilon_{4q+5}}{dz}(h) = (-1)^{q+1}\frac{h}{\varepsilon^{q+1}}, \qquad \frac{d\,v^\varepsilon_{4q+5}}{dz}(-h) = 0, \tag{2.2.30}$$

$$v^\varepsilon_{4q+5}\text{ is of class } C^3 \text{ at } z=0. \tag{2.2.31}$$

Thus, solving (2.2.28)–(2.2.31) is equivalent to solving a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (0,0,\ldots,0,-2(-1)^{q+1}/\varepsilon^{q+1/2},0) \in R^8$. We then obtain the explicit expression of

v^ε_{4q+5} of which we give an approximation to within an *e.l.t.*

Hence, we deduce thanks to (2.2.27) an approximation $\overline{v}^\varepsilon$ of v^ε in this case ($k = 4q+1$):

$$\begin{aligned}\overline{v}^\varepsilon(z) &= -\sum_{j=1}^{q}(-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \frac{(-1)^{q+1}}{2\ \varepsilon^{q+1/2}} \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \Big\{ \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\chi_{(0,h)} + \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\chi_{(-h,0)}\Big\}. \qquad (2.2.32)\end{aligned}$$

Case 5 : For $k = 4q+2, q \in N$, we define similarly

$$v^\varepsilon_2 = v^\varepsilon, \quad \text{and} \quad v^\varepsilon_{4j+2} = v^\varepsilon_{4j-2} + (-1)^j\ \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j}, \quad \forall j \geq 1. \qquad (2.2.33)$$

Using again the convention (2.2.19), we write (2.2.33) for $j = 1, 2, \ldots, q+1$ and summing these $(q+1)$ equations, we obtain

$$v^\varepsilon = -\sum_{j=1}^{q+1}(-1)^j\ \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + v^\varepsilon_{4q+6}, \qquad (2.2.34)$$

where v^ε_{4q+6} satisfies

$$\varepsilon\ \frac{d^4 v^\varepsilon_{4q+6}}{dz^4} + v^\varepsilon_{4q+6} = (-1)^{q+1}\ \frac{H_2}{\varepsilon^{q+1}}, \qquad in\ \ (-h,h), \qquad (2.2.35)$$

$$v^\varepsilon_{4q+6}(z=h) = (-1)^{q+1}\frac{h}{\varepsilon^{q+1}}, \qquad v^\varepsilon_{4q+6}(z=-h) = 0, \qquad (2.2.36)$$

$$\frac{d\, v^\varepsilon_{4q+6}}{dz}(h) = \frac{(-1)^{q+1}}{\varepsilon^{q+1}}, \qquad \frac{d\, v^\varepsilon_{4q+6}}{dz}(-h) = 0, \qquad (2.2.37)$$

$$v^\varepsilon_{4q+6} \text{ is of class } C^3 \text{ at } z = 0. \qquad (2.2.38)$$

As before, this leads to a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (0, \ldots, 0, (-1)^q\sqrt{2}/\varepsilon^{q+1/2}$, $0, 0)$, a vector of R^8.

Furthermore, we deduce an approximation of v^ε called $\overline{v}^\varepsilon$ and given by

$$\begin{aligned}\overline{v}^\varepsilon(z) &= -\sum_{j=1}^{q}(-1)^j \frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \frac{\sqrt{2}(-1)^{q+1}}{4\varepsilon^{q+3/4}} \Big\{ \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) \\ &\quad \Big[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\Big]\chi_{(0,h)} + \\ &\quad + \exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\Big[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\Big]\chi_{(-h,0)}\Big\}. \qquad (2.2.39)\end{aligned}$$

Case 6 : Similarly for $k = 4q+3, q \in N$, we write $v_3^\varepsilon = v^\varepsilon$ and we define recursively

$$v^\varepsilon_{4j+3} = v^\varepsilon_{4j-1} + (-1)^j \, \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j}, \quad \forall j \geq 1. \tag{2.2.40}$$

As in the above cases, by summing the equations (2.2.40), we deduce that

$$v^\varepsilon = -\sum_{j=1}^{q+1} (-1)^j \, \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + v^\varepsilon_{4q+7}, \tag{2.2.41}$$

where v^ε_{4q+7} satisfies the following equations

$$\varepsilon \, \frac{d^4 \, v^\varepsilon_{4q+7}}{dz^4} + v^\varepsilon_{4q+7} = (-1)^{q+1} \, \frac{H}{\varepsilon^{q+1}}, \qquad in \;\; (-h,h), \tag{2.2.42}$$

$$v^\varepsilon_{4q+7}(z=h) = \frac{(-1)^{q+1}}{\varepsilon^{q+1}}, \qquad v^\varepsilon_{4q+7}(z=-h) = 0, \tag{2.2.43}$$

$$\frac{d\, v^\varepsilon_{4q+7}}{dz}(h) = \frac{d\, v^\varepsilon_{4q+7}}{dz}(-h) = 0, \tag{2.2.44}$$

$$v^\varepsilon_{4q+7} \text{ is of class } C^3 \text{ at } z = 0. \tag{2.2.45}$$

We apply the same techniques to resolve the system of equations (2.2.42) to (2.2.45) (system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (0, \ldots, 0, -1/\varepsilon^{q+1}, 0, 0, 0) \in R^8$) and we derive an approximation for v^ε_{4q+7} and then for v^ε :

$$\begin{aligned} \overline{v}^\varepsilon(z) \;\; &= \;\; -\sum_{j=1}^{q} (-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \frac{(-1)^q}{2\;\varepsilon^{q+1}} \cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big) \\ & \Big\{ \exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\chi_{(0,h)} - \exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\chi_{(-h,0)} \Big\}. \end{aligned} \tag{2.2.46}$$

Finally, we gather all these results in the following proposition :

Proposition 2.2.2 *The solution v^ε of (2.2.16)-(2.2.17) possesses an approximation to within an e.s.t. denoted by $\overline{v}^\varepsilon$, i.e.,*

$$v^\varepsilon = \overline{v}^\varepsilon + e.s.t., \tag{2.2.47}$$

where $\overline{v}^\varepsilon$ is given by (2.2.24), (2.2.32), (2.2.39) and (2.2.46) according to the value of k, the e.s.t. term being exponentially small(er) in all classical spaces (such as $\mathcal{C}^k, H^m, \ldots$).

Remark 13 : We can obviously derive the full explicit expressions of u^ε or v^ε, but the results of Proposition 2.2.1 and Proposition 2.2.2, which make explicit their (simpler) approximations $\overline{u}^\varepsilon$ and $\overline{v}^\varepsilon$, will be sufficient to derive the corrector and prove the convergence results in higher dimensions.

2.3 Proof of Theorem 2.1.4.

This section is devoted to the proof of Theorem 2.1.4. Firstly, we start by defining the corrector and secondly we prove the estimates (2.1.5), (2.1.6) and (2.1.7).

2.3.1 Construction of the corrector θ^ε.

Based on the resolution of an analogous one-dimensional problem to (2.2.1)-(2.2.2) but with nontrivial boundary conditions, (for which we may prove a result analogous to Proposition 2.2.1), we propose the following corrector the validity of which will be rigourosly justified later:

$$\overline{\theta}^{\varepsilon,0} = \overline{U}^\varepsilon - u^0. \tag{2.3.1}$$

Here $\overline{U}^\varepsilon$ is an approximation of U^ε solution of the following system

$$(\mathcal{C}_\varepsilon) \quad \begin{cases} \varepsilon \dfrac{\partial^4 U^\varepsilon}{\partial z^4} + U^\varepsilon = H, & \text{in } (-h,h), \\ \\ U^\varepsilon = \Phi_0(x,y,.), & \text{at } z = \pm h, \\ \\ \dfrac{\partial U^\varepsilon}{\partial z} = -\Phi_1(x,y,.), & \text{at } z = \pm h. \end{cases}$$

The resolution of $(\mathcal{C}_\varepsilon)$ is based on the same techniques as those used to solve (2.2.1)-(2.2.2) in Section 2.2.1. Thus, the expression of $\overline{U}^\varepsilon$ can be made explicit.
After lengthy but straight forward calculations, the expression of $\overline{\theta}^{\varepsilon,0}$ is given by

$$\begin{aligned} \overline{\theta}^{\varepsilon,0}(x,y,z) &= \exp\big(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\Big[-(\beta_\varepsilon+\gamma_\varepsilon)+(\beta_\varepsilon+\gamma_\varepsilon)\Phi_0(h)+ \\ &\quad +\sqrt{2}\;\varepsilon^{1/4}\gamma_\varepsilon\Phi_1(h)\Big]\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)+\Big[\beta_\varepsilon-\gamma_\varepsilon- \\ &\quad -(\beta_\varepsilon-\gamma_\varepsilon)\Phi_0(h)-\sqrt{2}\;\varepsilon^{1/4}\beta_\varepsilon\Phi_1(h)\Big]\sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}- \\ &\quad -\frac{1}{2}\exp\big(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big), \end{aligned} \tag{2.3.2}$$

for all $(x,y,z) \in \Omega^+$ and

$$\overline{\theta}^{\varepsilon,0}(x,y,z) = \exp\big(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\{\Big[(\beta_\varepsilon+\gamma_\varepsilon)\Phi_0(-h)-$$

$$-\sqrt{2}\,\varepsilon^{1/4}\gamma_\varepsilon\Phi_1(-h)\Big]\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)+\Big[(\beta_\varepsilon-\gamma_\varepsilon)$$
$$\Phi_0(-h)-\sqrt{2}\,\varepsilon^{1/4}\beta_\varepsilon\Phi_1(-h)\Big]\sin\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\Big\}+$$
$$+\frac{1}{2}\exp\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big)\cos\big(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\big), \tag{2.3.3}$$

for all $(x,y,z)\in\Omega^-$. Here we have set

$$\Omega^+=\Big\{(x,y,z)\in\Omega\ /\ z>0\Big\},\qquad \Omega^-=\Big\{(x,y,z)\in\Omega\ /\ z<0\Big\}.$$

Note that in (2.3.2), (2.3.3) and hereafter, some terms such as $\Phi_i(\pm h)$ may depend on x or y, but for simplicity this dependence is not mentioned in our writing. Secondly, observe that the functions $\overline{\theta}^{\varepsilon,0}$ and their derivatives are not continuous at $z=0$. Hence, after some computations, the equation and boundary conditions for $\overline{\theta}^{\varepsilon,0}$ read

$$\begin{aligned}\varepsilon\frac{\partial^4\overline{\theta}^{\varepsilon,0}}{\partial z^4}+\overline{\theta}^{\varepsilon,0} &= -\varepsilon\,\delta^{(3)}+\varepsilon\alpha(\varepsilon)\lambda_3(x,y)\,\delta^{(3)}+\frac{\varepsilon^{3/4}}{\sqrt{2}}\alpha(\varepsilon)\lambda_2(x,y)\,\delta''+\\ &\quad+\frac{\varepsilon^{1/2}}{2}\alpha(\varepsilon)\lambda_1(x,y)\,\delta'+\frac{\varepsilon^{1/4}}{2\sqrt{2}}\alpha(\varepsilon)\lambda_0(x,y)\,\delta,\quad \text{in }\ \Omega,\end{aligned} \tag{2.3.4}$$

$$\overline{\theta}^{\varepsilon,0} = -u^0+\Phi_0-\frac{sign(z)}{4}\alpha(\varepsilon),\qquad \text{on }\ \Gamma, \tag{2.3.5}$$
$$\frac{\partial\overline{\theta}^{\varepsilon,0}}{\partial\nu} = \Phi_1-\frac{\varepsilon^{-1/4}}{4}\alpha(\varepsilon),\qquad \text{on }\ \Gamma, \tag{2.3.6}$$

where

$$\lambda_i(x,y)=\lambda_i^+(x,y)-\lambda_i^-(x,y),\quad i\in\{0,1,2,3\}, \tag{2.3.7}$$

and

$$\begin{aligned}\lambda_3^+(x,y) &= -(\beta_\varepsilon+\gamma_\varepsilon)+(\beta_\varepsilon+\gamma_\varepsilon)\Phi_0(h)+\sqrt{2}\gamma_\varepsilon\varepsilon^{1/4}\Phi_1(h),\\ \lambda_3^-(x,y) &= (\beta_\varepsilon+\gamma_\varepsilon)\Phi_0(-h)-\sqrt{2}\gamma_\varepsilon\varepsilon^{1/4}\gamma_\varepsilon\Phi_1(-h),\\ \lambda_2^+(x,y) &= \frac{1}{2}-2\gamma_\varepsilon+2\gamma_\varepsilon\Phi_0(h)-\sqrt{2}(\beta_\varepsilon-\gamma_\varepsilon)\varepsilon^{1/4}\Phi_1(h),\\ \lambda_2^-(x,y) &= \frac{1}{2}-2\gamma_\varepsilon\Phi_0(-h)+\sqrt{2}(\gamma_\varepsilon-\beta_\varepsilon)\varepsilon^{1/4}\Phi_1(-h),\\ \lambda_1^+(x,y) &= 2\Big[(\beta_\varepsilon-\gamma_\varepsilon)+(\gamma_\varepsilon-\beta_\varepsilon)\Phi_0(h)-\sqrt{2}\beta_\varepsilon\varepsilon^{1/4}\Phi_1(h)\Big],\\ \lambda_1^-(x,y) &= 2\Big[-(\beta_\varepsilon-\gamma_\varepsilon)\Phi_1(h)+\sqrt{2}\beta_\varepsilon\varepsilon^{1/4}\Phi_1(-h)\Big],\end{aligned}$$

$$\begin{aligned}
\lambda_0^+(x,y) &= -1+4\beta_\varepsilon-4\beta_\varepsilon\Phi_0(h)-2\sqrt{2}(\beta_\varepsilon+\gamma_\varepsilon)\varepsilon^{1/4}\Phi_1(h),\\
\lambda_0^-(x,y) &= -1+4\beta_\varepsilon\Phi_0(-h)-2\sqrt{2}(\gamma_\varepsilon+\beta_\varepsilon)\varepsilon^{1/4}\Phi_1(-h).
\end{aligned}$$

More precisely, the coefficients $\lambda_p^\pm$ are given by

$$\lambda_p^+(x,y)=\frac{\partial^p\overline{\theta}^{\varepsilon,0}}{\partial z^p}\Big|_{z=0^+},\qquad \lambda_p^-(x,y)=\frac{\partial^p\overline{\theta}^{\varepsilon,0}}{\partial z^p}\Big|_{z=0^-},\quad \forall\ p\in\{0,1,2\},$$

and $\lambda_3^\pm$ satisfy

$$\overline{\theta}^{\varepsilon,0}(x,y,0^+)=-\frac{1}{2}+\lambda_3^+(x,y)\alpha(\varepsilon),\qquad \overline{\theta}^{\varepsilon,0}(x,y,0^-)=\frac{1}{2}+\lambda_3^-(x,y)\alpha(\varepsilon).$$

We note the presence of the term $\varepsilon\,\delta^{(3)}$ in the right-hand side of the equation (2.3.4) of $\overline{\theta}^{\varepsilon,0}$. This term will absorb the main singularity produced by replacing, formally, u^ε by u^0 in (2.1.1). The other singularities $(-\varepsilon\alpha(\varepsilon)\lambda_3(x,y)\,\delta^{(3)},-\varepsilon^{3/4}\alpha(\varepsilon)\lambda_2(x,y)\,\delta'',\ldots)$ are of lower order since they are multiplied by $\alpha(\varepsilon)$ which is an *e.s.t.* in $H^m(\Omega)$ for all $m\in N$. However, we can not apply, for the moment, the standard energy method for $u^\varepsilon-u^0-\overline{\theta}^{\varepsilon,0}$ since the right-hand side of its equation does not belong to $L^2(\Omega)$. In order to resolve this difficulty, we define the main form of the corrector θ^ε as follows :

$$\theta^\varepsilon=\overline{\theta}^{\varepsilon,0}-\varphi^\varepsilon, \tag{2.3.8}$$

where φ^ε is designed to absorb the above mentioned singularities :

$$\varphi^\varepsilon(x,y,z)=\sum_{p=0}^{3}\Big[\lambda_{3-p}^+(x,y)\frac{z^p}{p!\ \varepsilon^{p/4}}\chi_{(0,h)}-\lambda_{3-p}^-(x,y)\frac{(-z)^p}{p!\ \varepsilon^{p/4}}\chi_{(-h,0)}\Big]\alpha(\varepsilon). \tag{2.3.9}$$

Thanks to (2.3.4)–(2.3.6) and (2.3.8), we deduce that θ^ε verifies the following equations

$$\varepsilon\,\frac{\partial^4\theta^\varepsilon}{\partial z^4}+\theta^\varepsilon = -\varepsilon\,\delta^{(3)}-\varphi^\varepsilon,\qquad \text{in }\ \Omega, \tag{2.3.10}$$

$$\theta^\varepsilon = -u^0+\Phi_0-\frac{sign(z)}{4}\alpha(\varepsilon)-\varphi^\varepsilon,\qquad \text{on }\ \Gamma, \tag{2.3.11}$$

$$\frac{\partial\,\theta^\varepsilon}{\partial\,\nu} = \Phi_1-\frac{\varepsilon^{-1/4}}{4}\alpha(\varepsilon)+\frac{\partial\varphi^\varepsilon}{\partial z},\qquad \text{on }\ \Gamma, \tag{2.3.12}$$

The corrector θ^ε is now completely defined.

2.3.2 The L^2, H^1, and H^2 estimates.

Let $w^\varepsilon = u^\varepsilon - u^0 - \theta^\varepsilon$ which satisfies

$$
\begin{aligned}
\varepsilon\,\Delta^2 w^\varepsilon + w^\varepsilon &= \varphi^\varepsilon - \varepsilon \sum_{\substack{i,j=1\\ i+j\neq 6}}^{3} \frac{\partial^4\,\theta^\varepsilon}{\partial x_i^2 \partial x_j^2}, \quad in\ \ \Omega, && (2.3.13)\\
w^\varepsilon &= \frac{sign(z)}{4}\alpha(\varepsilon) + \varphi^\varepsilon, \qquad \text{on}\ \ \Gamma, && (2.3.14)\\
\frac{\partial\, w^\varepsilon}{\partial\,\nu} &= \frac{\varepsilon^{-1/4}}{4}\alpha(\varepsilon) - \frac{\partial \varphi^\varepsilon}{\partial z}, \qquad \text{on}\ \ \Gamma; && (2.3.15)
\end{aligned}
$$

here we have denoted x, y, and z by x_1, x_2, and x_3.

Now, we intend to apply the standard energy estimates to (2.3.13)–(2.3.13). For this purpose, we transform the boundary conditions of w^ε into the homogeneous ones by setting

$$
\widetilde{w}^\varepsilon = w^\varepsilon - \psi^\varepsilon, \tag{2.3.16}
$$

where ψ^ε is a polynomial function in z "small" in all $H^m(\Omega)$ and given by

$$
\begin{aligned}
\psi^\varepsilon(x,y,z) \;=\; & \Big[-\frac{\alpha(\varepsilon)}{2} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h) + h(\varphi^\varepsilon_z(h) + \varphi^\varepsilon_z(-h) - \frac{\varepsilon^{-1/4}}{2}\alpha(\varepsilon))\Big]\frac{z^3}{4h^3} - \\
& -\Big[\varphi^\varepsilon_z(-h) - \varphi^\varepsilon_z(h)\Big]\frac{z^2}{4h} - \Big[\frac{3}{h}\Big(-\frac{\alpha(\varepsilon)}{2} + \varphi^\varepsilon(-h) - \varphi^\varepsilon(h)\Big) + \\
& + \varphi^\varepsilon_z(h) + \varphi^\varepsilon_z(-h) - \frac{\varepsilon^{-1/4}}{2}\alpha(\varepsilon)\Big]\frac{z}{4} + \frac{\varphi^\varepsilon(-h) + \varphi^\varepsilon(h)}{2} + \\
& + \frac{h}{4}\Big(\varphi^\varepsilon_z(-h) - \varphi^\varepsilon_z(h)\Big), \qquad \forall (x,y,z) \in \Omega. \qquad (2.3.17)
\end{aligned}
$$

Precise estimates for ψ^ε are given below, see (2.3.24).

Hence, $\widetilde{w}^\varepsilon$ satisfies

$$
\begin{aligned}
\varepsilon\Delta^2 \widetilde{w}^\varepsilon + \widetilde{w}^\varepsilon &= \varphi^\varepsilon - \varepsilon \sum_{\substack{i,j=1\\ i+j\neq 6}}^{3} \frac{\partial^4\,\theta^\varepsilon}{\partial x_i^2 \partial x_j^2} - \\
& \quad - \psi^\varepsilon - \varepsilon \sum_{\substack{i,j=1\\ i+j\neq 6}}^{3} \frac{\partial^4\,\psi^\varepsilon}{\partial x_i^2 \partial x_j^2}, \quad in\ \ \Omega, && (2.3.18)\\
\widetilde{w}^\varepsilon = \frac{\partial \widetilde{w}^\varepsilon}{\partial \nu} &= 0, \qquad on\ \Gamma, && (2.3.19)
\end{aligned}
$$

and

$$\widetilde{w}^\varepsilon \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{2.3.20}$$

We multiply (2.3.18) by $\widetilde{w}^\varepsilon$, integrate over Ω and we apply the Green formula to the fourth order term. Thanks to (2.3.19) and (2.3.20), all the boundary terms cancel and we obtain

$$\begin{aligned}\varepsilon\|\Delta\widetilde{w}^\varepsilon\|^2_{L^2(\Omega)}+\|\widetilde{w}^\varepsilon\|^2_{L^2(\Omega)} &= \Big(\varphi^\varepsilon+\varepsilon\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\frac{\partial^4\,\theta^\varepsilon}{\partial x_i^2\partial x_j^2},\widetilde{w}^\varepsilon\Big)_{L^2(\Omega)}+\\ &+\Big(-\psi^\varepsilon+\varepsilon\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\frac{\partial^4\,\psi^\varepsilon}{\partial x_i^2\partial x_j^2},\widetilde{w}^\varepsilon\Big)_{L^2(\Omega)}.\end{aligned} \tag{2.3.21}$$

Notice that the differentiation in the tangential direction (x) or (y) cancels the singularities δ and δ', produced by the discontinuity of θ^ε at $z=0$, in the term $\partial^2\theta^\varepsilon/\partial z^2$. Consequently the right-hand side of (2.3.21) is regular since the term $\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\Big(\partial^4\,\theta^\varepsilon/\partial x_i^2\partial x_j^2\Big)$ is in $L^2(\Omega)$.

By applying the Cauchy-Schwarz and Young inequalities for the right-hand side of (2.3.21), we deduce that

$$\begin{aligned}\varepsilon\|\Delta\widetilde{w}^\varepsilon\|^2_{L^2(\Omega)}+\|\widetilde{w}^\varepsilon\|^2_{L^2(\Omega)} &\le c\,\|\psi^\varepsilon\|^2_{L^2(\Omega)}+c\,\|\varphi^\varepsilon\|^2_{L^2(\Omega)}+\\ &+c\,\varepsilon^2\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\Big\|\frac{\partial^4\,\theta^\varepsilon}{\partial x_i^2\partial x_j^2}\Big\|^2_{L^2(\Omega)}+\\ &+c\,\varepsilon^2\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\Big\|\frac{\partial^4\,\psi^\varepsilon}{\partial x_i^2\partial x_j^2}\Big\|^2_{L^2(\Omega)}.\end{aligned} \tag{2.3.22}$$

On the other hand, we infer from the explicit expression of φ^ε given by (2.3.9) that

$$\|\varphi^\varepsilon\|_{L^2(\Omega)}\le c\,\varepsilon^{-3/4}\,\alpha(\varepsilon), \tag{2.3.23}$$

and from (2.3.17) follows the following estimate

$$\|\psi^\varepsilon\|_{H^m(\Omega)}\le c\,\varepsilon^{-3/4}\,\alpha(\varepsilon),\qquad \forall m\in N. \tag{2.3.24}$$

Hence, taking into account (2.3.24) and differentiating the expression of θ^ε ((2.3.2), (2.3.3) and (2.3.8)) with respect to x_i and x_j two times, we deduce that

$$\begin{aligned}\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\Big\|\frac{\partial^4\theta^\varepsilon}{\partial x_i^2\partial x_j^2}\Big\|_{L^2(\Omega)} &\leq C\,\Big\{\Big\|\frac{\partial^2\theta^\varepsilon}{\partial z^2}\Big\|_{L^2(\Omega^+)}+\Big\|\frac{\partial^2\theta^\varepsilon}{\partial z^2}\Big\|_{L^2(\Omega^-)}\Big\}.\\ &\leq c\,\varepsilon^{-3/8}. \end{aligned}\tag{2.3.25}$$

Combining (2.3.23), (2.3.24) and (2.3.25) into (2.3.22), we obtain the estimates (2.1.5) and (2.1.7). Finally, the estimate (2.1.6) is obtained by interpolation.
This ends the proof of Theorem 2.1.4.

2.4 Proof of Theorem 2.1.5.

In this section, we will show how Theorem 2.1.4 can be extended to the solution u_k^ε of an equation with a stronger singularity for the source function as in (2.1.8). We will derive the correctors corresponding to the problem (2.1.8) with the boundary conditions (2.1.2) and then prove the convergence results as in Section 2.3. For this purpose and based on the well known technique in boundary layers theory which consists in neglecting the tangential derivatives if compared to the normal ones, we start in a first step by considering the space dimension one. In a second step, we reduce the singularity of the right-hand side of the equation (2.1.8). Thus, we propose to look for v_k^ε solution of this system

$$(\mathcal{C}_\varepsilon^k)\quad\begin{cases}\varepsilon\,\dfrac{\partial^4 v_k^\varepsilon}{\partial z^4}+v_k^\varepsilon=\delta^{(k)}, & \text{in } (-h,h),\\[2ex] v_k^\varepsilon=\Phi_0(x,y,.), & \text{at } z=\pm h,\\[2ex] \dfrac{\partial v_k^\varepsilon}{\partial z}=-\Phi_1(x,y,.), & \text{at } z=\pm h.\end{cases}$$

At this stage, we intend to use the techniques applied in Section 2.2.2. Similarly, we have to consider four cases according to the value of k, and we will give the form of the corrector for the four cases. However, we will prove the convergence results of Theorem 2.1.5 just for one case and the others can be treated in a similar way.

Case 1 : For $k = 4q+3, q \in N$, as in Subsection 2.2.2, we may reduce the singularity of the source function. By applying a simple recursive formula and by taking into account the convention (2.2.19), we deduce that

$$v^{\varepsilon}_{4q+3} = -\sum_{j=1}^{q+1}(-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \tilde{v}^{\varepsilon}_{4q+3}, \tag{2.4.1}$$

where $\tilde{v}^{\varepsilon}_{4q+3}$ verifies the following system

$$\begin{cases} \varepsilon \dfrac{\partial^4 \tilde{v}^{\varepsilon}_{4q+3}}{\partial z^4} + \tilde{v}^{\varepsilon}_{4q+3} = \dfrac{(-1)^{q+1}}{\varepsilon^{q+1}} H, & \text{in } (-h,h), \\ \tilde{v}^{\varepsilon}_{4q+3} = \Phi_0(x,y,h) + \dfrac{(-1)^{q+1}}{\varepsilon^{q+1}}, & \text{at } z = h, \\ \tilde{v}^{\varepsilon}_{4q+3} = \Phi_0(x,y,-h), & \text{at } z = -h, \\ \dfrac{\partial \tilde{v}^{\varepsilon}_{4q+3}}{\partial z} = -\Phi_1(x,y,.), & \text{at } z = \pm h. \end{cases}$$

Let us recall that the resolution of the above system leads to a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (\Phi_0(x,y,h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x,y,h), \Phi_0(x,y,-h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x,y,-h),$ $(-1)^q/\varepsilon^{q+1}, 0, 0, 0)$. The explicit expression of v^{ε}_{4q+3} is then deduced and as usual we only need to define an approximation of v^{ε}_{4q+3}, in the sense of Proposition 2.2.1 and Proposition 2.2.2, denoted by $\overline{v}^{\varepsilon}_{4q+3}$.

We are now ready to define the primary form of the corrector, $\overline{\theta}^{\varepsilon,0}_{4q+3} = \overline{v}^{\varepsilon}_{4q+3} - u^0_{4q+3}$, as follows

$$\begin{aligned} \overline{\theta}^{\varepsilon,0}_{4q+3}(x,y,z) \; = \; & -\sum_{j=0}^{q}(-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ & \left\{\nu_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} - \\ & -\frac{(-1)^{q+1}}{2\,\varepsilon^{q+1}} \exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right), \end{aligned} \tag{2.4.2}$$

for all $(x, y, z) \in \Omega^+$ and

$$\begin{aligned}
\overline{\theta}^{\varepsilon,0}_{4q+3}(x,y,z) &= -\sum_{j=0}^{q}(-1)^j \frac{\delta^{(4(q-j)+3)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\
&\left\{\eta_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} + \\
&+ \frac{(-1)^{q+1}}{2\ \varepsilon^{q+1}}\exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right),
\end{aligned} \tag{2.4.3}$$

for all $(x, y, z) \in \Omega^-$, where we have set

$$\begin{aligned}
\nu_\varepsilon(x,y) &= \beta_\varepsilon + \gamma_\varepsilon + \Phi_0(h) + \sqrt{2}\ \varepsilon^{1/4}\gamma_\varepsilon\Phi_1(h), \\
\xi_\varepsilon(x,y) &= -(\beta_\varepsilon - \gamma_\varepsilon)\Phi_0(h) - \sqrt{2}\ \varepsilon^{1/4}\beta_\varepsilon\Phi_1(h), \\
\eta_\varepsilon(x,y) &= (\beta_\varepsilon + \gamma_\varepsilon)\Phi_0(-h) - \sqrt{2}\ \varepsilon^{1/4}\gamma_\varepsilon\Phi_1(-h), \\
\sigma_\varepsilon(x,y) &= (\beta_\varepsilon - \gamma_\varepsilon)\Phi_0(-h) - \sqrt{2}\ \varepsilon^{1/4}\beta_\varepsilon\Phi_1(-h).
\end{aligned}$$

Recall that the above function is not continuous at $z = 0$. More precisely, $\overline{\theta}^{\varepsilon,0}_{4q+3}$ contains two kind of singularities; the first one is aimed at absorbing the main singularity produced by u^0_k at $z = 0$ and the second one is a consequence of the approximations used previously, where $\tilde{v}^\varepsilon_{4q+3}$ was approximated by $\overline{v}^\varepsilon_{4q+3}$. As in Section 2.3, we define the final form of the corrector by setting

$$\theta^\varepsilon_{4q+3} = \overline{\theta}^{\varepsilon,0}_{4q+3} - \varphi^\varepsilon_k, \tag{2.4.4}$$

where φ^ε_k is a correcting function which is "small" in all the spaces $H^m(\Omega)$, $\forall m \in N$ and given by

$$\begin{aligned}
\varphi^\varepsilon_k(x,y,z) &= \sum_{p=0}^{3}\left[\lambda^{p,+}_k(x,y)\frac{z^{3-p}}{(3-p)!\ \varepsilon^{p/4}}\chi_{(0,h)}\right]\alpha(\varepsilon) - \\
&- \sum_{p=0}^{3}\left[\lambda^{p,-}_k(x,y)\frac{(-z)^{3-p}}{(3-p)!\ \varepsilon^{p/4}}\chi_{(-h,0)}\right]\alpha(\varepsilon).
\end{aligned} \tag{2.4.5}$$

Here we have set, for all $k \in N$:

$$\begin{aligned}
&\lambda^{3,+}_k(x,y) = \nu_\varepsilon(x,y), && \lambda^{3,-}_k(x,y) = \eta_\varepsilon(x,y), \\
&\lambda^{2,+}_k(x,y) = \nu_\varepsilon(x,y) + \xi_\varepsilon(x,y), && \lambda^{2,-}_k(x,y) = -\eta_\varepsilon(x,y) + \sigma_\varepsilon(x,y), \\
&\lambda^{1,+}_k(x,y) = \xi_\varepsilon(x,y), && \lambda^{1,-}_k(x,y) = -\sigma_\varepsilon(x,y), \\
&\lambda^{0,+}_k(x,y) = 2[\xi_\varepsilon(x,y) - \nu_\varepsilon(x,y)], && \lambda^{0,-}_k(x,y) = 2[\eta_\varepsilon(x,y) + \sigma_\varepsilon(x,y)].
\end{aligned}$$

After some computations, we see that $\theta^\varepsilon_{4q+3}$ verifies the following equations

$$\varepsilon\,\frac{\partial^4\,\theta^\varepsilon_{4q+3}}{\partial z^4}+\theta^\varepsilon_{4q+3} \;=\; -\varepsilon\,\delta^{(k+4)}-\varphi^\varepsilon, \qquad \text{in}\ \ \Omega, \tag{2.4.6}$$

$$\theta^\varepsilon \;=\; \Phi_0-\frac{sign(z)}{4\varepsilon^{q+1}}\alpha(\varepsilon)-\varphi^\varepsilon, \qquad \text{on}\ \ \Gamma, \tag{2.4.7}$$

$$\frac{\partial\,\theta^\varepsilon}{\partial\,\nu} \;=\; \Phi_1-\frac{1}{4\varepsilon^{q+5/4}}\alpha(\varepsilon)+\frac{\partial\varphi^\varepsilon}{\partial z}, \qquad \text{on}\ \ \Gamma. \tag{2.4.8}$$

Now we define $w^\varepsilon_{4q+3}=u^\varepsilon_{4q+3}-u^0_{4q+3}-\theta^\varepsilon_{4q+3}$. Thanks to (2.1.2), (2.1.9), (2.4.12) and (2.4.13), we deduce the boundary values of w^ε_{4q+3} which read

$$\begin{aligned} w^\varepsilon_{4q+3} &= \frac{sign(z)}{4\varepsilon^{q+1}}\alpha(\varepsilon)+\varphi^\varepsilon, && \text{on}\ \ \Gamma,\\ \frac{\partial\,w^\varepsilon_{4q+3}}{\partial\,\nu} &= \frac{1}{4\varepsilon^{q+5/4}}\alpha(\varepsilon)-\frac{\partial\varphi^\varepsilon}{\partial z}, && \text{on}\ \ \Gamma. \end{aligned}$$

Afterwards, in order to apply the standard energy methods, we transform the boundary conditions of w^ε_{4q+3} into the homogeneous ones by setting

$$\widetilde{w}^\varepsilon_{4q+3}=w^\varepsilon_{4q+3}-\psi^\varepsilon_{4q+3}, \tag{2.4.9}$$

where ψ^ε_{4q+3} is a"small" polynomial function in z, given by

$$\begin{aligned} \psi^\varepsilon_{4q+3}(x,y,z) \;=\; & \Big[-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+1}}+\varphi^\varepsilon(-h)-\varphi^\varepsilon(h)+h(\varphi^\varepsilon_z(h)+\varphi^\varepsilon_z(-h)-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}})\Big]\frac{z^3}{4h^3}-\\ & -\Big[\varphi^\varepsilon_z(-h)-\varphi^\varepsilon_z(h)\Big]\frac{z^2}{4h}-\Big[\frac{3}{h}\Big(-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}}+\varphi^\varepsilon(-h)-\varphi^\varepsilon(h)\Big)+\\ & +\varphi^\varepsilon_z(h)+\varphi^\varepsilon_z(-h)-\frac{\alpha(\varepsilon)}{2\varepsilon^{q+5/4}}\Big]\frac{z}{4}+\frac{\varphi^\varepsilon(-h)+\varphi^\varepsilon(h)}{2}+\\ & +\frac{h}{4}\Big(\varphi^\varepsilon_z(-h)-\varphi^\varepsilon_z(h)\Big), \qquad \forall(x,y,z)\in\Omega, \end{aligned} \tag{2.4.10}$$

Hence, $\widetilde{w}^\varepsilon_{4q+3}$ verifies the following equations

$$\begin{aligned} \varepsilon\Delta^2\widetilde{w}^\varepsilon_{4q+3}+\widetilde{w}^\varepsilon_{4q+3} \;=\;& \varphi^\varepsilon_{4q+3}-\varepsilon\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\frac{\partial^4\,\theta^\varepsilon_{4q+3}}{\partial x_i^2\partial x_j^2}-\\ & -\psi^\varepsilon_{4q+3}-\varepsilon\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\frac{\partial^4\,\psi^\varepsilon_{4q+3}}{\partial x_i^2\partial x_j^2}, \qquad in\ \ \Omega, \end{aligned} \tag{2.4.11}$$

$$\widetilde{w}^\varepsilon_{4q+3}=\frac{\partial\widetilde{w}^\varepsilon_{4q+3}}{\partial\nu} \;=\; 0, \qquad on\ \Gamma, \tag{2.4.12}$$

and

$$\widetilde{w}^{\varepsilon}_{4q+3} \text{ is periodic with periods } L_1 \text{ and } L_2 \text{ in the } x \text{ and } y \text{ directions.} \tag{2.4.13}$$

We multiply (2.4.11) with $\widetilde{w}^{\varepsilon}_{4q+3}$, integrate over Ω, apply the boundary and periodicity conditions ((2.4.12) and (2.4.13)) and obtain

$$\begin{aligned}
\varepsilon \|\Delta \widetilde{w}^{\varepsilon}_{4q+3}\|^2_{L^2(\Omega)} + \frac{1}{2}\|\widetilde{w}^{\varepsilon}_{4q+3}\|^2_{L^2(\Omega)} \quad \leq \quad & \|\psi^{\varepsilon}_{4q+3}\|^2_{L^2(\Omega)} + \|\varphi^{\varepsilon}_{4q+3}\|^2_{L^2(\Omega)} + \\
& + \varepsilon^2 \sum_{\substack{i,j=1 \\ i+j\neq 6}}^{3} \Big\| \frac{\partial^4 \theta^{\varepsilon}_{4q+3}}{\partial x_i^2 \partial x_j^2} \Big\|^2_{L^2(\Omega)} + \\
& + \varepsilon^2 \sum_{\substack{i,j=1 \\ i+j\neq 6}}^{3} \Big\| \frac{\partial^4 \psi^{\varepsilon}_{4q+3}}{\partial x_i^2 \partial x_j^2} \Big\|^2_{L^2(\Omega)}.
\end{aligned} \tag{2.4.14}$$

Furthermore, observe that

$$\|\varphi^{\varepsilon}_{4q+3}\|_{L^2(\Omega)} \leq c\, \varepsilon^{-q-7/4}\ \alpha(\varepsilon), \tag{2.4.15}$$

and, that from (2.4.10), $\psi^{\varepsilon}_{4q+3}$ verifies

$$\|\psi^{\varepsilon}_{4q+3}\|_{H^m(\Omega)} \leq c\, \varepsilon^{-q-7/4}\ \alpha(\varepsilon), \qquad \forall m \in N. \tag{2.4.16}$$

Using the expression of $\theta^{\varepsilon}_{4q+3}$ given by (2.4.2), (2.4.3) and (2.4.4), we deduce that

$$\begin{aligned}
\sum_{\substack{i,j=1 \\ i+j\neq 6}}^{3} \Big\| \frac{\partial^4 \theta^{\varepsilon}_{4q+3}}{\partial x_i^2 \partial x_j^2} \Big\|_{L^2(\Omega)} \quad & \leq \quad C \Big\{ \Big\| \frac{\partial^3 \theta^{\varepsilon}_{4q+3}}{\partial x \partial z^2} \Big\|_{L^2(\Omega +)} + \Big\| \frac{\partial^3 \theta^{\varepsilon}_{4q+3}}{\partial x \partial z^2} \Big\|_{L^2(\Omega -)} \Big\}. \\
& \leq \quad c\, \varepsilon^{-3/8}.
\end{aligned} \tag{2.4.17}$$

Combining (2.4.15), (2.4.16) and (2.4.17) into (2.4.14) and since $\alpha(\varepsilon)$ is given by (2.2.7), we infer the estimates (2.1.10) and (2.1.12); the estimate (2.1.11) is then obtained by a simple interpolation.

Case 2 : Similarly, for $k = 4q + 2$, $q \in N$, we start by reducing the singularity of the right-hand side of the first equation in $(\mathcal{C}^k_\varepsilon)$. By induction, the right-hand side $\delta^{(k)}$

will be transformed to $(-1/\varepsilon)^{q+1}H_2$ and we get a linear system of type $(\mathcal{S}_\varepsilon)$ with $\mathcal{B}_\varepsilon = (\Phi_0(x,y,h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x,y,h), \Phi_0(x,y,-h), -\sqrt{2}\varepsilon^{1/4}\Phi_1(x,y,-h), 0, (-1)^q\sqrt{2}/\varepsilon^{q+3/4}$ $, 0, 0)$. The resolution of this system leads to the definition of an approximate solution from which we remove the limit solution u^0_{4q+3}. Finally, we obtain the primary form of the corrector as follows

$$\begin{aligned}\overline{\theta}^{\varepsilon,0}_{4q+2}(x,y,z) = & -\sum_{j=0}^{q}(-1)^j\frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right)\\ & \left\{\nu_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} + \\ & + \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+3/4}}\exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) - \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right],\end{aligned} \tag{2.4.18}$$

for all $(x,y,z)\in\Omega^+$ and,

$$\begin{aligned}\overline{\theta}^{\varepsilon,0}_{4q+2}(x,y,z) = & -\sum_{j=0}^{q}(-1)^j\frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right)\\ & \left\{\eta_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} + \\ & + \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+3/4}}\exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right],\end{aligned} \tag{2.4.19}$$

for all $(x,y,z)\in\Omega^-$.

We note that $\overline{\theta}^{\varepsilon,0}_{4q+2}$ contains two kinds of discontinuities at $z=0$; the first one is aimed at absorbing the main singularity produced by the discontinuity of u^0_k at $z=0$, namely $\varepsilon\delta^{(k+4)}$, that we will keep here; the second one results from the approximations used above. In order to eliminate the second singularity, we introduce another corrector function and obtain the final form of the corrector

$$\theta^\varepsilon_{4q+2} = \overline{\theta}^{\varepsilon,0}_{4q+2} - \varphi^\varepsilon_k,$$

where φ^ε_k is given by (2.3.9).

Hence, the equation for $\theta^\varepsilon_{4q+2}$ reads

$$\varepsilon\Delta^2\theta^\varepsilon_{4q+2} + \theta^\varepsilon_{4q+2} = -\varepsilon\,\delta^{(k+4)} - \varphi^\varepsilon_k + \varepsilon\sum_{\substack{i,j=1\\ i+j\neq 6}}^{3}\frac{\partial^4\,\theta^\varepsilon_{4q+2}}{\partial x_i^2\partial x_j^2}. \tag{2.4.20}$$

Finally, since the right-hand side of (2.4.20) is regular ($\in L^2(\Omega)$), we may apply the standard energy estimates for $\widetilde{w}^\varepsilon_{4q+2} = w^\varepsilon_{4q+2} - \psi^\varepsilon_{4q+2}$, where $w^\varepsilon_{4q+2} = u^\varepsilon_{4q+3} - u^0_{4q+3} - \theta^\varepsilon_{4q+3}$ and, ψ^ε_{4q+2} is a "small" polynomial function in z.

Case 3 : For $k = 4q+1, q \in N$, we deduce from the resolution of the corresponding one-dimensional problem the primary form of the corrector

$$\begin{aligned}
\overline{\theta}^{\varepsilon,0}_{4q+1}(x,y,z) \;=\; & -\sum_{j=0}^{q}(-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\
& \left\{\nu_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} + \\
& + \frac{(-1)^{q+1}}{2\ \varepsilon^{q+1/2}}\exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right),
\end{aligned} \tag{2.4.21}$$

for all $(x,y,z) \in \Omega^+$ and

$$\begin{aligned}
\overline{\theta}^{\varepsilon,0}_{4q+1}(x,y,z) \;=\; & -\sum_{j=0}^{q}(-1)^j \frac{\delta^{(4(q-j)+1)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\
& \left\{\eta_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} + \\
& + \frac{(-1)^{q+1}}{2\ \varepsilon^{q+1/2}}\exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right),
\end{aligned} \tag{2.4.22}$$

for all $(x,y,z) \in \Omega^-$.

Now, we define as follows the final form of the corrector by removing the singularities produced by the approximations used above

$$\theta^\varepsilon_{4q+1} = \overline{\theta}^{\varepsilon,0}_{4q+1} - \varphi^\varepsilon_k.$$

Finally, the corrector $\theta^\varepsilon_{4q+1}$ verifies an equation similar to (2.4.20) and then we may apply the same techniques, as in the first case, to prove the estimates stated in Theorem 2.1.5.

Case 4 : For $k = 4q$, $q \in N$, we follow the same steps as in the above cases and we obtain firstly the primary form of the corrector which is given by

$$\begin{aligned}
\overline{\theta}^{\varepsilon,0}_{4q}(x,y,z) \;=\; & -\sum_{j=0}^{q}(-1)^j \frac{\delta^{(4(q-j))}}{\varepsilon^j} + \exp\left(\frac{z-h}{\sqrt{2}\varepsilon^{1/4}}\right) \\
& \left\{\nu_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \xi_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} - \\
& - \frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+1/4}}\exp\left(-\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right],
\end{aligned} \tag{2.4.23}$$

for all $(x,y,z) \in \Omega^+$ and

$$\begin{aligned}\overline{\theta}^{\varepsilon,0}_{4q}(x,y,z) \;=\; & -\sum_{j=0}^{q}(-1)^j\frac{\delta^{(4(q-j)+2)}}{\varepsilon^j} + \exp\left(-\frac{z+h}{\sqrt{2}\varepsilon^{1/4}}\right) \\ & \left\{\eta_\varepsilon(x,y)\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right) + \sigma_\varepsilon(x,y)\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right\} - \\ & -\frac{(-1)^{q+1}\sqrt{2}}{4\varepsilon^{q+1/4}}\exp\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\left[\cos\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)-\sin\left(\frac{z}{\sqrt{2}\varepsilon^{1/4}}\right)\right], \quad (2.4.24)\end{aligned}$$

for all $(x,y,z) \in \Omega^-$.

Secondly, we define the final form of the corrector which reads in this case

$$\theta^\varepsilon_{4q} = \overline{\theta}^{\varepsilon,0}_{4q} - \varphi^\varepsilon_k,$$

where φ^ε_k is still given by (2.3.9) small in all $L^2(\Omega)$.

Hence, the corrector is now well defined and it verifies an equation similar to (2.4.20).

This completes the study of all the cases and ends the proof of Theorem 2.1.5.

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Deuxième partie

Boundary layers for the regularised minimal surface problem

Chapitre 3

Asymptotic analysis for the regularised minimal surface problem in the radial symmetric case

Abstract

In this article, we derive the first term in the asymptotic expansion of the regularised minimal surface solution in the radial symmetric case when the domain is a pair of concentric circles in R^2. General domains and time-dependent problems will be considered elsewhere.

3.1 Introduction.

The calculus of variations constitute a fundamental tool in the study of optimization problems. Such problems arise e.g. for the optimization (either maximization or minimization) of a fundamental physical or mechanical quantity such as length, area, volume, time, work, or energy. More recent preoccupations have added to this list cost and efficiency, in relation with economical issues. References related to this subject include e.g. Ekeland-Temam [ET], Fomenko [F3], Kinderheher-Stampacchia [KS], Lyusternik [Ly] and

Marcellini [Ma]. One of the oldest and most famous problems in the calculus of variations is the Plateau problem which still occupies an important place in modern mathematics. This problem which consists in finding the surface of minimal area spaned by a given fixed boundary, relates also to many different problems including surfaces with prescribed mean curvature, and image processing problems; for further details see e.g. Almgren [A2], Fomenko [F2], Mumford [Mu], Radó [R] and Struwe [St]. Minimal surfaces appear e.g. in soap films, soap bubbles or in general interfaces between two media, as encountered in many problems of chemistry, physics and biology, for example, membranes in living cells, capillary phenomena, etc.. Other applications arise in nature, in computer simulation and in architecture. Many approaches are used in the study of minimal surfaces, such as the differential geometry and topology (see for instance Fomenko [F1] and the bibliography therein); the numerical simulations are used to find and discover new minimal surfaces (see e.g. Astruc, Demengel and Goubet [ADG], Callahan, Hoffman D. and Hoffman J. [CHH] and Rosenberg and Hoffman [RH], etc.). But the existence of such surfaces remains in general an open problem which has been intensively studied by Almgren [A1], Courant [C], Schoen and Yau [SY], Wente [W1], [W2]; see also the bibliography therein. As we may see in the long article of Taylor [Ty], minimal surfaces present in general singularities called branch points or vertical points, see for instance Atkinson and Peletier [AP] and Atkinson, Peletier and Serrin [APS]. A special attention was given by Hardt and Simon [HS], Hildebrandt and Nitsche [HN] and others to the study of the behavior of minimal surfaces at the boundary where singularities are generally located. To study the problem of existence of generalized minimal surfaces when the usual non parametric minimal surface does not exist, Temam [Te1] combined the utilization of duality methods with an elliptic regularization method (which has been used in other applications, see e.g. Baouendi [Ba], Kohn and Nirenberg [KN] and Lions [Li2]). However, this regularisation leads to an unusual singular perturbation problem for which the usual methods as in e.g. Lions [Li1], O'Malley [O] and Vishik-Lyusternik [VL] do not apply. Indeed, in most singular perturbation problems the limit solution is regular, but this is not the case here since the limit solution (the generalized minimal surface; see for instance Bombieri, De

Giorgi and Miranda [BDM]) has an infinite gradient on some parts of the boundary; see e.g. [Te1].

Our main object in this article is to give the first term in the asymptotic expansion of the regularized minimal surface solution in a suitable Sobolev space; namely we define an approximation, in some sense to be specified, for the boundary layer function and we confirm this choice by a convergence result.

The rest of this article is organized as follows : In the next section, we will state the problem and recall the relevant results described above. In Section 3.3, we study the boundary layer produced by the regularisation and we define an approximation to the regularized minimal surface solution. Section 3.4 is devoted to the statement and the proof of the main result. Finally, we give in Section 3.5 a numerical simulation for both the minimal surface solution and the regularised one.

3.2 Position of the problem

The minimal surfaces are a particular case of the surfaces with prescribed mean curvature, more precisely, their mean curvature is everywhere equal to zero. From the mathematical point of view, they are extremals of the area functional. It turns out that the converse is true : A surface M that is a critical point of the area functional is minimal *i.e.*, it has a zero mean curvature.

Hence, in the non parametric case, *i.e.* for surfaces that are graph of a function $u : \Omega \to R$, the surfaces with prescribed mean curvature are solutions of the following variational problem

$$\inf_{u=\Phi \text{ } on \text{ } \partial\Omega} \int_\Omega \{\sqrt{1+|\nabla u|^2} - Hu\} \, dx, \tag{3.2.1}$$

where Φ is a function given on $\partial\Omega$, H is the (prescribed) mean curvature of the surface and Ω is a bounded open set of R^n.

As we said $H = 0$ corresponds to the minimal surface problem of which the corresponding

Euler-Lagrange equation reads (see e.g. Troutman [Tr] for further details)

$$-\operatorname{div}(T(\nabla u)) = 0, \quad \text{in } \Omega, \tag{3.2.2}$$

$$u = \Phi, \quad \text{on } \Gamma, \tag{3.2.3}$$

where

$$T(v) = \frac{v}{\sqrt{1+v^2}},$$

and $\Gamma = \partial\Omega$.

The uniqueness of solution for (3.2.2)-(3.2.3) is easy but the existence is not always guaranteed. Elementary examples (including the one studied in this article) show that this problem does not have necessarily a solution, neither in the strong sense ($u \in C^2(\Omega) \cap C^0(\overline{\Omega})$), nor in the weak sense ($u \in W^{1,1}(\Omega)$); see for instance Serrin [Se] and Lawson-Osserman [LO]. In fact, the existence of u solution of (3.2.2)-(3.2.3) fails to be true because, in general, u does not satisfy the boundary condition (3.2.3) and we say that u may "detach from the desired boundary values" on some parts of Γ, see for that the paper of Marcellini and Miller [MM] where, with theory and numerical simulations, they have determined singular sets for surfaces with prescribed mean curvature on the disk.

On the other hand, Temam [Te1] obtained, by duality argument and as limit of sequences, a "generalized solution" that we will define subsequently, and showed that, at the boundary of Ω, two possibilities can occur for the solution u of the problem (3.2.2)-(3.2.3), namely

$$\frac{\partial u}{\partial \nu} = \pm\infty, \qquad \text{or} \quad u = \Phi. \tag{3.2.4}$$

Thus, the main difficulty for the existence of a solution of (3.2.1) is located at the boundary. This is in full agreement with the results of Bombieri, De Giorgi and Miranda [BDM] who proved that if u is a bounded solution of class C^2 of (3.2.2), then ∇u is bounded on $\Omega' \subset\subset \Omega$[1] ; see also Bernstein [Be], Leary [L] for the dimension 2 and Ladyzenskaya & Uralceva [LU] for more general variational problems.

[1] $\Omega' \subset\subset \Omega$ means $\Omega' \subset \overline{\Omega'} \subset \Omega$.

3.2.1 The concept of generalized solution.

Firstly, we start by giving the definition of the "generalized solution" for the minimal surface equations which was introduced by Temam in [Te1] and [Te2] :

Definition 3.2.1 : *A function $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ is called generalized solution of (3.2.1) or (3.2.2)-(3.2.3) if it satisfies :*

(i) u is a solution of (3.2.2), the Euler equation of (3.2.1).

(ii) Any bounded minimizing sequence $\{v_m\}_m$ of (3.2.1) converges to u in the following sense

$$\begin{aligned} v_m &\to u, \quad \text{in} \quad L^1(\Omega)/R, \\ \frac{\partial v_m}{\partial x_i} &\to \frac{\partial u}{\partial x_i}, \quad \text{in} \quad L^1(\Omega'), \quad \forall\, \Omega' \subset\subset \Omega, \; i = 1, .., n. \end{aligned}$$

The existence of these generalized solutions is proved by considering the dual problem of (3.2.1) in the sense of convex analysis. We recall that Bombieri, De Giorgi and Miranda [BDM] and Giusti [G] have also obtained this solution by a purely geometric argument, namely, by extending the area functional appearing in (3.2.1) to the class of functions of bounded variations. However, the construction of such solution will be described in Section 3.2.2.

In his article on some equations of type minimal surfaces, Temam proved a result, that we recall here, related to the trace of u at the boundary :

Proposition 3.2.1 *(Temam [Te1]). Suppose that there exists a solution $v \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ of (3.2.2) which satisfies :*

$$\exists x_0 \in \partial\Omega \textit{ such that } \limsup_{\substack{x \to x_0 \\ x \in \Omega}} |\nabla v(x)| < +\infty. \tag{3.2.5}$$

Then, for $\Phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, there exists $u \in W^{1,1}(\Omega) \cap W^{1,\infty}(\Omega) \cap L^\infty(\Omega'), \forall \Omega' \subset\subset \Omega$, solution of (3.2.2) such that

$$u = \Phi, \quad a.e., \tag{3.2.6}$$

on some parts γ of Γ with positive measure, and more precisely on

$$\{x/x \in \Gamma;\ \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| < +\infty\}.$$

Moreover, u is a generalized solution which satisfies

$$v_m \to u, \quad \text{in} \quad L^1(\Omega),$$

for all bounded minimizing sequence $\{v_m\}_m$ of (3.2.1).

Moreover, Temam proved in [Te2] that if $\gamma \subset \Gamma$ such that $\gamma \neq \emptyset, \gamma$ is a C^3-manifold of nonnegative mean curvature and $\Phi \in L^\infty(\Omega)$ is continuous on γ, then $u = \Phi$ on γ.

3.2.2 The regularized elliptic problem.

In this part, we follow the procedure of construction of the sequence $\{v_m\}_m$ given by Temam [Te1] and consequently obtaining the generalized minimal surface solutions. We proceed by regularisation and introduce, for $\varepsilon > 0$, the regularized problem defined as follows :

$$-\varepsilon\,\Delta u^\varepsilon - \operatorname{div}(T(\nabla u^\varepsilon)) = 0, \quad \text{in} \quad \Omega, \tag{3.2.7}$$

$$u^\varepsilon = \Phi, \quad \text{on} \quad \Gamma. \tag{3.2.8}$$

The addition of the flux term $-\varepsilon\Delta u^\varepsilon$ in (3.2.7) makes this equation uniformly elliptic, and the functional to be minimized is strictly convex. Hence, the existence is obtained by standard methods (cf. Ekeland and Temam [ET]). The regularity $u^\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ follows also from general results; see Ladyzenskaya & Uralceva [LU].

Now, equations (3.2.7)-(3.2.8) form a singular perturbation problem and, here, we are interested in the behavior of u^ε as $\varepsilon \to 0$. Formally, for $\varepsilon = 0$, this problem is exactly the minimal surfaces problem (3.2.2)-(3.2.3) which may not have solutions because of the boundary condition (3.2.8) (not necessarily satisfied by the generalized solution u^0). However, Temam proved in [Te1] that u^ε converges to the generalized solution that does not necessary verify (3.2.3). More precisely, he proved the following result which is a summary of some Lemmas in [Te1] :

Proposition 3.2.2 *(cf. [Te1]). For $\Phi \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, there exists an unique u^0 generalized solution of (3.2.2)-(3.2.3) such that :*

$$u^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u^0 \text{ strongly in } L^\alpha(\Omega), \quad 1 < \alpha < \frac{n}{n-1}, \tag{3.2.9}$$

$$u^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u^0 \text{ strongly in } H^1(\Omega'), \quad \forall\, \Omega' \subset\subset \Omega. \tag{3.2.10}$$

The above result gives a first description of the behavior of u^ε as $\varepsilon \to 0$. However, neither (3.2.9) nor (3.2.10) provides an information on the behavior of u^ε on Γ. This will be the main objective of this article.

3.2.3 The revolution minimal surfaces : The catenoïd.

In the radial symmetric case, the minimal surface problem is already a non obvious problem; it appears in many physical and natural phenomenas. From the mathematical point of view, a great attention is devoted to the surfaces of revolution with prescribed mean curvature, see e.g. Delaunay [D], Fomenko [F1], Troutman [Tr] and the bibliography therein. A special study is carried out by Atkinson and Peletier [AP] and Atkinson, Peletier and Serrin [APS] on the asymptotic properties of "vertical points" for the surfaces of revolution with prescribed mean curvature.

For the rest of this paper, we consider for Ω a pair of concentric circles :

$$\Omega = \Big\{ x \in R^2 / 0 < R_1 < |x| < R_2 \Big\}, \quad (R_1 < R_2),$$

and, in view of the geometry of the domain, we look for radial solutions of (3.2.2)-(3.2.3). Hence, the minimal surface equations (3.2.2)-(3.2.3) are now ODE's

$$\frac{d}{dr}\Big(rT(u_r^0) \Big) = 0, \qquad \text{in } \Omega, \tag{3.2.11}$$

$$u^0 = \Phi, \qquad \text{on } \Gamma; \tag{3.2.12}$$

here $\Gamma = \Gamma_1 \cup \Gamma_2, \Gamma_i = \{|x| = R_i, i = 1, 2.\}$, and

$$\Phi(r) = \begin{cases} K_1 & \text{if} \quad r = R_1, \\ K_2 & \text{if} \quad r = R_2, \end{cases} \qquad (K_1 \neq K_2), \tag{3.2.13}$$

where K_1, K_2 are two given constants.

Similarly, the equations for the radially symmetric regularized solutions u^ε read

$$\frac{d}{dr}\left\{r\left(\varepsilon u_r^\varepsilon + T(u_r^\varepsilon)\right)\right\} = 0, \qquad \text{in } \Omega, \tag{3.2.14}$$

$$u^\varepsilon = \Phi, \qquad \text{on } \Gamma. \tag{3.2.15}$$

Our main objective is to find an approximation for u^ε. We recall here that Marcellini and Miller (cf.[MM]) have studied a similar problem; they have described (among other things) the limit behavior, in the intervals of boundary layers, of the regularised surfaces with prescribed mean curvature on a ball of R^n with homogeneous boundary conditions.

Remark 14 : Since both (3.2.11)-(3.2.12) and (3.2.14)-(3.2.15) are stable by translation and symmetry, we can assume without loss of generality, that Φ is of the form

$$\Phi(r) = \begin{cases} K & \text{if} \quad r = R_1, \\ 0 & \text{if} \quad r = R_2, \end{cases} \tag{3.2.16}$$

where $K > 0$.

We note that the only figures of revolution, *i.e.* which can be obtained by simply spanning a curved line around an axis, and that have no curvature (minimal surfaces) are the catenoïd and the plane; see Proposition 3.2.3 below for the details.

Formulation of the minimal surface solution.

In this paragraph, we will see, by explicit resolution of (3.2.11)- (3.2.12), that for appropriate values of K, the solution of the minimal surface of revolution contains a singularity or a vertical point at the boundary.

Proposition 3.2.3 : *There exists $K_* > 0$ depending on R_1 and R_2 such that :*

(i) For all $K \leq K_*$, there exists a minimal surface solution of (3.2.11)-(3.2.12).

(ii) For all $K > K_*$, the problem (3.2.11)-(3.2.12) does not have a solution and the generalized solution u^0 satisfies

$$\frac{du^0}{dr} = -\infty, \quad at \;\; r = R_1.$$

Proof . By integrating (3.2.11), we obtain

$$\frac{du^0}{dr} = \pm \frac{C_1}{\sqrt{r^2 - (C_1)^2}}, \tag{3.2.17}$$

where C_1 is a positive constant.
Since we require that u^0 verify (3.2.12)-(3.2.16), then du^0/dr must be < 0 and u^0 is a decreasing function; this yields

$$\frac{du^0}{dr} = -\frac{C_1}{\sqrt{r^2 - (C_1)^2}}. \tag{3.2.18}$$

By integrating (3.2.18), we deduce the explicit expression of u^0 :

$$u^0(r) = -C_1 \ln(r + \sqrt{r^2 - (C_1)^2}) + C_2, \tag{3.2.19}$$

for all $r \in [R_1, R_2]$, where C_2 is a real constant.
Knowing that we have necessarily

$$u^0 = \Phi, \quad \text{on} \;\; \Gamma_2, \quad i.e., \quad u^0(R_2) = 0, \tag{3.2.20}$$

we would like u^0 to satisfy (3.2.12)-(3.2.16) which is equivalent to finding C_1 and C_2 solutions of the following system

$$(\mathcal{S}) \quad \begin{cases} -C_1 \ln(R_1 + \sqrt{R_1^2 - (C_1)^2}) + C_2 = K, \\ \\ -C_1 \ln(R_2 + \sqrt{R_2^2 - (C_1)^2}) + C_2 = 0. \end{cases}$$

Thus, necessarily, $0 \le C_1 \le R_1$, and we observe that the function

$$x \longmapsto f(x) = \ln(\frac{R_2 + \sqrt{R_2^2 - x^2}}{R_1 + \sqrt{R_1^2 - x^2}}) - \frac{K}{x},$$

is increasing on $]0, R_1]$ $(x = C_1)$.

On the other hand,

$$\lim_{x \to 0^+} f(x) = -\infty, \qquad \text{and} \qquad f(R_1) = \frac{K_* - K}{R_1},$$

where

$$K_* = R_1 \ln\Big(\frac{R_2 + \sqrt{R_2^2 - R_1^2}}{R_1}\Big). \tag{3.2.21}$$

Hence, if $K \le K_*$, there exists a solution of $(\mathcal{S})$ and the function u^0 given by (3.2.19) is the graph of the minimal surface solution which satisfies (3.2.12).

Now, if $K > K_*$, the equation $f(x) = 0$ does not have a solution and consequently neither $(\mathcal{S})$. Thanks to (3.2.4), the nonpositivity of u_r^0 (see (3.2.18)), and the nonegativity of the mean curvature of Γ_2, we deduce that, for $K > K_*$,

$$\frac{du^0}{dr} = -\infty, \quad \text{on} \quad \Gamma_1. \tag{3.2.22}$$

This implies that we have necessarily $C_1 = R_1$.

Finally, we have the explicit expression of u^0

$$u^0(r) = -R_1 \ln\Big(\frac{r + \sqrt{r^2 - R_1^2}}{R_2 + \sqrt{r^2 - R_2^2}}\Big), \quad \forall\, r \in [R_1, R_2]. \tag{3.2.23}$$

This ends the proof. •

Properties of u^0 and u^ε.

In this section, we derive some analytic properties for u^0 and u^ε. Our purpose is to better understand their behaviors, this will help us in the derivation of the approximation of u^ε.

Lemma 3.2.1 *The function $v = u^\varepsilon$ or u^0 verifies*

$$v \ge 0, \qquad v_r < 0, \qquad \textit{and} \qquad v_r \nearrow. \tag{3.2.24}$$

Proof . The Lemma for u^0 follows readily from (3.2.18) and (3.2.20).

For u^ε, we integrate (3.2.14) in r and we obtain

$$\varepsilon u_r^\varepsilon + T(u_r^\varepsilon) = \frac{A_\varepsilon}{r}, \quad \text{in } \Omega,$$

where A_ε is a real constant.

This proves that u_r^ε has a constant sign. Due to (3.2.15) and (3.2.16), $u_r^\varepsilon < 0 \, \forall \, r \in [R_1, R_2]$ and consequently $A_\varepsilon < 0$. We set then $A_\varepsilon = -R_\varepsilon$, $R_\varepsilon > 0$, and now (3.2.14) is equivalent to

$$\varepsilon u_r^\varepsilon + T(u_r^\varepsilon) = -\frac{R_\varepsilon}{r}, \quad \text{in } \Omega. \tag{3.2.25}$$

To conclude, we differentiate (3.2.14) and obtain

$$\varepsilon u_r^\varepsilon + T(u_r^\varepsilon) + r u_{rr}^\varepsilon \Big(\varepsilon + \frac{1}{(1 + (u_r^\varepsilon)^2)^{3/2}} \Big) = 0; \tag{3.2.26}$$

hence, $u_{rr}^\varepsilon > 0, \forall r \in [R_1, R_2]$. •

In the next result, we prove that u_r^ε is bounded in $L^\infty(\Omega')$, $\forall \Omega' \subset\subset \Omega$ and that , as ε goes to zero, it converges to u_r^0 in $L^\infty(\Omega')$ weak-star for all such Ω'. The convergence up to the boundary fails to be true because of the difference between boundary values of u^ε and u^0. This is a particular case of the results in [Te1] but the proof is elementary in this case.

Lemma 3.2.2 *For all $\delta > 0$, we have*

$$u_r^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u_r^0, \quad \textit{strongly in} \quad L^\infty(R_1 + \delta, R_2). \tag{3.2.27}$$

Proof . Firstly, we prove that u_r^ε is bounded on $[R_1 + \delta, R_2], \forall \delta > 0$.

We proceed by contradiction. Suppose otherwise that $u_r^\varepsilon(R_1 + \delta_0)$ is not bounded for some $\delta_0 > 0$. Hence, we have necessarily, after extracting a subsequence :

$$\lim_{\varepsilon \to 0} u_r^\varepsilon(R_1 + \delta_0) = -\infty. \tag{3.2.28}$$

Thanks to Lemma 3.2.1, we deduce that

$$u_r^\varepsilon(R_1+\delta_0) \geq u_r^\varepsilon(r), \qquad \forall\, r \in [R_1, R_1+\delta]. \tag{3.2.29}$$

By integrating (3.2.29) on $[R_1, R_1+\delta]$, we obtain

$$\begin{aligned} \delta_0 u_r^\varepsilon(R_1+\delta_0) &\geq u^\varepsilon(R_1+\delta_0) - K \\ &\quad \text{(Thanks to Lemma 3.2.1)} \\ &\geq -K. \end{aligned} \tag{3.2.30}$$

Combining (3.2.28) and (3.2.30), we obtain a contradiction.

Thus, we deduce that $u_r^\varepsilon(R_1+\delta)$ is bounded, $\forall \delta > 0$. Moreover, thanks to Lemma 3.2.1, we have

$$u_r^\varepsilon(R_1+\delta) \leq u_r^\varepsilon(r) \leq 0, \quad \forall\, r \in [R_1+\delta, R_2].$$

Hence, u_r^ε is uniformly bounded on $[R_1+\delta, R_2]$, $\forall \delta > 0$.

From (3.2.27), we prove that u_{rr}^ε is also uniformly bounded on $[R_1+\delta, R_2], \delta > 0$.

Secondly, we pass to the limit as ε goes to zero in (3.2.27) and thanks to the uniqueness of the derivative of the solution of (3.2.11)- (3.2.12), we conclude the result (3.2.27).

•

From (3.2.25), it is obvious that the study of the asymptotic expansion of u_r^ε is strongly related to the behavior of R_ε. Hence, it is convenient to know, at least, the limit of R_ε as ε goes to zero.

Lemma 3.2.3 : *The constant (in r) R_ε decreases as $\varepsilon \searrow 0$, and it satisfies*

$$\lim_{\varepsilon\to 0} R_\varepsilon = R_1. \tag{3.2.31}$$

Proof . We define

$$F_\varepsilon(\xi) = \varepsilon\xi + \frac{\xi}{\sqrt{1+\xi^2}}, \quad \forall\, \xi \in R.$$

Observe that the function F_ε is strictly increasing and therefore it is invertible. We note G_ε the inverse of F_ε, defined on R, we have

$$G_\varepsilon(F_\varepsilon(\xi)) = \xi, \quad \forall\, \xi \in R. \tag{3.2.32}$$

Now we differentiate (3.2.32) in ξ; since $\partial F_\varepsilon/\partial\xi > 0, \forall \xi \in R$, we deduce that

$$\frac{\partial G_\varepsilon}{\partial \eta} > 0, \quad \forall\ \eta \in R. \tag{3.2.33}$$

On the other hand, by differentiating (3.2.32) in ε, we obtain

$$\frac{\partial G_\varepsilon}{\partial \varepsilon}(\eta) + \frac{\partial G_\varepsilon}{\partial \eta}\frac{\partial F_\varepsilon}{\partial \varepsilon}(\xi) = 0, \quad \forall\ \xi \in R, \tag{3.2.34}$$

where $\eta = F_\varepsilon(\xi)$ (ξ and η have the same sign). This combined with (3.2.33) and the fact that $\partial F_\varepsilon(\xi)/\partial\varepsilon < 0, \forall \xi < 0,$ yields

$$\frac{\partial G_\varepsilon}{\partial \varepsilon}(\eta) > 0, \quad \forall\ \eta < 0. \tag{3.2.35}$$

Let $0 < \varepsilon < \varepsilon'$. By contradiction, we suppose that

$$R_\varepsilon > R_{\varepsilon'}, \tag{3.2.36}$$

and rewrite (3.2.25) as follows

$$F_\varepsilon(u_r^\varepsilon) = -\frac{R_\varepsilon}{r}, \quad i.e. \quad u_r^\varepsilon = G_\varepsilon(-\frac{R_\varepsilon}{r}). \tag{3.2.37}$$

Integrating (3.2.37) on (R_1, R_2) and using (3.2.15), we obtain

$$-K = \int_{R_1}^{R_2} G_\varepsilon(-\frac{R_\varepsilon}{r})dr. \tag{3.2.38}$$

From (3.2.33), (3.2.35) and (3.2.36), we conclude that

$$G_\varepsilon(-\frac{R_\varepsilon}{r}) < G_{\varepsilon'}(-\frac{R_\varepsilon}{r}) < G_{\varepsilon'}(-\frac{R_{\varepsilon'}}{r}). \tag{3.2.39}$$

Combining now (3.2.38) and (3.2.39), we deduce that

$$-K = \int_{R_1}^{R_2} G_\varepsilon(-\frac{R_\varepsilon}{r})dr < \int_{R_1}^{R_2} G_{\varepsilon'}(-\frac{R_{\varepsilon'}}{r})dr = -K, \tag{3.2.40}$$

which is a contradiction. Consequently, we have $R_\varepsilon < R_{\varepsilon'}$, for all $0 < \varepsilon < \varepsilon'$. Hence, R_ε decreases as $\varepsilon \searrow 0$, and R_ε is bounded. Let R_0 its limit.

Now, writing (3.2.25) at $r = R_2$ and using Lemma 3.2.2, we obtain

$$-\frac{R_\varepsilon}{R_2} = (\varepsilon u_r^\varepsilon + T(u_r^\varepsilon))(r = R_2) \underset{\varepsilon \to 0}{\longrightarrow} T(u_r^0)(r = R_2) = -\frac{R_1}{R_2}.$$

Hence, $R_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} R_0 = R_1$. •

3.3 Study of the boundary layer at $r = R_1$.

3.3.1 The auxiliary approximation for u^ε near the boundary $r = R_1$ and the approximate value of R_ε.

We infer from the previous analysis that, at $r = R_1$, $T(u_r^0) = -1$, $T(u_r^\varepsilon) \approx -1$, and u^ε decreases very rapidly near $r = R_1$, from K to K_*. In a boundary layer region (R_1, r_*), yet to be determined, we roughly approximate equation (3.2.25) by replacing $T(u_r^\varepsilon)$ by -1, and R_ε by C_ε. We choose $r_* = R_1(1 + A\varepsilon^{1/2})$, where $A > 0$ is a constant to be fixed later on, and on (R_1, r_*), we consider the rough (first) approximation $\widetilde{\varphi}^\varepsilon$ of u^ε defined by

$$\frac{d}{dr}\{r(\varepsilon\widetilde{\varphi}_r^\varepsilon - 1)\} = 0, \quad \text{in } [R_1, r_*], \tag{3.3.1}$$

$$\widetilde{\varphi}^\varepsilon(R_1) = K, \quad \text{and} \quad \widetilde{\varphi}^\varepsilon(r_*) = K_* - \Delta K_*. \tag{3.3.2}$$

Here ΔK_* is a constant that will be choosen subsequently such that $\Delta K_* \to 0$ as $\varepsilon \to 0$). Better approximation of $T(u_r^\varepsilon)$ will be introduced later on but at this point we infer from (3.3.1) that

$$\widetilde{\varphi}_r^\varepsilon = \frac{1}{\varepsilon}(1 - \frac{C_\varepsilon}{r}), \quad \forall\, r \in (R_1, r_*). \tag{3.3.3}$$

The total variation of $\widetilde{\varphi}^\varepsilon$ on (R_1, r_*), which is deduced from (3.3.2), allows us to find the value of C_ε:

$$\int_{R_1}^{r_*} \widetilde{\varphi}_r^\varepsilon \mathrm{d}r = K_* - \Delta K_* - K, \tag{3.3.4}$$

i.e.,

$$r_* - R_1 - C_\varepsilon \ln(\frac{r_*}{R_1}) = (K_* - \Delta K_* - K)\,\varepsilon.$$

This implies that

$$C_\varepsilon = \frac{R_1 A\varepsilon^{1/2} + (K_* - \Delta K_* - K)\varepsilon}{\ln(1 + A\varepsilon^{1/2})}, \tag{3.3.5}$$

and, for ε small enough, we may approximate the constant C_ε, by asymptotic expansion as follows :

$$\begin{aligned} C_\varepsilon &= R_1\Big(1 + \frac{K + \Delta K_* - K_*}{AR_1}\varepsilon^{1/2}\Big)\Big(1 - \frac{A}{2}\varepsilon^{1/2} + O(\varepsilon)\Big)^{-1}, \\ &= R_1\Big[1 + (\frac{A}{2} + \frac{K + \Delta K_* - K_*}{AR_1})\varepsilon^{1/2}\Big] + O(\varepsilon). \end{aligned} \tag{3.3.6}$$

Now we choose a particular value for A such that the coefficients of $\varepsilon^{1/2}$ in $r_* = R_1(1 + A\varepsilon^{1/2})$ and in the expansion of C_ε are equal. Namely (assuming $|\Delta K_*| \ll K - K_* = O(1)$) :

$$\frac{A}{2} + \frac{K + \Delta K_* - K_*}{AR_1} = A, \quad i.e., \quad A = \sqrt{\frac{2}{R_1}(K + \Delta K_* - K_*)}. \tag{3.3.7}$$

We note that C_ε which is given by (3.3.5) and approximated in (3.3.6) is an approximation of R_ε in some sense to be specified subsequently. Let us mention also that the constant C_ε will play a fundamental role in the derivation of the global approximation of u^ε.

3.3.2 Improving the approximation of u^ε near the boundary $r = R_1$.

In the above paragraph, we have defined an approximation $\tilde{\varphi}^\varepsilon$ for u^ε near $r = R_1$, but this will not be a sufficiently good approximation and its role is only limited to deriving the value of the constant C_ε. Now, we look for a function φ_r^ε which will (hopefully) better approximate u_r^ε by taking the next term in the asymptotic expansion of $T(u_r^\varepsilon)$ *i.e.*,

$$T(u_r^\varepsilon) = -1 + \frac{1}{2(u_r^\varepsilon)^2} + O(\frac{1}{(u_r^\varepsilon)^4}),$$

for $u_r^\varepsilon < 0$ and large.

Hence, we impose to φ_r^ε to verify

$$\varepsilon\ \varphi_r^\varepsilon + \frac{1}{2(\varphi_r^\varepsilon)^2} = 1 - \frac{C_\varepsilon}{r}, \tag{3.3.8}$$

we denote by $\sigma_\varepsilon(r)$ the right-hand side of (3.3.8).

Determination of φ_r^ε.

The resolution of (3.3.8) is equivalent to finding $X = \varphi_r^\varepsilon$ solution of

$$X^3 - \frac{\sigma_\varepsilon(r)}{\varepsilon}X^2 + \frac{1}{2\varepsilon} = 0. \tag{3.3.9}$$

To solve (3.3.9), we use the following result which make explicit the expression of the roots of a third order polynomial function :

Lemma 3.3.1 *We consider the following equation*

$$(E) \qquad X^3 + a_1X^2 + a_2X + a_3 = 0.$$

We denote by

$$Q = \frac{3a_2 - a_1^2}{9}, \qquad R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}.$$

If a_1, a_2 and a_3 are real and if $D = Q^3 + R^2$ is the discriminant, then

(i) One root of (E) is real and two are complex conjugate if $D > 0$. The real solution is given by

$$X = S + T - \frac{1}{3}a_1, \tag{3.3.10}$$

where $S = (R + \sqrt{D})^{1/3}$ and $T = (R - \sqrt{D})^{1/3}$.

(ii) If $D \leq 0$, all roots are real and given by

$$X_k = 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{2k\pi}{3}\right) - \frac{1}{3}a_1, \tag{3.3.11}$$

where $\cos\theta = R/\sqrt{-Q^3}$ and $k \in \{0, 1, 2\}$.

We apply now Lemma 3.3.1 to equation (3.3.9). Its discriminant reads

$$D_\varepsilon(r) = \frac{1}{16\varepsilon^2} - \frac{1}{2\varepsilon}\left(\frac{\sigma_\varepsilon(r)}{3\varepsilon}\right)^3. \tag{3.3.12}$$

Observe that

$$D_\varepsilon(r) = 0 \iff r = r(C_\varepsilon) = \frac{C_\varepsilon}{1 - \frac{3}{2}\varepsilon^{2/3}}. \tag{3.3.13}$$

Moreover, we have

$$\begin{aligned} D_\varepsilon(r) &> 0, \quad \forall\, r \in [R_1, r(C_\varepsilon)], \\ D_\varepsilon(r) &< 0, \quad \forall\, r \in]r(C_\varepsilon), R_2]. \end{aligned}$$

Indeed, $r \longmapsto D_\varepsilon(r)$ is a decreasing function, since

$$D_\varepsilon'(r) = -\frac{1}{2\varepsilon}\frac{\sigma_\varepsilon'}{3\varepsilon}\left(\frac{\sigma_\varepsilon(r)}{3\varepsilon}\right)^2 \leq 0, \quad \forall\, r \in [R_1, R_2];$$

therefore we distinguish for the moment $R_1 \leq r \leq r(C_\varepsilon)$ and $r(C_\varepsilon) \leq r \leq R_2$.

On $[R_1, r(C_\varepsilon)]$, thanks to Lemma 3.3.1 and the fact that $D_\varepsilon(r) > 0$, we deduce that

$$\begin{aligned}\varphi_r^\varepsilon(r) \;\; = \;\; & \left(-\frac{1}{4\varepsilon} + (\frac{\sigma_\varepsilon(r)}{3\varepsilon})^3 + \sqrt{\frac{1}{16\varepsilon^2} - \frac{1}{2\varepsilon}(\frac{\sigma_\varepsilon(r)}{3\varepsilon})^3} \right)^{1/3} + \\ & + \left(-\frac{1}{4\varepsilon} + (\frac{\sigma_\varepsilon(r)}{3\varepsilon})^3 - \sqrt{\frac{1}{16\varepsilon^2} - \frac{1}{2\varepsilon}(\frac{\sigma_\varepsilon(r)}{3\varepsilon})^3} \right)^{1/3} + \frac{\sigma_\varepsilon(r)}{3\varepsilon}, \qquad (3.3.14)\end{aligned}$$

$\forall r \in [R_1, r(C_\varepsilon)]$.

On the other hand, on $]r(C_\varepsilon), R_2]$, where $D_\varepsilon(r) < 0$, we have three real roots but we will be interested only in the negative one since we hope that φ_r^ε approximate $u_r^\varepsilon < 0$. Thus, we have

$$\varphi_r^\varepsilon(r) = \frac{\sigma_\varepsilon(r)}{3\varepsilon}\left\{1 + 2\cos\left[\frac{1}{3}\arccos\left(1 - \frac{3^3}{4}\frac{\varepsilon^2}{(\sigma_\varepsilon(r))^3}\right) + \frac{2\pi}{3}\right]\right\}, \qquad (3.3.15)$$

for all $r \in]r(C_\varepsilon), R_2]$. This choice will be confirmed in the proof of Lemma 3.3.2.

Finally, since we look for an approximation near the boundary $r = R_1$, it is sufficient to define φ^ε on $[R_1, r_\gamma]$, where

$$r_\gamma = R_1(1 + \varepsilon^{1/2-\gamma}), \qquad (3.3.16)$$

and $0 < \gamma < 1/2$ is to be defined later on. At this point we should note that with the choice (3.3.7) of A, we have, for ε small enough,

$$R_1 < r_* < r(C_\varepsilon) < r_\gamma < R_2. \qquad (3.3.17)$$

Lemma 3.3.2 *We have $\varphi_r^\varepsilon < 0$ and φ_r^ε is increasing in r.*

Proof . To determine the sign of φ_r^ε, we need to study the roots of the function in (3.3.9) :

$$X \longmapsto f_\sigma^\varepsilon(X) = X^3 - \frac{\sigma_\varepsilon(r)}{\varepsilon}X^2 + \frac{1}{2\varepsilon}.$$

We have

$$\frac{\partial f_\sigma^\varepsilon}{\partial X}(X) = 3X(X - \frac{2\sigma_\varepsilon(r)}{3\varepsilon}).$$

Hence, f_σ^ε is increasing on $]-\infty, \min(0, X_0)[$ where $X_0 = \frac{2\sigma_\varepsilon(r)}{3\varepsilon}$, decreasing on $]\min(0, X_0), \max(0, X_0)[$ and increasing on $]\max(0, X_0), +\infty[$.

For all $r \in [R_1, r(C_\varepsilon)]$, we have

$$f^\varepsilon_\sigma(0) = \frac{1}{2\varepsilon} > 0, \quad \text{and} \quad f^\varepsilon_\sigma(X_0) = -\frac{2^2}{3^3}(\frac{\sigma_\varepsilon}{\varepsilon})^3 + \frac{1}{2\varepsilon} > 0,$$

since $\sigma_\varepsilon(r) < 3/2\ \varepsilon^{2/3}, \forall r \in [R_1, r(C_\varepsilon)]$.

This implies that f^ε_σ has an unique real root with negative sign which is given by (3.3.10). For $r \in [r(C_\varepsilon), r_\gamma]$, we have $\sigma_\varepsilon(r) \geq 3/2\ \varepsilon^{2/3}$. Hence, we find $f^\varepsilon_\sigma(X_0) \leq 0$ and $X_0 > 0$ in this case. Therefore, there exists three real solutions of (3.3.8)-(3.3.9), but only one of the three roots is negative. This is given by (3.3.11) for $k = 1$. Indeed, we infer from (3.3.15) that

$$\varphi^\varepsilon_r(r(C_\varepsilon)) = -\frac{1}{2}\varepsilon^{-1/3},$$

which implies that the two other solutions given by (3.3.11) for $k = 0, 2$ are necessarily positive.

To conclude, we differentiate (3.3.8) in r and we obtain

$$\varphi^\varepsilon_{rr}(\varepsilon - \frac{1}{(\varphi^\varepsilon_r)^3}) = \sigma'_\varepsilon(r). \tag{3.3.18}$$

Thanks to the fact that $\sigma'_\varepsilon(r) > 0$ and $\varphi^\varepsilon_r < 0$, we deduce that $\varphi^\varepsilon_{rr} > 0$. This ends the proof of Lemma 3.3.2. •

Remark 15 : The function φ^ε_r solution of (3.3.8)-(3.3.9) and those expression is given by (3.3.10) and (3.3.15) is a regular (C^∞) function of r on $[R_1, r_\gamma]$. Indeed, we have $\widetilde{F}_\varepsilon(\varphi^\varepsilon_r) = \sigma_\varepsilon(r) \in C^\infty(R_1, r_\gamma)$, where $\widetilde{F}_\varepsilon(X) = \varepsilon X + 1/(2X^2)$ is a one-to-one on $\{X < 0\}$.

The total variation of φ^ε_r on (R_1, r_γ).

In order to obtain the total variation of φ^ε_r on (R_1, r_γ), we need to estimate the order of magnitude (in ε) of φ^ε_r and $\widetilde{\varphi}^\varepsilon_r$ at the boundary of each of these intervals : (R_1, r_*), $(r_*, r(C_\varepsilon))$ and $(r(C_\varepsilon), r_\gamma)$.

Firstly, we start by the point $r = R_1$. Since

$$\widetilde{\varphi}^\varepsilon_r = \frac{\sigma_\varepsilon}{\varepsilon}, \quad \text{and} \quad \sigma_\varepsilon(R_1) = 1 - \frac{C_\varepsilon}{R_1} = -A\varepsilon^{1/2} + O(\varepsilon),$$

we deduce that

$$\tilde{\varphi}_r^\varepsilon(r = R_1) = O^-(\varepsilon^{-1/2}), \tag{3.3.19}$$

where $O^-(\xi)$ (respectively $O^+(\xi)$) denotes a negative (respectively positive) $O(\xi)$.

Thanks to (3.3.14) and the asymptotic expansion of $\sigma_\varepsilon(R_1)$ with respect to ε, we conclude that

$$\varphi_r^\varepsilon(r = R_1) = O^-(\varepsilon^{-1/2}). \tag{3.3.20}$$

Secondly, at $r = r_*$, we have

$$\sigma_\varepsilon(r_*) = 1 - \frac{C_\varepsilon}{r_*} = O^-(\varepsilon). \tag{3.3.21}$$

By inserting (3.3.21) into (3.3.14), we obtain

$$\varphi_r^\varepsilon(r = r_*) = O^-(\varepsilon^{-1/3}). \tag{3.3.22}$$

From (3.3.21) and the expression of $\tilde{\varphi}_r^\varepsilon$, we deduce that

$$\tilde{\varphi}_r^\varepsilon(r = r_*) = O^-(1). \tag{3.3.23}$$

Thirdly, at $r = r(C_\varepsilon)$,

$$\sigma_\varepsilon(r(C_\varepsilon)) = 1 - \frac{C_\varepsilon}{r(C_\varepsilon)} = \frac{3}{2}\varepsilon^{2/3}.$$

This combined with (3.3.14) implies that

$$\varphi_r^\varepsilon(r = r(C_\varepsilon)) = O^-(\varepsilon^{-1/3}). \tag{3.3.24}$$

On the other hand we easily observe that

$$\tilde{\varphi}_r^\varepsilon(r = r(C_\varepsilon)) = O^+(\varepsilon^{-1/3}). \tag{3.3.25}$$

Finally, at the point $r = r_\gamma$, we have

$$\sigma_\varepsilon(r_\gamma) = 1 - \frac{C_\varepsilon}{r_\gamma} = O^-(\varepsilon^{1/2-\gamma}).$$

Hence, from (3.3.15) we infer that

$$\varphi_r^\varepsilon(r = r_\gamma) = O^-(\varepsilon^{\gamma/2-1/4}). \tag{3.3.26}$$

Now, we are able to prove the following proposition

Proposition 3.3.1 *: Let φ_r^ε be the solution of (3.3.8)-(3.3.9) satisfying $\varphi^\varepsilon(R_1) = K$. There exists $0 < \gamma < 1/6$ such that*

$$\int_{R_1}^{r_\gamma} \varphi_r^\varepsilon \mathrm{d}r = K_* - \Delta K_* - K + O^-(\varepsilon^{1/6-\gamma}). \tag{3.3.27}$$

Proof . We have the following relation between φ_r^ε and $\tilde{\varphi}_r^\varepsilon$:

$$\varphi_r^\varepsilon = \tilde{\varphi}_r^\varepsilon - \frac{1}{2\varepsilon(\varphi_r^\varepsilon)^2}, \quad \forall\ r \in [R_1, r_\gamma]. \tag{3.3.28}$$

By integrating (3.3.28) on $(R_1, r(C_\varepsilon))$, we obtain

$$\int_{R_1}^{r(C_\varepsilon)} \varphi_r^\varepsilon \mathrm{d}r = \int_{R_1}^{r_*} \tilde{\varphi}_r^\varepsilon \mathrm{d}r + \int_{r_*}^{r(C_\varepsilon)} \tilde{\varphi}_r^\varepsilon \mathrm{d}r - \frac{1}{2\varepsilon} \int_{R_1}^{r(C_\varepsilon)} \frac{1}{(\varphi_r^\varepsilon)^2} \mathrm{d}r. \tag{3.3.29}$$

We intend to estimate the last two integrals in the right-hand side of (3.3.29). For this purpose, we remark firstly that $\tilde{\varphi}_r^\varepsilon$ is increasing in r ($\tilde{\varphi}_{rr}^\varepsilon = \frac{\sigma'\varepsilon}{\varepsilon} > 0$); this implies that

$$\tilde{\varphi}_r^\varepsilon(r = r_*) \leq \tilde{\varphi}_r^\varepsilon(r) \leq \tilde{\varphi}_r^\varepsilon(r = r(C_\varepsilon)), \quad \forall\ r\ \in [r_*, r(C_\varepsilon)].$$

From these inequalities, (3.3.23) and (3.3.25), we find

$$|\tilde{\varphi}_r^\varepsilon(r)| \leq |\tilde{\varphi}_r^\varepsilon(r = r(C_\varepsilon))| \leq C\varepsilon^{-1/3},$$

for all $r\ \in [r_*, r(C_\varepsilon)]$, where C will denote, hereafter, a generic constant that depend on all data except ε.

Therefore

$$\begin{aligned} \left| \int_{r_*}^{r(C_\varepsilon)} \tilde{\varphi}_r^\varepsilon \mathrm{d}r \right| &\leq C\ \varepsilon^{-1/3}(r(C_\varepsilon) - r_*), \\ &\leq C\ \varepsilon^{1/3}. \end{aligned} \tag{3.3.30}$$

On the other hand, thanks to Lemma 3.3.2,

$$\varphi_r^\varepsilon(r = R_1) \leq \varphi_r^\varepsilon(r) \leq \varphi_r^\varepsilon(r(C_\varepsilon)) < 0, \quad \forall\ r\ \in [R_1, r(C_\varepsilon)].$$

We infer from the above inequalities, (3.3.20) and (3.3.24) that

$$|\varphi_r^\varepsilon(r)| \geq C\ \varepsilon^{-1/3}, \quad \forall\ r\ \in [R_1, r(C_\varepsilon)]. \tag{3.3.31}$$

Hence, we deduce that

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{R_1}^{r(C_\varepsilon)}\frac{1}{(\varphi_r^\varepsilon)^2}\mathrm{d}r &\leq C\frac{\varepsilon^{2/3}}{\varepsilon}(r(C_\varepsilon)-R_1), \\ &\leq C\ \varepsilon^{1/6}. \end{aligned} \tag{3.3.32}$$

By inserting (3.3.30), (3.3.32) in (3.3.29), we conclude that

$$\begin{aligned}\int_{R_1}^{r(C_\varepsilon)}\varphi_r^\varepsilon \mathrm{d}r &= \int_{R_1}^{r_*}\tilde{\varphi}_r^\varepsilon \mathrm{d}r + O^-(\varepsilon^{1/6}), \\ & \quad \text{(Thanks to (3.3.4))} \\ &= K_* - \Delta K_* - K + O^-(\varepsilon^{1/6}). \end{aligned} \tag{3.3.33}$$

Now, we intend to derive an estimation for the total variation of φ_r^ε on $[r(C_\varepsilon), r_\gamma]$. Thanks to Lemma 3.3.2 and (3.3.24),

$$|\varphi_r^\varepsilon(r)| \leq |\varphi_r^\varepsilon(r = r(C_\varepsilon))| \leq C\ \varepsilon^{-1/3}, \quad \forall r \in [r(C_\varepsilon), r_\gamma].$$

Hence, we deduce that

$$|\int_{r(C_\varepsilon)}^{r_\gamma}\varphi_r^\varepsilon \mathrm{d}r| \leq C\ \varepsilon^{1/6-\gamma}. \tag{3.3.34}$$

Finally, by taking into account the sign of φ_r^ε and by combining (3.3.33) and (3.3.34), we obtain the total variation of φ_r^ε on $[R_1, r_\gamma]$ in the form (3.3.27). From now on we fix γ such that $0 < \gamma < 1/6$.

This completes the proof of Proposition 3.3.1. •

In summary, for the approximation φ^ε defined on the boundary layer interval $[R_1, r_\gamma]$, we have

$$\varphi^\varepsilon(R_1) = K, \qquad \varphi^\varepsilon(r_\gamma) = K_* - \Delta K_* - K + O^-(\varepsilon^{1/6-\gamma}), \tag{3.3.35}$$

and

$$|\varphi_r^\varepsilon| \leq |\varphi_r^\varepsilon(r = r_\gamma)| \leq c\ \varepsilon^{\gamma/2-1/4}, \quad \forall r \in [R_1, r_\gamma]. \tag{3.3.36}$$

We recall, here, that $K_* = u^0(R_1)$ and $|\Delta K_*| \ll K - K_*$.

From (3.3.8) and the asymptotic expansion of $T(\varphi_r^\varepsilon)$ for $\varphi_r^\varepsilon < 0$ and large, we deduce that

φ_r^ε satisfies, on (R_1, r_γ), the following equation

$$\begin{aligned} r\Big(\varepsilon\varphi_r^\varepsilon + T(\varphi_r^\varepsilon)\Big) &= -C_\varepsilon\chi_{(R_1,r_\gamma)} - \frac{3}{8}\frac{r}{(\varphi_r^\varepsilon)^4}\chi_{(R_1,r_\gamma)} + \\ &\quad + O(\frac{1}{(\varphi_r^\varepsilon)^8})\chi_{(R_1,r_\gamma)}. \end{aligned} \tag{3.3.37}$$

In the next section, we match the function φ^ε, defined by (3.3.14)-(3.3.15), to another function which will approximate u^ε in the rest of the domain. For this purpose, we consider just a continuous matching; this is sufficient to define the global approximation of u^ε.

3.3.3 The global approximation of u^ε on the whole of the domain.

In this subsection, we continue the construction of an approximation of u^ε solution of the perturbed minimal surface problem (3.2.14)-(3.2.15).
Now, we define the continuous function v^ε, which will be the global approximation of u^ε, as follows

$$v^\varepsilon = \varphi^\varepsilon\chi_{(R_1,r_\gamma)} + \psi^\varepsilon\chi_{(r_\gamma,R_2)}, \tag{3.3.38}$$

where ψ^ε is a curve close to the chain (u^0) defined by

$$T(\psi_r^\varepsilon) = -\frac{C_\varepsilon}{r} - \varepsilon u_r^0, \tag{3.3.39}$$

i.e.,

$$\psi_r^\varepsilon = -\frac{C_\varepsilon + \varepsilon r u_r^0}{\sqrt{r^2 - C_\varepsilon^2 - 2r\varepsilon C_\varepsilon u_r^0 - (\varepsilon r u_r^0)^2}}. \tag{3.3.40}$$

Furthermore ψ_r^ε is required to verify the continuous matching condition

$$\psi^\varepsilon(r_\gamma) = \varphi^\varepsilon(r_\gamma) = K_* - \Delta K_* + O^-(\varepsilon^{1/6-\gamma}), \tag{3.3.41}$$

and the boundary condition

$$\psi^\varepsilon(R_2) = 0. \tag{3.3.42}$$

To prove the validity of the choice of ψ^ε, we firstly approximate the expression of ψ_r^ε (3.3.40). Of course all these "proofs" are formal. The fully rigouros results will be obtained in Section 3.4.

Since

$$r^2 - C_\varepsilon^2 \geq r_\gamma^2 - C_\varepsilon^2 = O^+(\varepsilon^{1/2-\gamma}),$$

and

$$-r\varepsilon C_\varepsilon u_r^0 \leq -r_\gamma \varepsilon C_\varepsilon u_r^0(r_\gamma), \qquad \forall r \in [r_\gamma, R_2],$$

we deduce that

$$r^2 - C_\varepsilon^2 \geq -r\varepsilon C_\varepsilon u_r^0, \quad \forall \in [r_\gamma, R_2]. \tag{3.3.43}$$

Consequently,

$$\psi_r^\varepsilon = -\frac{C_\varepsilon + \varepsilon r u_r^0}{\sqrt{r^2 - C_\varepsilon^2}}\Big(1 + \frac{r\varepsilon C_\varepsilon u_r^0 + 1/2(r\varepsilon u_r^0)^2}{r^2 - C_\varepsilon^2}\Big) + O(\varepsilon^{2\gamma}),$$

which implies that

$$\psi_r^\varepsilon = \widetilde{\psi}_r^\varepsilon + O(\varepsilon^{2\gamma}), \tag{3.3.44}$$

where

$$\widetilde{\psi}_r^\varepsilon = -\frac{C_\varepsilon + \varepsilon r u_r^0}{\sqrt{r^2 - C_\varepsilon^2}} - \frac{R_1^2 \varepsilon r u_r^0}{(r^2 - C_\varepsilon^2)^{3/2}}. \tag{3.3.45}$$

We integrate (3.3.44) in r and with the help of (3.3.42) we deduce then the expression of ψ^ε :

$$\begin{aligned}\psi^\varepsilon(r) \quad = \quad & K_* - C_\varepsilon \ln(r + \sqrt{r^2 - C_\varepsilon^2}) + \frac{\varepsilon R_1}{2} + \varepsilon R_1^3 \frac{\sqrt{r^2 - R_1^2}}{\sqrt{r^2 - C_\varepsilon^2}(R_1^2 - C_\varepsilon)^2} + \\ & + \ln|2(r^2 - C_\varepsilon^2) - R_1^2 + \sqrt{(r^2 - C_\varepsilon^2)(r^2 - R_1^2)}| + O(\varepsilon^{2\gamma} + \varepsilon^{1/2}).\end{aligned} \tag{3.3.46}$$

Thanks to the continuous matching condition (3.3.41) and (3.3.46), we deduce the equation of ΔK_*,

$$\begin{aligned}\Delta K_* \quad = \quad & C_\varepsilon \ln(r_\gamma + \sqrt{r_\gamma^2 - C_\varepsilon^2}) - \frac{\varepsilon R_1}{2} - \varepsilon R_1^3 \frac{\sqrt{r_\gamma^2 - R_1^2}}{\sqrt{r_\gamma^2 - C_\varepsilon^2}(R_1^2 - C_\varepsilon)^2} - \\ & - \ln|2(r_\gamma^2 - C_\varepsilon^2) + R_1^2 - \sqrt{(r_\gamma^2 - C_\varepsilon^2)(r_\gamma^2 - R_1^2)}| + O(\varepsilon^{2\gamma} + \varepsilon^{1/6-\gamma}).\end{aligned} \tag{3.3.47}$$

Hence, we deduce an estimate value for ΔK_* :

$$\Delta K_* = O(\varepsilon^{2\gamma} + \varepsilon^{1/6-\gamma}). \tag{3.3.48}$$

This is in full argument with our initial hypothesis that $\Delta K_* \to 0$ as $\varepsilon \to 0$.

Remark 16 The role of the constant ΔK_* is to adjust the matching of φ^ε to ψ^ε which that makes (3.3.42) available.

Remark 17 We can easily see that the function ψ^ε given by (3.3.46) is a curve close to the chain u^0 *i.e.* $\psi^\varepsilon \approx u^0$.

3.4 The main result: Convergence in $H^1(\Omega)$.

This section is devoted to state and prove a rigouros convergence result of u^ε towards. Hence it proves confirmation of the validity of the approximation introduced in the above section and denoted by v^ε.

We start by stating the main theorem :

Theorem 3.4.6 *For u^ε solution of the perturbed minimal surface problem (3.2.14)-(3.2.15) and its approximation v^ε defined by (3.3.38), there exists a constant C independent of ε and $\gamma \in (0, 1/6)$ such that :*

$$\|u^\varepsilon - v^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^\gamma \sqrt{|\ln \varepsilon|}. \tag{3.4.1}$$

Proof . The variational formulation of (3.2.14)-(3.2.15) reads :

$$\varepsilon\ (ru^\varepsilon_r, \theta_r)_{L^2(\Omega)} + (rT(u^\varepsilon_r), \theta_r)_{L^2(\Omega)} = 0, \qquad \forall\theta \in H^1_0(\Omega). \tag{3.4.2}$$

For v^ε defined in (3.3.38), we write the analogous variational equation in the form :

$$\varepsilon\ (rv^\varepsilon_r, \theta_r)_{L^2(\Omega)} + (rT(v^\varepsilon_r), \theta_r)_{L^2(\Omega)} = (h^\varepsilon(r), \theta_r)_{L^2(\Omega)}, \tag{3.4.3}$$

$$\forall\theta \in H^1_0(\Omega),$$

where $h^\varepsilon(r)$ is a function to be made precise later on.

The equation (3.3.37) together with (3.3.39) yields

$$\begin{aligned} h^\varepsilon(r) &= -C_\varepsilon \chi_{(R_1,R_2)} - \frac{3}{8}\frac{r}{(\varphi^\varepsilon_r)^4}\chi_{(R_1,r_\gamma)} + \\ &\quad + O(\frac{1}{(\varphi^\varepsilon_r)^8})\chi_{(R_1,r_\gamma)} + \varepsilon(\psi^\varepsilon_r - u^0_r)\chi_{(r_\gamma,R_2)}. \end{aligned} \tag{3.4.4}$$

By taking the difference (3.4.1)−(3.4.3) and denoting $w^\varepsilon = u^\varepsilon - v^\varepsilon$, we obtain

$$\varepsilon\ (rw_r^\varepsilon, \theta_r)_{L^2(\Omega)} + \Big(r(T(u_r^\varepsilon) - T(v_r^\varepsilon)), \theta_r\Big)_{L^2(\Omega)} = -(h^\varepsilon(r), \theta_r)_{L^2(\Omega)}, \tag{3.4.5}$$

$$\forall \theta \in H_0^1(\Omega).$$

Thanks to (3.2.15), (3.3.36), (3.3.42) and the definition (3.3.38)of v^ε , we deduce that

$$w^\varepsilon = 0, \qquad \text{on} \ \ \Gamma. \tag{3.4.6}$$

Hence, we can replace θ by w^ε in (3.4.5) and we obtain

$$\varepsilon\ (rw_r^\varepsilon, w_r^\varepsilon)_{L^2(\Omega)} + \Big(r(T(u_r^\varepsilon) - T(v_r^\varepsilon)), w_r^\varepsilon\Big)_{L^2(\Omega)} = -(h^\varepsilon(r), w_r^\varepsilon)_{L^2(\Omega)}. \tag{3.4.7}$$

To derive an estimate on w_r^ε, we firstly begin by bounding from below the left-hand side of (3.4.7). On one hand, we have

$$\varepsilon\ (rw_r^\varepsilon, w_r^\varepsilon)_{L^2(\Omega)} \geq \varepsilon R_1 \|w_r^\varepsilon\|_{L^2(\Omega)}^2. \tag{3.4.8}$$

On the other hand, we have

$$\begin{aligned} T(u_r^\varepsilon) - T(v_r^\varepsilon) &= \frac{u_r^\varepsilon}{\sqrt{1+(u_r^\varepsilon)^2}} - \frac{v_r^\varepsilon}{\sqrt{1+(v_r^\varepsilon)^2}}, \\ &= \frac{u_r^\varepsilon + v_r^\varepsilon}{I_{u_r^\varepsilon} I_{v_r^\varepsilon}(u_r^\varepsilon I_{v_r^\varepsilon} + v_r^\varepsilon I_{u_r^\varepsilon})} w_r^\varepsilon, \\ &= g_\varepsilon(r) w_r^\varepsilon, \end{aligned} \tag{3.4.9}$$

where $I_u = \sqrt{1+u^2}$. Thus,

$$(r(T(u_r^\varepsilon) - T(v_r^\varepsilon)), w_r^\varepsilon)_{L^2(\Omega)} = \int_{R_1}^{R_2} r g_\varepsilon(r)(w_r^\varepsilon)^2 \mathrm{d}r. \tag{3.4.10}$$

Thanks to Lemma 3.2.1, Lemma 3.3.2 and (3.3.40) which imply that u_r^ε and v_r^ε are < 0 (see (3.3.40)), we deduce that

$$g_\varepsilon(r) \geq 0, \qquad \forall r \in [R_1, R_2].$$

This implies the the positivity of the minimal surface operator *i.e.*,

$$(r(T(u_r^\varepsilon) - T(v_r^\varepsilon)), w_r^\varepsilon)_{L^2(\Omega)} \geq 0. \tag{3.4.11}$$

Secondly, we derive an estimate for the right-hand side of (3.4.7) which reads

$$\begin{aligned}(-h^\varepsilon(r), w_r^\varepsilon)_{L^2(\Omega)} &= -C_\varepsilon \int_{R_1}^{R_2} w_r^\varepsilon \mathrm{d}r - \frac{3}{8}\int_{R_1}^{r_\gamma} \frac{r}{(\varphi_r^\varepsilon)^4} w_r^\varepsilon \mathrm{d}r + \\ &\quad + \int_{R_1}^{r_\gamma} O(\frac{1}{(\varphi_r^\varepsilon)^8}) w_r^\varepsilon \mathrm{d}r + \varepsilon \int_{r_\gamma}^{R_2} (\psi_r^\varepsilon - u_r^0) w_r^\varepsilon \mathrm{d}r. \end{aligned} \tag{3.4.12}$$

We denote by $I_1, \ldots, I_4$ the integrals in the right-hand side of (3.4.12) counted from left to right. We note, here, that the function integrated in I_3 verifies

$$C\ \varepsilon^4 \leq O(\frac{1}{(\varphi_r^\varepsilon)^8}) \leq C\ \varepsilon^{2-4\gamma}. \tag{3.4.13}$$

Then, thanks to (3.4.6), we have

$$I_1 = 0. \tag{3.4.14}$$

We infer from Lemma 3.3.2 and (3.3.15) that

$$|\varphi_r^\varepsilon(r)| \geq |\varphi_r^\varepsilon(r_\gamma)| \geq C\varepsilon^{\gamma/2-1/4}, \quad \forall r \in [R_1, r_\gamma]. \tag{3.4.15}$$

Thanks to the Cauchy-Schwarz inequality and (3.4.15), we deduce that

$$\begin{aligned} I_2 &\leq \frac{3}{8}\frac{1}{\min(\varphi_r^\varepsilon)^4}\ (r_\gamma - R_1)^{1/2}\ \|w_r^\varepsilon\|_{L^2(R_1,r_\gamma)}, \\ &\leq C\varepsilon^{1-2\gamma}\ (r_\gamma - R_1)^{1/2}\ \|w_r^\varepsilon\|_{L^2(R_1,r_\gamma)}, \\ &\leq C\varepsilon^{5/4-5\gamma/2}\ \|w_r^\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{3.4.16}$$

From (3.4.13) and the Cauchy-Schwarz inequality,

$$I_3 \leq C\varepsilon^{9/4-9\gamma/2}\|w_r^\varepsilon\|_{L^2(\Omega)}. \tag{3.4.17}$$

Finally, we have to estimate I_4 :

$$\begin{aligned} I_4 &= \varepsilon \int_{r_\gamma}^{R_2} r(\psi_r^\varepsilon - u_r^0) w_r^\varepsilon \mathrm{d}r, \\ &\leq \varepsilon R_2 \|\psi_r^\varepsilon - u_r^0\|_{L^2(r_\gamma,R_2)} \|w_r^\varepsilon\|_{L^2(r_\gamma,R_2)} \end{aligned} \tag{3.4.18}$$

Thanks to (3.3.44) and (3.3.45), we have on $[r_\gamma, R_2]$:

$$\begin{aligned} \psi_r^\varepsilon &= -\frac{C_\varepsilon}{\sqrt{r^2 - C_\varepsilon^2}} + O(\varepsilon^{2\gamma}), \\ &= -\frac{R_1}{\sqrt{r^2 - R_1^2}}\Big(1 + \frac{R_1^2 - C_\varepsilon^2}{r^2 - R_1^2}\Big)^{-1/2} + O(\varepsilon^{2\gamma} + \varepsilon^{1/4+\gamma/2}), \\ &= u_r^0(1 + O(\varepsilon^\gamma)) + O(\varepsilon^{2\gamma} + \varepsilon^{1/4+\gamma/2}). \end{aligned}$$

Hence,

$$\psi_r^\varepsilon = u_r^0 + O(\varepsilon^\gamma)u_r^0. \tag{3.4.19}$$

This further implies that

$$\|\psi_r^\varepsilon - u_r^0\|_{L^2(r_\gamma,R_2)} \leq C\varepsilon^\gamma \|u_r^0\|_{L^2(r_\gamma,R_2)}. \tag{3.4.20}$$

Recall from Section 3.2 that

$$\|u_r^0\|_{L^2(r_\gamma,R_2)} = \Big(\int_{r_\gamma}^{R_2} (\frac{R_1}{\sqrt{r^2-R_1^2}})^2 \mathrm{d}r\Big)^{1/2} \leq C\sqrt{|\ln\varepsilon|}. \tag{3.4.21}$$

Therefore,

$$\|\psi_r^\varepsilon - u_r^0\|_{L^2(r_\gamma,R_2)} \leq C\varepsilon^\gamma\sqrt{|\ln\varepsilon|}. \tag{3.4.22}$$

Combining now (3.4.18) and (3.4.22), we obtain

$$I_4 \leq C\varepsilon^{1+\gamma}\sqrt{|\ln\varepsilon|}\ \|w_r^\varepsilon\|_{L^2(\Omega)}. \tag{3.4.23}$$

Finally, by inserting (3.14), (3.16), (3.17) and (3.24) into (3.13), we deduce the following estimation

$$-(h^\varepsilon(r), w_r^\varepsilon)_{L^2(\Omega)} \leq C\{\varepsilon^{5/4-5\gamma/2} + \varepsilon^{1+\gamma}\sqrt{|\ln\varepsilon|}\}\|w_r^\varepsilon\|_{L^2(\Omega)}. \tag{3.4.24}$$

To conclude, we bound from left the left-hand side of (3.4.7) with the help of (3.4.8) and (3.4.11) and we bound from right the right-hand side using (3.4.24). So, we obtain

$$\varepsilon R_1\|w_r^\varepsilon\|_{L^2(\Omega)}^2 \leq C\{\varepsilon^{5/4-5\gamma/2} + \varepsilon^{1+\gamma}\sqrt{|\ln\varepsilon|}\}\|w_r^\varepsilon\|_{L^2(\Omega)},$$

i.e.,

$$\|w_r^\varepsilon\|_{L^2(\Omega)}^2 \leq C\{\varepsilon^{1/4-5\gamma/2} + \varepsilon^\gamma\sqrt{|\ln\varepsilon|}\}. \tag{3.4.25}$$

Hence, we have

$$\|w_r^\varepsilon\|_{L^2(\Omega)}^2 \leq C\varepsilon^\gamma\sqrt{|\ln\varepsilon|}.$$

This ends the proof of Theorem 3.4.6. •

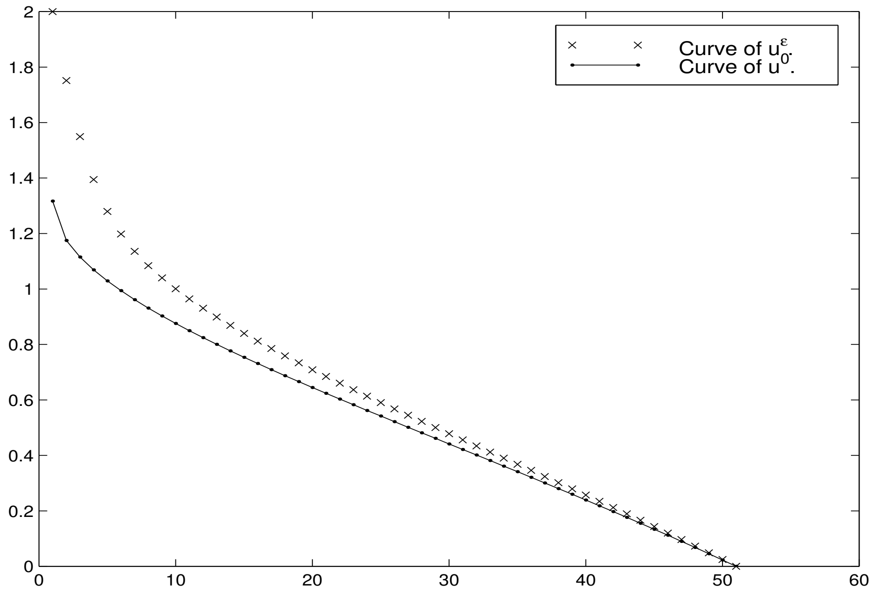
Remark 18 The best value of γ is given by

$$\frac{1}{4} - \frac{5\gamma}{2} = \gamma \quad \Leftrightarrow \quad \gamma = \frac{1}{14}.$$

We observe that the above value of γ is in the interval $(0, 1/6)$ which is in agreement with our initial choice.

3.5 Numerical simulation.

In this part, we produce calculations which confirm our theorical result by comparing the simulation of u^ε and u^0.



Here, we have considered the following numerical data :

$$\varepsilon = 10^{-3}, \quad K = 2, \quad R_1 = 1 \quad and \quad R_2 = 2.$$

We note that these simulations are obtained by the use of uniform meshes; however, we can improve this result by using an adaptative scheme taking into account the structure of the boundary layers, as e.g. in Cheng, Temam & Wang [CTW].

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