Diverses méthodes pour des problèmes de stabilisation

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Je dédie cette thèse à Clémentine
et à celui (ou celle?) qui va bientôt naître!
Patience plus que quelques semaines...

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Articles en appendice

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Introduction générale

Beaucoup de phénomènes physiques reposent sur l'étude d'une équation différentielle. Lorsque nous pouvons agir sur ces phénomènes, par exemple en appliquant une force, nous obtenons un système commandé que nous modélisons alors par une équation différentielle dépendant d'un paramètre noté u appelé contrôle.

La théorie du contrôle s'intéresse, entre autres, à l'étude du contrôle qu'il faut appliquer pour que le système contrôlé ait de bonnes propriétés. Avoir un équilibre asymptotiquement stable est l'une des propriétés que nous cherchons à obtenir, c'est-à-dire comment trouver une loi de commande pour un système commandé Σ telle que nous ayons la réunion des deux propriétés suivantes :

- l'équilibre est attractif, c'est-à-dire tel que toutes les trajectoires convergent vers l'équilibre du système (par exemple, tel que quelle que soit la température initiale d'une pièce, une climatisation assure que la température tende vers 19 degrés Celsius).
- l'équilibre est stable, c'est-à-dire tel que les trajectoires ne s'écartent pas trop pendant la période transitoire (par exemple, réussir à garer sa voiture en créneau sans être obligé d'aller très loin des positions de départ et d'arrivée).

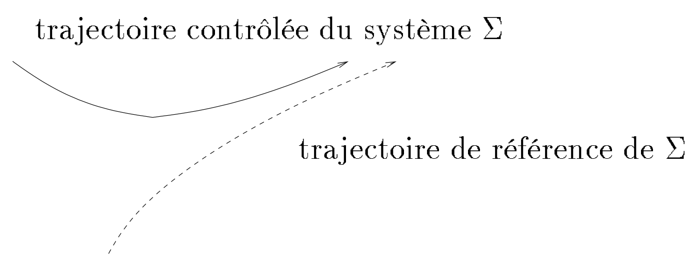

FIG. 1 – *Convergence d'une trajectoire, en trait plein, vers une trajectoire de référence, en pointillés.*

Outre ce problème de stabilisation de trajectoires vers un équilibre ou vers une trajectoire fixée (voir figure 1), nous étudierons la sensibilité de ces trajectoires aux erreurs de mesures, de modélisation et d'implémentation en cherchant à trouver un contrôle qui assure que, quelles que soient les erreurs (petites) ε, les trajectoires du système perturbé Σ_ε correspondant convergent vers la trajectoire de référence ou, au moins, restent proches de celle-ci (voir figure 2). Nous disons que nous avons une propriété de stabilisation robuste lorsque nous avons une relative insensibilité des propriétés d'attractivité et de stabilité par rapport aux petites erreurs (voir figure 2).

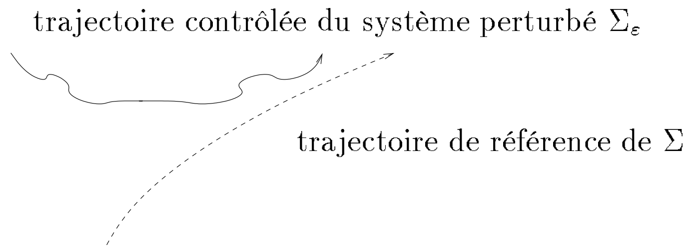

FIG. 2 – *Stabilisation robuste d'une trajectoire perturbée par une petite erreur, en trait plein, vers une trajectoire de référence, en pointillés.*

Nous allons étudier ces questions dans trois classes de systèmes dynamiques :

Partie I : systèmes non linéaires en dimension finie.
Partie II : systèmes non linéaires en dimension infinie.
Partie III : systèmes linéaires en dimension finie.

Expliquons plus précisément les questions et les résultats de chaque partie.

Partie I :

Nous considérons des systèmes contrôlés de dimension finie de la forme :

$$\dot{x} = f(x,u) \ ,$$

localement Lipschitz et tel que l'équilibre est globalement asymptotiquement contrôlable, c'est-à-dire tel que pour toute condition initiale, il existe un contrôle $t \mapsto u(t)$ dépendant du temps tel que la trajectoire du système bouclé tende vers l'origine et satisfasse une propriété de stabilité. Nous nous intéressons aux questions suivantes :

Question I.1 : *Existe-t-il un feedback, c'est-à-dire une fonction $u(x)$ de l'état, tel que l'origine du système bouclé soit un équilibre globalement asymptotiquement stable ?*

Question I.2 : *Pouvons-nous imposer une relative insensibilité aux bruits, même si ce feedback n'est pas régulier ?*

Question I.3 :*Pouvons-nous expliciter ce contrôle ou en donner une expression relativement simple ?*

Nous savons (voir [SS:80, B:83]) qu'il existe des systèmes dont l'origine est un équilibre globalement asymptotiquement contrôlable et tels qu'il n'existe pas de feedback continu qui réponde à la question I.1. En revanche, il est possible de stabiliser asymptotiquement tout équilibre globalement asymptotiquement contrôlable avec des contrôles discontinus (voir [S:79, CLSS:97, AB:99]) en considérant différents types de solutions. Mais tous ces feedbacks sont très sensibles aux petites erreurs de mesure et ne répondent donc pas à la question I.2, c'est-à-dire qu'il existe des fonctions $t \mapsto e(t)$ de norme infinie aussi petites (et non nulles) que nous voulons telle que l'équilibre de

$$\dot{x} = f(x,u(x+e(t))\ ,$$

ne soit plus globalement pratiquement asymptotiquement stable (c'est-à-dire qu'il existe des conditions initiales de π-solutions, calculées à partir d'un échantillonnage du contrôle, qui ne rentrent pas vers un voisinage fixé de l'origine). Les auteurs de [S:99, CLRS:00] prouvent l'existence de feedbacks qui répondent à la question I.1 en considérant les π-solutions et qui sont robustes aux bruits s'annulant à l'origine dès que la partition du temps permettant d'implémenter le contrôle n'est pas trop fine. Cependant, ces contrôles restent très sensibles aux erreurs si le pas de cette partition est trop petit. Ils ne répondent donc que partiellement à la question I.2. Ces travaux (voir aussi [LS:97]) prouvent la nécessité de considérer d'autres types de feedbacks. Ici nous proposons d'utiliser des feedbacks hybrides, *i.e.* des contrôles dépendant d'un état mixte discret/continu pour obtenir une robustesse par rapport à un ensemble d'erreurs plus large.

Le théorème suivant (Theorem 1, page 92) répond aux questions I.1 et I.2 pour les π-solutions et les solutions d'Euler (définies comme étant les limites des π-solutions lorsque le pas de la partition tend vers 0).

Théorème I.1 : *Il existe un contrôle avec état mixte discret/continu et un diamètre maximal de perturbations ρ satisfaisant*

$$\forall x \in \mathbb{R}^n \setminus \{0\}, \rho(x) > 0 \quad , \qquad \rho(0) = 0\ , \tag{1}$$

tels que, quelles que soient les erreurs de mesure $e(x,t)$, d'implémentation $a(x,t)$ et de modélisation $d(x,t)$, toutes trois continues par rapport à x et satisfaisant

$$\|e(x,t)\| \leq \rho(x) \quad , \qquad \|a(x,t)\| \leq \rho(x) \quad , \qquad \|d(x,t)\| \leq \rho(x), \forall x, \forall t \geq 0\ , \tag{2}$$

l'équilibre des systèmes perturbés

$$\dot{x} = f(x,u(x+e(x,t))+a(x,t))+d(x,t)\ ,$$

est globalement asymptotiquement stable pour les π-solutions et les solutions d'Euler.

La classe de solutions est plus grande que celle considérée dans [S:99] et [CLRS:00] car le pas de l'échantillonnage utilisé pour l'implémentation du contrôle peut être pris aussi petit que l'on veut, et est complètement indépendant de la norme infinie des perturbations.

La démonstration de ce théorème est relativement constructive, nous répondons donc aussi à la question I.3. Cependant, en général, ce contrôle dépend d'une infinité de variables discrètes, il reste par conséquent assez difficile à implémenter.

C'est pourquoi nous avons étudié plus spécifiquement le problème de stabilisation robuste pour les systèmes chaînés en dimension quelconque avec deux commandes qui forment une classe de systèmes possédant une large palette d'applications (voir [MS:93] et [PA:01] par exemple) et qui s'écrivent :

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_2 u_1, \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1. \end{aligned}$$

Nous prouvons alors (voir Theorem 1, page 114) :

Théorème I.2 : *Pour le système chaîné, nous pouvons* expliciter *un contrôle dépendant de l'état et d'une seule variable discrète tel que, quel que soit le segment fermé I donné contenant l'origine et pouvant être de longueur strictement positive aussi petite que l'on veut, il existe un diamètre maximal de perturbations ρ tel que*

$$\forall x \in \mathbb{R}^n \setminus I, \rho(x) > 0 \quad , \qquad \forall x \in I, \rho(x) = 0 \ , \tag{3}$$

tel que quelles que soient les erreurs $e(x,t)$, $a(x,t)$ et $d(x,t)$, continues par rapport à x et satisfaisant (2), l'équilibre du système chaîné perturbé en boucle fermée est globalement asymptotiquement stable.

Ce théorème répond donc entièrement à la question I.3 puisque le feedback est explicite et simple à implémenter. Mais il répond seulement partiellement à la question I.2 puisque l'erreur ne s'annule pas seulement en l'origine (comparez (1) et (3)).

Notons qu'il existe une littérature importante étudiant ces questions et considérant le cas de la stabilisation par retour d'état instationnaire, *i.e.* par des contrôles continus dépendant de l'état et du temps et périodiques en temps. Ce type de méthode a été introduite par [SS:80] et [S:91]. L'auteur de [C:95] prouve l'existence d'une commande répondant à la question I.1 pour tout système dont l'origine est un équilibre localement contrôlable. Par continuité, ce type de contrôle assure une robustesse par rapport aux petits bruits et donc est solution de la question I.2. Cependant la propriété de contrôlabilité locale autour de l'origine est plus forte que celle de la contrôlabilité asymptotique locale. En fait, il existe un système (les cercles d'Artstein) dont l'origine est un équilibre localement asymptotiquement stable pour lequel il n'existe pas de contrôle continu et instationnaire (ni même dynamique) tel que l'équilibre du système bouclé soit localement asymptotiquement stable (voir [BR:01, Proposition 3.5]). Or nous avons prouvé, pour ce système précisèment, qu'il existe un feedback à état mixte discret/continu tel que l'équilibre du système bouclé est globalement asymptotiquement stable avec une robustesse par rapport aux petits bruits (voir [P:00]). Cela justifie bien l'apport des contrôles hybrides par rapport aux contrôles instationnaires.

Notons aussi que pour étudier le problème de stabilisation des systèmes chaînés, les contrôles instationnaires apportent aussi des réponses positives à nos questions. Voir par exemple [C:92] où la "méthode du retour" est appliquée pour prouver que pour tout système sans dérive, comme c'est le cas pour les systèmes chaînés, il existe un retour d'état périodique en temps tel que l'origine du système bouclé est un équilibre globalement asymptotiquement stable. Des fonctions de Lyapunov sont utilisées pour *construire* une telle commande dans [P:92]. Des techniques d'homogénéisation ont été appliquées dans [MPS:99].

Dans la partie I, nous nous intéressons aussi au cas où, pour un système contrôlé localement Lipschitz, nous connaissons deux commandes continues u_g et u_l, telles que l'équilibre du système bouclé correspondant soit asymptotiquement stable globalement pour u_g, et seulement localement asymptotiquement stable pour u_l. Soit Ω un ouvert d'adhérence strictement incluse dans l'ensemble de définition du contrôleur local u_l. Nous nous intéressons aux questions suivantes :

Question I.4 : *Pouvons-nous trouver un troisième contrôle u qui soit égal à la commande locale*

u_l sur Ω, égal à la commande globale u_g loin de l'origine et tel que l'origine soit un équilibre globalement asymptotiquement stable ?

Question I.5 : *Pouvons-nous imposer une robustesse par rapport aux erreurs de mesure, d'implémentation et de modélisation supposées petites par rapport à l'état ?*

Nous prouvons le théorème suivant (voir Theorem 2.6 page 45) :

Théorème I.3 : *Il existe un système contrôlé et deux contrôles, tel qu'aucun feedback, ni continu ni discontinu, ne réponde simultanément aux questions I.4 et I.5.*

Par conséquent, il est nécessaire de considérer des contrôles instationnaires ou dépendant d'un état discret/continu. Nous prouvons que ces deux types de contrôles peuvent répondre aux questions I.4 et I.5. En effet, nous avons (voir Theorem 4.1, page 49 et Theorem 5.1, page 52) :

Théorème I.4 : *Il existe*

- *un contrôle avec état mixte discret/continu,*
- *et un contrôle instationnaire et continu, périodique par rapport au temps,*

qui soient égaux à la commande locale u_l sur Ω, égaux à la commande globale u_g sur le complémentaire d'un certain compact, et tels que l'équilibre des systèmes bouclés soient globalement asymptotiquement stables avec une robustesse par rapport aux erreurs de mesure, d'implémentation et de modélisation.

Partie II :

Nous considérons dans cette partie un bac longitudinal rempli de fluide sur lequel nous pouvons exercer une force pour le déplacer suivant la longueur du récipient. Nous modélisons le problème à l'aide des équations de Saint-Venant (*shallow water equations*). Dans cette partie le système commandé est donc donné par un système d'équations aux dérivées partielles hyperboliques non linéaires avec un terme source qui sera le contrôle. Cette classe de systèmes commandés est encore peu étudiée et nous nous intéressons au problème de la stabilisation du bac et du fluide avec contrôle de la force.

D'après [C:01], l'origine des équations de Saint-Venant est localement contrôlable autour de tout équilibre. Les auteurs de [DPR:99] ont par ailleurs démontré que le système linéarisé autour de tout équilibre n'est pas asymptotiquement contrôlable, c'est pourquoi nous nous intéressons à la question suivante :

Question II.1 : *Pouvons-nous trouver un contrôle par retour d'état tel que l'équilibre du modèle non linéarisé soit localement asymptotiquement stable ?*

Nous cherchons à appliquer ce contrôle aux problèmes plus concrets posés par exemple dans [G:00, M:00b]. Il est donc important, pour le calcul de ce contrôle, de ne pas avoir à mesurer la vitesse du fluide dans tout le bac car c'est difficile à faire en pratique. Il faut par conséquent répondre à la question :

Question II.2 : *Pouvons-nous trouver un contrôle stabilisant l'origine et qui soit fonction uni-*

quement de variables du système faciles à mesurer, comme la hauteur de l'eau aux bords et la position du bac?

Pour tester nos contrôleurs et voir si les objectifs sont atteints, nous avons écrit un programme en C++ (voir page 146) pour simuler le système. Nous avons discrétisé les équations de Saint-Venant en utilisant le schéma de Godunov et le schéma de Preissmann. Tous deux ont déjà été utilisés pour approcher les équations de Saint-Venant (voir [DPR:99] pour le schéma de Godunov et [M:94, G:98] pour celui de Preissmann).

En utilisant des couples flux-entropie associés aux équations hyperboliques, une approche Lyapunov, et notre programme, nous avons abouti au résultat suivant :

Résultat numérique II.1 : *Il existe des contrôles, fonctions de l'état du système, tel que l'équilibre des équations (non linéaires) modélisant le problème de la stabilisation du bac soit localement asymptotiquement stable.*

Notons que ce résultat numérique ne prouve pas que l'origine soit bien un équilibre localement asymptotiquement stable et nous n'avons donc pas répondu à la question II.1, mais nous proposons une approche, sur laquelle nous travaillons encore, permettant de prouver la

Conjecture II.1 : *Les fonctions de l'état du système que nous avons calculées sont des contrôles tels que l'équilibre des équations (non linéaires) modélisant le problème de la stabilisation du bac soit localement asymptotiquement stable.*

Par ailleurs, en ce qui concerne la question II.2, nous proposons des contrôleurs dynamiques dont la loi dynamique ne dépend que de la hauteur du fluide au bord du bac et de la trajectoire du bac, mais dont le contrôle au temps initial dépend, lui, de tout l'état du système. Cela constitue une première étape pour prouver la

Conjecture II.2 : *Il existe des contrôles, fonctions des variables physiquement mesurables, tels que l'équilibre des équations (non linéaires) modélisant le problème de la stabilisation du bac soit un équilibre localement asymptotiquement stable.*

Partie III :

Le problème général considéré dans cette partie est celui de la stabilisation de l'origine d'un système linéaire en dimension finie lorsque nous avons une incertitude sur les données du système.

Les propriétés des systèmes linéaires en dimension finie sont maintenant bien connues. Nous avons aussi bien des caractérisations de propriétés théoriques (voir par exemple la propriété de la stabilité dans [K:80, Chapter 3]) que des outils numériques efficaces pour les résoudre (voir par exemple la résolution de problèmes SDP en temps polynômial dans [BEFB:94]). En revanche, les transitions possibles vers les études de systèmes non linéaires ou les introductions des non linéarités partielles dans des systèmes linéaires restent des problèmes difficiles et encore peu étudiés. Un exemple où il est nécessaire de passer d'un système linéaire à un système non linéaire, est le cas de l'étude d'un système linéaire avec une incertitude sur les données du système ou sur la condition initiale.

Dans ce cadre, le problème n'est plus aussi facile qu'avec le système entièrement linéaire mais nous pouvons, dans certains cas, transformer ce problème non linéaire en un nouveau problème linéaire, mais de plus grande dimension, puis appliquer les techniques classiques des systèmes linéaires à ce nouveau problème. Il reste maintenant à considérer l'application numérique de ces méthodes à des problèmes concrets. C'est l'objet de cette partie où nous étudions des techniques de relaxation permettant de passer d'un problème semi-défini positif avec incertitudes sur les données, à un problème semi-défini positif en dimension supérieure mais sans incertitudes sur les données. Nous appliquons ces techniques à un problème concret, en l'occurrence au problème du contrôle de production d'électricité par une centrale thermique. Nous nous donnons une approximation linéaire de ce système autour d'un certain régime. Les matrices définissant ce système linéaire ne sont donc connues qu'approximativement et nous cherchons

Question III.1 : *Peut-on savoir s'il existe un contrôle tel que, quelles que soient les incertitudes sur les coefficients des matrices, la sortie du système bouclé satisfasse des inégalités imposées par les contraintes physiques du système réel? Si oui, peut-on le calculer numériquement?*

Ce problème peut être transformé en un problème de résolution d'inégalités linéaires matricielles avec des incertitudes sur les données.

Nous nous sommes d'abord intéressés au cas sans incertitude, et nous avons appliqué des techniques classiques de résolution de problème semi-défini pour obtenir (voir paragraphe 4, page 220) :

Résultat numérique III.1 : *Dans le cas où le système linéaire approchant le système réel est parfaitement connu,* i.e. sans incertitude*, nous savons numériquement dire s'il existe un contrôle tel que les sorties du système suivent un certain régime. Si un tel contrôle existe, nous pouvons le calculer numériquement.*

Pour le problème industriel avec incertitudes, la variété des erreurs de modélisation et la taille des systèmes modélisant le problème ne nous ont permis de répondre que partiellement à la question III.1. En revanche pour un système plus petit de dimension deux avec certaines incertitudes, la résolution a été complète. En effet, nous avons (voir paragraphe 1.5 pour le problème réel et Theorem 3, page 211 pour le problème en dimension deux)

Résultat numérique III.2 : *Dans le cas où le système linéaire approchant le problème réel n'a pas de structures sur les incertitudes,* i.e. avec des incertitudes pleines*, nous pouvons savoir numériquement s'il existe un contrôle tel que la sortie du système bouclé satisfasse des inégalités données. Si un tel contrôle existe, nous pouvons alors le calculer numériquement sur de petits horizons de temps.*

En revanche, pour un système plus simple de dimension deux *et* quelles que soient les incertitudes considérées*, nous pouvons faire la même analyse sur de grands horizons de temps.*

Étudions maintenant plus précisèrent les questions et les résultats pour chaque partie.

Première partie

Systèmes dynamiques à état mixte continu/discret. Applications à la commande

Motivations

Souvent en théorie du contrôle, lorsque nous cherchons à définir une commande $x \mapsto u(x)$, qui soit une fonction de l'état x d'un système commandé de la forme :

$$\dot{x} = F(x,u) \ , \tag{4}$$

et tel que le système bouclé $\dot{x} = F(x,u(x))$ ait certaines propriétés (stabilisation, optimisation, ...), nous sommes conduits à considérer des fonctions $u(x)$ discontinues (voir par exemple, pour le problème de la stabilisation, [S:79, SS:80, B:83, CLSS:97, AB:99]). Ainsi, même si le système contrôlé (4) est continu en la variable (x,u) où x est *l'état* et u *le contrôle*, le système bouclé par la fonction $u(x)$ est discontinu. Nous pouvons alors avoir une grande sensibilité aux petits bruits et avoir des comportements inattendus si nous essayons de calculer ou d'implémenter le système bouclé. Pour éviter cette grande sensibilité, nous devons changer notre notion de trajectoires. En effet, le contrôleur satisfait ces propriétés mais pour un ensemble de trajectoires qui est trop petit si nous souhaitons tenir compte de la sensibilité aux bruits. C'est pourquoi nous allons maintenant considérer des notions de trajectoires plus générales, et résoudre le problème de la synthèse de contrôle avec cette nouvelle classe de trajectoires. Ceci peut s'avérer très difficile, voir même impossible. Citons [H:67, Exemple 2] où le même problème d'optimisation (ou [S:99, Cercles d'Artstein] pour un problème de stabilisation) est étudié pour deux ensembles de trajectoires différentes et aboutit à deux solutions totalement différentes.

Lorsque le problème devient trop compliqué pour être résolu, ou même impossible à résoudre avec un contrôleur discontinu (ou continu) et avec une classe de trajectoires suffisamment riche pour rendre compte d'un très grand nombre de comportements possibles en présence de petits bruits, il est naturel de chercher une nouvelle classe de contrôleurs qui assurerait une insensibilité aux bruits.

Une telle classe de contrôleurs peut être construite à partir des lois de commandes discontinues en remplaçant chaque surface de discontinuité par une couche sur laquelle le contrôleur peut être égal à toutes les valeurs prises dans un voisinage de cette surface de discontinuité. Le choix de la valeur du contrôleur dépend du passé d'une variable discrète. Cette connaissance partielle du passé assure une robustesse du contrôleur car la valeur du contrôleur ne dépend plus uniquement de la valeur de l'état bruité.

Plus précisément nous introduisons une classe de contrôleurs dynamiques avec un état mixte continu/discret de la forme :

$$\begin{aligned} u &= k(x, s_d) \ , \\ s_d &= k_d(x, s_d^-) \ , \end{aligned} \tag{5}$$

où la variable s_d vit dans un ensemble discret $\mathcal{D}$, k (resp. k_d): $\mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^m$ (resp. $\mathbb{R}^n \times \mathcal{D} \to \mathcal{D}$) est une application et s_d^- est défini, pour le moment informellement, par

$$s_d^-(t) = \lim_{s \nearrow t} s_d(s) \ .$$

Nous savons qu'en présence de bruits sur l'état, il est nécessaire de considérer toutes les valeurs prises par le contrôle dans un petit voisinage du point considéré (ce qui peut donner un ensemble très varié pour un contrôle discontinu, mais qui est inclus dans une petite boule pour un contrôle continu). Par contre, au moins heuristiquement, pour une commande dépendant d'une variable discrète et d'une information partielle du passé, et pas uniquement de la valeur de l'état (connue à un bruit prés), les valeurs du contrôle bruité, forment un ensemble beaucoup plus restreint que celui obtenu pour un contrôle discontinu. Ainsi avec un tel contrôle mixte continu/discret nous avons une relative insensibilité aux bruits.

Cela justifie l'étude au chapitre 1 des systèmes (4) bouclés par des contrôles mixtes (5), que nous appelons systèmes mixtes, et qui sont donc de la forme

$$\begin{aligned} \dot{x} &= f(x, s_d)\ , \\ s_d &= k_d(x, s_d^-)\ . \end{aligned} \tag{6}$$

Cette classe de système est considérée dans [T:87] par exemple qui est un cas particulier des systèmes hybrides étudiés dans [BM:97] notamment. Cependant nous avons préféré nous limiter à l'étude des ces sytèmes car cela nous suffit (au chapitre 2) pour résoudre des problèmes, issus de la théorie du contrôle, qui sont trop difficiles ou impossibles à résoudre avec des contrôles discontinus (ou continus) mais qui résolubles en considérant la classe de commandes mixtes (5). Les formulations précises et les démonstrations complètes des résultats annoncés sont données en appendice.

Chapitre 1

Systèmes mixtes perturbés

Dans ce chapitre, nous nous donnons un système différentiel de la forme :

$$\begin{array}{rcl} \dot{x} & = & f(x, s_d) , \\ s_d & = & k_d(x, s_d^-) . \end{array} \tag{1.1}$$

La variable s_d vit dans un ensemble discret $\mathcal{D}$ muni de la topologie discrète (i.e. toutes les parties de $\mathcal{D}$ sont ouvertes), $f : \mathbb{R}^n \times \mathcal{D} \to \mathbb{R}^n$ est une application localement lipschitzienne et $k_d : \mathbb{R}^n \times \mathcal{D} \to \mathcal{D}$ est une application. Dans (1.1), s_d^- est défini, pour le moment informellement, par

$$s_d^-(t) \;=\; \lim_{s \nearrow t} s_d(s) \; . \tag{1.2}$$

Nous disons que ce système est mixte car nous avons une loi dynamique sur une variable mixte continue/discrète.

Notons que ces systèmes ont été étudiés initialement dans [T:87] et que ce sont des cas particuliers des systèmes hybrides considérés par exemple dans [BM:97, BM:00, M:01]. Cependant, d'après (6), ces systèmes décrivent les boucles fermées des systèmes commandés par les contrôle du type (5) à état mixte discret/continu. Ces contrôles suffisent (et sont indispensables) pour résoudre les problèmes de théorie du contrôle considérés au chapitre 2 ci-dessous. Il aurait été donc superflu pour notre étude d'étudier des systèmes hybrides plus généraux que ceux décrits par (1.1).

Nous explicitons au paragraphe 1.1 la notion de solutions de (1.1) la plus naturelle possible dans notre contexte. Puis dans le paragraphe 1.2, nous rappelons un cas particulier du système (1.1). Nous introduisons les systèmes mixtes perturbés au paragraphe 1.3 et nous étudions ensuite un exemple simple au paragraphe 1.4. Nous montrons que la notion de solutions la plus naturelle n'est pas appropriée pour l'étude de systèmes avec petites perturbations. Nous proposons une autre notion de solutions au paragraphe 1.5 avec des propriétés satisfaisantes et nous finissons ce chapitre par des généralisations de cette notion de solutions.

1.1 Notion de solutions la plus naturelle possible

Nous allons définir une notion de solutions en cherchant le cadre le plus naturel possible et donner quelques propriétés de ces solutions.

Remarquons que pour que $\dot{x}$ existe dans (1.1), il est logique d'imposer que x soit absolument continue.

De plus si nous imposons à s_d la plus grande régularité possible, à savoir s_d continu alors, du fait de la topologie discrète, s_d est une fonction constante et donc

$$s_d^-(t) = s_d(t) \ ,$$

pour tous les temps t. Donc (1.1) devient une équation différentielle ordinaire et nous n'avons rien de nouveau. Il faut donc demander moins de régularité sur la variable s_d. C'est pourquoi nous imposons s_d continu à droite ce qui est équivalent, du fait du choix de la topologie discrète sur $\mathcal{D}$, à la propriété suivante :

$$\forall t_0, \exists \varepsilon > 0, \forall t \in [t_0, t_0 + \varepsilon), s_d(t) = s_d(t_0), \tag{1.3}$$

ce qui implique :

$$\forall t_0, \exists \varepsilon > 0, \forall t \in [t_0, t_0 + \varepsilon), \dot{x}(t) = f(x(t), s_d(t_0)) \ ,$$

et donc x est une solution de Carathéodory par morceaux. Dans ce contexte, la théorie classique des équations différentielles ordinaires est un cas particulier de notre étude. Cela nous encourage à imposer effectivement s_d continue à droite.

Remarquons les points suivants à propos du calcul de (1.2) :

- Il est nécessaire d'orienter le temps, ce qui assez inusuel. Par conséquent étant donnée une solution (x, s_d) définie sur $[0, T)$, $t \mapsto (x(-t), s_d(-t))$ ne sera pas nécessairement une solution sur $(-T, 0]$.
- Il faut calculer le passé de s_d, autrement dit (1.2) peut être vu comme un opérateur sur la fonction s_d toute entière. Donc nous pouvons dire que l'équation (1.1) est un opérateur qui n'admet pas de représentation en dimension finie puisqu'on calcule la solution dans un espace de dimension infinie.
- L'existence de la limite dans (1.2) n'est pas acquise. En effet soit, par exemple, s_d une fonction continue à droite à valeurs dans $\{1{,}2\}$ définie sur $[0{,}1]$ avec un point d'accumulation de discontinuités en 3/4. Nous avons donc $s_d^-(3/4)$ n'existe pas. Voir figure 1.1.

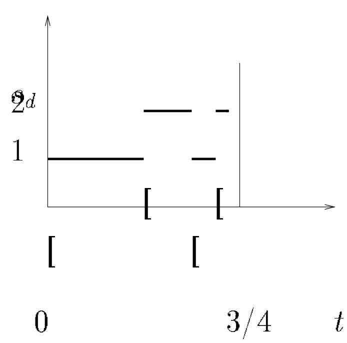

FIG. 1.1 – *La limite en 3/4 n'existe pas : l'équation (1.2) n'a pas de sens en 3/4.*

Nous résumons les différentes propriétés d'une solution dans la définition suivante :

Définition 1.1.1 *Soit $T > 0$ et (x_0, s_0) dans $\mathbb{R}^n \times \mathcal{D}$. Nous disons que $(X, S_d) : [0, T) \to \mathbb{R}^n \times \mathcal{D}$ est une* solution *de (1.1) si nous avons les propriétés suivantes:*

1. *La fonction X est absolument continue sur $[0,T)$ et vérifie, pour presque tout t dans $[0,T)$,*

$$\dot{X}(t) = f(X(t), S_d(t)) \ . \tag{1.4}$$

2. *La fonction S_d est continue à droite sur $[0, T)$.*
3. *Pour tout t de $(0, T)$, où $S_d^-(t)$ existe, nous avons*

$$S_d(t) = k_d(X(t), S_d^-(t)) \ . \tag{1.5}$$

Cette définition est sous-entendue ou explicite dans beaucoup d'articles, voir par exemple [T:87, BM:97, B:98, LNP:00].

Discutons maintenant de la notion d'état du système (1.1).

Nous allons montrer que, contrairement à ce que nous pourrions penser, l'état au temps t n'est pas le couple $(X(t), S_d(t))$.

Pour bien comprendre ce qu'est l'état du système (1.1), nous allons nous inspirer de [S:90, Section 2.1]. Donnons-nous deux solutions (X_-, S_{d-}) et (X_+, S_{d+}) définies sur $[-T, 0]$ et $[0, T]$ respectivement, avec $T > 0$.

La notion usuelle d'état d'un système dynamique est l'information qu'il faut connaître de (X_-, S_{d-}) pour pouvoir concaténer (X_-, S_{d-}) et (X_+, S_{d+}). Au vu des équations (1.1), c'est possible si la condition suivante est satisfaite :

$$X_-(0) = X_+(0) \quad , \qquad S_{d+}(0) = k_d(X_-(0), S_{d-}^-(0)) \ ,$$

si $S_{d-}^-(0)$ existe, ou

$$X_-(0) = X_+(0) \ ,$$

si $S_{d-}^-(0)$ n'existe pas.

Par conséquent, étant donnée une solution (X, S_d) de (1.1), l'état du système au temps t est le couple $(X(t), S_d^-(t))$ si $S_d^-(t)$ existe, et sinon l'état à l'instant t est simplement $X(t)$.

Une condition initiale est un état à un instant initial. Nous pouvons donc poser la définition suivante (voir aussi [BM:97]) :

Définition 1.1.2 *Soit $T > 0$ et (x_0, s_0) dans $\mathbb{R}^n \times \mathcal{D}$. Nous disons que $(X, S_d) : [0, T) \to \mathbb{R}^n \times \mathcal{D}$ est une* solution *de (1.1) avec condition initiale (x_0, s_0) si (X, S_d) est une solution de (1.1) au sens de la définition 1.1.1 et si les égalités*

$$X(0) = x_0 \quad , \qquad S_d(0) = k_d(x_0, s_0) \ ,$$

sont satisfaites.

Remarque 1.1.3 Notons que cette notion de conditions initiales est utilisée dans [P:01b] et [PA:01] mais que ce n'est pas le cas pour [P:01a] pour lequel la notion de conditions initiales choisie est

$$(X(0), S_d(0)) \ ,$$

parce que la discussion précédente n'avait pas encore été faite. Voir aussi la remarque 1.5.5 ci-dessous. ◇

Remarque 1.1.4 Notons que pour une solution (X,S_d) de (1.1), définie sur $[0,T)$, l'ensemble des points de discontinuité est au plus dénombrable et donc presque partout l'état est $(X(t),S_d^-(t))$ et (1.5) a lieu presque partout. ◇

Démonstration :

Considérons $\sim$ la relation d'équivalence sur $[0,T)$ définie par

$$t \sim t' \quad \Leftrightarrow \text{sur le segment } [t,t'] \text{ (ou } [t',t] \text{ si t'<t) } S_d \text{ est constant.}$$

C'est bien une relation d'équivalence (réflexivité, transitivité et symétrie). Les classes d'équivalence sont de la forme $[t,t')$ (ou $[t',t)$ si t'<t) avec (t,t') dans $[0,T) \times [0,T]$ car S_d est continue à droite. La fonction S_d est discontinue en t ssi t appartient à une borne d'une classe d'équivalence et donc l'ensemble des points de discontinuité est de cardinal inférieur à celui des classes d'équivalence de $\sim$. Distinguons deux cas :

- Si $T < +\infty$ alors le nombre de classe d'équivalence sur $[0,T)$ de longueur supérieure ou égale à $\frac{T}{n}$ est au plus n, et toutes les classes d'équivalence ont une longueur strictement positive. Donc il y un nombre dénombrable de points de discontinuité.
- Si $T = +\infty$, on procède de la même façon en comptant les classes d'équivalence sur $[0,N)$ avec $N \in \mathbb{N} \setminus \{0\}$ quelconque. □

Dans cette partie nous nous intéresserons souvent au problème de Cauchy :

Problème de Cauchy : Étant donnée une condition initiale, est-ce qu'il existe une solution partant de cette condition initiale, et quel est l'ensemble de définition maximal de cette solution?

Étudions un cas particulier de système à état mixte.

1.2 Un cas particulier important : l'hystérésis

Nous introduisons brièvement dans ce paragraphe l'hystérésis qui est un cas particulier des systèmes de la forme (1.1). Nous nous plaçons dans le cadre de [P:01a] où ce type de système mixte est très précisément étudié et appliqué au problème du rapiècement de deux contrôleurs (voir paragraphe 2.1 ci-dessous).

Étant donnés deux systèmes différentiels

$$\dot{x} = f_1(x) \ , \tag{1.6}$$

et

$$\dot{x} = f_2(x) \ , \tag{1.7}$$

nous pouvons réaliser un hystérésis entre (1.6) et (1.7) sur une partie[1] A de $\mathbb{R}^n$ d'intérieur non vide séparant $\mathbb{R}^n$ en deux parties fermées C_1 et C_2 en étudiant (1.1) où s_d est dans $\{1,2\}$, k_d: $\mathbb{R}^n \times \{1, 2\} \to \{1, 2\}$ est défini par :

$$\begin{aligned} k_d(x, s_d) &= 1 \quad \forall x \in C_1 \ , \\ &= 2 \quad \forall x \in C_2 \ , \\ &= s_d \quad \forall x \in A \ , \end{aligned}$$

1. A comme anneau.

et f: $\mathbb{R}^n \times \{1,2\} \to \mathbb{R}^n$ par :

$$\begin{array}{rcl} f(x,1) & = & f_1(x) \quad \forall x \in \mathbb{R}^n , \\ f(x,2) & = & f_2(x) \quad \forall x \in \mathbb{R}^n . \end{array}$$

Nous avons représenté la fonction k_d sur la figure 1.2 en fonction de x. Notons que

$$\forall x \in A, k_d(x,\{1, 2\}) = \{1, 2\} .$$

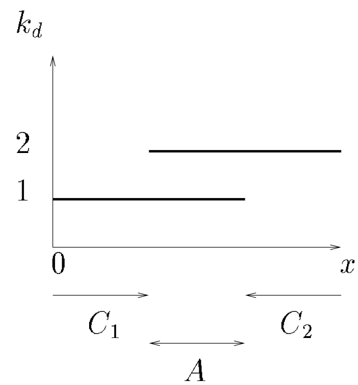

FIG. 1.2 – *L'hystérésis entre deux systèmes. Les ensembles décrivant l'hystérésis sont donnés en trait gras (A, C_1 et C_2).*

Intuitivement sur A, nous avons la superposition des deux systèmes (1.6) et (1.7), tandis que sur C_1 (resp. C_2), s_d ne peut valoir que 1 (resp. 2) donc le système (1.1) est en fait (1.6) (resp. (1.7)). (Pour passer d'un système à un autre on utilisera la proposition 1.3.2 donnée plus bas.)

Nous allons donner un cadre précis à toutes ces idées au paragraphe 1.3 mais avant tout, nous étudions quelques travaux sur ce cas particulier dans la remarque suivante.

Remarque 1.2.1 Il existe une littérature abondante pour les systèmes avec hystérésis. Voir par exemple les livres [BS:96] et [V:91]. Un exemple d'application de ces études peut être trouvé dans les systèmes mécaniques (voir [BKS:95, BKS:96]) ou magnétiques (voir [B:93]).

Mais le contexte dans lequel nous nous plaçons et celui étudié ci-dessus n'ont rien à voir. En effet, les conditions initiales sur la variable S_d ne sont pas spécifiées et les objectifs classiques des études citées ci-dessus consistent à étudier l'opérateur suivant :

$$\begin{array}{rrcl} \mathcal{E} : & \mathcal{C}^0([0,T], \mathbb{R}^n) \times \mathcal{D} & \to & \mathcal{F}([0,T], \mathcal{D}) \\ & (X, S_d(0)) & \mapsto & (t \mapsto S_d(t)) \end{array}$$

où X désigne la trajectoire continue, $S_d(0)$ la variable discrète au temps initial, $\mathcal{F}([0,T], \mathcal{D})$ est l'ensemble des fonctions de $[0,T]$ à valeurs dans $\mathcal{D}$, et $\mathcal{C}^0([0,T], \mathbb{R}^n)$ est l'ensemble des fonctions continues de $[0,T]$ à valeurs dans $\mathbb{R}^n$.

Par exemple nous pouvons donner, sous certaines conditions, des propriétés de cet opérateur pour un hystérésis "marche" généralisé (voir [V:91, Theorem III.2.2]) ou pour un hystérésis "butée" généralisé (voir [V:91, Theorem III.3.2]).

En revanche, ici, nous intéressons au problème de Cauchy défini à la fin du paragraphe 1.1, c'est-à-dire, que partant d'une condition initiale (x_0,s_0), nous laissons la variable S_d évoluer en la calculant à l'instant t à partir de sa limite à gauche et en appliquant (1.1) pour calculer $X(t)$ et $S_d(t)$. En particulier pour donner un sens à la notion de variable S_d, il n'est pas nécessaire de connaître la solution X entièrement. ◇

1.3 Introduction de perturbations

Replaçons-nous dans le cadre du système (1.1). Nous allons motiver l'introduction de perturbations dans (1.1) et nous allons voir en fait que la notion de solutions de la définition 1.1.2 implique un comportement assez particulier pour certains systèmes très simples et que ce n'est pas le bon cadre d'étude pour les systèmes avec petites perturbations. Cette discussion n'est pas donnée explicitement dans [P:00] mais elle a justifié [P:00, Definition 1].

Le système (1.1) nécessite d'être écrit avec des perturbations dans les cas suivants :

- Lorsque nous nous intéressons aux applications, ou que les états sont mal connus et ne sont donnés qu'à une petite erreur près de mesure. Ainsi il est nécessaire de remplacer x par $x+e$ dans (1.1) où e est une fonction inconnue dépendant du temps mais de norme plus petite qu'un réel ρ donné. Les systèmes eux-mêmes ne sont souvent connus qu'à une (petite) erreur de modélisation près, il est alors naturel de remplacer f dans $f+d$ où d est une fonction inconnue dépendant du temps mais de norme plus petite que ρ.
- Lorsque nous nous intéressons aux équations différentielles avec second membre discontinu comme c'est le cas du système (1.1) à cause de la présence de la variable s_d. En effet dans ce cadre d'équations différentielles il n'y a pas de notion de solutions qui puisse être *a priori* privilégiée. Voir aussi les discussions dans le livre [F:88, Chapter 2], [AB:99, page 457 et suivantes][2] ou [H:67, S:78].

Motivés par ces deux points, nous nous donnons un réel positif ρ et nous étudions les solutions des systèmes perturbés suivants :

$$\begin{aligned} \dot{x} &= f(x+e,\, s_d)+d \;, \\ s_d &= k_d(x+e,\, s_d^-) \;, \end{aligned} \tag{1.8}$$

où e et $d\colon [0,+\infty) \to \mathbb{R}^n$ sont deux fonctions inconnues mais vérifiant :

$$\sup_{t\geq 0}|e(t)| \leq \rho \quad , \qquad \operatorname{esssup}_{t\geq 0}|d(t)| \leq \rho \;. \tag{1.9}$$

Ici $\sup(g)$ et $\operatorname{esssup}(g)$ désignent respectivement la borne supérieure et essentiellement supérieure d'une fonction g.

Notons qu'il suffit d'avoir une information sur la borne essentielle de d car l'équation en $\dot{x}$ dans (1.8) n'a de sens que presque partout. En revanche pour e, il faut impérativement connaître sa borne supérieure (voir [P:01a, Theorem 4.2] pour une discussion sur ce sujet).

2. Noter comme le fait F. Wirth dans [BW:00], que l'ensemble des solutions de Filippov donné n'est pas donné complètement, contrairement à ce qui est affirmé. Il manque les solutions qui restent sur (0,0) sur un intervalle $[1,r]$ donné et qui suivent la moitié droite de la parabole.

Introduisons les ensembles suivants[3], pour toute fonction $S_d : [0,T] \to \mathcal{D}$:

$$S_d^m(t) = \{s, \exists t_n \to_{\leq} t, S_d(t_n) \to s\} \ , \tag{1.10}$$
$$S_d^p(t) = \{s, \exists t_n \to_{\geq} t, S_d(t_n) \to s\} \ , \tag{1.11}$$

et la définition suivante :

Définition 1.3.1 *Nous disons qu'une fonction S_d :$[0,T] \to \mathcal{D}$ a une* commutation *(ou* switch*) en $t \in (0,T)$ si $S_d^m(t) \neq S_d^p(t)$.*

Pour localiser les points où nous avons une commutation, nous avons besoin des ensembles suivants. Étant donnés i et j deux points de $\mathcal{D}$,

$$C_{i \to j} = \{x, k_d(x,i) = j\} \ , \tag{1.12}$$

et

$$\Sigma_{i \to j} = (C_{i \to j} + B(0,\rho)) \cap (C_{i \to i} + B(0,\rho)) \cap (C_{j \to j} + B(0,\rho)) \ , \tag{1.13}$$

où $B(0,\rho)$ désigne la boule ouverte de $\mathbb{R}^n$ de centre 0 et de rayon ρ. L'ensemble des $C_{i \to j}$ porte le nom de *set-interface* en théorie des systèmes hybrides (voir par exemple [BM:97]).

Nous décrivons ces ensembles à la figure 1.3 dans le cadre de la fonction k_d définie par la figure 1.2 (exemple de l'hystérésis).

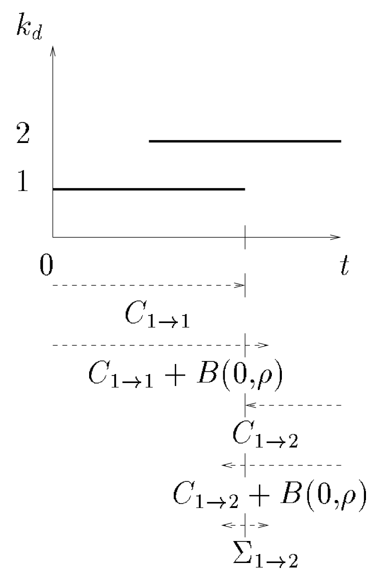

FIG. 1.3 – *Ensembles $C_{i \to j}$ et $\Sigma_{i \to j}$ définis par (1.12) et (1.13) pour la fonction k_d donnée à la figure 1.2 (exemple de l'hystérésis).*

Nous avons la propriété suivante qui permet de localiser les commutations :

Proposition 1.3.2 *Soit $t \in (0,T)$ un instant tel qu'une solution (X, S_d) de (1.8), définie sur $[0,T]$, a une commutation et $i \neq j$ dans $\mathcal{D}$ tels que i appartient à $S_d^m(t)$ et j appartient à $S_d^p(t)$. Alors nous avons*

$$X(t) \in \Sigma_{i \to j} \ .$$

3. m comme *moins* et p comme *plus*.

Nous ne donnons pas la démonstration de cette proposition (nous pouvons trouver une forme généralisée de ce résultat dans [P:01b]).

Les utilisations de cette proposition sont multiples. Par exemple, étant donnés trois éléments i, j et k de $\mathcal{D}$ tels que

$$\text{dist}(\Sigma_{i\to j},\Sigma_{j\to k}) > 0 \ ,$$

(c'est le cas de la figure 1.3) où dist désigne la distance usuelle dans $\mathbb{R}^n$, alors nous pouvons montrer très facilement, en utilisant l'inégalité des accroissements finis, que pour toute solution $(X,\, S_d)$ telle que S_d passe successivement de la variable i à j puis à k, nous avons un temps minimum entre ces commutations.

Nous pourrons utiliser ce temps minimum pour prouver certaines propriétés (attractivité, stabilité, ...). C'est l'idée générale de [P:01a], [P:01b] et [PA:01]. Mais nous allons voir qu'il faut se méfier de ces raisonnements trop intuitifs. Pour cela étudions le cas des cercles d'Artstein avec hystérésis.

1.4 Les cercles d'Artstein

Dans ce paragraphe, nous étudions les cercles d'Artstein avec un hystérésis et les solutions du système bouclé. Nous montrons qu'en présence de petites perturbations, nous avons des propriétés des solutions qui ne sont pas satisfaisantes. Cela justifiera l'introduction d'une autre définition de solutions au paragraphe suivant.

Les cercles d'Artstein, système de dimension deux, ont été initialement introduits dans [A:83]. C'est un exemple très utilisé en théorie du contrôle (voir par exemple [S:99, C:00]) car c'est un des systèmes les plus simples où la condition de Brockett ([B:83, Theorem 1, (iii)]), nécessaire pour avoir l'existence d'un contrôle continu rendant l'origine du système bouclé globalement asymptotiquement stable, est satisfaite, alors qu'un tel contrôle n'existe pas. D'autre part les courbes intégrales sont très faciles à étudier et permettent d'étudier l'impact d'un bruit sur un système discontinu.

Nous avons étudié plus précisément ce système dans [P:00].

Soit le système suivant :

$$\left\{ \begin{array}{rcl} \dot{x_1} & = & u(-x_1^2 + x_2^2) \ , \\ \dot{x_2} & = & -2ux_1x_2 \ , \end{array} \right. \tag{1.14}$$

avec u dans $\mathbb{R}$. Les ensembles suivants sont globalement invariants pour (1.14) :

- l'origine,
- tous les cercles centrés sur l'axe x_2 et tangents à l'axe x_1,
- l'axe x_1.

Avec $u > 0$ les cercles sont suivis dans le sens direct si $x_1 < 0$ et indirect si $u < 0$ (voir figure 1.4).

Remarque 1.4.1 Notons que les solutions de Carathéodory de (1.14) avec $u = 1$ si $x_1 \geq 0$, et $u = -1$ sinon, tendent vers (0,0), mais que chaque point de l'axe x_2 est un équilibre pour les solutions de Krasovskii (voir [P:00] et la définition 1.5.9 ci-dessous pour la notion de solutions de Krasovskii). Nous réutiliserons cette remarque au paragraphe 2.2 ci-dessous. $\diamond$

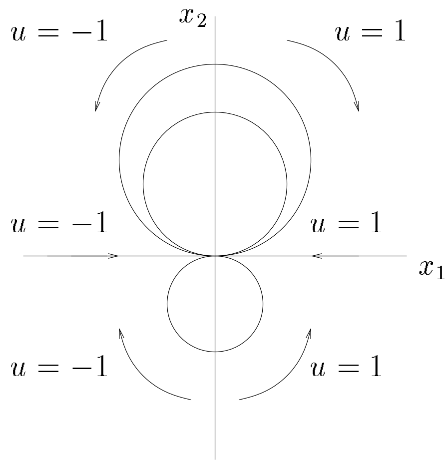

FIG. 1.4 – *Les courbes intégrales des cercles d'Artstein.*

Définissons les ensembles suivants

$$C_1 = \{x \in \mathbb{R}^2 : -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}\} \cup \{(0,0)\} \ ,$$

et

$$C_{-1} = \{x : \frac{3\pi}{4} \leq \theta \leq \pi \text{ ou} -\pi < \theta \leq -\frac{3\pi}{4}\} \cup \{(0,0)\} \ ,$$

où θ désigne dans $(-\pi,\pi]$ l'angle polaire d'un point $x \neq 0$.

Étudions l'hystérésis de (1.14) avec $u=1$ et de (1.14) avec $u=-1$ grâce à $u: \mathbb{R}^2 \times \{-1,1\} \to \{-1,1\}$ défini par :

$$u(x,s_d) \ = \ s_d, \forall (x,\, s_d) \in \mathbb{R}^2 \times \{-1,1\} \ , \qquad (1.15)$$

et la fonction k_d: $\mathbb{R}^2 \times \{-1,1\} \to \{-1,1\}$ définie par :

$$k_d(x,s_d) = \begin{cases} 1 & \text{si } x \in C_1 \setminus \{(0,0)\} \ , \\ -1 & \text{si } x \in C_{-1} \setminus \{(0,0)\} \ , \\ s_d & \text{sinon} \ . \end{cases} \qquad (1.16)$$

Considérons une condition initiale (x_0,s_0) et étudions la solution (X,S_d) associée. Nous pouvons montrer facilement que nous avons nécessairement l'un des cas suivants :

1. Si x_0 est dans C_1, alors, pour tout t dans $[0,+\infty)$, $S_d(t)=1$ et X est solution de

$$\begin{cases} \dot{x_1} &= & -x_1^2 + x_2^2 \ , \\ \dot{x_2} &= & -2x_1x_2 \ . \end{cases} \qquad (1.17)$$

2. Si x_0 est dans C_{-1}, alors, pour tout t dans $[0,+\infty)$, $S_d(t)=-1$ et X est solution de

$$\begin{cases} \dot{x_1} &= & x_1^2 - x_2^2 \ , \\ \dot{x_2} &= & 2x_1x_2 \ . \end{cases} \qquad (1.18)$$

3. Si x est dans $\mathbb{R}^2 \setminus (C_1 \cup C_{-1})$, alors, pour tout t dans $[0,+\infty)$, $S_d(t)=s_0$, et X est une solution de (1.17) si $s_0=1$ ou de (1.18) si $s_0=-1$.

Par conséquent, nous n'avons pas de commutation des solutions (au sens de la définition 1.3.1) et le contrôleur (1.15)-(1.16) n'a rien fait d'autre que de superposer les systèmes (1.17) et (1.18) sur $\mathbb{R}^2 \setminus (C_1 \cup C_{-1})$.

Fixons maintenant un réel $\rho > 0$ “petit”[4], et considérons maintenant e_1 et $e_2 : [0,+\infty) \to \mathbb{R}$ inconnus mais tels que :

$$\sup_{t\in\mathbb{R}_{\geq 0}} |e_1(t)| + |e_2(t)| \leq \rho \ . \tag{1.19}$$

Considérons le système (1.14)-(1.16) perturbé par ces erreurs de mesure, c'est-à-dire :

$$\left\{ \begin{array}{rcl} \dot{x_1} & = & s_d(-(x_1+e_1)^2 + (x_2+e_2)^2) \ , \\ \dot{x_2} & = & -2s_d(x_1+e_1)(x_2+e_2) \ , \\ s_d & = & k_d(x+e, s_d^-) \ . \end{array} \right. \tag{1.20}$$

Considérons les ensembles $\Sigma_{1\to-1}$ et $\Sigma_{-1\to1}$ associés au réel ρ et définis par (1.13) et une condition initiale $(x_0,1)$ où x_0 est dans $\Sigma_{1\to-1}$.

Nous allons étudier quelques solutions perturbées pour montrer que l'ensemble des solutions perturbées n'a pas de propriétés satisfaisantes.

Soit $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ une solution de (1.17) avec condition initiale x_0 et définie sur $[0, +\infty)$.

Soit une erreur $e = (e_1, e_2)$: $\mathbb{R}_{\geq 0} \to \mathbb{R}^2$ satisfaisant (1.19) et telle que

$$e(t) = 0 \ ,$$

pour tout t différent d'un petit réel $t_{1\to-1}$ fixé, et, en $t_{1\to-1}$, nous avons

$$k_d(\chi(t_{1\to-1}) + e(t_{1\to-1}), 1) = -1 \ . \tag{1.21}$$

Une telle fonction e existe si $t_{1\to-1}$ est choisi assez petit (voir l'allure des courbes intégrales à la figure 1.4). Soit $S_d : [0, +\infty) \to \mathcal{D}$ définie par

$$\begin{array}{rcl} S_d(t) & = & 1, \forall t \in [0, t_{-1\to1}) \ , \\ S_d(t) & = & -1, \forall t \in [t_{-1\to1}, +\infty) \ , \end{array}$$

et $X : [0,+\infty) \to \mathbb{R}^2$ étant, sur $[0, t_{-1\to1})$, une solution de (1.17), tandis qu'étant, sur $[t_{-1\to1}, +\infty)$, une solution de (1.18). Nous avons alors que (X, S_d) est une solution de (1.20) sur $[0,+\infty)$. En effet

- entre 0 et $t_{1\to-1}$, il n'y a pas de bruit donc (X, S_d) est dans le cas 3. de la page 27 avec condition initiale $(x_0,1)$ et S_d vaut 1.
- En $t_{1\to-1}$, nous avons $S_d^-(t_{1\to-1}) = 1$ et, à cause de (1.21) et de (1.5), nous avons $S_d(t_{1\to-1}) = -1$. Donc la solution commute en $t_{1\to-1}$ et au delà (X, S_d) est une solution non perturbée et nous retrouvons le cas 3. de la page 27.

Par conséquent les bruits e_1 et e_2 ont introduit un nouveau comportement de la variable S_d des solutions, puisque maintenant ils ont créé une commutation, mais la variable X converge tout de même vers (0,0).

4. par exemple, plus petit que 1.

Considérons la même condition initiale mais avec un bruit différent :

$$e(t) = 0 \ , \tag{1.22}$$

pour tout t dans $[0,\, t_{1\to-1}]$ où $t_{1\to-1}$ est encore un réel fixé, et, pour t dans $(t_{1\to-1}, t_{1\to-1} + \widetilde{\varepsilon})$, où $\widetilde{\varepsilon} > 0$ est un réel fixé, et, pour tout x dans un voisinage fixé de $\chi(t_{1\to-1})$, nous avons

$$k_d(x + e(t),\, 1) = -1 \ . \tag{1.23}$$

Comme précédemment une solution $(X,\, S_d)$ existe sur $[0,\, t_{1\to-1})$ et nous avons

$$S_d(t) = 1 \ , \forall t \in [0,\, t_{1\to-1}) \ . \tag{1.24}$$

Supposons que $(X,\, S_d)$ soit définie pour les temps supérieurs ou égaux à $t_{1\to-1}$. Alors, d'après (1.22), nous n'avons pas d'erreur en $t_{1\to-1}$ et donc pas de commutation en $t_{1\to-1}$ et nous avons $S_d(t_{1\to-1}) = 1$. De plus, d'après la définition 1.1.2, S_d doit être continue à droite et donc $S_d(t) = 1$ pour t dans $[t_{1\to-1}, t_{1\to-1} + \varepsilon)$. Soit $t^\star$ dans $(t_{1\to-1},\, t_{1\to-1} + \frac{\min(\widetilde{\varepsilon},\varepsilon)}{2}]$. Nous avons $S_d(t^\star) = 1$ et $S_d^-(t^\star) = 1$. Or d'après (1.23), nous devons avoir, si $t^\star$ est suffisamment proche de $t_{1\to-1}$,

$$S_d(t^\star) = k_d(X(t^\star) + e(t^\star),\, S_d^-(t^\star)) = k_d(X(t^\star) + e(t^\star),1) = -1 \ .$$

Cette égalité contredit $S_d(t^\star) = 1$. Nous en déduisons donc que la solution maximale $(X,\, S_d)$ est définie sur $[0,\, t_{1\to-1})$.

Ceci est un résultat surprenant puisque, pour une équation différentielle ordinaire, une solution, maximalement définie sur un ensemble borné, explose, c'est-à-dire tend vers l'infini au bord de l'ensemble de définition. Or nous avons exhibé une solution maximale de (1.14)-(1.16) qui s'arrête. Cela signifie que notre notion de solutions n'est pas bonne et qu'il faut agrandir l'ensemble des solutions. Cette discussion n'a pas été faite dans [P:00] mais elle a motivé la notion de solutions choisie et justifie l'introduction d'un ensemble CD plus petit que l'espace $\mathbb{R}^2 \times \{1, -1\}$ sur lequel nous imposons à S_d d'être continue à droite[5]. (La fonction S_d n'est alors plus continue à droite sur l'espace $\mathbb{R}^2 \times \{1, -1\}$ tout entier comme l'exigeait la définition 1.1.2.)

Le choix que nous allons faire pour cet ensemble est le suivant :

Si $S_d(t) = 1$ et si la solution a tendance à commuter à -1, c'est-à-dire si $X(t)$ est dans $\Sigma_{1\to-1}$ (comme c'était le cas pour la solution perturbée précédente), alors nous ne demandons pas à la solution S_d d'être continue à droite. Donc nous choisissons

$$CD = \mathbb{R}^2 \times \{-1,\, 1\} \setminus (\Sigma_{1\to-1} \times \{1\} \ \cup \ \Sigma_{-1\to1} \times \{-1\}) \ . \tag{1.25}$$

1.5 La bonne définition d'une solution

Replaçons-nous dans le cadre de l'équation (1.8).

Nous allons maintenant définir une nouvelle notion de solutions avec des propriétés satisfaisantes. Cette étude nous sera utile pour établir les applications à la théorie du contrôle données au chapitre 2.

5. CD comme Continu à Droite.

Soit $\rho > 0$ et (e,d) deux fonctions inconnues satisfaisant (1.9), et les ensembles $\Sigma_{i\to j}$, pour $i \neq j$ dans $\mathcal{D}$ définis par (1.13).

La discussion faite au paragraphe précédent justifie l'introduction, pour chaque i dans $\mathcal{D}$, de l'ensemble CD_i strictement inclus dans $\mathbb{R}^n \times \mathcal{D}$ suivant :

$$CD_i \;=\; \mathbb{R}^n \times \mathcal{D} \setminus \left(\bigcup_{j\in\mathcal{D}, j\neq i} \Sigma_{i\to j} \times \{i\} \right)$$

qui généralise (1.25). Introduisons également la notion suivante :

Définition 1.5.1 *Soit $T > 0$ et $(x_0,\, s_0)$ dans $\mathbb{R}^n \times \mathcal{D}$. Nous disons que $(X,\, S_d) : [0,T) \to \mathbb{R}^n \times \mathcal{D}$ est une* CD-solution *de (1.8) avec condition initiale $(x_0,\, s_0)$ si nous avons les propriétés suivantes :*

1. *La fonction X est absolument continue sur $[0,T)$ et vérifie, pour presque tout t dans $[0,T)$,*
$$\dot{X}(t) = f(X(t) + e(t),\, S_d(t)) + d(t) \ .$$
2. *Soit i dans $\mathcal{D}$ et t dans $[0,T)$ tel que $S_d(t) = i$. Si $(X(t),\, S_d(t))$ est dans CD_i alors la fonction S_d est continue à droite.*
3. *Pour tout t de $[0,T)$, où $S_d^-(t)$ existe, nous avons*
$$S_d(t) = k_d(X(t) + e(t),\, S_d^-(t)) \ .$$
4. *Nous avons*
$$X(0) = x_0 \quad , \qquad S_d(0) = k_d(x_0 + e(0), s_0) \ . \tag{1.26}$$

Remarque 1.5.2 Notons que si $\rho = 0$, c'est-à-dire si nous considérons le système (1.8) sans erreur, alors, d'après (1.13), pour tout i, j, $\Sigma_{i\to j}$ est vide et donc les CD-solutions de (1.8) sont exactement les solutions (au sens de la définition 1.1.1) de (1.1). La définition 1.5.1 est donc bien la généralisation de la définition 1.1.1 aux systèmes avec état mixte discret/continu en présence de perturbations.

Notons par ailleurs que nous traduisons la notion de CD-solution par $\mathcal{RC}$-solution en anglais comme dans [P:01b]. $\diamond$

Nous pouvons prouver que cette notion de solutions est la “bonne”, plus exactement que nous avons les résultats suivants

Proposition 1.5.3 *[voir, dans un contexte analogue, le Lemme 4.2 page 96 ou [P:01b]]*

Soit t un instant où nous avons une commutation de S_d, pour (X,S_d) une CD-solution de (1.8) et $i \neq j$ dans $\mathcal{D}$ tels que $i \in S_d^m(t)$ et $j \in S_d^p(t)$. Alors nous avons

$$X(t) \in \Sigma_{i\to j} \ .$$

Cette proposition est le pendant de la proposition 1.3.2 pour les solutions au sens de la définition 1.1.2. Soit Ω un ouvert de $\mathbb{R}^n$ tel que

$$\forall i,\, j,\, k \in \mathcal{D},\, i \neq j,\, j \neq k,\, \mathrm{dist}(\Sigma_{i\to j} \cap \Omega, \Sigma_{j\to k} \cap \Omega) > 0 \ ,$$

comme ce sera le cas dans nos systèmes à état mixte discret/continu considérés dans les applications du cahpitre 2 et dans [BM:97, eq. (1.21)], alors nous avons aussi le résultat d'existence suivant :

Proposition 1.5.4 *[voir, dans un contexte analogue, le Lemme 4.3 page 96 ou [P:01b]]*
Soit une condition initiale (x_0, s_0) *dans* $\Omega \times \mathcal{D}$. *Il existe une CD-solution de (1.8) partant de cette condition définie sur* $[0,T)$ *avec* $T > 0$.

Remarque 1.5.5 Nous pouvons vérifier facilement que si la notion de conditions initiales donnée par la définition 1.1.2 avait été choisie dans [P:01a] alors le [P:01a, Théorème 6.6] aurait eu la forme plus simple suivante :

> *[P:01a, Théorème 6.6 modifié] : Pour tout couple de condition initiale, il existe une solution partant de cette condition initiale.*

Remarquons aussi que l'affirmation [P:01a, Claim 6.11] aurait, elle aussi, été légèrement modifiée. ◇

Enfin énonçons le résultat d'explosion en temps fini qui est faux pour les solutions "naturelles" de la définition 1.1.2 comme nous l'avons vu au paragraphe 1.1 :

Proposition 1.5.6 *[voir, dans un contexte analogue, le Lemme 4.4 page 97 ou [P:01b]]*
Soit une CD-solution de (1.8) maximalement définie sur $[0,T)$ *avec* $T < +\infty$. *Alors nous avons*

$$\lim_{t \to T} |X(t)| = +\infty \ .$$

Ces propriétés, qui sont habituelles dans le cadre des solutions ordinaires d'une équation différentielle, justifient de poursuivre dans cette direction.

Nous cherchons à étudier l'influence des bruits sur les CD-solutions. C'est pourquoi nous introduisons la notion de solutions généralisées comme les limites possibles de CD-solutions perturbées (voir [H:67, S:78]). Plus précisément nous avons la

Définition 1.5.7 *Soit* $T > 0$ *et* x_0 *dans* $\mathbb{R}^n$. *Nous disons que* $X : [0, T) \to \mathbb{R}^n$ *est une* solution généralisée *de (1.8) avec condition initiale* x_0 *si, pour tout compact* J *de* $[0, T)$, *il existe une suite* $(e^k, d^k)_{k\in\mathbb{N}} : [0, +\infty) \to \mathbb{R}^n$ *et une suite* $(X^k, S_d^k)_{k\in\mathbb{N}}$ *de CD-solutions de*

$$\begin{array}{rcl} \dot{x} & = & f(x + e + e^k, s_d) + d + d^k \ , \\ s_d & = & k_d(x + e + e^k, s_d^-) \ , \end{array} \tag{1.27}$$

telle que

$$\lim_{k \to +\infty} \left(\text{esssup}_J|d^k| + \sup_J|e^k| + \text{esssup}_J|X^k - X|\right) = 0 \ ,$$

et telle que

$$X(0) = x_0 \ . \tag{1.28}$$

Remarquons qu'en utilisant [H:79, CR:94, F:88], nous pouvons prouver la proposition suivante :

Proposition 1.5.8 *Si nous n'avons pas de dynamique discrète alors l'ensemble des solutions généralisées est exactement l'ensemble des solutions de Krasovskii.*

Rappelons, pour être complet, la notion de solutions de Krasovskii, voir [F:88, Chapter 2]

Définition 1.5.9 *Soit* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *une fonction localement bornée. Nous disons que* $X :$ $[0,T) \to \mathbb{R}^n$ *est une* solution de Krasovskii *de* $\dot{x} = \Phi(x)$ *si* X *est absolument continue et satisfait, pour presque tout* t *dans* $[0,T)$,

$$\dot{X}(t) \in K(\Phi)(X(t)) \ ,$$

où $K(\Phi)(x) = \cap_{\varepsilon>0}\overline{Conv}\,\Phi(\{x\} + \varepsilon B)$ *(*B *est la boule unité de* $\mathbb{R}^n$ *et* $\overline{Conv}\, S$ *le plus petit convexe fermé contenant* S*).*

Introduisons maintenant les π-solutions de (1.8) qui sont les approximations polygonales des équations différentielles. Elles correspondent aux solutions calculées lorsque nous cherchons à implémenter une équation différentielle ou un contrôleur, comme ce sera le cas au chapitre 2. Voir les références dans [P:01b, S:98] et aussi la notion de jeux différentiels de Krasovskii et Subbotin [KS:88].

Pour cela introduisons une partition de $[0, +\infty)$, c'est-à-dire une suite $\pi = \{0 = t_0 < t_1 < \cdots < t_\infty = +\infty\}$, dont le diamètre $\overline{d}(\pi)$ est défini par (voir [S:98])

$$\overline{d}(\pi) = \sup_{i\in\mathbb{N}}(t_{i+1} - t_i) \ .$$

Définition 1.5.10 *Soit* $T > 0$ *et* x_0 *dans* $\mathbb{R}^n$*, nous disons que* $(X,\, S_d)$*:* $[0,\, T) \to \mathbb{R}^n \times \mathcal{D}$ *est une* π-solution *de (1.8) sur* $[0,T)$ *avec condition initiale* (x_0,s_0) *si*

1. *La fonction* X *est absolument continue sur* $[0,T)$ *et vérifie, pour tout* i *dans* $\mathbb{N}$ *et pour presque tout* t *dans* $[\min(t_i,T),\ \min(t_{i+1},T))$*,*

$$\dot{X}(t) = f(X(t_i) + e(t_i),\, S_d(t_i)) + d(t) \ .$$

2. *Nous avons, pour tout* t *dans* $[t_0, \min(t_1,\, T))$*,*

$$S_d(t) \ = \ S_d(t_0) \ ,$$

et, pour tout i *dans* $\mathbb{N}^\star$ *et pour tout* t *dans* $[\min(t_i,T), \min(t_{i+1},T))$*, nous avons*

$$S_d(t) \ = \ k_d(X(t_i) + e(t_i),\, S_d(t_{i-1})) \ .$$

3. *Nous avons (1.26).*

Enfin nous définissons de façon usuelle les solutions d'Euler comme les limites de π-solutions lorsque le diamètre des partitions tend vers 0, plus précisément :

Définition 1.5.11 *Soit* $T > 0$ *et* $x_0 \in \mathbb{R}^n$*, nous disons que* $X : [0,\, T) \to \mathbb{R}^n$ *est une* solution d'Euler *de (1.8) sur* $[0,\, T)$ *avec condition initiale* x_0*, si, pour tout compact* J *de* $[0,\, T)$*, il existe une suite de partitions* π^k *de* $\mathbb{R}$ *et une suite* $(X^k,\, S_d^k)$ *de* π^k*-solutions de (1.8) définies sur* J *telles que*

$$\lim_{k\to\infty} \sup_J |X^k - X| + \overline{d}(\pi^k) = 0 \ ,$$

et telle que nous avons (1.28).

Nous pouvons prouver, pour ces quatre types de solutions, des propriétés analogues aux propriétés 1.5.3, 1.5.4 et 1.5.6 (voir pour une démonstration dans un contexte analogue les Lemmes 4.2, 4.3, 4.4 à partir de la page 96, ou [P:01b]).

Maintenant nous allons pouvoir utiliser l'étude des solutions de (1.8) pour les applications du chapitre 2.

Chapitre 2

Applications à la commande

Nous appliquons dans cette partie les résultats formulés au chapitre précédent à trois problèmes différents issus de la théorie du contrôle. Nous montrons que des contrôleurs du type (5) apportent des solutions à ces problèmes, respectivement :

- le rapiècement de deux contrôles (paragraphe 2.1),
- la stabilisation globale asymptotique robuste aux bruits de tout système asymptotiquement contrôlable (paragraphe 2.2),
- la stabilisation globale robuste du système chaîné (paragraphe 2.3).

Notons que dans le paragraphe 2.1 nous utilisons l'approche *CD-solutions* et *solutions généralisées*, tandis que les notions de π*-solutions* et de *solutions d'Euler* pour des systèmes avec un terme discret sont appliquées aux paragraphes 2.2 et 2.3.

2.1 Rapiècement de deux contrôles

Nous considérons un système contrôlé pour lequel, nous connaissons deux commandes u_g et u_l, telles que l'origine du système bouclé correspondant soit un équilibre asymptotiquement stable globalement pour u_g et seulement localement asymptotiquement stable pour u_l. Soit Ω un ouvert d'adhérence strictement incluse dans l'ensemble de définition du contrôleur local. Nous nous intéressons au problème suivant :

> $(\mathcal{P}_1)$: *Trouver un troisième contrôle u qui soit égal au contrôle local u_l sur Ω, égal au contrôle global u_g loin de l'origine et tel que l'origine soit un équilibre globalement asymptotiquement stable en boucle fermée avec u.*

En général, étant données certaines *propriétés intéressantes* (minimisation de coût, convergence exponentielle, ...), nous pouvons trouver un contrôle satisfaisant ces critères (étude du Lagrangien, linéarisation, ...) mais stabilisant l'origine uniquement localement. Si par ailleurs nous connaissons un contrôleur dont l'origine du système bouclé correspondant est un équilibre globalement asymptotiquement stable mais n'ayant pas les *propriétés intéressantes* définies ci-dessus, nous pouvons en rapiéçant les deux contrôleurs, trouver une commande dont l'origine du système bouclé correspondant soit un équilibre globalement asymptotiquement stable et satisfaisant les *propriétés* souhaitées localement. C'est pourquoi nous nous intéressons au problème $(\mathcal{P}_1)$.

Nous montrons (voir Theorem 2.3 page 43) que ce problème n'a pas toujours une solution lorsque nous nous restreignons aux contrôles continus à cause de l'existence d'une obstruction topologique. De même lorsque nous étendons la classe de contrôleurs admissibles aux contrôleurs discontinus, et que nous imposons une robustesse aux bruits de mesure, d'implémentation et de modélisation (qui était automatiquement acquise pour un contrôleur continu), nous montrons que le problème de rapiècement ($\mathcal{P}_1$) peut ne pas avoir de solutions (voir Theorem 2.6 page 45). Pour cela nous utilisons des résultats issus de l'analyse non-lisse et de la théorie des fonctions de Lyapunov (en particulier [CLS:98]).

Cela nous conduit à élargir la classe de contrôles considérée. Nous avons explicité deux solutions à ce problème :

- Un contrôle avec terme discret comme celui de (5) (voir [P:01a, Theorem 4.1]) ou page 49).
- Un contrôle continu dépendant du temps et de l'état $u(x,t)$ périodique par rapport au temps ([P:01a, Theorem 5.1] ou page 52).

2.2 Stabilisation robuste des systèmes contrôlables

Nous étudions des systèmes dont l'équilibre est globalement asymptotiquement contrôlable, c'est-à-dire tels que pour toute condition initiale, il existe un contrôle dépendant du temps tel que la trajectoire du système bouclé avec ce contrôle tende vers l'origine avec une propriété de stabilité (voir Définition 2.7 ci-dessous, page 92 pour une définition précise). Nous prouvons ([P:01b] ou Theorem 1 page 92) qu'il existe un contrôle de la forme (5) tel que pour toutes les π-solutions ou les solutions d'Euler définies au paragraphe 1.1, l'origine du système bouclé (1.1) est globalement et asymptotiquement stable avec une robustesse pour des erreurs (petites) de mesure, d'implémentation et de modélisation.

Ce résultat peut être vu comme la généralisation de [P:00] (voir pages 69 et suivantes) à tout système dont l'origine est un équilibre globalement asymptotiquement contrôlable.

Notons que l'étude du système (1.14), dont l'origine est un équilibre globalement asymptotiquement contrôlable, mais pour lequel il n'existe pas de loi de commande $u(x)$ continue avec laquelle l'origine du système bouclé est un équilibre globalement asymptotiquement stable, prouve qu'il faut envisager la classe des contrôles discontinus pour stabiliser tout système dont l'origine est globalement asymptotiquement contrôlable. Voir aussi [B:83, SS:80].

La propriété suivante est prouvée dans [S:79]

($\mathcal{P}$) Tout système dont l'origine est globalement asymptotiquement contrôlable peut être asymptotiquement stabilisé par un contrôleur discontinu.

Dans ce papier "par solutions du système bouclé", il faut comprendre "solutions de Carathéodory satisfaisant une «exit rule»". Dans [CLSS:97] la même propriété ($\mathcal{P}$) est en considérant les π-solutions (voir la définition 1.5.10 ci-dessus). Récemment dans [AB:99], la propriété ($\mathcal{P}$) est prouvée pour toutes les solutions de Carathéodory grâce à un "patchy feedback".

Les contrôleurs dans [CLSS:97, AB:99] sont robustes par rapport aux erreurs d'implémentation et de modélisation mais ne sont pas robustes pour toutes erreurs de mesure aussi petites soient-elles (voir remarque 1.4.1 et [P:00] ou page 69 et suivantes).

Cela justifie d'élargir encore une fois la classe de contrôleur admissible et de considérer des lois de commande de la forme (5).

Nous comparons (voir page 92) ce résultat avec d'autres résultats donnant d'autres contrôleurs discrets [CLRS:00, S:99]. Nous remarquons que la notion de solutions considérée est plus générale, par conséquent la propriété de stabilisation dans [P:01b] est plus riche. Soulignons que la technique utilisée est tout à fait comparable avec celle de [S:99], où l'auteur introduit de la *mémoire du temps* en imposant que la discrétisation du contrôle est assez lente, tandis que nous, en faisant un hystérésis entre les différentes composantes continues du contrôle, nous créons une *mémoire spatiale*.

Notons que nous pouvons stabiliser l'équilibre par des retours d'etat instationnaires continus et périodiques en temps. L'auteur de [C:95] prouve l'existence d'une telle commande pour tout système dont l'équilibre est localement contrôlable. Cependant, d'après [BR:01, Proposition 3.5], il n'existe pas de contrôles continus instationnaires (ou même dynamiques) qui rendent l'équilibre des cercles d'Artstein localement asymptotiquement stable, alors que nous avons prouvé dans [P:00] qu'il existe un contrôle avec état mixte discret/continu tel que l'équilibre du système bouclé est globalement asymptotiquement stable avec une robustesse par rapport aux petits bruits. Cela justifie donc d'utiliser le type de contrôles que nous avons introduit. Notons, par ailleurs, que ces feedbacks instationnaires ont été utilisés dans [MSPJ:95] pour le problème de la stabilisation d'un satellite avec contrôle des couples de torsion. Plus récemment, les auteurs de [MS:97] ont explicité des commandes plus simples pour ce même problème.

2.3 Stabilisation robuste des systèmes chaînés

Nous nous intéressons aux systèmes chaînés en dimension n quelconque avec 2 commandes (voir [MS:93]).

Nous prouvons (Theorem 1, page 114) qu'il existe un contrôle de la forme (5) tel que l'origine soit un équilibre globalement asymptotiquement stable pour les $\mathcal{RC}$-solutions perturbées par une erreur ayant une certaine structure (voir page 109 et suivantes ou [PA:01] pour plus de détails).

En fait l'origine des systèmes chaînés est un équilibre globalement asymptotiquement contrôlable. Donc ce résultat peut être vu comme une simple application du paragraphe 2.2. Cependant il donne *l'existence* d'un contrôle avec une dynamique sur une *infinité* de variables discrètes et ne propose pas de feedback explicite. Le résultat présenté spécifiquement pour les systèmes chaînés utilise *une seule* variable discrète dynamique et est totalement *explicite*.

Nous utilisons [VA:01] qui donne un contrôleur discontinu tel que l'origine est un équilibre globalement asymptotiquement stable avec une robustesse locale par rapport aux erreurs de mesure, d'implémentation et de modélisation.

Remarque 2.3.1 Le résultat de [VA:01] cité ci-dessus peut paraître surprenant puisque un système chaîné est un système affines sans dérive et donc, d'après [CR:94], si un tel feedback existe alors il existe un contrôle continu tel que l'origine du système bouclé est un équilibre globalement asymptotiquement stable. Or un tel contrôle n'existe pas pour l'intégrateur de Brockett (d'après la condition de Brockett [B:83, Theorem 1,(iii)]) qui est équivalent à un système chaîné particulier. En fait il n'y pas de contradiction car les notions de contrôle discontinu dans [CR:94] et [VA:01] ne sont pas les mêmes. (Comparer le contrôle de [VA:01] et [CR:94, Definition 1.2].) ◇

Nous rajoutons au contrôleur de [VA:01] une variable discrète pour créer un hystérésis entre les composantes continues du contrôle de [VA:01]. La géométrie de cet hystérésis n'est pas aussi claire que [P:01a] ce qui impose seulement une robustesse partielle et une structure des erreurs. Voir page 109 et suivantes pour plus de détails.

Notons qu'il existe de nombreux résultats concernant les systèmes affines sans dérive, *i.e.* les systèmes de la forme :

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x) \ , \tag{2.1}$$

asymptotiquement contrôlables en l'origine (comme c'est le cas pour les systèmes chaînés). Voir par exemple [C:92] où la "méthode du retour" est appliquée pour prouver qu'alors l'origine du système (2.1) est un équilibre globalement asymptotiquement stabilisable par un retour d'état périodique en temps. Des fonctions de Lyapunov sont utilisées pour *construire* une telle commande dans [P:92]. Des techniques d'homogénéisation ont été appliquées dans [MPS:99]. Voir aussi [S:91] et [PS:94] pour des contrôleurs instationnaires pour le système non-holonome plus spécifiquement.

Notons que [MS:01] étudie plus spécifiquement le problème de la stabilisation pratique (le contrôle fait converger tous les points dans un voisinage donné de l'équilibre) avec une propriété de robustesse pour les systèmes sans dérive.

Appendice

Dans cet appendice nous donnons les résultats précis et les preuves du chapitre 2. En ce qui concerne les résultats donnés au chapitre 1, nous avons donné à chaque fois des éléments des preuves qui sont le plus souvent généralisées dans les chapitres suivants et ont été présentés à une conférence [P:01c].

Nous trouvons donc dans cet ordre

1. À partir de la page 39, le problème du rapiècement défini au paragraphe 2.1 de la partie I, ses obstructions et ses solutions. Ce chapitre a fait objet de la publication [P:01a], et des présentations [PP:99] et [P:99].
2. À partir de la page 69, la stabilisation robuste des cercles d'Artstein. Cet exemple nous a permis de comprendre beaucoup d'idées du chapitre 1 de la partie I est a fait l'objet de la publication [P:00].
3. À partir de la page 87, la stabilisation asymptotique de tout système asymptotiquement contrôlable [P:01b] et a été présenté dans [P:01d].
4. À partir de la page 109, la stabilisation des systèmes sous forme chaînée [PA:01].

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Uniting Local and Global Controllers with Robustness to Vanishing Noise*

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Abstract. We consider control systems for which we know two stabilizing controllers. One is globally asymptotically stabilizing, the other one is only locally asymptotically stabilizing but for some reason we insist on using it in a neighborhood of the origin. We look for a uniting control law being equal to the local feedback on a neighborhood of the origin, equal to the global one outside of a larger neighborhood and being a globally stabilizing controller. We study several solutions based on continuous, discontinuous, hybrid, time-varying controllers. One criterion of the selection of a controller is the robustness of the stability to vanishing noise. This leads us in particular to consider a kind of generalization of Krasovskii trajectories for hybrid systems.

Key words. Nonlinear stabilization, Lyapunov functions, Disturbance, Errors measurements, Generalized trajectories.

1. Problem Statement and Related Results

1.1. Introduction

In nonlinear control system theory, we now have numerous tools (backstepping, forwarding, feedback linearization, passivation, ...) to design (globally) asymptotically stabilizing feedbacks. However, if such feedbacks give a satisfactory answer to the global asymptotic stabilization problem, they are not necessarily intended to address the performance problem. As opposed to this fact, for instance via linearization, one may design controllers addressing satisfactorily both the asymptotic stabilization and the performance problems but only locally. A pratical example of such a framework is given in [TKM]. This leads us to the idea of uniting a local controller with a global controller, i.e. given (1) a controller u_l able to stabilize locally while providing better performance and (2) a controller u_g providing global asymptotic stability, we are looking for a maybe time-varying, possibly hybrid, dynamic controller providing uniform global asymptotic stability for the overall system while matching exactly the local controller u_l when the system

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state component is in a neighborhood of the origin and matching the global controller u_g when this component is outside a compact set containing the origin.

1.2. Problem Statement

Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function such that $f(0,0) = 0$. We consider the system

$$\dot{x} = f(x, u). \tag{1}$$

We call the following the uniting problem:

Let Ω be a bounded open connected neighborhood of the origin (in $\mathbb{R}^n$) and two continuous controllers $u_l\colon \mathscr{D} \to \mathbb{R}^m$ and $u_g\colon \mathbb{R}^n \to \mathbb{R}^m$, where $\mathscr{D} \subset \mathbb{R}^n$ is an open set containing $\mathtt{clos}(\Omega)$, which make the systems $f_l\colon \mathscr{D} \to \mathbb{R}^n$, $x \mapsto f(x, u_l(x))$ and $f_g\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(x, u_g(x))$ admit the origin as an asymptotically stable equilibrium, globally on their respective domain of definition.

We look for:

1. A bounded closed set $A \subset \mathbb{R}^n$ (e.g. an annulus) which separates $\mathbb{R}^n$ in two connected open sets C_l and C_g (e.g. an open ball and the complement of a closed ball) such that we have
$$\Omega \subset C_l \subset \mathscr{D}.$$
2. A control law $\varphi(x)$ (at this stage we restrict to a continuous static time-invariant control law), satisfying, for all j in $\{l, g\}$,
$$\varphi(x) = u_j(x), \qquad \forall x \in C_j, \tag{2}$$
and such that the origin of the system $\dot{x} = f(x, \varphi(x))$ is globally asymptotically stable.

1.3. Related Results and the Organization of This Paper

Studies on the uniting problem have already been reported, in particular in [MMP] and [TK]. In [MMP] the solution is given in the form of a continuous static time-invariant controller φ as requested above. It assumes the existence of a continuous path of stabilizing controllers between u_l and u_g. Unfortunately we show by means of an example that this assumption can be violated. Actually, for this particular example, there is no continuous (and even discontinuous) static time-invariant controller. This shows that dynamic extension of the control φ may be necessary, this being via time variations, discrete or continuous state, i.e. the problem formulation formulated above has to be considered with a more general class of controllers. Note that in [TK] a dynamic time-invariant controller $\varphi(x, s)$ is proposed but it does not satisfy our requirement (2). Specifically, along the trajectories, the proposed control converges with time to the global one u_g.

We first return, in Section 2.1, to the static time-invariant continuous controller proposed in [MMP] but we show, in Section 2.2, a system to which it cannot be applied. In fact this system motivates us to look at obstructions for solving the problem via (dis)continuous static time-invariant controllers (Section 2.3). This negative result leads us to reformulate, in Section 3, in terms of more general controllers. A first class of such controllers is dynamic hybrid control. We show that indeed the uniting problem can be solved in terms of weak generalized trajectories (Section 4.1) but unfortunately not in terms of strong generalized trajectories (Section 4.2). So finally, in Section 5, we propose a periodic static continuous controller solving the problem in its whole generality.

2. Static Time-Invariant Controllers

2.1. A Solution to the Uniting Problem

Following the arguments and ideas of [MMP], we get:

Theorem 2.1. *Let Ω be a bounded open connected neighborhood of the origin in $\mathbb{R}^n$, and let u_l and u_g be two continuous functions on $\mathbb{R}^n$. We assume the existence of a triple (ψ, V, c) as:*

- ψ: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^m$, *a continuous path connecting u_l to u_g, i.e. for all x in $\mathbb{R}^n$, we have*
$$\psi(0,x) = u_l(x), \qquad \psi(1,x) = u_g(x),$$
- V: $[0,1] \times \mathbb{R}^n \to \mathbb{R}_+$, *a $\mathscr{C}^1$ function which, for all $s \in [0,1]$, is positive definite and radially unbounded,*
- *c a positive real number such that Ω is contained in $\{x\colon V(0,x) < c\}$*

with the following properties:

1. *For each s in $[0,1]$ and each x in $\mathbb{R}^n \backslash \{0\}$, we have*
$$\frac{\partial V}{\partial x}(s,x) f(x, \psi(s,x)) < 0. \tag{3}$$
2. *For each (s,x) satisfying $V(s,x) = c$, we have*
$$\frac{\partial V}{\partial s}(s,x) < 0. \tag{4}$$

Under these conditions, we can find a bounded closed set A and a continuous function φ defined as
$$\varphi(x) = \psi(\gamma(x), x), \qquad \forall x \in \mathbb{R}^n, \tag{5}$$
where γ is a locally Lipschitz function, which solves the uniting problem.

Remark 2.2. 1. With (3) holding for $s = 1$, we impose that $\mathscr{D}$ is actually $\mathbb{R}^n$. In fact this restriction is too strong. We need only that Ω be sufficiently small inside $\mathscr{D}$. This "sufficiently" is linked to the stability properties provided by u_g. To avoid making our statement of Theorem 2.1 too complicated, we have preferred to impose $\mathscr{D} = \mathbb{R}^n$.

2. The existence of some c such that Ω is contained in $\{x, V(0,x) < c\}$ follows from the fact that V is radially unbounded. However, with (4) one asks more of such c.

Proof. Let A be the closed set $\{x: c \leq V(0,x), V(1,x) \leq c\}$. Condition (4) implies that this set is nonempty, bounded and contains the set $\{x: V(0,x) = c\}$. Also the sets

$$C_l = \{x: V(0,x) < c\}, \qquad C_g = \{x: c < V(1,x)\}$$

are open and connected, with an empty intersection and satisfy $\mathbb{R}^n = C_l \cup A \cup C_g$. This means that A has the required properties.

We now define the controller $\varphi(x)$. To do this, we introduce a function γ on $\mathbb{R}^n$ as follows:

- for x in C_l, we let $\gamma(x) = 0$,
- for x in C_g, we let $\gamma(x) = 1$,
- for x in A, we choose $\gamma(x)$ as the solution of

$$V(\gamma, x) = c. \tag{6}$$

Condition (4) and the Implicit Function Theorem imply that γ is well defined and Lipschitz on $\mathbb{R}^n$ and $\mathscr{C}^1$ on $\mathbb{R}^n \backslash (\partial C_l \cup \partial C_g)$ and we have

$$x \in \mathtt{int}(A) \quad \Leftrightarrow \quad \gamma(x) \in (0,1). \tag{7}$$

Then we let φ be the function defined by (5), and we readily have (2), for j in $\{l, g\}$. The global asymptotic stability of the origin for the closed-loop system is a consequence of the following three properties:

1. For each x in $\mathtt{clos}(C_g)$, we have (from (3))

$$\frac{\partial V}{\partial x}(1,x) f(x, \varphi(x)) < 0.$$

   With continuity, it follows that there exists ε_g such that any trajectory X with initial condition x verifying $V(1,x) \geq c - \varepsilon_g$ satisfies, for x and all $t \geq t_0$, $V(1, X(t)) \leq V(1,x)$, reaches in finite time, depending only on $V(1,x)$, and then remains in the set

$$\{x: V(1,x) \leq c - \varepsilon_g\} \subsetneq C_l \cup A.$$

2. Similarly, using $V(0,x)$, we get the existence of ε_l such that the origin is asymptotically stable with a basin of attraction containing the set

$$C_l \subsetneq \{x: V(0,x) \leq c + \varepsilon_l\}.$$

3. The function γ is $\mathscr{C}^1$ on $\mathtt{int}(A)$ and, from (3), (4) and (6), we get

$$\begin{aligned}\frac{\partial \gamma}{\partial x}(x) f(x, \varphi(x)) &= -\frac{(\partial V/\partial x)(\gamma(x), x)}{(\partial V/\partial s)(\gamma(x), x)} f(x, \varphi(x)) \\ &= -\frac{1}{(\partial V/\partial s)(\gamma(x), x)} \frac{\partial V}{\partial x}(\gamma(x), x) f(x, \psi(\gamma(x), x)) \\ &< 0.\end{aligned}$$

   It follows that any trajectory with initial condition x in the set

$$\{x: V(1,x) \leq c - \varepsilon_g, V(0,x) \geq c + \varepsilon_l\} \subset \mathtt{int}(A)$$

satisfies, for all $t \geq 0$, $V(1, X(t)) \leq c$, converges to the set $\{x: \gamma(x) \leq 0\} = \mathtt{clos}(C_l)$, and therefore reaches the set $\{x: V(0,x) \leq c + \varepsilon_l\}$ in finite time independent of x. ■

2.2. A Topological Obstruction

Theorem 2.1 provides a solution to the uniting problem via a static time-invariant continuous controller. We show in this section that we must not restrict our attention to only such kinds of feedbacks.

Let the system be

$$\dot{x} = -y^2 x, \qquad \dot{y} = u. \tag{8}$$

The data of the uniting problem we consider are

$$u_l = -y + x, \qquad u_g = -y - x, \qquad \Omega = \{(x,y): x^2 + y^2 < \tfrac{1}{2}\}, \qquad \mathscr{D} = \mathbb{R}^2.$$

The fact that u_l and u_g are global asymptotic stabilizers can be checked with LaSalle's invariance theorem and the Lyapunov function $2x^2 + y^4$.

Let A be any closed set which separates $\mathbb{R}^2$ into two connected open sets C_l and C_g with C_l containing Ω. There exists $0 < c_l < c_g$ such that

$$A \subsetneq \{(x,y): c_l^2 \leq x^2 + y^2 \leq c_g^2\}.$$

Assume the existence of a static time-invariant continuous controller $\varphi(x,y)$ solving the uniting problem. Then we have

$$\varphi(x,y) = \begin{cases} -y + x & \text{if} \quad x^2 + y^2 \leq c_l^2, \\ -y - x & \text{if} \quad c_g^2 \leq x^2 + y^2, \end{cases}$$

and, in particular,

$$\varphi(c_l, 0) = c_l, \qquad \varphi(c_g, 0) = -c_g.$$

Since c_l and c_g are positive, the continuity of φ implies the existence of c, strictly positive, such that $\varphi(c, 0) = 0$. It follows that $(c, 0)$ is an equilibrium of the closed-loop system contradicting the fact that $\varphi(x,y)$ is globally asymptotically stabilizing the origin.

We have established that the conclusion of Theorem 2.1 does not hold and so its assumptions are violated. Actually the same argument as above shows that there is no continuous function $\psi(s, (x,y))$ connecting u_l to u_g and providing a globally asymptotically stabilizing controller for each s in $[0,1]$.

The obstruction observed with the system (8) leads to the following necessary condition for the solvability of the uniting problem via static time-invariant continuous feedback, where we impose $\mathscr{D} = \mathbb{R}^n$ to avoid making our statement too complicated.

Theorem 2.3. *Let (u_l, u_g, Ω) be the data of a uniting problem. If there exists a static time-invariant continuous control as a solution for this problem, then there exists $0 < c_l < c_g$ such that the functions $\tilde{u}_l$ and $\tilde{u}_g$ below are homotopic*:

$$\forall j \in \{l, g\}, \qquad \tilde{u}_j: \mathbb{S}^{n-1} \to \Sigma := \{(x,u), f(x,u) \neq 0\} \subset \mathbb{R}^n \times \mathbb{R}^m,$$
$$\xi \mapsto (c_j \xi, u_j(c_j \xi)).$$

Proof. Since A is a bounded closed set not containing the origin, there exists $0 < c_l < c_g$ such that A is included in $\{(x, y)\colon c_l \leq |x| \leq c_g\}$. The function $H\colon [0,1] \times \mathbb{S}^{n-1} \mapsto \Sigma$ defined as

$$H(s, \xi) = ([(c_g - c_l)s + c_l]\xi, u([(c_g - c_l)s + c_l]\xi))$$

provides the required homotopy. Indeed, there is no equilibrium in $\{(x, y)\colon c_l \leq |x| \leq c_g\}$, so we have, for all (s, ξ) in $[0,1] \times \mathbb{S}^{n-1}$, $f(H(s, \xi)) \neq 0$. ■

The necessary condition given in Theorem 2.3, written in terms of homotopy, can also be expressed in terms of homology as in [C].

For the system (8), the set Σ is $\mathbb{R}^3$ without the x- and y-axis. The image of $\mathbb{S}^1$ by $\tilde{u}_l$ in $\mathbb{R}^3$ is an ellipsis, intersection of the plane $u + y - x = 0$ with the cylinder $x^2 + y^2 = c_l^2$. Similarly, the one by $\tilde{u}_g$ is an ellipsis in the plane given by $u + y + x = 0$. We can see that there is no continuous deformation allowing us to go from one ellipsis to the other without crossing the x- or y-axis. So the necessary condition is not met.

We conclude that the class of static time-invariant continuous controllers is not rich enough to address the uniting problem. Before investigating a richer class, we show that, in some cases, the class of static time-invariant discontinuous controllers is also not rich enough.

2.3. Obstruction for a Solution with a Discontinuous Static Time-Invariant Controller

In this section we want to prove a necessary condition for having a solution to the uniting problem with a discontinuous static time-invariant controller. The closed-loop system with such a controller has a discontinuous right-hand side. The notion of trajectories for such a differential equation will be the Krasovskii trajectories:

Definition 2.4. Let t_0 be in $\mathbb{R}$ and $T > t_0$. Let Φ be a locally bounded function. We say that X defined on $[t_0, T)$ is a *Krasovskii trajectory* of $\dot{x} = \Phi(x)$ if X is absolutely continuous and, for almost all t in $[t_0, T)$, we have

$$\dot{X}(t) \in K(\Phi)(X(t)),$$

where $K(\Phi)(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{Conv}}\, \Phi(\{x\} + \varepsilon B)$ (B denotes the unit ball of $\mathbb{R}^n$ and $\overline{\mathrm{Conv}}\, S$ the smaller closed convex set containing S).

Usually this distinction is not made because we have:

Lemma 2.5 [F, p. 50]. *When Φ is continuous each Krasovskii trajectory of $\dot{x} = \Phi(x)$ is a Peano trajectory, i.e. X is C^1 and $\dot{X} = \Phi(X(t))$, for all t.*

It follows that Theorems 2.1 and 2.3 hold for Krasovskii trajectories.

In this section we impose that f is affine in u, i.e.

$$f(x, u) = a(x) + \sum_{i=1}^{m} b_i(x)u_i,$$

where a and b are locally Lipschitz, and we have

Theorem 2.6. *Let $(u_l, u_g, \Omega, \mathscr{D})$ be the data of a uniting problem for f affine in u. If the uniting problem is solvable, in terms of Krasovskii trajectories, with a locally bounded static time-invariant controller, then, for any bounded open connected set $\tilde{\Omega}$, with a neighborhood of the origin such that*

$$\mathtt{clos}(\tilde{\Omega}) \subset \Omega,$$

the uniting problem, with data $(u_l, u_g, \tilde{\Omega}, \mathscr{D})$, is also solvable in terms of Krasovskii trajectories by a static time-invariant continuous controller.

This theorem implies that, when f is affine in u, if the uniting problem cannot be solved with a continuous static time-invariant controller, then it cannot be solved with a discontinuous static time-invariant controller either. Note that this is the case for system (8).

To prove Theorem 2.6, we need the following result:

Lemma 2.7 [CLS, Theorem 1.3]. *Let Φ be a locally bounded function. If the origin is globally asymptotically stable for $\dot{x} = \Phi(x)$, in terms of the Krasovskii trajectories, then there exist a proper positive definite C^∞ function V and a positive definite continuous function W such that, for all $x \in \mathbb{R}^n$, we have*

$$\max_{v \in K(\Phi)(x)} \frac{\partial V}{\partial x}(x) v \leq -W(x).$$

Proof of Theorem 2.6. Let A and φ solve the uniting problem with φ a static time-invariant locally bounded controller. Let $\tilde{A} \subset \mathbb{R}^n$ be a compact set separating $\mathbb{R}^n$ in two connected open sets $\tilde{C}_l$ and $\tilde{C}_g$ such that we have

$$\{0\} \in \tilde{\Omega} \subset \tilde{C}_l \subset \mathtt{clos}(\tilde{C}_l) \subset C_l, \qquad \mathtt{clos}(C_g) \subset \tilde{C}_g. \tag{9}$$

Let V and W be given by Lemma 2.7. We have, for all x in $\mathbb{R}^n \backslash \{0\}$,

$$L_a V(x) + \varphi(x) \cdot L_b V(x) \leq -W(x),$$

where we note $\varphi \cdot L_b V(x)$ instead of $\sum_i L_{\varphi_i b_i} V$ with L_b the Lie derivative along b. Since $\tilde{A}$ is compact and does not contain the origin and W is positive definite and continuous, for all x in $\mathtt{int}(\tilde{A})$, there exists $\rho_x > 0$, such that, for all y in $B(x, \rho_x) \subset \mathtt{int}(\tilde{A})$, we have

$$L_a V(y) + \varphi(x) \cdot L_b V(y) \leq -\frac{W(y)}{2}, \qquad \rho_x \leq \tfrac{1}{2} d(x, \partial \tilde{A}).$$

From Theorem V.4.4 of [B], there exists a $\mathscr{C}^\infty$ partition of unity $(\Psi_i)_{i \in \mathbb{N}}$ subordinate to the open covering $\{B(x, \rho_x)\}_{x \in \mathtt{int}(\tilde{A})}$ of $\mathtt{int}(\tilde{A})$. For each i, let x_i be one arbitrary of the points such that

$$\mathtt{supp}(\Psi_i) \subset B(x_i, \rho_{x_i}).$$

We let $\varphi_i = \varphi(x_i)$. By construction, we have, for all x in $\mathtt{supp}(\Psi_i)$,

$$L_a V(x) = \varphi_i \cdot L_b V(x) \leq -\frac{W(x)}{2}. \tag{10}$$

Now we define the function $\tilde{\varphi}$ as follows:

1. For all x in $\mathtt{int}(\tilde{A})$, we let $\tilde{\varphi}(x) = \sum_i \Psi_i(x)\varphi_i$.
2. For all x in $\mathtt{clos}(\tilde{C}_j)$, for all j in $\{l, g\}$, we let $\tilde{\varphi}(x) = \varphi_j(x)$.

Assuming, for the time being, that $\tilde{\varphi}$ is continuous, (10) implies that the origin is a globally asymptotically stable equilibrium of $\dot{x} = a(x) + b(x)\tilde{\varphi}(x)$. So $(\tilde{A}, \tilde{\varphi})$ solves the uniting problem in terms of Peano trajectories.

We check that $\tilde{\varphi}$ is continuous on $\mathbb{R}^n$. By construction, $\tilde{\varphi}$ is continuous on $\mathtt{int}(\tilde{A}) \cup \tilde{C}_l \cup \tilde{C}_g$. To prove that $\tilde{\varphi}$ is continuous on $\partial\tilde{C}_l$ (resp. $\partial\tilde{C}_g$), we fix x_0 in $\partial\tilde{C}_l$. Now let $\varepsilon > 0$ be arbitrary. From (9) and the continuity of u_l, there exists $\eta_1 > 0$ such that, for each $\eta \in (0, \eta_1]$,

$$B(x_0, \eta) \subset C_l \quad \text{and} \quad |u_l(x_0) - u_l(x)| < \varepsilon, \qquad \forall x \in B(x_0, \eta). \tag{11}$$

Now, for each x in $B(0, \frac{1}{2}\eta)$

- either x is in $\mathtt{clos}(\tilde{C}_l)$ and then $|\tilde{\varphi}(x_0) - \tilde{\varphi}(x)| = |u_l(x_0) - u_l(x)| \leq \varepsilon$
- or x is in $\mathtt{int}(\tilde{A})$ and then

$$|\tilde{\varphi}(x) - \tilde{\varphi}(x_0)| = \left| \sum_{i \in I(x)} \Psi_i(x)[u_l(x_i) - u_l(x_0)] \right|, \tag{12}$$

where $I(x)$ is the finite set such that, for all i in $I(x)$, $\Psi_i(x) \neq 0$. Now, for each $\eta \in (0, \eta_1]$ and each i in $I(x)$, we have

$$\begin{aligned} d(x_i, \partial\tilde{C}_l) < |x_i - x_0| < \rho_{x_i} + \tfrac{1}{2}\eta < \tfrac{1}{2}\, d(x_i, \partial\tilde{A}) + \tfrac{1}{2}\eta \\ < \tfrac{1}{2}\min\{d(x_i, \partial\tilde{C}_l), d(x_i, \partial\tilde{C}_g)\} + \tfrac{1}{2}\eta. \end{aligned}$$

This implies successively

$$d(x_i, \partial\tilde{C}_l) < \eta, \qquad \rho_{x_i} < \tfrac{1}{2}\eta, \qquad |x_i - x_0| < \eta.$$

So with (11) and (12), we get $|\tilde{\varphi}(x) - \tilde{\varphi}(x_0)| \leq \varepsilon$. ■

3. A Larger Class of Controllers and Notions of Trajectories

Theorems 2.3 and 2.6 prove that we cannot restrict ourselves to static time-invariant discontinuous controller to solve the uniting problem. Therefore we now consider controllers admitting the following description (see [T]):

$$u = \varphi(x, s_d, t), \qquad s_d = k_d(x, s_d^-, t),$$

where s_d evolves in some finite set $\mathscr{F}$, the functions φ and k_d are locally bounded, and s_d^- is defined as

$$s_d^-(t) = \lim_{s \nearrow t} s_d(s).$$

For this to make sense we equip $\mathscr{F}$ with discrete topology (i.e. every set is an open set). The above controller is

- dynamic with the presence of s_d,
- time varying due to the presence of t,
- hybrid due to the presence of the discrete dynamics of s_d.

This class of hybrid controllers has received much attention in many different contexts. However, we have not seen it considered for the uniting problem. Nevertheless, it has been studied in the closely related case of achieving high performance in the presence of input saturation (see [DG] for instance).

This controller with a switching strategy gives rise to a nonclassical ordinary differential equation describing the dynamics of the closed-loop system. In particular, this system is infinite-dimensional since to evaluate $s_d^-(t)$ at time t, we need to know s_d at time instants t_n taken in (at least) one infinite sequence converging from the left to t. As a consequence we have to make precise what we mean by trajectory. The most natural definition of trajectory is

Definition 3.1. Given (x, s_d, t_0) in $\mathbb{R}^n \times \mathscr{F} \times \mathbb{R}$ and $T > t_0$. A function (X, S_d) defined on $[t_0, T)$ is said to be a *classical trajectory* of

$$\dot{x} = f(x, \varphi(x, s_d, t)), \qquad s_d = k_d(x, s_d^-, t) \tag{13}$$

with initial condition (x, s_d) at time t_0 if:

1. X is absolutely continuous on $[t_0, T)$ and, for each t in $[t_0, T)$, there exists $\varepsilon > 0$ such that S_d is constant on $[t, t+\varepsilon)$.
2. For almost all t in $[t_0, T)$, we have
$$\dot{X}(t) = f(X(t), \varphi(X(t), S_d(t), t)),$$
and, for all t in (t_0, T), where S_d has a limit as s tends to t, $s < t$, we have
$$S_d(t) = k_d(X(t), S_d^-(t), t). \tag{14}$$
3. We have
$$(X(t_0), S_d(t_0)) = (x, s_d). \tag{15}$$

Note that by constant for S_d on $[t, t+\varepsilon)$, we mean S_d right-continuous at t because $\mathscr{F}$ is finite. In general, given (x, s_d), there may be several solutions admitting this point as the initial condition. However, there may be also none. Note that we do not ask for (14) to hold at $t = t_0$ and at all t such that $S_d^-(t)$ does not exist. In the following we denote by $\mathtt{sup}_J$ the bound of the function on J and by $\mathtt{esssup}_J$ the essential bound. Actually, we are interested in a notion of trajectories which is robust with respect to disturbances. For this reason, we introduce the notion of generalized trajectory (see also [H2], [H1], and [CR]).

Definition 3.2. Given (x, s_d, t_0) in $\mathbb{R}^n \times \mathscr{F} \times \mathbb{R}$ and $T > t_0$. A function (X, S_d) defined on $[t_0, T)$ is said to be a *weak generalized trajectory* (resp. a *strong generalized trajectory*) of (13) with initial condition (x, s_d) at time t_0, if X: $[t_0, T) \to \mathbb{R}^n$ is continuous, and with S_d: $[t_0, T) \to \mathscr{F}$, we have (15) and, for each $J = [\tau_0, \tau_1]$, a compact subinterval of $[t_0, T)$, and, for each n in $\mathbb{N}$, we can find three functions

e_n, a_n and d_n in $L^\infty_{\text{loc}}([t_0, T))$, a point (x_n, s_{dn}) in $\mathbb{R}^n \times \mathscr{F}$, and a classical trajectory (X_n, S_{dn}) starting from (x_n, s_{dn}) at time τ_0 of

$$\begin{aligned} \dot{x} &= f(x, \varphi(x + e_n(t), s_d, t) + a_n) + d_n(t), \\ s_d &= k_d(x + e_n(t), s_d^-, t) \end{aligned} \tag{16}$$

defined on a right open interval containing J and satisfying

$$\sup_J (X - X_n) + \sup_J (e_n) + \operatorname{esssup}_J (a_n) + \operatorname{esssup}_J (d_n) \underset{n\to\infty}{\longrightarrow} 0 \tag{17}$$

$$\left(\text{resp. } \sup_J (X - X_n) + \operatorname{esssup}_J (e_n) + \operatorname{esssup}_J (a_n) + \operatorname{esssup}_J (d_n) \underset{n\to\infty}{\longrightarrow} 0\right) \tag{18}$$

and such that, for all t in J, there exists N satisfying

$$S_{dn}(t) = S_d(t), \qquad \forall n \geq N. \tag{19}$$

In the above definition e_n plays the role of a measurement noise on x which disturbs the control computation, a_n is an actuator error and d_n is an external disturbance of the dynamics. A generalized trajectory is a limit, when the noise vanishes, of disturbed classical trajectories. (For other motivations for considering generalized trajectories, see pp. 164–165 of [H2].) As explained in Remark 1.4 of [LS1], with the presence of d_n we can omit any explicit reference on actuator errors because f is supposed to be locally Lipschitz. So in the following we suppose that in Definition 3.2, for all n in $\mathbb{N}$, $a_n \equiv 0$.

Of course a classical trajectory is a weak generalized trajectory and a weak generalized trajectory is a strong generalized trajectory. Moreover, the weak generalized trajectories are a generalization of Krasovskii trajectories introduced in Definition 2.4. Indeed, since f is assumed to be locally Lipschitz, by combining the technique of the proof of Proposition 1.4 of [CR] with Theorem 5.5 of [H1], we get

Lemma 3.3. *In the case without discrete dynamics* (*i.e. without* s_d) *and when* φ *is locally bounded, any Krasovskii trajectory of* $\dot{x} = f(x, \varphi(x))$ *is a weak generalized trajectory.*

Remark 3.4. By invoking Zorn's lemma exactly as in the proof of Proposition 1 of [R], we can prove that every generalized trajectory can be extended into a maximal generalized trajectory (X, S_d) defined on an interval $[t_0, T)$ with $T \leq +\infty$ (i.e. for which there exists no generalized trajectory on a $[t_0, T')$ with $T' > T$ and whose restriction is (X, S_d) on $[t_0, T)$).

In this context, if we denote a norm by $|\cdot|$, our definition of global asymptotic stability is

Definition 3.5. The origin is said to be a weakly (resp. strongly) globally asymptotically stable equilibrium of the system (13) if there exists a weak (resp. strong) generalized trajectory and the following properties hold for all t_0:

1. All the weak (resp. strong) maximal generalized trajectories are defined on $[t_0, +\infty)$.
2. There exists a function β of class $\mathscr{KL}$ such that each weak (resp. strong) generalized trajectory (X, S_d) satisfies, for all $t \geq t_0$,

$$|X(t)| \leq \beta(|X(t_0)|, t - t_0). \tag{20}$$

Remark 3.6. We observe, with Remark 2.4 and Proposition 2.5 of [LSW], that (20) is equivalent to the set of the following two properties:

1. There exists a class-$\mathscr{K}_\infty$ function α such that we have

$$|X(t)| \leq \alpha(|X(t_0)|), \qquad \forall t \geq t_0. \tag{21}$$

2. For any $r > 0$ and $\varepsilon > 0$, there exists $T > t_0$ such that

$$|X(t_0)| \leq r \quad \Rightarrow \quad |X(t)| \leq \varepsilon, \qquad \forall t \geq T. \tag{22}$$

In this paper we make the distinction of solving the uniting problem considering only the classical trajectories or by also taking into account the strong or weak generalized trajectories. Usually this distinction is not made since we have the following result proved in the Appendix.

Theorem 3.7. *In the case without discrete dynamics (i.e. without s_d) and when φ is continuous, each strong generalized trajectory is a classical trajectory.*

It follows that Theorems 2.1, 2.3 and 2.6 also give a result in terms of strong generalized trajectories.

The problem of global asymptotic stabilization for weak generalized trajectories has been considered per se in [LS1] and [LS2]. There are strong connections with the problem of uniting local and global controllers that we are considering here, both in the technicalities and the results. In particular, as we shall see, we also have in our context:

- the necessity of having a smooth Lyapunov function,
- the need for using an hybrid controller.

4. Dynamic Time-Invariant Controller with Hysteresis

4.1. A Solution to the Uniting Problem

A very natural way to overcome the difficulties encountered with static time-invariant continuous or discontinuous controllers is to introduce hysteresis, taking advantage of the existence of a region where both controllers u_l and u_g are appropriate.

Theorem 4.1. *Let $(u_l, u_g, \Omega, \mathscr{D})$ be the data of a uniting problem. There exists an appropriate bounded closed set A such that the controller below solves the uniting* *problem in terms of weak generalized trajectories*

$$u = \varphi(x, s_d), \qquad s_d = k_d(x, s_d^-), \tag{23}$$

where s_d is in $\{0,1\}$ and the functions φ and k_d satisfy

$$\begin{aligned} \varphi(x,0) &= u_l(x) \qquad \textit{if} \quad x \in \mathtt{clos}(\mathbb{R}^n \backslash C_g), \\ \varphi(x,1) &= u_g(x) \qquad \textit{if} \quad x \in \mathbb{R}^n, \end{aligned} \tag{24}$$

and

$$k_d(x, s_d) = \begin{cases} 0 & \textit{if} \quad x \in \mathtt{clos}(C_l), \\ s_d & \textit{if} \quad x \in \mathtt{int}(A), \\ 1 & \textit{if} \quad x \in \mathtt{clos}(C_g). \end{cases} \tag{25}$$

The proof of Theorem 4.1 is technical and requires us to set down some machinery to handle generalized trajectories. We postpone it until Section 6.

4.2. A Problem with Strong Generalized Trajectories

By the fact that, with strong generalized trajectories, noise with very large amplitude is allowed, Theorem 4.1 is not true. Precisely, for j in $\{l, g\}$ and t_0 in $\mathbb{R}$, let X_j be the solution of

$$\forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq t_0}, \qquad \frac{\partial X_j(x,t)}{\partial t} = f(X_j(x,t), u_j(X_j(x,t))), \tag{26}$$

$$\forall x \in \mathbb{R}^n, \qquad X_j(x, t_0) = x. \tag{27}$$

We have

Theorem 4.2. *Let A be the compact set and let (φ, k_d) be the controller given by Theorem* 4.1 *as a solution to the uniting problem in terms on weak generalized trajectories. If there exist a strictly positive real number ε and two compact sets E_l and E_g, subsets of A, such that, for all x in E_l (resp. E_g), there exists $\tau_{xl} \geq \varepsilon$ (resp. $\tau_{xg} \geq \varepsilon$) satisfying*

$$X_l(x, \tau_{xl}) \in E_g \qquad (resp.\ X_g(x, \tau_{xg}) \in E_l), \tag{28}$$

$$X_l(x,t)\ (resp.\ X_g(x,t)) \in \mathtt{int}(A), \qquad \forall t \in [0, \tau_{xl}] (resp.\ \forall t \in [0, \tau_{xg}]), \tag{29}$$

then $(A, (\varphi, k_d))$ does not solve the uniting problem in terms of strong generalized trajectories.

Proof. For k in $\mathbb{N}$ we denote $i_k = l$ if k is odd and $i_k = g$ if k is even.

Now to prove Theorem 4.2 we exhibit recursively a cyclic strong generalized trajectory (X, S_d) and noise e, d. Let $d \equiv 0$ on $\mathbb{R}_{\geq 0}$.

- Let $(x, 1)$ be an initial condition in $E_g \times \{0,1\}$. From (28), there exists a time $\tau_0 \geq \varepsilon$ such that $X_g(x, \tau_0)$ is in E_l. Let $t_1 = \tau_0$ and, for $t \in [0, t_1)$, we define

$$X(t) = X_g(x,t), \qquad S_d(t) = 1, \qquad e(t) = 0,$$

and, for $t = t_1$, we let

$$X(t_1) = X_g(x, t_1), \qquad S_d(t_1) = 0,$$

and we choose $e(t_1)$ such that $X(t_1) + e(t_1)$ is in C_l.

- Now assume that (X, S_d, e) has been defined on $[0, t_k]$ with $X(t_k) \in E_{i_k}$. We extend the definition of (X, S_d, e) to $[0, t_{k+1}]$ with $t_{k+1} > t_k$ as follows:

  There exists a time $\tau_k \geq \varepsilon$ such that $X_{i_k}(X(t_k), \tau_k)$ is in $E_{i_{k+1}}$. So, we let $t_{k+1} = t_k + \tau_k$ and, for $t \in [t_k, t_{k+1})$, we define

$$X(t) = X_{i_k}(X(t_k), (t - t_k)), \qquad S_d(t) = \begin{cases} 0 & \text{if} \quad i_k = l, \\ 1 & \text{if} \quad i_k = g, \end{cases} \qquad e(t) = 0,$$

  and, for $t = t_{k+1}$, we let

$$X(t_{k+1}) = X_{i_k}(X(t_k), \tau_k), \qquad S_d(t_{k+1}) = \begin{cases} 0 & \text{if} \quad i_k = g, \\ 1 & \text{if} \quad i_k = 1, \end{cases}$$

  and we choose $e(t_{k+1})$ such that $X(t_{k+1}) + e(t_{k+1})$ is in $C_{i_{k+1}}$.
- We go on this way defining ultimately (X, S_d, e) on $\mathbb{R}_{\geq 0}$.

By construction (X, S_d) is a classical trajectory of the disturbed equation:

$$\dot{x} = f(x, \varphi(x + e(t), s_d)) + d(t), \qquad s_d = k_d(x + e(t), s_d^-). \tag{30}$$

In particular, note that $X(t)$ remains in $\mathtt{int}(A)$ so that S_d can have switches only at times t_k. Also e is zero except at a countable number of times. This implies that e is in $L^\infty(\mathbb{R}_{\geq 0})$, and

$$\underset{\mathbb{R}_{\geq 0}}{\mathtt{esssup}}(d) + \underset{\mathbb{R}_{\geq 0}}{\mathtt{esssup}}(e) = 0.$$

So (X, S_d) is a strong generalized trajectory of the system (1) with the controller given by Theorem 4.1 which does not tend to the origin. ■

We illustrate Theorem 4.2 by considering the following system in $\mathbb{R}^2$:

$$(\dot{x}, \dot{y}) = u. \tag{31}$$

The feedback $u_l(x, y) = -(x, y)$ makes the closed-loop system globally asymptotically stable. Moreover, the following trajectory defined on $[0, +\infty)$ is a trajectory of the closed-loop system (31) with $u = u_l$:

$$\forall t \geq 0, \qquad (x(t), y(t)) = (2 \exp(-t), 0).$$

Let F be the closed set $F = \{(x, 0) \colon x \in [1, 2]\}$. Let s be a $\mathscr{C}^\infty$ function on $\mathbb{R}^2$ such that

$$s(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in F, \\ 0 & \text{if} \quad d((x, y), F) \geq \frac{1}{2}. \end{cases}$$

Let θ and u_g be the functions defined, for all $(x, y) \in \mathbb{R}^2$, by

$$\theta(x, y) = s(x, y)^\pi, \qquad u_g(x, y) = \mathscr{R}((x, y), \theta(x, y)) u_l(x, y),$$

where, for all (x, y) in $\mathbb{R}^2$ and for all θ in $[0, 2\pi]$, $\mathscr{R}((x, y), \theta)$ denotes the rotation with center (x, y) and angle θ. Namely, $u_g(x, y) = u_l(x, y)$ for (x, y) far enough of the segment F and $u_g(x, y) = -u_l(x, y)$ for (x, y) in the segment F.

Lemma 4.3. *u_g makes the origin of* (31) *globally asymptotically stable.*

Proof. There exist classical trajectories because u_g is continuous. Denoting B the unit ball in $\mathbb{R}^2$, for any radius $R > 3$ the ball RB is forward invariant and attractive by u_g. Moreover, in a neighborhood of the origin, u_g is equal to u_l. So we have Lyapunov stability. We now show the global attractivity of the origin. For any radius $R > 3$ the ball RB is forward invariant, so for any (x, y) in $\mathbb{R}^2$ the trajectory through (x, y) is bounded. In this case, Poincaré–Bendixon's theorem implies that any trajectory must tend either to the origin or to a periodic trajectory which must encircle the origin. However, this periodic trajectory does not exist since the set $\mathbb{R}_{\geq 0} \times \mathbb{R}$ is forward invariant by u_g. So we have attractivity. We conclude with Corollary 1 of [ABB]. ■

We now prove that the hypotheses of Theorem 4.2 are verified by taking $E_l = \{(2, 0)\}$ and $E_g = \{(1, 0)\}$. We note that, with u_g (resp. u_l), the trajectory with initial condition $(1, 0)$ (resp. $(2, 0)$) is

$$\forall t \in [0, \log(2)], \qquad (x(t), y(t)) = (\exp(t), 0) \qquad (\text{resp.} = (2\exp(-t), 0)).$$

So (28) and (29) hold. Hence, the controller given by (23) does not solve the uniting problem in terms of strong generalized trajectories for all closed sets $A \subset \mathbb{R}^2$ which separate $\mathbb{R}^2$ in two connected open sets C_l and C_g and such that we have F included in A.

5. Static Periodic Continuous Controller

Instead of enriching the class of controllers with nonsmooth components, we state here that it is sufficient to introduce time-dependence.

Theorem 5.1. *Let $(u_l, u_g, \Omega, \mathscr{D})$ be the data of the uniting problem. Suppose the existence of four bounded open connected sets Γ_j and Σ_j, for all j in $\{l, g\}$, such that*:

- $\Omega \subset \Gamma_l \subset \mathtt{clos}(\Gamma_l) \subset \Sigma_g \subset \mathtt{clos}(\Sigma_g) \subset \Sigma_l \subset \mathtt{clos}(\Sigma_l) \subset \Gamma_g \subset \mathscr{D}$.
- *For all j in $\{l, g\}$, $\mathtt{clos}(\Gamma_j)$ and $\mathtt{clos}(\Sigma_j)$ are forward invariant for the closed-loop system* (1) *obtained with $u = u_j$.*

Under these conditions, we can find an appropriate bounded closed set A and a continuous time-periodic function φ such that the controller $u = \varphi(x, t)$ solves the uniting problem in terms of strong generalized trajectories.

To prove Theorem 5.1 we use the technique of the "wave" between two controllers. This construction is quite similar to that in [SS], where a time-varying continuous controller for a one-dimensional system is exhibited.

Proof. *The set A*: Let $C_l = \Gamma_l$, $C_g = \mathtt{int}(\mathbb{R}^n \backslash \Gamma_g)$ and let finally $A = \mathbb{R}^n \backslash (C_l \cup C_g)$.
The controller: Let $u_-\colon \mathbb{R}^n \to \mathbb{R}^n$ be any continuous function such that

$$u_-(x) = \begin{cases} u_g(x) & \text{if} \quad x \in \mathbb{R}^n \backslash \Gamma_g, \\ u_l(x) & \text{if} \quad x \in \Sigma_l. \end{cases} \tag{32}$$

Similarly, let u_+ be any continuous function such that

$$u_+(x) = \begin{cases} u_g(x) & \text{if} \quad x \in \mathbb{R}^n \backslash \Sigma_g, \\ u_l(x) & \text{if} \quad x \in C_l. \end{cases} \tag{33}$$

Let τ_j be the real numbers defined as

$$\tau_g = \max_{\{x \in \Gamma_g \backslash \Sigma_g\}} \inf_{\{s\colon X_g(x,t) \in \Sigma_g, \forall t > s\}} \{s\}, \qquad \tau_l = \max_{\{x \in \Sigma_l \backslash C_l\}} \inf_{\{s\colon X_l(x,t) \in C_l, \forall t > s\}} \{s\},$$

where for j in $\{l, g\}$, X_j is defined by (26)–(27). These τ_j's are finite because u_l (resp. u_g) is asymptotically stabilizing the origin with the basin of attraction containing $\mathtt{clos}(\Sigma_l)$ (resp. $\mathtt{clos}(\Gamma_g)$) which is a compact set. Let $\mathscr{E}$ be the following compact set:

$$\mathscr{E} = \{(x, u)\colon x \in \mathtt{clos}(\Sigma_l \backslash \Sigma_g), u \in \overline{\mathrm{Conv}}(\{u_l(x), u_g(x)\})\}.$$

We define M and τ_c as follows:

$$M = \sup_{(x,u) \in \mathscr{E}} |f(x, u)|, \qquad \tau_c = \frac{\mathtt{dist}(\Sigma_g, \mathbb{R}^n \backslash \Sigma_l)}{2M}.$$

With these notations, we choose τ as any real number satisfying $\tau > (\tau_g + 2\tau_c + \tau_l)$. Let γ be a τ-periodic $\mathscr{C}^\infty$ function with value 1 on $[0, \tau_g]$ and on $[\tau_g + 2\tau_c + \tau_l, \tau]$, and 0 on $[\tau_g + \tau_c, \tau_g + \tau_c + \tau_l]$. We define the controller as

$$u = \varphi(x, t) = \gamma(t) u_+(x) + (1 - \gamma(t)) u_-(x).$$

It is continuous, τ-periodic and satisfies, for all t and for all j in $\{l, g\}$,

$$\varphi(x, t) = u_j(x), \qquad \forall x \in \mathtt{clos}(C_j). \tag{34}$$

Let $x \in \mathbb{R}^n$ and let X be a strong generalized trajectory starting from x at t_0 of

$$\dot{x} = f(x, \varphi(x, t)) \tag{35}$$

right maximally defined on $[t_0, T)$. This is a classical trajectory because of Theorem 3.7. We show that $T = +\infty$. Suppose not. Then $|X(x, t)|$ tends to $+\infty$ for t going to T. So there exists t_1 such that, for all $t \geq t_1$, $X(t) \in \mathbb{R}^n \backslash \Gamma_g$ and therefore $\varphi(X(t), t) = u_g(X(t))$. This shows that, for $t \geq t_1$, $X(t)$ is a trajectory of $\dot{x} = f(x, u_g(x))$ which tends to $+\infty$. This contradicts the hypothesis that u_g is globally stabilizing. So we must have $T = +\infty$.

Now, the fact that we have global asymptotic stability of the origin follows from Remark 3.6 and the following four points:

1. Let x be in $\mathtt{int}(C_g)$ and let X be a strong generalized trajectory of (35) starting from x. There exists t_1 in $(t_0, +\infty]$ such that X with values in $\mathtt{int}(C_g)$ is maximally defined on $[t_0, t_1)$. It follows that the restriction of X

on $[t_0, t_1]$ is a trajectory of

$$\dot{x} = f(x, u_g(x)). \tag{36}$$

However, u_g being globally asymptotically stabilizing, and $\texttt{clos}(\Gamma_g) = \mathbb{R}^n \backslash C_g$ being forward invariant and containing the origin, this set is stable and attractive. So in particular t_1 is finite and $X(t_1)$ is in $\texttt{clos}(\Gamma_g)$.

2. Let x be in $\texttt{clos}(\Gamma_g)$ and let X be a strong generalized trajectory of (35) starting from x. Then X cannot leave $\texttt{clos}(\Gamma_g)$ since this set is supposed to be forward invariant for $u = u_g$ and we have (34).
3. Let k be in $\mathbb{Z}$, let t_1 be in $[k\tau, (k+1)\tau)$ and let X be a strong generalized trajectory of (35) such that $x_1 = X(t_1)$ is in $\texttt{clos}(\Gamma_g)$. Let $t_2 = (k+1)\tau$. We have $X(t_2)$ in $\texttt{clos}(\Gamma_g)$, and:
   (a) For t in $[t_2, t_2 + \tau_g]$, γ is 1 so $\varphi(x,t) = u_+(x)$. It follows that there exists t_3 in $[t_2, t_2 + \tau_g]$ such that $X(t_3)$ is in $\texttt{clos}(\Sigma_g)$. Indeed suppose not. Then for t in $[t_2, t_2 + \tau_g]$, $\varphi(x,t) = u_g(x)$ and the restriction of X on $[t_2, t_2 + \tau_g]$ is a trajectory of (36). The definition of τ_g implies that $X(t_2 + \tau_g)$ is in $\texttt{clos}(\Sigma_g)$ which is a contradiction. So there exists t_3 in $[t_2, t_2 + \tau_g]$ such that $X(t_3)$ is in $\texttt{clos}(\Sigma_g)$. Then X cannot leave $\texttt{clos}(\Sigma_g)$ since this set is supposed to be forward invariant for $u = u_g$ and we have (33). So $X(t_2 + \tau_g)$ is in $\texttt{clos}(\Sigma_g)$.
   (b) For t in $[t_2 + \tau_g, t_2 + \tau_g + \tau_c]$, we have no specific properties given by the controller except that it is in $\overline{\text{Conv}}(\{u_l, u_g\})$. The definition of τ_c implies that $X(t_2 + \tau_g + \tau_c)$ is in Σ_l.
   (c) For t in $[t_2 + \tau_g + \tau_c, t_2 + \tau_g + \tau_c + \tau_l]$, γ is 0 so $\varphi(x,t) = u_-(x)$. Then X cannot leave $\texttt{clos}(\Sigma_l)$ between $t_2 + \tau_g + \tau_c$ and $t_2 + \tau_g + \tau_c + \tau_l$ because this set is supposed to be forward invariant for $u = u_l$ and we have (32). So for t in $[t_2 + \tau_g + \tau_c, t_2 + \tau_g + \tau_c + \tau_l]$, we have $\varphi(x,t) = u_l(x)$ and X is a trajectory of

$$\dot{x} = f(x, u_l(x)). \tag{37}$$

   The definition of τ_l implies that $X(t_2 + \tau_g + \tau_c + \tau_l)$ is in $\texttt{clos}(\Sigma_l)$.
4. Let x be in $\texttt{clos}(C_l)$ and let X be a generalized trajectory starting from x. Then X cannot leave $\texttt{clos}(C_l)$ since this set is supposed to be forward invariant for $u = u_l$ and we have (34). So X is a trajectory of (37). ■

6. Basic Properties of Weak Generalized Trajectories and Proof of Theorem 4.1

6.1. Basic Properties of Weak Generalized Trajectories Given by a Controller with Hysteresis

Understanding the behavior of weak generalized trajectories is of interest on its own. In this section we study some of the properties of such trajectories for the system

$$\dot{x} = f(x, \varphi(x, s_d)), \qquad s_d = k_d(x, s_d^-),$$

with φ and k_d satisfying (24) and (25) but without relying on our specific problem of uniting a local and a global controller. In particular, in this paragraph, we do not suppose that u_l and u_g are asymptotically stabilizing. We impose only that u_l and u_g are two continuous functions $\mathbb{R}^n \to \mathbb{R}^m$.

Definition 6.1. A function (X, S_d) defined on $[t_0, T)$ is said to have a switch at time $t \in [t_0, T)$ if S_d is not continuous at t.

We start by locating the points where a weak generalized trajectory may have a switch:

Theorem 6.2. *Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$ with a switch at time $t \in [t_0, T)$. Consider the sets*

$$S_d^p(t) = \{s \colon \exists t_n \in [t, T), t_n \xrightarrow[n\to\infty]{} t, S_d(t_n) \xrightarrow[n\to\infty]{} s\}, \tag{38}$$

$$S_d^m(t) = \{s \colon \exists t_n \in [t_0, t), t_n \xrightarrow[n\to\infty]{} t, S_d(t_n) \xrightarrow[n\to\infty]{} s\}. \tag{39}$$

- *If the switch is such that $1 \in S_d^m(t)$ and $0 \in S_d^p(t)$, then $X(t)$ is in ∂C_l,*
- *or if the switch is such that $0 \in S_d^m(t)$ and $1 \in S_d^p(t)$, then $X(t)$ is in ∂C_g.*

Proof. Suppose that, for $t \in [t_0, T)$, we have $1 \in S_d^m(t)$ and $0 \in S_d^p(t)$, i.e. for each $\varepsilon \in (t_0, T - t)$ there exists t' and $t'' \in [t', t' + \varepsilon]$ such that $S_d(t') = 1$, $S_d(t'') = 0$ and $t \in [t', t' + \varepsilon]$. Let J_ε be the compact interval $[t', t' + \varepsilon]$. There exists a sequence (X_n, S_{dn}) of classical trajectories such that, from (19), for n sufficiently large, we have $S_{dn}(t') = S_d(t')$ and $S_{dn}(t'') = S_d(t'')$. So, for n sufficiently large, each classical trajectory must have a switch at some time $t_n \in [t', t'']$ with $1 \in S_{dn}^m(t_n)$ and $0 \in S_{dn}^p(t_n)$. Due to (39), there exists a sequence (t_n^p), $t_n^p < t_n$, which tends to t_n as p tends to infinity and such that $S_{dn}(t_n^p) = 1$. Therefore, due to (38), there exists a maximal ε_n^p such that, for all s in $[t_n^p, t_n^p + \varepsilon_n^p)$, we have $S_{dn}(s) = 1$ and such that $S_d(\varepsilon_n^p + t_n^p) = 0$. This implies with (14) that

$$0 = S_{dn}(\varepsilon_n^p + t_n^p) = k_d(X_n(t_n^p + \varepsilon_n^p) + e_n(t_n^p + \varepsilon_n^p), 1).$$

So, from the definition of k_d, we have $X_n(t_n^p + \varepsilon_n^p) + e_n(t_n^p + \varepsilon_n^p)$ in $\texttt{clos}(C_l)$ and $X_n(s) + e_n(s)$ in $\texttt{int}(A \cup C_g)$, for all s in $[t_n^p, t_n^p + \varepsilon_n^p)$. By taking the limit as p and n tend to the infinity, we get $X(t) \in \partial C_l$.

The other case is established in exactly the same way. ■

The switches cannot happen too often:

Theorem 6.3. *There exists a strictly positive number η such that, for every weak generalized trajectory and for every time t, there exists at most one switch in the interval $(t, t + \eta)$.*

Before proving Theorem 6.3, we establish a similar result for disturbed classical trajectories:

Lemma 6.4. *For any $D > 0$, there exist strictly positive numbers ε and η such that, for any disturbed classical trajectory with $\mathtt{sup}(e) \le \varepsilon$ and $\mathtt{esssup}(d) \le D$, there is at most one switch in $(t, t+\eta)$.*

Proof. Let Σ_l and Σ_g be two open sets such that we have $\partial C_l \subset \Sigma_l$, $\partial C_g \subset \Sigma_g$ and $d(\Sigma_l, \Sigma_g) > 0$. Let

$$M = \sup_{x \in \Sigma_g \cup A \cup \Sigma_l, s \in \{l,g\}, d \in D.\mathtt{clos}(B)} |f(x, u_s(x)) + d| < +\infty. \tag{40}$$

Let $\varepsilon > 0$ be such that, for all j in $\{l, g\}$,

$$\{x \in \partial C_j, |e| \le \varepsilon\} \Rightarrow x + e \in \Sigma_j. \tag{41}$$

Let (X, S_d) be a disturbed classical trajectory, defined on $[t_0, T)$, of

$$\dot{x} = f(x, \varphi(x + e(t), s_d)) + d(t), \qquad s_d = k_d(x + e(t), s_d^-),$$

with $\mathtt{sup}_{[t_0,T)}(e) \le \varepsilon$ and $\mathtt{esssup}_{[t_0,T)}(d) \le D$. Since S_d is by definition constant on $[t, t+\varepsilon)$ for some $\varepsilon > 0$, we have $S_d^p(t) = \{S_d(t)\}$ and, in each compact sub-interval of $[t_0, T)$, we can have at most a finite number of switches. So, suppose this trajectory has two consecutive switches, at times $t_1 < t_2$ in $[t_0, T)$. We must have

$$0 \in S_d^m(t_1), \qquad S_d^p(t_1) = \{1\}, \qquad X(t_1) + e(t_1) \in \partial C_g$$
$$(\text{resp.} \quad 1 \in S_d^m(t_1), \qquad S_d^p(t_1) = \{0\}, \qquad X(t_1) + e(t_1) \in \partial C_l),$$
$$1 \in S_d^m(t_2), \qquad S_d^p(t_2) = \{0\}, \qquad X(t_2) + e(t_2) \in \partial C_l$$
$$(\text{resp.} \quad 0 \in S_d(t_2), \qquad S_d^p(t_2) = \{1\}, \qquad X(t_2) + e(t_2) \in \partial C_g).$$

Note that, with (41), we also have $X(t_1) \in \Sigma_g$ (resp. $X(t_1) \in \Sigma_l$) and $X(t_2) \in \Sigma_l$ (resp. $X(t_2) \in \Sigma_g$). The continuity of X implies the existence of $\tilde{t}_1$ and $\tilde{t}_2$ such that we have the following three properties (resp. the other cases):

$$t_1 \le \tilde{t}_1 < \tilde{t}_2 \le t_2, \qquad X(\tilde{t}_1) \in \Sigma_g, \qquad X(\tilde{t}_2) \in \Sigma_l, \tag{42}$$
$$\forall t \in [\tilde{t}_1, \tilde{t}_2], \qquad (X(t), X(t) + e(t)) \in (\Sigma_l \cup A \cup \Sigma_g) \times (\Sigma_l \cup A \cup \Sigma_g).$$

Then the definition of M and φ implies

$$d(\Sigma_g, \Sigma_l) \le |X(\tilde{t}_1) - X(\tilde{t}_2)| \le M(\tilde{t}_2 - \tilde{t}_1) \le M(t_2 - t_1).$$

So, by letting $\eta = (d(\Sigma_g, \Sigma_l))/M$, we get that any two consecutive times of switch t_1 and t_2 must satisfy $t_2 - t_1 \ge \eta$. ∎

Proof of Theorem 6.3. Let $D = 1$ and let ε and η be given by Lemma 6.4. Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$. For the sake of getting a contradiction, suppose it has two switches at times t_1 and $t_2 < t_3$, with $t_3 = \min\{t_1 + \eta/4, T\}$. Let t_4 in $[\max(t_0, t_1 - \eta/4), t_1]$ be such that S_d has two switches on $[t_4, t_3]$. Let $J = [t_4, t_3]$. Let (X_n, S_{dn}) be a sequence of classical disturbed trajectories which tends to (X, S_d) on J, with $\mathtt{sup}_J(e_n) \le \varepsilon$ and with $\mathtt{esssup}_J(d_n) \le D$. So, as in the proof of Theorem 6.2, for n sufficiently large, the classical disturbed trajectories must also have a first switch at some time larger than or equal to t_4 and another one in the neighborhood of t_2. This contradicts Lemma 6.4. ■

Between two consecutive switches, a weak generalized trajectory is a classical trajectory. Precisely, we have the following:

Theorem 6.5. *Let t_0 be in $\mathbb{R}$ and let (X, S_d) be a weak generalized trajectory such that $S_d \equiv 0$ (resp. 1) on $[t_0, T)$ with $T > t_0$. Then X is an undisturbed classical trajectory of $\dot{X} = f(X, u_l(X))$ (resp. $\dot{X} = f(X, u_g(X))$) on $[t_0, T)$.*

Proof. Let $T' \in (t_0, T)$. Let (X_n, S_{dn}) be a sequence of disturbed classical trajectories which tends to (X, S_d) on $[t_0, T']$. Let $t_n \in [t_0, T']$ be defined by

$$t_n = \sup\{t \le T' \colon S_{dn}(s) \equiv 0, \forall s \in [t_0, t]\}.$$

Since $S_d(t_0) = 0$, (19) implies, for n sufficiently large, $S_{dn}(t_0) = 0$ and therefore that t_n exists. Since S_{dn} is constant on $[t, t + \varepsilon)$, for all t in $[t_0, T)$, we have either $t_n = T'$ or $S_{dn}(t_n) = 1$. We can find a subsequence such that $t_n \to \tilde{t} \in [t_0, T']$. We show that $\tilde{t} = T'$. Suppose the contrary, i.e. $\tilde{t} < T'$. Then for n sufficiently large we have $t_n < T'$, so $S_{dn}(t_n) = 1$. Then, due to Lemma 6.4, $S_{dn} \equiv 1$ on $[t_n, \min(t_n + \frac{1}{2}\eta, T'))$. By taking the limit when n goes to infinity, $S_d \equiv 1$ on $(\tilde{t}, \min(\tilde{t} + \frac{1}{2}\eta, T'))$, which is a nonempty set. This contradicts the assumption of Theorem 6.5. So we have $S_{dn} \equiv 0$ on $[t_0, t_n)$ with $t_n \to T'$.

Let $s < t$ in $[t_0, T')$. With a similar argument of the proof of Theorem 3.7, we prove that

$$X(t) - X(s) = \int_s^t f(X(\tau), u_l(X(\tau)))\, d\tau.$$

This proves that X is a classical trajectory of (37) on $[t_0, T']$, for all $T' \in (t_0, T)$. So X is a classical trajectory of (37) on $[t_0, T)$. ■

With the properties we know now for the weak generalized trajectories, we can write exactly when such trajectories exist.

Theorem 6.6. *Let t_0 be in $\mathbb{R}$. There exists a weak generalized trajectory starting from (x_0, s_0) in $\mathbb{R}^n \times \{0, 1\}$ and defined on $[t_0, T)$ for some $T > t_0$ if and only if we have one of the following configurations:*

- *x_0 in C_l and $s_0 = 0$.*
- *x_0 in A and s_0 in $\{0, 1\}$.*
- *x_0 in C_g and $s_0 = 1$.*

Proof.

- Let x_0 be in C_l and $s_0 = 1$. There is no classical (slightly) disturbed trajectory starting from $(x_0, 1)$. Indeed, for such a trajectory, for all sufficiently small nonnegative times t, we must have S_d constant and X in C_l (neighborhood of x_0). However, for all x in a neighborhood of x_0 and all e sufficiently small,

(25) yields $s_d = k_d(x+e, s_d^-) = 0$. So S_d must be 0. As a consequence, we cannot find $\tau > t_0$ and a sequence of classical disturbed trajectories (X_n, S_{dn}) defined at least on $[t_0, \tau]$ and with noise e_n, d_n such that $\mathtt{sup}_{[t_0,\tau]}(e_n) + \mathtt{esssup}_{[t_0,\tau]}(d_n)$ goes to 0, $X_n(t_0)$ goes to x_0 and $S_{dn}(t_0)$ goes to 1. So there is no weak generalized trajectory starting from $(x_0, 1)$ when x_0 is in C_l.

- Let x_0 be in $\mathtt{clos}(C_l)$ and $s_0 = 0$. Then, due to the continuity of f and $\varphi(\cdot, 0)$, there exists a C^1 function Y: $[t_0, T) \to \mathtt{int}(\mathbb{R}^n \backslash C_g)$ with $T > t_0$, satisfying

$$\dot{Y} = f(Y(t), \varphi(Y(t), 0)), \qquad Y(t_0) = x_0.$$

By letting

$$X(t) = Y(t), \qquad S_d(t) = 0,$$

for $t \in [t_0, T)$, we get an (undisturbed) classical trajectory starting from $(x_0, 0)$. Such a trajectory is also a weak generalized trajectory.

- Let x_0 in ∂C_l and $s_0 = 1$. There exists a sequence $\{x_n\}$ of points in $\mathtt{int}(A)$ which tends to x_0 and satisfying

$$d(x_n, \partial C_g) \geq \Delta > 0.$$

With M defined in (40) and η given by Lemma 6.4, let τ be equal to $\min\{\frac{1}{2}\eta, \Delta/2M\}$. Now, for each n, there exists a C^1 function Y_n with values in $\mathtt{int}(A)$, right maximally defined on $[0, T_n)$ and satisfying

$$\dot{Y}_n = f(Y_n(t), \varphi(Y_n(t), 1)), \qquad Y_n(0) = x_n.$$

Then for n sufficiently large, we can build an undisturbed classical trajectory (X_n, S_{dn}) defined on $[t_0, t_0 + \tau)$ with initial condition $(x_n, 1)$. Indeed

– either $T_n > \tau$, then by letting

$$X_n(t) = Y(t - t_0), \qquad S_{dn}(t) = 1,$$

for $t \in [t_0, T_n + t_0)$, we get an undisturbed classical trajectory, defined on $[t_0, t_0 + \tau)$ starting from $(x_n, 1)$,

– or $T_n \leq \tau$, then, when t tends to T_n, $Y_n(t)$ tends to z_n in ∂A and, more precisely in ∂C_l, since $T_n < \Delta/M$. In this case, there exists a C^1 function Z_n with values in $\mathtt{int}(\mathbb{R}^n \backslash C_g)$, right maximally defined on $[0, T_n')$ and satisfying

$$\dot{Z}_n = f(Z_n(t), \varphi(Z_n(t), 0)), \qquad Z_n(0) = z_n.$$

Since $\mathbb{R}^n \backslash C_g$ is a compact set and its boundary is ∂C_g, the argument used in the proof of Lemma 6.4 shows that we must have $T_n' \geq \frac{1}{2}\eta \geq \tau$. Then by letting

$$X_n(t) = Y_n(t - t_0), \qquad S_{dn}(t) = 1,$$

for $t \in [t_0, t_0 + T_n)$, and

$$X_n(t) = Z_n(t - T_n - t_0), \qquad S_{dn}(t) = 0,$$

for $t \in [t_0 + T_n, t_0 + T_n' + T_n)$, we get an undisturbed classical trajectory, defined on $[t_0, t_0 + \tau)$ and starting from $(x_n, 1)$.

The sequence X_n takes values in the compact set $\mathbb{R}^n \backslash C_g$ and is equicontinuous. So due to Ascoli's theorem, we can extract a subsequence converging as n goes to ∞ to a continuous function Y, uniformly on $[t_0, (t_0 + \tau)/2]$. Then, in the subsequence, either one or the other of the following occurs:

- There is subsubsequence such that $S_{dn}(t) = 1$ for all t in $[t_0, (t_0 + \tau)/2]$. In this case, by letting

$$X(t) = Y(t), \qquad S_d(t) = 1,$$

for $t \in [t_0, (t_0 + \tau)/2)$, we get a weak generalized trajectory starting from $(x_0, 1)$,

- From the construction of S_{dn}, for all n sufficiently large, there exists $\varepsilon_n \in (t_0, (t_0 + \tau)/2]$ such that $S_{dn}(t) = 1$ for t in $[t_0, \varepsilon_n)$ and 0 for t in $[\varepsilon_n, (t_0 + \tau)/2]$. There exists a subsubsequence such that ε_n tends monotonically to $\varepsilon \in [t_0, (t_0 + \tau)/2]$. In this case, by letting

$$X(t) = Y(t), \qquad S_d(t) = \begin{cases} 1 & \text{if} \quad t < \varepsilon, \\ 1 & \text{if} \quad t = \varepsilon \quad \text{and} \quad \varepsilon_n > \varepsilon, \qquad \forall n, \\ 0 & \text{if} \quad t = \varepsilon \quad \text{and} \quad \varepsilon \geq \varepsilon_n, \qquad \forall n, \\ 0 & \text{if} \quad \varepsilon < t, \end{cases}$$

for $t \in [t_0, (t_0 + \tau)/2)$, we get a weak generalized trajectory starting from $(x_0, 1)$. Indeed, in particular, note that we have

$$\varepsilon_n > \varepsilon \quad \Rightarrow \quad S_{dn}(\varepsilon) = 1; \qquad \varepsilon \geq \varepsilon_n \quad \Rightarrow \quad S_{dn}(\varepsilon) = 0.$$

- Let x_0 be in $\texttt{int}(A)$, and s_0 in $\{0, 1\}$. Then, as in the previous cases, we can construct an undisturbed classical trajectory. So there exists a weak generalized trajectory.
- The other cases:
  - x_0 in C_g and $s_0 = 0$ is studied similarly to the case x_0 in C_l and $s_0 = 1$,
  - x_0 in $\texttt{clos}(C_g)$ and $s_0 = 1$ is studied similarly to the case x_0 in $\texttt{clos}(C_l)$ and $s_0 = 0$,
  - x_0 in ∂C_g and $s_0 = 0$ is studied similarly to the case x_0 in ∂C_l and $s_0 = 1$. ∎

With this theorem of existence at hand, we remark:

Remark 6.7. If (X, S_d) is a weak generalized trajectory defined on $[t_0, T)$ starting from (x_0, s_0), then, for any s in $[t_0, T)$, $(\tilde{X} = X(\cdot + s), \tilde{S}_d = S_d(\cdot + s))$ is a weak generalized trajectory defined on $[0, T - s)$, starting from $(X(s), S_d(s))$.

We conclude our general study of weak generalized trajectories by noting that, as classical trajectories, weak generalized trajectories can be maximally extended (see Remark 3.4) and such an extension must blow up if its domain of definition is bounded:

Theorem 6.8. *Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$ with $T < +\infty$. Then we have $\lim_{t \to T} |X(t)| = +\infty$.*

Proof. We start with a claim:

Claim 6.9. *If, for a weak generalized trajectory* (X, S_d), *we have that* $X(t)$ *is in a given compact set, for all* t *in* $[t_2, t_2]$, *then there exists* ξ *such that, for all* (s, t) *in* $[t_1, t_2]$, *we have*

$$|X(s) - X(t)| \leq \xi|s - t|. \tag{43}$$

Claim 6.9 is a direct consequence of the fact that this holds for the disturbed classical trajectories tending to (X, S_d) on $[t_1, t_2]$, since $f(x, \varphi(x + e, s_d))$ is bounded in this case.

Suppose the conclusion of Theorem 6.8 does not hold, i.e. there exists K a compact set of $\mathbb{R}^n$ and times t_n in $[t_0, T)$ tending monotonically to $T < +\infty$ such that $(X(t_n), S_d(t_n))$ is in $K \times \{0, 1\}$ for all n. We show the following:

Claim 6.10. *If* $X(t_n)$ *is in* K *for all* n, *with* t_n *converging to* T, *then* $(X(t_n), S_d(t_n))$ *has a limit* (x_0, s_0^-) *when* t *tends to* T.

Proof of Claim 6.10. We show first that, for n sufficiently large, $X(t)$ is in the bounded open set $K + B$ for all $t \in [t_n, T)$. Indeed if this is not true, the continuity of X implies the existence of $s_n \in (t_n, T)$ such that

$$|X(t_n) - X(s_n)| = 1 \quad \text{and} \quad |X(t_n) - X(t)| < 1, \qquad \forall t \in [t_n, s_n).$$

It follows that $X(t)$ is in the compact set $K + \mathtt{clos}(B)$, for all t in $[t_n, s_n]$. So, from Claim 6.9, there exists ξ such that

$$1 = |X(t_n) - X(s_n)| \leq \xi|s_n - t_n| \leq \xi|T - t_n|.$$

This cannot hold since t_n converges to t. So, for n sufficiently large, $X(t)$ in $K + B$ for all t in $[t_n, T)$. From Claim 6.9 this implies that there exists ξ such that, for all (s, t) in $[t_n, T)$, we have (43). It follows (by invoking the Cauchy criterion) that $X(t)$ has a limit x_0 when t tends to T. Finally to conclude the proof of Claim 6.10 we invoke Theorem 6.3. It guarantees the existence of $\sigma < \eta$ such that there is no switch in $[T - \sigma, T)$. This implies that S_d is constant on $[T - \sigma, T)$ and so $S_d(t)$ has a limit, denoted s_0^- when t tends to T. ∎

From this point we want to show that the weak generalized trajectory can be extended beyond T. We do this by extending the approximating disturbed classical generalized trajectories:

Extension of the Approximating Disturbed Classical Generalized Trajectories to $[t_0, T + \rho/(4M))$. Let $X(T) = x_0$, $K = X([t_0, T])$ and

$$M = \sup_{x \in K + \mathtt{clos}(B), s \in \{l, g\}} |f(x, u_s(x))| < +\infty. \tag{44}$$

Note that we have, for all t in $[t_0, T]$, $|X(t) - x_0| \leq M(T - t)$. Let s_0 be equal to $k_d(x_0, s_0^-)$. Note that, from (25), this definition of s_0 implies the existence of

$\rho \in (0, 1]$ such that we have

$$s_0 = k_d(x, s_0), \qquad \forall x \in \{x_0\} + \rho B.$$

From Theorem 6.3, we know that the weak generalized trajectory has a finite number of switches occurring at times s_i in $[t_0, T - \sigma]$ with σ introduced in the proof of Claim 6.10. Then let

$$\mathscr{T}_n = \{s_i\} \cup \left\{ t_0, \max\left(t_0, \frac{1}{n}\left(T - \frac{1}{n}\right)\right), \max\left(t_0, \frac{2}{n}\left(T - \frac{1}{n}\right)\right), \dots, \max\left(t_0, \left(T - \frac{1}{n}\right)\right)\right\}. \tag{45}$$

Now from the definition of a weak generalized trajectory, (19), Lemma 6.4 and the convergence of $X(t)$ to x_0, for n sufficiently large, there exists a disturbed classical trajectory (X_n, S_{dn}) defined at least on $[t_0, T - 1/n]$ and such that

$$\left| X_n\left(T - \frac{1}{n}\right) - x_0 \right| \le \frac{\rho}{2}, \tag{46}$$

$$\sup_{[t_0, T-1/n]} (X - X_n) + \sup_{[t_0, T-1/n]} (e_n) + \operatorname{esssup}_{[t_0, T-1/n]}(d_n) \le \frac{1}{n}, \tag{47}$$

$$S_{dn}(\tau) = S_d(\tau), \qquad \forall \tau \in \mathscr{T}_n, \tag{48}$$

$$S_{dn}(t) = s_0^-, \qquad \forall t \in \left[T - \sigma, T - \frac{1}{n}\right]. \tag{49}$$

The latter following from the fact that we cannot have two switches in $[T - \sigma, T - 1/n]$. Then let Y_n with values in $\{x_0\} + \rho B$ be a trajectory of

$$\dot{Y}_n = f(Y_n, \varphi(x_0, s_0)), \qquad Y_n(0) = X_n\left(T - \frac{1}{n}\right),$$

right maximally defined on $[0, T_n)$. So $|Y_n(T_n) - x_0| = \rho$. Using (44), (46) and (47) we have

$$\rho \le |Y_n(T_n) - Y_n(0)| + \left| X_n\left(T - \frac{1}{n}\right) - X\left(T - \frac{1}{n}\right)\right| + \left| X\left(T - \frac{1}{n}\right) - x_0 \right|$$
$$\le MT_n + \frac{1}{n} + \frac{\rho}{2}.$$

Therefore, with n large enough, $T_n > \rho/(2M)$. This proves that Y_n is defined on $[0, \rho/(2M)]$. Now, we define a function (X_n', S_{dn}', e_n') on $[t_0, T + \rho/(2M) - 1/n]$ by letting, for t in $[t_0, T - 1/n)$,

$$X_n'(t) = X_n(t), \qquad S_{dn}'(t) = S_{dn}(t), \qquad e_n'(t) = e_n(t), \qquad d_n'(t) = d_n(t),$$

and, for t in $[T - 1/n, T + \rho/(2M) - 1/n]$,

$$X_n'(t) = Y_n\left(t - T + \frac{1}{n}\right), \qquad S_{dn}'(t) = s_0, \qquad e_n'(t) = 0, \qquad d_n'(t) = 0.$$

This function gives rise to a disturbed classical trajectory on $[t_0, T+\rho/(2M)-1/n]$. This is clear for the interval $[t_0, T-1/n)$. For the remainder of the interval, this follows from the equalities:

$$f(X_n'(t), \varphi(X_n'(t)+e_n'(t), S_{dn}'(t))) = f\left(X_n'(t), \varphi\left(Y_n\left(t-T+\frac{1}{n}\right), s_0\right)\right) = f(X_n'(t), s_0),$$

$$S_{dn}'(t) = s_0 = k_d(x_0, s_0) = k_d(X_n'(t)+e_n'(t), S_{dn}'^{-}(t)),$$

for t in $(T-1/n, T+\rho/(2M)-1/n]$ and, with (49),

$$S_{dn}'\left(T-\frac{1}{n}\right) = s_0 = k_d(x_0, s_0^-) = k_d\left(X_n'\left(T-\frac{1}{n}\right)+e_n'\left(T-\frac{1}{n}\right), S_{dn}'^{-}\left(T-\frac{1}{n}\right)\right).$$

Now, on the interval $[T, T+\rho/(4M)]$, the sequence of functions $X_n'(t)$ takes values on $\{x_0\}+\rho B$ so is bounded and equicontinuous. Then there exists a subsequence such that X_n' converges uniformly to a continuous function Y. We define a function (X', S_d') on $[t_0, T+\rho/(4M)]$ by letting, for t in $[t_0, T)$,

$$X'(t) = X(t), \qquad S_d'(t) = S_d(t),$$

and, for t in $[T, T+\rho/(4M)]$,

$$X'(t) = Y(t), \qquad S_d'(t) = s_0.$$

We prove that the above subsequences provide an appropriate approximating sequence of disturbed classical trajectories converging to (X', S_d') in the sense of Definition 3.2:

1. We have that X_n' converge uniformly to X on $[t_0, T+\rho/(4M)]$. Indeed:
   - On $[T, T+\rho/4M]$ this follows from the convergence of X_n' to Y.
   - On $[t_0, T-1/n]$ this follows from the convergence of X_n' to X.
   - Finally, for t in $[T-1/n, T]$, we have

$$X_n'(t) - X'(t) = X_n'\left(T-\frac{1}{n}\right) - X\left(T-\frac{1}{n}\right) + X\left(T-\frac{1}{n}\right) - X(t) + \int_{T-1/n}^{t} f(X_n'(s), \varphi(x_0, s_0)). \tag{50}$$

   This implies, for t in $[T-1/n, T]$,

$$|X_n'(t) - X(t)| \le \sup_{[t_0, T-1/n]} (X_n' - X) + M\left(t-T+\frac{1}{n}\right) + M\left(t-T+\frac{1}{n}\right).$$

2. On the interval $[0, \rho/(4M)]$, the sequence $S_{dn}'(t+T)$ is constant and equal to s_0 as the function $S_d'(t+T)$. Also pick t in $[t_0, T]$.
   - Either t is not a time s_i of a switch of the weak generalized trajectory, i.e. S_d is continuous at t, and is constant on some neighborhood of t. This implies that for all n sufficiently large, we can find m such that
     (1) t is in the interval $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,
     (2) S_d is constant on $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,

(3) using Lemma 6.4, S_{dn} has at most one switch in $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,
(4) and, using (48),

$$S_{dn}\left(\frac{m}{n}\left(T-\frac{1}{n}\right)\right) = S_d\left(\frac{m}{n}\left(T-\frac{1}{n}\right)\right),$$

$$S_{dn}\left(\frac{m+1}{n}\left(T-\frac{1}{n}\right)\right) = S_d\left(\frac{m+1}{n}\left(T-\frac{1}{n}\right)\right).$$

This implies that S_{dn} is constant on $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$ and therefore $S_{dn}(t) = S_d(t)$.

- Or $t = s_i$, for some i. Then, from (48) and the definition of $\mathscr{T}_n$ in (45), we have, for n sufficiently large, $S_{dn}(t) = S_d(t)$.

3. We have that (47) implies that e'_n and d'_n converge uniformly to zero on the interval $[t_0, T + \rho/(4M)]$.

So (X', S'_d) is a weak generalized trajectory on $[t_0, T+\rho/(4M))$ and its restriction to $[t_0, T)$ is (X, S_d). So this contradicts the fact that (X, S_d) is a maximal weak generalized trajectory. ■

6.2. *Proof of Theorem 4.1*

The Set A. Since the origin is asymptotically stable for (37) with domain of attraction $\mathscr{D}$ and $f(\cdot, u_l(\cdot))$ is continuous, there exists a C^∞ function V which is positive definite and proper on $\mathscr{D}$ such that, for all x in $\mathscr{D}\backslash\{0\}$, we have

$$\frac{\partial V}{\partial x}(x) f(x, u_l(x)) < 0. \tag{51}$$

Then, since $\mathtt{clos}(\Omega)$ is a subset of $\mathscr{D}$, there exists c_l such that $\Omega \subsetneq C_l := \{x \in \mathscr{D}\colon V(x) < c_l\}$. Let c_g, C_g and A be defined by

$$c_g = 2c_l, \qquad C_g = \{x\colon V(x) > c_g \text{ or } x \notin \mathscr{D}\}, \qquad A = \mathbb{R}^n \backslash (C_l \cup C_g). \tag{52}$$

Maximality of Trajectories. Let $(x, s_d) \in \mathbb{R}^n \times \{0, 1\}$ be any point so that there exists a weak generalized trajectory, maximally defined on $[t_0, T)$ (see Theorem 6.6). We show that, for each such trajectory, we have $T = +\infty$. Suppose not. Then, with Theorem 6.8, we know that $|X(t)|$ tends to $+\infty$ when t goes to T. So there exists t_1 such that, for all $t \geq t_1$, $X(t) \in C_g$ and in particular $X(t) \notin \partial C_g \cup \partial C_l$. With Theorem 6.2, we know that, in this case, this trajectory cannot have switches in $[t_1, T)$. So S_d is constant on $[t_1, T)$ and, by using an approximating sequence of disturbed classical trajectories, with (25), we can claim that its value is 1. So, with Theorem 6.5, X restricted to $[t_1, T)$ is a classical trajectory of the closed-loop system with the global controller $u = u_g$ which tends to $+\infty$. This is a contradiction of the fact that u_g is a global asymptotic stabilizer. So we must have $T = +\infty$.

Now in order to prove the global asymptotic stability of the origin, we first establish

Claim 6.11. *For any weak generalized trajectory, only two cases are possible*:

1. *There exists no switch and the generalized trajectory is a classical trajectory of* (37) *on* $[t_0, +\infty)$ *and remains in the set* $A \cup C_l$.
2. *There exists at least one switch at a nonnegative time* σ *and the generalized trajectory is such that*:
   - $X(\sigma)$ *is in* ∂C_l.
   - *For all* t *in* (t_0, σ), X *is a classical trajectory of* (36) *and it is not in* C_l.
   - *For all* t *in* $[\sigma, +\infty)$, X *is a classical trajectory of* (37).

Proof of Claim 6.11. We consider five cases depending on the location of the initial condition:

1. Let $x \in \mathtt{int}(A) \cup \mathtt{clos}(C_l)$ and $s_d = 0$. From Theorem 6.6 and the above, there exists (X, S_d), a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d). We show that it has no switch. Indeed, if not, since the possible number of switches is finite on each compact interval of time (see Theorem 6.3), there exists a time τ for a first switch. At this time, S_d must change from 0 to 1. However, since $x_0 \notin \partial C_g$ and $s_d = 0$, Theorem 6.2 implies that τ cannot be 0 and that $X(\tau)$ is in ∂C_g. On the other hand, Theorem 6.5 says that X restricted to $[t_0, \tau)$ is a classical trajectory of (37). Then, from (51), we have, for all t in $[t_0, \tau)$,
$$V(X(t)) \leq V(x) < c_g. \tag{53}$$
So, for t going to τ, we get $V(X(\tau)) < c_g$ which contradicts the fact that $X(\tau)$ is in ∂C_g, from the definition of C_g (52). Consequently, there is no switch, $S_d \equiv 0$ on $[t_0, +\infty)$ and X is a classical trajectory of (37) on $[t_0, +\infty)$ and remains in $A \cup C_l$.
2. Let $x \in C_l$ and $s_d = 1$. There is no weak generalized trajectory (see Theorem 6.6).
3. Let $x \in C_g$ and $s_d = 0$. There is no weak generalized trajectory (see Theorem 6.6).
4. Let $x \in \partial C_g$ and $s_d = 0$. From Theorem 6.6 and the above, there exists (X, S_d), a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d).
   (a) If there is no switch at $t = t_0$, there exists $t_1 > t_0$ such that $S_d(t) = 0$ for all t in $[t_0, t_1)$. It follows that the restriction of X to $[t_0, t_1)$ is a classical trajectory of (37) and satisfies, for t in (t_0, t_1), (53). So, by Remark 6.7, the restriction of (X, S_d) to $[(t_0 + t_1)/2, +\infty)$ is one of the weak generalized trajectories studied in Case 1 above.
   (b) If there is a switch at $t = t_0$, then there is another one. Indeed, if not, for all $t_1 > t_0$, the restriction of X to $[t_1, +\infty)$ is a classical trajectory of (36) and the restriction of S_d to $[t_1, +\infty)$ is the constant 1. However, in this case, since u_g is globally asymptotically stabilizing, there exists s such that $X(t)$ is in C_l for all $t \geq s$. So, by Remark 6.7, the restriction of

(X, S_d) to $[s, +\infty)$ is a weak generalized trajectory with initial condition $X(s)$ in C_l and $S_d(s) = 1$, which is impossible in view of Theorem 6.6. So there is another switch. Let σ be the first one. We must have $1 \in S_d^m(\sigma)$ and $0 \in S_d^p(\sigma)$, and, from Theorem 6.2, $X(\sigma)$ is in ∂C_l. Then:

- Since, for each $t_1 \in (t_0, \sigma)$, the restriction of S_d to $[t_1, \sigma)$ is the constant 1, it follows from Case 2 and the continuity of X that, for all t in $[t_1, \sigma)$, X is not in C_l.
- By the continuity of X and the definition of S_d^p, there exists t_2, close to σ, such that $X(t_2)$ is in $\mathtt{int}(A) \cup \mathtt{clos}(C_l)$ and $S_d(t_2) = 0$. So the restriction of (X, S_d) to $[t_2, +\infty)$ is one of the weak generalized trajectories studied in Case 1 above.

5. Let $x \notin C_l$ and $s_d = 1$. As in Case 4(b) above, there is a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d). It has one switch at $\sigma > 0$ and satisfies the property stated in Claim 6.11. ■

With the help of Claim 6.11 and Remark 3.6, we can now prove the global asymptotic stability:

1. We have proved that all the trajectories are defined on $[t_0, +\infty)$.
2. *Establishing* (21): Let α_l be a class-$\mathcal{K}_\infty$ function such that

$$\alpha_l(s) \geq s \quad \text{and} \quad (V(x) \leq c_g \Rightarrow |x| \leq \alpha_l(V(x))).$$

Let also v be the positive definite function defined on $[0, c_g]$ as

$$v(s) = \inf_{\{x:\, V(x) \geq s \text{ or } x \notin \mathscr{D}\}} |x|, \qquad \forall s \in [0, c_g].$$

Finally, because for the system (36) the origin is globally asymptotically stable, there exists a class-$\mathcal{K}_\infty$ function α_g such that, for all x, we have

$$|X_g(x, t)| \leq \alpha_g(|x|), \qquad \forall t \geq t_0,$$

where $X_g(x, t)$ is defined by (26)–(27). Without loss of generality, we can impose

$$\alpha_g(v(s)) \geq s, \qquad \forall s \in [0, c_g].$$

We show that, given (x, s_d) an initial condition, a generalized trajectory (X, S_d) satisfies

$$|X(t)| \leq \alpha_l(\alpha_g(|x|)), \qquad \forall t \geq t_0. \tag{54}$$

Indeed:

- If X has no switch, then X is a classical trajectory of (37) and is in $A \cup C_l$. So, from (51), we have

$$V(X(t)) \leq V(x) \leq c_g, \qquad \forall t \geq 0,$$

and therefore, for all $t \geq 0$,

$$|X(t)| \leq \alpha_l(\alpha_g(v(V(x)))) \leq \alpha_l(\alpha_g(|x|)).$$

- If X has one switch at time $\sigma \geq 0$. Then the restriction of X to $[t_0, \sigma)$ is a classical trajectory of (36) not in C_l. So we have

$$|X(t)| \le \alpha_g(|x|) \le \alpha_l(\alpha_g(|x|)) \quad \text{and} \quad \{V(X(t)) \ge c_l \text{ or } X(t) \notin \mathscr{D}\}.$$

The restriction of X to $[\sigma, +\infty)$ is a classical trajectory of (37). So we have

$$V(X(t)) \le V(X(\sigma)) = c_l \le \alpha_g(v(c_l)) \le \alpha_g(|x|),$$

and therefore we have (54).

3. *Establishing* (22): Let $r > 0$ and $\varepsilon > 0$. Let r_l and r_g be defined by

$$r_l = \mathtt{sup}\{|x|, x \in C_l \cup A\} < +\infty, \qquad r_g = \tfrac{1}{2}\inf\{|x|, x \in \partial C_l\} > 0.$$

Due to the global asymptotic stability of the systems (37) and (36), there exists $T_g, T_l < +\infty$ such that

$$|x| \le r \quad \Rightarrow \quad |X_g(x,t)| \le r_g, \qquad \forall t \ge T_g, \tag{55}$$

and such that

$$|x| \le r_l \quad \Rightarrow \quad |X_l(x,t)| \le \varepsilon, \qquad \forall t \ge T_l. \tag{56}$$

Let $T = T_l + T_g$. We show that

$$|x| \le r \quad \Rightarrow \quad |X(t)| \le \varepsilon, \qquad \forall t \ge T, \tag{57}$$

where (X, S_d) is any generalized trajectory starting from (x, s_d). Indeed:

- If X has no switch, then X is a classical trajectory of (37) and is contained in $A \cup C_l$. So (57) is a consequence of the definition of r_l and (56).
- If X has one switch at time σ, then $X(\sigma)$ is in ∂C_l, and X is a classical trajectory of (36) on (t_0, σ) and a classical trajectory of (37) on $[\sigma, +\infty)$. So, for all t in $[T_l + \sigma, +\infty)$, we have

$$|X(t)| \le \varepsilon. \tag{58}$$

Also we have $\sigma \le T_g$. Suppose not. Then (55) implies $0 < |X(\sigma)| \le \frac{1}{2}\inf\{|x|, x \in \partial C_l\}$, which is a contradiction to $X(\sigma)$ in ∂C_l. So we have (58) for all t in $[T, +\infty)$.

This achieves the proof of Theorem 4.1. ■

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Appendix. Proof of Theorem 3.7

Proof. Let X, defined on $[t_0, T)$, be a strong generalized trajectory of

$$\dot{x} = f(x, \varphi(x,t)), \tag{59}$$

where the functions f and φ are continuous. Let $J = [\tau_0, \tau_1]$ be a compact subinterval of $[t_0, T)$. Let (X_n, S_n) be disturbed classical trajectories of

$$\dot{x} = f(x, \varphi(x + e_n, t)) + d_n,$$

with

$$\sup_J (X - X_n) + \mathtt{esssup}(e_n) + \mathtt{esssup}(d_n) \underset{n\to\infty}{\longrightarrow} 0. \tag{60}$$

Let $\sigma < t$ be in J. We have, for all n in $\mathbb{N}$,

$$\begin{aligned} X(t) - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau \\ = X(t) - X_n(t) + X_n(t) - X_n(\sigma) + X_n(\sigma) \\ - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau. \end{aligned} \tag{61}$$

From (16), for all n in $\mathbb{N}$, we have

$$X_n(t) - X_n(\sigma) = \int_\sigma^t (f(X(\tau), \varphi(X(\tau) + e_n(\tau), \tau)) + d_n(\tau))\, d\tau.$$

With (60) and the continuity of f and φ, we have, for almost all τ in $[\sigma, t]$,

$$\lim_{n\to\infty} f(X_n(\tau), \varphi(X_n(\tau) + e_n(\tau), \tau)) + d_n(\tau) = f(X(\tau), \varphi(X(\tau), \tau)),$$

and the sequence $(f(X_n(\tau), \varphi(X_n(\tau) + e_n(\tau), \tau)))$ is essentially bounded on $[\sigma, t]$. So with the Dominated Convergence Theorem, we obtain with (61)

$$\lim_{n\to\infty} \left(X(t) - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau \right) = 0.$$

Since X is independent of n, and since f is continuous, we conclude that X is in $\mathscr{C}^1$ and is a classical trajectory of (59) on all J compact subintervals of $[t_0, T)$. So X is a trajectory of (59) on $[t_0, T)$. ■

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A Robust Globally Asymptotically Stabilizing Feedback: The Example of the Artstein's Circles

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Abstract: We study a two dimensional system which is globally asymptotically stabilizable with a discontinuous feedback but for which there exists no smooth stabilizing feedback. However this asymptotic stability is not robust to measurement, actuator or external noise. We show that such a robustness property can be achieved with an hybrid controller. In doing so we need to introduce an appropriate notion of solutions for hybrid systems.

1 Problem statement and related results

1.1 Introduction

The general problem under consideration in this paper is the asymptotic stabilization via hybrid state feedback. Let us recall that *asymptotic stabilization* means the satisfaction of two properties: stability of the origin of the closed-loop system and convergence to this point of all the solutions of the closed-loop system.

We focus our study on the example of the Artstein's circles, i.e. the following two dimensional system, see [A:83]:

$$\left\{ \begin{array}{rcl} \dot{x_1} & = & u(-x_1^2 + x_2^2) \\ \dot{x_2} & = & -2ux_1x_2 \end{array} \right. \tag{1}$$

with u in $\mathbb{R}$. Let g be defined for all (x_1,x_2) in $\mathbb{R}^2$:

$$g(x) = (-x_1^2 + x_2^2, -2x_1x_2)'.$$

In (1), all motions are allowed along the integral curves of g i.e.:

- the origin
- all circles centered on the x_2-axis and tangent to the x_1-axis
- the x_1-axis

With $u > 0$ the circle is followed clockwise if $x_2 > 0$ and anticlockwise if $x_2 < 0$.

In the complex plane ($z = x_1 + ix_2$) we can rewrite the system (1) as: $\dot{z} = -z^2u$. The restriction of any neighborhood of $z = u = 0$ of the map $(z,u) \mapsto -z^2u$ takes value into a neighborhood of the $z = 0$. So the necessary condition [B:83, Theorem 1, (iii)] for the existence of a continuous control law which makes the origin globally asymptotically stable is satisfied. However it is proved in [S:99] that it can not exist a continuous stabilizing feedback.

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Nevertheless there are many obvious discontinuous stabilizing feedbacks e.g. the following:

$$u(x) = \begin{cases} -1 & \text{if} \quad x_1 \leq 0 \\ 1 & \text{if} \quad x_1 > 0 \end{cases} \tag{2}$$

which makes the origin of the closed-loop system a globally asymptotically stable equilibrium when we restrict our attention only to Carathéodory solutions. The meaning of the solution of the discontinuous right-hand side differential equation (1) with u given by (2) is an important issue. Indeed the origin of this system is not locally attractive for Filippov solutions (in particular not locally attractive for Krasovskii solutions i.e. the limit of the perturbed Carathéodory solutions as the perturbations tend to 0, see [CR:94].). The reason is that every point of the x_2-axis is an equilibrium for Filippov solutions. So the system (1) in closed loop with (2) is very sensitive to measurement noises.

However in [LS:99] when we consider the π-solutions (i.e. (1) with the feedback (2) computed with an arbitrary small sampling schedule) the origin of the closed-loop system is a globally asymptotically stable equilibrium. Moreover this controller is robust with respect to external disturbances but not with respect to measurement noise.

We study in this paper the robust stabilization i.e. the insensitivity of the feedback's performance with respect to measurement errors, actuator errors and external errors. We know by [A:83] that there exists no smooth control Lyapunov function for (1). Then, due to [LS:99, Theorem 1], there exists no robust stabilizing feedback $u = u(x)$. So we must enlarge the class of controller if we want to robustly stabilize the system (1). In [LS:97] the authors introduce the notion of "dynamic hybrid controller" which is computed with an "external model". This controller has the following form:

$$u = k(x',z)$$

where z has the same size as x and denotes the state of the external model, and x' is the measured estimate of state vector x. The origin is a robustly globally asymptotically stable equilibrium for the π-solutions. But as remarked in [S:99] it requires a resetting of the controller which may be difficult to construct. Moreover we prove in this paper the same result for a larger class of solutions.

In [S:99], E.D. Sontag transforms the controller of [CLSS:97] in a controller which is robust to measurement noise and makes the origin of the system (1) a semiglobal practical stable equilibrium (i.e. driving all states in a given compact set of initial conditions into a specified neighborhood of the origin). We exhibit here a robust global asymptotic stabilizing controller.

In this paper we propose a robust stabilizing (dynamic hybrid) controller u such that:

1. The controller is easy to compute and does not differ substantially from (2).
2. The solutions are a generalization of Carathéodory solutions.

A very natural way to overcome the nonrobustness to noise encountered with static time-invariant discontinuous controllers is to enlarge the surface of discontinuities and to introduce hysteresis. See [P:01a] and [P:99], where the authors introduce hysteresis between two different controllers: one local and one global. Here we introduce hysteresis between two controllers, but we do not consider one local and one global controller but rather one controller which is defined on the right-hand side of the plane $\mathbb{R}^2$ and another one which is defined on the left-hand side. In [P:01a], the hysteresis technique allow us to join the local and the global controller with robustness to noise. Here we join the right-hand and the left-hand side of the plane with robustness to noise.

1.2 Organization of this paper

In Section 1.3 we define the class of admissible controllers and the notions of solutions. In Section 2, we introduce two controllers which are basic components of the robust stabilizing hybrid controller presented in Section 3.1. In Sections 3.2 and 3.3 we study solutions of the closed loop-system and we prove the main Theorem 1 in Section 3.4. In Section 4, we prove the Propositions 2.1, 2.3 and 3.3.

1.3 Class of controllers and notions of solutions

In this section we make more precise the notions of controller and solutions under consideration. We dot not restrict the system under consideration to be (1) but, in this section only, we allow us to study a larger class of systems.

Let D be a subset of $\mathbb{R}^n$ containing the origin. Let $f : D \times \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function such that $f(0,0) = 0$. We consider the system

$$\dot{x} \; = \; f(x,u) \; . \tag{3}$$

The controllers under consideration in this paper admit the following description (see [T:87])

$$u \; = \; k(x,s_d) \quad , \qquad s_d \; = \; k_d(x,s_d^-) \tag{4}$$

where s_d evolves in some finite set $\mathcal{F}$, $k : \mathbb{R}^n \times \mathcal{F} \to \mathbb{R}^m$ is continuous in x for each fixed s_d, $k_d : \mathbb{R}^n \times \mathcal{F} \to \mathcal{F}$ is a function and s_d^- is defined, at this stage only formally, as

$$s_d^-(t) \; = \; \lim_{s<t} s_d(s) \; .$$

For this to make sense, we equip $\mathcal{F}$ with the discrete topology, i.e. every set is an open set. The above controller is hybrid due to the presence of the discrete dynamics of s_d. It gives rise to a non classical ordinary differential equation describing the dynamics of the closed loop system. In particular this system is infinite dimensional since to evaluate $s_d^-(t)$ at time t, we need to know the past values of $s_d(t)$.

In this paper we are interested in a notion of robustness to small noise. Let three functions

- e and d in $L^\infty_{loc}(D \times [0,+\infty);\mathbb{R}^n)$, which are continuous in x for each t,
- a in $L^\infty_{loc}(D \times [0,+\infty);\mathbb{R}^m)$, which is continuous in x for each t.

We introduce these functions as a measurement noise e, an actuator noise a and an external noise d of (3) and study the following perturbed system:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),k(x(t)+e(x,t),s_d(t)) + a(x,t)) + d(x,t) \\ s_d(t) & = & k_d(x(t)+e(x,t),s_d^-(t)) \end{array} \right. \tag{5}$$

As noted in [LS:97, Remark 1.4], we can omit any explicit reference to actuator errors because f is supposed to be locally Lipschitz. So in the following we suppose that, for all x in D and for all $t \geq 0$, we have:

$$a(x,t) \; = \; 0 \; .$$

We have to make precise what we mean by solution of the corresponding differential equation. We want to study the implementation of the controller (4). A natural framework is the π-solutions. These π-solutions are studied in [S:99, LS:99, S:98] in the case of a ordinary differential equation. In our context a natural definition is

Definition 1.1 *Let π be a sampling schedule of $[0,T)$ with $T > 0$ (i.e. a partition $\pi = \{t_0 = 0 < t_1 < \cdots < t_\infty\}$). Given $(x_0,s_{-1}) \in \mathbb{R}^n \times \mathcal{F}$.*

We say that (X,S_d) is a sampled solution*, starting from (x_0,s_{-1}), of (3) on $[0,T)$ if*

1. *X is absolutely continuous on $[0,T)$*
2. *For all i in $\mathbb{N}$ and for almost all t in $[t_i,t_{i+1})$, we have*[2]

$$\dot{X}(t) \;=\; f\left(X(t),U(X(t_i),S_d(t_i))\right) \;,$$

3. *For all i in $\mathbb{N}$ and for all t in $[t_i,t_{i+1})$, we have*

$$S_d(t) \;=\; k_d(X(t_i),S_d(t_{i-1})) \;. \tag{6}$$

4. *We have:*

$$X(0) \;=\; x_0 \;. \tag{7}$$

We say that X is a π-solution*, starting from x_0, of (3) on $[0,T)$ if there exists a sequence $(X_n,S_{d,n})$ of sampled solutions of (3) defined on $[0,T)$ such that $\sup_{n\to\infty}|X_n - X| = 0$ and such that we have (7).*

Remark 1.2 By invoking Zorn's Lemma exactly as in the proof of [R:99, Proposition 1], we can prove that every sampled solution (resp. π-solution) can be extended to a maximal sampled solution (resp. π-solution) (X,S_d) defined on an interval $[0,T)$ with $T \leq +\infty$ (i.e. for which there exists no solution defined on an interval $[0,T')$ with $T' > T$ and whose restriction is (X,S_d) on $[0,T)$). ◇

In this context our definition of global asymptotic stability is

Definition 1.3 *Let e,d be two functions with our standing regularity assumption. The origin is said to be a* globally asymptotically stable equilibrium *of the system (5) on D if the following three properties hold*

1. *For every x_0 in D, there exists s_0 in $\mathcal{F}$ and a sampled solution starting from (x_0,s_0).*
2. *All the maximal sampled solutions are defined on $[0,+\infty)$.*
3. *There exists a function β of class $\mathcal{KL}$ such that each maximal sampled solution $(X(t),S_d(t))$ satisfies for all $t \geq 0$:*

$$|X(t)| \;\leq\; \beta(|X(0)|,t) \;. \tag{8}$$

Note that (8) holds for all maximal π-solutions if the origin is a globally asymptotically stable equilibrium.

Remark 1.4 We observe, with [LSW:96, Remark 2.4 and Proposition 2.5], that (8) is equivalent to the set of following two properties:

1. There exists a class-$\mathcal{K}_\infty$ function α such that we have

$$|X(t)| \leq \alpha(|X(0)|) \;,\forall t \geq 0 \;. \tag{9}$$

2. we denote $S_d(t_{-1}) = s_{-1}$.

2. For any $r > 0$ and $\varepsilon > 0$, there exists $T > 0$ such that

$$|X(0)| \leq r \quad \Rightarrow \quad |X(t)| \leq \varepsilon \ , \forall t \geq T \ . \tag{10}$$

◇

Actually, we are interested in a notion of the robustness with respect to small noise. For this reason, we introduce the notion of robust stabilizing controller:

Definition 1.5 *We say that the controller (k,k_d) is a robust global asymptotically stabilizing controller if there exists two continuous functions ρ_e, ρ_d defined on D such that $\rho_j(x) > 0$ for any $x \neq 0$ and for any j in $\{e,d\}$ and such that for any perturbed system (5) with, for all x in D*

$$\sup_{\mathbb{R}_{\geq 0}} |e(x,.)| \leq \rho_e(x) \quad , \qquad \sup_{\mathbb{R}_{\geq 0}} |d(x,.)| \leq \rho_d(x) \ , \tag{11}$$

where e,d are two functions with our standing regularity assumption, the origin is a globally asymptotically stable equilibrium on D as characterized in Definition 1.3.

2 A local continuous controller

In this section we define two controllers of the control system (1) which are defined respectively on the right-hand and the left-hand side of the plane $\mathbb{R}^2$. We overlap the domain of definition of these controllers to define the robust global asymptotic stabilizing controller in Section 3.3.

For any α in $(\frac{\pi}{2},\frac{3\pi}{4})$, let us define the following set

$$\begin{aligned} D_1 \ &= \ \left\{x \in \mathbb{R}^2 : \ -\alpha \leq \theta \leq \alpha\right\} \cup \{(0,0)\} \ , \\ (\text{resp. } \ D_{-1} \ &= \ \left\{x : \ \pi - \alpha \leq \theta \leq \pi \text{ or } -\pi < \theta \leq -\pi + \alpha\right\} \cup \{(0,0)\}) \ , \end{aligned}$$

where θ in $(-\pi,\pi]$ denotes the polar angle of the point $x \neq 0$. On this set, we consider the controllers:

$$u \ = \ 1 \qquad (\text{resp. } u \ = \ -1) \ .$$

Proposition 2.1 *There exist two continuous functions ρ_e and ρ_d defined on $\mathbb{R}^2$ such that $\rho_e(x) > 0$ and $\rho_d(x) > 0$ for any $x \neq 0$ and α in $(\frac{\pi}{2},\frac{3\pi}{4})$ such that the origin of the system*

$$\dot{x} = g(x) + d \qquad (\textit{resp. } \dot{x} = -g(x) + d) \tag{12}$$

is globally asymptotically stable on D_1 (resp. D_{-1}), for all noises satisfying (13), for all x in $\mathbb{R}^2$.

Remark 2.2 In this proposition the notion of solutions is the classical one i.e. the Carathéodory solutions. Moreover we must take care of the fact that the origin is on the boundary of the domains D_1 and D_{-1}. Thus to prove this proposition, we cannot use

- every continuous stabilizing feedback is a robust stabilizing feedback ([K:63, Theorem 19.1]).

– a Lyapunov function and the characterization of the robust stabilization given by [LSW:96, Theorem 2.9]. ◇

To explicit the robust controller, we need to define the angles in $(\frac{\pi}{2},\alpha)$

$$\beta = \arctan(\tan(\alpha) - 1) \quad , \qquad \gamma = \arctan(3(\tan(\alpha) - 1)) \ . \tag{13}$$

The following proposition collects all technical properties that we use in Section 3.1 to study the robust stabilizing controller. We denote $|.|$ the norm $|x| = |x_1| + |x_2|$.

Proposition 2.3 *We find two functions ρ_d and ρ_e such that the statements of Proposition 2.1 hold and we have the property:*
For all x_2 in $\mathbb{R}$ and for all $e = (e_1,e_2)$ satisfying $|e| \leq 2\rho_e(x)$, we have

$$\rho_d(0,x_2) \leq \frac{x_2^2}{2} \ , \tag{14}$$

for $x \neq 0$, we have

$$x + e \ \neq \ 0 \ , \tag{15}$$

and the following implications hold

$$|x_2| \ \leq \ \tan(\alpha)\, x_1, \quad \Rightarrow \quad |x_2 + e_2| \ < \ \tan(\beta)(x_1 + e_1) \ , \tag{16}$$

$$|x_2| \ \geq \ \tan(\gamma)\, x_1, \quad \Rightarrow \quad |x_2 + e_2| \ > \ \tan(\beta)(x_1 + e_1) \ , \tag{17}$$

$$|x_2| \ \leq \ \tan(\gamma)\, x_1, \quad \Rightarrow \quad x_1 + e_1 \ < \ 0 \ . \tag{18}$$

The proofs of Propositions 2.1 and 2.3 are technical. We postpone them to Section 4.

3 Dynamic time-invariant controller with hysteresis

In this section we use the stabilizing controllers expressed in the above section and we join the domain of definition of these functions by making an hysteresis.

3.1 A robust stabilizing controller

Our main result is:

Theorem 1 *With β given by (13), the controller below makes the origin of the system (1) a robust globally asymptotically stable equilibrium on $\mathbb{R}^2$:*

$$u \ = \ s_d \quad , \qquad s_d \ = \ k_d(x,s_d^-) \tag{19}$$

where s_d is in $\{-1,1\}$, the function k_d satisfies:

$$k_d(x,s_d) = \begin{cases} -1 & \text{if } -\pi < \theta \leq -\beta \quad \text{or} \quad \beta \leq \theta \leq \pi \\ s_d & \text{if } -\beta < \theta < -\pi + \beta \quad \text{or} \quad \pi - \beta < \theta < \beta \\ 1 & \text{if } -\pi + \beta \leq \theta \leq \pi - \beta \end{cases} , \tag{20}$$

for $x \neq 0$ and

$$k_d(0,s_d) = s_d \ . \tag{21}$$

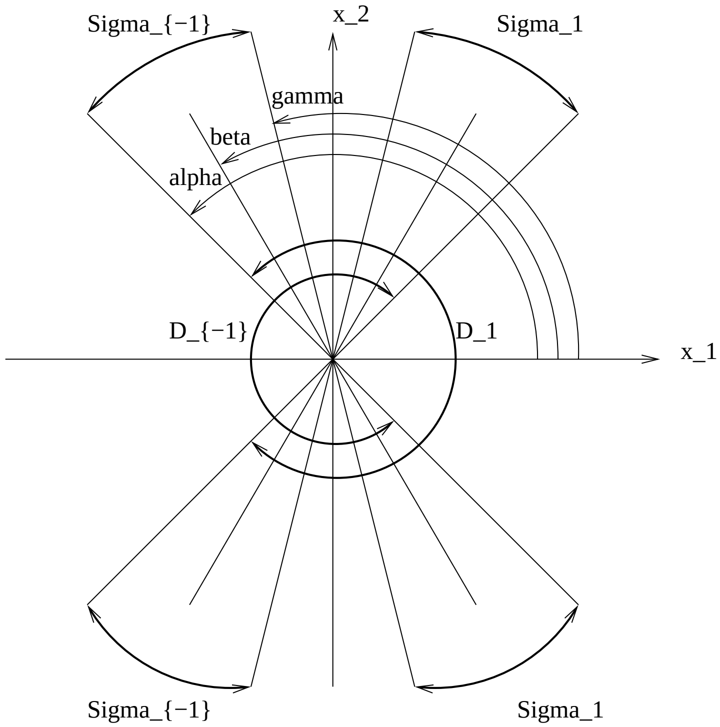

FIG. 1 – *Definitions of the angles* α, β, γ *and the sets* D_{s_0}, Σ_{s_0} *for* s_0 *in* $\{-1,1\}$

Remark 3.1 The function k_d makes an hysteresis between the two functions 1 and -1 on appropriate subsets of $\mathbb{R}^2$. For any s_d in $\{-1,1\}$, the function $k_d(.,s_d)$ is continuous except on the boundary of the sets defining the hysteresis. This remark is important to establish Proposition 3.5. $\diamond$

To prove Theorem 1 we need to introduce a class of solutions which contain the sampled solutions with a sufficiently fast schedule (see Section 3.2) then we prove preliminary results on the solutions of the closed-loop in Section 3.3. Finally we prove Theorem 1 in Section 3.4.

3.2 Definition of the $\mathcal{RC}$-solutions

The perturbed system under consideration is

$$\dot{x} = s_d g(x) + d(x,t) \quad , \qquad s_d \;=\; k_d(x + e(x,t), s_d^-) \ , \tag{22}$$

In this section we define a class of solutions which contain the solution defined in Definition 1.1. For the statement of this definition we need the following closed sets (see Figure 1):

$$\Sigma_{-1} \;=\; \{x :\, -\alpha \leq \theta \leq -\gamma \,\text{or}\, \gamma \leq \theta \leq \alpha\} \,\cup\, \{(0,0)\}, \tag{23}$$
$$\Sigma_{1} \;=\; \{x :\, -\pi + \gamma \leq \theta \leq -\pi + \alpha \,\text{or}\, \pi - \alpha \leq \theta \leq \pi - \gamma\} \,\cup\, \{(0,0)\}, \tag{24}$$

and the open set

$$\mathcal{R}C = \mathbb{R}^2 \times \{-1,1\} \setminus (\Sigma_1 \times \{-1\} \cup \Sigma_{-1} \times \{1\}) \ . \tag{25}$$

Definition 3.2 *$(X(t),S_d(t))$ defined on $[0,T)$ is a $\mathcal{RC}$-*solution *of (22) if*

1. *X is absolutely continuous on $[0,T)$ and takes values in D.*
2. *For each t in $[0,T)$ such that $(X(t),S_d(t))$ is in $\mathcal{RC}$, S_d is right continuous.*
3. *For almost all t in $[0,T)$, we have*

$$\dot{X}(t) \;=\; s_d g(X(t)) + d(X(t),t) \;, \tag{26}$$

and, for all t in $(0,T)$ where $S_d(s)$ has a limit as s tends to t from the left, we have[3]*:*

$$S_d(t) \;=\; k_d(X(t) + e(X(t),t), S_d^-(t)) \;. \tag{27}$$

Note that we can make Remark 1.2 for the $\mathcal{RC}$-solutions. Now we define the notion of sampled solutions with a sufficiently fast schedule. Let p: $\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that, for all $x \neq 0$, $p(x) > 0$. We say that a sampled solution (X,S_d) has a *sampling rate less* than p if for all i in $\mathbb{N}$ and for all t in $[t_i,t_{i+1})$, we have

$$t_{i+1} - t_i \leq p(X(t) + e(X(t),t) \;.$$

Every sampled solution is a $\mathcal{RC}$-solution. More precisely we have the following

Proposition 3.3 *There exists ρ_e, ρ_d and p three continuous functions such that $\rho_e(x) > 0$, $\rho_d(x) > 0$ and $p(x) > 0$ for all $x \neq 0$ and such that, for any e, d: $\mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$ with our regularity assumption and satisfying (11) for all x in $\mathbb{R}^2$, every sampled solution (X,S_d) starting from (x_0,s_{-1}) of (22) with a sampling rate less than p is a $\mathcal{RC}$-solution of (22) on $[t_0,T)$ starting from (x_0,s_0) with $s_0 = k_d(x_0 + \xi(x_0,t_0),s_{-1})$.*

The proof of Proposition 3.3 requires technical properties of $\mathcal{RC}$-solutions and of sampled solutions. Therefore we postpone it in Section4.

3.3 Basics properties of the $\mathcal{RC}$-solutions of (1) with u given by (19)

In this section we study properties of $\mathcal{RC}$-solutions of the closed-loop system (1) with the controller stated in Theorem 1. The unperturbed system under consideration is:

$$\dot{x} = s_d g(x) \quad , \qquad s_d \;=\; k_d(x,s_d^-) \;, \tag{28}$$

where k_d is defined by (20)-(21). Given two function e and d with our standing regularity assumption, the perturbed system under consideration is (22).

Definition 3.4 *A function (X,S_d) defined on $[0,T)$ is said to have a switch at time $t \in [0,T)$ if S_d is not continuous at t.*

We start by locating the points where a $\mathcal{RC}$-solution may have a switch:

Proposition 3.5 *Let (X,S_d) be a $\mathcal{RC}$-solution of (22) defined on $[0,T)$ with a switch at time $t \in [0,T)$. Consider the following sets.*

$$S_d^m(t) \;=\; \{s : \exists t_n \in [0,t], t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s\} \;, \tag{29}$$
$$S_d^p(t) \;=\; \{s : \exists t_n \in [t,T), t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s\} \;. \tag{30}$$

– *If the switch is such that $-1 \in S_d^m(t)$ and $1 \in S_d^p(t)$, then $X(t)$ is in Σ_1,*

3. Note that we do not ask for (27) to hold at $t = 0$.

– *If the switch is such that* $1 \in S_d^m(t)$ *and* $-1 \in S_d^p(t)$*, then* $X(t)$ *is in* Σ_{-1}.

Note that $S_d^p = \{S_d(t)\}$, for all t such that $(X(t),S_d(t))$ is in $\mathcal{RC}$.

Proof : Suppose that, for $t \in [0,T)$, we have $-1 \in S_d^m(t)$ and $1 \in S_d^p(t)$. From (29), there exists a sequence $t_n \leq t$ converging to t such that $S_d(t_n) = -1$. We have to consider two cases.

1. Suppose that there *exists* a sequence $t_n \leq t$ converging to t such that $S_d(t_n) = -1$ and such that S_d is *not right-continuous* at each t_n. Therefore $(X(t_n),S_d(t_n))$ is not in $\mathcal{RC}$. Thus due to (25), $X(t_n)$ is in Σ_1. Therefore by continuity and closedness of Σ_1, $X(t)$ is in Σ_1.
2. Suppose that there *does not exist* a sequence $t_n \leq t$ converging to t such that $S_d(t_n) = -1$ and such that S_d *is not right-continuous* at each t_n. Then there *exists* a sequence t_n converging to t such that S_d *is right-continuous* at each t_n and such that $S_d(t_n) = -1$. Thus there exists a maximal $\varepsilon_n > 0$ such that $S_d(s) = -1$ for s in $[t_n,t_n + \varepsilon_n)$. This implies $t_n + \varepsilon_n \leq t$, since if not we should have $S_d(t) = -1$ and S_d right-continuous at t which contradicts $1 \in S_d^p(t)$. $(t_n + \varepsilon_n)$ is a sequence converging to t, so we have three (non exclusive) cases
   (a) there exists a subsequence such that $S_d(t_n + \varepsilon_n) = 1$
   (b) or there exists a subsequence such that $S_d(t_n+\varepsilon_n) = -1$ and S_d is right-continuous at each $t_n + \varepsilon_n$.
   (c) or there exists a subsequence such that $S_d(t_n + \varepsilon_n) = -1$ and S_d is not right-continuous at each $t_n + \varepsilon_n$.

   The case 2b is not possible because ε_n is supposed to be maximally defined. The case 2c is not possible because we have supposed that there does not exist a sequence $t_n \leq t$ converging to t such that $S_d(t_n) = -1$ and such that S_d is not right-continuous at each t_n.

   Thus $S_d(t_n + \varepsilon_n) = 1$. This implies with (27) that
   $$1 \ = \ S_d(t_n + \varepsilon_n) \ = \ k_d(X(t_n + \varepsilon_n) + e(X(t_n + \varepsilon_n),t_n + \varepsilon_n), -1) \ . \qquad (31)$$
   So, from the definition of k_d, the polar angle of $X(t_n + \varepsilon_n) + e(X(t_n + \varepsilon_n),t_n + \varepsilon_n)$ is $[-\pi + \beta,\pi - \beta]$. From (16)-(18), this implies that the polar angle of $X(t_n + \varepsilon_n)$ is in $[-\pi + \gamma,\pi - \gamma]$. Similarly, for all s in $[t_n,t_n + \varepsilon_n)$, the polar angle of $X(s) + e(X(s),s)$ is in $(\pi - \beta,\pi] \cup (-\pi, -\pi + \beta)$, and therefore $X(s)$ is in D_{-1}. By continuity this implies that $X(t_n + \varepsilon_n)$ is in the closed set Σ_1 and $X(t)$ must also be in Σ_1.

The case $1 \in S_d^m(t)$ and $-1 \in S_d^p(t)$ is established in the same way. □

We are now in order to study the regularity of the function S_d.

Proposition 3.6 *Let* (X,S_d) *be a* $\mathcal{RC}$*-solution of (22) defined on* $[0,T)$*. Then, for all* t *in* $(0,T)$ *such that* $X(t) \neq 0$*,* $X(t)$ *is right-continuous at* t *or left-continuous at* t.

Proof : Let t in $(0,T)$ such that S_d is *not right-continuous* at t and such that S_d is *not left-continuous* at t. Therefore there exists two sequences $s_n < t$ and $t_n > t$ converging to t such that $S_d(s_n) = S_d(t_n) \neq S_d(t)$. Therefore $S_d^m(t) = \{-1,1\}$ and $S_d^p(t) = \{-1,1\}$. Thus due to Proposition 3.5, $X(t)$ is in $\Sigma_1 \cap \Sigma_{-1}$ which implies with (23) and (24) that $X(t) = 0$. □

Consider the sets: Let M be the set of the origin and the pairs (x_0,s_0) such that

1. s_0 is 1 or -1 and θ_0, the polar angle of x_0, is in $(-\beta, -\pi + \beta)$ or in $(\pi - \beta,\beta)$,

2. s_0 is -1 and θ_0 is in $(-\pi,-\beta]$ or in $[\beta,\pi]$,
3. s_0 is 1 and θ_0 is in $[-\pi+\beta,\pi-\beta]$,

and M' be the set of the origin and the pairs (x_0,s_0) such that

1. s_0 is 1 or -1 and θ_0 is in $(-\gamma,-\pi+\gamma)$ or in $(\pi-\gamma,\gamma)$,
2. s_0 is -1 and θ_0 is in $(-\pi,-\gamma]$ or in $[\gamma,\pi]$,
3. s_0 is 1 and θ_0 is in $[-\pi+\gamma,\pi-\gamma]$,

Note that $M' \subset M$ and that, for every x_0 is in $\mathbb{R}^2$, there exists s_0 in $\{-1,1\}$ such that (x_0,s_0) in M'. We prove the following result of existence of $\mathcal{RC}$-solutions.

Proposition 3.7 *For every (x_0,s_0) in M, there exists a $\mathcal{RC}$-solution of (28) starting from (x_0,s_0). Similarly, for every (x_0,s_0) in M' and every noise (e,d) satisfying (11) for all x in $\mathbb{R}^2$, there exists a $\mathcal{RC}$-solution of (22) starting from (x_0,s_0).*

Proof : We prove the result for M'. The case M is analogous. For $x_0 = 0$ and any s_0, we get from (21) that $X(t) \equiv 0, S_d \equiv s_0$ is a $\mathcal{RC}$-solution of (22).

Let (x_0,s_0) in M', $x_0 \neq 0$. From our standing regularity assumption on f, e, and d, the Carathéodory conditions are met for the system

$$\dot{Y} = s_0 g(Y) + d \quad , \qquad Y(0) = x_0 \ .$$

So, with the specific definition of M', there exists $T > 0$ and an absolutely continuous function Y defined on $[0,T)$ and such that $(Y(t),s_0)$ is in M'. Due to (16)-(18), we get $k_d(Y(t) + e(Y(t),t),s_0) = s_0$, for all t in $[0,T)$. This implies that, by letting:

$$X(t) \ = \ Y(t) \quad , \quad S_d(t) \ = \ s_0 \ ,$$

for $t \in [0,T)$, we get a $\mathcal{RC}$-solution of (22) starting from (x_0,s_0). □

Remark 3.8 If $(X(t),S_d(t))$ is a $\mathcal{RC}$-solution defined on $[0,T)$, then for any s in $[0,T)$, $(X(t+s),S_d(t+s))$ is a $\mathcal{RC}$-solution defined on $[0,T-s)$. ◇

Proposition 3.9 *For every noise satisfying (11) for all x in $\mathbb{R}^2$, every $\mathcal{RC}$-solution of (22) can be extended to a maximal $\mathcal{RC}$-solution (X,S_d) defined on an interval $[0,T)$ with $T \leq +\infty$. Moreover if $T < +\infty$ then $\lim_{t\to T} |X(t)| = +\infty$.*

Proof :
Part 1: Maximal extension: See Remark 1.2. *Part 2: Explosion in finite time:* Let (e,d) satisfy (11), for all x in $\mathbb{R}^2$, and $(X(t),S_d(t))$ be a $\mathcal{RC}$-solution maximally defined on $[0,T)$. Suppose the conclusion of Proposition 3.9 does not hold, i.e. there exists K a compact set of $\mathbb{R}^n$ and times t_n in $[0,T)$ tending monotonically to T such that $(X(t_n),S_d(t_n))$ is in $K \times \{-1,1\}$ for all n. We first establish

Claim 3.10 *For some n sufficiently large, for all $t \in [t_n,T)$, $X(t)$ is in the bounded open set $K+B$ with $B = \{x \in \mathbb{R}^2, |x| < 1\}$.*

Proof of Claim 3.10: If the conclusion of this claim is not true, the continuity of X implies the existence of $s_n \in (t_n,T)$ such that

$$|X(t_n) - X(s_n)| = 1 \quad \text{and} \quad |X(t_n) - X(t)| < 1, \forall t \in [t_n,s_n) \ . \qquad (32)$$

It follows that $X(t)$ is in the compact set $K + \text{clos}(B)$, for all t in $[t_n,s_n]$. Let

$$\rho = \max_{x \in K+\text{clos}(B)} (|\rho_e(x)|,|\rho_d(x)|)\,,\ \xi = \max_{x \in K+\text{clos}(B),\ d\in \rho\text{clos}(B)} |g(x)+d|\ .$$

Then we have for all (s,t) in $[t_n,s_n]$, $|X(t) - X(s)| \leq \xi|t-s|$. Therefore:

$$1\ =\ |X(t_n) - X(s_n)|\ \leq\ \xi\,|s_n - t_n|\ \leq\ \xi\,|T - t_n|\ .$$

This cannot hold for n sufficiently large and proves Claim 3.10. □

The Claim 3.10 implies that there exists ξ such that for all (s,t) in $[t_n,T)$, we have $|X(s) - X(t)|\ \leq\ \xi\,|s-t|$. It follows (by invoking Cauchy criterion) that $X(t)$ has a limit x_0 when t tends to T.

Part 1: If $S_d^m(T)$ is not a singleton

Then there exists a sequence of times of switch $t_n < T$ tending to T from the left such that:

$$1 \in S_d^m(t_{2n})\ ,\ -1 \in S_d^p(t_{2n})\,,\quad -1 \in S_d^m(t_{2n+1})\ ,1 \in S_d^p(t_{2n+1})\ .$$

Then due to Proposition 3.5, $X(t_{2n})$ is in Σ_{-1} and $X(t_{2n+1})$ is in Σ_1, for all n in $\mathbb{N}$. Then by continuity and closedness, $x_0 = \lim_{n\to+\infty} X(t_n)$ in $\Sigma_{-1}\cap\Sigma_1$. So $x_0 = (0{,}0)$. We define a function (X',S'_d) on $[0,+\infty)$, by letting

$$\begin{array}{rcl} \forall t \in [0,T), & X'(t) = X(t) \quad , & S'_d(t)\ =\ S_d(t)\ , \\ \forall t \in [T,+\infty), & X'(t)\ = 0 \quad , & S'_d(t)\ =\ 1\ , \end{array}$$

We have no limit of S_d at $t = T$, so we do not need to have (27) to hold at time $t = T$. So the function (X',S'_d) is a $\mathcal{RC}$-solution of (22) defined on $[0,+\infty)$ whose restriction on $[0,T)$ is (X,S_d).

Part 2: If $S_d^m(T)$ is a singleton $\{s_0\}$

We have to consider two cases:

1. if $x_0 = (0{,}0)$. We define a function (X',S'_d) on $[0,+\infty)$, by letting

$$\begin{array}{rcl} \forall t \in [0,T), & X'(t)\ =\ X(t) \quad , & S'_d(t)\ =\ S_d(t)\ , \\ \forall t \in [T,+\infty), & X'(t)\ = 0 \quad , & S'_d(t)\ =\ s_0\ , \end{array}$$

   It follows from (21) that the function (X',S'_d) is a $\mathcal{RC}$-solution of (22) defined on $[0,+\infty)$ whose restriction on $[0,T)$ is (X,S_d).

2. if $x_0 \neq (0{,}0)$. Let

$$s_1 = k_d(x_0 + e(x_0,T),s_0)\ . \tag{33}$$

   There exists Y defined on $[0,T_1)$ a solution of

$$\dot{Y}(t) = s_1 g(Y(t)) + d(Y(t),t+T)$$

   starting from x_0. We have to consider two cases:

   (a) If there exists $0 < T_0 \leq T_1$ such that, for all t in $[0,T_0)$, we have

$$k_d(Y(t) + e(Y(t),t+T),s_1) = s_1\ . \tag{34}$$

Then we define a function (X',S'_d) on $[0,T+T_0)$, by letting

$$\begin{array}{rrcll} \forall t \in [0,T), & X'(t) & = & X(t) \quad , & S'_d(t) = S_d(t) \ , \\ \forall t \in [T,T+T_0), & X'(t) & = & Y(t-T) \quad , & S'_d(t) = s_1 \ . \end{array}$$

(34) implies that $k_d(X'(T)+e(X'(T),T),\lim_{t<T} S'_d(T)) = S'_d(T)$. Thus the function (X',S'_d) is a $\mathcal{RC}$-solution of (22) defined on $[0,T+T_0)$ whose restriction on $[0,T)$ is (X,S_d).

(b) If there does not exist T_0, $0 < T_0 \leq T_1$, such that, for all t in $[0,T_0)$, we have (34). Let us prove the following

Claim 3.11 *We have* $(x_0,s_1) \notin \mathcal{RC}$.

Proof of Claim 3.11: Suppose that $s_1 = 1$ (the case $s_1 = -1$ is studied in exactly the same way). Then there exists $t_n \searrow 0$ such that $k_d(Y(t_n)+e(Y(t_n),t_n+T),1) = -1$. So from the definition of k_d, $Y(t_n)+e(Y(t_n),t_n+T)$ is in the set defined by

$$-\pi < \theta \leq -\beta \quad \text{or} \quad \beta \leq \theta \leq \pi$$

Due to (16)-(18), we obtain that $Y(t_n)$ is in the closed set defined by

$$-\pi < \theta \leq -\gamma \quad \text{or} \quad \gamma \leq \theta \leq \pi \tag{35}$$

By continuity we obtain that x_0 is in the set defined by (35) or $x_0 = (0,0)$. But under the assumption of case 2, we have $x_0 \neq (0,0)$. Thus x_0 is in the set defined by (35).

Moreover (33) implies that the polar angle of $x_0 + e(x_0,T)$ is in $(-\beta,\beta)$ or $x_0 + e(x_0,T) = (0,0)$. But if $x_0 + e(x_0,T) = (0,0)$ then due to (15) we obtain $x_0 = (0,0)$ which is a contradiction with the assumption of the case 2. Therefore the polar angle of $x_0 + e(x_0,T)$ is in $(-\beta,\beta)$. Due to (16)-(18), we obtain that the polar angle of x_0 is in $[-\alpha,\alpha]$.

Therefore the polar angle of x_0 is in $[-\alpha,-\gamma] \cup [\gamma,\alpha]$ and we conclude the proof of Claim 3.11 with (25). □

With our standing regularity assumption on e and d, there exists a solution Z defined on $[0,T_2)$ with $T_2 > 0$ of

$$\dot{Z}(t) = -s_1 g(Z(t)) + d(Z(t),t+T) \ . \tag{36}$$

Then we define a function (X',S'_d) on $[0,T+T_2)$, by letting

$$\begin{array}{rrcll} \forall t \in [0,T), & X'(t) & = & X(t) \quad , & S'_d(t) = S_d(t) \ , \\ & X'(T) & = & x_0 \quad , & S'_d(T) = s_1 \ , \\ \forall t \in (T,T+T_2), & X'(t) & = & Z(t-T) \quad , & S'_d(t) = -s_1 \ . \end{array}$$

(33) implies that $k_d(X'(T)+e(X'(T),T),\lim_{t<T} S'_d(T)) = S'_d(T)$, whereas Claim 3.11 implies that we do not need S_d to be right-continuous at $t = T$. So (X',S'_d) is a $\mathcal{RC}$-solution of (22) defined on $[0,T+T_2)$ whose restriction on $[0,T)$ is (X,S_d).

In the above various cases, we have obtained a contradiction with the fact that (X,S_d) is a maximal $\mathcal{RC}$-solution. □

Now we exhibit points of $\mathbb{R}^2 \times \{-1,1\}$ for which there exist no $\mathcal{RC}$-solution of the initial condition problem (22) in positive time.

Proposition 3.12 *For all s_0 in $\{-1,1\}$, for all x_0 in $\mathbb{R}^2 \setminus D_{s_0}$ and for all functions e,d satisfying (11) for all x in $\mathbb{R}^2$, there is no $\mathcal{RC}$-solution of (22).*

Proof : Let x_0 be in $\mathbb{R}^2 \setminus D_{-1}$ and $s_0 = 1$. There is no $\mathcal{RC}$-solution starting from $(x_0,1)$. Indeed for such a , we have $(x_0,1)$ is in $\mathcal{RC}$ and then, due to Definition 3.2, there exists $\varepsilon > 0$ such that S_d is constant on $[0,\varepsilon)$, and X being continuous, X is in the open set $\mathbb{R}^2 \setminus D_{-1}$ (neighborhood of x_0). But, for all x in a neighborhood of x_0 and all e satisfying (11), (16)-(18) yield $s_d = k_d(x + e,s_d^-) = -1$. So S_d must be -1. So there is no $\mathcal{RC}$-solution. The other case is established in a same way. □

As a conclusion of Proposition 3.12, and due to Remark 3.8, we can claim that for all $\mathcal{RC}$-solution (X,S_d) of (22) defined on $[0,T_0)$ and, for all t in $[0,T)$, by denoting $s_0 = S_d(t)$, we have: $X(t)$ is in D_{s_0}.

We end this section by counting the number of switches:

Proposition 3.13 *Let s_0 in $\{-1,1\}$, let (e,d) be noises satisfying (11) for all x in $\mathbb{R}^2$. Let (x_0,s_0) be an initial condition such that there exists a $\mathcal{RC}$-solution (X,S_d) of (22). Only two cases can occur:*

1. *There exists no switch i.e. $S_d \equiv s_0$ and X is a solution of*
$$\dot{x} = s_0 g(x) + d \tag{37}$$
and is contained in D_{s_0}.
2. *There exists a switch at the time $\sigma > 0$. Then:*
   (a) *$X(\sigma)$ is in Σ_{-s_0},*
   (b) *For all t in $[0,\sigma)$, X is a of (37) and, for all t in $[0,+\infty)$, X is a solution of:*
$$\dot{x} = -s_0 g(x) + d \ . \tag{38}$$
   (c) *If there exist two switches at times $0 < \sigma < \sigma'$ then for all $t \geq \sigma'$, $X(t) = 0$.*

Proof : Let us start with a remark: If there exists t_0 such that $X(t_0) = 0$ then, for all $t \geq t_0$, $X(t) = 0$. We prove now the following:

Claim 3.14 *If $X(t) \neq 0$, for all t in an open interval I, then there is at most one switch of S_d in I.*

Proof of Claim 3.14: Suppose that $s_0 = 1$ and there exist two switches at times $\sigma' > \sigma$ in I. In view of Proposition 3.5, we can suppose that $X(\sigma)$ is in Σ_{-1} and $X(\sigma')$ is in Σ_1. Then by continuity there exists a time t in (σ,σ') such that:
$$x_1(t) = 0 \quad , \qquad x_2(t) \neq 0 \quad , \qquad S_d(t) = -1 \quad , \qquad \dot{x}_1(t) \geq 0 \ . \tag{39}$$
But (14) implies that if $S_d(t) = -1$ then $\dot{x}_1(t) \leq -\frac{x_2^2(t)}{2}$. Therefore if $S_d(t) = -1$ then $\dot{x}_1(t) < 0$ which is a contradiction with (39). □

The Proposition 3.13 is a consequence of Claim 3.14 and Proposition 3.12. □

Now we are in order to prove Theorem 1.

3.4 Proof of Theorem 1

Maximality of trajectories: Let $(x,s_d) \in \mathbb{R}^2 \times \mathcal{F}$ be any point so that there exists a $\mathcal{RC}$-solution maximally defined on $[0,T)$ (see Propositions 3.7 and 3.9). Let us show that, for each such $\mathcal{RC}$-solution, we have $T = +\infty$. Suppose not. Then with Proposition 3.13 there exists t_0 such that, for some s_0 in $\{-1,1\}$, with $X(t_0) \in D_{s_0}$, X is a on $[t_0,T)$ of (37) which tends to $+\infty$. This is a contradiction with Proposition 2.1. So we must have $T = +\infty$.

With the help of Proposition 3.13 and Remark 1.4, we can now prove that the controller (19) is a *robust global asymptotic stabilizing* controller:

From Proposition 3.7, for any x_0 there exists s_0 such that, for any noise (e,d) satisfying (11), there exists a $\mathcal{RC}$-solution of (22) starting from (x_0,s_0). Also, from the previous paragraph, all the s of (22) are defined on $[0,+\infty)$.

Establishing (9): Let (e,d) be a noise satisfying (11). Due to Proposition 2.1 there exists for s_0 in $\{-1,1\}$, a class-$\mathcal{K}_\infty$ function α_{s_0} such that, for all x_0 in D_{s_0}, we have

$$|X_{s_0}(t)| \leq \alpha_{s_0}(|x_0|) \ , \qquad \forall t \geq 0 \ , \tag{40}$$

where X_{s_0} is any $\mathcal{RC}$-solution of (37) starting from x_0. Let us show that, given a noise (e,d) and an initial condition (x_0,s_0), a $\mathcal{RC}$-solution (X,S_d) satisfies:

$$|X(t)| \leq \max(\alpha_{-1}(\alpha_1(|x_0|)),\alpha_1(\alpha_{-1}(|x_0|))) \ , \quad \forall t \geq 0 \ . \tag{41}$$

Indeed

- if X has no switch then, due to Proposition 3.13, X is a solution of (37). So, from (40) we have (41).
- if X has a first switch at time $\sigma > 0$, then for all t in $[0,\sigma)$:

  $$|X(t)| \leq \alpha_{s_0}(|x_0|) \leq \alpha_{s_0}(\alpha_{-s_0}(|x_0|))$$

  and, from Proposition 3.13, for all t in $[\sigma,+\infty)$:

  $$|X(t)| \leq \alpha_{-s_0}(|X(\sigma)|) \leq \alpha_{-s_0}(\alpha_{s_0}(|x_0|) \ ,$$

  and therefore we have (41).

Establishing (10): Let $r > \varepsilon > 0$. Let $R \geq r$ and $0 < \varepsilon' \leq \varepsilon$ be defined by

$$R = \max(\alpha_{-1}(r),\alpha_1(r)) \quad , \qquad \varepsilon' = \min(\alpha_{-1}^{-1}(\varepsilon),\alpha_1^{-1}(\varepsilon)) \ .$$

For all s_0 in $\{-1,1\}$, for all x in D_{s_0} and for all $X_{s_0}(t)$ of (37), we obtain:

$$|X_{s_0}(0)| \leq r \qquad \Rightarrow \qquad |X_{s_0}(t)| \leq R \ ,\forall t \geq 0 \ , \tag{42}$$

$$|X_{s_0}(0)| \leq \varepsilon' \qquad \Rightarrow \qquad |X_{s_0}(t)| \leq \varepsilon \ ,\forall t \geq 0 \ , \tag{43}$$

Due to the global asymptotic stability of the systems (37), for s_0 in $\{-1,1\}$, there exists $T_{s_0} < +\infty$ such that:

$$x \in D_{s_0}, |X_{s_0}(0)| \leq R \quad \Rightarrow \quad |X_{s_0}(t)| \leq \varepsilon', \forall t \geq T_{s_0} \ . \tag{44}$$

Let $T = T_{-1} + T_1$. Let us show that:

$$|X(0)| \leq r \quad \Rightarrow \quad |X(t)| \leq \varepsilon \ ,\forall t \geq T \ , \tag{45}$$

where $X(t)$ is any solution of (22) with initial condition (x,s_0). Indeed

- if X has no switch then, due to Proposition 3.13, X is a solution of (37) and is contained in D_{s_0}. So (45) is a consequence of (44).

- if X has one switch at time σ, due to Proposition 3.13, then $X(\sigma)$ is in Σ_{-s_0}, X is a solution of (37) on $[0,\sigma)$ and a solution of (38) on $[\sigma,+\infty)$. Two cases may occur
  1. suppose $\sigma < T_{s_0}$. Due to (42), since $X(0) \leq r$, we have $|X(\sigma)| \leq R$ and X is a of (38) on $[\sigma,+\infty)$. So due to (44), we have, for all $t \geq \sigma + T_{-s_0}$, $|X(t)| \leq \varepsilon' \leq \varepsilon$. In particular we have (45).
  2. if $T_{s_0} \leq \sigma$ then due to (44) we have, for all t in $[T_{s_0},\sigma]$, $|X(t)| \leq \varepsilon' \leq \varepsilon$. Therefore due to (43), for all t in $[\sigma,+\infty)$, we have $|X(t)| \leq \varepsilon$. So in particular we have (45).

This achieves the proof of Theorem 1. □

4 Technical proofs

4.1 Proof of Proposition 2.1

Let $a > 0$, $b > \frac{1}{2}$ be such that

$$3a^2 \ < \ 2b - 1 \ . \tag{46}$$

Part 1: Choice of α

Claim 4.1 *Let $P(w) = w^3 - 3aw^2 + (2b-1)w + a$. There exists w_0 in $(-1,0)$ such that for any $w \geq w_0$, we have $P(w) > 0$.*

The proof of the Claim 4.1 results from the following three remarks:

1. Under the condition (46), P is strictly nondecreasing.
2. P is continuous.
3. $P(-1) < 0$ and $P(0) > 0$.

Let α in $(\frac{\pi}{2},\frac{3\pi}{4})$ be defined by $\tan(\alpha) = w_0^{-1}$. A consequence of the Claim 4.1 is

$$\forall \theta \in (0,\alpha], P((\tan\theta)^{-1}) > 0 \tag{47}$$

Part 2: Computation of a Lyapunov function for (1) with $u = 1$

Let V_1 be defined for all x in D_1 by

$$V_1(x_1,x_2) = \frac{1}{2}x_1^2 - ax_1|x_2| + \frac{b}{2}x_2^2 \ .$$

Since (46) implies $a^2 < b$, we have, for all x in D_1, V_1 is non negative and $V_1(x) = 0 \Leftrightarrow x = 0$. Along a solution of (1) with $u = 1$, we have

$$\dot{V}_1(x_1,x_2) = -x_1^3 + (-2b+1)x_1x_2^2 + 3a|x_2|x_1^2 - a|x_2|^3 \ .$$

If $x_2 \neq 0$ then it follows

$$\dot{V}_1(x_1,|x_2|) = -P(\frac{x_1}{|x_2|})|x_2|^3 \ ,$$

and if $x_2 = 0$ then we have $\dot{V}_1(x_1,|x_2|) = -x_1^3$. Therefore with (47) we deduce that for all x in $D_1 \setminus \{(0,0)\}$, we have $\dot{V}_1(x) < 0$. Let K be the compact set $K = \{x \in D_1, V_1(x) = 1\}$. Let $c_1 = \max_{x\in K} -\dot{V}_1(x)$. Thus, for all x in K,

$$\dot{V}_1(x) \leq -c_1(V_1(x))^{\frac{3}{2}} \tag{48}$$

But by homogeneity this inequality holds for all x in D_1. We use now this Lyapunov function to study the perturbed system $\dot{x} = g(x) + d(x,t)$. Let

$$c_2 = \max_{x\in K}\left\{|-x_1^2+x_2^2||\frac{\partial V_1}{\partial x_1}| + |2x_1x_2||\frac{\partial V_1}{\partial x_2}|, |\frac{\partial V_1}{\partial x_1}| + |\frac{\partial V_1}{\partial x_2}|\right\} .$$

By denoting $\dot{V}_1$ the derived function of V_1 along a solution of this system we deduce from (48) that for all x in $D_1 \setminus \{(0,0)\}$

$$\dot{V}_1 \leq -c_1(V_1(x))^{\frac{3}{2}} + c_2(V_1(x))^{\frac{1}{2}}|d(x,t)| .$$

Therefore by letting $c_3 = \frac{c_1}{2c_2}$, we get that for any (e,d) satisfying

$$\forall x \in D_1, \sup_{\mathbb{R}_{\geq 0}}|e(x,.)| \leq c_3 V_1(x) , \quad \sup_{\mathbb{R}_{\geq 0}}|d(x,.)| \leq c_3 V_1(x) .$$

every solution satisfies $\dot{V}_1(x) \leq -\frac{c_1}{2}(V_1(x))^{\frac{3}{2}}$. Then $u = 1$ is a robust global asymptotically stabilizing controller for the restriction of the system (1) on D_1.

Part 3: solution of (1) with $u = -1$

We remark that for every solution $(x_1(t),x_2(t))$ of a perturbed system of (1) with $u = -1$, the pair $(-x_1(t),x_2(t))$ is a of a perturbed system of (1) with $u = 1$. We deduce that $u = -1$ is a robust global asymptotically stabilizing controller for the restriction of the system (1) on D_{-1}. Correspondingly, the Lyapunov functions is

$$V_{-1}(x_1,x_2) = \frac{1}{2}x_1^2 + a x_1|x_2| + \frac{b}{2}x_2^2 .$$

Part 4: ρ_e and ρ_d can be defined on $\mathbb{R}^2$

Let

$$\rho_e(x) = \frac{1}{2}\min\left\{\frac{1}{(1-\tan(\alpha))^2}, c_3\left(\left(1-\frac{a}{\sqrt{b}}\right)\frac{\min\{1,b\}}{4}\right)^{\frac{1}{2}}\right\}(|x|), \tag{49}$$

$$\rho_d(x) = \min\left\{\frac{1}{4}, c_3\left(\left(1-\frac{a}{\sqrt{b}}\right)\frac{\min\{1,b\}}{4}\right)^{\frac{3}{2}}\right\}(|x|)^2 . \tag{50}$$

On D_1 and on D_{-1}, we have $\rho_e(x) \leq c_3 V_1(x)^{\frac{1}{2}}$ and $\rho_d(x) \leq c_3 V_1(x)$. Finally, from the above, for s_0 in $\{-1,1\}$, the origin of the system $\dot{x} = s_0 g(x) + d$ is globally asymptotically stable on D_{s_0} for every noise (e,d) satisfying (11). □

4.2 Proof of Proposition 2.3

Note that $1-\tan(\alpha) \geq 2$ and (49) imply that $\rho_e \leq \frac{|x|}{8}$. Also for all e such that $|e| \leq 2\rho_e(x)$ we have

$$0 = x + e \Rightarrow |x| = |e| \leq 2\rho_e(x) \leq \frac{|x|}{4} \Rightarrow x = 0 ,$$

Thus (15) holds. Moreover (14) holds since we have, for $x_1 = 0$, $\frac{x_2^2}{4} \geq \rho_d(0,x_2)$. The implications (16)-(18) result from (49) and the inequality:

$$2\,\rho_e(x_1,x_2) \leq \frac{1}{(1-\tan(\alpha))^2}(|x_1| + |x_2|)$$

□

4.3 Proof of Proposition 3.3

Let p: $\mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

1. for all $x \neq 0$, we have $p(x) > 0$.
2. for all e and d satisfying (11) for all x in $\mathbb{R}^2$ and for all x such that the polar angle of $x + e(x,0)$ is in $(-\pi,-\beta] \cup [\beta,\pi]$ (resp. $[-\beta,\beta]$, resp. $(-\pi,-\beta] \cup [\pi-\beta,\pi]$, resp. $[-\pi+\beta,\pi-\beta]$) we have

$$\begin{aligned} p(x+e(x,0)) < \frac{1}{2}\min_{u\in\{-1,1\}}\{T,\ X(0)=x,&\\ \forall t \in [0,T], \text{the polar angle of } X(t) \in (-\pi,-\gamma] \cup [\gamma,\pi]&\\ (\text{resp. } [-\alpha,\alpha], \text{resp. } (-\pi,-\alpha] \cup [\pi-\alpha,\pi], \text{resp. } [-\pi+\gamma,\pi-\gamma])&\\ X \text{ is a solution of } \dot{X} = ug(X) + d(X,t)\}& \end{aligned}$$

The existence of such function p results from (11) and (16)-(18).

Let us prove Proposition 3.5 in the case of the sampled solutions. We can remark that for every (X,S_d) sampled solution of (22) on $[0,T)$, due to (6), if there is a switch at time t in $[0,T)$, then there exists i in $\mathbb{N}_{>0}$ such that $t = t_i$, $S_d^m(t_i) = \{S_d(t_{i-1})\}$ and $S_d^p(t_i) = \{S_d(t_i)\}$.

Lemma 4.2 *Let (X,S_d) be a sampled solution of (22) on $[0,T)$ such that its sampling rate is less than p and such that S_d has a switch at time $t_i \in (0,T)$.*

- *If the switch is such that $S_d(t_{i-1}) = -1$ and $S_d(t_i) = 1$, then, for all t in $[t_i,t_{i+1})$, $X(t)$ is in Σ_1.*
- *If the switch is such that $S_d(t_{i-1}) = 1$ and $S_d(t_i) = -1$, then, for all t in $[t_i,t_{i+1})$, $X(t)$ is in Σ_{-1}.*

Proof : Let i in $\mathbb{N}_{>0}$ such that $S_d(t_{i-1}) = 1$ and $S_d(t_i) = -1$. Then due to (20), the polar angle of $X(t_i) + e(X(t_i),t_i)$ is in $(-\pi,-\beta] \cup [\beta,\pi]$. From the Assumption 2 this implies that for all t in $[t_i,t_{i+2})$, the polar angle of $X(t)$ is in $(-\pi,-\gamma] \cup [\gamma,\pi]$.

Similarly the polar angle of $X(t_{i-1}) + e(X(t_{i-1}),t_{i-1})$ is in $(-\beta,\beta)$. From the Assumption 2 this implies that for all t in $[t_{i-1},t_{i+1})$, the polar angle of $X(t)$ is in $[-\alpha,\alpha]$.

Therefore for all t in $[t_i,t_{i+1})$, the polar angle of $X(t)$ is in $\{(-\pi,-\gamma] \cup [\gamma,\pi]\} \cap \{[-\alpha,\alpha]\}$, and thus, for all t in $[t_i,t_{i+1})$, $X(t)$ is in Σ_{-1}. The case $S_d(t_{i-1}) = -1$ and $S_d(t_i) = 1$ is established in the same way. □

Now we are in order to prove Proposition 2.3.

Proof of Proposition 2.3: Let (X,S_d) be a sampled solution of (22) with a sampling rate less than p. Let i in $\mathbb{N}$. Note that if $i > 0$, then we have $S_d^-(t_i) = S_d(t_{i-1})$. Therefore we have (27) at time $t = t_i$.

If there is no switch at time t_i, then for all t in (t_i,t_{i+1}), $S_d^-(t) = S_d(t_i) = S_d(t_{i-1})$. And therefore we have (27) for all t in (t_i,t_{i+1}).

Suppose that there is a switch at time t_i such that $S_d(t_{i-1}) = -1$ and $S_d(t_i) = 1$. Then, for all t in (t_i,t_{i+1}), $S_d^-(t) = S_d(t_i) = 1$ and due to Lemma 4.2, $X(t)$ is in Σ_1 and due to (16)-(18), the polar angle of $X(t) + e(X(t),t)$ is in $[-\pi+\gamma,\gamma]$. Thus (20) implies that we have (27), for all t in (t_i,t_{i+1}).

The case $S_d(t_{i-1}) = 1$ and $S_d(t_i) = -1$ is established in the same way.

Therefore we have (27) for all t. Moreover for all i in $\mathbb{N}$ and for all t in (t_i,t_{i+1}), we have $S_d(t) = S_d(t_i)$. Therefore we have (26) for almost all t. □

Asymptotic Controllability and Robust Asymptotic Stabilizability

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Abstract: We study asymptotic controllable systems for which there exists no smooth stabilizing state feedback. We consider a class of hybrid feedback. We define a class of solutions which is a generalization of Carathéodory solutions and Euler solutions. We prove that, in terms of such solutions, the origin of all globally asymptotically controllable systems can be globally asymptotically stabilized via a hybrid patchy feedback with robustness to measurement noises, actuator errors and external disturbances.

Key Words: Control systems, feedback stabilization, controllability, measurement noises.

AMS subject classifications: 93B52, 93D15.

1 Introduction

Let us consider the following system:

$$\dot{x} = f(x,u) \tag{1}$$

assuming that control set $K \subset \mathbb{R}^m$ is a compact subset of $\mathbb{R}^m$ and that the map $f : \mathbb{R}^n \times K \to \mathbb{R}^n$ is locally lipschitz. We focus our study on systems which are asymptotically controllable i.e. which satisfy:

- For every initial point x_0 in $\mathbb{R}^n$, there exists a measurable $u : [0, +\infty) \to K$ such that the (Carathéodory) solution of the Cauchy problem

  $$\dot{x} = f(x,u(t)) \quad , \qquad x(0) = x_0 \ ,$$

  is defined for all $t \geq 0$ and tends to the origin as t tends to infinity.
- A stability property (see Definition 2.6 below).

The general problem under consideration in this paper is the asymptotic stabilization via state feedback. Let us recall that *asymptotic stabilization* means that the following two properties hold:

- Stability of the origin of the closed-loop system.
- Convergence to this point of all the solutions.

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There exists a necessary condition [B:83, Theorem 1, (iii)] for the existence of a continuous control law which makes the origin globally asymptotically stable. But there are asymptotically controllable systems which do not satisfy this necessary condition hence for which there can not exist a continuous stabilizing feedback (see [SS:80, B:83]).

Therefore we must consider discontinuous controllers to stabilize all asymptotically controllable systems. The following property is proved in [S:79]

($\mathcal{P}$) Any asymptotically controllable systems can be asymptotically stabilized by a discontinuous controller.

In that paper by solutions of (1) in closed-loop with a discontinuous controller the author means the Carathéodory solutions satisfying an "exit rule". In [CLSS:97] the same property ($\mathcal{P}$) was proved and the author mean by solution of (1) a π-solution (i.e. (1) with a feedback computed with an arbitrary small sampling schedule). Recently in [AB:99], the authors prove the property ($\mathcal{P}$) for all Carathéodory solutions by exhibiting a "patchy feedback".

The controllers in [CLSS:97, AB:99] are robust to actuator and external disturbances (i.e. all systems perturbed by small actuator and external disturbances are asymptotically stable) but are not robust to arbitrary small measurement noises. One way to robustly stabilize the system (1) is to enlarge the class of controllers as done in [LS:97] where the authors introduce the notion of "dynamic hybrid controller" which is computed with an "external model". This controller has the following form:

$$u = k(x',z)$$

where z has the same size as x and denotes the state of the external model, and x' is the measured estimate of state vector x. The origin is a robustly globally asymptotically stable equilibrium for the π-solutions. But as remarked in [S:99] it requires a resetting of the controller which may be difficult to construct. Moreover we prove in this paper the same result for a larger class of solutions.

In [S:99], E.D. Sontag proves the existence for all asymptotic controllable systems of a controller which is robust to measurement noises and makes the origin of the system (1) a semiglobal *practical* stable equilibrium (i.e. driving all states in a given compact set of initial conditions into a specified neighbourhood of the origin). It is proved in [S:98, sec. 5.4] that one can in fact get a more general result: one can prove the existence of a sampling feedback making the origin a robust global asymptotic equilibrium for all π-solutions with a sampling rate *sufficiently slow*. We exhibit in this paper a robust global asymptotic stabilizing controller for the π-solutions with *any fast* enough sampling schedule.

The main result of this paper is Theorem 1: if (1) is asymptotically controllable then there exists a "hybrid patchy feedback" which makes the origin a globally asymptotically stable equilibrium and which is robust to measurement noises, actuator errors and external disturbances.

To prove it we use some techniques of [AB:99] to build a family of nested "patchy vector fields" and we introduce hysteresis between an infinite number of controllers as it is done in [P:00] for two controllers. It gives rise to a non classical ordinary differential equation (ODE) so we have to introduce the notion of solutions for differential equation with a hybrid term. We insist on the fact that the considered notion of solutions is a generalization of the Carathéodory solutions and the π-solutions for the ODEs.

2 Statement and Definitions

In this section we make more precise the notions of controller and solutions under consideration.

Let $\mathcal{A}$ be a non empty totally ordered index set. The controllers under consideration in this paper admit the following description (see [T:87])

$$u = u(x,s_d) \quad , \qquad s_d = k_d(x,s_d^-) \tag{2}$$

where s_d evolves in the set $\{1,2\}^{\mathcal{A}}$, $u : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to K$ is continuous in x for each fixed s_d, $k_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to \{1,2\}^{\mathcal{A}}$ is a function and s_d^- is defined, at this stage only formally, as

$$s_d^-(t) = \lim_{s<t,\, s\to t} s_d(s) \ . \tag{3}$$

For this to make sense, we equip $\{1,2\}^{\mathcal{A}}$ with the discrete topology, i.e. every set is an open set. The above controller is hybrid due to the presence of the discrete dynamics of s_d. It gives rise to a non classical ordinary differential equation describing the dynamics of the closed loop system. In particular this system is infinite dimensional since to evaluate $s_d^-(t)$ at time t, we need to know the past values of $s_d(t)$. Note that time can not be reversed.

In this paper we are interested in a notion of robustness to small noise. Let us consider three functions satisfying our *standing regularity assumptions* i.e.

– ξ and ζ in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}_{\geq 0}; \mathbb{R}^n)$ which are continuous in x in $\mathbb{R}^n$ for each t in $\mathbb{R}_{\geq 0}$,

– ψ in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}_{\geq 0}; \mathbb{R}^m)$ which is continuous in x in $\mathbb{R}^n$ for each t in $\mathbb{R}_{\geq 0}$.

We introduce these functions as a measurement noise ξ, an actuator noise ψ and an external noise ζ of (1) and study the following perturbed system:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(x(t)+\xi(x,t),s_d(t))+\psi(x,t))+\zeta(x,t) \\ s_d(t) & = & k_d(x(t)+\xi(x,t),s_d^-(t)) \end{array} \right. \tag{4}$$

As noted in [LS:97, Remark 1.4], with the presence of ζ, we can omit any explicit reference to actuator errors because f is supposed to be locally Lipschitz. So in the following we suppose that, for all x in $\mathbb{R}^n$ and for all $t \geq 0$, we have:

$$\psi(x,t) = 0 \ .$$

We have to precise what we mean by solution of the corresponding differential equation. We want to study the implementation of the controller (2). A natural framework is the π-solutions which have a meaningful physical interpretation: it is an accurate model of the process in computer control. These π-solutions are studied in [CLSS:97, S:99, LS:99, S:98] in the case of an ordinary differential equation. Let π be a sampling schedule of $\mathbb{R}$, i.e. a sequence $(t_n)_{n\in\mathbb{Z}}$ such that, for all n in $\mathbb{Z}$, we have $t_n < t_{n+1}$ and $\lim_{n\to+\infty} t_n = \lim_{n\to-\infty} -t_n = +\infty$. Note that the upper and lower diameters of the sampling schedule are defined by (see [S:99])

$$\overline{d}(\pi) = \sup_{i\in\mathbb{Z}}(t_{i+1} - t_i) \quad , \qquad \underline{d}(\pi) = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) \ .$$

In our context, we have

Definition 2.1 *Given t_0 in π and $T > t_0$, we say that (X,S_d): $[t_0,T) \to \mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ is a* π-solution *of (4) on $[t_0,T)$ if*

1. The map X is absolutely continuous on $[t_0,T)$.

2. *We have, for all t in $[t_0,\min(t_1,T))$,*

$$S_d(t) \;=\; S_d(t_0)\ , \tag{5}$$

and, for all i in $\mathbb{N}_{>0}$ and for all t in $[\min(t_i,T),\min(t_{i+1},T))$, we have

$$S_d(t) \;=\; k_d(X(t_i)+\xi(X(t_i),t_i),S_d(t_{i-1}))\ . \tag{6}$$

3. *For all i in $\mathbb{N}$ and for almost all t in $[\min(t_i,T),\min(t_{i+1},T))$, we have*

$$\dot X(t) \;=\; f\,(X(t),u(X(t_i)+\xi(X(t_i),t_i),S_d(t_i)))+\zeta(X(t),t)\ .$$

Now we discuss the meaning of the notion of the initial condition of a solution. To understand this, note that a necessary and sufficient condition to make the concatenation of two π-solutions X_- and X_+ of an ordinary differential equation without hybrid term (i.e. (4) without s_d) defined respectively on $(t_{-1},t_1]$ and $[t_0,t_2)$ is

$$X_-(t_0) = X_+(t_0)\ .$$

In the context of Definition 2.1, due to (5) and (6), we note that we can concatenate $(X_-,S_{d,-})$ and $(X_+,S_{d,+})$ if we have

$$k_d(X_-(t_0)+\xi(X_-(t_0),t_0),S_{d,-}(t_{-1})) = S_{d,+}(t_0) \quad , \qquad X_-(t_0) = X_+(t_0)\ . \tag{7}$$

It follows that if we let $S_{d,+}(t_{-1})$, which is a priori not defined be equal to $S_{d,-}(t_{-1})$, then (7) yields a constraint on $S_{d,+}(t_0)$. But initial conditions are usually free of constraints. Hence this motivates us to call initial condition of a solution the couple $(X(t_0),S_d(t_{-1}))$. More specifically, we have the following

Definition 2.2 *Given t_0 in π, $T>t_0$, and $(x_0,s_0)\in\mathbb{R}^n\times\{1,2\}^{\mathcal{A}}$, we say that (X,S_d): $[t_0,T)\to\mathbb{R}^n\times\{1,2\}^{\mathcal{A}}$ is a* π-solution, *with initial condition* (x_0,s_0) *of (4) on $[t_0,T)$ if (X,S_d) is a π-solution of (4) on $[t_0,T)$ and if we have*

$$X(t_0) = x_0 \quad , \qquad S_d(t_0) = k_d(x_0+\xi(x_0,t_0),s_0))\ , \tag{8}$$

In [P:00], (8) was not imposed and the existence of a solution for every initial condition and for all disturbances was not guaranteed. In this paper, with this notion of initial condition, we prove that, for every initial condition and for all small disturbances, we have the existence of a solution starting from this point. (Compare Lemma 4.3 below and [P:00, Proposition 6].)

Given an initial condition (x_0,s_0) in $\mathbb{R}^n\times\{1,2\}^{\mathcal{A}}$ and $T>t_0$, the Cauchy problem under consideration in this paper is (4) and (8). And as usual we define

Definition 2.3 *Given t_0 in $\mathbb{R}$, $T>t_0$ and $x_0\in\mathbb{R}^n$, we say that $X:[t_0,T)\to\mathbb{R}^n$ is an* Euler solution, *starting from x_0, of (4) on $[t_0,T)$ if, for all compact subinterval J of $[t_0,T)$, there exists a sequence π^n of sampling schedule of $\mathbb{R}$ and a sequence (X^n,S_d^n) of π^n-solutions of (4) defined on J such that $\lim_{n\to\infty}\sup_J|X^n-X|+\overline{d}(\pi^n)=0$ and such that we have*

$$X(t_0) = x_0\ . \tag{9}$$

Actually we are interested in a notion of solutions which is robust with respect to disturbances. For this reason we introduce a notion of generalized solutions (see [H:67, H:79, P:01a]).

Definition 2.4 *Let t_0 in $\mathbb{R}$, $T > t_0$ and x_0 in $\mathbb{R}^n$. We say that X: $[t_0,T) \to \mathbb{R}^n$ is a* generalized solution *starting from x_0 of (4) if we have (9) and if, for each J compact subinterval of $[t_0,T)$, there exists two sequences $(e^n)_{n\in\mathbb{N}}$ and $(d^n)_{n\in\mathbb{N}}$ of measurable functions $[t_0,+\infty) \to \mathbb{R}^n$ and a sequence $(X^n,S_d^n)_{n\in\mathbb{N}}$ of π-solutions of*

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(x(t)+\xi(x(t),t),s_d(t)) + \zeta(x,t) + d^n(t) \\ s_d(t) & = & k_d(x(t)+\xi(x,t)+e^n(t),s_d^-(t)) \end{array} \right. \tag{10}$$

such that, we have

$$\lim_{n\to+\infty} (\sup_J |X^n - X| + \sup_J |e^n| + \mathrm{esssup}_J |d^n|) = 0 \ . \tag{11}$$

Remark 2.5 By invoking Zorn's Lemma exactly as in the proof of [R:99, Proposition 1], one can prove that every π-solution (X,S_d) (resp. Euler solution X, resp. generalized solution X) can be extended to a maximal π-solution (X,S_d) (resp. Euler solution X, resp. generalized solution X) defined on an interval $[t_0,\tau^{max})$ with $\tau^{max} \leq +\infty$ (i.e. for which there exists no solution defined on an interval $[t_0,T)$ with $T > \tau^{max}$ and whose restriction is (X,S_d) (resp. X) on $[t_0,\tau^{max})$). $\diamond$

Let us recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0,+\infty) \to \mathbb{R}^+$ which is continuous, strictly increasing, satisfies $\delta(0) = 0$ and $\lim_{\varepsilon\to+\infty} \delta(\varepsilon) = +\infty$. In our context our definition of robust global asymptotic stability is (see [ABB:97])

Definition 2.6 *The origin is said to be a* robust globally asymptotically stable equilibrium *of the system (4) if the following properties hold*

1. The system is complete: *For all $\delta > 0$, there exists $\chi_0 > 0$ and $d_0 > 0$, such that for all ξ, ζ satisfying our regularity assumptions and*

$$\sup_{x\in\mathbb{R}^n,\, t\geq 0} |\xi(x,t)| \leq \chi_0 \quad , \qquad \mathrm{esssup}_{x\in\mathbb{R}^n,\, t\geq 0} |\zeta(x,t)| \leq \chi_0 \ , \tag{12}$$

   for all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$, such that $|x_0| \leq \delta$ there exists a π-solution of (4) (resp. an Euler solution, resp. a generalized solution) starting from (x_0,s_0) (resp. starting from x_0) at $t_0 = 0$, and all the maximal π-solutions satisfying

$$\overline{d}(\pi) \leq d_0 \tag{13}$$

   (resp. Euler solutions, resp. generalized solutions obtained as limit of π-solutions satisfying (13)) of (4) are defined on $[0,+\infty)$.
2. Global stability: *There exists δ of class $\mathcal{K}_\infty$ such that, for all $\varepsilon > 0$, there exists $\chi_0 > 0$ and $d_0 > 0$ such that, for all ξ, ζ satisfying our regularity assumptions and (12), for all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ with $|x_0| < \delta(\varepsilon)$ and for every maximal π-solution (X,S_d) of (4) satisfying (13) (resp. Euler solution X, resp. generalized solution obtained as limit of π-solutions satisfying (13)) starting from (x_0,s_0) (resp. starting from x_0) at $t_0 = 0$, one has*

$$|X(t)| < \varepsilon, \forall t \geq 0 \ . \tag{14}$$

3. Global attractivity: *For all* $\varepsilon > 0$ *and for all* $\delta > 0$*, there exists* $T > 0$*,* $\chi_0 > 0$ *and* $d_0 > 0$ *such that, for all* ξ*,* ζ *satisfying our regularity assumptions and (12), for each* (x_0,s_0) *in* $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ *with* $|x_0| < \delta$ *and for every maximal* π*-solution* (X,S_d) *of (4) satisfying (13) (resp. Euler solution* X*, resp. generalized solution obtained as limit of* π*-solutions satisfying (13)) starting from* (x_0,s_0) *(resp. starting from* x_0*) at* $t_0 = 0$*, one has*

$$|X(t)| < \varepsilon, \forall t \geq T \ . \tag{15}$$

We recall the definition of global asymptotic controllability of the system (1):

Definition 2.7 *The system (1) is said to be* globally asymptotically controllable to the origin *if the following properties hold:*

1. *For each* x_0 *in* $\mathbb{R}^n$*, there exists an admissible control* u_0 *(i.e. a measurable function* $[0,+\infty) \to K$*) such that the maximal Carathéodory solution* X *starting from* x_0 *of*

$$\dot{x} = f(x,u_0) \tag{16}$$

   is defined for all $t \geq 0$ *and satisfies* $X(t) \to 0$ *as* $t \to +\infty$*.*
2. *For each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* x_0 *in* $\mathbb{R}^n$ *with* $|x_0| < \delta$*, there is an admissible control* u_0 *such that the maximal Carathéodory solution* X *of (16) starting from* x_0 *at* $t = 0$ *is defined for all* $t \geq 0$ *and satisfies* $|X(t)| < \varepsilon$*, for all* $t \geq 0$*.*

Our main result is

Theorem 1 *Let (1) be a globally asymptotically controllable system to the origin. Then there exists a feedback control,* $u : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to K$*,* $k_d : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ *such that the origin is a robust globally asymptotically stable equilibrium for the system (4).*

Remark 2.8

1. Note that in Theorem 1 we have the robust global asymptotic stability for the π-solutions for any fast enough sampling rate and not only for the π-solutions with a sampling rate sufficiently slow as in [S:99]. Compare (12) and the inequality in [S:99, Theorem 1]:

$$|\xi(t)| \leq \underline{d}(\pi), \forall t \geq 0 \ .$$

   Given a sampling schedule whose lower diameter is close to zero, this restriction imposes that the measurement noise is close to zero. In our context the measurement noise and the lower diameter are completely independent.

   Note that there exist π-solutions of the example of the Artstein's circles in closed-loop with the controller given by [S:99] which does not tend to the origin, if the measurement noise is not sufficiently small compared to the lower diameter $\underline{d}(\pi)$. See the discussion given in [S:98, Section 4].
2. Note that Theorem 1 is false if in (12) *sup* is relaxed by *esssup* (see [P:01a, Theorem 4.2] in an analogous situation). ◇

To prove Theorem 1 we need to introduce a class of hybrid patchy feedbacks (see Section 3). Then we give basics properties for the π-solutions of system (1) with a hybrid patchy feedback in Section 4 and we prove Theorem 1 in Section 5.

3 Definition of the hybrid patchy feedbacks

In the following, by $B(x,r)$, we denote the closed ball centered at x with radius r. Let Ω be a non empty open connected subset of $\mathbb{R}^n$. The closure, the interior and the boundary of Ω are written as $\mathrm{clos}(\Omega)$, $\mathrm{int}(\Omega)$ and $\partial\Omega$, respectively. Let $\mathcal{F}$ be the finite subset of $\mathbb{N}$: $\mathcal{F} = \{1,\dots,7\}$. Let $\mathcal{A}$ be a non empty totally ordered index set. Given $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ a set-valued map, we can define the solutions X of the differential inclusion

$$\dot{x} \in F(x)$$

as all absolutely continuous function satisfying $\dot{X}(t) \in F(X(t))$ almost everywhere. We follow the ideas of [AB:99, Definition 2.1], but we extend the definition to allow nested sets (as in [P:00]):

Definition 3.1 *We say that* $(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$ *is a* family of nested patchy vector fields *if:*

- *For all* α *in* $\mathcal{A}$ *and for all* l *in* $\mathcal{F}$, $\Omega_{\alpha,l}$ *is an open bounded subset of* $\mathbb{R}^n$.
- *For all* $m > l$ *in* $\mathcal{F}$ *and for all* α *in* $\mathcal{A}$,

$$\Omega_{\alpha,l} \subsetneq \mathrm{clos}(\Omega_{\alpha,l}) \subsetneq \Omega_{\alpha,m} \ . \tag{17}$$

- *For all* α *in* $\mathcal{A}$, g_α *is a smooth vector field defined in a neighbourhood of* $\mathrm{clos}(\Omega_{\alpha,7})$.
- *For all compact* C *of* $\mathbb{R}^n$, *there exists* $r > 0$ *and* $T > 0$, *such that for all* α *in* $\mathcal{A}$ *and for all* l *in* $\mathcal{F}$ *such that* $\Omega_{\alpha,l} \subset C$, *and for all solutions of*

$$\dot{x} \in g_\alpha(x) + B(0,r) \ . \tag{18}$$

starting in $\partial\Omega_{\alpha,l} \setminus \bigcup_{\beta>\alpha} \Omega_{\beta,1}$ *satisfy* $X(t)$ *is in* $\mathrm{clos}(\Omega_{\alpha,l})$, *for all* t *in* $[0,T]$.
- *For all* l *in* $\mathcal{F}$, *the sets* $(\Omega_{\alpha,l})_{\alpha\in\mathcal{A}}$ *form a locally finite covering of* Ω.

Remark 3.2 Note that we can characterize the property given by (18) in terms of proximal normal by [CLSW:98, Theorem 4.3.8] and we can redefine the notion of the patchy vector fields by using this concept of smooth analysis as done in [BW:00]. $\diamond$

An example of a family of nested patchy vector fields is given by Figure 1 for $\mathcal{A} = \{a,b\}$ with the lexicographic order $a < b$. To make the figure clearer only four open sets are shown.

With such a family of nested patchy vector fields we can define a specified hybrid controller as those considered in Section 2. Indeed we have

Definition 3.3 *Let* $(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$ *be a family of nested patchy vector fields. Assume that, for each* α *in* $\mathcal{A}$, *we can find a point* k_α *in* K *such that, for each* x *in* $\Omega_{\alpha,7}$, *we have*

$$g_\alpha(x) = f(x,k_\alpha) \ . \tag{19}$$

Let k_0 *be an arbitrary point in* K. *Then we can define the map* (u,k_d) *by*

$$\begin{array}{rcll} u : \{1,2\}^{\mathcal{A}} & \to & K & \\ s_d & \mapsto & k_0 & \textit{if} \quad \{\beta \in \mathcal{A},\, s_{d,\beta} = 1\} \textit{ is empty or infinite} \\ & & k_\alpha & \textit{if} \quad \alpha = \max\{\beta \in \mathcal{A}, s_{d,\beta} = 1\} \ , \end{array} \tag{20}$$

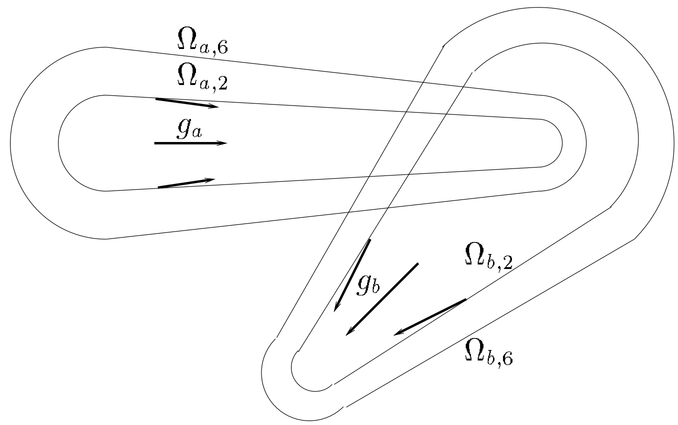

FIG. 1 – *A family of nested patchy vector fields for $\mathcal{A} = \{a,b\}$.*

and

$$\begin{array}{rcl} k_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} & \to & \{1,2\}^{\mathcal{A}} \\ (x,s_d) & \mapsto & t_d \end{array} \tag{21}$$

where t_d is the sequence defined, for all α in $\mathcal{A}$, by:

$$\begin{array}{rll} t_{d,\alpha} = 1 & \text{if} & x \in \operatorname{clos}(\Omega_{\alpha,2}), \\ t_{d,\alpha} = s_{d,\alpha} & \text{if} & x \in \Omega_{\alpha,6} \setminus \operatorname{clos}(\Omega_{\alpha,2}), \\ t_{d,\alpha} = 2 & \text{if} & x \notin \Omega_{\alpha,6}. \end{array} \tag{22}$$

Remark 3.4 The function k_d makes a hysteresis between the controllers $(k_\alpha)_{\alpha\in\mathcal{A}}$ on appropriate subsets of $\mathbb{R}^n$. For any s_d in $\{1,2\}^{\mathcal{A}}$, the function $k_d(.,s_d)$ is continuous except on the boundary of the sets defining the hysteresis. This remark is very helpful in particular to establish Lemma 4.2. ◇

Definition 3.5 *Let u: $\{1,2\}^{\mathcal{A}} \to K$ and $k_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to \{1,2\}^{\mathcal{A}}$. The map (u,k_d) is said to be a* hybrid patchy feedback *on Ω if there exists a family of nested patchy vector fields*

$$(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$$

on Ω such that (19) to (22) hold.

Figure 2 shows the hybrid patchy feedback associated to the family of nested patchy vector fields of Figure 1. The k_d component is at the left hand-side and the u at the right hand-side of Figure 2.

To investigate the robustness to noises with a family of nested patchy vector fields we generalize [AB:99, Definition 2.3] to a family of nested patchy vector fields and we introduce

Definition 3.6 *Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a continuous map such that, for all $x \neq 0$, $\chi(x) > 0$.*

- *We say that χ is an admissible radius for the measurement noises, if, for all x in $\mathbb{R}^n$ and for all α in $\mathcal{A}$ such that x in $\Omega_{\alpha,7}$, we have*

$$\chi(x) < \frac{1}{2} \min_{l\in\{1,\dots,6\}} d(\mathbb{R}^n \setminus \Omega_{\alpha,l+1},\Omega_{\alpha,l}) \ . \tag{23}$$

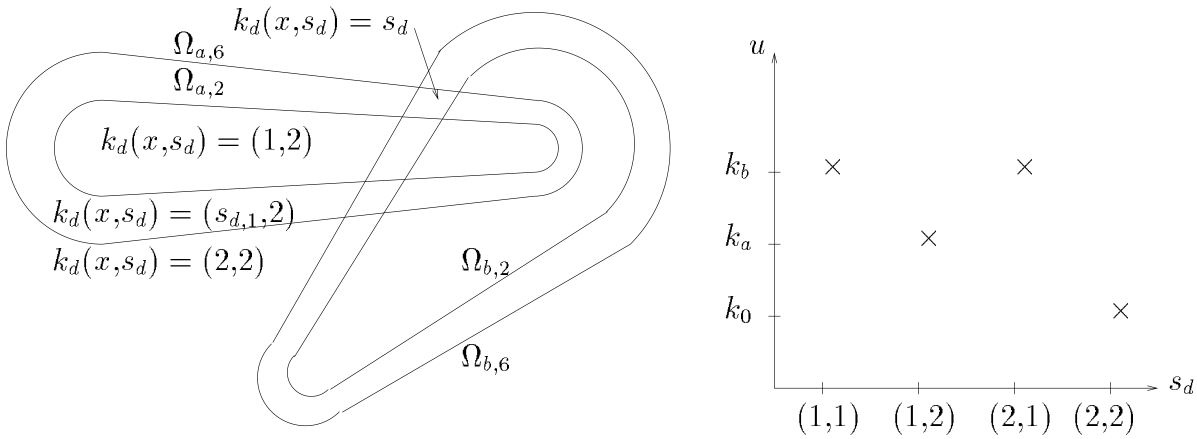

FIG. 2 – *A hybrid patchy feedback.*

– *We say that χ is an admissible radius for the external disturbances if, for all x in $\mathbb{R}^n$, by denoting C a compact subset of $\mathbb{R}^n$ containing all $\Omega_{\alpha,7}$, for all α in $\mathcal{A}$ such that x is in $\Omega_{\alpha,7}$, we have $\chi(x) \leq r$ where r is given by (18).*

Note that due to Definition 3.1, there are such functions χ. Indeed, for all x in $\mathbb{R}^n$, $\{\alpha \in \mathcal{A}, x \in \Omega_{\alpha,7}\}$ is finite. Therefore with (17) the right-hand side of the inequality (23) is strictly positive. Moreover, for all x in $\mathbb{R}^n$, such a C exists because the sets $\Omega_{\alpha,l}$ are bounded, for all α in $\mathcal{A}$ and for all l in $\mathcal{F}$.

Note that in Definition 3.3, u does not depend on x. Therefore only the function k_d depend on the measurement noises. Thus the notions of the admissible radius for the measurement noises and for the external disturbances are completely independent.

We need to consider sufficiently fast π-solutions. To define this *sufficiently fast*, let us introduce the following

Definition 3.7 *Let p: $\mathbb{R}^n \to \mathbb{R}_{>0}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$. We say that a π-solution (X,S_d) of (4) defined on $[0,T)$ has a* sampling rate less than p *if, for all i in $\mathbb{N}$ and for all t in $[\min(t_i,T), \min(t_{i+1},T))$, we have*

$$t_{i+1} - t_i \leq p(X(t_i) + \xi(X(t_i),t_i)) \ . \tag{24}$$

Now we study properties of π-solutions.

4 Properties of the π-solutions

In this section we study properties of π-solutions of a system with a hybrid patchy feedback. Let Ω be a non empty open connected subset of $\mathbb{R}^n$ and

$$(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$$

be a family of nested patchy vector fields such that (19) hold and let (u,k_d) be the hybrid patchy feedback on Ω defined by (20)-(22). Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be an admissible radius for the measurement noises and the external disturbances. Let ξ and ζ satisfying our standing regularity assumptions and

$$\forall x \in \mathbb{R}^n, \quad \sup_{t\geq 0}|\xi(x,t)| \leq \chi(x) \quad , \qquad \mathrm{esssup}_{t\geq 0}|\zeta(x,t)| \leq \chi(x) \ . \tag{25}$$

The perturbed system under consideration is

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,u(s_d)) + \zeta \ , \\ s_d & = & k_d(x+\xi,s_d^-) \ . \end{array} \right. \tag{26}$$

Now we introduce a function p as considered in Definition 3.7 and we study the properties of the π-solutions with a sampling rate less than p. Let p: $\mathbb{R}^n \to \mathbb{R}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$ and such that, for all (ξ,ζ) with our standing regularity assumptions and (25), the following assumptions hold

A1 For all x in $\mathbb{R}^n$, $p(x) > 0$.

A2 For all x in $\mathbb{R}^n$, $p(x+\xi(x,0)) < \frac{1}{4}\min_{l\in\{1,\dots,6\}} \min_{\alpha\in\mathcal{A},\, x+\xi(x,0)\in\Omega_{\alpha,7}} \dfrac{d(\mathbb{R}^n \setminus \Omega_{\alpha,l+1},\Omega_{\alpha,l})}{\sup_{y\in\Omega_{\alpha,7},\, u\in K}|f(y,u)|}$.

A3 For all x in $\mathbb{R}^n$, by denoting C a compact subset of $\mathbb{R}^n$ containing all $\Omega_{\alpha,7}$, for all α in $\mathcal{A}$ such that x is in $\Omega_{\alpha,7}$, we have $p(x+\xi(x,0)) < T$, where T is defined by (18).

The existence of such function p results from the fact that, for all l in $\{1,\dots,7\}$, $(\Omega_{\alpha,l})_{\alpha\in\mathcal{A}}$ is locally finite, (17), (23) and (25).

Definition 4.1 *A map $S_{d,\alpha}$: $[0,T) \to \{1,2\}$ is said to have a* switch *at time t if $S_{d,\alpha}$ is not continuous at t.*

Given a π-solution (X,S_d) of (26) we start by locating the points where there exists α in $\mathcal{A}$ such that $S_{d,\alpha}$ may have a switch. Note that, due to (5)-(6), if there is a switch at time t, then there exists i in $\mathbb{N}_{>0}$ such that $t = t_i$, $S_{d,\alpha}(t_i) \neq S_{d,\alpha}(t_{i-1})$.

Lemma 4.2 *Let (X,S_d) be a π-solution of (26) such that its sampling rate is less than p and such that $S_{d,\alpha}$ has a switch at time $t_i \in (0,T)$.*

- *If the switch is such that $S_{d,\alpha}(t_{i-1}) = 1$ and $S_{d,\alpha}(t_i) = 2$, then, for all t in $[t_i, \min(t_{i+1},T))$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,7}) \setminus \Omega_{\alpha,5}$.*
- *If the switch is such that $S_{d,\alpha}(t_{i-1}) = 2$ and $S_{d,\alpha}(t_i) = 1$, then, for all t in $[t_i, \min(t_{i+1},T))$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,3}) \setminus \Omega_{\alpha,1}$.*

Proof : Let α in $\mathcal{A}$ and i in $\mathbb{N}_{>0}$ such that $S_{d,\alpha}(t_{i-1}) = 1$ and $S_{d,\alpha}(t_i) = 2$. Then due to (21)-(22), $X(t_{i-1}) + \xi(X(t_{i-1}),t_{i-1})$ is in $\Omega_{\alpha,6}$. Thus with (23), Assumption A2 and (25), we prove that, for all t in $[t_{i-1}, \min(t_{i+1},T))$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,7})$. Similarly we prove that, for all t in $[t_i, \min(t_{i+2},T))$, $X(t) \notin \Omega_{\alpha,5}$. Thus we obtain that, for all t in $[t_i,t_{i+1})$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,7}) \setminus \Omega_{\alpha,5}$.

The case $S_d(t_{i-1}) = 2$ and $S_d(t_i) = 1$ is established in the same way. □

Let us claim a result of existence:

Lemma 4.3 *For all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$, there exists a π-solution starting from (x_0,s_0) of (26).*

Proof : Let (x_0,s_0) be in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$. Let $s_1 = k_d(x_0 + \xi(x_0,0),s_0)$ and α be in $\mathcal{A}$ such that $k_\alpha = u(s_1)$. From our standing regularity assumptions on f and ζ, the Carathéodory conditions are met for the system

$$\dot{X} = f(X,k_\alpha) + \zeta \quad , \qquad X(0) = x_0 \ . \tag{27}$$

Let X defined on $[0,T)$ with $0 < T \leq t_1$ a Carathéodory solution of (27). Let S_d be defined, for all t in $[0,T)$, by $S_d(t) = s_1$, for all t in $[0,T)$. (X,S_d) is a π-solution of (26) starting from (x_0,s_0). □

We note that, as usual, maximal π-solutions of (26) must blow up if their domains of definition are bounded:

Lemma 4.4 *Let ξ and ζ be satisfying our standing regularity assumptions and (25) and (X,S_d) be a maximal π-solution of (26) defined on $[0,T)$. Suppose that $T < +\infty$, then*

$$\text{limsup}_{t\to T}|X(t)| = +\infty \ .$$

Proof : Let ξ,ζ be satisfying our standing regularity assumptions and (25) and a maximal π-solution $(X(t),S_d(t))$ be defined on $[0,T)$. Suppose that the conclusion of Lemma 4.4 does not hold, i.e. there exists C a compact set of $\mathbb{R}^n$ and times t_n in $[0,T)$ tending monotonically to T such that $(X(t_n),S_d(t_n))$ is in $C \times \{1,2\}^{\mathcal{A}}$ for all n. We first establish

Claim 4.5 *For some n sufficiently large, for all $t \in [t_n,T)$, $X(t)$ is in the bounded open set $C + \text{int}(B)$, where $B = \{x \in \mathbb{R}^n,\ |x| \leq 1\}$.*

Proof of Claim 4.5: If the conclusion of Claim 4.5 is not true, the continuity of X implies the existence of $s_n \in (t_n,T)$ such that

$$|X(t_n) - X(s_n)| = 1 \quad \text{and} \quad |X(t_n) - X(t)| < 1, \forall t \in [t_n,s_n) \ .$$

It follows that $X(t)$ is in the compact set $C + B$, for all t in $[t_n,s_n]$. Let

$$\rho = \max_{x\in C+B} |\chi(x)| \quad , \qquad \sigma = \sup_{\zeta\in\rho B,\, x\in C+B,\, u\in K}|f(x,u) + \zeta| \ .$$

Then we have, for all t,s in $[t_n,s_n]$, $|X(t) - X(s)| \leq \sigma|t-s|$. Therefore, for n sufficiently large,

$$1 \ = \ |X(t_n) - X(s_n)| \ \leq \ \sigma\,|s_n - t_n| \ \leq \ \sigma\,|T - t_n| \ .$$

This cannot hold for n large enough and proves Claim 4.5. □

Claim 4.5 implies that there exists σ in $\mathbb{R}_{\geq 0}$ such that, for all (s,t) in $[t_n,T)$, we have

$$|X(s) - X(t)| \ \leq \ \sigma\,|s-t| \ .$$

It follows (by invoking Cauchy criterion) that $X(t)$ has a limit x_0 when t tends to T. Note moreover that by Definition 2.1, there exists i in $\mathbb{N}$ such that T is in $(t_i,t_{i+1}]$ and thus, for all α in $\mathcal{A}$, $\lim_{t\to T,\, t<T} S_{d,\alpha}(t)$ exists. We denote $s_0 = S_d^-(T)$. Due to Lemma 4.3, there exists $(\widetilde{X},\widetilde{S_d})$ a π-solution starting from (x_0,s_0) and defined on $[0,\widetilde{T})$ with $\widetilde{T} > 0$. Due to Definition 2.2, we have that (X',S_d') defined by

$$\begin{aligned} \forall t \in [0,T), X'(t) \ &= \ X(t) \quad , \quad S_d'(t) \ = \ S_d(t) \ , \\ \forall t \in (T,T+\widetilde{T}), X'(t) \ &= \ \widetilde{X}(t-T) \quad , \quad S_d'(t) \ = \ \widetilde{S_d}(t-T) \ , \end{aligned}$$

is a π-solution of (26) defined on $[0,T+\widetilde{T})$ for the sampling schedule $\pi \cup \{T\}$ whose restriction on $[0,T)$ is (X,S_d). So we have obtained a contradiction with the fact that (X,S_d) is a maximal π-solution. □

Now we can study the behaviour of the π-solutions between two switches. Let, for all α in $\mathcal{A}$,

$$\tau_\alpha = \sup_{X \text{being a Carathéodory solution of } \dot{x}=f(x,k_\alpha)+\zeta} \{T,\, \forall t \in [0,T),\, X(t) \in \Omega_{\alpha,7}\} \ . \tag{28}$$

Note that they may exist α in $\mathcal{A}$, such that $\tau_\alpha = +\infty$. Let M be the subset of $\Omega \times \{1,2\}^{\mathcal{A}}$ defined by

$$M = \left\{ (x,s_d), \text{s.t.} \left\{ \begin{array}{c} \{\beta \in \mathcal{A},\, s_{d,\beta} = 1\} \text{ is empty or infinite} \\ \text{or} \\ x \in \Omega_{\alpha,5}, \text{ where } \alpha = \max\{\beta,\, s_{d,\beta} = 1\} \end{array} \right\} \right\} \tag{29}$$

Note that we have the property:

$$\forall x_0 \in \Omega,\, \exists s_0 \in \{1,2\}^{\mathcal{A}},\, (x_0,s_0) \in M \ . \tag{30}$$

Lemma 4.6 *Let $(X(t),S_d(t))$ be a π-solution of (26) such that its sampling rate is less than p, defined on $[0,T)$ and starting in M. Then, there exists m in $\mathbb{N} \cup \{+\infty\}$, an increasing sequence of switches $(T_j)_{j \in \{1,\ldots,m\}}$ ($(T_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in $[0,T)$, a sequence $(\alpha_j)_{j\in\{1,\ldots,m\}}$ ($(\alpha_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in $\mathcal{A}$ and a sequence $(k_j)_{j\in\{1,\ldots,m\}}$ ($(k_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in K such that if we let $T_0 = 0$ and $T_{m+1} = T$ (if $m < +\infty$), we have, for all j in $\{0,\ldots,m\}$ (for all j in $\mathbb{N}$, if $m = +\infty$)*

1. *For all t in (T_j,T_{j+1}), $u(S_d(t)) = k_{\alpha_j}$.*
2. *The map X is a Carathéodory solution of $\dot{x} = f(x,k_{\alpha_j}) + \zeta$ on (T_j,T_{j+1}).*
3. *For all t in $[T_0,T_1)$, $X(t)$ is in $\Omega_{\alpha_0,5}$.*
4. *For all t in $[T_j,T_{j+1})$, $X(t)$ is in $\operatorname{clos}(\Omega_{\alpha_j,3})$.*
5. *The sequence $\alpha_0,\ldots,\alpha_{m+1}$ ($(\alpha_j)_{j\in\mathbb{N}}$ if $m = +\infty$) is strictly increasing.*
6. *The inequality $T_{j+1} - T_j < \tau_{\alpha_j}$ holds.*

Proof : To make the presentation of the proof clearer, we suppose there exists a finite number of switches in $[0,T)$. Thus let m be in $\mathbb{N}$ and $(T_j)_{j\in\{1,\ldots,m\}}$ be the sequence of switches in $[0,T)$. There exists a sequence $\alpha_0,\ldots,\alpha_{m+1}$ in $\mathcal{A}$ and a sequence of admissible controls $k_0,\ldots,k_{m+1}$ in K such that, by letting $T_0 = 0$ and $T_{m+1} = T$ (if $m < +\infty$), we have statements 1. and 2. of Lemma 4.6. We can suppose that, for all j in $\{0,\ldots,m\}$ (for all j in $\mathbb{N}$, if $m = +\infty$)

$$\alpha_j \neq \alpha_{j+1} \ . \tag{31}$$

Let us prove the statement 3. and $\alpha_0 < \alpha_1$.

Due to Definition 3.1, there exists α in $\mathcal{A}$ such that $X(T_0)$ is in $\Omega_{\alpha,1}$, then due to (25), (23) and (21), there exists α in $\mathcal{A}$, such that $S_{d,\alpha}(T_0) = 1$, thus by letting α_0 in $\mathcal{A}$ such that, for all t in $[T_0,T_1)$, $u(S_d(t)) = k_{\alpha_0}$, we have $\alpha_0 = \max\{\alpha,\, S_{d,\alpha}(T_0) = 1\}$.

This implies with (29) that $X(T_0)$ is in $\Omega_{\alpha_0,5}$. Similarly we can prove that, for all β in $\mathcal{A}$ such that $\alpha < \beta$, we have $X(T_0)$ is not in $\Omega_{\beta,1}$. Thus with (25), the fact that χ is an admissible radius for the external noises, (24) and Assumption A3 on the function p we deduce that, for all t in $[T_0,T_1)$, $X(t)$ is in $\Omega_{\alpha_0,5}$. Therefore with Lemma 4.2, we deduce that S_{d,α_0} can not switch at time T_1 and, for all t in $[T_1,T_2)$, $S_{d,\alpha_0}(t) = 1$.

Moreover due to (20), for all t in (T_1,T_2), we have $S_{d,\alpha_1}(t) = 1$. So, due to (20) and (31), $\alpha_0 < \alpha_1$.

Let us prove the following Claim 4.7 which implies the statements 4. and 5. of Lemma 4.6.

Claim 4.7 *For all j in $\{1,\ldots,m\}$ and for all t in $[T_j,T_{j+1})$, $X(t)$ is in $\operatorname{clos}(\Omega_{\alpha_j,3})$ and $\alpha_j < \alpha_{j+1}$.*

Proof of Claim 4.7: Let us prove Claim 4.7 by induction.

The inequality $\alpha_0 < \alpha_1$ implies with (20) that $S_{d,\alpha_1}(T_0) = 2$. Thus with Lemma 4.2, (24) and Assumption A3 we have, for all t in $[T_1,T_2)$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha_1,3})\backslash\Omega_{\alpha_1,1}$. Thus with Lemma 4.2, S_{d,α_1} can not switch at time T_2 and we have, for all t in $[T_2,T_3)$, $S_{d,\alpha_1}(t) = 1$. Moreover due to (20), for all t in (T_2,T_3), we have $S_{d,\alpha_2} = 1$. So, due to (20) and (31), $\alpha_1 < \alpha_2$.

One can inductively deduce the statements 4. and 5., for $j \geq 2$ similarly. □

To achieve the proof of Lemma 4.6, note that the statement 6. is a consequence of (28) and statements 2. and 4. □

With Lemma 4.6 at hand we can make a remark:

Remark 4.8 M is forward invariant for the system (26) for all ξ and ζ satisfying our regularity assumptions and (25). ◇

The Figure 3 shows two different π-solutions of the hybrid patchy vector field of Figure 1 by taking account of Lemma 4.2 and Lemma 4.6. The π-solution denoted by (-) starts in M, has two switches at times T_1 and T_2 and, with the notations of Lemma 4.6, the sequence α_j is strictly increasing. The other π-solution (-) does not start in M, has only one switch and its sequence α_j is strictly decreasing.

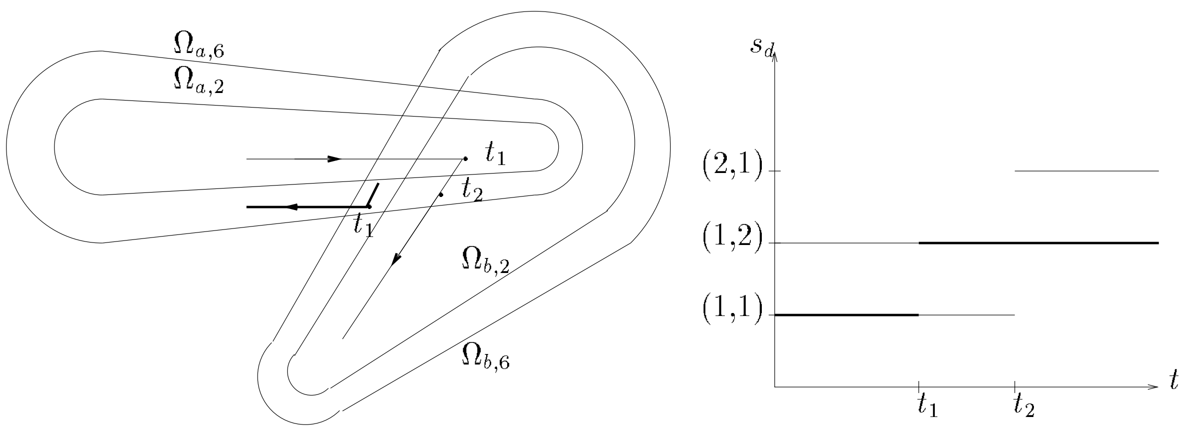

FIG. 3 – *Two $\mathcal{RC}$-solutions of with a hybrid patchy feedback.*

Due to the property (30), we can add a switch to make all π-solutions enter in M. More precisely, let $(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$ be a family of nested patchy vector fields. Assume that we have (19). Then we can define a map $u : \{1,2\}^{\mathcal{A}} \to K$ by (20) and $\widetilde{k}_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to \{1,2\}^{\mathcal{A}}$ by

$$\begin{array}{rcl} \widetilde{k}_d(x,s_d) & = & k_d(x,s_d) \text{ if } \left\{ \begin{array}{c} \{\beta \in \mathcal{A},\, s_{d,\beta} = 1\} \text{ is empty or infinite} \\ \text{or} \\ x \in \Omega_{\alpha,4}, \text{ where } \alpha = \max\{\beta, k_{d,\beta}(x,s_d) = 1\} \end{array} \right. \\ & \text{else} & \\ & = & s_0, \text{ where } s_0 \text{ is such that } x \in \Omega_{\alpha,2}, \text{ where } \alpha = \max\{\beta, s_{0,\beta} = 1\} \end{array} \tag{32}$$

Consider now the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),u(s_d(t))) + \zeta(x,t) \\ s_d(t) & = & \widetilde{k}_d(x(t) + \xi(x,t), s_d^-(t)) \end{array} \right. . \tag{33}$$

We rewrite Lemma 4.6 for all initial conditions:

Lemma 4.9 *Let $(X(t),S_d(t))$ be a π-solution of (33) such that its sampling rate is less than p, defined on $[0,T)$ and starting in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$. Then, there exists m in $\mathbb{N} \cup \{+\infty\}$, a increasing sequence of switches $(T_j)_{j\in\{1,\dots,m\}}$ ($(T_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in $[0,T)$, a sequence $(\alpha_j)_{j\in\{1,\dots,m\}}$ ($(\alpha_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in $\mathcal{A}$ and a sequence $(k_j)_{j\in\{1,\dots,m\}}$ ($(k_j)_{j\in\mathbb{N}}$ if $m = +\infty$) in K such that if we let $T_0 = 0$ and $T_{m+1} = T$ (if $m < +\infty$), we have, for all j in $\{0,\dots,m\}$ (for all j in $\mathbb{N}$ if $m = +\infty$),*

1. *For all t in (T_j,T_{j+1}), we have $u(S_d(t)) = k_{\alpha_j}$.*
2. *The map X is a Carathéodory solution of $\dot{x} = f(x,k_{\alpha_j}) + \zeta$ on (T_j,T_{j+1}).*
3. *For all t in $[T_0,T_1)$, $X(t)$ is in $\Omega_{\alpha_0,4}$.*
4. *For all t in $[T_j,T_{j+1})$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha_j,3})$.*
5. *The sequence $\alpha_1,\dots,\alpha_{m+1}$ ($(\alpha_j)_{j\in\mathbb{N}}$ if $m = +\infty$) is strictly increasing.*
6. *The inequality $T_{j+1} - T_j < \tau_{\alpha_j}$ holds.*

Proof : The proof of statements 1. and 2. of Lemma 4.9 are completely analogous of the proof of statements 1. and 2. of Lemma 4.6.

Due to (32), $X(T_0) + \xi(X(T_0),T_0)$ is in $\Omega_{\alpha_0,4}$. Then due to (25) and (23), we have $X(T_0)$ is in $\Omega_{\alpha_0,5}$. Similarly we can prove that, for all β in $\mathcal{A}$ such that $\alpha < \beta$, we have $X(T_0)$ is not in $\Omega_{\beta,1}$. Therefore with (25), the fact that χ is an admissible radius for the external noises, (24) and Assumption A3 on the function p, we have statements 3.

This implies that $X(T_1)$ is in $\Omega_{\alpha_0,4}$ and therefore $(X(T_1),S_d(T_1))$ is in M and we deduce the statements 4. to 6. of Lemma 4.9 from statements 4. to 6. of Lemma 4.6. □

Remark 4.10 Note that after the first switch, we have $\widetilde{k}_d(X(t) + \xi(X(t),t)) = ;k_d(X(t) + \xi(X(t),t))$, for all π-solutions. And thus after the first switch, every π-solution of (33) is a π-solution of (26) and in particular we have the conclusion of Lemma 4.4. ◇

5 Use of the asymptotically controllability

Now we use properties of a differential system in closed-loop with a hybrid patchy feedback. The purpose of this section is to prove Theorem 1. Let us prove the following measurement noises robust version of [AB:99, Proposition 4.1]:

Proposition 5.1 *Let (1) be globally asymptotically controllable to the origin. Then for every $0 < r < s$, there exists $T,R,\chi > 0$, an open subset of $\mathbb{R}^n$, $D^{r,s}$, and a feedback control, $u = u^{r,s} : \{1,2\}^{\mathbb{N}} \to K$, $k_d = k_d^{r,s} : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ satisfying:*

$$B(0,s) \setminus \mathrm{int}(B(0,r)) \subset D^{r,s} \subset B(0,R) \tag{34}$$

such that, for any measurable maps ζ, $\xi : [0,+\infty) \to \mathbb{R}^n$ satisfying,

$$\mathrm{sup}_{t\geq 0}|\xi(t)| \leq \chi \quad , \qquad \mathrm{esssup}_{t\geq 0}|\zeta(t)| \leq \chi \ , \tag{35}$$

and for any initial state x_0 in $D^{r,s} \setminus \mathrm{int}(B(0,r))$, and for any s_0 in $\{1,2\}^{\mathbb{N}}$ the perturbed system:

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x,u(s_d)) + \zeta \ , \\ s_d &=& k_d(x + \xi, s_d^-) \end{array} \right. \tag{36}$$

admits a π-solution (X,S_d) starting from (x_0,s_0). Moreover for all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathbb{N}}$ and for any maximal π-solution (X,S_d) starting from (x_0,s_0), there exists $t_{X,S_d} \leq T$, $t_{X,S_d} < \tau^{\max}(X,S_d)$ such that:

$$|X(t_{X,S_d})| < r \tag{37}$$

Proof : We follow the proof of [AB:99, Proposition 4.1] and we prove Proposition 5.1 in four steps.

Step 1. Fix $0 < r < s$. For each x_0 in $B(0,s)$, there exists a piecewise constant admissible control $u_0 = u_{x_0}$ and some constant $T_0 = T_{x_0}$ such that there exists a solution $X_0 = x(.;x_0,u_0)$ of $\dot{x} = f(x,u_0)$ for which the inequality

$$|X_0(T_0)| < \frac{r}{2} \tag{38}$$

holds. Moreover, on the one hand by continuity we can suppose that we have

$$m := \inf_{t\in[0,T_0]}|X_0(t)| > \frac{r}{4} \tag{39}$$

and, on the other hand, by possibly redefining u_0, we may assume that X_0 takes different values at any two different points t,t' in $[0,T_0]$. Let $\tau_{0,0} = 0 < \ldots < \tau_{0,N_0} = T_0$ be the points of discontinuities for u_0 on $[0,T_0]$ and $k_{0,j}$ in K the corresponding values of u_0, i.e. we suppose that, for all j in $\{0,\ldots,N_0-1\}$ and for all t in $(\tau_{0,j},\tau_{0,j+1})$, we have $u_0(t) = k_{0,j}$. Set

$$M_0 = M_{x_0} = \sup_{t\in[0,T_0]}|X_0(t)| \ . \tag{40}$$

There exists some strictly positive constants $c_0 = c_{x_0}$, $\overline{\rho}_0 = \overline{\rho}_{x_0}$ and $\overline{\chi}_0 = \overline{\chi}_{x_0}$ such that, for any fixed τ in $[0,T_0]$, any strictly positive radius $\rho \leq \overline{\rho}_0$ and $\chi \leq \overline{\chi}_0$, any initial point $\overline{x}$ in $B(X_0(\tau),\rho)$, and any Carathéodory solution $X_{\rho,\chi}(.)$, of

$$\dot{x} \in f(x,u_0(t)) + B(0,\chi) \tag*{(41)$_\chi$}$$

starting from $\overline{x}$ at time $t = \tau$, there holds:

$$\sup_{[\tau,T_0+\rho]}|X_{\rho,\chi}(t) - X_0(t)| < c_0(\rho + \chi) \ . \tag{42}$$

Let two strictly positive reals $\rho_0 \leq \overline{\rho}_0$ and $\chi_0 \leq \overline{\chi}_0$ be such that by letting

$$\rho_{x_0,1} = \rho_{0,1} = \rho_0 \ , \tag{43}$$

and, for all j in $\{2,\ldots,N_0+1\}$,

$$\rho_{x_0,j} = \rho_{0,j} = \Sigma_{k=3}^{j} 7^{k-3} c_0^{k-2} 2\chi_0 + 7^{j-2} c_0^{j-1}(7\rho_0 + 2\chi_0) \ ,$$

we have

$$7\rho_{x_0,N_0+1} < \frac{r}{2} \ ,$$

and

$$m - 7\max_j \rho_{x_0,j} > \frac{r}{8} \ , \tag{44}$$

where m is defined by (39). Thus one can inductively deduce that for any fixed $j = 1,\dots,N_0$, if for any $\overline{x}$ such that:

$$\overline{x} \in B(X_0(\tau_{0,j}),7\rho_{0,j}) \ ,$$

we let $X_{\rho_{0,j},2\chi_0}(.)$ be any Carathéodory solution of $(41)_{2\chi_0}$ starting from $\overline{x}$ at time $\tau_{0,j}$, then one has

$$\sup_{t\in[\tau_{0,j},T_0+\rho_{0,j}]}|X_{\rho_{0,j},\chi_0}(t) - X_0(t)| < \rho_{0,j+1}. \tag{45}$$

Step 2. For j in $\{0,\dots,N_0-1\}$ and for any $\overline{x}$ in $\mathbb{R}^n$, denote $\mathcal{A}_j(\overline{x},t)$ the attainable set in time t for the Carathéodory solution of $\dot{x} \in f(x,k_{0,j}) + B(0,2\chi_0)$ starting from $\overline{x}$. Let us define, for all l in $\{1,\dots,7\}$ and for all j in $\{1,\dots,N_0-1\}$, the open sets

$$\Gamma_{x_0,j,l} = \bigcup_{\substack{\overline{x} \in \operatorname{int}(B(X_0(\tau_{0,j-1}),l\rho_{0,j})) \\ 0 \le t \le \tau_{0,j} - \tau_{0,j-1}}} \mathcal{A}_j(\overline{x},t)$$

and

$$\Gamma_{x_0,N_0,l} = \bigcup_{\substack{\overline{x} \in \operatorname{int}(B(X_0(\tau_{0,N_0-1}),l\rho_{0,j})) \\ 0 \le t \le T_0 + \rho - \tau_{0,N_0-1}}} \mathcal{A}_{N_0}(\overline{x},t) \ .$$

Note that we have, for all j in $\{1,\dots,N_0\}$ and for all $l < l'$ in $\{1,\dots,7\}$,

$$\Gamma_{x_0,j,l} \subsetneq \operatorname{clos}(\Gamma_{x_0,j,l}) \subsetneq \Gamma_{x_0,j,l'} \ , \tag{46}$$

and, due to (40) and (45), we have

$$\Gamma_{x_0,j,l} \subset B(0,7\rho_{x_0,j+1} + M_{x_0}) \ , \tag{47}$$

and, due to (39), (45) and (44), we have

$$B(0,\frac{r}{8}) \subset \mathbb{R}^n \setminus \Gamma_{x_0,j,l} \ . \tag{48}$$

Finally, for all s a N_0-tuple taking value in $\{1,\dots,7\}$, define

$$\Delta_{x_0,s} \ = \ \bigcup_{j=1}^{N_0} \Gamma_{x_0,j,s(j)} \quad , \qquad \Delta_{x_0} = \Delta_0 = \Delta_{x_0,\mathbf{1}_{x_0}}.$$

where, for any point x_0 in $B(0,s)\setminus\operatorname{int}(B(0,r))$, we denote $\mathbf{1}_{x_0}$ the following constant sequence:

$$\begin{array}{rcl} \mathbf{1}_{x_0} = \mathbf{1}_0 : \{1,\dots,N_0\} & \to & \{1,\dots,7\} \ , \\ l & \mapsto & 1 \ . \end{array}$$

Let $g_{0,j} = g_{x_0,j}$ be the vector field on $\mathbb{R}^n$ defined, for all j in $\{1,\dots,N_0-1\}$, by

$$g_{0,j}(x) = f(x,k_{0,j}) \ . \tag{49}$$

We can claim that

$$(\Delta_0,((\Gamma_{0,j,l})_{l\in\mathcal{F}},g_{0,j})_{j\in\{1,\dots,N_0\}})$$

is a family of nested patchy vector fields as considered in Definition 3.1. Indeed note in particular that we have (17) due to (46), and we have (18) because there exists $T > 0$ such that all solutions of $(41)_{2\chi_0}$ starting in $\partial\Gamma_{x_0,j,l} \setminus \bigcup_{j'>j} \Gamma_{x_0,j',1}$ stay in $\mathrm{clos}(\Gamma_{x_0,j,l})$, for all t in $[0,T)$. Moreover, due to (49), we can define a hybrid patchy feedback (u^0,k_d^0) as considered in Definition 3.3 and thus $(u^0,\widetilde{k}_d^0)$ a feedback control defined by (32). We can take χ_0 smaller and suppose that

$$0 < \chi_0 < \frac{1}{2} \min_{j\in\{1,\dots,N_0\}} \min_{l\in\{1,\dots,6\}} d(\mathbb{R}^n \setminus \Gamma_{0,j,l+1}, \Gamma_{0,j,l}) \ .$$

and note that χ_0 is an admissible radius for the external disturbances. Let two measurable maps $\zeta, \xi : [0,+\infty) \to \mathbb{R}^n$ satisfying,

$$\sup_{t\geq 0}|\xi(t)| \leq \chi_0 \quad , \qquad \mathrm{esssup}_{t\geq 0}|\zeta(t)| \leq \chi_0 \ ,$$

and let (x_0,s_0) be an initial condition in $\Delta_0 \setminus B(0,r) \times \{1,2\}^{N_0}$. Due to Lemma 4.3, there exists (X,S_d) a maximal π-solution of (36) in closed-loop with $(u^0,\widetilde{k}_d^0)$ starting from (x_0,s_0) and defined on $[0,\tau^{max})$. Moreover due to properties established Lemma 4.9, there exists H in $\mathbb{N} \cup \{+\infty\}$, a sequence of points $t_0 = 0 < \cdots < t_H \leq 2T_0$ and a sequence of indices $j_0,\dots,j_H$ in $\{1,\dots,N_0\}$, such that there holds, for all h in $\{1,\dots,H-1\}$,

$$\forall t \in [t_h,t_{h+1}),\ X(t) \in \Gamma_{x_0,j_h,7} \ , \tag{50}$$

$$t_{h+1} - t_h \leq \tau_{0,j_h} \ . \tag{51}$$

Note that due to Lemma 4.9, the sequence $j_1,\dots,j_H$ described below is strictly increasing. Due to (50) and (47), (X,S_d) can not blow up in $\Gamma_{x_0,j_h,7}$, for all h in $\{0,\dots,H-1\}$, and due to Lemma 4.4, (50) and (47), there exists $T^0_{X,S_d} \leq 2T_0$ such that, we have the inequalities

$$\forall t \in [0,T^0_{X,S_d}),\ |X(t)| \leq 7 \max_{j\in\{1,\dots,N_0\}} \rho_{0,j+1} + M_0 \ , \tag{52}$$

$$|X(T^0_{X,S_d})| < r \ . \tag{53}$$

Step 3. Since x_0 is in Δ_{x_0}, the family of open tubes $\{\Delta_{x_0},\ r \leq |x_0| \leq s\}$ form an open covering of the compact set $B(0,s) \setminus \mathrm{int}(B(0,r))$. Let

$$\{\Delta_i,\ i \in \{1,\dots,N(r,s)\}\}, \quad \Delta_i = \bigcup_{j=1}^{N_i} \Gamma_{i,j,1}, \quad \Gamma_{i,j,1} = \Gamma_{x_i,j,\mathbf{1}_{x_i}(j)} \ ,$$

be a finite subcover. Denote

$$k_{i,j} = k_{x_i,j} \ ,$$

and the vector field

$$g_{i,j}(x) = f(x,k_{i,j}) \tag{54}$$

defined on $\mathbb{R}^n$. The index set:

$$\mathcal{A} = \{(i,j),\ i \in \{1,\dots,N(r,s)\},\ j \in \{1,\dots,N_i\}\}$$

can be totally ordered by letting:

$$(i,j) < (h,k) \quad \text{if either} \quad i < h \quad \text{or else} \quad i = h,\ j < k \ . \tag{55}$$

Let

$$D^{r,s} = \bigcup_{i=1}^{N(r,s)} \Delta_i \ .$$

We can now define a family of nested patchy vector fields on $\Omega = D^{r,s}$. Let $\mathcal{F}$ be the finite set $\mathcal{F} = \{1,\dots,7\}$ and, for all α in $\mathcal{A}$ and for all l in $\mathcal{F}$, $\Omega_{\alpha,l}$ be the open set $\Gamma_{i,j,l}$ where $(i,j) = \alpha$. Note that due to (46), for all $m > l$ in $\mathcal{F}$ and for all α in $\mathcal{A}$, we have (17) and (18) can be proved as in Step 2. Therefore we can claim that

$$(\Omega,((\Omega_{\alpha,l})_{l\in\mathcal{F}},g_\alpha)_{\alpha\in\mathcal{A}})$$

is a family of nested patchy vector fields. Moreover, due to (54), we can define a hybrid patchy feedback $(u^{r,s},k_d^{r,s})$ and thus a feedback control $(u^{r,s},\widetilde{k}_d^{r,s})$ as in (32). We let

$$\chi^{r,s} = \min_{1\leq i\leq N(r,s)} \chi_{x_i} \ ,$$

which is an admissible radius for the external disturbances. We can choose $\chi^{r,s}$ smaller and suppose that:

$$0 < \chi^{r,s} < \frac{1}{2} \min_{(i,j)\in\mathcal{A}} \ \min_{l\in\{1,\dots,6\}} d(\mathbb{R}^n \setminus \Gamma_{i,j,l+1},\, \Gamma_{i,j,l}) \ .$$

Then $\chi^{r,s}$ is an admissible radius for the measurement noises.

Step 4 For all x_0 in $B(0,s) \setminus B(0,r)$, let $T_{x_i} > 0$ be defined in Step 1 and $\chi^{r,s} > 0$ as in Step 3. Let

$$T = 2 \sum_{i=1}^{N(r,s)} T_{x_i} \ ,$$

and two measurable maps ξ and ζ: $[0,+\infty) \to \mathbb{R}^n$ be such that

$$\sup_{t\geq 0}|\xi(t)| \leq \chi^{r,s} \quad , \qquad \operatorname{esssup}_{t\geq 0}|\zeta(t)| \leq \chi^{r,s} \ ,$$

Let (x_0,s_0) be an initial condition in $D^{r,s} \setminus B(0,r) \times \{1,2\}^{\mathcal{A}}$. Due to Lemma 4.3, there exists (X,S_d) a maximal π-solution of (36) in closed-loop with $(u^{r,s},\widetilde{k}_d^{r,s})$ starting from (x_0,s_0). Moreover due to properties established in Step 3 and Lemma 4.6, there exists H in $\mathbb{N}\cup\{+\infty\}$, a sequence of points $t_0 = 0 < \cdots < t_H \leq T$ and a sequence of indices $\alpha_1,\dots,\alpha_H$ in $\mathcal{A}$, such that there holds, for all h in $\{0,\dots,H-1\}$,

$$\forall t \in [t_h,t_{h+1}),\ X(t) \in \Gamma_{\alpha_h,7} \ , \tag{56}$$

$$t_{h+1} - t_h < \tau_{\alpha_h} \ . \tag{57}$$

Note that due to Lemma 4.9, the sequence $\alpha_1,\dots,\alpha_H$ described below is strictly increasing. Due to (56) and (47), (X,S_d) can not blow up in $\Gamma_{\alpha_h,7}$, for all h in $\{0,\dots,H-1\}$, and due to Lemma 4.4, (56), there exists $T_{X,S_d} \leq T$ such that, we have the inequalities

$$\forall t \in [0,T],\ |X(t)| < R \ , \tag{58}$$

$$|X(T)| < r \ . \tag{59}$$

where R is defined by

$$R = \sup_{1\leq i\leq N(r,s),1\leq j\leq N_i}\{7\rho_{x_i,j} + M_{x_i}\} \ .$$

This achieves the proof of Proposition 5.1. □

Proposition 5.2 *Let (1) be globally asymptotically controllable to the origin. Then for any fixed* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for every* $0 < r < s \leq \delta$*, there exists* $T,R,\chi > 0$*, an open subset of* $\mathbb{R}^n$*,* $D^{r,s}$ *and a feedback control,* $u = u^{r,s} : 2^{\mathbb{N}} \to K$*,* $k_d = k_d^{r,s} : \mathbb{R}^n \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *as in Proposition 5.1 with*

$$R < \varepsilon .$$

Proof : The proof is similar to the proof of [AB:99, Proposition 4.2] and consists in choosing the piecewise constant admissible control u_{x_0} for each point x_0 in $B(0,s)\setminus \mathrm{int}(B(0,r))$ properly. □

We are now ready to prove Theorem 1:

Proof of Theorem 1:

Part 1: Definition of the feedback control

Let $(r_n,s_n)_{n\in\mathbb{Z}}$ be a decreasing sequence of strictly positive numbers such that

- For all n in $\mathbb{Z}$, we have $r_{n-1} < s_n$
- s_n converges to zero as $n \to +\infty$,
- r_{-n} converges to infinity as $n \to +\infty$.

Let $T_n = T(r_n,s_n)$, $R_n = R(r_n,s_n)$, $\chi_n = \chi(r_n,s_n)$ be three sequence of strictly positive numbers and a sequence of hybrid patchy feedback

$$(D^{r_n,s_n},u^{r_n,s_n},k_d^{r_n,s_n},((\Gamma_{i,l}^n)_{l\in\{1,\ldots,7\}},k_i^n)_{i\in\{1,\ldots,N_n\}})$$

such that

$$R_n < \frac{1}{n}, \forall n \in \mathbb{N}_{>0} \tag{60}$$

as in Proposition 5.2. The index set

$$\mathcal{B} = \{(n,i),\, n \in \mathbb{Z},\, i \in \{1,\ldots,N_n\}\}$$

can be totally ordered with the same relation of order as (55) i.e. by letting

$$(n,i) < (m,j) \quad \text{if either} \quad n < m \quad \text{or else} \quad n = m,\, i < j\ .$$

Then we have the following family of nested patchy vector fields on $\mathbb{R}^n \setminus \{0\}$,

$$(\mathbb{R}^n \setminus \{0\},((\Gamma_{i,l}^m)_{l\in\{1,\ldots,7\}},k_i^m)(m,i) \in \mathcal{B})\ .$$

We can define a hybrid patchy feedback (u,k_d) on $\mathbb{R}^n \setminus \{0\}$ as in Definition 3.3. Let $\chi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_{>0}$ be a continuous map satisfying

$$\chi(x) \leq \min(\chi_n,\frac{|x|}{2}) \quad \text{if} \quad x \in D^{r_n,s_n} \setminus \bigcup_{m>n} D^{r_m,s_m}\ . \tag{61}$$

We define $\chi(0) = 0$. The map χ is continuous at 0 and then χ is an admissible radius for the measurement noises and the external disturbances. Let $(u,\widetilde{k}_d)$ be the feedback control defined by (32) for the hybrid patchy feedback (u,k_d). Let us prove that $(u,\widetilde{k}_d)$ is a global robust stabilizing controller on $\mathbb{R}^n$, i.e. that the origin of the system (33) is a robust globally asymptotically stable equilibrium.

Part 2: Theorem 1 for the π-solutions Let p: $\mathbb{R}^n \to \mathbb{R}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$ satisfying the properties A1, A2, and A3.

Existence of π-solutions

Let ξ, ζ satisfying our standing regularity assumptions. Let (x_0,s_0) be in $\mathbb{R}^n \times \{1,2\}^{\mathcal{B}}$. Let $s_1 = \widetilde{k}_d(x_0 + \xi(x_0,0),s_0)$ and α be in $\mathcal{B}$ such that $k_\alpha = u(s_1)$. From our standing regularity assumptions on f and ζ, the Carathéodory conditions are met for the system (27). Let X defined on $[0,T)$ with $0 < T \leq t_1$ a Carathéodory solution of (27). Let S_d be defined by $S_d(t) = s_1$, for all t in $[0,T)$. (X,S_d) is a π-solution of (33) starting from (x_0,s_0).

Maximality and global stability of the π-solutions

Let $\varepsilon > 0$. Let n in $\mathbb{N}$ such that $\varepsilon < R_{-n}$. Such R_{-n} exists because we have $r_{-n} \leq R_{-n}$ and r_{-n} tends to infinity as $n \to +\infty$. Let d_0 and p_0 respectively defined by

$$d_0 = \min\left(\inf_{x\in B(0,R_{-n})\setminus B(0,r_{-n}), y\in B(0,\chi(x))} p(x+y), \frac{d(\mathbb{R}^n \setminus B(0,s_{-n}),B(0,r_{-n}))}{\max_{x\in B(0,s_{-n}),\, u\in K} |f(x,u)|}\right) \tag{62}$$

$$\chi_0 = \inf_{x\in B(0,R_{-n})\setminus B(0,r_{-n})} \chi(x) \ . \tag{63}$$

Note that due to (61), we have $d_0 > 0$ and $\chi_0 > 0$. Let ξ, ζ satisfying our standing regularity assumptions and (12). Let (X,S_d) be a π-solution of (33) maximally defined on $[0,\tau^{max})$ starting from (x_0,s_0), with $|x_0| < s_{-n}$ and (13).

Note that due to (62) and (63), for all i in $\mathbb{N}$ such that $X(t_i)$ is in $B(0,R_{-n}) \setminus B(0,r_{-n})$, we have (24) and, for all t such that $X(t)$ is in $B(0,R_{-n}) \setminus B(0,r_{-n})$, we have (25).

Therefore, due to Proposition 5.1 and the definition of the feedback control, if there exists i in $\mathbb{N}$ such that $X(t_i)$ is in $B(0,s_{-n}) \setminus B(0,r_{-n})$, then there exists $j > i$ such that $X(t_j)$ is in $B(0,r_{-n})$ and for all t in $[i,j]$, $X(t)$ is in $B(0,R_{-n})$. Moreover due to (62), if there exists i in $\mathbb{N}$ such that $X(t_i)$ is in $B(0,r_{-n})$, then for all t in $[t_i,t_{i+1}]$, we have $X(t)$ is in $B(0,s_{-n})$.

Therefore we have, for all t in $[0,\tau^{max})$,

$$|X(t)| \leq R_{-n} \ . \tag{64}$$

Note that $\delta(\varepsilon) = s_{-n}$ tends to $+\infty$ as ε tends to infinity because when ε tends to infinity, n tends to infinity, r_{-n} tends to infinity and we have $r_{-n-1} < s_{-n}$.

The maximality and the stability result form the following:

$$\tau^{max} = +\infty \ . \tag{65}$$

We prove (65) by exhibiting a contradiction. Let us suppose that $\tau^{max} < +\infty$. Let I in $\mathbb{N}$ such that $\tau^{max} \in (t_I,t_{I+1}]$. Let us prove the following

Claim 5.3 $t_{I+1} = \tau^{\max}$.

Proof of Claim 5.3: Suppose that $t_{I+1} > \tau^{max}$, then the interval of definition of S_d can be extended to $[0,t_{I+1})$, by letting $S_d(t) = S_d(t_I)$, for all t in $[t_I,t_{I+1})$. Moreover by letting α_I

such that $k_{\alpha_I} = u(X(t_I) + \xi(X(t_I),t_I),S_d(t_I))$, we have that, for all t in $[0,\tau^{max} - t_I)$, $X(t_I + t)$ is a Carathéodory solution of

$$\dot{Y} = f(Y,k_{\alpha_I}) + \zeta \quad , \qquad Y(0) = X(t_I) \ , \tag{66}$$

and, for all Carathéodory solution Y of (66) defined on an interval $[0,T_0)$ with $T_0 < t_{I+1} - t_I$, we have that (X',S_d) is a π-solution on $[0,T_0 + t_I)$ with X' defined by, for all t in $[0,t_I)$

$$X'(t) = X(t)$$

and, for all t in $[t_I,T_0 + t_I)$,

$$X'(t) = Y(t - t_I) \ .$$

Therefore X is a maximal Carathéodory solution of (66) defined on $[t_I,\tau^{max})$ with $\tau^{max} < t_{I+1}$. Therefore $\lim_{t \nearrow \tau^{max}} |X(t)| = +\infty$. This is a contradiction with (64). □

Note that due to Definition 2.1 and (64), $\lim_{t \to t_{I+1}} X(t)$ exists. Let

$$x_0 = \lim_{t \to t_{I+1}} X(t) \quad , \qquad s_0 = k_d(x_0 + \xi(x_0,t_{I+1}),S_d(t_I)) \ .$$

Let $(\widetilde{X},\widetilde{S_d})$ be a π'-solution starting from (x_0,s_0) at time t_{I+1} and defined on $[t_{I+1},T_0)$ with $t_{I+2} > T_0 > t_{I+1}$. Such π-solution exists due to the beginning of the Part 2 of the proof of Theorem 1. Let (X,S_d) obtained as the concatenation of (X,S_d) and $(\widetilde{X},\widetilde{S_d})$. This is a π-solution defined on $[0,T_0)$ with $T_0 > \tau^{max}$. This a contradiction with the maximality of (X,S_d).

Therefore we have (65) and the maximality and stability properties.

Global attractivity

Let $\varepsilon > 0$ and $\delta > 0$. Let n in $\mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and such that $\delta < r_{-n}$. Let $d_0 > 0$ and $\chi_0 > 0$ respectively defined by

$$d_0 = \inf_{x \in B(0,R_{-n}) \setminus B(0,r_n)} p(x) \ , \tag{67}$$

$$\chi_0 = \inf_{x \in B(0,R_{-n}) \setminus B(0,r_n)} \chi(x) \ . \tag{68}$$

Let ξ, ζ satisfying our standing regularity assumptions and (12). Let (X,S_d) be a π-solution defined on $[0, +\infty)$, starting from (x_0,s_0) with $\overline{d}(\pi) < d_0$ and $|x_0| < \delta$. Moreover due to Proposition 5.1, there exists $\widetilde{T}$ in $[0,T_{-n} + T_{-n+1} + \cdots + T_n]$ such that $|X(t)| < r_n$. Let $T' = \inf\{t \in [0,\widetilde{T}], |X(t)| < s_n\}$. Then due to the stability property and $R_n < \frac{1}{n}$, we have

$$\forall t \geq \widetilde{T}, |X(t)| \leq \frac{1}{n} \ .$$

Therefore we have (15) with $T = T_{-n} + \cdots + T_n$.

Part 3: Theorem 1 for the generalized solutions

Let us prove the statements of Theorem 1 for the generalized solutions.

The system is complete for the generalized solutions

This results from the fact that every π-solution of (33) is a generalized solution of (33).

Global stability and global attractivity

Let $\varepsilon > 0$. Let $\delta > 0$, $\chi_0 > 0$ and $d_0 > 0$ such that we have the stability property (14) for all π-solutions of (33) such that we have (13) and for all ξ, ζ satisfying our standing regularity assumptions and (12).

Let X be a generalized solution of (33) starting from x_0 with ξ, ζ satisfying our standing regularity assumptions and

$$\sup_{x\in\mathbb{R}^n,\, t\geq 0}|\xi(x,t)| \leq \frac{\chi_0}{2} \quad , \qquad \operatorname{esssup}_{x\in\mathbb{R}^n,\, t\geq 0}|\zeta(x,t)| \leq \frac{\chi_0}{2} \ , \tag{69}$$

and obtained as limit of π-solutions (X^n,S_d^n) satisfying (13). Let us prove that we have (14).

For n sufficiently large, we have

$$\sup_J|e_n(t)| + \operatorname{esssup}_J|d_n(t)| < \frac{\chi_0}{2} \ , \tag{70}$$

for all J compact subinterval of $[0,T)$. Then for n sufficiently large, (X^n,S_d^n) is a π-solutions of (33) such that we have (13) with a disturbance satisfying (12). Then we have (14) for this sequence of π-solutions. Therefore we have (14) for the generalized solution X.

The global attractivity property can be similarly proved.

Part 4: Theorem 1 for the Euler solutions

Existence and maximality of the Euler solutions

Let x_0,s_0 in $\mathbb{R}^n \times \{1,2\}^{\mathcal{B}}$ and π_n be a sequence of sampling schedule such that $\overline{d}(\pi_n) \to 0$ as n tends to infinity. Let (X^n,S_d^n) be a maximal π-solution of (33), starting from (x_0,s_0) and defined on $[0,+\infty)$. Due to the Part 2 of the proof of Theorem 1, this sequence exists for n sufficiently large and there exists R such that, for all t in $[0,+\infty)$ and for n sufficiently large, we have

$$|X_n(t)| < R \ .$$

Therefore with Ascoli's Theorem, we can define X an Euler solution defined on $[0,+\infty)$ and starting from x_0.

Global stability and global attractivity

Let $\varepsilon > 0$. Let $\delta > 0$, χ_0 and $d_0 > 0$ such that we have the stability property (14) for all π-solutions of (33) such that we have (13) and for all ξ, ζ satisfying our standing regularity assumptions and (12).

Let X be a Euler solution of (33) starting from x_0 with ξ, ζ satisfying our standing regularity assumptions and (12) and obtained as limit of π-solutions (X^n,S_d^n) satisfying $\overline{d}(\pi_n) \to 0$ as n tends to infinity.

Let us prove that we have (14).

For n sufficiently large, we have $\overline{d}(\pi_n) < d_0$. Then for n sufficiently large, (X^n,S_d^n) is a π-solution of (33) such that we have (13) with a disturbance satisfying (12). Then we have (14) for this sequence of π-solutions. Therefore we have (14) for the generalized solution X.

The global attractivity can be similarly proved.

This achieves the proof of Theorem 1. $\square$

Robust stabilization of chained systems via hybrid control

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Abstract: A hybrid state feedback control law for nonholonomic chained systems is proposed. It is shown that the control law yields global asymptotic stability and global robustness against a class of small measurement errors and exogenous disturbances. These perturbations are supposed to vanish in a set of measure zero strictly containing the origin and with a diameter that can be arbitrarily reduced by a proper selection of the controller parameters.
Keywords: Chained systems, hybrid control, robust stabilization, measurement noise, exogenous disturbances.

1 Introduction

The problem of asymptotic stabilization of chained systems has been widely studied in the last decade and several control laws, yielding diverse asymptotic properties, have been proposed, see [MPS:98, KM:95] and references therein. It is therefore fair to conclude that the stabilization problem is well understood from a mathematical point of view, and that various design tools are available for applications.

On the contrary, the robust stabilization problem for nonholonomic systems is not yet completely solved. Several attempts have been made to study the robustness properties of existing control laws or to robustify given controllers [ALM:99, MMPS:99, J:00]. Most of the robust stabilization results and investigations focus on the problems of parametric uncertainties or model errors, see *e.g.* [MS:00] where the problem of local robust stabilization by means of time-varying control laws have been studied; [J:00], where a similar problem has been addressed using the class of discontinuous control laws introduced in [A:95], and [LG:95, HLM:99, MS:00], where several types of hybrid control laws have been used to achieve local robustness against unknown parameters or unmodelled dynamics. On the other hand, the fundamental problems of robustness in the presence of sensor noise, external disturbances and actuator disturbances have been only partially addressed, see *e.g.* [ALM:99, VA:01]. These problems are of special interest and relevance whenever discontinuous control laws are employed, as for such control laws classical robustness results and Lyapunov theory are not directly applicable. Note that, discontinuous control laws yield (at least in the ideal case of known parameters, zero disturbances and no unmodelled dynamics) superior convergence properties then time-varying control laws. However, most of the existing discontinuous control laws are non-robust against exogenous disturbances and measurement noise. This means that for any initial condition $\bar{x}$ the trajectory of the closed–loop system with the nominal control $u(x)$ converges exponentially to zero, whereas, for any non-zero e, the trajectory of the system with the perturbed control law $u(x+e)$ is in general not close to the nominal trajectory and may diverge. Moreover,

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the presence of arbitrarily small exogenous disturbances may generate unbounded trajectory, see *e.g.* [ALM:99]. Hence, it makes sense to study the robustness properties of (a class of) discontinuous control laws and to discuss possible robustification procedures.

Following the line of research started in [A:95], we make use of a special class of discontinuous control laws, and we show how, adding a proper modification together with a *hybrid variable*, it is possible to obtain a closed–loop system with global stability properties and which is globally robust against a class[3] of measurement noises and exogenous disturbances. The controller construction proposed in the present paper takes inspiration partly from the results in [P:00], and partly from the results in [VA:01]. In the former, a hybrid control law achieving robust stabilization of the so-called Artstein circle has been proposed; whereas the latter provides some basic tools to construct two control laws, the so-called *local controller* and *global controller* that, together with the hybrid dynamics, yield a robust closed–loop system. Note finally, that the main result of the paper relies on the general results in [P:01a], which is however not directly applicable because of the special nature of the system considered.

The robustness property that is investigated in the sequel is strongly related to the notion of ISS [S:95]. Namely, it will be assumed that the measurement noise and the external disturbances are upper bounded by functions of the state and vanishes in a non-empty set with zero measure and with a diameter that can be arbitrarily reduced by a proper selection of the controller parameters. Note that the more difficult and challenging problem of constructing a control law achieving ISS is still open, and the present paper has to be understood as a (possible) step in the solution of such a problem.

Before concluding the introduction we mention a simple physical and often encountered situation in which a robust control law is required. Consider the kinematic model of a simple wheeled mobile robot (see [CS:92] for detail), *i.e.*

$$\begin{aligned} \dot{x} &= \cos\theta \, v \,, \\ \dot{y} &= \sin\theta \, v \,, \\ \dot{\theta} &= \omega \,. \end{aligned} \tag{1}$$

Assume that the measurement device used to determine the angle θ is prone to errors which increase whenever the robot is not pointing in the direction of the x-axis. This is the case if the robot orientation is determined via a camera (or laser device) mounted in front of the robot and which relies on the knowledge of some (fixed) reference direction, namely the direction of the x-axis. Suppose moreover the robot is equipped with elastic (pneumatic) tires. As a result the robot may violate the nonholonomic constraint $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$ whenever $\omega \neq 0$. This phenomenon, known as transverse creep (see [NF:72, Chapter 6]), is associated to the presence of non-ideal wheels and cannot be neglected in real applications.

Consider now the feedback transformation

$$\begin{aligned} x_1 &= \theta \,, \\ x_2 &= x\cos\theta + y\sin\theta \,, \\ x_3 &= -x\sin\theta + y\cos\theta \,, \\ u_1 &= \omega \,, \\ u_2 &= v + (x\sin\theta - y\cos\theta)\omega \,. \end{aligned}$$

In the new coordinates, and taking into consideration the effect of the transverse creep, the

3. See later for detail.

system (1), with the new control inputs u_1 and u_2, is described by equations of the form

$$\begin{aligned} \dot{x}_1 &= u_1 , \\ \dot{x}_2 &= u_2 , \\ \dot{x}_3 &= x_2 u_1 + \delta(u_1) , \end{aligned}$$

in which $\delta(u_1)$ is a (unknown) function describing the transversal creep and such that $\delta(0) = 0$ and the variables available for feedback (*i.e.* the measurement variables) are

$$\begin{aligned} y_1 &= x_1 + e_1(x_1) \\ y_2 &= x_2 + e_2(x_1,x_2,x_3) \\ y_3 &= x_3 + e_3(x_1,x_2,x_3) \end{aligned}$$

with $e_1(0) = 0$, $e_2(0,x_2,x_3) = 0$ and $e_3(0,x_2,x_3) = 0$. It is therefore obvious that, even in this very simple situation, it is necessary to design a control law with guaranteed robustness against a class of (structured) external disturbances and measurement errors.

Starting from the above motivating example, in the rest of the paper we focus on n-dimensional chained systems with two controls (see [MS:93] for detail), *i.e.* systems described by equations of the form

$$\begin{aligned} &\dot{x}_1 = u_1, \\ &\dot{x}_2 = u_2, \\ &\dot{x}_3 = x_2 u_1, \\ &\vdots \\ &\dot{x}_n = x_{n-1} u_1, \end{aligned} \tag{2}$$

and we address the problem of robust stabilization in the presence of measurement errors, external disturbances and actuators errors. Note that the main results of the paper can be applied to the much more general classes of systems considered in [J:00, LA:99], however, for simplicity we do not pursue such generalizations.

Chained systems are asymptotically controllable, thus, by the general results in [P:01b], there exists a hybrid feedback rendering the origin a robustly globally asymptotically stable equilibrium. In the present work such a feedback is *explicitly constructed* and it is shown that only a *finite* number of hybrid variables is needed in the construction. However, we obtain only a *partially robust* globally asymptotically stabilizing controller, compare Theorem 1 in Section 3 with [P:01b, Theorem 1].

The paper is organized as follows. In Section 2 the class of controllers used in the paper is introduced and the notion of solution of the resulting closed–loop system is discussed. Section 3 presents the main result of the paper, while in Sections 4 and 5 a control law robustly stabilizing system (2) is described. Finally, Section 6 contains the proof of the main result and Section 7 summarizes the main contributions of the work and points to some open problems and future research directions.

2 Class of controllers and notion of solutions

In this section we introduce the notions of controller and of solutions of differential equations that will be used throughout the paper.

The controllers under consideration admit the following description (see [T:87])

$$u = k(x,s_d) \quad , \qquad s_d = k_d(x,s_d^-) \tag{3}$$

where s_d evolves in the finite set $\{1,2\}$, $k: \mathbb{R}^n \times \{1,2\} \to \mathbb{R}^2$ is continuous in x for each fixed s_d, $k_d: \mathbb{R}^n \times \{1,2\} \to \{1,2\}$ is a function and s_d^- is defined, at this stage only formally, as

$$s_d^-(t) = \lim_{s<t} s_d(s) \ . \tag{4}$$

For this to make sense, we equip $\{1,2\}$ with the discrete topology, *i.e.* every set is an open set. The above controller is hybrid due to the presence of the discrete dynamics of s_d. It gives rise to a non-classical ordinary differential equation describing the dynamics of the closed–loop system. In particular this system is infinite dimensional since to evaluate $s_d^-(t)$ at time t, we need to know the past values of $s_d(t)$. Denoting with $f: \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ the function defining the right hand-side of the differential equation (2), we can rewrite (2) as

$$\dot{x} = f(x,u) \ . \tag{5}$$

In this paper we are interested in a notion of robustness to small noise. For, consider three functions

- e and d in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times [0,+\infty);\mathbb{R}^n)$, and continuous in $x \in \mathbb{R}^n$ for all $t \in [0,+\infty)$,
- a in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times [0,+\infty);\mathbb{R}^2)$, and continuous in $x \in \mathbb{R}^n$ for all $t \in [0,+\infty)$.

We introduce these functions as a measurement noise e, an actuator noise a and an external noise d and define the perturbed system with u given by equation (3), *i.e.*

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t),k(x(t)+e(x(t),t),s_d(t))+a(x(t),t))+d(x(t),t) \\ s_d(t) & = & k_d(x(t)+e(x(t),t),s_d^-(t)). \end{array} \right. \tag{6}$$

As noted in [LS:97, Remark 1.4], we can omit any explicit reference to actuator noise because f is locally Lipschitz. So in the following we suppose that

$$a(x,t) = 0 \ ,$$

for all x in $\mathbb{R}^n$ and for all $t \geq 0$, and we study the perturbed system

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x(t),k(x(t)+e(x(t),t),s_d(t)))+d(x(t),t), \\ s_d(t) & = & k_d(x(t)+e(x(t),t),s_d^-(t)). \end{array} \right. \tag{7}$$

The notion of initial condition for system (7) is discussed in details in [P:01b], however for completeness we recall that a pair (x_0,s_0) is said to be an initial condition of a solution (X,S_d) of (7) if

$$X(0) = x_0 \quad , \qquad S_d(0) = k_d(x_0 + e(x_0,0),s_0) \ . \tag{8}$$

Given an initial condition (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}$ and $T > 0$, the Cauchy problem under consideration in this paper is given by (7) and (8). However, for this to be meaningful, we have to make precise what we mean by solution of the corresponding differential equation. To this end, we adapt the definition of [T:87] to the present context of a perturbed hybrid system (see also [B:98, LNP:00]). Hence, we rewrite the definition given in [P:00] and we introduce a non-empty set $\mathcal{RC}$ strictly included in $\mathbb{R}^n \times \{1,2\}$.

Definition 2.1 *Given $T > 0$, $(x_0,s_0) \in \mathbb{R}^n \times \{1,2\}$, and a non-empty set $\mathcal{RC}$ strictly included in $\mathbb{R}^n \times \{1,2\}$, we say that (X,S_d) is a* solution*, starting from (x_0,s_0), of (7) on $[0,T)$ if the following holds.*

1. *The map X is absolutely continuous on $[0,T)$.*
2. *For almost all t in $[0,T)$, we have*
$$\dot{X}(t) \ = \ f\left(X(t),k(X(t)+e(X(t),t),S_d(t))\right) + d(X(t),t) \ .$$
3. *For all $t \in [0,T)$ such that $(X(t),S_d(t))$ is in $\mathcal{RC}$, the map S_d is right-continuous at t.*
4. *For all $t \in (0,T)$ such that $S_d^-(t)$ exists, one has*
$$S_d(t) \ = \ k_d(X(t)+e(X(t),t),S_d^-(t)) \ . \tag{9}$$
5. *$X(0) \ = \ x_0$ and $S_d(0) = k_d(x_0 + e(x_0,0),s_0)$.*

Remark 2.2 Invoking Zorn's Lemma exactly as in the proof of [R:99, Proposition 1], we can prove that every solution (X,S_d) can be extended to a maximal solution (X,S_d) defined on an interval $[0,T)$ with $T \leq +\infty$ (*i.e.* for which there exists no solution defined on an interval $[0,T')$ with $T' > T$ and whose restriction is (X,S_d) . ◇

In this context the definition of global asymptotic stability can be given as follows.

Definition 2.3 *Let e,d be two functions satisfying our standing regularity assumptions. The origin of the system (7) is said to be a* globally asymptotically stable equilibrium *on $\mathbb{R}^n$ if the following three properties hold.*

1. The system is complete: *For every (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}$, there exists a solution of (7) starting from (x_0,s_0). Moreover all maximal solutions of (7) are defined on $[0,+\infty)$.*
2. Global stability: *There exists δ of class $\mathcal{K}_\infty$ such that, for all $\varepsilon > 0$ and for all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}$ with $|x_0| \leq \delta(\varepsilon)$ and for all maximal solutions (X,S_d) of (7) starting from (x_0,s_0), one has*
$$|X(t)| \leq \varepsilon, \forall t \geq 0 \ . \tag{10}$$
3. Global attractivity: *For all $r > 0$ and for all $C > 0$, there exists $T = T(r,C) > 0$, such that, for all (x_0,s_0) in $\mathbb{R}^n \times \{1,2\}$ with $|x_0| \leq C$ and for all solutions (X,S_d) of (7) starting from (x_0,s_0), one has*
$$|X(t)| \leq r, \forall t \geq T \ . \tag{11}$$

Moreover, as we are interested in a notion of robustness with respect to small noise, we complement the above definition with the usual notion of robustly stabilizing controller, see [S:98, Section 4.1].

Definition 2.4 *Let e and d be two functions satisfying our standing regularity assumptions. The controller (k,k_d) is a* robustly globally asymptotically stabilizing controller *for (2) if there exists a continuous function ρ: $\mathbb{R}^n \to \mathbb{R}$, such that*
$$\rho(x) > 0 \tag{12}$$
for all $x \neq 0$, and such that for any perturbed system (7) with
$$\sup_{\mathbb{R}_{\geq 0}} |e(x,.)| \leq \rho(x) \quad , \qquad \operatorname{esssup}_{\mathbb{R}_{\geq 0}} |d(x,.)| \leq \rho(x) \ , \tag{13}$$
for all x in $\mathbb{R}^n$, the origin is a globally asymptotically stable equilibrium on $\mathbb{R}^n$ as characterized in Definition 2.3.

3 Main result

In this section we state our main result, namely a theorem showing that the problem of global robust stabilization, in the presence of small measurement and external noises, is partially solvable.

Theorem 1 *For all $R > 0$, there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$, such that*

$$\rho(x) > 0, \forall x \in \mathbb{R}^n \setminus \{x_1 = x_3 = \cdots = x_n = 0, |x_2| \leq R\} \ , \tag{14}$$

and a globally asymptotically stabilizing controller (k,k_d) for the system (7), and for all perturbations e, d satisfying our standing regularity assumptions and (13).

Remark 3.1 Some observations are in order.

1. Theorem 1 does not provide a robustly globally asymptotically stabilizing controller for (2) as stated in Definition 2.4, because the class of perturbations considered in Theorem 1 is not so large as in Definition 2.4. Compare equation (12) with equation (14). For a three dimensional chained system, the set where the perturbations are zero is depicted in Figure 1.
2. We can formulate Theorem 1 is terms of generalized solution as defined in [H:67, P:01a] or in terms of π-solutions with a sufficiently small sampling schedule and Euler solutions as defined in *e.g.* [S:98]. See also [P:01b].
3. In [VA:01, Proposition 3], for any sufficiently small disturbances, the controller renders the origin of the perturbed closed–loop system attractive and locally stable. In Theorem 1, imposing more structure on the perturbations we obtain a global result. $\diamond$

To prove Theorem 1 we need to define two controllers, which are called the "local controller" and the "global controller", respectively. We define the local controller in Section 4.1 and the global one in Section 4.2. We join the domain of definition of this feedbacks by means of a hysteresis, as detailed in Section 5, and finally we prove Theorem 1 in Section 6.

4 The components of the hysteresis

4.1 The local controller

In this section we define the "local controller" and we give some properties of the Carathéodory solutions of (7) with such control law.

Consider the system (2) and the control law u_l: $\mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\begin{cases} u_{1l}(x) = & -x_1, \\ u_{2l}(x) = & p_2 x_2 + p_3 \dfrac{x_3}{x_1} + p_4 \dfrac{x_4}{x_1^2} + \cdots + p_n \dfrac{x_n}{x_1^{n-2}}, \end{cases} \tag{15}$$

with the p_i such that the matrix

$$A = \begin{bmatrix} p_2 & p_3 & p_4 & p_5 & \cdots & p_{n-1} & p_n \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & n-2 \end{bmatrix} \tag{16}$$

is Hurwitz. Let $P = P^T > 0$ be such that

$$A^T P + PA < 0 \tag{17}$$

and let z be a variable in $\mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} z \;=\; z(x) \;&=\; Y^T\ P\ Y \quad \text{if } x_1 \neq 0 \ , \\ &=\; +\infty \quad\quad \text{if } x_1 = 0 \ , \end{aligned} \tag{18}$$

for all x in $\mathbb{R}^n$, with $Y \in \mathbb{R}^{n-1}$ defined by

$$Y \;=\; Y(x) \;=\; \begin{bmatrix} x_2 \\ \frac{x_3}{x_1} \\ \vdots \\ \frac{x_n}{x_1^{n-2}} \end{bmatrix} \ , \tag{19}$$

for all x in $\mathbb{R}^n$, $x_1 \neq 0$.

Let e and d: $\mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ be two perturbations satisfying our standing regularity assumptions. The closed–loop system in consideration in this section is the system

$$\dot{x} = f(x, u_l(x+e)) + d \ . \tag{20}$$

It has been proved in [A:95] that the closed–loop system (2)-(15) admits a unique Carathéodory solution for any initial condition $x(0)$ such that $x_1(0) \neq 0$, and that if

$$\sigma(A) \subset \mathbb{C}^- \ , \tag{21}$$

such a Carathéodory solution converges *exponentially* to zero. A similar result holds for the perturbed system (20).

Lemma 4.1 *There exists a continuous function $\rho_l : \mathbb{R} \to \mathbb{R}$ satisfying*

$$\rho_l(\xi) > 0 \ , \forall \xi \neq 0, \tag{22}$$

such that for all e, d: $\mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ satisfying our standing regularity assumptions and

$$\sup_{\mathbb{R}_{\geq 0}} |e(x,.)| \leq \rho_l(x_1) \quad , \qquad \operatorname{esssup}_{\mathbb{R}_{\geq 0}} |d(x,.)| \leq \rho_l(x_1) \ , \tag{23}$$

for all x in $\mathbb{R}^n$, and for all x_0 satisfying $|z(x_0)| \leq k$, there exists a Carathéodory solution X of (20) starting from x_0 and all such Carathéodory solutions are maximally defined on $[0, +\infty)$ and satisfy $|z(X(t))| < k$, for all $t > 0$.

Moreover there exists a function δ_l of class $\mathcal{K}_\infty$ such that, for all ε, r and k, and for all x_0 satisfying $|x_0| \leq \delta_l(\varepsilon)$ and $|z(x_0)| \leq k$, there exists $T_l = T_l(r, \delta_l(\varepsilon), k)$ such that $|X(t)| \leq \varepsilon\sqrt{k}$ for all $t \geq 0$ and $|X(t)| \leq r$ for all $t \geq T_l$.

Remark 4.2 Lemma 4.1 states that, for any $k > 0$, the region $|z(x)| \leq k$ is robustly forward invariant, *i.e.* it is positively invariant in the presence of a class of measurement and external noise. Moreover, any trajectory in such a region converges exponentially to the origin. Finally, note that a somewhat simpler and local version of Lemma 4.1 has been given in [VA:01]. $\diamond$

Proof of Lemma 4.1: To begin with note that the Carathéodory conditions are met for system (20). Therefore we have the existence of a unique forward Carathéodory solution of (20) for any initial condition.

Let x in $\mathbb{R}^n$ be such that $x_1 \neq 0$. Let us impose that

$$\rho_l(\xi) \leq \frac{|\xi|}{2} \ , \forall \xi \in \mathbb{R} \ . \tag{24}$$

A simple computation gives, along every Carathéodory solution of (7),

$$\begin{aligned} \dot{z} \ = \ & \left[\ p_2(x_2+e_2) + p_3 \frac{x_3+e_3}{x_1+e_1} + \cdots + d_2 \quad \frac{(-x_2(x_1+e_1)+d_3)x_1 + x_3(x_1+e_1+d_1)}{x_1^2} \quad \cdots \ \right] P \ Y \\ & + Y^T \ P \begin{bmatrix} p_2(x_2+e_2) + p_3 \dfrac{x_3+e_3}{x_1+e_1} + d_2 \\ \dfrac{(-x_2(x_1+e_1)+d_3)x_1 + x_3(x_1+e_1+d_1)}{x_1^2} \\ \vdots \end{bmatrix} . \end{aligned} \tag{25}$$

Moreover the function $x \mapsto Y(x)^T \ A^T \ P \ Y(x)$ is continuous on $\mathbb{R}^n \setminus \{x, x_1 = 0\}$. As a result, there exists ρ_l and $C_1 > 0$ such that for all perturbations satisfying (23),

$$\dot{z} \leq C_1 \left(Y^T(x) \ A^T \ P \ Y(x) \ + \ Y^T(x) \ P \ A \ Y(x) \right) \ . \tag{26}$$

Thus, by equations (17) and (26) and the positivity of P, we infer the existence of two real numbers $C_2 > 0$ and $C_3 > 0$, such that

$$\dot{z} \leq -C_2 |Y|^2 \leq -C_3 z \ .$$

Therefore, there exists a function ρ_l such that, for all e and d satisfying our standing regularity assumptions and (23), along all Carathéodory solutions of (7) the variable z, hence the variable Y, converges (exponentially) to zero.

Furthermore, due to (18) and the positivity of P, there exists $C_4 > 0$ such that, for all z,

$$|x_2| \leq C_4 \sqrt{z} \quad , \qquad \frac{|x_3|}{|x_1|} \leq C_4 \sqrt{z} \quad , \qquad \cdots \quad , \qquad \frac{|x_n|}{|x_1|^{n-2}} \leq C_4 \sqrt{z} \ ,$$

and, by (2) and (15), along all Carathéodory solutions, x_1 tends (exponentially) to zero.

We conclude that choosing the function $\rho_l \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ sufficiently small and satisfying equation (22), the properties of stability and attractivity, for all perturbations satisfying (23), are established, and this completes the proof of Lemma 4.1. $\square$

4.2 The global controller

In this section we define the second component of the hysteresis, which is called the "global controller" and is denoted by u_g. Moreover we give some basic properties of the Carathéodory solutions of the closed–loop system (7) with such a control law u_g.

Let $\mu > 0$ and consider the control law u_g defined on $\mathbb{R}^n$ by

$$\begin{cases} u_{1g} = sign(x_1) , \\ u_{2g} = -\mu x_2 , \end{cases} \tag{27}$$

where we use the somewhat non-standard definition of the sign function

$$sign(x_1) = \begin{cases} 1 & x_1 \geq 0 \\ -1 & x_1 < 0 \end{cases} . \tag{28}$$

The closed–loop system in consideration in this section is

$$\dot{x} = f(x,u_g(x+e)) + d, \tag{29}$$

and for such a system the following statement holds.

Lemma 4.3 *There exists a continuous function ρ_g: $\mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\rho_g(x) > 0 \ , \forall x \neq 0 \ , \tag{30}$$

such that, for any initial condition, all perturbed systems (29), where e and d are two functions satisfying our standing regularity assumptions and equations (13) with $\rho = \rho_g$, admit a unique Carathéodory solution, defined for all $t \geq 0$.

Moreover there exists a function δ_g of class $\mathcal{K}_\infty$ such that, for any $\varepsilon > 0$ and for any $k > 0$, there exists a time $T_g = T_g(k,\delta_g(\varepsilon))$ such that, for all Carathéodory solutions X of (29) with initial condition x_0 with $|x_0| \leq \delta_g(\varepsilon)$, $|z(X(t))| \leq k$ for all $t \geq T_g$, and $|X(t)| \leq \varepsilon$ for all $t \geq 0$.

Remark 4.4 Lemma 4.3 states that, for any $k > 0$, the trajectories of the system (29) enter the region $|z(x)| \leq k$ in finite time, while remaining bounded. ◇

Proof of Lemma 4.3: To begin with note that, by the regularity assumptions on e and d, system (29) satisfies the Carathéodory conditions. Therefore there exists a unique forward Carathéodory solution of (29) for any initial condition. Moreover a direct integration of the closed loop system (5)-(27) yields

$$\begin{cases} x_1(t) &=& x_{10} + t \ , \\ x_2(t) &=& x_{20}e^{-\mu t} \ , \\ x_3(t) &=& x_{30} + \dfrac{x_{20}}{\mu}(1-e^{-\mu t}) \ , \\ & \vdots & \\ x_n(t) &=& \sum_{j=3}^n \dfrac{x_{j0}\ t^{n-j}}{(n-j)!} \\ & & + \sum_{j=4}^n \dfrac{(-1)^{j-4}\ x_{20}\ t^{n-j+1}}{\mu^{j-3}\ (n-j+1)!} \\ & & + \dfrac{(-1)^{n-3}\ x_{20}}{\mu^{n-2}}(1-e^{-\mu t}) \ . \end{cases} \tag{31}$$

when $x_1(0) \geq 0$, and

$$\begin{cases} x_1(t) &=& x_{10} - t, \\ x_2(t) &=& x_{20} e^{-\mu t}, \\ x_3(t) &=& x_{30} - \dfrac{x_{20}}{\mu}(1 - e^{-\mu t}), \\ & \vdots & \\ x_n(t) &=& \sum_{j=3}^{n} \dfrac{(-1)^{n-j}\ x_{j0}\ t^{n-j}}{(n-j)!} \\ && + \sum_{j=4}^{n} \dfrac{(-1)^{n-j}\ x_{20}\ t^{n-j+1}}{\mu^{j-3}\ (n-j+1)!} \\ && - \dfrac{x_{20}}{\mu^{n-2}}(1 - e^{-\mu t}), \end{cases} \tag{32}$$

when $x_1(0) < 0$.

As a result, in both cases $\lim_{t \to \infty} z(t) = 0$, so there exists a finite time $t^*_{x(0)} \geq 0$ (minimally defined) such that $|z(t^*_{x(0)})| < k$. Moreover, for any $C > 0$, and by (18) and (31)-(32) $T = \sup\{t^*_{x(0)}, |x(0)| \leq C\}$ is finite, and this implies the stability property as stated in Lemma 4.3. Finally, insensitivity with respect to small perturbations e and d satisfying (13) results from the right-continuity of (29). □

5 Definition of the hybrid controller

In this section we define the hybrid controller, robustly stabilizing system (2), by using a hysteresis to join the controllers defined in Sections 4.1 and 4.2. To this end, for any strictly positive real number k, we define the subset Γ_k of $\mathbb{R}^n$ as

$$\Gamma_k = \{x, x_1 \neq 0, |z| < k\} , \tag{33}$$

where z is defined by (7). Moreover, let $R > 0$ be given by Theorem 1 and let $k_6 > \cdots > k_1 > 0$ be such that

$$\sqrt{k_6 |p_2|} < R , \tag{34}$$

where p_2 is defined by (16). For simplicity, in what follows, for all $i \in \{1, \ldots, 6\}$, we define $\Gamma_i := \Gamma_{k_i}$.

The hybrid controller (k, k_d) is defined making a hysteresis between u_l and u_g on Γ_5 and Γ_2, *i.e.*

$$\begin{array}{rcllll} k : \{1,2\} \times \mathbb{R}^n & \to & \mathbb{R}^2 & & & \\ (s_d, x) & \mapsto & u_l(x) & \text{if} & s_d = 1 & \text{and} \quad x_1 \neq 0 , \\ & & 0 & \text{if} & s_d = 1 & \text{and} \quad x_1 = 0 , \\ & & u_g(x) & \text{if} & s_d = 2 , & \end{array} \tag{35}$$

and

$$\begin{array}{rclll} k_d : \mathbb{R}^n \times \{1,2\} & \to & \{1,2\} & & \\ (x, s_d) & \mapsto & 1 & \text{if} & x \in \Gamma_2 \cup \{0\}, \\ & & s_d & \text{if} & x \in \Gamma_5 \setminus \Gamma_2, \\ & & 2 & \text{if} & x \in \mathbb{R}^n \setminus (\Gamma_5 \cup \{0\}) . \end{array} \tag{36}$$

The sets used in the definition of the hybrid controller (35)-(36), in the case of a three dimensional chained system, are depicted in Figure 1. Note that this sets are symmetric with respect to $\{x_1 = x_3 = 0\}$ and that the region in "bold" is the region where, according to Theorem 1, the perturbation ρ vanishes. Finally, to avoid confusion, only the intersections of the sets Γ_2 and Γ_5 with the half-space $\{x_1 \geq 0\}$ are shown.

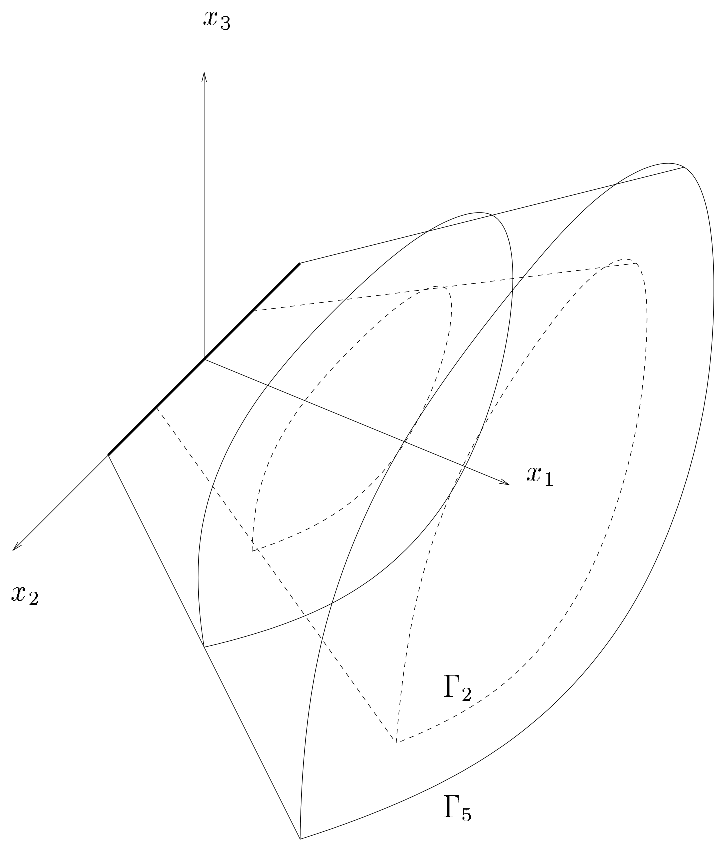

FIG. 1 – *Sketches of the sets Γ_2 and Γ_5 defining the hybrid controller in $\mathbb{R}^3$ (Γ_2 dashed, Γ_5 solid). In "bold" the set where the perturbation ρ vanishes. Only intersections with $\{x_1 \geq 0\}$ are shown.*

Remark 5.1 The function k_d makes a hysteresis between the controllers u_l and u_g on appropriate subsets of $\mathbb{R}^n$. For any $s_d \in \{1,2\}$, the function $k_d(.,s_d)$ is continuous except on the boundary of the sets defining the hysteresis. This observation will be used in the proof of Lemma 6.6. $\diamond$

6 Properties of the solutions and proof of Theorem 1

6.1 Properties of the solutions

In this section we study some properties of the solutions of all perturbed systems (7) in closed–loop with the hybrid controller (35) and (36). To this end, consider the sets[4]

$$\Sigma_{1\to 2} = \mathrm{clos}(\Gamma_3) \setminus \Gamma_1 \ , \tag{37}$$
$$\Sigma_{2\to 1} = \mathrm{clos}(\Gamma_6) \setminus \Gamma_4 \ , \tag{38}$$

the set (see [P:00] for a similar, yet simpler situation)

$$\mathcal{RC} = \mathbb{R}^n \times \{1{,}2\} \setminus \left(\Sigma_{1\to 2} \times \{2\} \bigcup \Sigma_{2\to 1} \times \{1\}\right) \ , \tag{39}$$

and the solutions of the controlled perturbed system as defined in Definition 2.1.

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

1. Equation (14) holds.
2. For all $x \in \Gamma_6$,
$$\rho(x) \leq \rho_l(x_1) \ , \tag{40}$$
and, for all $x \in \mathbb{R}^n \setminus \Gamma_1$,
$$\rho(x) \leq \rho_g(x) \ , \tag{41}$$
where ρ_l and ρ_g are defined in Lemma 4.1 and Lemma 4.3, respectively.
3. The following implication holds
$$\forall i \in \{1,\cdots,5\}, \forall e \in \mathbb{R}^n, |e| \leq \rho(x),\ |z(x+e)| \leq k_i \Rightarrow |z(x)| \leq k_{i+1} \ . \tag{42}$$
4. The following implication holds
$$\forall i \in \{2,\cdots,6\}, \forall e \in \mathbb{R}^n, |e| \leq \rho(x),\ |z(x+e)| \geq k_i \Rightarrow |z(x)| \geq k_{i-1} \ . \tag{43}$$

Remark 6.1 Note that it is not possible to have a property stronger than (14) because, for all $i \in \{1,\cdots,5\}$ and for all $x \in \{x_1 = x_3 = \cdots = x_n = 0, |x_2 p_2| \leq \sqrt{k_i}\}$, $\delta(x,\mathbb{R}^n \setminus \Gamma_{i+1}) = 0$, where δ denotes the usual distance in $\mathbb{R}^n$. See, for an illustration, the geometry of the sets Γ_i in Figure 1 and recall equation (34). $\diamond$

Let e and d be two functions satisfying our standing regularity assumptions and (13). We now establish a series of preliminary results which are instrumental to prove Theorem 1. To begin with, we show that system (7) has a solution for any initial condition.

Lemma 6.2 *For all (x_0,s_0) in $\mathbb{R}^n \times \{1{,}2\}$, there exists a solution of (7) starting from (x_0,s_0).*

4. In the forthcoming discussion, for a given subset Ω of $\mathbb{R}^n$, the closure, the interior and the boundary of Ω are denoted as $\mathrm{clos}(\Omega)$, $\int(\Omega)$ and $\partial\Omega$, respectively.

Proof : Let (x_0,s_0) be in $\mathbb{R}^n \times \{1,2\}$ and $s_1 = k_d(x_0 + e(x_0,0),s_0)$.

If $x_0 = 0$, then $e(x_0,t) = 0$, for all t, and by (35) and (36), the function (X,S_d): $[0,+\infty) \to \mathbb{R}^n \times \{1,2\}$, defined by

$$X(t) = 0 \quad , \quad S_d(t) = s_1 \ ,$$

for all t in $[0,+\infty)$ is a solution of (7) starting from (x_0,s_0) on $[0,+\infty)$.

Assume now $x_0 \neq 0$. From our standing regularity assumptions on f and d, the system

$$\dot{X} = f(X,k(X + e(X,t),s_1)) + d(X,t) \quad , \qquad X(0) = x_0 \tag{44}$$

satisfies the Carathéodory conditions. Let X, defined on $[0,T)$ with $0 < T$, be a Carathéodory solution of (44), and consider the following two cases.

1. $s_1 = 1$.
   In this case, in view of (36) and (42)-(43), only two possibilities may occur.
   (a) If x_0 is in Γ_4 then, by continuity, there exists $0 < T' < T$ such that $X(t)$ is in Γ_4, for all t in $[0,T')$, and thus by (42) and (33), $X(t)+e(X(t),t)$ is in Γ_5, for all t in $[0,T')$. Therefore $1 = k_d(X(t)+e(X(t),t),1)$, for all t in $[0,T')$ and by letting $S_d(t) = 1$, we have that (X,S_d) is a solution of (7) starting from (x_0,s_0) and defined on $[0,T')$.
   (b) If x_0 is in $\Gamma_6 \setminus \Gamma_4$ then, by continuity, there exists $0 < T' < T$ such that $X(t)$ is not in Γ_3, for all t in $[0,T')$ and thus $2 = k_d(X(t) + e(X(t),t),2)$. Therefore, letting $S_d(t) = 2$, for all t in $(0,T')$ and $S_d(0) = 1$, we have that (X,S_d) is a solution of (7) starting from (x_0,s_0) and defined on $[0,T')$. (Note that $(x_0,1) \notin \mathcal{RC}$ hence, according with Definition 2.1, S_d may not be right-continuous at $t = 0$.)
2. The case $s_1 = 2$ is established in the same way. □

As a second result, we prove that maximal solutions of (7) must blow up if their domains of definition are bounded.

Lemma 6.3 *Let (X,S_d) be a maximal solution of (7) defined on $[0,T)$. Suppose $T < +\infty$, then* $\limsup_{t\to T}|X(t)| = +\infty$.

Proof : Let (X,S_d) be a maximal solution defined on $[0,T)$ and suppose the conclusion of Lemma 6.3 does not hold, *i.e.* there exists a compact set $C \subset \mathbb{R}^n$ and times $t_n \in [0,T)$, tending monotonically to T, such that $(X(t_n),S_d(t_n))$ is in $C \times \{1,2\}$ for all n. Then the following claim holds.

Claim 6.4 *For some n sufficiently large, for all $t \in [t_n,T)$, $X(t)$ is in the bounded open set $C + \mathring{B}$, where $B = \{x \in \mathbb{R}^n, |x| \leq 1\}$.*

Proof of Claim 6.4: If the conclusion of Claim 6.4 is not true, by continuity of X there exists $s_n \in (t_n,T)$ such that

$$|X(t_n) - X(s_n)| = 1$$

and

$$|X(t_n) - X(t)| < 1, \forall t \in [t_n,s_n) \ .$$

It follows that $X(t)$ is in the compact set $C + B$, for all t in $[t_n,s_n]$. Let

$$\rho = \max_{x\in C+B} |\chi(x)|$$

and define

$$\sigma = \sup_{d\in\rho B,\, x\in C+B,\, i\in\{l,g\}} |f(x,u_i(x)) + d| \ .$$

Then, for all t and s in $[t_n,s_n]$, one has $|X(t) - X(s)| \leq \sigma|t-s|$. Therefore, for n sufficiently large,

$$1 \ = \ |X(t_n) - X(s_n)| \ \leq \ \sigma\,|s_n - t_n| \ \leq \ \sigma\,|T - t_n| \ .$$

However, this cannot hold for n large enough, hence Claim 6.4 is established. □

Claim 6.4 implies that there exists σ in $\mathbb{R}_{\geq 0}$ such that, for all (s,t) in $[t_n,T)$

$$|X(s) - X(t)| \ \leq \ \sigma\,|s-t| \ .$$

It follows (invoking Cauchy criterion) that $X(t)$ has a limit x_0 when t tends to T, and only two cases may occur.

1. Suppose $\lim_{t\to T,\, t<T} S_{d,\alpha}(t)$ exists. Let $s_0 = S_d^-(T)$. By Lemma 6.2, there exists a solution $(\widetilde{X},\widetilde{S_d})$ starting from (x_0,s_0) and defined on $[0,\widetilde{T})$ with $\widetilde{T} > 0$. Let (X',S_d') be defined on $[0,T+\widetilde{T})$ by

$$X'(t) \ = \ X(t),\, S_d'(t) \ = \ S_d(t) \quad , \quad \forall t \in [0,T), \tag{45}$$

$$X'(t) \ = \ \widetilde{X}(t-T),\, S_d'(t) \ = \ \widetilde{S_d}(t-T) \quad , \quad \forall t \in (T,T+\widetilde{T}) \ . \tag{46}$$

2. Suppose $\lim_{t\to T,\, t<T} S_{d,\alpha}(t)$ does not exist. Hence, by Lemma 6.6,

$$x_0 \in (\mathrm{clos}(\Gamma_6) \setminus \Gamma_3) \cap (\mathrm{clos}(\Gamma_4) \setminus \Gamma_1) \ .$$

   Therefore $x_0 = 0$. Let now $(\widetilde{X},\widetilde{S_d})$: $\mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \{1,2\}$ be defined by

$$X(t) = 0,\, S_d(t) = 1 \ ,\forall t \geq 0,$$

   and (X',S_d') be defined on $[0,T+\widetilde{T})$ by (45) and (46).

We conclude that, in both cases, we have defined a solution of (7) on $[0,T+\widetilde{T})$ whose restriction on $[0,T)$ is (X,S_d), and this is in contradiction with the assumption that (X,S_d) was a maximal solution. □

To describe further properties of the trajectories of the considered system, we need to recall the definition of *switch* time.

Definition 6.5 *A map S_d: $[0,T) \to \{1,2\}$ is said to have a* switch *at time t if S_d is not continuous at t.*

Let (X,S_d) be a solution of (7) and consider the problem of locating the points where S_d may have a switch. To address this problem, for all t in $(0,T)$, consider the sets

$$S_d^p(t) \ = \ \{s : \exists t_n \in [t,T), t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s\} \ , \tag{47}$$

$$S_d^m(t) \ = \ \{s : \exists t_n \in [t_0,t], t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s\} \ . \tag{48}$$

With the aid of such sets it is possible to provide some properties of trajectories with a *switch*, as detailed in the following statement.

Lemma 6.6 *Let (X,S_d) be a solution of (7) such that S_d has a switch at time $t \in (0,T)$.*

– *If the switch is such that $2 \in S_d^m(t)$ and $1 \in S_d^p(t)$, then $X(t)$ is in* $\mathrm{clos}(\Gamma_3) \setminus \Gamma_1$.

– *If the switch is such that* $1 \in S_d^m(t)$ *and* $2 \in S_d^p(t)$*, then* $X(t)$ *is in* $\mathrm{clos}(\Gamma_6) \setminus \Gamma_4$.

Proof : Let t be such that $2 \in S_d^m(t)$ and $1 \in S_d^p(t)$. Then, by (48), there exists a sequence of time $t_n < t$, such that $t_n \to t$ as n tends to infinity, and $S_d(t_n) = 2$. Consider now the following cases.

1. There exists a sequence $t_n < t$ converging to t such that $S_d(t_n) = 2$ and S_d is not right-continuous at t_n. If this is the case, by Definition 2.1, $(X(t_n), S_d(t_n))$ is not in $\mathcal{RC}$, hence, by (39), $X(t_n)$ is in $\Sigma_{1\to 2}$ and, by continuity, $X(t)$ is in $\mathrm{clos}(\Gamma_3) \setminus \Gamma_1$.
2. There exists a sequence $t_n < t$ converging to t such that $S_d(t_n) = 2$ and S_d is right-continuous at t_n. If this is the case, by (47), there exists a maximal ε_n in $(0, t - t_n]$, such that, for all t in $[t_n, t_n + \varepsilon_n)$, $S_d(t) = 2$. Note now that $(t_n + \varepsilon_n)$ is a sequence converging to t, so we may have only the following three (non-exclusive) cases.
   (a) There exists a subsequence such that $S_d(t_n + \varepsilon_n) = 1$.
   (b) There exists a subsequence such that $S_d(t_n + \varepsilon_n) = 2$ and S_d is right continuous at each $t_n + \varepsilon_n$.
   (c) There exists a subsequence such that $S_d(t_n + \varepsilon_n) = 2$ and S_d is not right continuous at each $t_n + \varepsilon_n$.

   Note that the case (2b) is not possible because ε_n is maximally defined and that the case (2c) has already been studied in case 1.
   As a result, we can suppose $S_d(t_n + \varepsilon_n) = 1$, for all n in $\mathbb{N}$. This implies, by (9), that

$$1 = S_d(t_n + \varepsilon_n) = k_d(X(t_n + \varepsilon_n) + e(X(t_n + \varepsilon_n), t_n + \varepsilon_n), 2) \ .$$

   Hence, by (36), $X(t_n + \varepsilon_n) + e(X(t_n + \varepsilon_n), t_n + \varepsilon_n) \in \Gamma_2$ and $X(t) + e(X(t), t) \in \mathbb{R}^n \setminus \Gamma_2$, for all t in $(t_n, t_n + \varepsilon_n)$. Therefore, by (42) and (43), $X(t_n + \varepsilon_n) \in \Gamma_3$ and $X(t) \in \mathbb{R}^n \setminus \Gamma_1$, for all t in $(t_n, t_n + \varepsilon_n)$, and finally, by continuity, $X(t) \in \mathrm{clos}(\Gamma_3) \setminus \Gamma_1$.

This concludes the proof of the first claim. The second claim, *i.e.* the case $1 \in S_d^m(t)$ and $2 \in S_d^p(t)$, can be established using similar considerations. □

We conclude this series of preliminary results, by discussing the behavior of the solutions between switches.

Lemma 6.7 *Let* (X, S_d) *be a maximal solution of (7) defined on* $[0, T)$*. Then* $T = +\infty$ *and only three cases are possible.*

1. *There exists no switch and* X *is a Carathéodory solution of (20) on* $[0, +\infty)$ *and remains in* Γ_6.
2. *There exists a time* σ *in* $(0, +\infty)$ *such that*
   (a) *The map* X *is a Carathéodory solution of (29) on* $[0, \sigma)$ *and is not in* Γ_1.
   (b) *The map* X *is a Carathéodory solution of (20) on* $[\sigma, +\infty)$ *and remains in* Γ_6.
3. *There exists two switches* σ_1 *and* σ_2 *in* $(0, +\infty)$*, such that,*
   (a) *The map* X *is a Carathéodory solution of (20) on* $[0, \sigma_1)$ *and remains in* Γ_6.
   (b) *The map* X *is a Carathéodory solution of (29) on* $[\sigma_1, \sigma_2)$ *and is not in* Γ_1.
   (c) *The map* X *is a solution of (20) on* $[\sigma_2, +\infty)$ *and remains in* Γ_6.

Remark 6.8 The proof of Lemma 6.7 is conceptually similar to the proof of [P:01a, Claim 6.11]. However, it is more involved because of the more complex geometry of the sets Γ_i associated with the chained system (2). ◇

Proof of Lemma 6.7: To begin with, we establish the following fact.

Claim 6.9 *There can not exist 2 consecutive switches σ_1,σ_2 such that $2 \in S_d^m(\sigma_1)$, $1 \in S_d^p(\sigma_1)$ and $2 \in S_d^p(\sigma_2)$.*

Proof of Claim 6.9: Suppose the conclusion of Claim 6.9 does not hold. Then, by Lemma 6.6, $X(\sigma_1) \in \text{clos}(\Gamma_3) \setminus \Gamma_1$, $X(\sigma_2) \in \text{clos}(\Gamma_6) \setminus \Gamma_4$ and, by (33),

$$k_1 < |z(X(\sigma_1))| \le k_3 \text{ or } X_1(\sigma_1) = 0 \ , \tag{49}$$

and

$$k_4 < |z(X(\sigma_2))| \le k_6 \text{ or } X_1(\sigma_2) = 0 \ . \tag{50}$$

Moreover, by Definition 2.1, $S_d(t) = 1$ for all t in (σ_1,σ_2). Therefore by (35), X is a solution of (20) on (σ_1,σ_2) and, by (36), $X(t) + e(X(t),t)$ is in Γ_5. Thus, by (42), $X(t)$ is in Γ_6, for all t in (σ_1,σ_2).

Note now that condition (40) implies that the conclusions of Lemma 4.1 hold, hence, by (49), Γ_3 is forward invariant. Therefore $|z(X(\sigma_2))| \le k_3$, and this contradicts (50). □

We are now ready to prove that $T = +\infty$. For, suppose $T < +\infty$, then by Claim 6.9 and equation (35) there exists $\sigma < T$ such that X is a solution of (20) (if $S_d(\sigma) = 1$) or of (29) (if $S_d(\sigma) = 2$) on $[\sigma,T)$. However, by Lemma 6.3, $\limsup_{t\to T}|X(t)| = +\infty$, and this contradicts the conclusions of Lemma 4.1 (if $S_d(\sigma) = 1$) or of Lemma 4.3 (if $S_d(\sigma) = 2$). As a consequence, $T = +\infty$.

To complete the proof of Lemma 6.7, note that, for all t in $[0, +\infty)$, $S_d(t) = 1$ (resp. $S_d(t) = 2$) implies, by (36), that $X(t) + e(X(t),t)$ is in Γ_5 (resp. not in Γ_2). Thus, by (13), (42) and (43), $X(t)$ is in Γ_6 (resp. not in Γ_1). □

Remark 6.10 An interesting consequence of Claim 6.9 is that, along any trajectory of the perturbed system, there are strictly less than four switches. ◇

We have now all tools necessary to prove Theorem 1.

6.2 Proof of Theorem 1

We break up the proof in three parts.

Existence and maximality of solutions
Existence of solutions follows from Lemma 6.2 and maximality from Lemma 6.7.

Global stability
Let $\varepsilon > 0$ and consider perturbations e, d: $\mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ satisfying our standing regularity assumptions and (13), for all x in $\mathbb{R}^n$. Let $|x_0|$ be in $\mathbb{R}^n$, such that $|x_0| < \min(\delta_l(\varepsilon),\delta_g(\varepsilon))$, let

s_0 be in $\{1,2\}$, and let (X,S_d) be the corresponding solution of (7) starting from (x_0,s_0) and defined on $[0,+\infty)$.

We now show that

$$|X(t)| < \max\left(\varepsilon\sqrt{k_6},\varepsilon,\delta_l^{-1}(\varepsilon)\sqrt{k_6},\delta_g^{-1}(\sqrt{k_6}\varepsilon),\delta_l^{-1}(\delta_g^{-1}(\sqrt{k_6}\varepsilon))\sqrt{k_6}\right) \qquad \forall t \geq 0 \ . \tag{51}$$

To this end, note that, by Lemma 6.7, only three cases may occur.

1. Suppose Case 1 of Lemma 6.7 holds. Then, by Lemma 4.1 and (33), (51) holds.
2. Suppose Case 2 of Lemma 6.7 holds. Then, by Lemma 4.3, (51) holds for all t in $[0,\sigma)$. Moreover, for all t in $[\sigma,+\infty)$, X is a Carathéodory solution of (20) such that $|X(\sigma)| < \delta_l(\delta_l^{-1}(\varepsilon))$. Thus we conclude (51) by Lemma 4.1.
3. Suppose Case 3 of Lemma 6.7 holds. Then Lemma 4.1 implies (51), for all t in $[0,\sigma_1)$. Moreover, for all t in $[\sigma_1,\sigma_2)$, X is a Carathéodory solution of (29) such that $|X(\sigma)| < \delta_g(\delta_g^{-1}(\sqrt{k_6}\varepsilon))$. Thus, by Lemma 4.1, (51) holds for all t in $[\sigma_1,\sigma_2)$. Finally, for all t in $[\sigma_2,+\infty)$, X is a Carathéodory solution of (20) such that $|X(\sigma)| < \delta_l(\delta_l^{-1}(\delta_g^{-1}(\sqrt{k_6}\varepsilon)))$. Hence, one has (51) by Lemma 4.1.

Consider now a function $\delta\colon \mathbb{R} \to \mathbb{R}$ of class $\mathcal{K}_\infty$ defined, for all $\varepsilon > 0$, as

$$\delta(\varepsilon) = \min(\delta_l(\Delta^{-1}(\varepsilon)),\delta_g(\Delta^{-1}(\varepsilon))) \ ,$$

where $\Delta\colon \mathbb{R} \to \mathbb{R}$, is defined by

$$\Delta(\varepsilon) = \max\left(\varepsilon\sqrt{k_6},\varepsilon,\delta_l^{-1}(\varepsilon)\sqrt{k_6},\delta_g^{-1}(\sqrt{k_6}\varepsilon),\delta_l^{-1}(\delta_g^{-1}(\sqrt{k_6}\varepsilon))\sqrt{k_6}\right) \ , \forall \varepsilon > 0 \ ,$$

and note that, with the use of such functions, it is immediate to infer global stability (see Definition 2.3).

Global attractivity

Consider constants $r > 0$, $C > 0$ and perturbations e, $d\colon \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying our standing regularity assumptions and (13), for all x in $\mathbb{R}^n$. Let $|x_0|$ be in $\mathbb{R}^n$, such that $|x_0| \leq C$, let s_0 be in $\{1,2\}$ and let (X,S_d) be a solution of (7) starting from (x_0,s_0) and defined on $[0,+\infty)$. Consider $T > 0$ defined as

$$\begin{aligned} T \quad = \quad & \max\big(T_l(r,C,k_6),T_g(k_1,C) + T_l(r,\delta_g^{-1}(C),k_6),T_l(r,\delta_g^{-1}(\delta_l^{-1}(C)),k_6) \\ & +T_g(\delta_l^{-1}(C),k_1) + \tilde{T}\big) \ , \end{aligned}$$

where

$$\tilde{T} = T_l(\delta_l(\min(r,\delta_g(\min(r,\delta_l(r))))),C,k_6) \ .$$

We now shown that

$$|X(t)| \leq r \ , \forall t \geq T \ . \tag{52}$$

By Lemma 6.7, only three cases may occur.

1. Suppose Case 1 of Lemma 6.7 holds. Then, by Lemma 4.1 and (33), (52) holds.
2. Suppose Case 2 of Lemma 6.7 holds. Then, by Lemma 4.3, $|X(\sigma)| \leq \delta_g^{-1}(C)$, $z(X(\sigma)) \leq k_6$ and $\sigma \leq T_g(k_1,C)$. Moreover, for all t in $[\sigma,+\infty)$, X is a Carathéodory solution of (20). Thus (52) holds by Lemma 4.1.

3. Suppose Case 3 of Lemma 6.7 holds. Then, the inequality $|X(\sigma_1)| \leq \delta_l^{-1}(C)$ yields
$$\sigma_2 - \sigma_1 < T_g(\delta_l^{-1}(C), k_1) \ . \tag{53}$$
and we need to consider two sub-cases.
   (a) If $\sigma_1 \leq \tilde{T}$, then $|X(\sigma_1)| \leq \delta_l^{-1}(C)$, and $|X(\sigma_2)| \leq \delta_g^{-1}(\delta_l^{-1}(C))$. Thus, by Lemma 4.1 and (53), (52) holds.
   (b) If $\sigma_1 > \tilde{T}$, then by Lemma 4.1, for all t in $[\tilde{T}, \sigma_1)$, $|X(t)| \leq \min(r, \delta_g(\min(r, \delta_l(r))))$, and, by Lemma 4.3, for all t in $[\sigma_1, \sigma_2)$, $|X(t)| \leq \min(r, \delta_l(r))$. Moreover, $|X(\sigma_2)| \leq \delta_l(r)$, hence Lemma 4.1 yields $|X(t)| \leq r$, for all $t \geq \sigma_2$, which proves (52).

We have thus proved global attractivity, and this completes the proof of Theorem 1.

7 Conclusions

The problem of global robust stabilization of nonholonomic chained systems in the presence of sensor noise and external disturbances has been addressed. It has been shown that the problem is *partially* solvable by means of a simple hybrid control law, *i.e.* it is possible to achieve global asymptotic stability of the zero equilibrium in the presence of (small) perturbations vanishing in a region of measure zero (containing the origin) and with a non-zero diameter that can be arbitrarily reduced.

The hybrid control law is constructed starting from two controllers, a local controller and a global controller, and *joining* their domain of definition by means of a hysteresis. The properties of the trajectories of the controlled perturbed system are studied in detail, and it is shown that for any admissible perturbation only a finite number of switches may occur.

The proposed control law is very simple and can be easily implemented in practice. Note that, in the absence of perturbations, the control law guarantees exponential convergence to the origin and retain some of the basic properties of the discontinuous control laws proposed in [A:95], namely exponential convergence rate and lack of oscillatory behavior.

The results presented in this paper are based on the general theory developed in [P:01b]. However, the theory in [P:01b] is not constructive, *i.e.* it cannot be directly used to design robustly stabilizing control laws for a given system. In this respect, the main contribution of this work is to show that, for a large class of nonholonomic systems, a robustly stabilizing control law can be explicitly designed, and it is possible to obtain explicit bounds on the admissible perturbations.

It is worth noting that, to the best of the authors' knowledge, and with the exception of the result in [P:00], which applies to a special low-dimensional system, the main result of the paper is the first application of the general theory in [P:01b] to a class of systems which are not continuously stabilizable.

The present study leaves several issues open. The problem of designing a controller achieving robustness in the ISS sense is still open. This problem may be addressed using more complex hybrid controllers, *i.e.* controllers with several hybrid variables. Note however that, for three dimensional chained systems, which are globally feedback equivalent to the Brockett integrator, a simple application of the methodology presented in this work together with the results in [A:98], may yield control laws which are robust in the ISS sense. Applicability of our methodology to the class of dynamic nonholonomic systems, as studied in [LA:99, J:00], is under investigation, although no conceptual difficulty arise in this extension. Finally, robust stabilization of the zero equilibrium by means of dynamic output feedback controllers is a much more challenging problem, which is currently under investigation.

Deuxième partie

Stabilisation d'un bac d'eau

Dans cette partie nous étudions une forme simplifiée des équations d'Euler pour les fluides incompressibles, non-visqueux et irrotationnels : les équations de Saint-Venant initialement introduites dans [S:71].

Ces équations hyperboliques sont beaucoup étudiées car elles présentent de nombreuses applications intéressantes. Citons par exemple l'étude des avalanches de neige dense [V:86, Chapitre II. 3], l'irrigation [M:94] ou la navigation fluviale [G:98].

Signalons que les propriétés des équations de Saint-Venant linéarisées autour de l'équilibre sont relativement connues (voir ainsi [DPR:99, PR:02]) mais que l'influence des termes non-linéaires est encore peu étudiée (à part [C:01]).

Nous nous intéressons à ces équations pour décrire les écoulements lents de fluide et nous étudions le problème de la stabilisation d'un récipient contenant un fluide par le contrôle de la force extérieure exercée longitudinalement sur le récipient (ou directement par le contrôle du déplacement longitudinal du récipient). La dynamique du fluide est modélisée par les équations de Saint-Venant à une dimension d'espace (chapitre 1). Nous nous intéressons au problème de la stabilisation de l'état du fluide, de la position et de la vitesse du bac. Nous utilisons une approche Lyapunov pour trouver des commandes qui font baisser l'entropie entre l'équilibre et l'état du système (chapitre 2) et nous constatons au chapitre 3 que numériquement ces contrôleurs réalisent le problème de stabilisation souhaité.

Nous pouvons trouver de nombreuses applications industrielles de ce problème. Voir par exemple la conduite d'un camion citerne dont le mouvement du liquide transporté peut perturber sensiblement la trajectoire du camion (voir [M:00b]), ou le remplissage de récipients ouverts remplis de liquide (briques de lait) que nous déplaçons entre différents ateliers et dont nous souhaitons prévenir tout débordement (voir [G:00]).

Pour les calculs complets, voir en appendice de ce chapitre à partir de la page 163 ou [PH:01].

Chapitre 1

Présentation du modèle

1.1 Les équations du fluide

Dans ce paragraphe nous donnons les équations du fluide. Elles seront complétées au paragraphe suivant par l'équation du bac (1.13).

Nous considérons un récipient contenant un liquide non visqueux irrotationnel et incompressible. Nous supposons que le récipient est rectangulaire et que le mouvement se fait longitudinalement. Nous supposons enfin que l'accélération du bac est petite par rapport à la constante gravitationnelle et que la hauteur d'eau est petite par rapport à la longueur du bac. C'est pourquoi nous utilisons les équations de Saint-Venant (*Shallow water equations* pour les anglo-saxons) (voir [D:94, Section 4.2] et voir également [CF:76, Page 32], [S:92], [PR:02], ou [S:71] pour une référence historique) qui s'écrivent, pour tout $t \geq 0$ et pour tout x dans $[0,L]$,

$$\begin{aligned} H_t(t,x) + (HV)_x(t,x) &= 0 \,, &(1.1)\\ V_t(t,x) + (gH + \frac{V^2}{2})_x(t,x) &= -\ddot{D}(t) \,, &(1.2)\\ V(t,0) &= 0 \,, &(1.3)\\ V(t,L) &= 0 \,, &(1.4)\end{aligned}$$

où

1. L est la longueur du bac.
2. $H(t,x)$ désigne la hauteur du fluide à l'instant $t \geq 0$ et au point d'abscisse x dans $[0,L]$.
3. $V(t,x)$ est la vitesse horizontale à l'instant t de la surface d'abscisse x. Du fait du mouvement longitudinal tous les points d'une même surface verticale perpendiculaire à la longueur du bac ont une même vitesse horizontale.
4. $D(t)$ est la position du centre du bac à l'instant t par rapport à un repère absolu.
5. g est la constante de gravité.

Voir la figure 1.

Un autre modèle possible, avec des hypothèses différentes, pour étudier un récipient contenant un fluide peut être trouvé dans [M:00a].

Les équations de Saint-Venant (1.1)-(1.4) sont écrites dans un repère mobile attaché au bac mais nous pouvons bien sûr les écrire facilement dans un repère absolu en faisant le changement

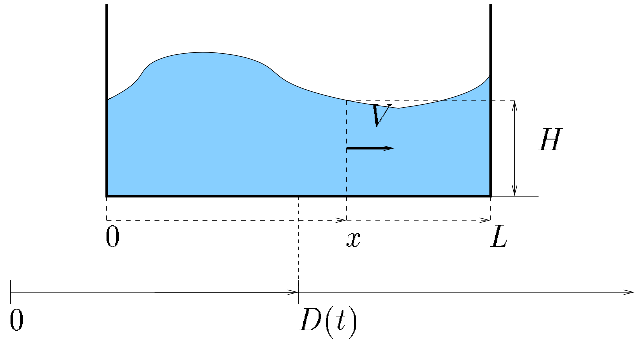

FIG. 1 – *Récipient contenant un liquide.*

de variables :

$$\begin{aligned} \tilde{x} &= x + D(t) - \frac{L}{2} , \\ \tilde{h}(t,\tilde{x}) &= h(t,x) , \\ \tilde{v}(t,\tilde{x}) &= v(t,x) + \dot{D}(t) . \end{aligned} \tag{1.5}$$

Nous trouvons alors, pour tout $\widetilde{x}$ dans $\mathbb{R}$ et pour tout $t \geq 0$,

$$\tilde{H}_t(t,\tilde{x}) + (\tilde{H}\tilde{V})_{\tilde{x}}(t,\tilde{x}) = 0 , \tag{1.6}$$

$$\tilde{V}_t(t,\tilde{x}) + (g\tilde{H} + \frac{\tilde{V}^2}{2})_{\tilde{x}}(t,\tilde{x}) = 0 , \tag{1.7}$$

$$\tilde{V}(t,0) = \dot{D}(t) , \tag{1.8}$$

$$\tilde{V}(t,L) = \dot{D}(t) . \tag{1.9}$$

Remarquons que c'est la paramétrisation choisie par exemple dans [PR:02, DPR:99]. La principale différence entre (1.1)-(1.4) et (1.6)-(1.9) est la position de la fonction extérieure au fluide $D(t)$, qui est soit au second membre des équations aux dérivées partielles soit dans les conditions aux bords.

Dans [GP:00], les auteurs démontrent que ces équations sont des approximations de l'équation d'Euler pour les fluides parfaits irrotationnels et incompressibles. Notons par ailleurs qu'elles forment un système d'équations aux dérivées partielles hyperboliques. Remarquons enfin comme démontré dans [L:90, Section 5.2] que les équations (1.6) et (1.7) correspondent aux équations d'un gaz isentropique lorsque le rapport (noté usuellement γ) de la chaleur spécifique à volume constant sur la chaleur spécifique à pression constante vaut 2.

Les équations (1.1)-(1.4) sont les équations du fluide. Si nous soumettons le bac à une force extérieure $F(t)$, nous pouvons décrire l'action de cette force sur le déplacement du bac dépendant de la masse M du bac et de la masse volumique ρ du fluide. Pour cela nous proposons deux méthodes :

- Un bilan d'énergie, en utilisant les équations dans le repère absolu, voir paragraphe 1.2.
- En appliquant l'équation d'Euler et en utilisant les équations dans le repère mobile, voir paragraphe 1.3.

1.2 Les équations du bac par un bilan d'énergie

Soit $E = E(t)$ l'énergie totale du récipient dans le repère absolu, c'est-à-dire la somme de l'énergie cinétique du bac vide, de l'énergie cinétique et potentielle du fluide, i.e.

$$E = \frac{1}{2}M\dot{D}^2 + \int_{D-\frac{L}{2}}^{D+\frac{L}{2}} \left(\rho\tilde{H}\frac{\tilde{V}^2}{2} + \rho g\frac{\tilde{H}^2}{2} \right) d\tilde{x} \ . \tag{1.10}$$

Nous utilisons la relation fondamentale de la dynamique ([LL: 88, Chapitre I.3]) qui s'écrit :

$$E_t = F\dot{D} \ ,$$

où $F = F(t)$ désigne la force extérieure exercée sur le bac à l'instant t. Ainsi, en utilisant (1.10), (1.6) et (1.7), nous obtenons :

$$\begin{aligned} M\ddot{D}\dot{D} &= F\dot{D} + \int_{D-\frac{L}{2}}^{D+\frac{L}{2}} \left(\rho\frac{\widetilde{V}^2}{2}(\widetilde{H}\widetilde{V})_{\widetilde{x}} + \rho\widetilde{H}\widetilde{V}(g\widetilde{H} + \frac{\widetilde{V}^2}{2})_{\widetilde{x}} + \rho g\widetilde{H}(\widetilde{H}\widetilde{V})_{\widetilde{x}} \right) d\widetilde{x} \\ & \quad -\dot{D}\left[\rho g\frac{\widetilde{H}^2}{2} + \rho\widetilde{H}\frac{\widetilde{V}^2}{2} \right]_{D-\frac{L}{2}}^{D+\frac{L}{2}} \ . \end{aligned}$$

En utilisant une intégration par parties, (1.8) et (1.9), nous calculons :

$$M\ddot{D}\dot{D} = F\dot{D} + \frac{\rho g\dot{D}}{2}\left(\widetilde{H}(D + \frac{L}{2}) - \widetilde{H}(D - \frac{L}{2}) \right) \ ,$$

qui, en utilisant (1.5) et les notations,

$$H_L(t) = \widetilde{H}(D(t) + \frac{L}{2},t) = H(L,t) \ , \tag{1.11}$$

$$H_0(t) = \widetilde{H}(D(t) - \frac{L}{2},t) = H(0,t) \ , \tag{1.12}$$

se réécrit

$$M\ddot{D}(t) = F(t) + \frac{\rho g}{2}(H_L^2(t) - H_0^2(t)) \ . \tag{1.13}$$

Nous allons maintenant retrouver cette équation par une autre méthode.

1.3 Les équations du bac en utilisant l'équation d'Euler

Dans ce paragraphe nous redémontrons (1.13) en utilisant l'équation d'Euler.

Nous écrivons le lagrangien du système bac-fluide qui est égal à l'énergie cinétique totale moins l'énergie potentielle totale, soit :

$$\mathcal{L}(V,H,D,\dot{D}) = M\frac{\dot{D}^2}{2} + \int_0^L \left(\rho H\frac{(V + \dot{D})^2}{2} - \rho g\frac{H^2}{2} \right) dx \ . \tag{1.14}$$

Nous écrivons l'équation d'Euler ([LL: 88, Chapitre VI.36]) :

$$\frac{d}{dt}(\mathcal{L}_{\dot{D}}) - \mathcal{L}_D = F \ . \tag{1.15}$$

Prouvons (1.13) en explicitant (1.15). Nous trouvons avec (1.14)

$$\mathcal{L}_D = 0 \ ,$$

$$\mathcal{L}_{\dot{D}} = M\dot{D} + \int_0^L \rho H(V + \dot{D})dx \ ,$$

et donc

$$\frac{d}{dt}(\mathcal{L}_{\dot{D}}) = M\ddot{D} + \int_0^L \Big(\rho H_t(V + \dot{D}) + \rho H V_t + \ddot{D}\rho H\Big)\, dx \ .$$

Soit, en utilisant (1.1) et (1.2),

$$\frac{d}{dt}(\mathcal{L}_{\dot{D}}) = M\ddot{D} - \int_0^L \left(\rho(HV)_x(\dot{D} + V) - \rho H(gH_x + (\frac{V^2}{2})_x)\right) dx \ .$$

Nous intégrons par parties et nous utilisons (1.3), (1.4), (1.11) et (1.12) pour trouver :

$$\frac{d}{dt}(\mathcal{L}_{\dot{D}}) = M\ddot{D} - \frac{\rho g}{2}(H_l^2 - H_0^2) \ ,$$

Par conséquent, nous retrouvons (1.13) avec (1.15).

Maintenant que les équations complètes ((1.1)-(1.4) et (1.13)) sont écrites, nous définissons le problème de stabilisation auquel on va s'intéresser.

1.4 Le problème de la stabilisation du bac

Dans ce paragraphe, nous définissons le problème de stabilisation qui va nous intéresser jusqu'à la fin de cette partie.

Soit le système contrôlé Σ suivant :

- (H, V, D, S, A) est l'état du système fluide-bac[1].
- F est le contrôle.

Nous introduisons l'espace E des fonctions $Y = (H, V, D, S, A)$ dans $\mathcal{C}^1([0,L]) \times \mathcal{C}^1([0,L]) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, muni de la norme

$$|Y| \ = \ |H|_1 + |V|_1 + |D| + |S| + |A| \ ,$$

où $|.|_1$ est la norme dans $\mathcal{C}^1([0,L])$ définie, pour tout f dans $\mathcal{C}^1([0,L])$, par

$$|f|_1 = \max_{x\in[0,L]} |f(x)| + \max_{x\in[0,L]} |f'(x)| \ ,$$

et $|.|$ désigne la valeur absolue dans $\mathbb{R}$.

1. nous verrons, à l'équation (1.16) et à la définition 1.4.1, que S est la vitesse et A l'accélération.

Soit $\bar{E} = \{(\bar{H},\, \bar{V},\, \bar{D},\, \bar{S},\, \bar{A})\}$ le sous-espace affine de E formé des équilibres possibles de E, c'est-à-dire tel que, si nous choisissons une trajectoire du bac vérifiant

$$D(t) = \bar{D} + \bar{S}t + \frac{1}{2}\bar{A}t^2 \ , \forall t \geq 0 \ , \tag{1.16}$$

alors le fluide à l'intérieur du bac a une hauteur et une vitesse respectivement égales à :

$$H(x) = \bar{H}(x) \qquad , \quad V(x) = \bar{V}(x) \ , \forall x \in [0,L] \ .$$

Noter que nous trouvons, dans [PH:01] et en appendice page 167, une expression de cet espace affine. Choisissons un équilibre $\bar{Y}$ de $\bar{E}$. D'après la loi de conservation de la masse nous devons avoir

$$\forall t \geq 0, \ \int_0^L H(t,x)dx = \int_0^L \bar{H}(x)dx \ .$$

De plus, d'après (1.2), (1.3) et (1.4), nous avons

$$H_x(t,0) = H_x(t,L) = -\frac{\ddot{D}(t)}{g} \ .$$

Cela justifie d'introduire $\mathcal{E}$, le sous-espace affine de E formé des fonctions $Y = (H,\, V,\, D,\, S,\, A)$ telles que

$$\begin{aligned} \int_0^L H dx &= \int_0^L \bar{H} dx \ , \\ H_x(0) &= H_x(L) \ = \ -\frac{A}{g} \ . \end{aligned}$$

Nous définissons maintenant la notion de solution du système contrôlé Σ :

Définition 1.4.1 *Soit* $0 \leq T_1 \leq T_2$. *Nous disons que* $(Y,\mathcal{F})$, *avec* $Y = (H,\, V,\, D,\, S,\, A) : [T_1,T_2] \to \mathcal{E}$ *et* $\mathcal{F} : [T_1,T_2] \times \mathcal{E} \to \mathbb{R}$, *est une* solution *du système contrôlé* Σ *si :*

- *Les fonctions* H *et* V *sont dans* $\mathcal{C}^1([T_1,T_2] \times [0,L],\, \mathbb{R})$.
- *Les fonctions* D, S *et* A *sont respectivement dans* $\mathcal{C}^2([T_1,T_2],\, \mathbb{R})$, $\mathcal{C}^1([T_1,T_2],\, \mathbb{R})$ *et* $\mathcal{C}([T_1,T_2],\, \mathbb{R})$.
- *Nous avons*
$$\forall t \in [T_1,T_2], \quad \dot{D}(t) = S(t) \ , \quad \dot{S}(t) = A(t) \ .$$
- *La fonction* $\mathcal{F}$ *est une fonction continue de* $[T_1,T_2] \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathbb{R}^3$ *dans* $\mathbb{R}$.
- *Les équations (1.1)-(1.4) et (1.13) sont satisfaites, pour tout* t *dans* $[T_1,T_2]$ *et pour tout* x *dans* $[0,L]$, *en posant*

$$F(t) = \mathcal{F}(t,\, H(t,.),\, V(t,.),\, D(t),\, S(t),\, A(t)) \ . \tag{1.17}$$

Notons que nous demandons au contrôle d'être continu. Nous pouvons donc énoncer un résultat d'existence de solutions de Σ écrit en termes de condition de compatibilité des conditions initiales (voir [PH:01] ou en appendice page 168, c'est essentiellement [LY:85, Theorem 4.2, page 96]).

Le problème de stabilisation auquel nous nous intéressons est le *problème de la stabilisation locale vers $\bar{Y}$ avec contrôle de la force $\mathcal{F}$*, i.e.

Pouvons-nous trouver $\mathcal{F} : \mathbb{R}_{\geq 0} \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}$ continue telle que

- *Il existe $C > 0$ telle que, pour tout Y_0 dans $\mathcal{E}$ satisfaisant $|Y_0 - \bar{Y}| \leq C$, il existe une et une seule fonction $Y :[0,+\infty) \to \mathcal{E}$ telle que $(Y,\mathcal{F})$ soit une solution de Σ et telle que $Y(0) = Y_0$. De plus cette solution vérifie*

$$\begin{aligned} &|H(t,.) - \bar{H}|_1 + |V(t,.) - \bar{V}|_1 + |D(t) - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2| \\ &+|\dot{D}(t) - \bar{S} - \bar{A}t| + |\ddot{D}(t) - \bar{A}| \to_{t\to+\infty} 0 \ . \end{aligned}$$

- *Pour tout $\varepsilon > 0$, il existe $\eta > 0$ telle que, si Y_0 est dans $\mathcal{E}$ tel que $|Y_0 - \bar{Y}| \leq \eta$ et $Y :[0,+\infty) \to \mathcal{E}$ est telle que $(Y,\mathcal{F})$ soit une solution de Σ satisfaisant $Y(0) = Y_0$, alors nous avons :*

$$\begin{aligned} &|H(t,.) - \bar{H}|_1 + |V(t,.) - \bar{V}|_1 + |D(t) - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2| \\ &+|\dot{D}(t) - \bar{S} - \bar{A}t| + |\ddot{D}(t) - \bar{A}| \leq \varepsilon, \forall t \geq 0 \ . \end{aligned}$$

Remarque 1.4.2 Notons que dans (1.17), nous avons fait dépendre le contrôle de tout l'état du système fluide-bac à l'instant t, à savoir

$$(t,\, H(t,.),\, V(t,.),\, D(t),\, \dot{D}(t),\, \ddot{D}(t)) \ , \tag{1.18}$$

et qu'il aurait été plus intéressant de faire dépendre le contrôle que des variables faciles à mesurer, à savoir

$$(t,\, H_0(t),\, H_L(t),\, D(t),\, \dot{D}(t),\, \ddot{D}(t)) \ . \tag{1.19}$$

Nous avons, en fait, répondu partiellement au problème de la stabilisation vers $\bar{Y}$, par le contrôle de $\mathcal{F}$ dépendant uniquement des variables mesurées (1.19). Voir paragraphe 2.1. ◇

Notons aussi qu'un problème de contrôle est aussi posé dans [PH:01] et [DPR:99], en étudiant le même système Σ mais avec un contrôle pris égal à A, ce qui *a priori* a moins de sens physique. En effet, nous appliquons en général une force et non une accélération sur un corps.

En fait, ces deux problèmes sont équivalents car la relation qui lie $F(t)$ et $\ddot{D}(t)$ dans (1.13) ne fait intervenir que $H_0(t)$ et $H_L(t)$ et non tout l'état du fluide. Par conséquent contrôler le bac avec une force F dépendant des variables (1.18) (resp. dépendant uniquement des variables (1.19)) ou directement avec l'accélération A dépendant de la variable (1.18) (resp. dépendant uniquement de la variable (1.19)) revient au même, il suffit juste d'ajouter une fonction de $H_0(t)$ et de $H_L(t)$. C'est pourquoi pour être cohérent avec ces travaux, nous nous intéressons uniquement au système contrôlé Σ' suivant :

- $(H,\, V,\, D,\, S)$ est l'état du système fluide-bac.
- $\mathcal{A}$ est le nouveau contrôle,

et au *problème de la stabilisation locale vers $\bar{Y}$ avec contrôle de l'accélération $\mathcal{A}$* :

Pouvons-nous trouver $\mathcal{A}: \mathbb{R}_{\geq 0} \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathcal{C}^1([0,L],\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}$ *continue telle que*

- *Il existe* $C > 0$ *telle que, pour tout* Y_0 *dans* $\mathcal{E}$ *satisfaisant* $|Y_0 - \bar{Y}| \leq C$*, il existe une et une seule fonction* $Y :[0,+\infty) \to \mathcal{E}$ *telle que* $(Y,\mathcal{F})$ *soit une solution de* Σ *et telle que* $Y(0) = Y_0$*. De plus cette solution vérifie*

$$|H(t,.) - \bar{H}|_1 + |V(t,.) - \bar{V}|_1 + |D(t) - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2| \\ +|\dot{D}(t) - \bar{S} - \bar{A}t| \to_{t\to+\infty} 0 \ .$$

- *Pour tout* $\varepsilon > 0$*, il existe* $\eta > 0$ *telle que, si* Y_0 *est dans* $\mathcal{E}$ *tel que* $|Y_0 - \bar{Y}| \leq \eta$ *et* $Y :[0,+\infty) \to \mathcal{E}$ *est telle que* $(Y,\mathcal{F})$ *soit une solution de* Σ *satisfaisant* $Y(0) = Y_0$*, alors nous avons :*

$$|H(t,.) - \bar{H}|_1 + |V(t,.) - \bar{V}|_1 + |D(t) - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2| \\ +|\dot{D}(t) - \bar{S} - \bar{A}t| \leq \varepsilon, \forall t \geq 0 \ .$$

La notion de solution de Σ' est évidemment analogue à la définition 1.4.1 mais en posant dans (1.1)-(1.4)

$$\ddot{D}(t) = \mathcal{A}(t, H(t,.), V(t,.), D(t), S(t)) \ .$$

Remarque 1.4.3 Nous répondons partiellement au problème de la stabilisation par le contrôle $\mathcal{A}$ dépendant de la variable (1.19) (plus exactement de $(t,H_0(t),H_L(t),D(t),\dot{D}(t))$). Voir paragraphe 2.1. ◇

Remarque 1.4.4 Nous pouvons montrer (voir [DPR:99] et [C:01]) que les équations (1.1)-(1.4) *linéarisées* autour de l'équilibre :

$$\bar{H} = C_1 \quad , \qquad \bar{V} = 0 \ , \tag{1.20}$$

$$\bar{D} = C_2 \quad , \quad \bar{S} = 0 \ , \qquad \bar{A} = 0 \ , \tag{1.21}$$

où C_1 et C_2 sont deux constantes ne sont pas contrôlables (même localement). En effet il existe une solution (il suffit en fait de prendre H paire et V impaire, plus une condition de compatibilité) non nulle de norme aussi petite que nous voulons avec un contrôle nul (contrôle de $\mathcal{A}$ ou de $\mathcal{F}$). En revanche les équations de Saint-Venant non linéaires sont localement contrôlables autour de toute origine de la forme (1.20)-(1.21) (voir [C:01]) et la question de la stabilisation des équations (1.1)-(1.4) est donc une question dont nous espérons une réponse positive.

Nous reviendrons plus tard sur cette question lorsque nous vérifierons numériquement qu'avec les termes non linéaires des équations de Saint-Venant nous avons un résultat de stabilisation mais pas, bien sûr, avec les termes linéaires uniquement. Voir paragraphe 3.4.2 ci-dessous. ◇

Maintenant nous allons proposer une approche Lyapunov pour trouver des contrôles stabilisant.

Chapitre 2

Expressions des contrôleurs

Dans ce paragraphe nous fixons un équilibre $\bar{Y}$ et un contrôle à l'équilibre $\bar{A}$, nous nous intéressons au problème de la stabilisation par le contrôle $\mathcal{A}$.

Les équations (1.1)-(1.2) forment un système d'équations de conservation hyperbolique. Il est donc naturel de penser à utiliser un couple (E,F) d'entropie-flux associé à ce système (voir [S:96, Tome 1, page 96]).

Nous cherchons (E,F) telle que la fonction $R:[0,+\infty) \to \mathbb{R}$ définie par

$$R(t) = \int_0^L E(H(t,x),V(t,x))dx \ , \tag{2.1}$$

soit minimale à l'équilibre et telle que $\mathcal{A} - \bar{A}$ apparaisse en facteur dans l'expression de la dérivée temporelle de R le long des solutions de (1.1)-(1.4), *i.e.*

$$\dot{R}(t) = (\mathcal{A} - \bar{A})\ k(t,\, H(t,.),\, V(t,.),\, D(t),\, \dot{D}(t)) \ , \tag{2.2}$$

où k est une fonction de l'état du fluide-bac. Ainsi, si nous posons

$$\mathcal{A} = -k(t,\, H(t,.),\, V(t,.),\, D(t),\, \dot{D}(t)) + \bar{A} \ , \tag{2.3}$$

nous aurons $\dot{R}(t) \leq 0$ et nous pouvons donc espérer que l'état du système converge vers l'équilibre.

Notons que c'est exactement la même démarche adoptée dans [CAB:99] pour trouver un candidat au problème de stabilisation posé dans cet article.

Remarque 2.0.5 Notons qu'en relation avec la remarque 1.4.2, pour chercher un contrôle dépendant uniquement de la variable (1.19), il serait préférable de trouver R telle que nous ayons

$$\dot{R}(t) = (\mathcal{A} - \bar{A})\ k(t,\, H_0(t),\, H_L(t),\, D(t),\, \dot{D}(t)) \ ,$$

et poser

$$\mathcal{A} = -k(t,\, H_0(t),\, H_L(t),\, D(t),\, \dot{D}(t)) + \bar{A} \ . \tag{2.4}$$

En fait nous verrons au paragraphe 2.1, que nous avons adopté une démarche différente. En effet nous avons d'abord cherché un contrôle de la forme (2.3) et nous avons ensuite essayé de trouver une expression équivalente sous la forme (2.4). $\diamond$

Nous ne pouvons pas affirmer que si $\dot{R}(t) \leq 0$, pour tout $t \geq 0$, alors l'état converge vers l'équilibre car nous n'avons pas prouvé, pour l'instant, la condition de LaSalle (voir [L:74]) :

Si $\dot{R}(t) = 0$, pour tout t, alors nous avons $R(t) = 0$ à chaque instant.

En plus même si cette condition est vraie, nous nous ne pouvons pas conclure directement car il faudrait en plus prouver une propriété de précompacité des trajectoires pour une topologie adaptée (nous sommes en effet dans un espace de dimension infinie).

Cependant nous montrerons au chapitre 3 que cette approche donne de bons résultats sur une simulation numérique. Ceci laisse donc penser que la condition de LaSalle est vraie. Nous avons quelques pistes pour la démontrer par exemple en écrivant le système d'EDP hyperboliques en termes d'invariants de Riemann, en prouvant ce résultat pour les fonctions de Lyapunov trouvées et en s'inspirant de l'utilisation des conditions aux bords faite par [LY:85].

Nous allons maintenant expliciter les fonctions de Lyapunov en commençant par une stabilisation du fluide uniquement (*i.e.* des variables H et V) au paragraphe 2.1, puis une stabilisation du système fluide-vitesse du bac (voir paragraphe 2.2) et enfin une stabilisation complète de tout le système au paragraphe 2.3. Mais avant tout, commençons par rappeler la définition de la boucle ouverte.

Définition 2.0.6 *La boucle ouverte est simplement définie par*

$$\mathcal{A} = \bar{A} \ .$$

2.1 Stabilisation du fluide uniquement

Les fonctions de Lyapunov sont un outil important pour obtenir un résultat de stabilisation. L'énergie est souvent un bon candidat pour une fonction de Lyapunov. Ici elle ne convient pas directement. Nous la modifions pour avoir toujours (2.1) et pour avoir un minimum au point d'équilibre $H = \bar{H}$ et $V = 0$. C'est pourquoi nous posons

$$R_1(t) = \lambda_1 \int_0^L H(t,x) \left(\frac{V^2(t,x)}{2} + g\frac{(H-\bar{H})^2(t,x)}{2} \right) dx \ , \tag{2.5}$$

où λ_1 est un paramètre, que nous pourrons ajuster pour régler le contrôleur. Notons qu'il existe une infinité de couples entropie-flux d'entropie. Voir [S:96, tome II, paragraphe 9.3].

Nous calculons $\dot{R}_1$ le long des solutions de (1.1) et (1.2), et nous remarquons que nous sommes dans la configuration de (2.2) pour $R = R_1$ et donc nous avons un contrôleur faisant strictement décroître la fonction R_1, à savoir :

$$\mathcal{A} = \lambda_1 \int_0^L (HV)(t,x) dx + \bar{A} \ . \tag{2.6}$$

En fait ce contrôle utilise toute l'information de l'état du fluide-bac. Nous pouvons, en réutilisant (1.1) et (1.2), le réécrire sous la forme :

$$\forall t \geq 0,\ \dot{\mathcal{A}}(t) \ = \ \lambda_1 \left(\frac{g}{2}(H_0^2(t) - H_L^2(t)) - \text{vol}\ \mathcal{A}(t) \right) \ , \tag{2.7}$$

$$\mathcal{A}(0) \ = \ \lambda_1 \int_0^L (HV)(0,x) dx + \bar{A} \ , \tag{2.8}$$

qui utilise toute la connaissance de l'état du fluide seulement à l'instant $t = 0$. Cela répond partiellement à la question de la remarque 1.4.2.

Notons que si la vitesse relative du fluide par rapport au bac est nulle au temps initial, alors le contrôle initial est nul et nous n'avons pas besoin de connaître la hauteur du fluide dans le bac.

D'autre part un prolongement possible de ce travail est d'étudier l'influence de la condition initiale du contrôle (2.7)-(2.8), et de voir si quelle que soit $\mathcal{A}(0)$ dans $\mathbb{R}$ (ou plus vraisemblablement dans une boule à déterminer), le contrôle (2.7) stabilise les variables H et V. Plutôt que de prolonger cette piste de *contrôle dynamique par retour de sortie*, nous avons préféré simuler numériquement, au chapitre 3, le système fluide-bac, et voir si le contrôle (2.6) stabilise les variables H et V, ce qui n'a pas été démontré théoriquement (voir la discussion concernant la condition de LaSalle page 142).

2.2 Stabilisation du fluide et de la vitesse du bac

Nous procédons comme précédemment mais en ajoutant un terme à R_1 qui est une fonction strictement convexe en la vitesse avec un minimum pour la vitesse $\dot{D} = \bar{S} + \bar{A}t$.

C'est pourquoi nous choisissons :

$$R_2(t) = R_1(t) + \frac{\lambda_2}{2}(\dot{D}(t) - \bar{S} - \bar{A}t)^2 \ ,$$

où λ_2 est un nouveau paramètre pour ajuster le contrôleur. Nous calculons donc

$$\mathcal{A} = \lambda_1 \int_0^L (HV)(t,x)dx - \lambda_2(\dot{D}(t) - \bar{S} - \bar{A}t) + \bar{A} \ . \tag{2.9}$$

Nous pouvons trouver une forme analogue à (2.7)-(2.8) et faire la même discussion qu'au paragraphe 2.1.

2.3 Stabilisation du système complet

Pour stabiliser le système complet, c'est-à-dire à la fois l'état du fluide, la position et la vitesse du bac, nous allons chercher à contrôler D et réutiliser les fonctions de Lyapunov trouvées précédemment pour conserver la stabilisation du fluide et de la vitesse du bac. Pour cela nous remarquons que pour stabiliser D autour de $\bar{D}$, il faut stabiliser $\int \dot{D}dt$ autour de $\int(\bar{S}+\bar{A}t)dt$. C'est pourquoi nous utilisons une technique de forwarding (introduite initialement par [MP:96]) pour trouver une nouvelle fonction de Lyapunov. Plus précisément nous calculons une fonction R_3 dont la dérivée le long des solutions de (1.1)-(1.4) est proportionnelle à $\dot{R}_2$ et qui est minimale pour $D(t) = \bar{D} + \bar{S}t + \frac{1}{2}\bar{A}t^2$. Cela nous conduit à :

$$R_3(t) = R_2(t) + \frac{\lambda_3}{2}\left(\lambda_2(D(t) - \bar{D} - \bar{S}t - \frac{1}{2}\bar{A}t^2) + \lambda_1 \int_0^L (\int_0^x (H - \bar{H})(t,\xi)d\xi)dx\right)^2$$

Cette nouvelle fonction de Lyapunov fait intervenir un nouveau degré de liberté que nous avons noté λ_3.

Nous calculons donc

$$\begin{aligned}\mathcal{A} &= \lambda_1 \int_0^L (HV)(t,x)dx - \lambda_2(\dot{D}(t) - \bar{S} - \bar{A}t) - \lambda_2\lambda_3(D(t) - \bar{D} - \bar{S}t - \frac{1}{2}\bar{A}t^2) \\ &\quad -\lambda_1\lambda_3 \int_0^L \left(\int_0^x (H(t,\xi) - \bar{H}(\xi))d\xi \right) dx + \bar{A} \ . \qquad (2.10)\end{aligned}$$

Nous pouvons trouver une forme analogue à (2.7)-(2.8) et faire la même discussion qu'au paragraphe 2.1.

Nous allons maintenant vérifier numériquement que ces contrôles réalisent bien le problème de stabilisation souhaité et étudier l'influence des termes non-linéaires des équations de Saint-Venant.

Chapitre 3

Validation numérique

Dans ce chapitre nous allons exposer le schéma numérique choisi pour résoudre les équations de Saint-Venant (1.1)-(1.4) et nous donnons les résultats numériques des simulations.

Il existe de nombreux schémas qui ont été utilisé pour les équations de Saint-Venant. Citons

- Le schéma [ABP:00] qui est spécifique aux équations de Saint-Venant et qui utilise la topographie du domaine oǐes équations de Saint-Venant sont vérifiés.
- Le schéma [CL:99] qui est maintenant utilisé pour des applications pratiques [L:01].
- Le schéma de Godunov (voir [G:56]) utilisé pour résoudre les équations de Saint-Venant dans [V:86, Chapitre VII] et [DPR:99].
- Le schéma de Preissmann qui est trés utilisés par les hydrauliciens (voir par exemple [M:94, G:98]).

Nous avons utilisé les schémas de Preissmann et de Godunov. Nous ne présentons que le schéma de Preissmann (celui de Godunov a donné des résultats similaires) dont nous rappelons la définition au paragraphe 3.1 et quelques propriétés au paragraphe 3.2. Puis nous donnons quelques indications sur l'organisation du programme qui a permis de simuler les équations de Saint-Venant au paragraphe 3.3 et enfin (paragraphe 3.4) nous étudions les résultats de quelques simulations

- comparant les contrôleurs définis au chapitre 2 et permettant de comprendre l'approche du forwarding (paragraphe 3.4.1)
- et étudiant l'importance des termes non-linéaires dans les équations de Saint-Venant (paragraphe 3.4.2).

3.1 Définition du schéma

Choisissons deux pas de discrétisation. Un pas Δx pour l'espace et un autre Δt pour la discrétisation en temps.

Soit $u : [0, +\infty) \times [0,L] \to \mathbb{R} \times \mathbb{R}$ la fonction de classe $\mathcal{C}^2$ définie par $u(t,x) = (H,V)(t,x)$.

Nous allons utiliser le schéma suivant, dit schéma de Preissmann :

$$u(t,x) = \frac{\theta}{2}(u_{i+1}^{n+1} + u_i^{n+1}) + \frac{1-\theta}{2}(u_{i+1}^{n} + u_i^{n}) \ , \tag{3.1}$$

$$u_x(t,x) = \frac{\theta}{\Delta x}(u_{i+1}^{n+1} - u_i^{n+1}) + \frac{1-\theta}{\Delta x}(u_{i+1}^{n} - u_i^{n}) \ , \tag{3.2}$$

$$u_t(t,x) = \frac{\theta}{2\Delta t}(u_{i+1}^{n+1} + u_i^{n+1} - u_{i+1}^{n} - u_i^{n}) \ , \tag{3.3}$$

où θ est un coefficient pris entre 0 et 1, et u_i^n désigne, pour tous entiers relatifs n et i,

$$u_i^n = u(n\Delta t, i\Delta x) \ .$$

Remarquons que ce schéma est décentré et que, comme θ est choisi entre 0 et 1, il est semi-implicite. Remarquons, par ailleurs, la similitude du schéma de Preissmann avec celui de Crank-Nicholson [F:91, Chapter 9].

Nous allons étudier plus précisément ses propriétés au paragraphe suivant.

3.2 Propriétés du schéma

Nous trouvons dans [M:94] les propriétés numériques de ce schéma. Nous les rappelons ici :

Consistance

Le schéma est du premier ordre en x pour θ différent de 0,5 et du deuxième en x sinon. Il est du premier ordre en t quelque soit la valeur de θ.

Stabilité

Le schéma est inconditionnellement stable si θ est choisi dans $[0{,}5; 1]$ mais inconditionnellement instable sinon.

Il faut donc choisir θ entre 0,5 et 1 pour avoir la convergence du schéma d'après le théorème de Lax (voir [L:90, Section 10.5]) et [M:94].

Si $\theta = 0{,}5$ nous n'avons pas de dissipativité numérique mais le schéma est presque instable, et les solutions ne sont pas régulières. Il faut donc prendre $1 > \theta > 0{,}5$ mais pas trop grand pour éviter d'introduire trop de dissipativité numérique.

3.3 Le programme en *C++*

Nous trouvons sur le site :

`http://www.math.u-psud.fr/~prieur`

une documentation détaillée de la structure du programme, écrit en *C++*, qui a permis de simuler les équations de Saint-Venant et les contrôleurs du chapitre 2 en utilisant le schéma de Preissmann défini au paragraphe précédent.

Disons pour résumer que nous avons créé des “objets” pour chaque entité du programme, par exemple, un objet pour les schémas de Preissmann et de Godunov, un autre pour les équations linéarisées, différents objets pour chaque type de contrôleur, etc... et que ces objets utilisent le savoir-faire des autres objets et travaillent parallèlement entre eux. C'est le principe de la Programmation Orientée Objet.

Précisons brièvement la nature et l'utilité des principaux objets :

- `CTank` concerne les caractéristiques du fluide, en particulier H et V pour les temps proportionnels au pas de discrètisation et pour les points du bac d'abscisse proportionnelle au pas spatial de discrètisation.
- `CTrajectory` contient la trajectoire du bac $D(t)$. Avec l'objet `CTank`, il peut calculer, entre autres, la valeur des fonctions de Lyapunov.
- `CSolver` résoud les équations de Saint-Venant en utilisant un des deux schémas ci-dessous (au choix de l'utilisateur).
- `CGodunovSolver` résoud les équations de Saint-Venant en utilisant le schéma de Godunov.
- `CPreissmannSolver` résoud les équations de Saint-Venant en utilisant le schéma de Preissmann.
- `CController` utilise les différentes mèthodes des objets suivants pour calculer le contrôle.
- `COpenController` donne la boucle ouverte.
- `CForwardController` contient l'expression du contrôle en utilisant la fonction R_3 obtenue par un forwarding (voir paragraphe 2.3). Il existe un objet analogue pour la forme dynamique de ce contrôleur.
- `CFlatController` implémente le contrôle en utilisant l'approche "sortie plate" décrite dans [DPR:99].

3.4 Les résultats

Pour être cohérent avec la remarque faite à la fin du paragraphe 2.1, nous n'implémentons ici que la forme statique des contrôleurs (à savoir (2.6), (2.9) et (2.10)).

3.4.1 Une stabilisation du fluide et un suivi de trajectoire

Nous choisissons comme coefficient de Preissmann $\theta = 0{,}51$ et comme pas de discrétisation $\Delta t = 0{,}2$ et $\Delta x = 0{,}5$. Nous choisissons de simuler $Nx = 25$ pas d'espace et $Nt = 901$ pas de temps.

Partons de la condition initiale suivante :

$$H(x) = 0.02x + 0.88 \quad , \qquad V(x) = \sin^2 \frac{x\pi}{L} \ , \forall x \in [0,\, L] \ ,$$
$$D = 0 \quad , \qquad \dot{D} = 0 \quad , \qquad \ddot{D} = 0 \ ,$$

et intéressons-nous à la stabilisation du fluide sachant que l'on souhaite que le centre du bac de longueur $L = 12$ m reste le plus proche possible de la position initiale.

La trajectoire de référence et l'état du fluide à l'équilibre sont donc

$$\bar{D}(t) = 0 \quad , \qquad \bar{H}(x) = 1 \quad , \qquad \bar{V}(x) = 0, \forall x \in [0,\, L], \quad \forall t \geq 0 \ .$$

Commençons par appliquer le contrôle donné par le paragraphe 2.2. Nous choisissons $\lambda_1 = 0{,}01$ et $\lambda_2 = 0{,}05$.

Nous avons à la figure 1, de haut en bas, le graphe de la position, de la vitesse et de l'accélération du bac avec un contrôle nul et avec le contrôle du paragraphe 2.2. Nous avons donc en bas l'expression des contrôles.

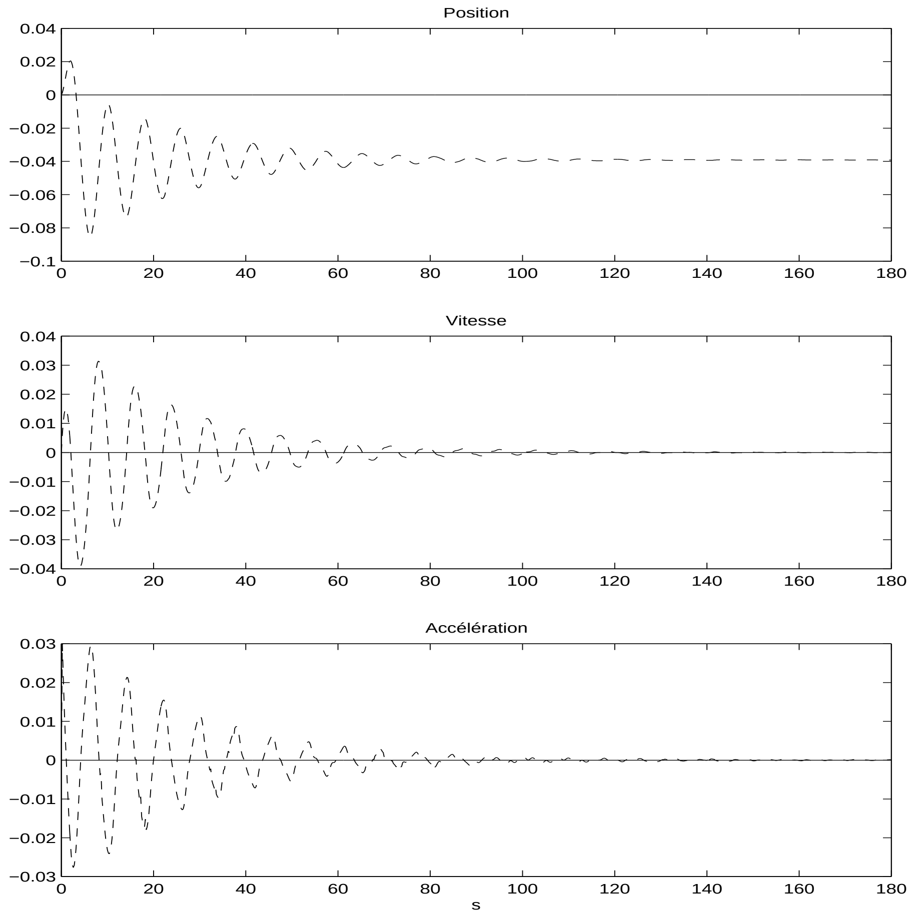

FIG. 1 – *Trajectoire de référence et boucle fermée par le contrôle du paragraphe 2.2. En haut la position, au milieu la vitesse, en bas l'accélération en fonction du temps. En trait plein, avec le contrôle nul et, en* $--$*, avec celui du paragraphe 2.2.*

Nous voyons aux figures 2 et 3 que le contrôle du paragraphe 2.2 réussit très bien à stabiliser la vitesse et la hauteur du fluide. Tandis que si nous n'appliquons pas de contrôle, le fluide continue à osciller même après 100 secondes.

Nous pouvons d'ailleurs vérifier que la fonction de Lyapunov définie au paragraphe 2.2 diminue bien à la figure 4.

Notons d'ailleurs que R_2 diminue même pour la boucle ouverte. Or d'après (2.2), nous avons

$$\dot{R}_2(t) = (\mathcal{A} - \bar{A})\,[\cdots]\ .$$

Par conséquent, si le contrôle $\mathcal{A}$ vaut $\bar{A}$, nous devrions avoir $\dot{R}_2(t) = 0$. Donc le schéma numérique dissipe l'énergie ce qui confirme les remarques du paragraphe 3.2 car dans cette simulation θ vaut 0,51 $>$ 0,5. Cet amortissement numérique est bien plus faible que la décroissance de R_2 avec la boucle fermée donc c'est bien le contrôleur qui fait diminuer R_2 et pas uniquement l'amortissement numérique.

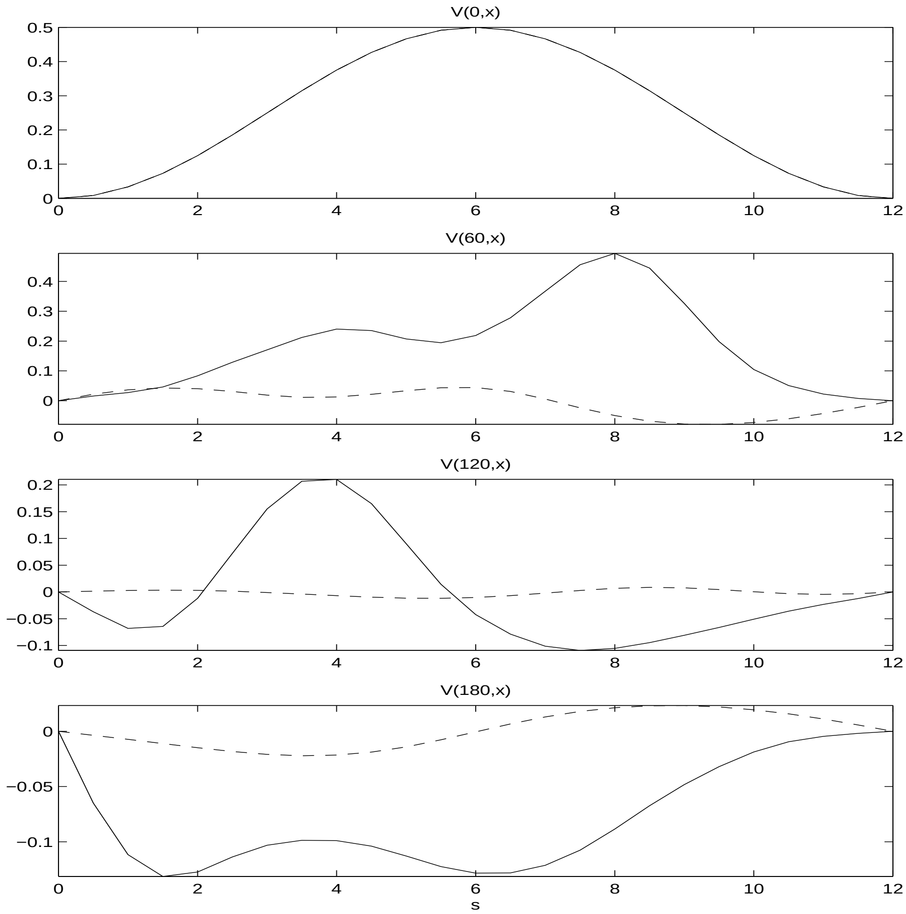

FIG. 2 – *Profil de vitesse du fluide par rapport au bac aux instants $t = 0$, $t = 60$, $t = 120$ et $t = 180$ secondes. En trait plein, avec un contrôle nul et, en $- \; -$, avec le contrôle du paragraphe 2.2. Noter que l'échelle n'est pas la même sur toutes les figures.*

En revanche, nous pouvons noter (voir figure 5) que l'erreur de la position du bac par rapport à la position de référence tend vers une constante non nulle, ce qui est prévisible car nous stabilisons uniquement la vitesse du bac et l'état du fluide, et que la position du bac à l'équilibre est indépendant des composantes de la vitesse et de la hauteur du fluide à l'équilibre. Cela justifie d'utiliser l'approche du paragraphe 2.3.

Nous calculons le contrôle du paragraphe 2.3 et nous le comparons avec les contrôles précédents. Voir figure 6. Nous choisissons $\lambda_3 = 0{,}04$.

Nous notons aux figures 7, 8 et 9 que le fluide converge toujours vers l'équilibre avec ce nouveau contrôle mais qu'en plus l'erreur de position tend vers 0. La technique du forwarding (voir [MP:96]) a donc réussit à ramener la limite de l'erreur de position du contrôleur précédent à 0 en contrôlant $\dot{D}$ pour que l'on ait $\int \dot{D} = 0$.

Nous vérifions enfin, à la figure 10, que la fonction de Lyapunov R_3 définie au paragraphe 2.3 diminue bien avec le contrôle du paragraphe 2.3.

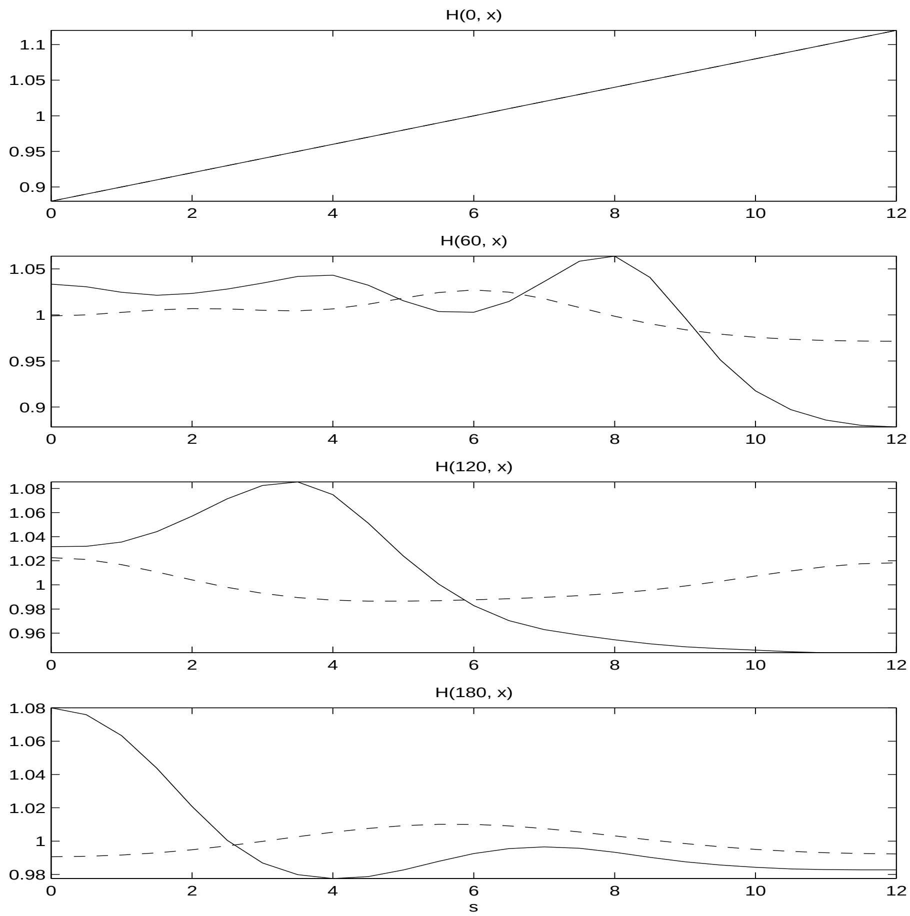

FIG. 3 – *Profil de la hauteur du fluide par rapport au bac aux instants $t = 0$, $t = 60$, $t = 120$ et $t = 180$ secondes. En trait plein, avec un contrôle nul et, en $- \, -$, avec le contrôle du paragraphe 2.2. Noter que l'échelle n'est pas la même sur toutes les figures.*

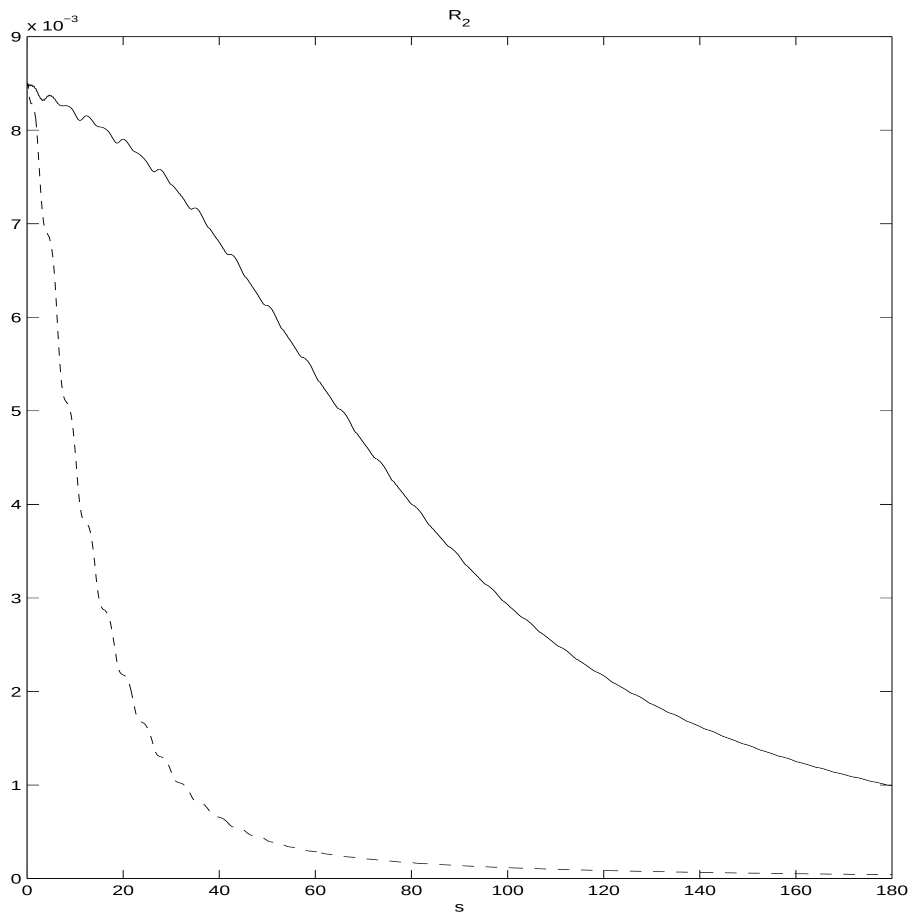

FIG. 4 – *Fonction de Lyapunov définie au paragraphe 2.2 au cours du temps. En trait plein, avec un contrôle nul et, en − −, avec le contrôle du paragraphe 2.2.*

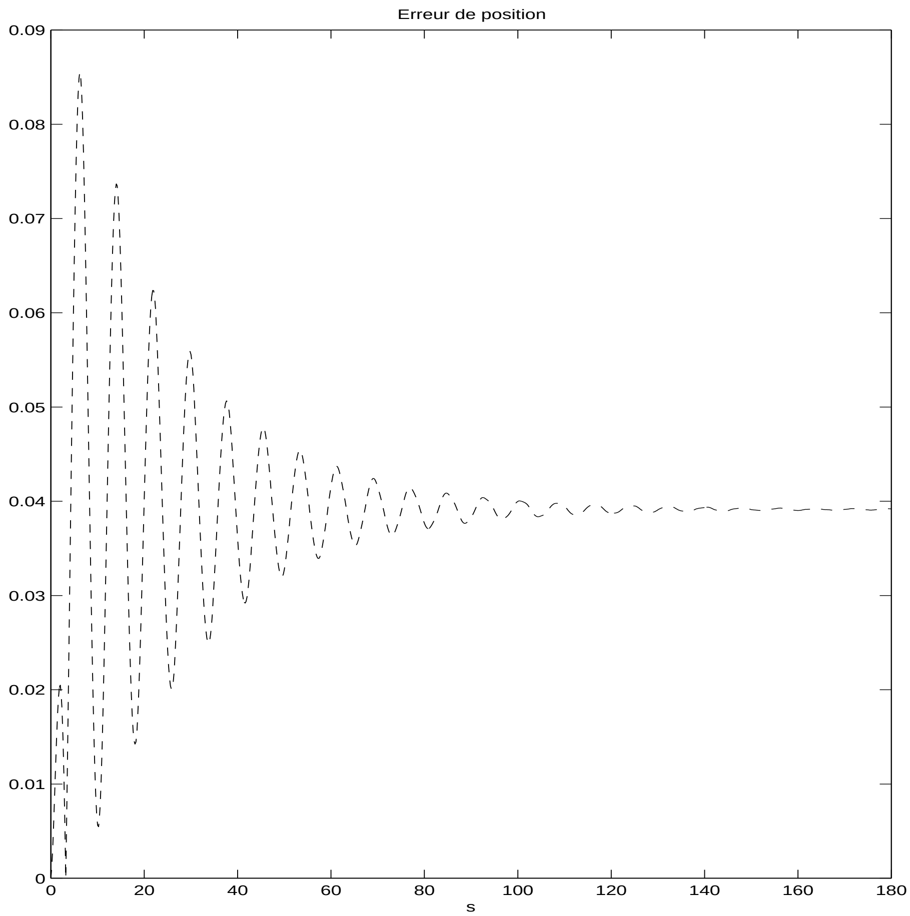

FIG. 5 – *Erreur de position du système bouclé avec le contrôle du paragraphe 2.2 par rapport à la position du bac sans contrôle au cours du temps.*

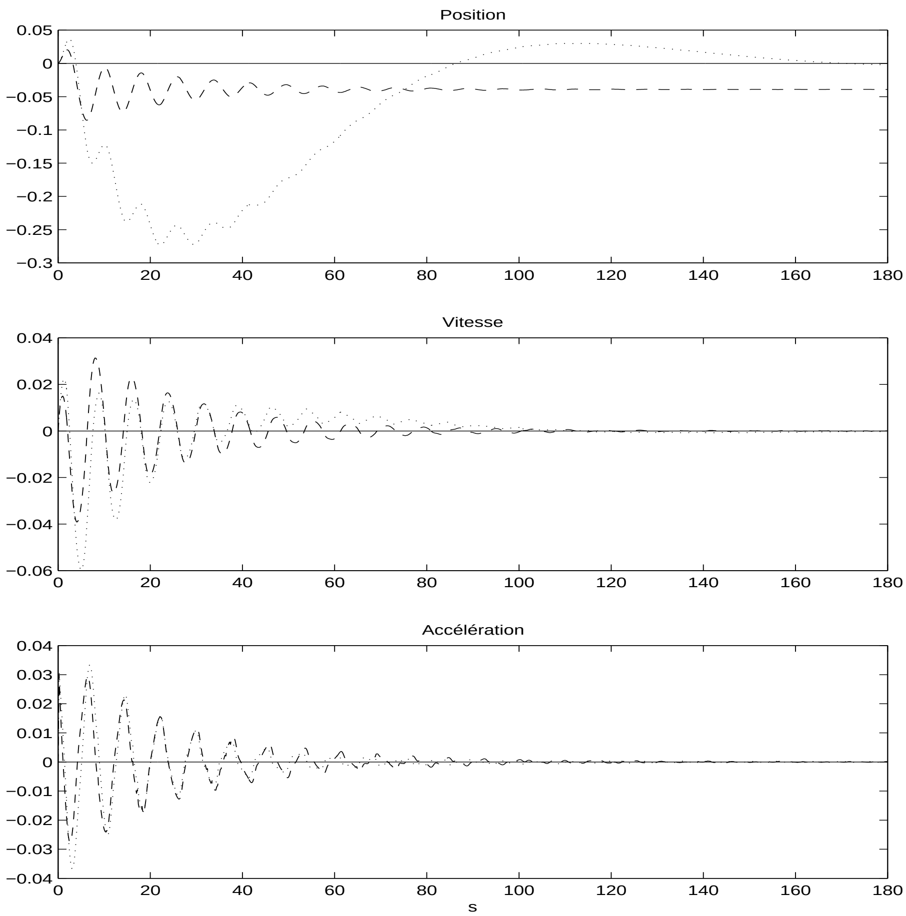

FIG. 6 – *Trajectoire de référence et boucle fermée par le contrôle du paragraphe 2.2 et par le contrôle du paragraphe 2.3. En haut la position, au milieu la vitesse, en bas l'accélération en fonction du temps. En trait plein, avec un contrôle nul, en* $--$, *avec le contrôle du paragraphe 2.2 et, en* $\cdots$, *avec le contrôle du paragraphe 2.3.*

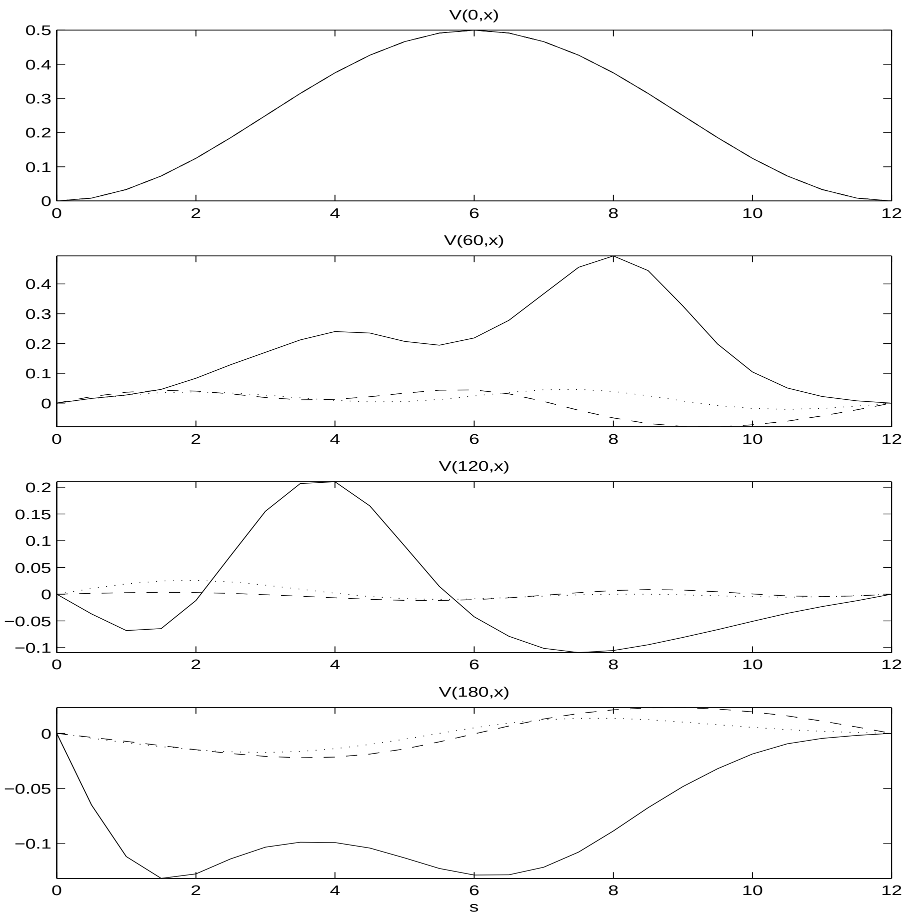

FIG. 7 – *Profil de vitesse du fluide par rapport au bac aux instants $t = 0$, $t = 60$, $t = 120$ et $t = 180$ secondes. En trait plein, avec un contrôle nul, en $--$, avec le contrôle du paragraphe 2.2 et, en $\cdots$, avec le contrôle du paragraphe 2.3. Noter que l'échelle n'est pas la même sur toutes les figures.*

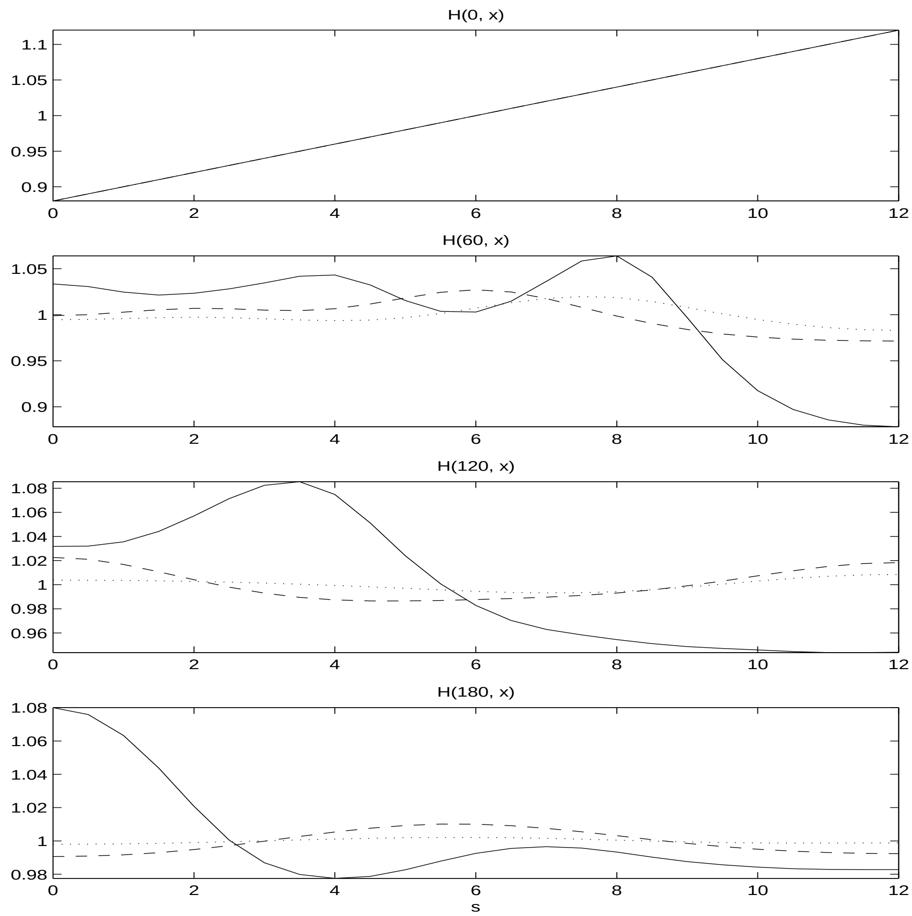

FIG. 8 – *Profil de la hauteur du fluide par rapport au bac aux instants* $t = 0$, $t = 60$, $t = 120$ *et* $t = 180$ *secondes. En trait plein, avec un contrôle nul, en* $- -$, *avec le contrôle du paragraphe 2.2 et, en* $\cdots$, *avec le contrôle du paragraphe 2.3. Noter que l'échelle n'est pas la même sur toutes les figures.*

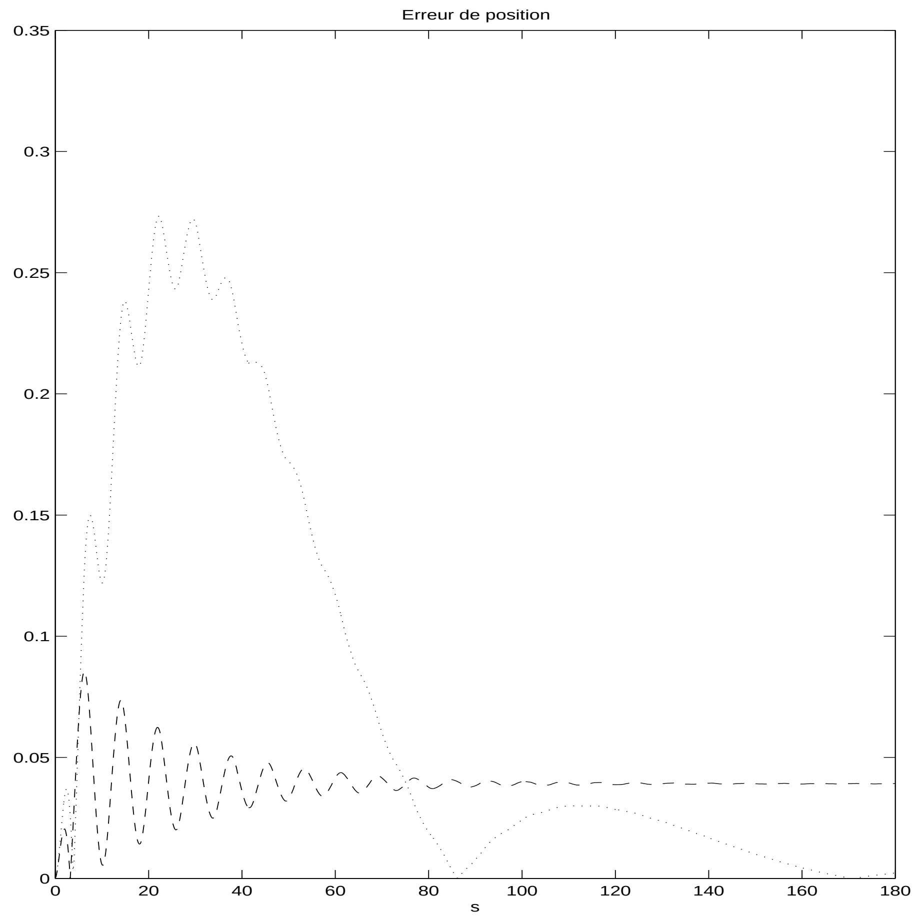

FIG. 9 – *Erreur de position du système bouclé par rapport à la boucle ouverte en fonction du temps. En $--$, la boucle fermée avec le contrôle du paragraphe 2.2 et, en $\cdots$, avec le contrôle du paragraphe 2.3.*

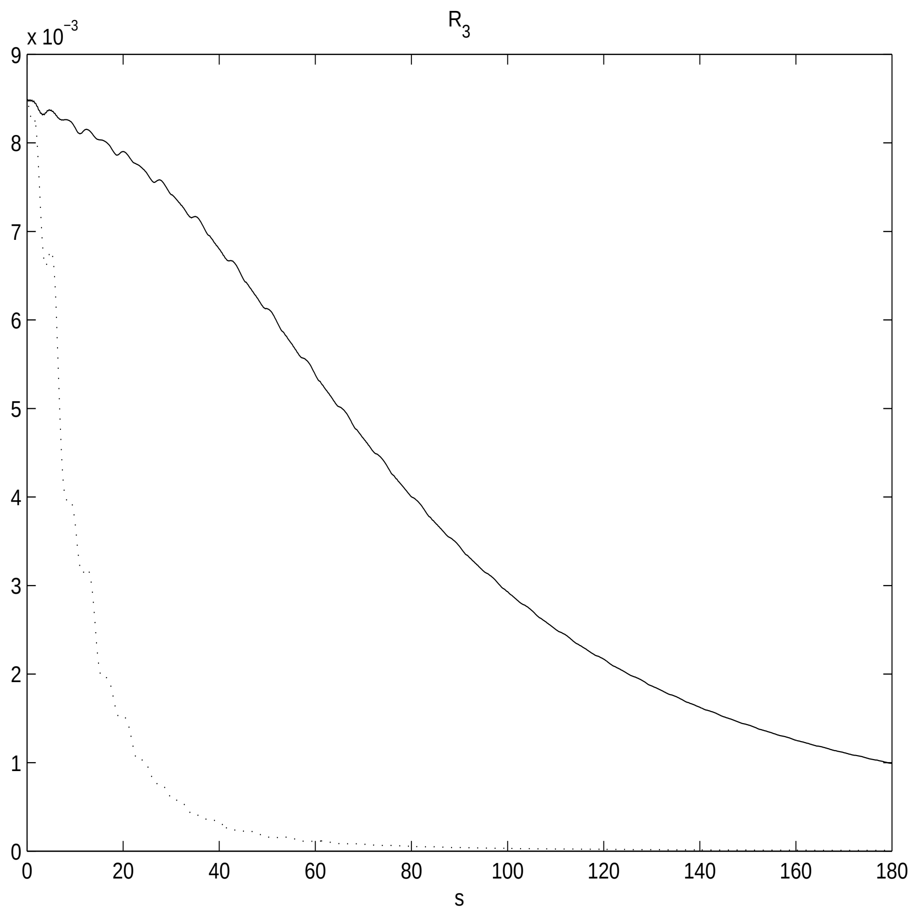

FIG. 10 – *Fonction de Lyapunov définie au paragraphe 2.3 au cours du temps. En trait plein, avec un contrôle nul, et, en* $\cdots$, *la boucle fermée avec le contrôle du paragraphe 2.3.*

3.4.2 Avec condition initiale dans la partie non-contrôlable du linéarisé autour de l'équilibre

Maintenant nous nous donnons une condition initiale dans la partie non-contrôlable du linéarisé autour de l'équilibre des équations de Saint-Venant, voir remarque 1.4.4. Nous allons vérifier que nous avons un résultat de stabilisation, mais très lente, car ce sont les termes non-linéaires qui vont faire converger la solution vers l'origine.

Pour cela nous considérons l'équilibre

$$\bar{D} = 0 \quad , \quad \bar{S} = 0 \quad , \quad \bar{A} = 0 \quad , \quad \bar{H} = 1{,}5 \quad , \quad \bar{V} = 0 \ ,$$

et notons que les fonctions H, $V\colon [0,+\infty) \times [0,L] \to \mathbb{R}$ et $D\colon [0,+\infty) \to \mathbb{R}$ définies par, pour tout $t \geq 0$ et pour tout x dans $[0,L]$,

$$D(t) = 0 \ , \quad H(t,x) = 1 + \sin^2(\frac{\pi x}{L}) \ , \quad V(t,x) = -2\frac{\pi}{gL} t \cos(\frac{\pi x}{L}) \sin(\frac{\pi x}{L}) \ ,$$

sont solutions des équations linéarisées avec le contrôle $\mathcal{A} = \bar{A} = 0$ c'est-à-dire, pour tout t dans $[0,+\infty)$ et pour tout x dans $[0,L]$,

$$\frac{\partial H}{\partial t}(t,x) = 0 \quad , \quad \frac{\partial V}{\partial t}(t,x) + g\frac{\partial H}{\partial x}(t,x) = 0$$
$$V(t,0) = 0 \quad , \quad V(t,L) = 0$$

Nous allons vérifier que, numériquement, si nous prenons comme condition initiale, la valeur de ces fonctions au temps $t = 0$, alors le contrôle du paragraphe 2.3 stabilise le système décrit par les équations non linéaires (1.1)-(1.4).

Nous nous donnons donc un bac d'une longueur $L = 7{,}5$ avec une hauteur d'eau de hauteur moyenne de 1,5 avec la condition initiale :

$$H(x) = 1 + \sin^2(\frac{\pi \times x}{L}) \quad , \quad V(x) = 0, \, \forall x \in [0,L] \ .$$

Comparons le contrôle nul et celui donné par le paragraphe 2.3. Choisissons $\lambda_1 = 0{,}4$, $\lambda_2 = 0{,}1$ et $\lambda_3 = 0{,}1$, comme coefficient de diffusion $\theta = 0{,}5001$, comme pas de discrétisation $\Delta x = 0{,}25$ et $\Delta t = 0{,}1$ et simulons les équations sur $Nx = 31$ pas d'espace et $Nt = 501$ pas de temps.

Avec ce choix de θ, nous aurons donc très peu de diffusion pour éviter que le système ne converge de lui-même numériquement. En revanche nous avons des solutions qui ne sont pas très régulières (voir en particulier la figure 13), comme nous l'avions prévu au paragraphe 3.2.

Nous prenons comme trajectoire de référence la position immobile du bac.

Les trajectoires avec un contrôle nul et avec le contrôle du paragraphe 2.3 sont données par la figure 11. Noter que le bac en boucle fermée dérive très légèrement mais reste proche de la position immobile.

Les figures 12 et 13 représentent l'évolution à différents instants de la hauteur et de la vitesse du fluide par rapport au bac.

Nous remarquons que l'état du fluide converge effectivement vers l'équilibre mais très lentement. Nous vérifions à la figure 14 que la fonction de Lyapunov définie au paragraphe 2.3 décroît effectivement au cours du temps mais beaucoup plus lentement qu'à la simulation précédente.

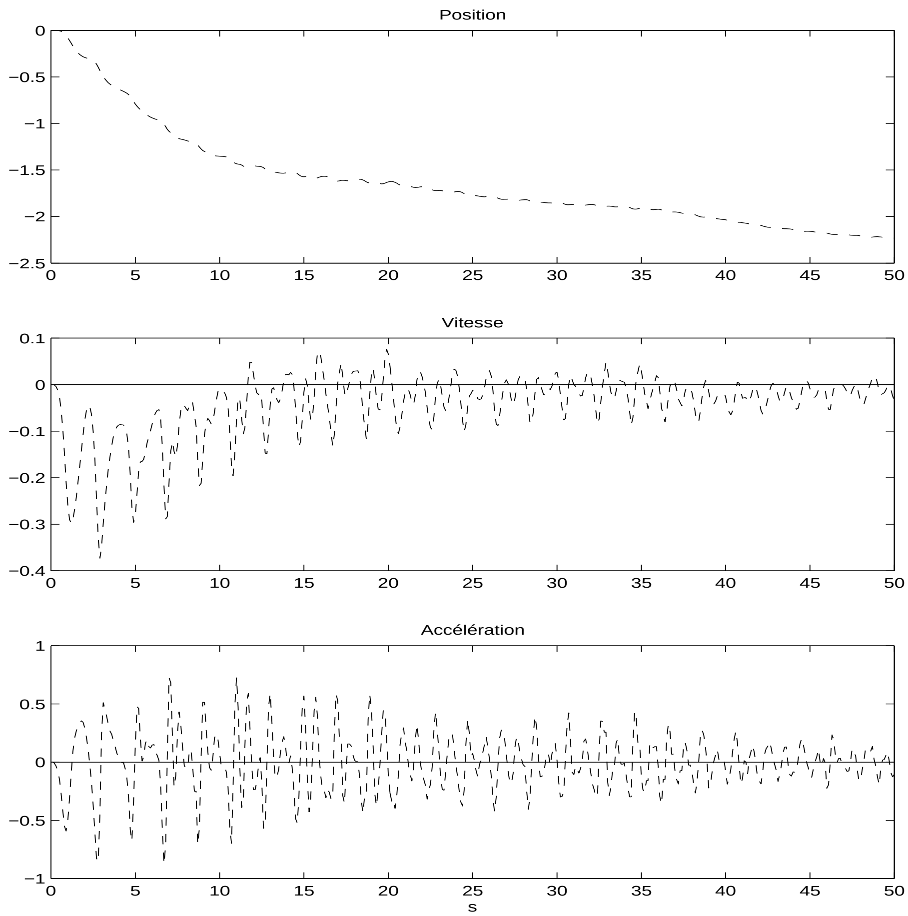

FIG. 11 – *Trajectoire de référence et boucle fermée par le contrôle du paragraphe 2.3 en fonction du temps. En haut la position, au milieu la vitesse, en bas la position. En trait plein, avec le contrôle nul et, en* $--$, *avec le contrôle du paragraphe 2.3.*

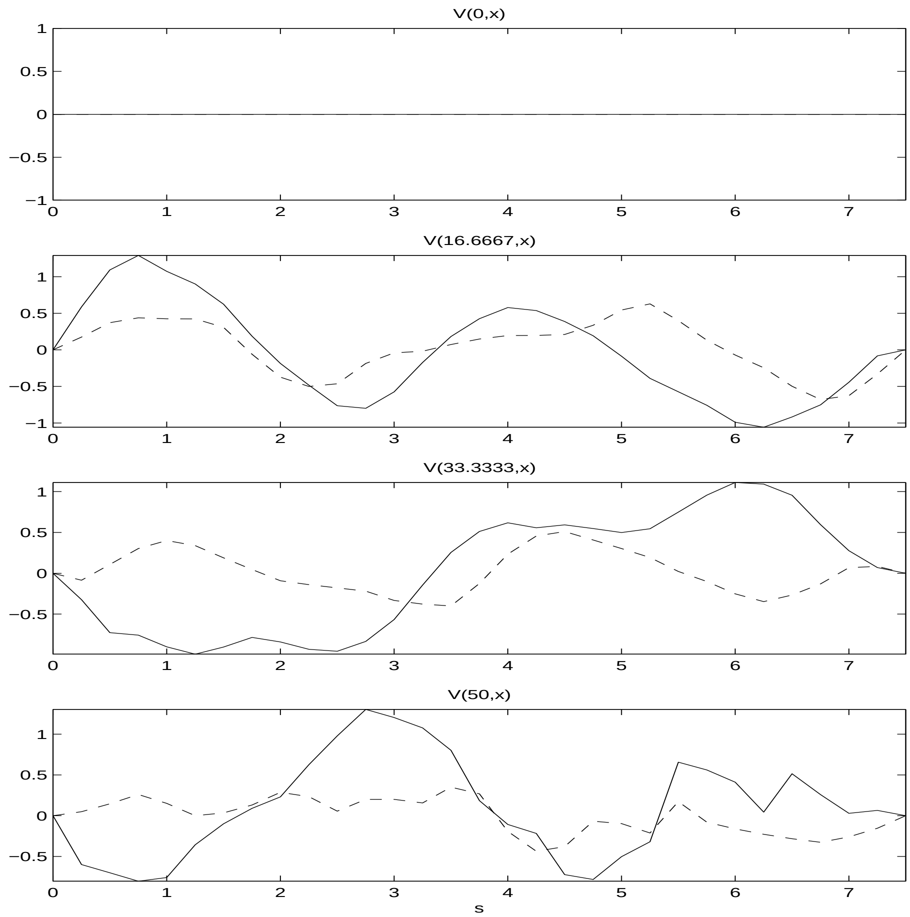

FIG. 12 – *Profil de vitesse du fluide par rapport au bac aux instants $t = 0$, $t = 16.6$, $t = 33.3$ et $t = 50$ secondes. En trait plein, le fluide avec un contrôle nul et, en $--$, avec le contrôle du paragraphe 2.3. Noter que l'échelle n'est pas la même sur toutes les figures.*

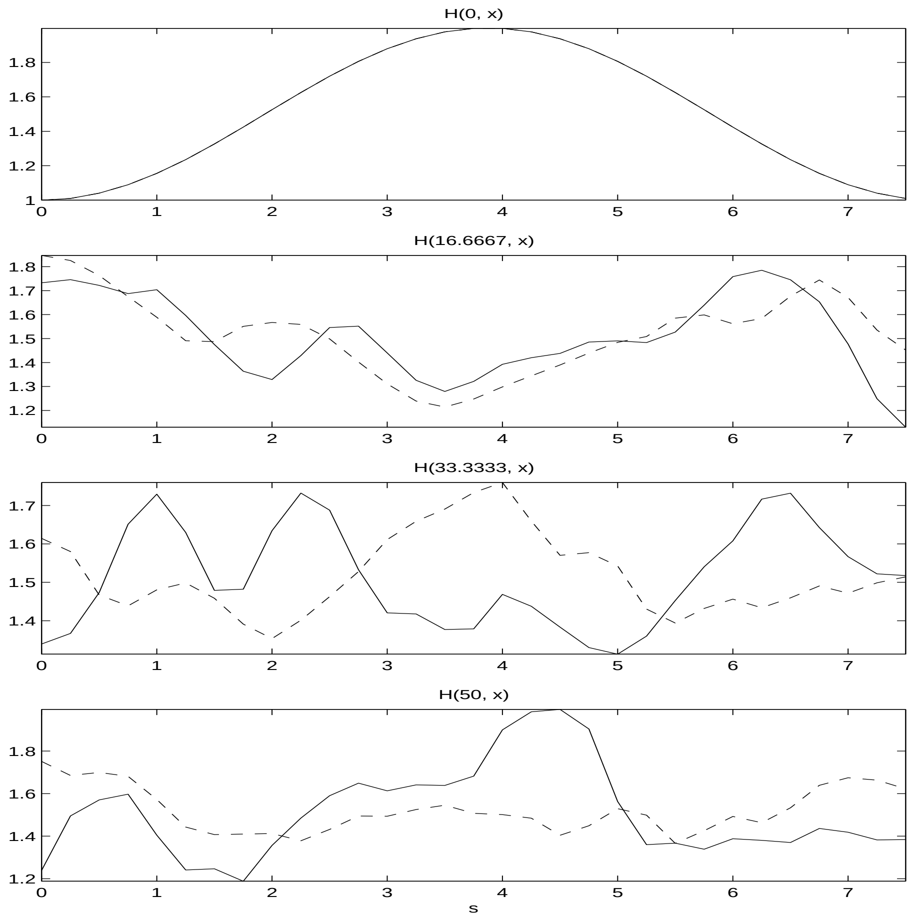

FIG. 13 – *Profil de la hauteur du fluide par rapport au bac aux instants* $t = 0$, $t = 16.6$, $t = 33.3$ *et* $t = 50$ *secondes. En trait plein, le fluide avec un contrôle nul et, en* $--$, *avec le contrôle du paragraphe 2.3. Noter que l'échelle n'est pas la même sur toutes les figures.*

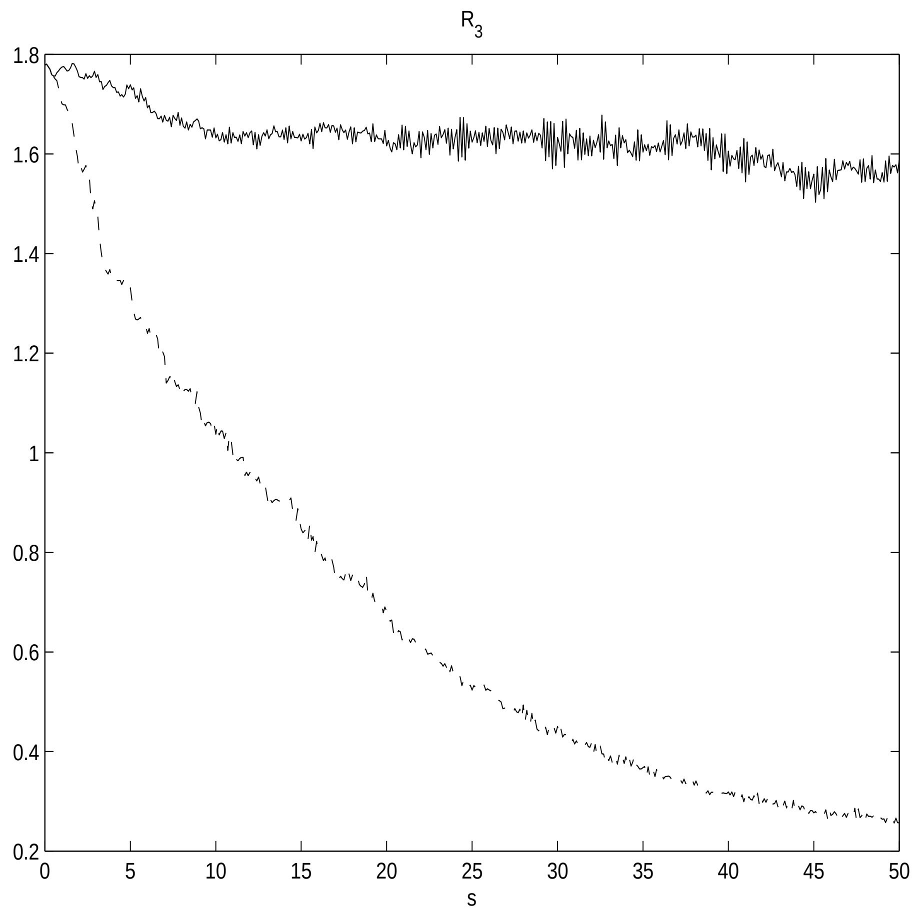

FIG. 14 – *Fonction de Lyapunov définie au paragraphe 2.3 au cours du temps. En trait plein, le fluide avec un contrôle nul et, en $--$, avec le contrôle du paragraphe 2.3.*

Appendice

Dans cet appendice nous donnons les démonstrations précises et les théorèmes des résultats annoncés dans le chapitre précédent.

Nous trouvons donc, à partir de la page 165, les calculs des contrôleurs du chapitre 2. Une version préliminaire de ce papier a été présentée en conférence [PH:01].

Stabilization of a tank

C. Prieur[1] and J. de Halleux[2]

Abstract: We consider a tank containing a fluid. The tank is subjected to a one-dimensional horizontal move and the motion of the fluid is described by the shallow water equations. By means of a Lyapunov approach, we deduce control laws to stabilize the fluid's state and the tank's speed. Although global asymptotic stability is yet to be proved, we numerically simulate the system and observe the stabilization for different control situations.
KeyWords: Quasilinear hyperbolic PDEs, shallow water equations, Lyapunov approach, boundary control, numerical resolution.

1 Introduction

We consider an 1-D tank containing an inviscid incompressible irrotational fluid. We are interested in the stabilization problem of the fluid state (height and speed relative to the tank) and the tracking problem of the trajectory of the tank (position, speed and acceleration) to a prescribed trajectory (e.g. a prescribed final position of the tank).

We suppose that the horizontal acceleration is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. Hence we describe the dynamic of the fluid by the shallow water equations (see [D:94, Section 4.2], see also [PR:02] and [S:71]).

The acceleration (or similarly an external force created by a mechanism applied to the tank, see Remark 2.1 below) defines the control variable. We exhibit a stabilizing feedback based on a Lyapunov approach (see Theorem 3). We emphasize that we proceed by increasing the complexity of the Lyapunov function. First we stabilize only the fluid's state (Section 3.1), then we stabilize also the tank's speed (Section 3.2) and then, we use a forward approach (see [MP:96]) to stabilize the entire state of the system fluid-tank in Theorem 3.

The control is a feedback of the entire state of the fluid and the tank. But many industrial motivations can be found in [G:00, M:00b] for looking such a feedback of the height of the fluid at the boundary of the tank, the time and the trajectory of the tank only. We answer partially to this question by expliciting a control whose dynamic law depends only on the measured variable but for which the computation of the initial value needs the knowledge of the complete state of the system fluid-tank. See Remark 3.2 below for more details.

Some results can be found in [M:00a] concerning the problem of the stabilization of a tank, but the input is defined as a flexible or a rigid wave generators and the equations are linearized around the equilibrium. Here we choose a different model of the control system.

The asymptotic stability is yet to be proved but we check numerically that the result is attained (see Section 4.1).

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Note that the shallow water equations, linearized around a suitable equilibrium, are uncontrollable (see [DPR:99]) but the non-linear shallow water equations are locally controllable around the equilibriums (see [C:01]). We check numerically that the stabilization property is achieved with the non-linear terms of the shallow water equations in Section 4.2.

2 Model description

The shallow water equations describe the motion of a perfect fluid under gravity g with a free boundary (the shallow water assumption) (see [S:71, D:94, PR:02]):

$$\frac{\partial H}{\partial t}(t,x) + \frac{\partial}{\partial x}(HV)(t,x) = 0 \ , \tag{1}$$

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial}{\partial x}(gH + \frac{V^2}{2})(t,x) = -\ddot{D}(t) \ , \tag{2}$$

where $x \in [0,L]$ is the spatial coordinate attached to the tank, $t \in [0,T]$ is the time coordinate, $T > 0$, $H(t,x)$ denotes the height of the liquid, $V(t,x)$ denotes the horizontal speed of the fluid in the referential attached to the tank, D is the position of the tank in the world coordinates, $\dot{D}$ and $\ddot{D}$ are respectively the first and second derivative of D with respect to the time t. See Figure 1.

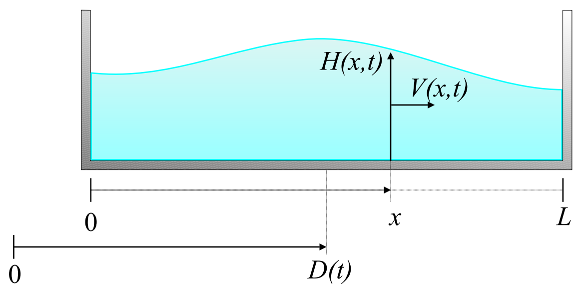

FIG. 1 – *A tank of length L containing a fluid.*

The boundary conditions are given by, for all t in $[0,L]$,

$$V(t,0) = 0 \quad , \qquad V(t,L) = 0 \ . \tag{3}$$

Note that the shallow water equations (1)-(2) can been derivated from the Euler equation for the perfect irrotational and incompressible fluids (see [GP:00] for more details).

Since we will consider small variations around the equilibrium, we introduce the following values, for all x in $[0,L]$ and for all t in $[0,T]$,

$$h(t,x) = H(t,x) - \bar{H}(x) \ , \tag{4}$$

$$v(t,x) = V(t,x) - \bar{V}(x) \ , \tag{5}$$

where $\bar{H}(x)$ and $\bar{V}(x)$ are the steady state values of (H,V) along the reach, i.e.:

$$\frac{\partial}{\partial x}(\bar{H}\bar{V}) = 0 \ , \tag{6}$$

$$\frac{\partial}{\partial x}(g\bar{H} + \frac{\bar{V}^2}{2}) = -\bar{A} \ , \tag{7}$$

where $\bar{A}$ is a constant number defining the constant acceleration of the tank. The above equations (6) and (7) can be rewritten as follows

$$\forall x \in [0,L], \quad \begin{array}{rcl} \bar{V}(x) & = & 0 \,, \\ \bar{H}(x) & = & \bar{H}(\frac{L}{2}) - (x - \frac{L}{2})\frac{\bar{A}}{g} \,. \end{array} \tag{8}$$

In fact, we can compute the (constant) volume of liquid in the tank:

$$\text{vol} \;=\; \int_0^L \bar{H} dx \;=\; L\bar{H}(\frac{L}{2}) \,. \tag{9}$$

We can rewrite (1) and (2) using (4)-(7):

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv + \bar{H}v) \;=\; 0 \,, \tag{10}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}(gh + \frac{v^2}{2}) \;=\; -u \,, \tag{11}$$

where we have defined our control variable u,

$$u = \ddot{D} - \bar{A}. \tag{12}$$

In the following we denote the measured height of the fluid at the boundary by H_0 and H_L, i.e., for all t in $[0,L]$,

$$H_0(t) = H(t,0) \quad , \qquad H_L(t) = H(t,L) \,.$$

Remark 2.1 In (12), we have chosen the control variable to be equal to $\ddot{D} - \bar{A}$ as in [C:01, DPR:99] but in many physical models the input is a force applied to the tank. But we can prove (see (1.13), Chapter 1) that the interaction between the fluid and the tank, as the tank is subject to a force $F(t)$ at time t, is described by the equation:

$$M\ddot{D}(t) = F(t) + \frac{\rho g}{2}(H_L^2(t) - H_0^2(t)) \,,$$

where M is the mass of the tank and ρ is the density of the fluid. Therefore the force and the control $u = \ddot{D} - \bar{A}$ are equal beside adding a function which depends only of the height of the fluid at the boundary.

Therefore the problem of the stabilization with the control variable $\ddot{D} - \bar{A}$ is equivalent to the problem of the stabilization with the control of F. This is a good reason to choose u defined by (12) as in [C:01, DPR:99]. ◇

Let $|.|$ be the usual norm of $\mathbb{R}$ and $|.|_1$ be the norm on $\mathcal{C}^1([0,L])$ defined by, for all f in $\mathcal{C}^1([0,L])$,

$$|f|_1 = \max_{x\in[0,L]} |f(x)| + \max_{x\in[0,L]} |f'(x)| \,,$$

where $'$ denotes the partial derivative with respect to x.

Given an initial condition $(\widetilde{H},\widetilde{V})$ for the fluid and an initial acceleration of the tank $\widetilde{A}$, note that there exist sufficient conditions for the existence of a solution of the Cauchy problem (1), (2) and (3) (see [LY:85, Theorem 4.2, page 96]):

Claim 2.2 *There exists a strictly positive constant ε such that, for any $(\widetilde{H},\widetilde{V})$ in $C^1([0,L])^2$ satisfying the compatibility conditions:*

$$2g\widetilde{H}(0)\widetilde{H}'(0)+\widetilde{V}(0)\widetilde{V}'(0)=-\widetilde{A}\ , \tag{13}$$

$$2g\widetilde{H}(L)\widetilde{H}'(L)+\widetilde{V}(L)\widetilde{V}'(L)=-\widetilde{A}\ , \tag{14}$$

and

$$|\widetilde{H}-\bar{H}|_1+|\widetilde{V}-\bar{V}|_1<\varepsilon\ , \tag{15}$$

the hyperbolic system (1) and (2) with initial conditions:

$$H(0,x)=\widetilde{H}(x)\ ,\quad V(0,x)=\widetilde{V}(x)\quad ,\forall x\in[0,\ L]\ ,$$

and with boundary conditions (3) has one and only one solution of class C^1 defined on $[0,L]\times[0,T)$, for some $T>0$.

Now let us define $\mathcal{E}=\{(H,\,V,\,D,\,S)\}$ the affine subspace of $\mathcal{C}^1([0,L])\times\mathcal{C}^1([0,L])\times\mathbb{R}\times\mathbb{R}$ such that we have

$$\mathrm{vol}=\int_0^L\widetilde{H}(x)dx\ ,$$

where vol is defined by (9), and

$$H_x(0)\ =\ H_x(L)\ =\ -\frac{u+\bar{A}}{g}\ .$$

We are interested in the problem of the local stabilization to the equilibrium $(\bar{H},\,\bar{V},\,\bar{D},\,\bar{S})$ with $\bar{A}$ in $\mathbb{R}$ fixed, satisfying (8) by the control u, *i.e.* we are looking for a function $u:[0,+\infty)\times\mathcal{E}$ such that

- There exists $C>0$ such that, for all $(\widetilde{H},\,\widetilde{V},\,\widetilde{D},\,\widetilde{S})$ in $\mathcal{E}$ satisfying the conditions (13)-(15) and

  $$|\widetilde{H}-\bar{H}|_1+|\widetilde{V}-\bar{V}|_1+|\widetilde{D}-\bar{D}|+|\widetilde{S}-\bar{S}|\leq C\ ,$$

  there exists one and only one $(H,\,V,\,D,\,S)$: $[0,+\infty)\to\mathcal{E}$ such that, we have (1)-(3) where, for all $t\geq 0$,

  $$\ddot{D}(t)-\bar{A}=u(t,H(t,.),V(t,.),D(t),S(t))\ , \tag{16}$$

  such that we have

  $$H(0,.)=\widetilde{H}\ ,V(0,.)=\widetilde{V}\ ,D(0)=\widetilde{D}\ ,S(0)=\widetilde{S}\ , \tag{17}$$

  and, for all $t\geq 0$,

  $$\dot{D}(t)=S(t)\ . \tag{18}$$

  Moreover this function satisfies

  $$|H(t,.)-\bar{H}|_1+|V(t,.)-\bar{V}|_1+|D(t)-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2| \\ +|\dot{D}(t)-\bar{S}-\bar{A}t|\to_{t\to+\infty}0\ .$$

– For all $\varepsilon > 0$, there exists $\eta > 0$ such that, if $(\widetilde{H}, \widetilde{V}, \widetilde{D}, \widetilde{S})$ in $\mathcal{E}$ satisfies the conditions (13)-(15) and

$$|\widetilde{H} - \bar{H}|_1 + |\widetilde{V} - \bar{V}|_1 + |\widetilde{D} - \bar{D}| + |\widetilde{S} - \bar{S}| \leq \eta \ ,$$

if (H, V, D, S): $[0, +\infty) \to \mathcal{E}$ is such that, (1)-(3), (16)-(18) hold, then we have

$$|H(t,.) - \bar{H}|_1 + |V(t,.) - \bar{V}|_1 + |D(t) - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2| \\ +|\dot{D}(t) - \bar{S} - \bar{A}t| \leq \varepsilon \ , \forall t \geq 0$$

In all the following we are interested in this problem and we propose a Lyapunov control design. Then we check that, numerically, the stabilization is attained.

3 Lyapunov control design

We want to build a Lyapunov candidate to stabilize both the fluid's state (h,v) and the tank's speed $\dot{D}$. The input is defined by (12). The idea of this section is to build a Lyapunov function based on an entropy of the fluid as in [CAB:99]. The Lyapunov functions are a general tool to prove, for a differential equation, that the origin is an asymptotic stable equilibrium. Note that usually this Lyapunov function does not have any physical sense for the control system. This method has been already developed in [CAB:99] to find the expression of the control law.

3.1 Stabilization of the fluid's state (h,v)

Let us consider first the stabilization of the fluid's state. We want to find an entropy $E(\bar{H},h,v)$ and an entropic flux $F(\bar{H},h,v)$. There is an infinite number of entropies for the shallow water equations (see [S:96, Volume II, Section 9.3]), one of them is derived from the moments of the fluid:

$$E(\bar{H},h,v) \ = \ (\bar{H} + h)\frac{v^2}{2} + g\frac{h^2}{2} \ , \tag{19}$$

$$F(\bar{H},h,v) \ = \ (\bar{H} + h)\frac{v^3}{2} + gv(\bar{H} + h)h \ . \tag{20}$$

We can define the Lyapunov candidate $R_1 : [0, +\infty) \to \mathbb{R}$

$$R_1(t) = \lambda_1 \int_0^L E(\bar{H}(x),h(t,x),v(t,x))dx \ , \tag{21}$$

for a constant $\lambda_1 > 0$ introduced for the tuning of the control. Note that R_1 is positive and is zero only at the point $(H,V) = (\bar{H},\bar{V})$. We can now exhibit a class of control laws for u, making R_1 decrease, as stated in the following

Theorem 1 *For any positive gain λ_1, the control law*

$$u_1(t) = \lambda_1 \int_0^L (HV)(t,x)dx \tag{22}$$

makes R_1 decrease, i.e. $\dot{R}_1 \leq 0$. Moreover $\dot{R}_1 = 0$ if $(h,v) = (0,0)$.

Remark 3.1 We can not apply LaSalle's Theorem since we do not know, if the fact that the equality $\dot{R}_1(t) = 0$ holds for all t, yields $(h,v) = (0,0)$. Note moreover that in an infinite dimensional space of functions, we have to prove a suitable compactness property. $\diamond$

Remark 3.2 Note that the control law (22) is given by a static form which depends on the entire state to the system fluid-tank. But we can find an equivalent expression of this feedback with a dynamic law which depends only on the height of the fluid at the boundary of the tank and therefore, except for the initial condition, the entire state of the system is not needed to implement the control law (23)-(24). In fact we only need depth measurements at $x = 0$ and $x = L$ and no speed measurements. More precisely, we have

$$\forall t \geq 0, \dot{u_1}(t) = \lambda_1 \left(\frac{g}{2}(H_0^2(t) - H_L^2(t)) - \mathrm{vol}(\bar{A} + u_1(t))\right) , \tag{23}$$

$$u_1(0) = \lambda_1 \int_0^L (HV)(0,x)dx , \tag{24}$$

We implement the static form of this control law u_1, and also of u_2 and u_3 defined respectively by Theorems 2 and 3 below, rather than the dynamic form. Indeed we want to check that the control law succeed in solving the stabilization problem for different control situations (which is yet to be proved, see Remark 3.1). Our goal, in this paper, is not to investigate the dynamic controller approach. See also the conclusion section. $\diamond$

Proof of Theorem 1 and Remark 3.2 We derive (21) with respect to t:

$$\dot{R}_1 = \lambda_1 \int_0^L \left(\frac{\partial E}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial E}{\partial h}\frac{\partial h}{\partial t}\right) dx .$$

Hence, using (1), (2) and (19), we have

$$\dot{R}_1 = -u_1\lambda_1 \left(\int_0^L Hv dx - [F]_0^L\right) . \tag{25}$$

Using the boundary conditions (3) and (20), we have $[F]_0^L = 0$, hence a natural expression for u_1 is

$$u_1(t) = \lambda_1 \int_0^L Hv dx . \tag{26}$$

In fact we see that $\dot{R}_1$ becomes

$$\dot{R}_1 = -\left(\lambda_1 \int_0^L Hv dx\right)^2 ,$$

and therefore we conclude that $\dot{R}_1 \leq 0$. If (h,v) is null, then trivially we have $\dot{R} = 0$.

Let us prove now Remark 3.2. To do this using (1) and (11), we note that

$$\frac{\partial Hv}{\partial t} + \frac{\partial}{\partial x}(g\frac{H^2}{2} + Hv^2) = -H(u + \bar{A}) . \tag{27}$$

We can now use the expression above to estimate u_1 by taking the first time derivate of (26),

$$\begin{aligned}\dot{u}_1 &= \lambda_1 \int_0^L \frac{\partial}{\partial t}(Hv)dx \ , \\ &= -\lambda_1 \int_0^L \frac{\partial}{\partial x}(g\frac{H^2}{2} + Hv^2) + H(u_1 + \bar{A})dx \ .\end{aligned}$$

Since u_1 and $\bar{A}$ are constant along x, we have

$$\dot{u}_1 = -\lambda_1 \left[g\frac{H^2}{2} + Hv^2 \right]_0^L - \lambda_1(u_1 + \bar{A}) \int_0^L H dx.$$

Using (9) and the boundary conditions (3), the above equation leads to (22) which is an ordinary differential equation in u_1 whose initial condition is defined by (26) and whose solution is (23).

This concludes the proof of Theorem 1 and Remark 3.2.□

3.2 Stabilization of the fluid's state (h,v) and of the tank's speed $\dot{D}$

In this section we want to stabilize also the tank's speed $\dot{D}$ around $\bar{S} + \bar{A}t$. In order to achieve this, we introduce a modified "kinetic energy" of the tank in (21),

$$R_2(t) = R_1(t) + \lambda_2 \frac{(\dot{D}(t) - \bar{S} - \bar{A}t)^2}{2} \ , \tag{28}$$

where R_1 is defined by (21) and λ_2 is a positive constant introduced for the tuning of the controller.

Note that R_2 is positive and is zero only at the point $(H,V,\dot{D}) = (\bar{H},\bar{V},\bar{S} + \bar{A}t)$.

Using the same approach as before, we can now propose a class of control laws for u, making R_2 decrease, as stated in the following

Theorem 2 *For any positive gains λ_1, λ_2, the control law*

$$u_2(t) = \lambda_1 \int_0^L (HV)(t,x)dx - \lambda_2(\dot{D}(t) - \bar{S} - \bar{A}t) \ , \tag{29}$$

makes R_2 decrease, i.e. $\dot{R}_2 \leq 0$. Moreover $\dot{R}_2 = 0$ if $(h,v,\dot{D}) = (0,0,\bar{S} + \bar{A}t)$.

Remark 3.3 As for Remark 3.2, we can find a (partially) dynamic form of the control law u_2. More precisely we have:

$$\begin{aligned}\forall t \geq 0, \dot{u}_2(t) &= -(\lambda_1 \mathrm{vol} + \lambda_2)u_2(t) + \lambda_1(\frac{g}{2}(H_0^2(t) - H_L^2(t)) - \bar{A}\mathrm{vol}), & (30) \\ u_2(0) &= \lambda_1 \int_0^L (HV)(0,x)dx - \lambda_2\dot{D}(0) + \lambda_2\bar{S} \ , & (31)\end{aligned}$$

◇

Proof of Theorem 2 and Remark 3.3 We compute the first derivative of (28) with respect to t:

$$\dot{R}_2 = \dot{R}_1 + \lambda_2(\dot{D} - \bar{S} - \bar{A}t)(\ddot{D} - \bar{A}) \ .$$

Hence, using (25) and (12), we have

$$\dot{R}_2 = -u_2\left(\lambda_1 \int_0^L Hv dx - \lambda_2(\dot{D} - \bar{S} - \bar{A}t)\right) . \tag{32}$$

Thus a natural expression for u_2 is

$$u_2 = \lambda_1 \int_0^L Hv dx - \lambda_2 \int_0^t u_2(\tau)d\tau - \lambda_2 \dot{D}(0) + \lambda_2 \bar{S} . \tag{33}$$

In fact, we see that $\dot{R}_2$ becomes

$$\dot{R}_2 = -\left(\lambda_1 \int_0^L Hv dx - \lambda_2 \int_0^t u_2(\tau)d\tau - \lambda_2 \dot{D}(0) + \lambda_2 \bar{S}\right)^2 ,$$

and therefore we conclude that $\dot{R}_2 \leq 0$.

As in the proof of Remark 3.2, we can rewrite (33) as (30)-(31) which is an ordinary differential equation of order one in u_2.

This ends the proof of Theorem 2 and Remark 3.3.□

3.3 Complete stabilization

In Section 3.2 we propose a candidate control law to stabilize the state of the fluid and the speed of the tank. In this section we want to stabilize the entire function D and not only its first derivative. Note that we may want to stabilize the tank to a fixed point $\bar{D}$ or to track the tank to a trajectory $\bar{D} + \bar{S}t + \frac{1}{2}\bar{A}t^2$.

To stabilize also the tank's position D to a prescribed trajectory $\bar{D} + \bar{S}t + \frac{1}{2}\bar{S}t^2$ we need to "stabilize" $\int \dot{D}dt$ around to $\int(\bar{S} + \bar{A}t)dt$.

To do this we can use a forward approach to find a modification of the Lyapunov function R_2 defined by (28). See e.g. [MP:96]. Thus we have to find a function whose time-derivative is proportional to $\dot{R}_2$. This leads to

$$R_3(t) = R_2(t) + \frac{\lambda_3}{2}\left(-\lambda_2(D(t) - \bar{D} - \bar{S}t - \frac{1}{2}\bar{A}t^2) - \lambda_1 \int_0^L (\int_0^x h(t,\xi)d\xi)dx\right)^2$$

where λ_1, λ_2 and λ_3 are three positive constants introduced for the tuning of the controller and R_2 is defined by (28). Note that R_3 is positive and is zero only at the point $(H,V,\dot{D},D) = (\bar{H},0,\dot{\bar{D}},\bar{D})$. We have the following

Theorem 3 *For any positive gains λ_1, λ_2 and λ_3 the control law u_3*

$$\begin{aligned} u_3 \;&=\; \lambda_1 \int_0^L HV - \lambda_2(\dot{D} - \bar{S} - \bar{A}t) - \lambda_2\lambda_3(D(t) - \bar{D} - \bar{S}t - \frac{1}{2}\bar{A}t^2) \\ &\quad -\lambda_1\lambda_3 \int_0^L \left(\int_0^x (h(t,\xi))d\xi\right) dx , \end{aligned} \tag{34}$$

makes R_3 decrease, i.e. $\dot{R}_3 \leq 0$. Moreover $\dot{R}_3 = 0$ if $(h,v,D,\dot{D}) = (0,0,\bar{D},\bar{S})$.

Remark 3.4 As for Remark 3.2, we can find a (partially) dynamic form of the control law u_2. More precisely u_3 is the solution of the differential equation (38) below of order 2 with initial conditions:

$$\begin{aligned} u_3(0) &= -\lambda_2\lambda_3(D(0)-\bar{D}(0)) - \lambda_1\lambda_3 \textstyle\int_0^L \int_0^x h - \lambda_2(\dot{D}(0)-\bar{S}) + \lambda_1 \int_0^L Hv \ , \\ \dot{u}_3(0) &= u_3(0)(-\lambda_2 - \lambda_1 \mathrm{vol}) + \lambda_1\lambda_3 \textstyle\int_0^L Hv - \frac{g}{2}\lambda_1(H_L^2 - H_0^2) \\ &\quad -\lambda_1 \bar{A}\mathrm{vol} - \lambda_2\lambda_3(\dot{D}(0)-\bar{S}) \ . \end{aligned}$$

◇

Proof of Theorem 3 and Remark 3.4 Note that due to (10) we have

$$\begin{aligned} \frac{d}{dt}\left(\int_0^L (\int_0^x h) dx\right) &= -\int_0^L \left(\int_0^x \frac{\partial(HV)}{\partial x}\right) dx \\ &= -\int_0^L HV \ , \end{aligned} \tag{35}$$

therefore the time-derivative of R_3 is

$$\dot{R}_3 = \dot{R}_2 + \lambda_3\left(-\lambda_2(\dot{D}-\bar{S}-\bar{A}t) + \lambda_1\int_0^L HV\right)\left(-\lambda_2(D-\bar{D}-\bar{S}t-\frac{1}{2}\bar{A}t^2) - \lambda_1\int\int h\right) \ ,$$

where $\dot{R}_2$ is given by (32). Thus we have

$$\dot{R}_3 = \left(\lambda_1\int_0^L Hv - \lambda_2(\dot{D}-\bar{S}-\bar{A}t)\right)\left(-u_3 - \lambda_2\lambda_3(D-\bar{D}-\bar{S}t-\frac{1}{2}\bar{A}t^2) - \lambda_1\lambda_3\int\int h\right) \ .$$

Thus a natural expression for u_3 is (34). In fact we see that $\dot{R}_3$ becomes

$$\dot{R}_3 = -\left(\lambda_1\int_0^L Hv dx - \lambda_2(\dot{D}-\bar{S}-\bar{A}t)\right)^2 \ , \tag{36}$$

and therefore we conclude that $\dot{R}_3 \leq 0$.

Let us prove now Remark 3.4. Using (27) and (35), the computation of $\dot{u}_3$ gives

$$\dot{u}_3 = -\lambda_2\lambda_3(\dot{D}-\bar{S}-\bar{A}t) + \lambda_1\lambda_3\int_0^L HV - \lambda_2 u_3 + \lambda_1\left(-\frac{g}{2}(H_L^2 - H_0^2) - (u_3+\bar{A})\int_0^L H\right) \ . \tag{37}$$

We compute the first derivative of (37) with respect to t

$$\begin{aligned} \ddot{u}_3 &= \dot{u}_3(-\lambda_2 - \lambda_1\mathrm{vol}) + u_3(-\lambda_2\lambda_3 - \lambda_1\lambda_3\mathrm{vol}) - \lambda_1 g(H_L\frac{\partial H_L}{\partial t} - H_0\frac{\partial H_0}{\partial t}) - \lambda_1\lambda_3\bar{A}\mathrm{vol} \\ &\quad -\lambda_1\lambda_3\frac{g}{2}({H_L}^2 - {H_0}^2) \ , \end{aligned} \tag{38}$$

where $\frac{\partial H_0}{\partial t}$ and $\frac{\partial H_L}{\partial t}$ denote respectively the time-derivative of H_0 and H_L. Note that we have used the following hypothesis stated in the computations of $(\bar{H},\bar{V})$ (see (8)): $\bar{A}$ is a constant value. The equation (38) is an ordinary differential equation of order 2 linear in u_3 and with constant coefficients for $\dot{u}_3$ and u_3. The initial conditions are given by (34) and (37).

This ends the proof of Theorem 3 and Remark 3.4.□

4 Numerical results

We discretize the shallow water equations with the semi-implicit Preissman scheme (see [M:94] or [G:98]). In fact, since Preissman scheme is a semi-implicit scheme, it is not subjected to the Courant condition.

When discretizing, it is possible to choose Preissman coefficient θ and Courant number C_r (namely $\theta = 0.5$ and $C_r = 1$) such that the discretization does not introduce numerical damping for the linear equations (see [M:94] or [G:98]). However, with this choice of parameters, the numerical errors are not damped and the solution obtained becomes non-smooth. Therefore we use a $\theta > 0.5$ even if it generates an artificial stabilization due to the numeric damping. To overcome this difficulty, we can compare the stabilization rate of open-loop to closed-loop systems.

4.1 Simulation with a complete stabilization

In this section, we set the Preissmann coefficient θ to the value 0.51 and the time-step $\Delta t = 0.2$ and the space-step $\Delta x = 0.5$. We consider the following initial conditions, for all x in $[0,\ L]$,

$$\widetilde{H}(x) = 0.02x + 0.88\, m\,,\ \widetilde{V}(x) = \sin^2(\frac{x\pi}{L})\, ms^{-1},$$
$$\widetilde{D} = 0\ m\ ,\quad \widetilde{S} = 0\ ms^{-1}\ ,\quad \widetilde{A} = 0\ ms^{-2}\ ,$$

and let us study the stabilization problem of the fluid and note that we want the tank of length $L = 12$ m to stay the most close as possible from its initial position.

Let us compare the three following control laws: the null control, $u = 0$, the control (29) given by Theorem 2 and the control (34) given by Theorem 3, with the gains $\lambda_1 = 0.01$, $\lambda_2 = 0.05$ and $\lambda_3 = 0.04$.

We note in Figures 2 and 3 that the controls defined in Sections 3.2 and 3.3 succeed in stabilizing the fluid's state contrary to the system without control, where some oscillations of the fluid stay even after 100 seconds.

In Figure 4, we check that the control of Section 3.2 stabilize the tank's speed around the value 0. We note that with this controller, the tank's position tends to a constant (≈ -0.05). This motives to use a forward approach as in Section 3.3 to track this value to 0. This control realize the complete stabilization of the tank. Note that at the bottom of Figure 4, we have the plot of the accelerations, therefore we have the controls.

We check, at Figure 5, that the Lyapunov function R_3 is a strictly decreasing function with control (34). Note that, due to (36), R_3 is decreasing with the null control although we should have a constant function. This decrease is due to the numerical damping because we have chosen in this simulation $\theta = 0.51 > 0.5$. But R_3 decreases faster with the control of section 3.3 than with a null control.

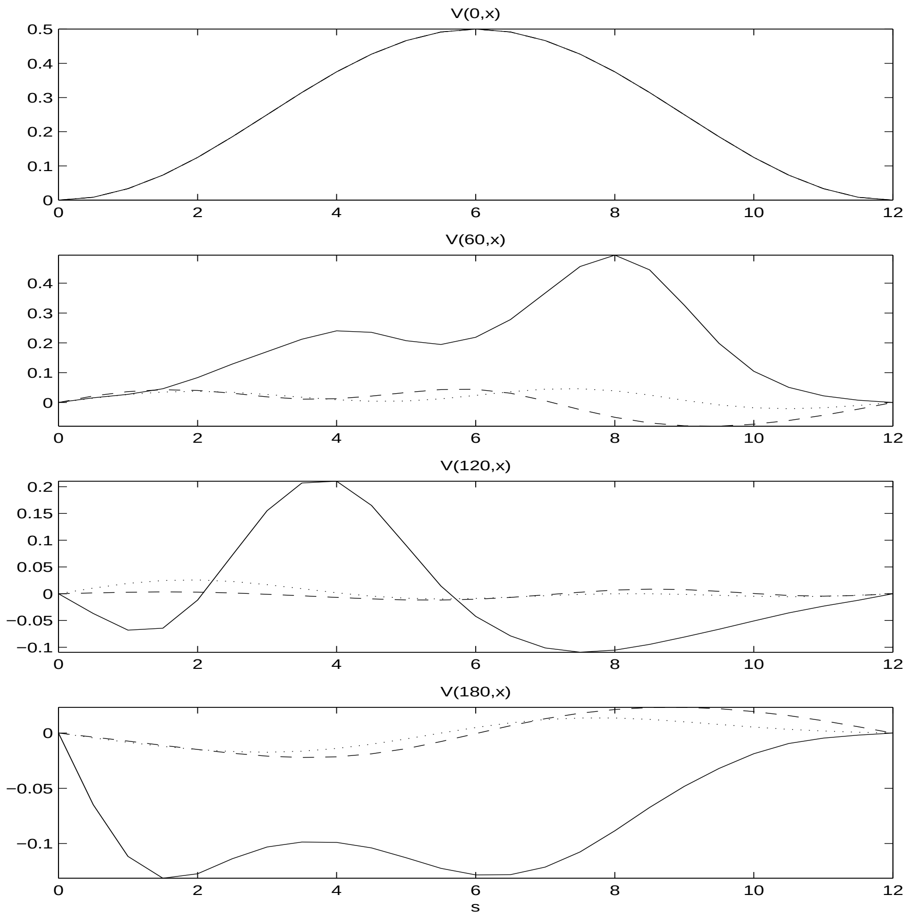

FIG. 2 – *Fluid's speed in the tank at time $t = 0$, $t = 60$, $t = 120$ and $t = 180$ seconds. The curve – is the fluid with the null control, – – with the control (29) of Section 3.2 and ⋯ with the control (34) of Section 3.3.*

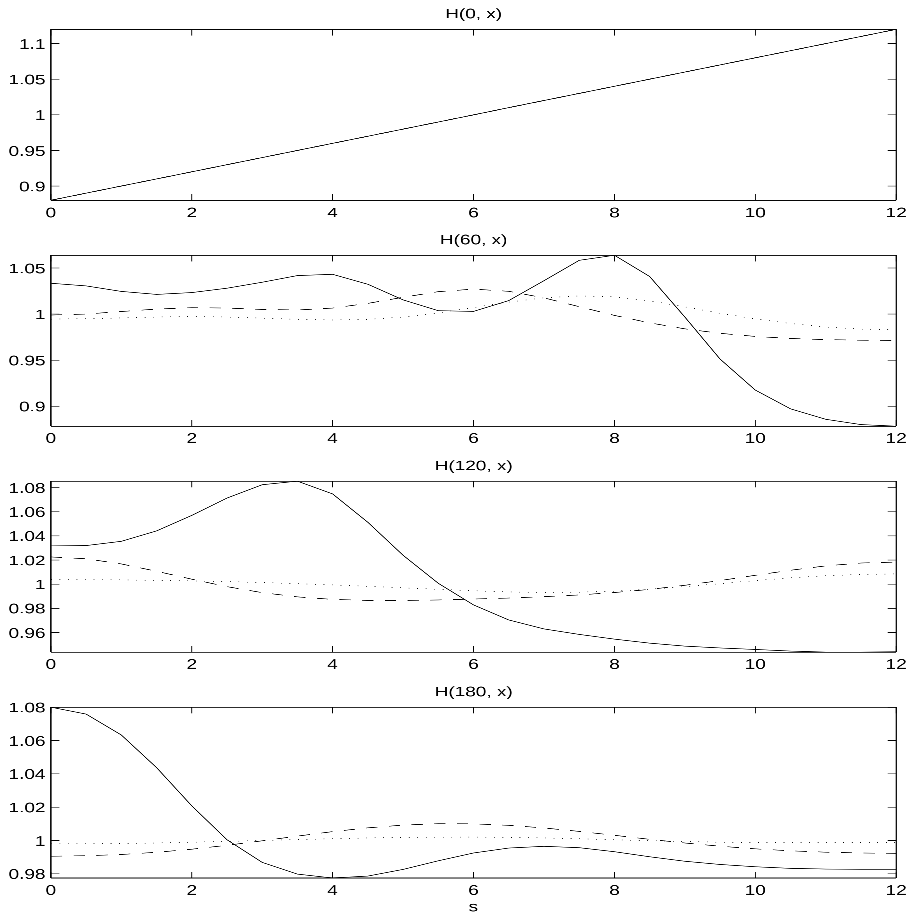

FIG. 3 – *Fluid's height in the tank at time $t = 0$, $t = 60$, $t = 120$ and $t = 180$ seconds. The curve – is the fluid with the null control, – – with the control (29) of Section 3.2 and ⋯ with the control (34) of Section 3.3.*

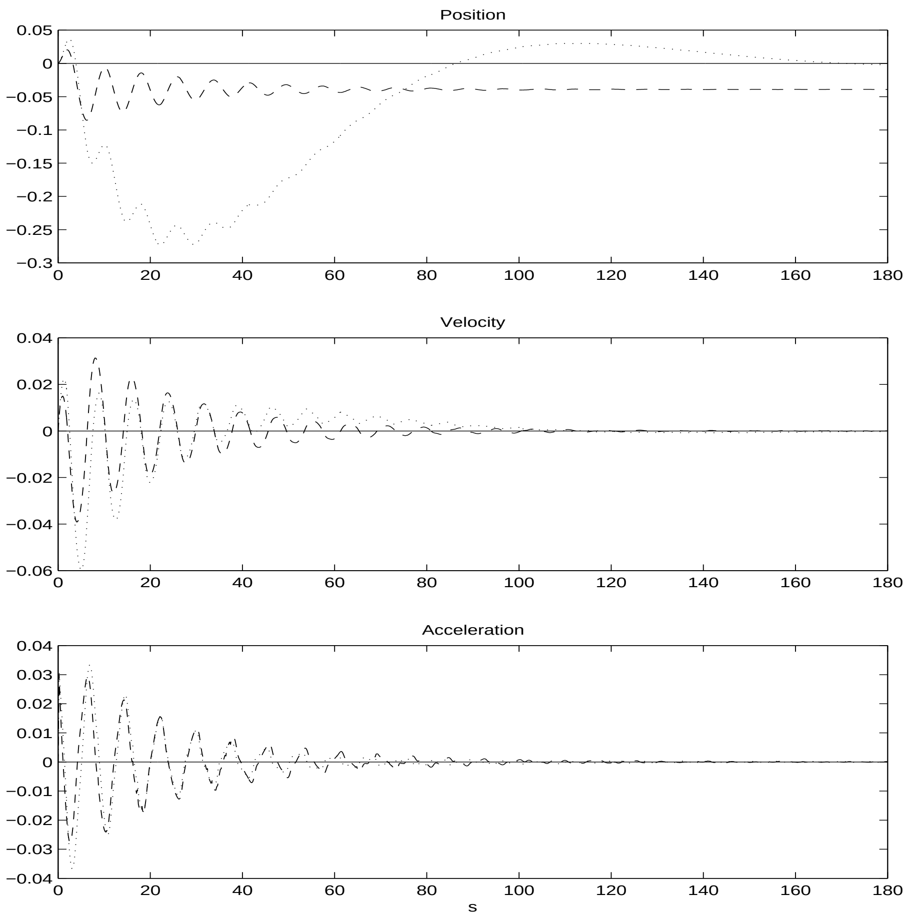

FIG. 4 – *Trajectory of the tank in closed-loop with the null control (–), with the (29) control of Section 3.2 (− −) and with the control (34) of Section 3.3 (· · ·). At the top, we have the plot of the position, in the middle, the velocity and at the bottom, the acceleration in function of the time given in seconds.*

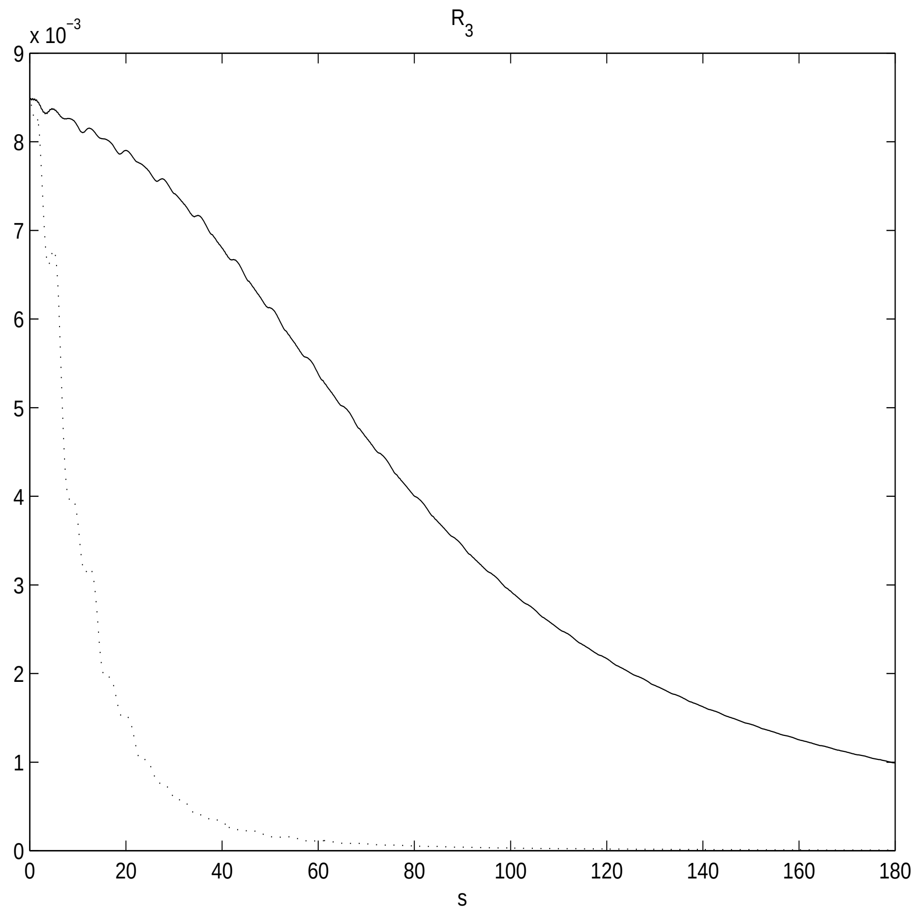

Fig. 5 – *Evolution of the Lyapunov function R_3 with the null control $(-)$ and with the control of Section 3.3 $(\cdots)$.*

4.2 Importance of the non-linear terms of the shallow water equations

In this section we consider the following equilibrium:

$$\bar{D} = 0 \quad , \qquad \bar{S} = 0 \quad , \qquad \bar{A} = 0 \quad , \qquad \bar{H} = 1.5 \quad , \qquad \bar{V} = 0$$

Note that the shallow water equations linearized around this equilibrium are uncontrollable, even locally (see [DPR:99]). Indeed the function $H, V : [0,+\infty) \times [0,L] \to \mathbb{R}$ and $D : [0,+\infty) \to \mathbb{R}$ defined by, for all $t \geq 0$ and for all x in $[0,L]$,

$$\ddot{D}(t) = 0 \, , \quad H(t,x) = 1 + \sin^2(\frac{\pi x}{L}) \, , \quad V(t,x) = -2\frac{\pi}{gL} t \cos(\frac{\pi x}{L}) \sin(\frac{\pi x}{L}) \, , \tag{39}$$

are solutions of the linearized equations with $u = 0$. However the nonlinear shallow water equations are locally controllable (see [C:01]), we expect (and we check) numerically that the nonlinear equations are stabilizable.

To do this, we consider as initial condition the value of the functions (39) at $t = 0$ and we implement the feedback of Section 3.3. More precisely let us introduce a tank of length $L = 7.5$ and the following initial conditions, for all x in $[0, L]$,

$$\widetilde{H}(x) = 1 + \sin^2(\frac{\pi x}{L}) \, , \, \widetilde{V}(x) = 0 \, , \, \widetilde{D} = 0 \, , \, \widetilde{S} = 0 \, , \, \widetilde{A} = 0 \, .$$

We set $\lambda_1 = 0.4$, $\lambda_2 = 0.1$ and $\lambda_3 = 0.1$. We choose $\theta = 0.5001$ which is very close to the critical value (namely 0.5). Therefore we have non-smooth numerical solutions (see Figure 8). The trajectories of the tank with the null control and with the control of Section 3.3 are given by Figure 6.

We observe in Figure 6 that the tank stays very close to the initial position but succeed in stabilizing the fluid's speed (see Figure 7) and the fluid's height (see Figure 8).

Note that the stabilization is reached but very slowly. See also the profile of the Lyapunov function R_3 at Figure 9. This fact could be understood by noting that the stabilization is attained only with the nonlinear terms. Note that in this simulation we choose $\theta = 0.5001$ to avoid too much numerical damping and the numerical stabilization of the system to the equilibrium.

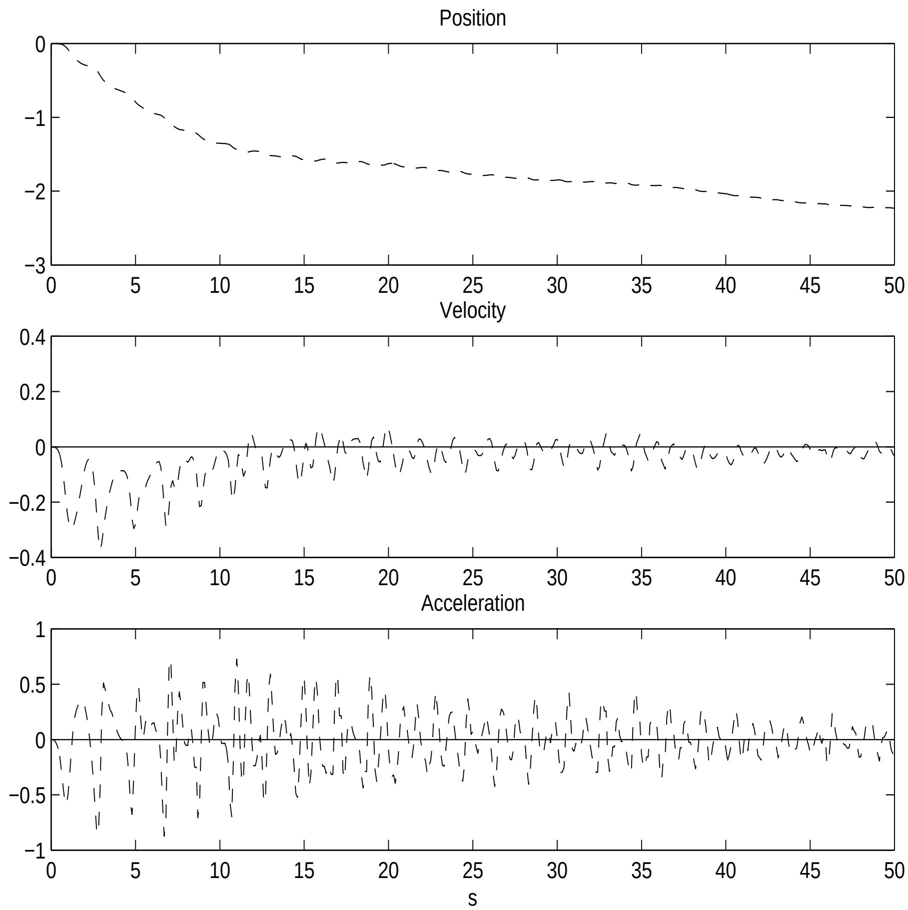

FIG. 6 – *Trajectory of the tank in closed-loop with the null control (–), with the control of Section 3.3 (− −). At the top, we have the plot of the position, in the middle, the velocity and at the bottom, the acceleration in function of the time given in seconds.*

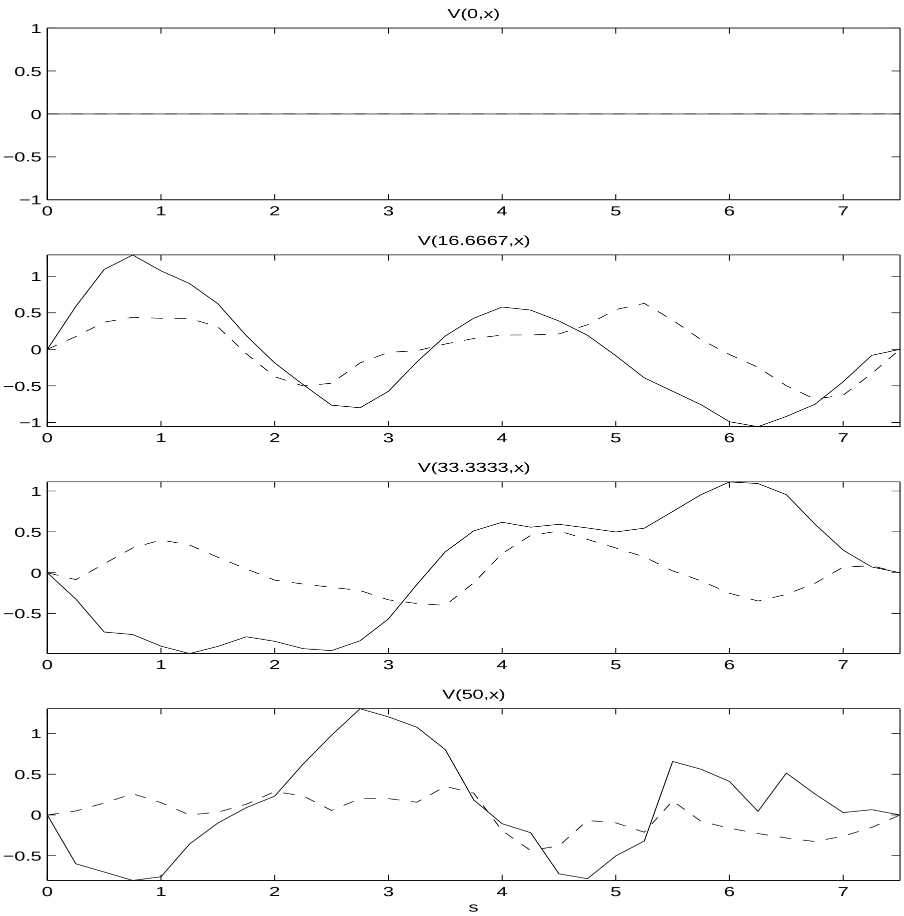

FIG. 7 – *Fluid's speed in the tank at time* $t = 0$, $t = 16.6$, $t = 33.3$ *and* $t = 50$ *seconds. The curve* – *is the fluid with the null control,* – – *with the control of Section 3.3.*

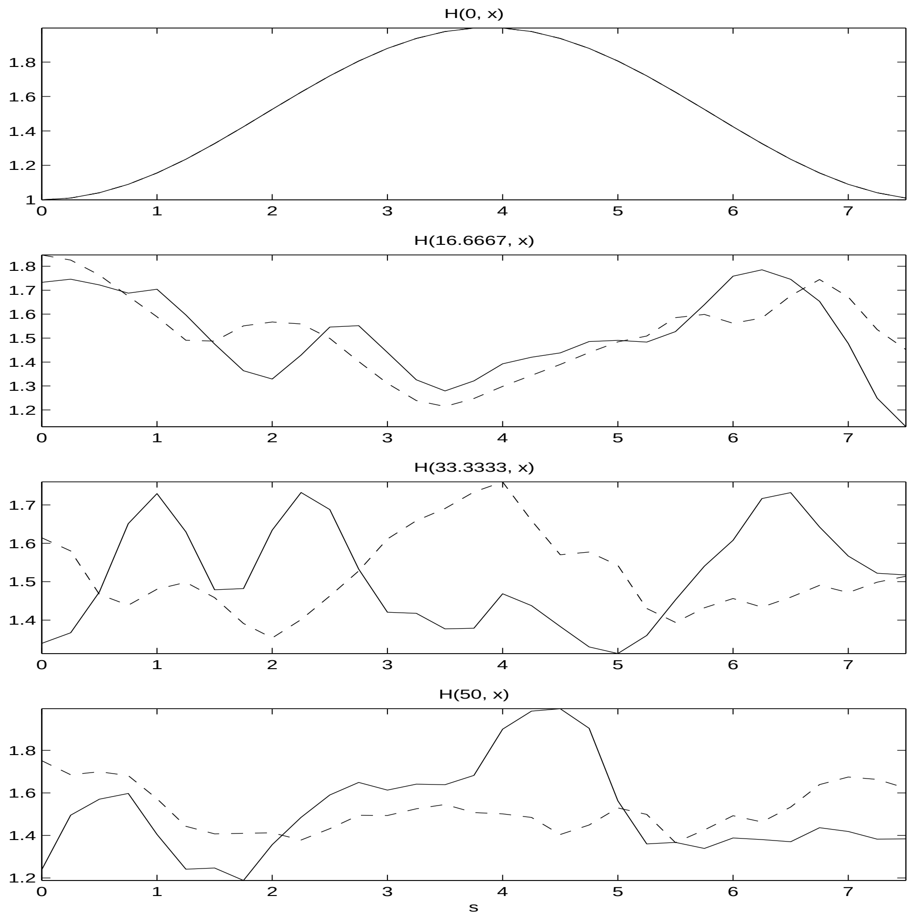

FIG. 8 – *Fluid's height in the tank at time* $t = 0$, $t = 16.6$, $t = 33.3$ *and* $t = 50$ *seconds. The curve* – *is the fluid with the null control,* $--$ *with the control of Section 3.3.*

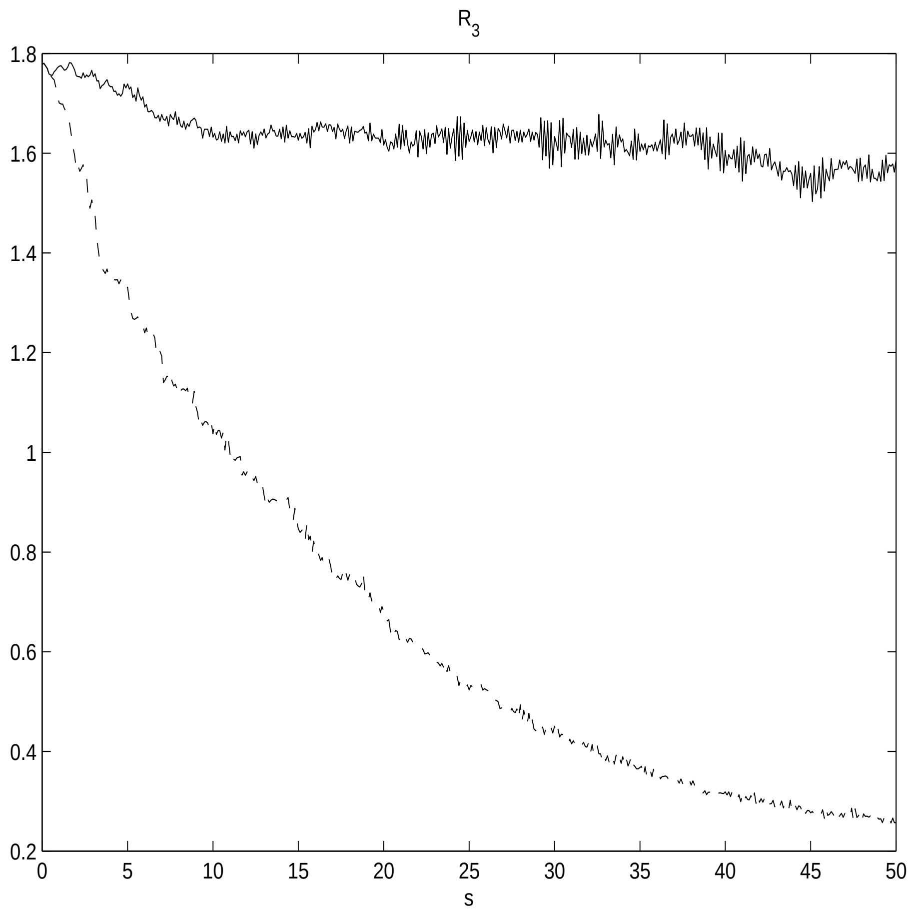

FIG. 9 – *Evolution of R_3 for the null control (−) and for the control of Section 3.3 (− −).*

Conclusion

A further work of this paper is to study the influence of the initial condition of the control law given by Theorems 1, 2 and 3. See Remark 3.2. In particular, one important question is the following: do we need to know the entire state of the fluid to compute an initial condition for the control law and to have a closed-loop asymptotically stable system? Rather than to study this question we have preferred to simulate the shallow water equations and to check numerically that the stabilization is attained which is yet to be theoritically proved. See Remark 3.1.

Troisième partie

Systèmes linéaires de dimension finie avec incertitudes

Le problème général considéré dans cette partie est celui de la stabilisation de l'origine d'un système linéaire en dimension finie lorsque nous avons une incertitude sur les données du système linéaire.

Les propriétés des systèmes linéaires en dimension finie sont maintenant bien connues. Nous avons aussi bien des caractérisations de propriétés théoriques (voir par exemple la propriété de la stabilité dans [K:80, Chapter 3]) que des outils numériques efficaces pour les résoudre (voir par exemple la résolution de problèmes SDP en temps polynomial dans [BEFB:94]). En revanche, les transitions possibles vers les études de systèmes non-linéaires ou les introductions des non-linéarités partielles dans des systèmes linéaires restent des problèmes difficiles et encore peu étudiés. Un exemple où il est nécessaire de passer d'un système linéaire à un système non linéaire, est le cas de l'étude d'un système linéaire avec une incertitude sur les données du système ou sur la condition initiale.

Dans ce cadre, le problème n'est plus aussi facile qu'avec le système entièrement linéaire mais nous pouvons, dans certains cas, transformer ce problème non-linéaire en un nouveau problème linéaire mais de plus grande dimension puis appliquer les techniques classiques des systèmes linéaires sur ce nouveau problème. Il reste maintenant à considérer l'application numérique de ces méthodes à des problèmes concrets. C'est l'objet de cette partie où nous étudions l'application de ces techniques à un problème d'origine industrielle et où nous trouvons une loi de commande telle que l'origine d'un système linéaire soit asymptotiquement stable avec une robustesse par rapport à des incertitudes sur les données.

Chapitre 1

Le problème et les résultats

Dans ce chapitre, nous allons étudier une technique pour construire un contrôleur pour un système linéaire, avec des incertitudes sur les données, tel qu'une certaine trajectoire étant fixée, le système bouclé correspondant converge vers cette trajectoire. Nous étudions aussi une application à un problème industriel. Nous présentons le problème au paragraphe 1.1 et sa formulation en termes d'inégalités linéaires matricielles au paragraphe 1.2. Au paragraphe 1.3, nous explicitons les méthodes numériques utilisées et nous donnons les résultats au paragraphe 1.4. Au paragraphe 1.5, nous introduisons une incertitude sur les données et nous détaillons des résultats partiels de la stabilisation robuste correspondante.

1.1 Position du problème

En France, une grande partie de la production d'électricité est produite dans des centrales nucléaires dont le principe de fonctionnement est tel qu'il est difficile d'adapter la production d'électricité à la demande. C'est pourquoi les centrales thermiques, au fonctionnement plus souple, jouent un rôle clé dans la production française d'électricité bien qu'il n'y ait qu'une faible partie d'électricité produite par les centrales thermiques.

Une centrale est un système contrôlé c'est-à-dire un système dont l'évolution dépend de variables extérieures qui peuvent être fixées par l'opérateur (voir figure 1).

- Comme variables d'entrée du système “centrale thermique”, i.e. les variables fixées par l'extérieur, nous avons :
  1. l'ouverture de la vanne à l'entrée de la turbine,
  2. le débit de charbon,
  3. le débit de surchauffe (eau injectée dans la vapeur pour contrôler la température).
- Comme variables de sortie, i.e. les variables que nous voulons contrôler :
  1. la puissance produite,
  2. la pression de la vapeur,
  3. la température de la vapeur.

Étant données des références de puissance $W^\star$, de température $T^\star$ et de pression $P^\star$, il existe un algorithme Proportionnel Intégral Dérivé (PID) qui détermine les contrôles qu'il faut suivre pour atteindre ces références désirées. Nous nous intéresserons uniquement au

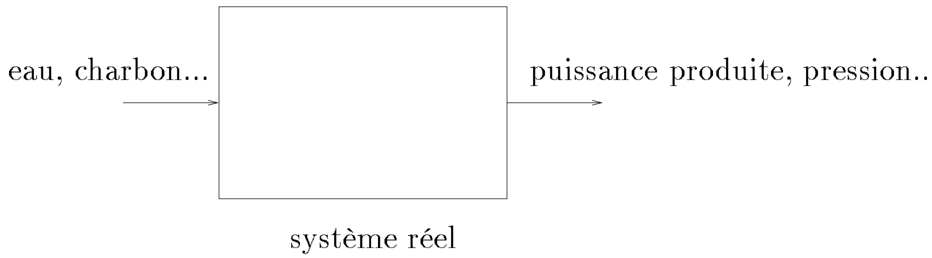

FIG. 1 – *Système réel d'une centrale thermique.*

système bouclé par le contrôleur PID sans chercher à connaître l'expression de ce contrôleur ni à le modifier (figure 2) et nous nous contenterons de trouver les fonctions $W^\star$, $T^\star$ et $P^\star$ telles que les sorties W, T et P suivent un certain profil, par exemple :

- La puissance fournie par la centrale doit décroître régulièrement en quelques heures pour passer de la puissance demandée aux heures de pointe à une puissance consommée de nuit (suivi d'une pente).
- Nous souhaitons passer soudainement d'un régime de fonctionnement bas à un régime haut (échelon).
- La température et la pression doivent rester plus petites que certaines valeurs critiques déterminées par les caractéristiques techniques de la centrale.

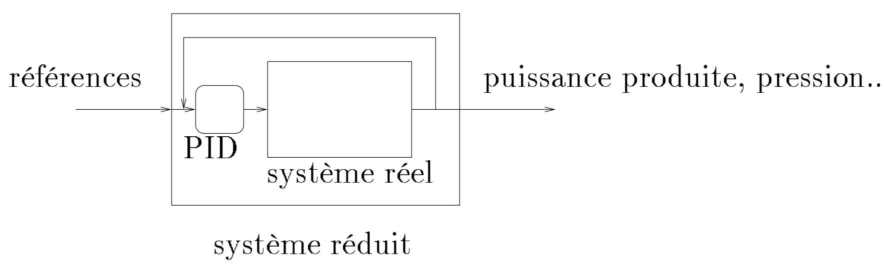

FIG. 2 – *Système réduit d'une centrale thermique.*

Étant donné un modèle linéaire, nous pouvons regarder quelle est la réponse à un échelon, c'est-à-dire comment le système réduit converge vers la nouvelle référence, lorsque nous passons soudainement d'une valeur de référence notée 0 à une valeur de référence 1. Les résultats sont donnés à la figure 3. Nous pouvons noter que la convergence des sorties est très lente (surtout pour la température, voir sur la diagonale) et que les termes croisés (figures hors de la diagonale) varient beaucoup (jusqu'à trois fois la valeur finale). Ces comportements nous encouragent à chercher à améliorer les sorties du système et motivent la recherche de références telles que nous ayons des variables de sortie qui convergent plus vite et plus régulièrement vers les valeurs souhaitées.

Posons plus spécifiquement le problème et étudions nos résultats.

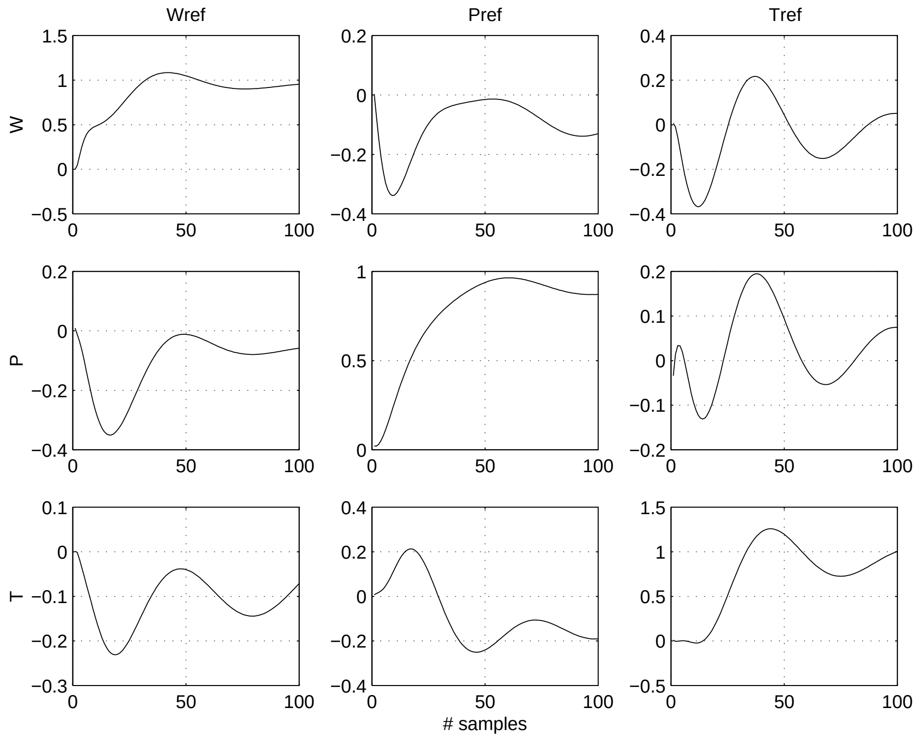

FIG. 3 – *Réponse du système réduit à un échelon.*

1.2 Le système réduit

Nous avons trouvé une modélisation relativement proche du système physique en simplifiant certains phénomèmes. Nous obtenons un modèle non-linéaire qui a été validé à partir de données expérimentales. Nous le linéarisons autour d'un certain régime de fonctionnement de la centrale après avoir choisi un certain pas de temps de discrétisation (ici 10 secondes). Nous trouvons alors, pour le système réduit de la figure 2, un système d'équations linéaires :

$$\begin{cases} x_{k+1} = Ax_k + Bu_k \ , \\ y_{k+1} = Cx_k + Du_k \ , \end{cases} \tag{1.1}$$

qui donne une représentation approximative (nous reviendrons sur cette question au paragraphe 1.5) de l'évolution de

$$y_k = \begin{pmatrix} \text{puissance produite} \\ \text{pression de la vapeur} \\ \text{température de la vapeur} \end{pmatrix} , \tag{1.2}$$

sortie du système (1.1) à l'itération k en fonction de la commande à la même itération, c'est-à-dire

$$u_k = \begin{pmatrix} \text{puissance de référence} \\ \text{pression de la vapeur de référence} \\ \text{température de la vapeur de référence} \end{pmatrix} , \tag{1.3}$$

et où x_k désigne l'état du système linéaire au temps t_k mais ne représente *a priori* pas du tout l'état du vrai système que nous ne savons même pas donner explicitement.

Soit x_0 une condition initiale de l'état du système (1.1). Soient $\underline{u_k} \leq \overline{u_k}$ et $\underline{y_k} \leq \overline{y_k}$ des suites réelles. Le problème que nous allons étudier est le suivant :

Trouver une suite u_k telle qu'étant donnée y_k la sortie du système (1.1), nous ayons, pour tout k

$$\begin{array}{ccccc} \underline{u_k} & \leq & u_k & \leq & \overline{u_k} \, , \\ \underline{y_k} & \leq & y_k & \leq & \overline{y_k} \, . \end{array} \tag{1.4}$$

Notons que savoir si une telle suite u_k existe ou pas fait partie du problème posé.

Ce problème est un problème classique de programmation linéaire (linear programming, LP), voir par exemple [W:97] pour une présentation de systèmes LP avec une méthode de résolution qui inspirera la nôtre. Mais nous tenons à utiliser une méthode plus générale en considérant ce problème comme un problème de résolution de LMI, c'est-à-dire sous la forme d'un problème d'optimisation de la forme

Trouver x tel que $c^T x$ soit minimal et tel que $F(x) \geq 0$.

où $F : \mathbb{R}^m \to \mathbb{R}^{N \times N}$ est une fonction matricielle affine, i.e.

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0 \ ,$$

avec $F_i = F_i^T$, des matrices symétriques fixées de $\mathbb{R}^{N \times N}$, m, un entier donné, et c, un vecteur de $\mathbb{R}^m$ appelé *vecteur objectif.*

En effet l'avantage ce type de problème est qu'il permet de rajouter facilement une incertitude comme ce sera le cas au paragraphe 1.5.

Nous allons maintenant détailler la partie numérique proprement dite de la résolution.

1.3 Méthodes numériques utilisées

Dans ce paragraphe nous décrivons brièvement les méthodes numériques utilisées pour résoudre le problème posé dans le paragraphe précédent.

Pour résoudre le problème LMI, nous avons utilisé des algorithmes basés sur des méthodes primal-dual par point intérieur (voir par exemple [HR:96] et [NT:98]). Ces algorithmes peuvent être utilisés comme package matlab et sont en accès libre. Voir par exemple dans [H:00] une introduction détaillée d'un tel package matlab.

Nous avons choisi d'utiliser différentes méthodes pour la résolution des LMI de notre problème. Le but étant de voir quel package était le plus adapté à nos objectifs. Nous avons utilisé les méthodes suivantes :

SP Cet algorithme a été développé par L. Vandenberghe et S. Boyd. Elle repose sur un principe de réduction en suivant la méthode primal-dual de Nesterov and Todd. Pour plus d'information ou pour utiliser ce package, voir le site

`http://www.ee.ucla.edu/~vandenbe/sp.html`

SDP Développé par J.-P. Alizadeh, M. Haeberly, M. Nayakkankuppam et M.L. Overton, ce package Matlab est aussi une méthode primal-dual mais basée, cette fois, sur un algorithme de prédiction-correction dû à Mehrotra. Voir la page de SDP :

`http://www.cs.nyu.edu/faculty/overton/sdppack/sdppack.html`

SDPHA Cette méthode a été développée par A. Florian, A. Potra, R. Sheng et N. Brixius. Elle adapte la méthode précédente en utlisant des techniques d'homogénéisation. Voir aussi la page de SDPHA :

`http://ceu.fi.udc.es/SAL/B/3/SDPHA.html`

Ces packages ont été développé sous une interface graphique rendant leur utilisation plus aisée. Il s'agit de Tklmitool développé par L. El-Ghaoui et J.-L. Commeau. Voir la page : `http://robotics.eecs.berkeley.edu/~elghaoui/lmitool/lmitool.html`

Exposons maintenant les résultats.

1.4 Les résultats

Nous détaillons maintenant les solutions du problème LMI posé.

Dans les simulations présentées dans ce paragraphe, nous avons utilisé l'algorithme SDP décrit au paragraphe précédent.

Nous avons réussi à résoudre la LMI du paragraphe 1.2 pour un grand horizon de temps (plus de 40 pas de temps), et donc d'imposer certains changements de régime très réalistes. Pour cela nous calculons $\underline{y_k}$ et $\overline{y_k}$ pour avoir le profil souhaité grâce à (1.4). Les figures 4 et 6 représentent respectivement deux échelons a $\pm 4\%$ et une rampe ($\pm 4\%$ par itération) pour les commandes représentées aux figures 5 et 7.

Nous remarquons que les références varient beaucoup, c'est pourquoi nous souhaitons imposer de plus la nouvelle contrainte :

$$|u_{k+1} - u_k| \leq varmax \ , \tag{1.5}$$

où

$$|u_{k+1} - u_k| = \begin{pmatrix} |u_{k+1}^{(1)} - u_k^{(1)}| \\ |u_{k+1}^{(2)} - u_k^{(2)}| \\ |u_{k+1}^{(3)} - u_k^{(3)}| \end{pmatrix} \ ,$$

et $varmax$ est un vecteur de $\mathbb{R}^3$ qui va définir le taux maximum de variation admissible et que nous avons choisi constant mais que nous aurions pu prendre variable (par exemple en imposant qu'aux changements de régime nous accordons plus de souplesse mais moins lorsque le régime est stable).

Nous nous intéressons donc maintenant au problème suivant :

Trouver une suite u_k telle que nous ayons (1.4) et (1.5) où y_k est calculée avec (1.1).

Les résultats sont donnés aux figures 8 et 10 pour deux échelons successifs et une rampe respectivement avec les commandes données aux figures 9 et 11.

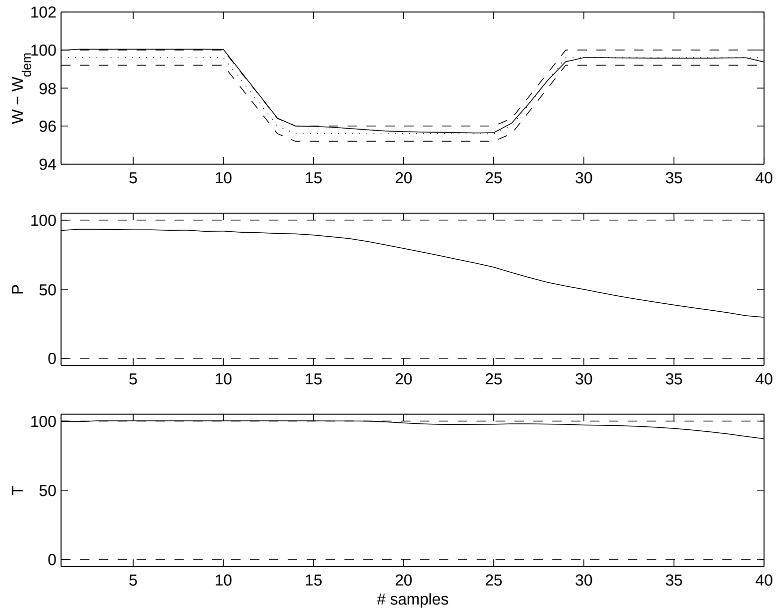

FIG. 4 – *Système bouclé par la commande de la figure 5. Nous réalisons deux échelons à ±4% par itération en satisfaisant les contraintes (1.4). Nous normalisons les quantités et nous n'étudions que l'évolution en pourcentage. De haut en bas, nous lisons la puissance produite, la pression de la vapeur et la température de la vapeur. Les lignes en − − désignent les contraintes hautes et basses, ⋯ la moyenne des contraintes de puissance, et le trait plein les sorties simulées.*

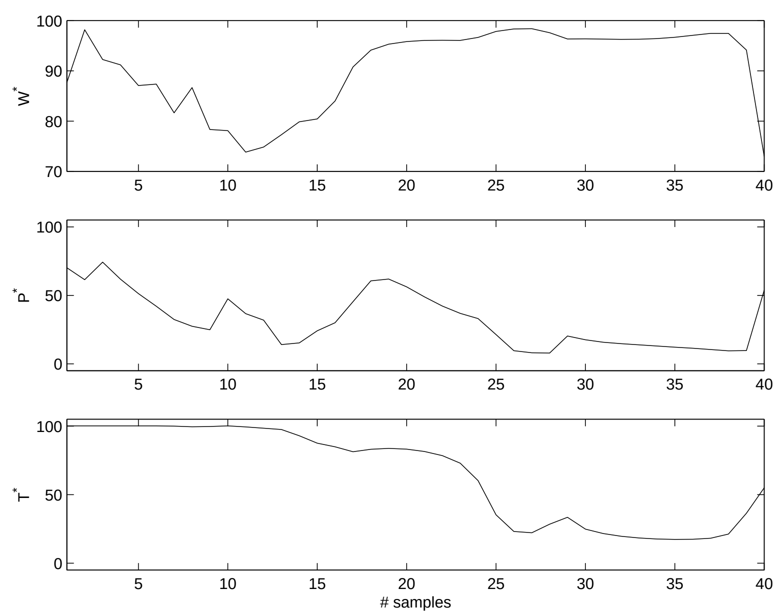

FIG. 5 – *Commandes réalisant les échelons de la figure 4 avec les contraintes (1.4).*

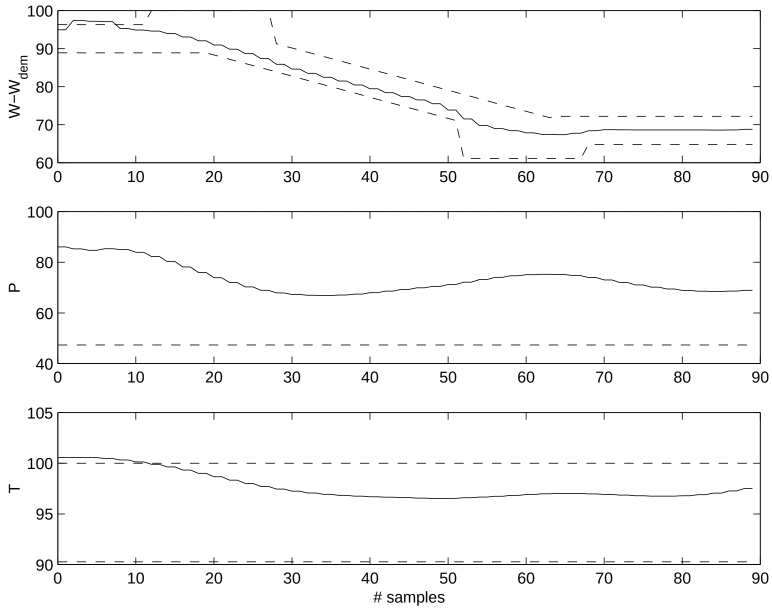

FIG. 6 – *Système bouclé par la commande de la figure 7. Nous réalisons une rampe à −4% en satisfaisant les contraintes (1.4). Nous normalisons les quantités et nous n'étudions que l'évolution en pourcentage. De haut en bas, nous lisons la puissance produite, la pression de la vapeur et la température de la vapeur. Les lignes en − − désignent les contraintes hautes et basses et le trait plein les sorties simulées.*

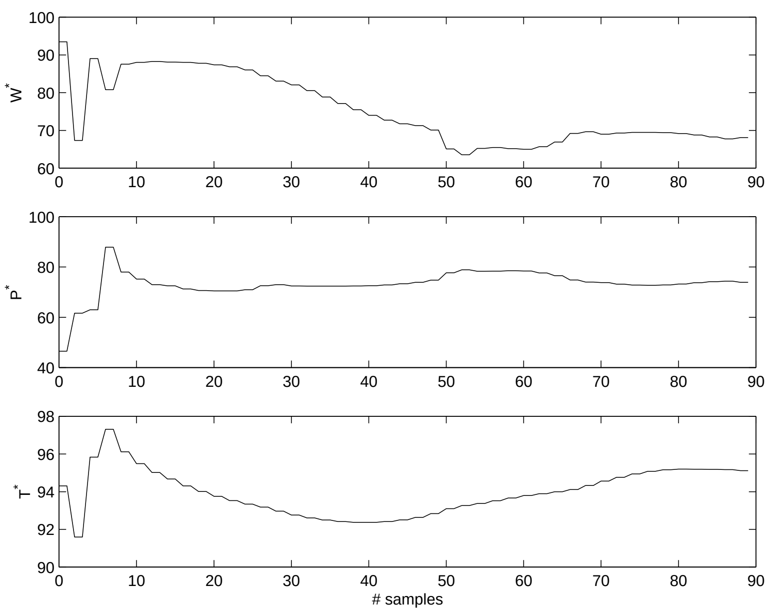

FIG. 7 – *Commandes réalisant la rampe de la figure 6 avec les contraintes (1.4).*

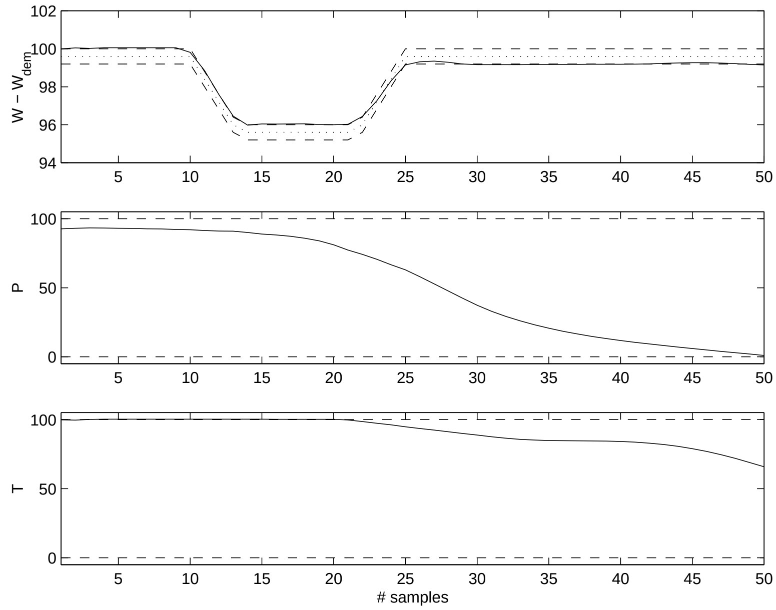

FIG. 8 – *Système bouclé par la commande de la figure 9. Nous réalisons deux échelons à ±4% en satisfaisant les contraintes (1.4) et (1.5). Nous normalisons les quantités et nous n'étudions que l'évolution en pourcentage. De haut en bas, nous lisons la puissance produite, la pression de la vapeur et la température de la vapeur. Les lignes en — — désignent les contraintes hautes et basses, ··· la moyenne des contraintes de puissance, et le trait plein les sorties simulées.*

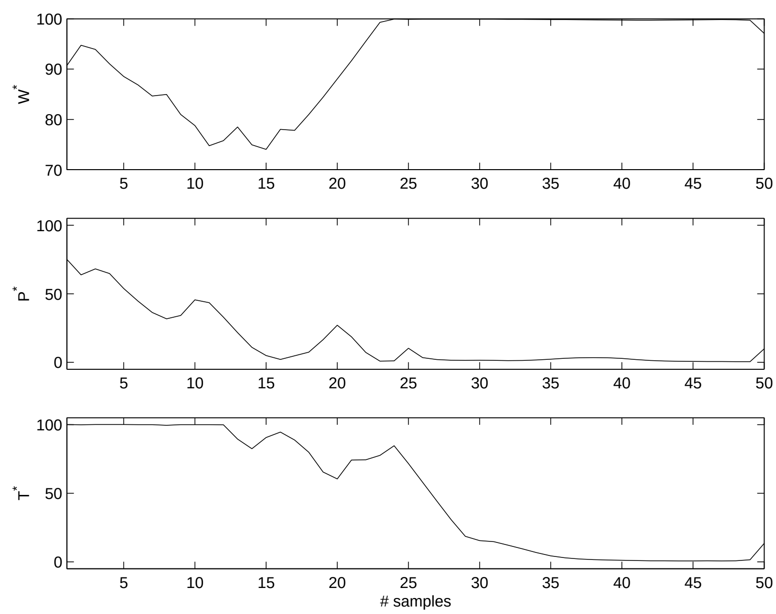

FIG. 9 – *Commandes réalisant les échelons de la figure 8 avec les contraintes (1.4) et (1.5).*

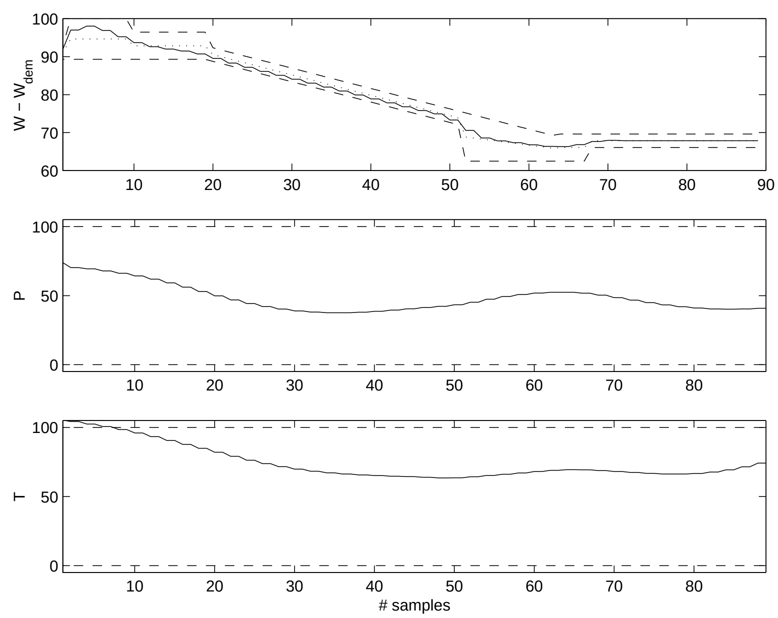

FIG. 10 – *Système bouclé par la commande de la figure 11. Nous réalisons une rampe à −4% en satisfaisant les contraintes (1.4) et (1.5). Nous normalisons les quantités et nous n'étudions que l'évolution en pourcentage. De haut en bas, nous lisons la puissance produite, la pression de la vapeur et la température de la vapeur. Les lignes en* − − *désignent les contraintes hautes et basses,* ··· *la moyenne des contraintes de puissance, et le trait plein les sorties simulées.*

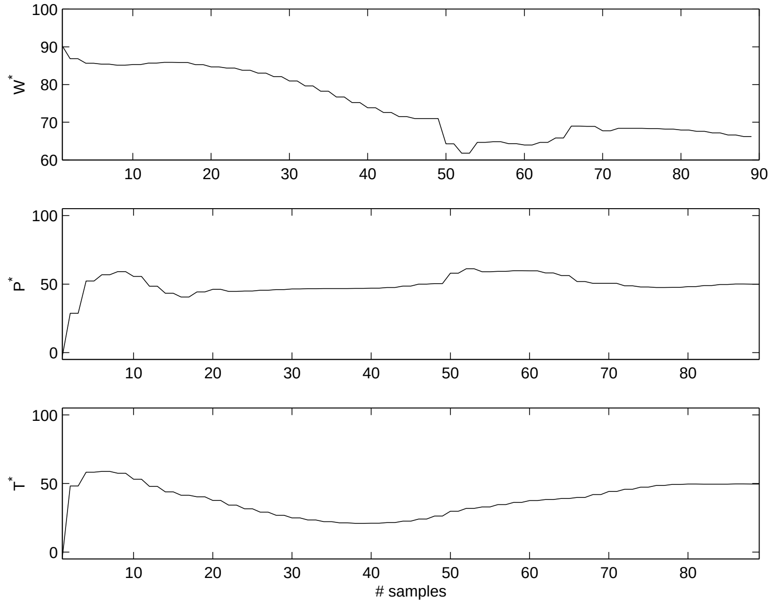

FIG. 11 – *Commandes réalisant la rampe de la figure 10 avec les contraintes (1.4) et (1.5).*

Nous allons maintenant nous intéresser à un problème de stabilisation robuste généralisant ces questions.

1.5 Vers une stabilisation robuste

Comme nous l'avons vu précédemment, le système (1.1) n'est qu'une approximation du système réduit de la centrale donné à la figure 2. De plus les matrices A, B, C et D sont souvent mal connues. Cela nous motive à remplacer le système (1.1) par

$$\left\{ \begin{array}{rcl} x_{k+1} & = & A(\Delta)x_k + B(\Delta)u_k \ , \\ y_{k+1} & = & C(\Delta)x_k + D(\Delta)u_k \ , \end{array} \right. \tag{1.6}$$

où A, B, C et D sont désormais des fonctions matricielles définies sur un ensemble borné de matrices noté $\boldsymbol{\Delta} = \{\Delta\}$.

Nous allons chercher à résoudre le problème suivant :

Trouver une suite u_k telle quelle que soit la suite Δ_k d'éléments de $\boldsymbol{\Delta}$, nous ayons (1.4) où y_k est calculé par (1.6) avec $\Delta = \Delta_k$.

Pour résoudre ce problème nous l'avons transformé en une LMI en utilisant une méthode de relaxation. Cette méthode permet de considérer le problème sans incertitude comme un cas particulier du problème avec robustesse (voir [BEN:00]). Il faut noter qu'elle utilise la structure de l'ensemble $\boldsymbol{\Delta}$, ainsi nous n'aurons pas la même structure de $\boldsymbol{\Delta}$ si l'incertitude porte sur tous les coefficients des matrices, par exemple avec A :

$$A(\Delta) = (a_{ij} + \varepsilon_{ij}) \ , \tag{1.7}$$

où la matrice A sans incertitude est donnée par $A(0) = (a_{ij})$, et nous avons, pour tous les couples (i,j), $|\varepsilon_{ij}| < \rho$ avec ρ désignant le rayon de l'incertitude.

Cette méthode n'a abouti, dans le cadre du système (1.6) avec $A(\Delta)$, $B(\Delta)$, $C(\Delta)$ et $D(\Delta)$ ayant une incertitude sur tous les coefficients (comme (1.7) pour A), que pour des très petits horizons de temps (3 itérations). Cela ne nous a pas permis de considérer des échelons ou des rampes complets. La raison est que le problème LMI correspondant après relaxation des incertitudes demandait beaucoup de mémoire et de temps de calcul.

Plusieurs pistes peuvent être envisagées pour contourner cette difficulté :

- Choisir une autre structure de l'ensemble d'incertitude $\boldsymbol{\Delta}$ qui donnerait une LMI plus facile à résoudre.
- Optimiser le code matlab pour utiliser moins de mémoire et de temps de calcul.
- Faire le calcul du contrôle (u_1,u_2,u_3) sur trois pas de temps seulement et considérer comme nouvelle condition initiale la condition finale trouvée après trois pas de temps. Puis recommencer le calcul pour trois pas de temps, et ainsi de suite... Cependant cette méthode risque d'aboutir à un problème insoluble si, après un certain nombre de séries de calculs sur trois pas de temps, il n'existe pas de contrôle qui réalise notre objectif avec la dernière condition finale calculée. C'est d'ailleurs un des avantages à utiliser un grand horizon de temps car nous assurons que le problème sera résoluble sur tout l'horizon de temps considéré (même si bien sûr nous n'avons aucune garantie de faisabilité au delà).

Nous avons cependant étudié entièrement un exemple en dimension 2 où le problème de mémoire n'existe pas pour d'aussi petit horizon de pas de temps mais subsiste pour plus de 12 itérations.

La formulation exacte de ce problème est donné à partir de la page 203 et a fait l'objet de l'article [PBE:00]. La figure 12 représente la sortie y du système avec incertitude et la commande correspondante.

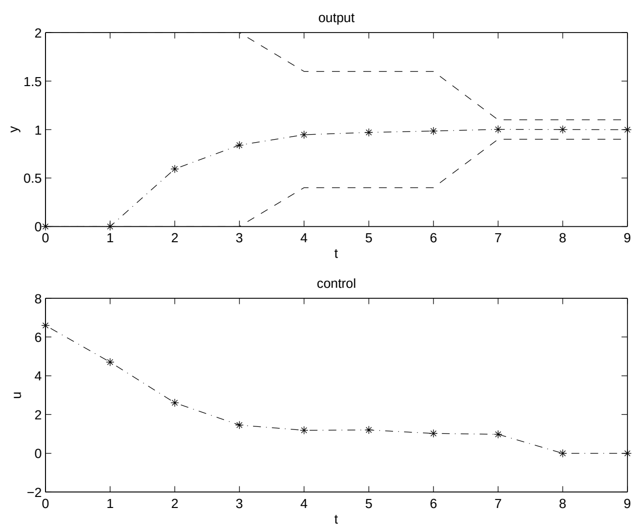

FIG. 12 – *Système analogue à celui de (1.6). En haut la sortie du système et en bas la commande correspondante au cours du temps. Les traits* $- \; - \; -$ *sont les contraintes du type (1.4) et les* $*$ *sont les variables calculées qui sont interpolées linéairement par les lignes* $-\cdot-$.

Nous remarquons que la commande ne varie pas beaucoup mais que la commande initiale est relativement grande (le double de la valeur moyenne) ce qui peut être gênant si, à la première itération, nous sommes à l'équilibre, c'est-à-dire avec $u = 0$ et $x = 0$. C'est pourquoi nous avons cherché à résoudre le même problème avec incertitude sur les matrices mais en plus en imposant

$$|u_0| < max \;, \tag{1.8}$$

où max est la valeur maximale admissible pour le contrôle initial (remarquez la similitude entre la condition (1.8) et (1.5)). Les résultats sont donnés à la figure 13.

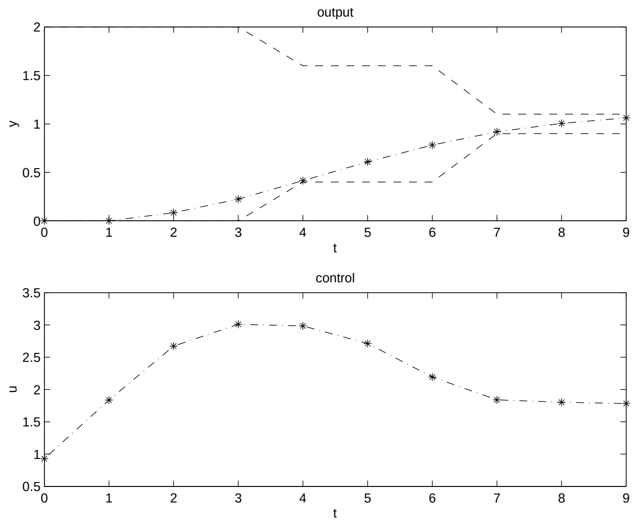

FIG. 13 – *Système analogue à celui de (1.6) avec en plus une contrainte du type (1.8). En haut la sortie du système et en bas la commande correspondante au cours du temps. Les traits* $---$ *sont les contraintes du type (1.4) et les* $*$ *sont les variables calculées qui sont interpolées linéairement par les lignes* $-\cdot-$.

Appendice

Dans cet appendice nous donnons les démonstrations précises et les théorèmes des résultats annoncés dans le chapitre précédent.

Nous trouvons donc dans cet ordre

1. Le papier [PBE:00] présenté à une conférence qui reprend les théories du chapitre 1.
2. Le rapport [BPFE:00] qui présente l'application de ces idées au problème de la simulation d'une centrale thermique.

Robust Optimization-based Control: an LMI Approach

C. Prieur[1] , P. Bendotti[2] and L. El-Ghaoui[3]

IEEE Conference on Decision and Control, Sydney, Australie, 2000.

Abstract: An optimization-based control technique is developed to defining feasible input trajectories with respect to operating constraints. The proposed approach is based on the construction of a semidefinite programming (SDP) problem equivalent to the constrained control problem. When the system is subject to structured uncertainty, the nominal SDP problem is extended to account for uncertain parameters evolving in a certain ellipsoidal set.

1 Introduction

The intend in this paper is to define feasible input trajectories that satisfy operating constraints. The proposed approach consists in defining an optimization problem equivalent to the constrained control problem. One possible way is to cast this problem in an SDP problem written in terms of Linear Matrix Inequalities that can be efficiently solved using available softwares (see [BEFB:94, A:95, VB:96]). An alternative way is to use model predictive control which has been widely used for tracking problems subject to constraints (see [GPM:89] for a comprehensive survey and [KBM:95] for technical details). This technique is also referred to as moving horizon control or receding horizon control. More particularly, it involves a two step-procedure: (1) a sequence of future control actions is chosen according to a prediction of the future evolution of the system and applied to the plant until new measurements are available, (2) a new sequence is determined which replaces the previous one. Although more than one control move is generally calculated, only the first one is implemented. The proposed approach is very similar except the sequence is computed only once over the whole horizon of prediction, thus all calculated control moves are implemented. In other words, the proposed approach is an *open-loop prediction* comparable to the first step of predictive control. In that respect, it could be referred to as predictive control using a *long range horizon*. Since the sequence of control does not take into account new measurements, it is assumed there is a feedback controller that guarantees stability along the trajectory. This assumption is fairly standard in the model predictive approach and has found to be very convenient from a practical point of view.

Given a nonlinear continuous-time closed loop system, linearized discrete-time system can be derived around an operating point. One way to proceed is to use a standard forward-Euler discretization scheme. This approximation leads to modelling errors that can be represented as structured uncertainties. Moreover, the original nonlinear system may be not well known, thus increasing the discrepancy between the actual controlled plant and the linear model. Such errors can be captured by an uncertain system. The proposed work is extended to cope

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with structured uncertainty. An alternative way is to use classical approaches like the robust receding horizon control (see e.g. [MM:93] and [PL:99]).

Section 2 is devoted to the nominal SDP while Section 3 extends the nominal SDP problem to account for uncertainty. In each section, the control approach employed is presented in distinct steps: system considered and associated SDP problem, construction of an equivalent SDP-problem, illustration using a second order system. Concluding remarks end the paper.

2 Nominal SDP-problem

2.1 A discrete-time system without uncertainty

The problem considered in this paper consists in the computation of a control sequence $u = (u_0, \dots, u_T)'$ for a discrete-time system of the form:

$$\forall k = 0, \dots, T, \left\{ \begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k + Du_k \end{array} \right. \tag{1}$$

such that the outputs satisfy the constraints:

$$\forall k = 0, \dots, T, \underline{y}_k \leq y_k \leq \overline{y}_k \ , \tag{2}$$

and the control satisfy the following constraints:

$$\forall k = 0, \dots, T, \underline{u}_k \leq u_k \leq \overline{u}_k \ , \tag{3}$$

with the objective:

$$\texttt{minimize } c'u \ . \tag{4}$$

where the time horizon T, the initial state x_0, the constraints $\underline{y}_k, \overline{y}_k, \underline{u}_k, \overline{u}_k$ and the objective c are given.

2.2 Semi-definite programming problem

We consider a semi-definite programming problem (SDP) of the form:

$$\texttt{min} c'_{\mathcal{F}}\xi \text{ subject to } \mathcal{F}(\xi) = \mathcal{F}_0 + \sum_{i=1}^{m} \xi_i \mathcal{F}_i \geq 0 \tag{5}$$

where $c_{\mathcal{F}} \in \mathbb{R}^m - \{0\}$, and the symmetric matrices $\mathcal{F}_i = \mathcal{F}'_i \in \mathbb{R}^{n \times n}, i = 0, \dots, m$ are given. SDPs are convex optimization problems and can be solved in polynomial-time with e.g. primal-dual interior-point methods [NN:94, VB:96, PS:95, KSS:95, AHO:98]. SDPs include linear programs and convex constrained programs, and arise in a wide range of engineering applications, see e.g. [BEFB:94, A:95, VB:96].

2.3 Construction of an equivalent SDP-problem

We can build an SDP problem which solves the problem of computing u such that (2)-(4) are satisfied using the following:

Theorem 1 *We can build a vector $c_{\mathcal{F}}$ and an affine function $\mathcal{F}$ such that the SDP problem (5) and the discrete-time problem (1)-(4) are equivalent.*

Proof : Let for all $0 \le k \le T$, $z_k = (y'_k\ u'_k)'$, $\overline{z}_k = (\overline{y}'_k\ \overline{u}'_k)'$, $\underline{z}_k = (\underline{y}'_k\ \underline{u}'_k)'$ and the matrices $\tilde{C} = (C'0)'$, $\tilde{D} = (D'I)'$. We can rewrite the problem as an equivalent form:

Find a control sequence $u_0, \ldots, u_T$ for the discrete-time dynamical system:

$$\forall k = 0, \ldots, T, \begin{cases} x_{k+1} &= Ax_k + Bu_k \\ z_k &= \tilde{C}x_k + \tilde{D}u_k \end{cases}$$

such that the output z satisfies the constraint:

$$\forall k = 0, \ldots, T, \underline{z}_k \le z_k \le \overline{z}_k\ . \tag{6}$$

Therefore we obtain a discrete-time dynamical system such that all constraints have being translated into the output z_k. Suppose that the vectors $(z_0, \ldots, z_T)$ are in the vector space $\mathbb{R}^{n_z}$. Define the integer $n_Z := (T+1)n_z$. We can write the constraints (6) as:

$$\mathtt{diag}\left(\begin{pmatrix} z^i_{rk} & z^i_k - z^i_{ck} \\ z^i_k - z^i_{ck} & z^i_{rk} \end{pmatrix} \begin{matrix} 1 \le i \le n_z \\ 0 \le k \le T \end{matrix}\right) \ge 0$$

where z^i_k is the i-th component of z_k and z_{rk} and z_{ck} define respectively the radius and the center of the polytope for the constraints. More precisely they are defined, for all by $0 \le k \le T$, by $z_{rk} = \frac{\overline{z}_k - \underline{z}_k}{2}$, $z_{ck} = \frac{\overline{z}_k + \underline{z}_k}{2}$. Let the generalized output vector Z, the generalized radius vector, the generalized center vector, the generalized state vector X and the generalized input vector U be defined respectively by $Z = (z_0, \ldots, z_T)'$, $Z_r = (z_{r0}, \ldots, z_{rT})'$, $Z_c = (z_{c0}, \ldots, z_{cT})'$, $X = (x_1, \ldots, x_{T+1})'$, $U = (u_0, \ldots, u_T)'$. We obtain $Z = M * U + Nx_0$, where M and N are defined respectively by $M = P * \mathtt{diag}(B) + \mathtt{diag}(\tilde{D})$, $N = (\tilde{C}', 0, \ldots, 0)' + P * (A', 0, \ldots, 0)'$, with

$$P = \begin{pmatrix} 0 & 0 & & & \\ \tilde{C} & \ddots & \ddots & & \\ & & \ddots & \ddots & 0 \\ & & & \tilde{C} & 0 \end{pmatrix} \begin{pmatrix} I & 0 & & & \\ -A & \ddots & \ddots & & \\ & & \ddots & \ddots & 0 \\ & & & -A & I \end{pmatrix}^{-1} .$$

The generalized output vector Z evolves in the vector space $\mathbb{R}^{n_Z}$ and define $(e_i)_{i \in [1, n_Z]}$ the canonical basis of the vector space $\mathbb{R}^{n_Z}$. So the problem (2)-(4) is equivalent to the following one:

Find U such that: $\mathcal{F}(U) \ge 0$, with the objective minimize $c'_{\mathcal{F}}U$, where $c_{\mathcal{F}} = (c', \ldots, c')'$ and $\mathcal{F}(U)$ is defined by

$$\mathcal{F}(U) = \mathtt{diag}_{i \in [1, n_Z]}$$

$$\begin{pmatrix} e'_i Z_r & e'_i(M * U + Nx_0 - Z_c) \\ e'_i(M * U + Nx_0 - Z_c) & e'_i Z_r \end{pmatrix} \tag{7}$$

So we have rewritten the problem (2)-(4) in a classical SDP-problem (5) in variable U; we can compute the generalized input-vector U solving only one SDP. □

Remark 2.1 We may use an algorithm which uses the structure of the affine function $\mathcal{F}$. Indeed $\mathcal{F}$ has a sparse form.

Moreover it is preferable to compute the control sequence $(u_0,\dots,u_T)$ in one step, like proposed in the proof, rather than to compute the control step by step, i.e. to build a SDP problem to compute u_0 and then another SDP problem to compute u_1 and so on. Indeed few number of steps can lead to an infeasible problem even though the global problem may be feasible. ◇

2.4 An example

Consider a second-order, continuous-time controlled system. We suppose that the data of this system have nominal values:

$$\left\{\begin{array}{rcl} x'' + a_1^{nom}x' + a_2^{nom}x &=& a_2^{nom}u \\ y &=& x \end{array}\right. \tag{8}$$

where the real parameters have the nominal value a_i^{nom}, $i = 1{,}2$. We want to find $u(t)$ such that, for all $t \geq 0$, we have $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$ and such that the output of the system (8) satisfies, for all $t \geq 0$, $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$. Note that we could take a higher order of constraint like a constraint on the derivative of x. By discretizing this system using a forward-Euler scheme with sampling frequency h, we obtain a system of the form (1)-(3):

Find a control sequence $(u_0,\cdots,u_T)$ *such that, for all* $0 \leq t \leq T$, $\underline{u}_k \leq u_k \leq \overline{u}_k$ *and such that the output* $(y_0,\cdots y_T)$ *of the two dimensional discrete-time dynamical system defined by (1) verifying the inequalities, for all* $0 \leq k \leq T$, $\underline{y}_k \leq y_k \leq \overline{y}_k$, *where*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 1 & h & 0 \\ -ha_2^{nom} & -ha_1^{nom} & ha_2^{nom} \\ \hline 1 & 0 & 0 \end{array}\right),$$

and where the constraints are defined, for $0 \leq k \leq T$, *by* $\underline{y}_k = \underline{y}(kh)$, $\overline{y}_k = \overline{y}(kh)$, $\underline{u}_k = \underline{u}(kh)$ *and* $\overline{u}_k = \overline{u}(kh)$.

Consider the open loop response to the constant input $u = 1$ with initial condition $x_0 = (0{,}0)'$. For $h = 0.1$, $a_1^{nom} = 3$, $a_2^{nom} = 9$, a time horizon $T = 50$ steps, Figure 1 shows the output y_k (i.e. the state x of (8)) obtained with the constant input $u = 1$.

We observe that it converges very slowly to the equilibrium $x = 1$. We can look for a new control such that the state converges to the equilibrium in less than 10 iterations of (1). More precisely, let the following problem:

Find a control input u *such that, for all* $t \geq 0$, $-10 \leq u(t) \leq 10$, *and such that the state* x *in (8) verifies, for all* $0 \leq t \leq 4h$, $0 \leq x(t) \leq 2$, *for all* $4h \leq t \leq 7h$, $0.4 \leq x(t) \leq 1.6$, *for all* $t \geq 7h$, $0.9 \leq x(t) \leq 1.1$.

The equivalent discrete-time problem is the following:

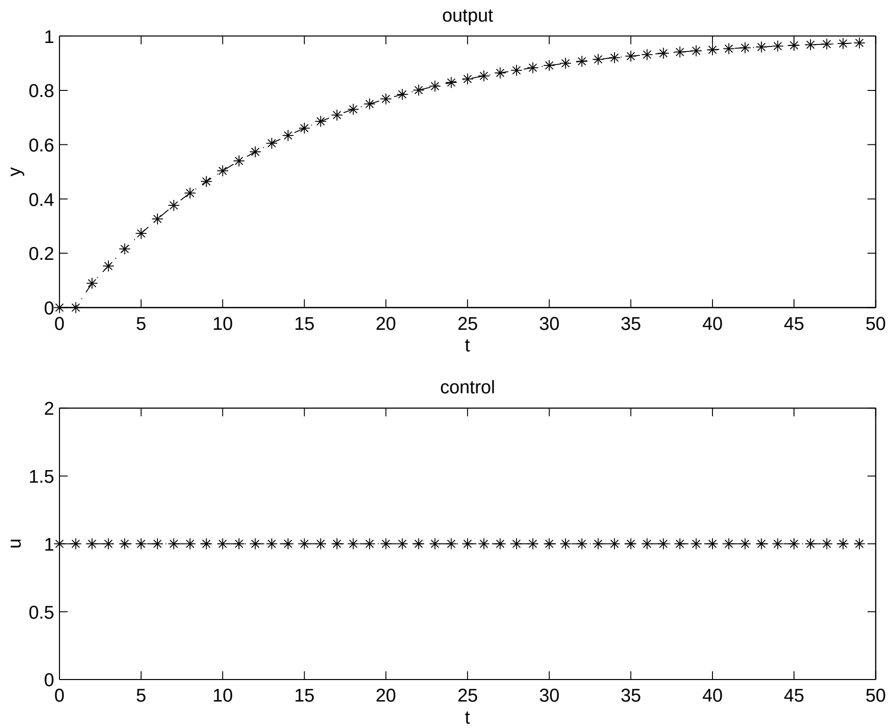

FIG. 1 – *The output of the discrete dynamical system without uncertainty for the input* $u = 1$.

Find a control sequence $(u_0, \ldots, u_T)$ *such that*

$$\forall k = 0, \ldots, T, \ -10 \leq u_k \leq \ 10 \quad , \tag{9}$$

and such that the output of the system (1) satisfies:

$$\forall k = 0, \ldots, 3, \qquad 0 \ \leq y_k \leq \ 2 \quad , \tag{10}$$

$$\forall k = 4, \ldots, 6, \quad 0.4 \ \leq y_k \leq \ 1.6 \quad , \tag{11}$$

$$\forall k = 7, \ldots, T, \quad 0.9 \ \leq y_k \leq \ 1.1 \quad . \tag{12}$$

We build the affine function $\mathcal{F}$ as shown in the proof of Theorem 1 and solve the corresponding SDP problem to obtain the input u and the output y satisfying (9)-(12) for a horizon $T = 9$ steps as shown in Figure 2. Note that the initial value u_0 is large with respect to the others values. Suppose that we have a transition between two equilibria at $t = 0$, more precisely suppose that for the non positive time t we have $u \equiv 0$ and so the equilibrium $x \equiv 0$, and that we want to reach the other equilibrium $x \equiv 1$ for the non negative time. So it may be essential to constraint the control sequence $(u_0, \ldots, u_T)$ to have small variations and the initial value u_0 to be small. This leads to the following new constraint:

$$|u_0| < maxvar \tag{13}$$

$$\forall k = 0, \cdots, T-1, \ |u_{k+1} - u_k| < maxvar \ , \tag{14}$$

where $maxvar$ is a new parameter which makes the control smoother and the value of $u(0)$ smaller. In Figure 2, the maximal variation ($maxvar$) of u is 6.5. But we can compute a new input u even smoother such that the maximal variation of u is less than $maxvar = 0.8$. We can build (see proof of Theorem 1) an affine function $\mathcal{F}$ so that the problem (9)-(12), (13)-(14) with $maxvar = 0.8$ and the SDP problem (5) are equivalent. Figure 3 shows the result of the computation of a possible control. Note that this new constraint makes the output much smoother.

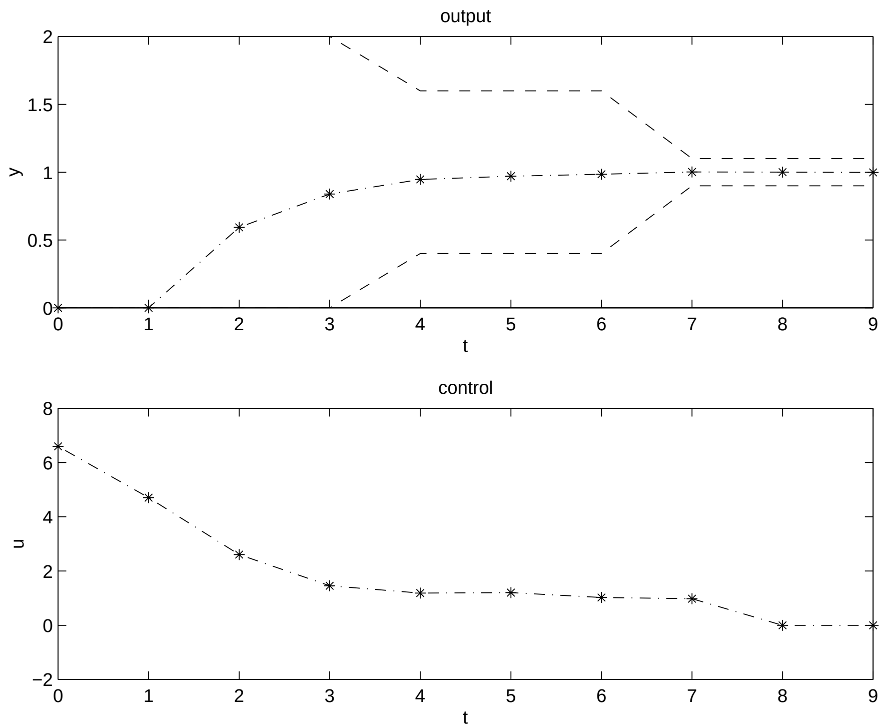

FIG. 2 – *The output and the optimal input of the discrete dynamical system.*

3 Robust SDP approach

3.1 A discrete-time system subject to structured uncertainty

In this paper, we will consider uncertain systems modelled as, for all $0 \leq k \leq T$,

$$\left\{ \begin{array}{rcl} x_{k+1} & = & \mathbf{A}(\Delta_k)x_k + \mathbf{B}(\Delta_k)u_k \\ y_k & = & \mathbf{C}(\Delta_k)x_k + \mathbf{D}(\Delta_k)u_k \end{array} \right. \tag{15}$$

where Δ_k is a (possible time-varying) uncertain matrix. We want to compute a control sequence $u = (u_0, \ldots, u_T)'$ such that the output of the system (15) satisfies the constraints:

$$\forall k = 0, \ldots, T,\ \underline{y}_k \leq y_k \leq \overline{y}_k\ , \tag{16}$$

and the control satisfies the following constraints:

$$\forall k = 0, \ldots, T,\ \underline{u}_k \leq u_k \leq \overline{u}_k\ , \tag{17}$$

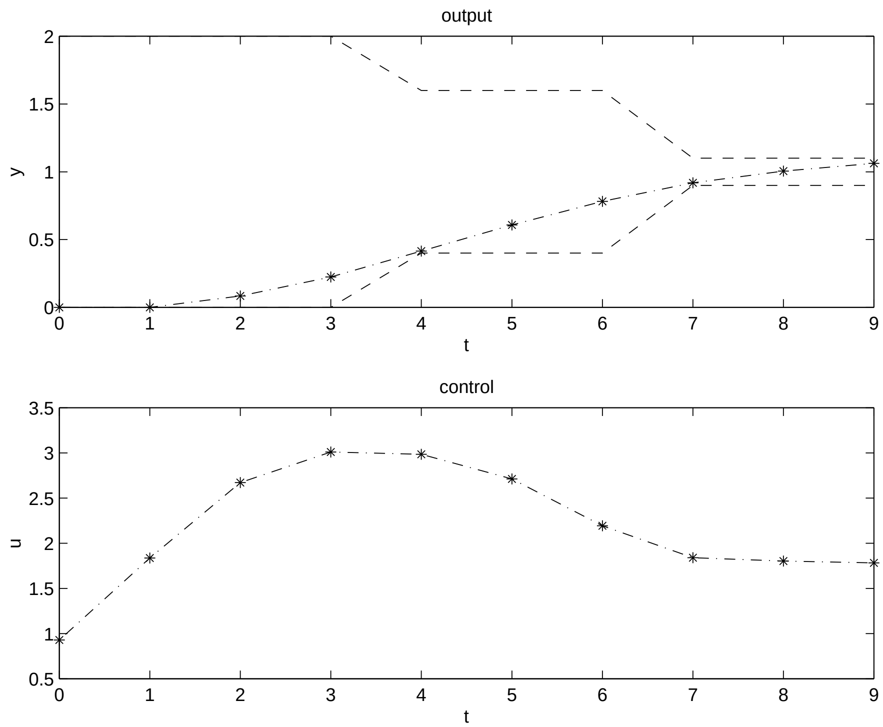

FIG. 3 – *The output and the optimal "smoother" input of the discrete dynamical system.*

with the objective:

$$\texttt{minimize}\ c'u\ . \tag{18}$$

where the time horizon T, the initial state x_0, the constraints $\underline{y}_k$,$\overline{y}_k$,$\underline{u}_k$,$\overline{u}_k$ and the objective c are given. We assume that the matrix-valued functions $\mathbf{A}(\Delta)$, $\mathbf{B}(\Delta)$, etc, are given by a *linear-fractional representation* (LFR):

$$\begin{pmatrix} \mathbf{A}(\Delta) & \mathbf{B}(\Delta) \\ \mathbf{C}(\Delta) & \mathbf{D}(\Delta) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} + L\Delta(I - H\Delta)^{-1}R$$

where A,B,C,D,L,R and H are constant matrices, while $\Delta \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is a bounded set of matrices. The subspace $\mathbf{\Delta}$ is called the structure set and defines the structure of the perturbation which must be norm-bounded. Together, the matrices A,B,C,D,L,R,H and $\mathbf{\Delta}$, constitute a *linear-fractional representation* (LFR) of our uncertain system. In this paper we will assume that the LFR is well-posed over $\mathbf{\Delta}$ meaning that we have $\forall\Delta \in \mathbf{\Delta}$, $\texttt{det}(I - H\Delta) \neq 0$. As it turns out, this is easy to cope with in our context, since our system is actually in closed loop with a stabilizing controller that will guarantee all external disturbances are asymptotically rejected. For the uncertainty set $\mathbf{\Delta}$, we can consider a very large class of matrix sets (see examples in [BEN:00]) but here we fix the class of sets $\mathbf{\Delta}$ to clarify the exposition. More precisely we consider only *ellipsoidal uncertainty*. This means that the perturbation consists of block vectors, each being subject to an Euclidean-norm bound, more precisely

$$\mathbf{\Delta} = \{\texttt{diag}(\delta_1 I_{r_1}, \cdots, \delta_l I_{r_l}),\ (\delta_1, \ldots, \delta_l)' \in \mathcal{D}\}\ , \tag{19}$$

where $\mathcal{D}$ is equal to

$$\left\{ \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_l \end{pmatrix} \in \mathbb{R}^l \,|\, \delta = \begin{pmatrix} \delta^1 \\ \vdots \\ \delta^N \end{pmatrix}, \begin{array}{c} \delta^k \in \mathbb{R}^{n_k}, \|\delta^k\|_2 \leq \rho, \\ 1 \leq k \leq N \end{array} \right\} ,$$

where $\rho \geq 0$ is a given parameter that determines the "size" of the uncertainty, the integers n_k denote the lengths of each block vector δ_k (we have of course $n_1 + \cdots + n_N = l$) and $r_1, \ldots, r_l$ are given integers.

3.2 The robust SDP problem

We consider here a robust semidefinite programming problem of the form:

$$\texttt{minimize } c_{\mathbf{F}}'\xi \text{ subject to } \mathbf{F}(\xi,\Delta) \geq 0 \ , \tag{20}$$

where $c_{\mathbf{F}} \in \mathbb{R}^m - \{0\}$, Δ is a "perturbation matrix" that is only known to belong to a set $\mathbf{\Delta_F}$, and $\mathbf{F}$ is a map from $\mathbb{R}^n \times \mathbf{\Delta_F}$ to $\mathcal{S}^n$, the set of the symmetric matrices in $\mathbb{R}^{n\times n}$. We assume that $\mathbf{F}$ is given by a "linear-fractional representation" (LFR):

$$\begin{array}{rcl} \mathbf{F}(\xi,\Delta) & = & F(\xi) + L_{\mathbf{F}}(\xi)\Delta(I - H_{\mathbf{F}}\Delta)^{-1}R_{\mathbf{F}} \\ & & +R_{\mathbf{F}}'(I - \Delta' H_{\mathbf{F}}')^{-1}\Delta' L_{\mathbf{F}}(\xi)' \ , \end{array} \tag{21}$$

where F is an affine function like (5), $L_{\mathbf{F}}(.)$ is an affine mapping taking values in $\mathbb{R}^{n\times p}$, $R_{\mathbf{F}} \in \mathbb{R}^{q\times n}$ and $H_{\mathbf{F}} \in \mathbb{R}^{q\times p}$ are given matrices, while $\Delta \in \mathbf{\Delta_F}$, where $\mathbf{\Delta_F}$ is a bounded set of matrices. We will assume that the LFR is well-posed over $\mathbf{\Delta}$ meaning that we have $\forall \Delta \in \mathbf{\Delta}$, $\texttt{det}(I - H_{\mathbf{F}}\Delta) \neq 0$. We consider only SDP problem with *ellipsoidal* perturbation sets $\mathbf{\Delta_F}$ equal to

$$\{\texttt{diag}(\delta_1 I_{r_{1\mathbf{F}}}, \cdots, \delta_{l_{\mathbf{F}}} I_{r_{l_{\mathbf{F}}\mathbf{F}}}), (\delta_1, \ldots, \delta_{l_{\mathbf{F}}})' \in \mathcal{D}_{\mathbf{F}}\} \ , \tag{22}$$

where $\mathcal{D}_{\mathbf{F}}$ is equal to

$$\left\{ \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_{l_{\mathbf{F}}} \end{pmatrix} \in \mathbb{R}^{l_{\mathbf{F}}} \mid \delta = \begin{pmatrix} \delta^1 \\ \vdots \\ \delta^{N_{\mathbf{F}}} \end{pmatrix}, \begin{array}{c} \delta^k \in \mathbb{R}^{n_{k\mathbf{F}}}, \\ \|\delta^k\|_2 \leq \rho_{\mathbf{F}}, \\ 1 \leq k \leq N_{\mathbf{F}} \end{array} \right\} ,$$

where $\rho_{\mathbf{F}} \geq 0$ is a given parameter that determines the "size" of the ellipsoidal uncertainty in the function $\mathbf{F}$, the integers $n_{k\mathbf{F}}$ denote the lengths of each block vector δ_k (we have of course $n_{1\mathbf{F}} + \cdots + n_{N_{\mathbf{F}}\mathbf{F}} = l_{\mathbf{F}}$) and $r_{1\mathbf{F}}, \ldots, r_{l_{\mathbf{F}}\mathbf{F}}$ are given integers. But the following Theorem 2 is available in a large class of set $\mathbf{\Delta}$ (see [BEN:00]). Let J_k be the set of indices $\nu_{k-1}+1, \ldots, \nu_k$ with $\nu_0 = 0$, $\nu_k = \sum_{i=1}^{k} n_{i\mathbf{F}}$. The following theorem can be shown with Lagrangian relaxations techniques see [BEN:00]:

Theorem 2 *Consider the uncertain semidefinite program with rational perturbation described by the LFR (21), where the perturbation matrix Δ lies in the set $\mathbf{\Delta_F}$ defined by (22) and assume the LFR is well-posed over $\mathbf{\Delta_F}$. Consider the semidefinite program*

$$\begin{array}{c} \texttt{minimize } c_{\mathbf{F}}'\xi \text{ subject to } S \geq 0, \\ \begin{pmatrix} F(\xi) & L_{\mathbf{F}}(\xi) \\ L_{\mathbf{F}}(\xi)' & 0 \end{pmatrix} \geq \begin{pmatrix} R_{\mathbf{F}} & H_{\mathbf{F}} \\ 0 & I \end{pmatrix}' \\ * \begin{pmatrix} T & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} R_{\mathbf{F}} & H_{\mathbf{F}} \\ 0 & I \end{pmatrix} , \end{array}$$

where $S = \mathtt{diag}(S_1,\dots,S_{N_{\mathbf{F}}})$, with each S_k of size $\sum_{i\in J_k} r_{i\mathbf{F}}$, and T is the block-diagonal matrix formed with the block-diagonal $r_{i\mathbf{F}}\times r_{i\mathbf{F}}$ blocks of S. Then the above semidefinite program in variables ξ,S is an approximation of the robust counterpart (20), i.e. the projection of the feasible set of (20) on the space of ξ-variables is contained in the set of robust feasible solutions.

3.3 Construction of a SDP-problem solving the problem subject to structured uncertainty.

We have Theorem 1 in the following *robust* form:

Theorem 3 *We can build a vector $c_{\mathcal{F}}$ and an affine function $\mathcal{F}$ such that the SDP problem (5) and the discrete-time problem (16)-(18) are equivalent.*

Proof : Following the proof of Theorem 1 to compute the LFR of the *generalized output matrix* of the equation (7), we obtain an SDP problem which is given by an LFR depending of the *generalized input vector* U defined by $U = (u_0',\dots,u_T')'$. So we can make a Lagrangian relaxation and use Theorem 2 to build an SDP problem (5) equivalent to (16)-(18). □

3.4 An example

We consider now a second-order, continuous-time controlled system with uncertainty in the data:

$$\left\{ \begin{array}{rcl} x'' + \mathbf{a}_1(t)x' + \mathbf{a}_2(t)x &=& \mathbf{a}_2(t)u \\ y &=& x \end{array} \right. \tag{23}$$

where the uncertain, time-varying parameters $\mathbf{a}_i$, $i = 1,2$ are subject to bounded variation of given relative amplitude ρ, more precisely $\mathbf{a}_i(t) = a_i^{nom}(1 + \rho\delta_i(t))$, $i = 1,2$, $t \geq 0$, where $-1 \leq \delta_i(t) \leq 1$ for every t, and a_i^{nom}, $i = 1,2$, is the nominal value of the parameters. We want to find $u(t)$ such that, for all $t \geq 0$, $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, and such that the output of the system (23), for any unknown function $\delta_i(t)$, $i = 1,2$ such that $-1 \leq \delta_i(t) \leq 1$ for every t, satisfy, for all $t \geq 0$, $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$. By discretizing this system using a forward-Euler scheme with discretization period h, we obtain a system of the form (15), with the following LFR:

$$\left(\begin{array}{c|c} \mathbf{A}(\Delta) & \mathbf{B}(\Delta) \\ \hline \mathbf{C}(\Delta) & \mathbf{D}(\Delta) \end{array} \right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) + L\Delta R \ ,$$

where $\Delta = \mathtt{diag}(\delta_1,\delta_2)$ and

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left(\begin{array}{cc|c} 1 & h & 0 \\ -ha_2^{nom} & -ha_1^{nom} & ha_2^{nom} \\ \hline 1 & 0 & 0 \end{array} \right) ,$$

$$L = -h\rho \left(\begin{array}{cc} 0 & 0 \\ a_1^{nom} & a_2^{nom} \\ \hline 0 & 0 \end{array} \right), R = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & -1 \end{array} \right).$$

With this choice of C and D, we have $y_k = y(kh)$ and we take, for all $0 \leq k \leq T$, $\underline{y}_k = \underline{y}(kh)$, $\overline{y}_k = \overline{y}(kh)$. We look for a control such that the state converges to the equilibrium in less than 10 iterations of (15). More precisely, let define the following problem:

Find a controller u such that, for all $t \geq 0$, $-10 \leq u(t) \leq 10$ and such that the state x in (23) verifies, for all $0 \leq t \leq 4h$, $0 \leq x(t) \leq 2$, for all $4h \leq t \leq 6h$, $0.4 \leq x(t) \leq 1.6$ and, for all $t \geq 6h$, $0.7 \leq x(t) \leq 1.3$.

The discrete-time equivalent problem is the following:

Find a control sequence $(u_0, \ldots, u_T)$ such that the output of the system (23) satisfies:

$$\forall k = 0, \ldots, 3, \quad (0;\ -10)' \leq y_k \leq (2;\ 10)' \ , \tag{24}$$

$$\forall k = 4,5, \quad (0.4;\ -10)' \leq y_k \leq (1.6;\ 10)' \ , \tag{25}$$

$$\forall k = 6, \ldots, T, \quad (0.7;\ -10)' \leq y_k \leq (1.3;\ 10)' \ . \tag{26}$$

We can build as in the proof of Theorem 3 an affine function $\mathcal{F}$ so that the problem (24), (25) and (26) and the SDP problem (5) are equivalent, and then we solve the corresponding SDP problem to obtain the control sequence $(u_0, \ldots, u_T)$ and the output y satisfying (24), (25) and (26) for a time horizon $T = 7$ steps as shown in Figure 4. Note that the initial value u_0 is large with respect to the others values. As explained in the nominal case in section 2.4, it may be essential to add the new constraints (13)-(14) where $maxvar$ is a new parameter which makes the control sequence smoother and the value of $u(0)$ smaller. In Figure 4, the maximal variation ($maxvar$) of u is 4.5. But we can compute a new control sequence $(u_0, \ldots, u_T)$ even smoother such that the maximal variation is less than $maxvar = 1$. We can build as in the proof of Theorem 3 an affine function $\mathcal{F}$ so that the problem (24)-(26), (13)-(14) with $maxvar = 1$ and the SDP problem (5) are equivalent. The figure 5 shows the result of the computation of a possible control. Note that this new constraint makes the output much smoother.

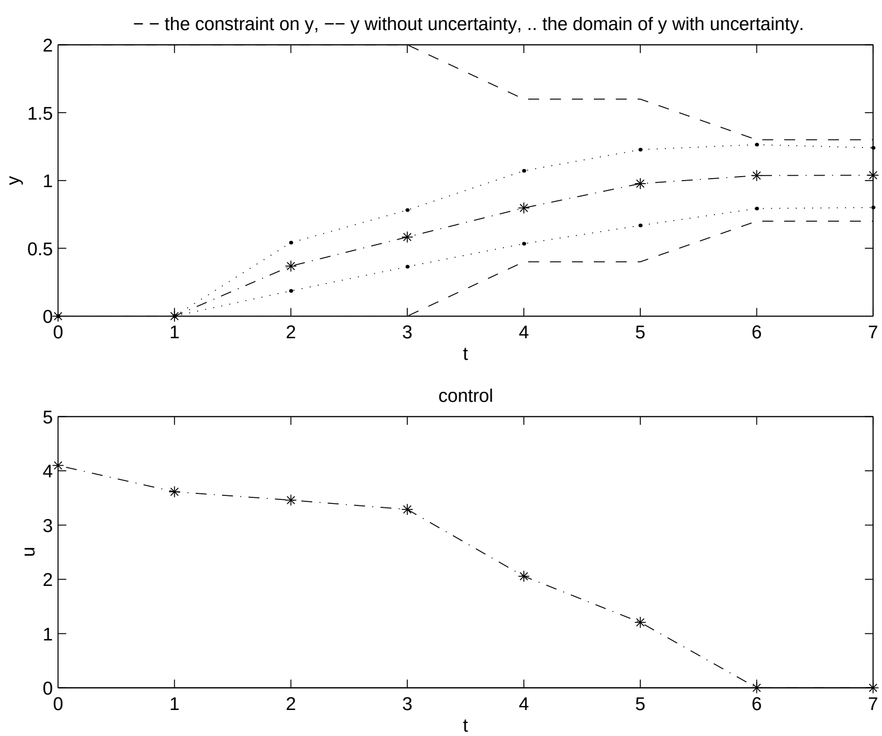

FIG. 4 – *The output and the optimal input of the discrete dynamical system with uncertainty.*

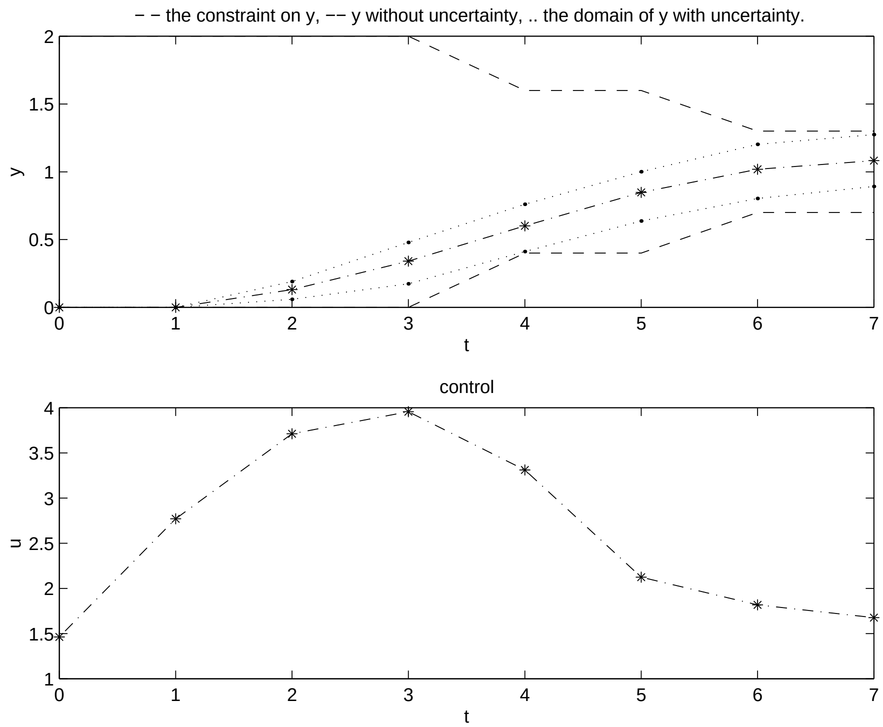

Fig. 5 – *The output and the optimal "smoother" input of the discrete dynamical system with uncertainty.*

4 Conclusion

An optimization-based control technique has been developed to defining feasible input trajectories with respect to operating constraints. This constrained problem is cast in an equivalent semidefinite programming (SDP) problem. When the system is subject to structured uncertainty, the nominal SDP problem is extended to account for uncertain parameters evolving in a certain ellipsoidal set. To illustrate the use of this technique, a constrained control problem involving a second order system is solved numerically with and without structured uncertainty. The simulations show that the constraints are satisfied. Furthermore, the results demonstrate that the control can be quite smooth when an additional constraint is used. This could be a crucial aspect from a practical prospective. This technique seems more appropriate to predict trajectories over a long range horizon. In the latter case, the computation is completed off line, but measurement cannot be used and a feedback controller is needed to guarantee stability along the trajectory. An application of such approach using the proposed technique is currently underway [BPFE:00].

An LMI Approach to Optimize References for a Fossil Power Plant

P. Bendotti [1] , C. Prieur [2] , C.-M. Falinower [3] and L. El-Ghaoui [4]

Research report, 2000.

Abstract: An optimization-based control technique is used to defining feasible reference trajectories with respect to operating constraints assuming a local controller can stabilize the system along the trajectory. This technique is applied to a fossil power plant (FPP) to optimize the references in electrical power, pressure, and superheat temperature.
Keywords: Optimization, LMI, reference trajectory, constrained control, input and output constraints, power plant.

1 Introduction

In the classical control design framework, control synthesis, reference trajectory and operating under constraints are three different problems that are solved independently. An appropriate control strategy should deal with the dynamical behaviour of the plant but also with operating conditions. A possible approach is to use model predictive control which has been widely used for tracking problems subject to constraints (see [GPM:89] for a comprehensive survey and [KBM:95] for technical details). In this paper, the intend is to derive reference trajectories satisfying operating constraints over long range horizons using optimization techniques. To this end, the computation is completed off line and feedback on measurement is not available. Since the adverse effect of disturbances cannot be accounted for, it is reasonable to assume that a local controller is used to stabilize the system along the trajectory. Such stabilizing controller can be designed using robust control techniques to guarantee satisfactory closed loop performance and stability with respect to a set of plant uncertainty. Obviously, classical PID controllers can be used instead. From a practical point of view, it is often very convenient to keep original PID loops and redefine references using advanced control techniques. This approach is very common when dealing with a model predictive control technique. In the present case, the implementation is even easier since the computation is done off line.

In [PBE:00], a robust optimization-based control technique is presented. It is based on the construction of a semidefinite programming problem (SDP) written in terms of Linear Matrix Inequalities (LMI). SDPs are convex optimization problems that can be solved in polynomial-time with primal-dual interior-point methods [NN:94] (see also references given in [PBE:00]). SDPs include linear programs and convex constrained programs, and arise in a wide range of engineering applications [BEFB:94]. The technique presented in [PBE:00]

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has been developed to defining feasible input trajectories that satisfy operating constraints assuming a local stabilizing controller is available. This approach shares some properties with model predictive control. In particular, the proposed optimization technique is comparable to the first step of predictive control where the prediction is completed with no feedback on the actual measurement. This technique will be applied to define appropriate references for a fossil power plant (FPP).

Even though the major contribution to electricity production in France and certain other countries is provided by nuclear power, fossil power plants still have a key role to play. Indeed, some fossil power plants are always connected to the grid and are more maneuverable to respond to unit load demand. Power plants are complex dynamic systems that need sophisticated control systems to achieve maximum performance. Advanced control algorithms, which have the potential to provide better control performance, are still not widely used in power plant control. But the need is greater than ever. Competition, tighter environmental regulations, and life extension are imposing new constraints on power plant operation. Expected benefits include improved plant response to load regulation demands, less variability in critical plant parameters leading to better plant efficiency and less wear and tear, and better control performance over a wider load range.

In this paper, we focus on determining optimal references for a FPP. A robust multivariable controller has already been developed. This controller will be used to stabilize the plant along the optimal trajectory. Hence, the closed loop system will be considered in place of the plant. Thus, the design model is necessarily stable regardless of the plant itself. The approach used in this paper is presented in [PBE:00]. A FPP reference optimization problem is derived and cast in an SDP problem.

Section 2 reviews the construction of the SDP problem. Section 3 is devoted to the problem statement. Section 4 presents the solution to the SDP problem with simulation results. Some concluding remarks end the paper.

2 Reference Optimization Problem as an equivalent SDP Problem

The reference optimization problem is as follows:
Consider a stable closed-loop system in discrete-time

$$\forall k = 0, \dots, N, \; \begin{cases} x_{k+1} & = & Ax_k + By_k^\star \\ y_k & = & Cx_k + Dy_k^\star \end{cases} \tag{1}$$

where $y_k \in \mathbb{R}^p$ denotes the output vector, $y_k^\star \in \mathbb{R}^m$ the reference and $x_k \in \mathbb{R}^n$ the state at sample k. Note that the reference in this formulation is the input of the considered system.

Determine a sequence of reference $y^\star$ that satisfies the following objective

$$\texttt{minimize } c'y^\star \tag{2}$$

under the following constraints, for $k = 0, \dots, N$:

$$\left\{ \begin{bmatrix} \underline{y}_k \\ \underline{y}_k^\star \end{bmatrix} \leq \begin{bmatrix} y_k \\ y_k^\star \end{bmatrix} \leq \begin{bmatrix} \overline{y}_k \\ \overline{y}_k^\star \end{bmatrix} \right. , \tag{3}$$

where the time horizon N, the initial state x_0, the constraints $\underline{y}_k$,$\overline{y}_k$,$\underline{y}^\star_k$,$\overline{y}^\star_k$ and the objective c are given.

Remark 2.1 Note that $y^\star$ is a vector containing the reference sequence obtained from $y^\star_k$ for $k = 0,\dots,N$. In the sequel, a vector appearing with no subscript will denote the sequence rather than the signal taken at a sampling time. ◇

Remark 2.2 Since the reference $y^\star$ denotes the input of the system, both inputs and outputs are subject to constraints in this optimization formulation. ◇

Let $\forall k = 0,\dots,N$,

$$z_k = \begin{bmatrix} y_k \\ y^\star_k \end{bmatrix}, \quad \overline{z}_k = \begin{bmatrix} \overline{y}_k \\ \overline{y}^\star_k \end{bmatrix}, \quad \underline{z}_k = \begin{bmatrix} \underline{y}_k \\ \underline{y}^\star_k \end{bmatrix} .$$

Suppose that the vectors $(z_k)_{k\in[0,N]} \in \mathbb{R}^{n_z}$, thus $z \in \mathbb{R}^{nZ}$ with $n_Z := (N+1)n_z$. We can write the constraints (3) as:

$$\texttt{diag}\left(\begin{bmatrix} {z^r}^i_k & z^i_k - {z^c}^i_k \\ z^i_k - {z^c}^i_k & {z^r}^i_k \end{bmatrix}, i = 1,\dots,n_z, k = 0,\dots,N\right) \geq 0 ,$$

where z^i_k is the i-th component of z_k, while z^r_k and z^c_k define respectively the radius and the center of the polytope of the constraints. More precisely they are defined by:

$$z^r_k = \frac{\overline{z}_k - \underline{z}_k}{2} , \qquad z^c_k = \frac{\overline{z}_k + \underline{z}_k}{2} .$$

The problem defined by (1)-(3) could be rewritten as follows (see [PBE:00] for further details):

Find $y^\star$ such that:

$$\texttt{minimize } c'_{\mathcal{F}} y^\star \text{ subject to } \mathcal{F}(y^\star) \geq 0 , \tag{4}$$

where

$$\mathcal{F}(y^\star) = \texttt{diag}\left(\begin{bmatrix} e'_i z^r & e'_i(Uy^\star + Vx_0 - z^c) \\ e'_i(Uy^\star + Vx_0 - z^c) & e'_i z^r \end{bmatrix}, i \in [1,n_Z]\right) \tag{5}$$

with $(e_i)_{i\in[1,n_Z]}$ is the canonical basis of the vector space $\mathbb{R}^{nZ}$, and M and N satisfy $z = Uy^\star + Vx_0$ and are given as follows:

$$U = \begin{bmatrix} 0_{m+p,n} & & \dots & 0_{m+p,n} \\ C & \ddots & & \vdots \\ 0_{m,n} & \ddots & \ddots & \\ \vdots & \ddots & C & 0_{p,n} \\ 0_{m,n} & \dots & 0_{m,n} & 0_{m,n} \end{bmatrix} \begin{bmatrix} I_n & 0_n & & \dots & 0_n \\ -A & \ddots & \ddots & & \vdots \\ 0_n & \ddots & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0_n \\ 0_n & \dots & 0_n & -A & I_n \end{bmatrix}^{-1} \texttt{diag}(B) + \texttt{diag}\begin{pmatrix} D \\ I \end{pmatrix} ,$$

and

$$V = \begin{bmatrix} C \\ 0_{m,n} \\ 0_{m+p,n} \\ \vdots \\ \\ 0_{m+p,n} \end{bmatrix} + \begin{bmatrix} 0_{m+p,n} & & \dots & 0_{m+p,n} \\ C & \ddots & & \vdots \\ 0_{m,n} & & \ddots & \\ \vdots & \ddots & C & 0_{p,n} \\ 0_{m,n} & \dots & 0_{m,n} & 0_{m,n} \end{bmatrix} \begin{bmatrix} I_n & 0_n & & \dots & 0_n \\ -A & \ddots & \ddots & & \vdots \\ 0_n & \ddots & & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0_n \\ 0_n & \dots & 0_n & -A & I_n \end{bmatrix}^{-1} \begin{bmatrix} A \\ 0_n \\ \vdots \\ \\ 0_n \end{bmatrix}$$

and

$$c_{\mathcal{F}} = \begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} . \tag{6}$$

where I_n denotes the $n \times n$ identity matrix and $O_{n,m}$ denotes the $n \times m$ zero matrix, if $n = m$ the second subscript is omitted.

The reference optimization problem has been cast in a classical SDP-problem that can be efficiently solved using available softwares.

3 Problem Statement

The main objective in controlling a fossil power plant is to improve the plant response to load regulation demands. The steam pressure and the superheat temperature are also needed to determine operating conditions. Indeed, the steam that drives the turbo-alternator generates electricity. Obviously, the steam should have its temperature and pressure within allowable limits. The generated electricity is expected to follow the dispatch load demand.

3.1 FPP Reference Optimization Problem

The objective of this optimization problem is to determine the sequence of reference for the electrical power, the pressure and the temperature such that the plant response to load regulation demand is improved, and the pressure and the temperature lie within allowable limits.

The three outputs are the electrical power W, the steam pressure P and the superheat temperature T. The control specification is that W follows the load demand W_{dem} as much as possible while P and T lie within allowable limits. The allowable region of evolution is described as follows

$$\begin{bmatrix} \underline{W}_k \\ \underline{P}_k \\ \underline{T}_k \end{bmatrix} \leq \begin{bmatrix} W_k \\ P_k \\ T_k \end{bmatrix} \leq \begin{bmatrix} \overline{W}_k \\ \overline{P}_k \\ \overline{T}_k \end{bmatrix} \quad \forall k \in [0, N] \tag{7}$$

$$|W_k - W_{\text{dem}k}| \leq \varepsilon_{W k} \tag{8}$$

where ε_W denotes thresholds characterizing a prescribed error, and $\underline{\bullet}$ denotes the lower limit and $\overline{\bullet}$ the upper limit. Note that Equation 8 specifies the performance objective, *i.e.* W should be close to the load demand.

The reference temperature $T^\star$ and the reference pressure $P^\star$ and the reference power $W^\star$ should be maintained within specified bounds. The preferred region of evolution is given as follows

$$\begin{bmatrix} \underline{W}^\star_k \\ \underline{P}^\star_k \\ \underline{T}^\star_k \end{bmatrix} \leq \begin{bmatrix} W^\star_k \\ P^\star_k \\ T^\star_k \end{bmatrix} \leq \begin{bmatrix} \overline{W}^\star_k \\ \overline{P}^\star_k \\ \overline{T}^\star_k \end{bmatrix} \quad \forall k \in [0, N] \tag{9}$$

where $\bullet^\star$ denote the sequence of reference to be optimized.

Another requirement is that the rate-of-change in the sequence of reference should be "reasonable". This could be translated into the following inequality:

$$\left| \begin{bmatrix} W^\star_{k+1} \\ P^\star_{k+1} \\ T^\star_{k+1} \end{bmatrix} - \begin{bmatrix} W^\star_{k} \\ P^\star_{k} \\ T^\star_{k} \end{bmatrix} \right| \leq \begin{bmatrix} \varepsilon_{W^\star_k} \\ \varepsilon_{P^\star_k} \\ \varepsilon_{T^\star_k} \end{bmatrix} \quad \forall k \in [0,\, N] \tag{10}$$

All constraints for the FPP are given in the inequalities (7), (8), (9) and (10) that can be written in the form of (3).

3.2 Design Model for FPP Reference Optimization

In this paper, we consider a linearized plant model in closed-loop with a robust multivariable controller as shown in Figure 1.

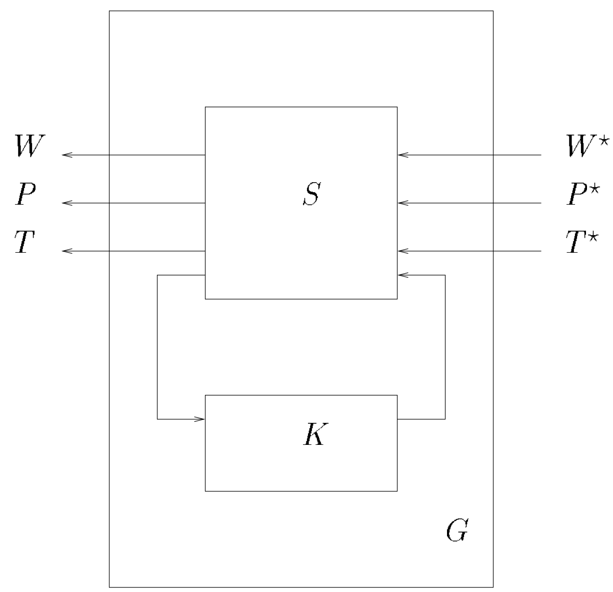

FIG. 1 – *Design Model*

For the optimization problem to be tractable, the state dimension of the closed-loop model is reduced using balanced truncation techniques [M:81]. Furthermore, the optimization technique results in reference sequences over a finite horizon. In fact, it is advisable to extend this horizon as much as possible since a feasible SDP problem with a large horizon could lead to infeasible SDP subproblems with shorter horizons. Therefore, the sampling frequency has been increased from that used for the multivariable controller. This is quite reasonable as the rate-of-variation of a reference signal is naturally lower compared to that of a control input signal. As stated earlier in the introduction, this is a very common practice, especially when dealing with model predictive control.

Let S denote the linearized plant model and K the multivariable controller, the closed loop model can be written as follows:

$$G = S \star K$$

where "$\star$" denotes the Redheffer star product (see *e.g.* [ZDG:95] for more details). The considered design model obtained via balanced truncation has 13 states and is denoted by $\widehat{G}$. Note that a state-space realization of $\widehat{G}$ in the form of that given in (1) will be used, thus using a partitioned matrix it can be written as

$$\widehat{G}(\mathbf{z}) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{11}$$

where $\mathbf{z}$ denotes the $\mathcal{Z}$-transform variable.

The step-responses of $\widehat{G}$ are shown in Figure 2. Note that the responses appearing on the diagonal are unitary but are quite different from one another in terms of dynamical behaviour, *i.e.* the response in temperature is much slower and lighter damped compared to that in pressure and power. Moreover, the cross couplings are significant (see non diagonal responses) with a maximum variation about three times smaller than that relative to the main term (see diagonal responses). This makes the reference optimization non trivial, and indeed particularly useful.

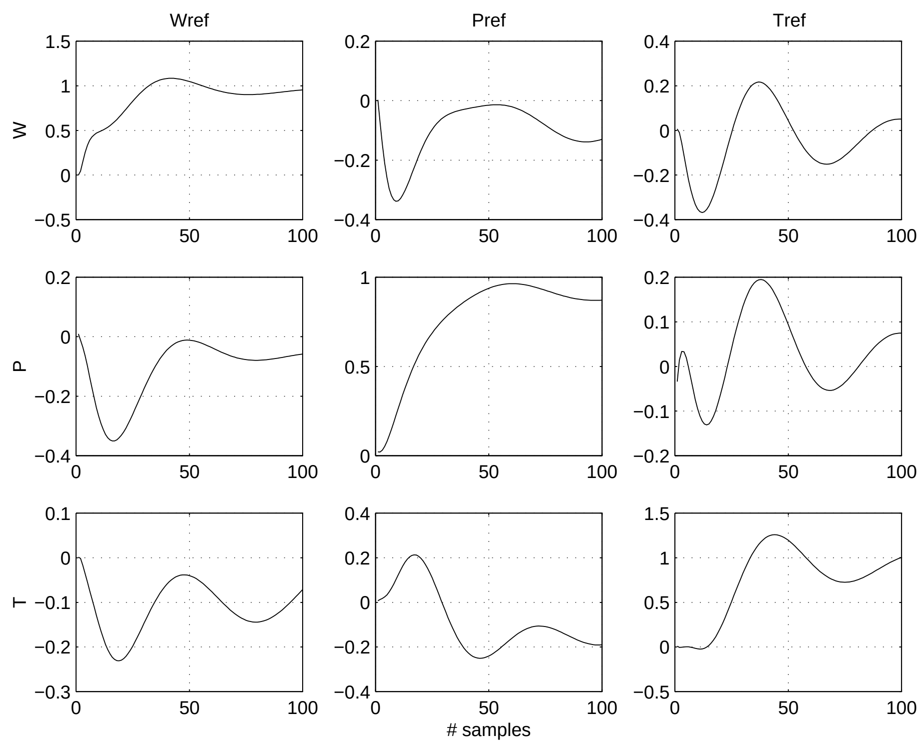

FIG. 2 – *Responses of* $\widehat{G}$ *to reference steps*

The reference optimization problem for the FPP as stated in (1), (2) and (3) can be derived using the model given in (11) and constraints in (7), (8), (9) and (10). This problem can be reformulated as an equivalent SDP problem given in (4) and (5).

4 Solving the SDP problem for a FPP

Consider the inequality constraints given in (7) and (9). All signals have been normalized such that they take value in percent between 0 and 100. In (8), the performance objective is also translated into a similar constraint where $\varepsilon_W = 1\%$. Note this constraint is the hardest to satisfy.

Two standard transients are considered for simulation: a -4% step at maximum load and a -4%/min ramp. For each transient, a first optimization is completed with constraints given in (7), (8) and (9). Recall a second type of constraint has been introduced in (10) to limit the rate-of-change in the optimized references. The maximum allowable variation is specified in the right hand side vector. This vector can be determined using the results from the first optimization. Hence a second optimization can be carried out taking into account both types of constraints (7)-(8)-(9) and (10).

4.1 Simulation results

Step changes in the load demand. A first optimization is completed as described previously in this section. It is worth noticing that the first step change cannot occur before the tenth iteration for the problem to be feasible. Indeed, if the prediction horizon is too short, there exist no allowable references, as defined in (9), for which the performance objective specified in (7)-(8) is satisfied. Moreover, the larger the step change, the longer the prediction horizon. The simulation results relative to this transient are shown in Figure 3. The allowable limits are shown in dashed lines while the actual outputs are in solid lines and the power demand is represented in dotted line. The optimized references used are shown in Figure 4. It appears that the optimization problem corresponding the first step change is very tight as the power is at its maximum admissible limit. For the second step change, the power is right between its allowable limits, thus following the power demand very closely. The resulting maximum rate-of-variation for each output is as follows:

$$\left| \begin{pmatrix} W_{i+1} \\ P_{i+1} \\ T_{i+1} \end{pmatrix} - \begin{pmatrix} W_i \\ P_i \\ T_i \end{pmatrix} \right| \leq \begin{pmatrix} 13 \\ 34 \\ 22 \end{pmatrix} \quad \forall k \in [0, 50] \tag{12}$$

The inequality (12) is used to determine the second type of constraint. The second optimization using both types of constraints (7)-(8)-(9) and (12) results in slightly smoother references in average while the maximum rate-of-variation is substantially reduced and satisfies now:

$$\left| \begin{pmatrix} W_{i+1} \\ P_{i+1} \\ T_{i+1} \end{pmatrix} - \begin{pmatrix} W_i \\ P_i \\ T_i \end{pmatrix} \right| \leq \begin{pmatrix} 4 \\ 11 \\ 14 \end{pmatrix} \quad \forall k \in [0, 50]$$

Figure 5 shows the plant response to such a step change. The allowable limits are shown in dashed lines while the actual outputs are in solid lines. The optimized references used are shown in Figure 6. Note that the large variation of $W^\star$ observed in Figure 4 at the end of the sequence has been drastically reduced in Figure 6. However, the optimization problem is again tighter as an additional constraint has been considered. Indeed, the power is now close to either the upper limit or the lower limit, hence it is at the maximum allowable distance from the power demand (as specified by ϵ_W given in (8)).

Ramp on the load demand. Such a ramp calls for a larger horizon and results in smaller rate-of-change of the outputs. Hence the sampling frequency can be increased. More precisely, one optimization is completed every other sample, the resulting references are applied and held over two samples, thus $y^{\star}_{2k+1} = y^{\star}_{2k}$. As previously, a first optimization has been carried out to define an appropriate rate-of-variation for each output, thus determining the second type of constraint given in (10). The simulation results are shown in Figure 7 and 8. Note that performance specifications have been relaxed to help make the problem feasible. However, very basic templates have been used at this point. More sophisticated templates can be considered to translate the specifications into convex constraints.

4.2 System with uncertainty

The technique used in this paper has been extended to account for uncertain systems as demonstrated in [PBE:00]. Indeed, the approach has been applied to a second order system with parametric uncertainty (see [PBE:00]). For the FPP model considered in this paper, it was not possible to solve the SDP using `lmitool` for horizons larger than 5 sample times. To overcome this problem, the complexity of the algorithm needs to be reduced. One way is to use an algorithm that accounts for sparse matrices. Investigations are currently underway to overcome such limitations.

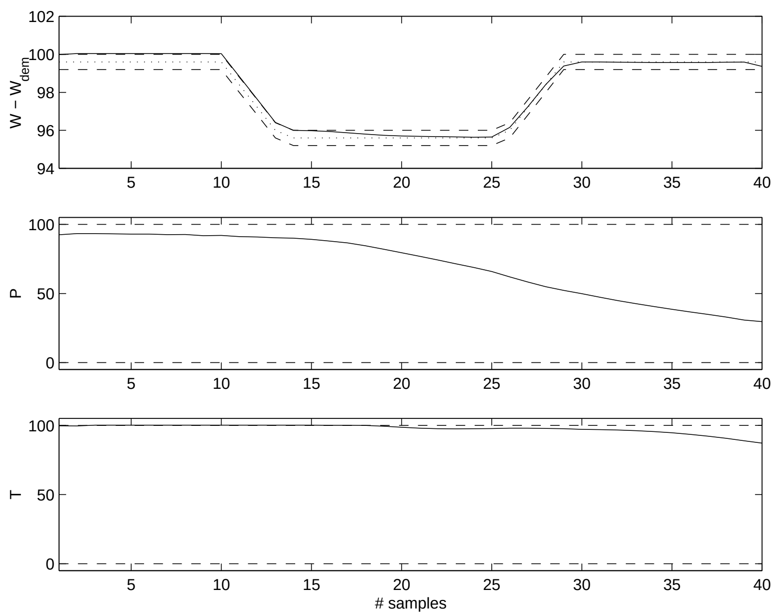

FIG. 3 – *Closed-loop responses of $\widehat{G}$ to $-/+4\%$ steps on load demand using optimal references satisfying first type of constraints*

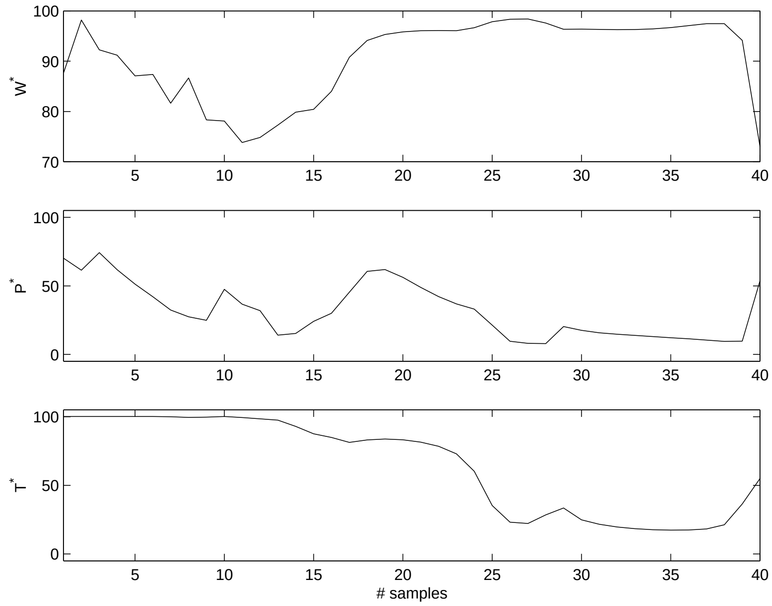

FIG. 4 – *Optimal references satisfying first type of constraints*

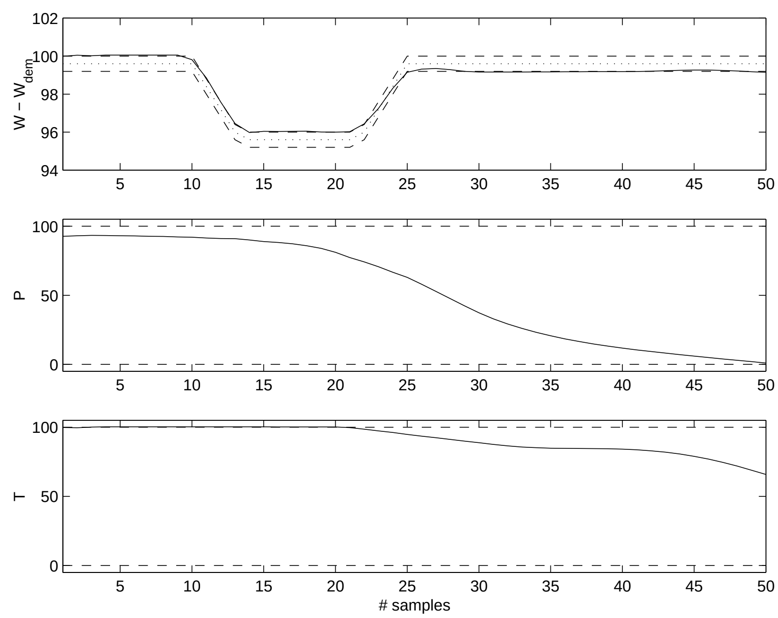

FIG. 5 – *Closed-loop responses of $\widehat{G}$ to a $-/+4\%$ step on load demand using optimal references satisfying both types of constraints*

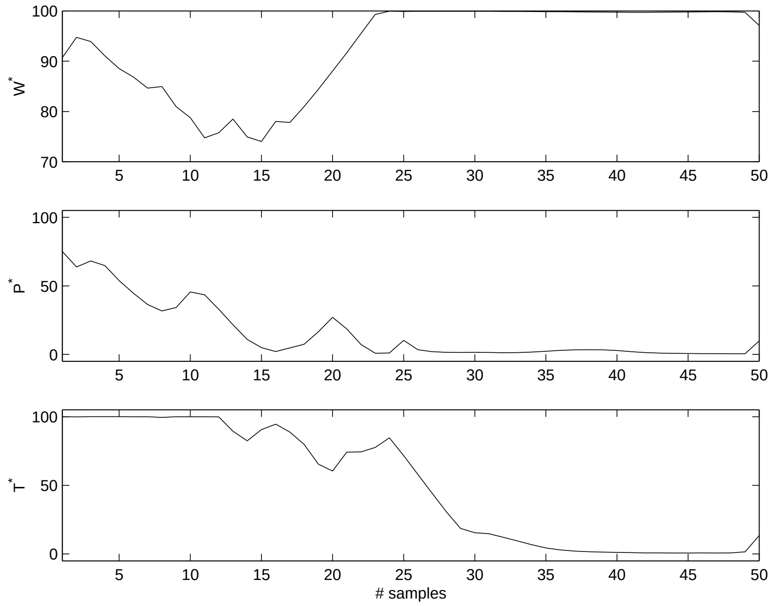

FIG. 6 – *Optimal references satisfying both types of constraints*

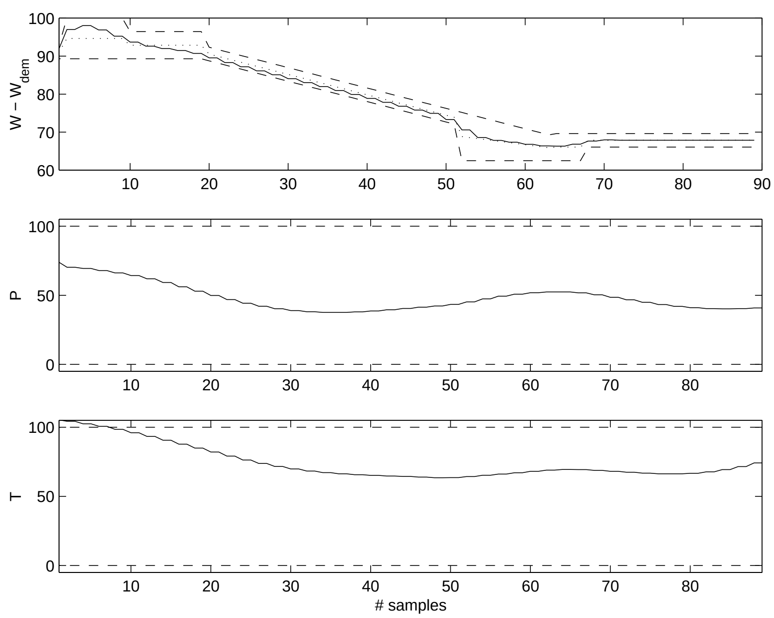

FIG. 7 – *Closed-loop responses of* $\widehat{G}$ *to a* $-4\%/\mathrm{min}$ *ramp on load demand using optimal references satisfying both types of constraints*

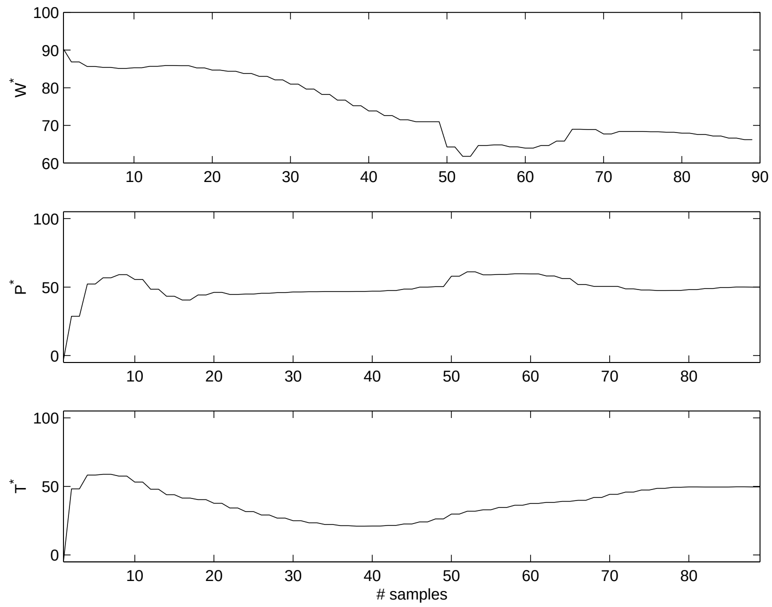

FIG. 8 – *Optimal references satisfying both types of constraints*

5 Concluding remarks

An optimization-based control technique has been applied to a fossil power plant to optimize the references in electrical power, pressure, and superheat temperature. Preliminary simulation results presented in this paper are very promising. For such results to be meaningful from a practical point of view, the range of prediction needs to be extended. Indeed, this proposed approach is more useful for large transients when operating constraints occur. At this point, the limiting factor for extending the prediction horizon lies in the complexity of the algorithm that should be reduced.

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Résumé : On étudie dans cette thèse des problèmes de stabilisation en théorie du contrôle pour trois types de systèmes différents.

Tout d'abord, on introduit, pour les systèmes non linéaires de dimension finie perturbés par des erreurs, une classe de contrôles dits hybrides, car dépendant d'un état mixte discret-continu. Étant donné un système dont l'équilibre est asymptotiquement contrôlable, on montre qu'il existe un contrôle tel que l'équilibre du système bouclé soit globalement asymptotiquement stable avec une robustesse par rapport aux petits bruits. On explicite pour les systèmes chaînés un tel contrôle robuste avec une seule dynamique discrète. On donne également un contrôle hybride et un contrôle par retour d'état continu et périodique en temps qui recollent robustement deux contrôles donnés tout en conservant une propriété de stabilité asymptotique.

Ensuite, on étudie le problème de stabilisation d'un bac de fluide par le contrôle du déplacement longitudinal. C'est un problème de théorie du contrôle en dimension infinie car on modélise le problème en utilisant les équations de Saint-Venant qui sont des équations aux dérivées partielles hyperboliques. On utilise une approche Lyapunov pour proposer des feedbacks qui, numériquement, stabilisent localement et asymptotiquement l'origine du système bouclé.

Enfin, on étudie le problème de stabilisation de l'origine d'un système linéaire en dimension finie lorsqu'on a une incertitude sur les données du système. On applique les méthodes de résolutions numériques des inégalités linéaires matricielles avec incertitudes à un problème industriel.

Mots-clés : Stabilisation asymptotique, robustesse aux bruits, contrôle hybride, contrôlabilité asymptotique, système chaîné, équations de Saint-Venant, équations aux dérivées partielles hyperboliques, approximation numérique, problème SDP, LMI robuste

Abstract : In this thesis, we study some problems of stabilization in control theory for three different class of systems.

First, for the non-linear finite-dimensional systems in presence of noise, we introduce a class of hybrid controllers with a mixed continuous/discrete state. Given a system with a globally asymptotically controllable equilibrium, we prove that there exists such a control such that the equilibrum is globally asymptotically stable with a robustness property with respect to small perturbations. For the chained systems we explicit such a feedback with only one discrete variable. We give also a hybrid control and a time-varying control which unit robustly any pair of continuous feedbacks and renders the origin a globally asymptotically stable equilibrium.

Secondly, we study the stabilization problem of the tank containing a fluid subject. It is subject to a horizontal move. It is a infinite-dimensional control problem because we describe the system by using the shallow water equations which are hyperbolic partial differential equations. We use a Lyapunov approach to propose some feedbacks which numerically stabilize locally and asymptotically the origin of the closed-loop system.

Finally, we study the problems of stabilization of the origin of a linear, finite-dimensional system with a uncertainty of the data of the system. We apply the methods of the numerical resolution of robust linear matrix inequalities to a industrial problem.

Keyword : Asymptotic stabilization, robustness, disturbance, hybrid control, asymptotic controlability, chained system, shallow-water equations, hyperbolic partial differential equations, numerical approximation, SDP problem, robust LMI