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Mauricio D. GARAY

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THÉORIE CLASSIQUE ET LEGENDRIENNE

DES POINTS D'APLATISSEMENTS ÉVANESCENTS

DES COURBES PLANES ET SPATIALES

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M. Vladimir ARNOLD	Directeur
M. Marc CHAPERON	
M. Alain CHENCINER	Président
M. Etienne GHYS	Rapporteur
M. Duco van STRATEN	Rapporteur
M. Bernard TEISSIER	

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Contents

Part I

THE CLASSICAL THEORY OF VANISHING FLATTENING POINTS

CHAPTER 1

Introduction.

1. Overview.

More than 30 years ago, V.I. Arnold introduced symplectic geometry in the realm of singularity theory ([**Arn1**]). Later on, he observed that many problems in projective geometry turn out to be problems in symplectic and contact geometries (see for example [**Arn6**]). Vice-versa new results in projective geometry motivate new developments in symplectic and contact geometries. A basic example of this relationship is given by the "generic" singularities of the Gauss map of a smooth hypersurface in Euclidean space $\mathbb{R}^n$ ([**Arn2**]). We shall come back to this example when dealing with the contactification of Kazarian's theory in the second part of this thesis.
The first part of this thesis deals with the classical theory of vanishing flattening points. Therefore, we avoid the language of contact and symplectic geometries.
In the second part, we shall see how the study of vanishing flattening points arises in the context of contact and symplectic geometries. For the reader's convenience the two parts are written independently one from the other once the reader is acquainted with the basic definitions given in chapter 2 section 1 and section 3.
So, our starting point is the study of special points of a curve (complex holomorphic or real C^∞). Roughly speaking, these special points called *flattening points* are defined as follows. A *flattening point* of a curve say in (real or complex) n-dimensional affine space is a point where the osculating hyperplane to the curve at the point has a higher order of tangency than at an usual point. For example for plane curves, they are commonly called inflection points[1].
At the end of the eighties', M.E. Kazarian developed in a series of works ([**Ka1**],[**Ka2**],[**Ka3**], [**AVGL2**]) the local theory of flattening points for families of rational curves that is families of curves which are locally the images of a family of say real C^∞ maps

$$f_\lambda : \mathbb{R} \longrightarrow \mathbb{R}^n$$

[1]That might seem odd to an analyst but for a geometer the curve $y = x^4$ has an inflection point of multiplicity 2 at the origin.

with C^∞ dependence on the parameter λ.
The subject was also treated independently from the algebro-geometric viewpoint by D. Eisenbud and J. Harris ([**EH1**],[**EH2**]).
M.E. Kazarian proved that the local study of rational curves with respect to their flattening points can always be reduced to the study of families of smooth curves (see [**Ka3**]).
When dealing with families of curves depending on a parameter, it may happen that the family cannot be considered as a family of rational curves. A simple example is given by the family of hyperbolas $\{xy = \varepsilon\}$ degenerating at $\varepsilon = 0$. The study of such families cannot be reduced to the study of families of smooth curves.
The behaviour of the flattening points (or more generally of the projective extrinsic invariants) of a family of smooth curves degenerating at a singular point is a subject of long study. The works of great mathematicians like Descartes, Newton, Poncelet, Plücker and Clebsch (e.g. [**Clebsch**], [**Pl**], [**Poncelet**]) where in included in the classical treatises (e.g. [**Salmon**], [**Klein1**], [**EC**]).
It seems that the Kazarian-Eisenbud-Harris theory does not include this setting. We shall develop a theory that does. Although the relationship with usual singularity theory (say for example with the Mixed Hodge structures) is still unclear to me.
There is not yet a contactification nor a symplectization of the "classical" theory that we shall develop. However, for smooth curves (that is for the case treated by M.E. Kazarian, D. Eisenbud and J. Harris), we did make such a theory and this will occupy our last chapter.
Surprisingly enough, at least for symplecticians, there are only a few cases for which the normal forms arising from the theory of flattening points of space curves coincide with the corresponding normal forms for wave fronts dual to the given curves. These cases are what we call the *excellent cases.* They are excellent because it suffices to study a simple problem on space curves in order to get results on families of wave fronts.
Before starting the theory, we would like to give an informal account on what is commonly called *Plücker's theorem.* The approach we will follow is somehow different from the one of the classics although all the results that we shall give are contained in the classical treatises (e.g. [**Klein1**], [**Jor**]).
This introduction will be informal because we are not going to be completely rigorous. However I hope that these informal considerations will help the reader for the formal ones.

2. Plücker's theorem.

By holomorphic curve in the affine plane $\mathbb{C}^2$, we mean an analytic subvariety of $\mathbb{C}^2$ with isolated singular points.
Denote by
$$D = \{t \in \mathbb{C} :\mid t \mid\leq 1\}$$

the unit disk centered at the origin.
A parameterization in a neighbourhood of a smooth point $p \in V$ is a holomorphic embedding $\gamma : D \longrightarrow \mathbb{C}^2$ whose image is contained in V and contains p.

DEFINITION 1.1. A smooth point $p \in \mathbb{C}^2$ of a plane holomorphic curve $V \subset \mathbb{C}^2$ is called an *inflection point* provided that for a given local holomorphic parameterization of V in a neighbourhood of $p \in V$ the two first derivatives of the parameterization are linearly dependent at the point p.

Remark. It can be easily proved that the inflection points of a holomorphic curve $V \subset \mathbb{C}^2$ are the points for which the multiplicity of intersection of the curve with its tangent line at the given point is at least equal to 3.
Example. The curve $\left\{(x, y) \in \mathbb{C}^2 : y = x^{k+2}\right\}$ has an inflection point at the origin if and only if $k > 0$.

Fix a coordinate system (x, y) (not necessarily affine) in $\mathbb{C}^2$.
Define the holomorphic function $H : \mathbb{C}^2 \longrightarrow \mathbb{C}$ by the formula

$$H(x, y) = xy.$$

The holomorphic curves:

$$H_\varepsilon = \left\{(x, y) \in \mathbb{C}^2 : H(x, y) = \varepsilon\right\}$$

have no inflection point.
Let $U \subset \mathbb{C}^2$ be a neighbourhood of the origin.
Denote by $\varphi : U \longrightarrow \mathbb{C}^2$ a holomorphic embedding in the affine plane $\mathbb{C}^2$ such that $\varphi(0) = 0$.
In general for a fixed value of ε, the image of the curve H_ε under φ have inflection points (see figure 1).

THEOREM 1.1. *There are* 6 *complex inflection points of the curve* $V_\varepsilon = \varphi(H_\varepsilon)$ *that converge towards the origin when* $\varepsilon \longrightarrow 0$ *provided that* φ *is generic*[2].

This theorem is due to Plücker ([**Pl**]). In chapter 2 section 2, we shall generalize this theorem to curves with arbitrary singularities in affine or projective spaces of arbitrary dimensions.

For the moment, we consider an example.
Let $\varphi : U \longrightarrow \mathbb{C}^2$ be defined by the polynomials

$$\varphi(x, y) = (x + y^2, y + x^2).$$

Denote by V_ε the image of the curve H_ε under φ. Consider the curve:

$$\dot{X} = \left\{p \in \mathbb{C}^2 : \varphi(p) \text{ is an inflection point of } V_{H(p)}\right\}.$$

[2]The variety of k-jets of map germs $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ that do not satisfy this property is a semi-algebraic variety of codimension at least one in the space of k-jets at the origin, provided that $k > 2$

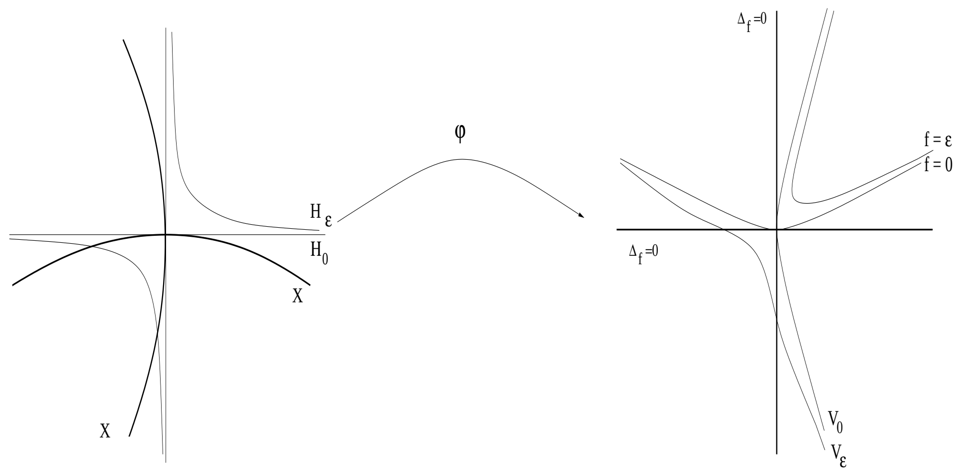

FIGURE 1. *The Plücker theorem. In the right part of the figure among the* 6 *inflection points of a curve* $V_\varepsilon = \varphi(H_\varepsilon)$ *only* 2 *have real coordinates (for a real value of the parameter* ε*). The preimages of the inflection points of* V_ε *under* φ *are the intersection points of the curve* H_ε *with the curve* $X = \{(x,y) \in \mathbb{C}^2 : \mathcal{D}[\varphi](x,y) = 0\}$.

We denote by X the closure of the curve $\dot{X}$.
Classically the curve $\varphi(X)$ is called the Hessian curve of the holomorphic function $H \circ \varphi^{-1}$ ([**Klein1**]).
We search for an equation of the curve X.
The so-called Hamilton vector field h of H is tangent to the level curves H_ε.
It is defined by the Hamilton differential equations

$$\begin{cases} \dot{x} &= \partial_y H, \\ \dot{y} &= -\partial_x H. \end{cases} \tag{1}$$

In our case $(\partial_y H)(x,y) = x$ and $(\partial_x H)(x,y) = y$.
Consequently the Hamilton vector field h is defined by

$$h(x,y) = x\partial_x - y\partial_y.$$

This field depends, of course, on the choice of the coordinate system (x,y) in $\mathbb{C}^2$.
Denote by $\dot{\varphi}$ the Lie derivative along the Hamilton vector-field h defined by

$$\dot{\varphi}(x_0,y_0) = \frac{d}{dt}_{|t=0} \varphi(x(t),y(t))$$

where $t \longrightarrow (x(t),y(t))$ is the solution of (1) satisfying the initial condition

$$(x(0),y(0)) = (x_0,y_0).$$

The first and second derivative of φ along the field h have the values

$$\begin{cases} \dot{\varphi}(x,y) & = & (x-2y^2, -y+2x^2), \\ \ddot{\varphi}(x,y) & = & (x+4y^2, y+4x^2). \end{cases}$$

The point $\varphi(x,y) \in \mathbb{C}^2$ is an inflection point of the image of H_ε, $\varepsilon \neq 0$, under φ if and only if $\dot{\varphi}$ and $\ddot{\varphi}$ are linearly dependent at $(x,y) \in \mathbb{C}^2$, provided that $(x,y) \neq 0$ is in a sufficiently small neighbourhood of the origin.
Denote by $\mathcal{D}[\varphi]$ the 2×2 determinant whose columns are the first and second derivative of φ along the Hamilton vector field h.
We have the equality

$$\mathcal{D}[\varphi](x,y) = 2xy + 2y^3 + 2x^3 - 16x^2y^2. \tag{2}$$

The curve X being given by

$$X = \left\{(x,y) \in \mathbb{C}^2 : \mathcal{D}[\varphi](x,y) = 0\right\}.$$

In other words, the inflection points of the image of the curve H_ε under φ when $\varepsilon \longrightarrow 0$ are the images under φ of the points of intersection of the curve

$$H_\varepsilon = \left\{(x,y) \in \mathbb{C}^2 : xy = \varepsilon\right\}$$

with the curve X provided that $(x,y) \neq 0$ is in a sufficiently small neighbourhood of the origin.
In figure 1, we depicted the curves H_ε, X and their images under φ for different values of ε.
When ε approaches 0 with real values, two real points of intersections of the curve X with H_ε converge towards the origin.
These points correspond to the 2 inflection points of $V_\varepsilon = \varphi(H_\varepsilon)$ with real coordinates that approach the origin as $\varepsilon \longrightarrow 0$. The other 4 inflection points are not real.
In order to find some approximate formula for the coordinates of the 6 inflection points, we approximate the plane holomorphic curve X in a neighbourhood of the origin.
The curve X "looks like" the union of two parabolas that intersect at the origin. Each one of these looking like parabolas is called a *branch* of the curve X.
Put $y = ax^2$ for the parabola tangent to the x-axis.
Substituting $y = ax^2$ in equation (2) we get the approximation

$$\mathcal{D}[\varphi](x, ax^2) = 2(a+1)x^3 + o(x^3)$$

Hence $a = -1$ provides the best approximation to this branch of X that we can have with parabolas. The curve X is not exactly a parabola. One can prove that it is given by an infinite convergent series $y = \sum_{n\geq 2} c_n x^n$ with $c_2 = -1$. This series is called the *Newton-Puiseux series* of the branch of the curve X ([**New**]).
Similarly, the other branch of X is given by

$$x = -y^2 + o(y^2)$$

We just drop the higher order terms and consider the approximation of the branches of the curves given by $x = -y^2$ and $y = -x^2$. There is a priori no justification for these approximations. This is what “informal account” means.
The approximated points of intersection of X with H_ε satisfy either the system of equations

$$\begin{cases} xy &= \varepsilon \\ y &= -x^2 \end{cases}$$

or the system

$$\begin{cases} xy &= \varepsilon \\ x &= -y^2 \end{cases}$$

Solving the first system of equations, we find that

$$x^3 = -\varepsilon.$$

These 3 solutions correspond to 3 approximated inflection points of the curve $V_\varepsilon = \varphi(H_\varepsilon)$. The other system gives also 3 solutions. Hence there are $3+3 = 6$ approximated inflection points of the curve V_ε in a neighbourhood of the origin. They are the images under φ of the points whose coordinates have the expressions

$$\begin{array}{lll} (-\varepsilon^{1/3}, -\varepsilon^{2/3}), & (-\omega\varepsilon^{1/3}, -\omega^2\varepsilon^{2/3}), & (-\omega^2\varepsilon^{1/3}, -\omega\varepsilon^{2/3}), \\ (-\varepsilon^{2/3}, -\varepsilon^{1/3}), & (-\omega^2\varepsilon^{2/3}, -\omega\varepsilon^{1/3}), & (-\omega\varepsilon^{2/3}, -\omega^2\varepsilon^{1/3}) \end{array} \tag{3}$$

Here $\varepsilon^{1/3}$ denotes a determination of the cube root of ε and $\omega = e^{2i\pi/3}$.
Remark that for ε real, only the points $(-\varepsilon^{1/3}, -\varepsilon^{2/3})$ and $(-\varepsilon^{2/3}, -\varepsilon^{1/3})$ are real. These are the approximations of the preimages of the real inflection points that are visible in figure 1.

Our example allows us to illustrate the relationship between asymptotics, monodromy and Galois groups. In the sequel, we shall in general avoid Galois theory because of its arithmetical subtleties. However Galois theory is inherent in our work because of this relationship that we explain now. The geometric Galois groups defined below is a classical object of study ([**Jor**], [**Seg**]).

Recall that V_ε is the image of the level-curve H_ε under φ.
For each value of ε, the curve V_ε has 6 inflection points that "vanish" at the origin when $\varepsilon \longrightarrow 0$.
Put $\varepsilon(t) = \delta^3 e^{2i\pi t}$, where δ is a small non-zero real number.
The inflection points of V_ε depend continuously on the parameter t. Hence, when t varies from 0 to 1 the inflection points of V_ε are permuted. The group generated by this permutation is called *the monodromy group with respect to inflection points* of the family of curves (V_ε).
We assert that, in the example we are studying, the monodromy group with respect to the approximated inflection points of the family of curves (V_ε) is the cyclic group $\mathbb{Z}/3\mathbb{Z}$. A generalization of this theorem is given in chapter

2 page 54 (see also theorem 2.6 on page 35).
For the rest of this subsection, the preimages of the inflection points of $V_\varepsilon = \varphi(H_\varepsilon)$ under φ will be called *preinflection points* of H_ε.
Project the preinflection points of H_ε on the complex line

$$L = \left\{(x,y) \in \mathbb{C}^2 : y = -x\right\}.$$

We take the first coordinate x times $\sqrt{2}$ for coordinate-system in L.
With these coordinate-systems, the projection on L is given by:

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & L \\ (x,y) & \mapsto & x-y. \end{array}$$

We consider only the leading term in ε when $\varepsilon \longrightarrow 0$ of the projected points. For example for the projection of the first point of the list (3) above we take $-\varepsilon^{1/3}$ instead of $-\varepsilon^{1/3} - \varepsilon^{2/3}$. Still there is no justification for this approximation but as we shall see later, we get the correct answer for the monodromy when we make it (see theorem 2.6 on page 35).
The approximated projections of the 6 preinflection points of the curve H_ε are given by $\pm\omega^k\varepsilon^{1/3}$ with $k \in \{1,2,3\}$. These are the six vertices of a regular hexagon (see figure 2 on page 12).
We put $\varepsilon(t) = \delta^3 e^{2i\pi t}$, then the 6 approximated projections of the preinflection points of $H_{\varepsilon(t)}$ are given by the expressions:

$$\left\{\begin{array}{ccc} z_1(t) = \delta e^{\frac{2i\pi t}{3}} & , \quad z_2(t) = \delta e^{\frac{2i\pi t}{3} + \frac{2i\pi}{3}} & , \quad z_3(t) = \delta e^{\frac{2i\pi t}{3} + \frac{4i\pi}{3}} \\ z_4(t) = -\delta e^{\frac{2i\pi t}{3}} & , \quad z_5(t) = -\delta e^{\frac{2i\pi t}{3} + \frac{2i\pi}{3}} & , \quad z_6(t) = -\delta e^{\frac{2i\pi t}{3} + \frac{4i\pi}{3}}. \end{array}\right.$$

When t moves from 0 to 1, the six vertices of the regular hexagon rotate by an angle $\frac{2\pi}{3}$.
The points are numbered such that the permutation corresponding to the loop

$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C} \\ t & \mapsto & \varepsilon(t) \end{array}$$

exchanges the numbering of the z_i's like the cyclic permutation

$$(123)(456).$$

This permutation generates the cyclic group $\mathbb{Z}_3$ of order 3.
This result is of course connected with the fact that the asymptotics of the approximated preinflection points of H_ε when $\varepsilon \longrightarrow 0$ are $\pm\omega^i\varepsilon^{1/3}$. If the asymptotics had been ε^2, there would have been no monodromy. If the asymptotics had been $\varepsilon^{1/6}$, the monodromy group would have exchanged cyclically the 6 points and so on. This is the relation between the asymptotics and the monodromy.
Finally remark that the asymptotics discussed above form continuous invariants related to a discrete one: the monodromy. But in fact the continuous invariants contain much more information as we shall see in chapter 6.

The connection with Galois theory is as follows. First I recall the definition of the Galois group of a polynomial ([**Gal1**]). Consider a polynomial:

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n,$$

where the a_k's are fixed complex numbers. Denote by $x_1, \ldots, x_n$ the roots of P over $\mathbb{C}$. Denote by L the field generated by the a_i's over $\mathbb{Q}$.

THEOREM 1.2. *(Galois) There exists one and only one permutation group G on $(1, \ldots, n)$ having the following property:*
for any polynomial function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ defined by a polynomial with coefficients in the field L, the complex number $f(x_1, \ldots, x_n)$ belongs to the field L if and only if $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, $\forall \sigma \in G$.

This theorem is proved in [**Gal1**] by an explicit construction. Namely the group G of the theorem is isomorphic to the automorphisms of the field K generated by the x_k's over $\mathbb{Q}$ that fix the elements of the subfield $L \subset K$.
The group G is called the *Galois group* of the polynomial. For example, the Galois group of $P(x) = x^2 - 1$ is trivial, whereas the Galois group of $P(x) = x^2 - 2$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$. In the second case the map:

$$\begin{array}{ccc} \mathbb{Q}[\sqrt{2}] & \longrightarrow & \mathbb{Q}[\sqrt{2}] \\ A + B\sqrt{2} & \mapsto & A - B\sqrt{2} \end{array} \tag{4}$$

is the generator of the automorphism group described above. The corresponding permutation exchanges the two roots of the polynomial.
To compute the Galois groups of the classical equations (e.g. the Galois groups of the modular equations) was a classical problem for the XIX^{th} century mathematicians. In fact this was one of Galois' motivations for creating his theory ([**Gal2**]).
We come back to the inflection points of the curves $V_\varepsilon = \varphi(H_\varepsilon)$.
Fix a complex number ε. Consider the polynomial P of degree 6 whose roots are the coordinates of the approximated projections of the 6 preinflection points of H_ε.
We have $P(x) = (x^3 + \varepsilon)(x^3 - \varepsilon)$. It is easily seen that the Galois group of the polynomial P is the product $\mathcal{S}_3 \times \mathcal{S}_3$ of two permutation groups on 3 elements provided that $\varepsilon^{1/3} \notin \mathbb{Q}$. The monodromy group $\mathbb{Z}/3\mathbb{Z}$ that we computed above is an invariant subgroup of this Galois group.
Of course, to construct this Galois group several choices were made: the affine coordinate-system in $\mathbb{C}^2$, the line of projection and so on.
However, it can be proved that for almost all possible choices we get the same result.
Following Jordan ([**Jor**]), given a projective plane curve, we consider the polynomial whose roots are the projections of the inflection points of the curve on a generic line. The Galois group of this polynomial is called the *geometric Galois group* of the curve. At the end of the XIX^{th} century and at the beginning of this century the geometric Galois groups were considered by Jordan, Weber, Dickson, Segre and others.

We now give few results concerning these geometric Galois groups.
The first of these results is proved in the classical treatises some other are not. For the other ones I do not know wether these are new results or not. I give them without proofs, just for the reader curiosity. The proofs are based on the local theory which is developed in the thesis but they can also be proved directly.

THEOREM 1.3. *The geometric Galois group of an elliptic curve is the group of affine transformation of the affine plane* $(\mathbb{Z}_3)^2$ *over the finite field* $\mathbb{Z}_3$.

This theorem is proved in [**Jor**], a beautiful exposition of it is also given in [**BrKn**]. Direct computations show that this group coincide with the monodromy group of the covering whose base is the set of (smooth) elliptic curves and the fibre consists of the inflection points of the corresponding elliptic curve.

THEOREM 1.4. *The geometric Galois group of a generic curve of degree* 4 *is the permutation group on its* 24 *inflection points.*
The geometric Galois group of a generic hyperelliptic curve of degree $d > 3$ *is the permutation group on its* $3d + 3$ *inflection points.*

Here again the geometric Galois group coincides with the monodromy group of the covering whose base is the set of smooth curves of degree 4 (resp. smooth hyperelliptic curves of degree $d > 3$) with non-degenerate inflection points[3] and the fibre consists of the inflection points of the corresponding curve.
Remark that the inflection points of a plane complex curve of degree 4 coincide with its Weierstrass points ([**GrHa**]). Theorem C.1 asserts that for a generic curve these points cannot be distinguished one from the other.
In this thesis, we shall rather be interested in the local analytic case than in the algebraic case. As a general rule we shall compute monodromy groups rather than geometric Galois groups. Although it seems that when dealing with projective extrinsic invariants a strange phenomenon occurs:
when the complexity increases the monodromy groups tend to become a full permutation group $\mathcal{S}_k$ (and consequently coincides with the corresponding Galois group).
This is not a theorem but rather an experimental fact based on the previous theorems and on some other examples. On the other hand, it seems that the exceptional varieties of $\mathbb{C}P^n$ might lead to unexpected geometric Galois groups which up to my knowledge have not been considered since Segre ([**Seg**]).
We have finished with Plücker's example. The next chapter presents most of our results, except for the results concerning the Legendrian theory which are given in the second part of the thesis.

[3]An inflection point p of a curve $V \subset \mathbb{C}^2$ is degenerate if there exists a line L intersecting V at p with intersection multiplicity at least 4.

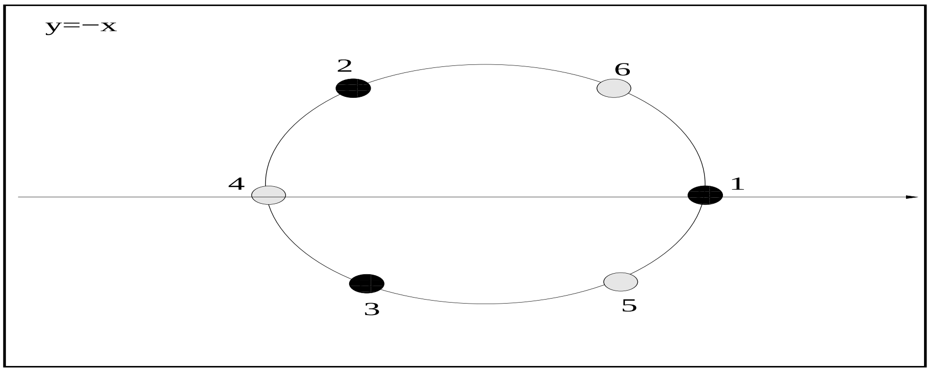

FIGURE 2. *Approximated projections of the preinflection points of the curves H_ε on the $y = -x$ complex-line. The line with an arrow represents the projection for real values of x and y. It contains the two real approximated preinflection points* 1 *and* 4.

CHAPTER 2

The classical theory of vanishing flattening points.

1. Basic definitions.

For notational reasons, we consider only the complex holomorphic case. The definitions of section 1 and of section 3 can be easily adapted to the real C^∞ case and for $\mathbb{K}$ formal or analytic series, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

1.1. The Young diagram of a map-germ.

We follow O.P. Scherback [**Sch**] and M.E. Kazarian [**Ka1**].
Consider a holomorphic map-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}^n$, where $\mathbb{C}^n$ denotes the n-dimensional complex affine space.

Definition 2.1. The map-germ f is called *triangular* provided that there exists an affine coordinate system in $\mathbb{C}^n$ such that:

$$\begin{array}{cccc} f: & (\mathbb{C},0) & \longrightarrow & \mathbb{C}^n \\ & t & \mapsto & (t^{\alpha_1}+(\dots), t^{\alpha_2}+(\dots), \dots, t^{\alpha_n}+(\dots)), \end{array}$$

The dots inside the parenthesis stand or higher order terms in the Taylor series and $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$ are integers.

Example 1. The map-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}^2$ defined by $f(t) = (t, t)$ is not triangular.
Example 2. The map-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}^2$ defined by $f(t) = (t^2, t^4)$ is triangular.
Example 3. The map-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}^3$ defined by $f(t) = (t, t^2, t^2)$ is not triangular.
Remark 1. Examples 1 and 3 are in some sense exceptional: the set of non-triangular holomorphic map-germs (for a fixed dimension n) form a variety of infinite codimension in the space of holomorphic map-germs.
Remark 2. The sequence $\alpha_1 < \dots < \alpha_n$ does not depend on the choice of the affine coordinate system in $\mathbb{C}^n$, nor on the choice of the coordinate in $\mathbb{C}$. We explain this fact.
Define an osculating k-plane of a curve $V \subset \mathbb{C}^n$ at a point p to be a complex k-plane intersecting V with maximal multiplicity at p.
For instance, the osculating 1-plane at a point $p \in V$ is the tangent line at p.
For simplicity, assume that the map $f : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$ is the germ of map $\bar{f}$ which defines a topological embedding.
Denote by V the curve germ parameterized by $\bar{f}$.

Since f is triangular, the osculating k-plane at the origin is unique. The number α_k is the multiplicity of intersection at the origin of V with this osculating k-plane.
Remark that this coordinate free definition of the sequence $\alpha_1, \dots, \alpha_n$ depends only of the local projective structure. In particular, it is well defined for a map-germ

$$f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}P^n.$$

Definition 2.2. Let j be the least number such that $\alpha_j - j > 0$. The sequence $a_1 = \alpha_n - n, \dots, a_{n-j+1} = \alpha_j - j$ is called *the anomaly sequence* of the map-germ f (the anomaly sequence can be empty).

Example 1. The anomaly sequence of the map-germ $t \mapsto (t, t^3, t^4, t^6)$ is $(2, 1, 1)$.
Example 2. The anomaly sequence at the origin of the map-germ $t \mapsto (t, t^2, t^6)$ is (3).
Remark. To each non-increasing sequence of positive integer numbers (a_j), $j \in \{1, \dots, k\}$ is associated a Young diagram. The number of squares in the j^{th} line of the Young diagram being equal to a_j. The Young diagrams of example 1 and example 2 are $\begin{smallmatrix}\square & \square \\ \square & \\ \square & \end{smallmatrix}$ and $\square\square\square$.

1.2. Conventions. By *holomorphic curve* in affine space $\mathbb{C}^n$, we mean the choice of a collection of holomorphic maps

$$f_j : U_j \longrightarrow \mathbb{C}^{n-1}$$

satisfying the 2 following conditions.
1) the analytic subvariety $V = \{p \in \mathbb{C}^n : \exists j,\ f_j(p) = 0\}$ of $\mathbb{C}^n$ is of dimension one,
2) In any coordinate system, the components of f_j form a reduced system of equations of $V \cap U_j$.
The points $p \in (V \cap U_j)$ for which $(Df_j)(p)$ is not surjective are called the *singular points* of the curve $V \subset \mathbb{C}^n$.
The points which are not singular points are called the *smooth points* of V.
The object that we call a curve is sometimes called a local complete intersection.
Let p be a point of a holomorphic curve $V \subset \mathbb{C}^n$. Let $B_r \subset \mathbb{C}^n$ be a small closed ball centered at p of radius r. The connected components of $(V \cap B_r) \setminus \{p\}$ for r small enough are called *the branches* of V at p. If p is a smooth point then V has only one branch.
The plane holomorphic curve of equation $xy = 0$ has two branches at the origin, namely the two coordinate axis.
A *parameterization* of a curve-germ $(V, 0)$ (resp. of a curve V) in $\mathbb{C}^n$ is a holomorphic map-germ $g : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$ (resp. a holomorphic map) such that:
1) the image of g is equal to $(V, 0)$ (resp. contained in V),
2) the map g is a topological embedding (i.e. a homeomorphism onto its

image).
For example, the map defined by the polynomials $t \mapsto (t^2, t^4)$ is not a parameterization of the lane curve given by the equation $y = x^2$.
Analogous conventions hold for curves in projective space $\mathbb{C}P^n$.

1.3. Flattening points of holomorphic curves.

DEFINITION 2.3. A point $p \in \mathbb{C}^n$ (or in $\mathbb{C}P^n$) of a holomorphic curve V in affine space $\mathbb{C}^n$ is called *triangular* if for any branch V_j of V at p the germ of V_j at p can be parameterized by a triangular map-germ. The curve V is called *triangular* if all its points are triangular.

Example 1. A line $V = \{(x, y) \in \mathbb{C}^2 : y = ax + b\}$ is not a triangular curve.
Example 2. The plane curve $V = \{(x, y) \in \mathbb{C}^2 : xy + x^3 = 0\}$ is not triangular. The plane curve $W = \{(x, y) \in \mathbb{C}^2 : xy + x^3 + y^3 = 0\}$ is triangular.

In the sequel, all the curves that we consider are triangular unless we mention explicitly the contrary.

DEFINITION 2.4. *The anomaly sequence* at a point p of a curve V in affine space $\mathbb{C}^n$ is the anomaly sequence of a triangular map-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}^n$ parameterizing the germ of V at p.

Remark 1. The anomaly sequence at a point of a curve does not depend on the choice of the triangular parameterization of the germ of the curve at the point.
Remark 2. The anomaly sequence of a point of a curve is also well-defined for curves in projective space $\mathbb{C}P^n$.

DEFINITION 2.5. A point p of a holomorphic curve $V \subset \mathbb{C}^n$ is called a *flattening point* of the curve V provided that:
- p is a smooth point of V,
- the anomaly sequence of V at p is not empty.

Example. For $n = 2$, the flattening points are the so-called inflection points. For instance, the plane holomorphic curve:

$$V_k = \left\{(x, y) \in \mathbb{C}^2 : y - x^{k+2} = 0\right\}$$

has an inflection point at the origin provided that $k > 0$.

DEFINITION 2.6. A flattening point p of a holomorphic curve $V \subset \mathbb{C}^n$ is called a *degenerate flattening point* of the curve V provided that the anomaly sequence of V at p is not equal to (1).

1.4. The Milnor number of a complete intersection map-germ.

We recall the definition of the Milnor number of a one-dimensional complete intersection ([**Mil**], [**Hamm**]).

Definition 2.7. A holomorphic map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ is called a *complete intersection map-germ* provided that the following three conditions are satisfied
1) the variety[1] $f^{-1}(0)$ of dimension $n - p$,
2) the origin is either a smooth point or an isolated critical point of $f^{-1}(0)$,
3) the equations f define a reduced system of equations of V.

Remark. With respect to the usual definition, we have added the conditions 2 and 3 for convenience.
Example. The map-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ defined by $f(x, y) = (x, x + xy)$ is not a complete intersection map-germ.

Consider a complete intersection map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$.
Let $\bar{f} : U \longrightarrow \mathbb{C}^{n-1}$ be a representative of f. Consider the family of curves (V_ε) defined by:

$$V_\varepsilon = \left\{ p \in U : \bar{f}(p) = \varepsilon \right\}.$$

Assume that V_0 has a singular point at the origin.
Denote by B_δ the closed ball of radius δ centered at the origin. Choose δ small enough so that V_0 intersects transversally the boundary of the ball B_r for any $0 < r \leq \delta$. Choose ε such that $V_\varepsilon \cap B_\delta$ is smooth.
Then $V_\varepsilon \cap B_\delta$ is homotopically equivalent to a bouquet of circles $S^1 \vee \cdots \vee S^1$ provided that ε is small enough. Moreover, the number of circles depends only on f. These facts are proved in [**Mil**] for function-germs and in [**Hamm**] for complete intersection map-germs.

Definition 2.8. The number of circles in the bouquet described above is called the *Milnor number* , denoted $\mu(f)$, of the complete intersection map-germ f.

Remark. When there is no possible misunderstanding, we simply write μ instead of $\mu(f)$.
Example. An explicit formula was found by Milnor ([**Mil**]) and completed by Greuel ([**Gre**]) and Hamm ([**Hamm**]. For plane holomorphic curves this formula is as follows.
Consider a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ with an isolated critical point at the origin.
Choose coordinates x, y in $\mathbb{C}^2$.
Denote by:
- $\mathcal{O}_{x,y}$ the ring of holomorphic function-germs,
- J_f the Jacobian ideal of f in $\mathcal{O}_{x,y}$ generated by the partial derivatives $\partial_x f$, $\partial_y f$.
Then the Milnor number of f is given by the formula

$$\mu(f) = dim_{\mathbb{C}}[\mathcal{O}_{x,y}/J_f].$$

[1] More precisely the germ at the origin of the zero-level set of a representative of f. Here and in the sequel, we, abusively, use this "shortened" formulation.

The assumption that the singularity is isolated is equivalent to the assumption $dim_{\mathbb{C}}[\mathcal{O}_{x,y}/Jf] < +\infty$ ([**AVG**]).
For example, take $f(x, y) = x^p + y^q$. Then J_f is generated by x^{p-1} and y^{q-1}. Consequently, in this particular case, the Milnor number of f is equal to

$$\mu = (p-1)(q-1).$$

2. Vanishing flattening points and generalized Plücker formulas.

2.1. The number of vanishing flattening points. Consider a holomorphic function $f : U \longrightarrow \mathbb{C}$ where U denotes an open neighbourhood of the affine plane $\mathbb{C}^2$.
Assume that f has an only critical point in U. Then, some inflection points of the curves:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\},$$

converge towards the critical point of f when ε approaches the corresponding critical value.
Following Arnold ([**Arn7**]), we say that the inflection points "vanish" at the critical point. For example Plücker proved that the number of vanishing inflection points at a generic Morse critical point (the meaning of generic will be explained in the sequel) of a holomorphic function is equal to 6 ([**Pl**]). Among these 6 vanishing inflection points no more than two can be real (see figure 1).
More generally consider a one parameter family (V_ε) of holomorphic curves in affine space $\mathbb{C}^n$ of arbitrary dimension $n > 1$.

Definition 2.9. A flattening point $p_1 \in V_{\varepsilon_1}$ *vanishes* at a singular point $p_0 \in V_{\varepsilon_0}$ provided that there exists a holomorphic function $h : S_\theta \longrightarrow \mathbb{C}^n$ such that:
- $h(1) = p_1$,
- $h(t)$ is a flattening point of the curve $V_{\varepsilon(t)}$ where $\varepsilon(t) = t\varepsilon_1 + (1-t)\varepsilon_0$.
Here S_θ denotes the closure of the sector $\{t \in \mathbb{C} : 0 <\mid t \mid \leq 1, \mid Arg t \mid \leq \theta\}$ with $\theta > 0$.

Remark. A similar definition holds in projective space $\mathbb{C}P^n$.

2.2. Generalized Plücker formula. We denote by $\mathbb{C}^n$ the n-dimensional vector space.
In this subsection and in the next one the flattening points are counted multiplicities: if a flattening point of a curve has anomaly sequence $a_1, \dots, a_k$ then it is counted $\sum_{j=1}^k a_j$ times.
Consider a complete intersection map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ with an isolated critical point at the origin. Denote by $\bar{f} : U \longrightarrow \mathbb{C}^{n-1}$ a representative of f. Let $\bar{f}_\varepsilon : U \longrightarrow \mathbb{C}^{n-1}$ be a one-parameter family of complete intersection map germs such that $f_0 = f$.
Denote by V_ε the curve in affine space $\mathbb{C}^n$ which is the zero-level set of f_ε.

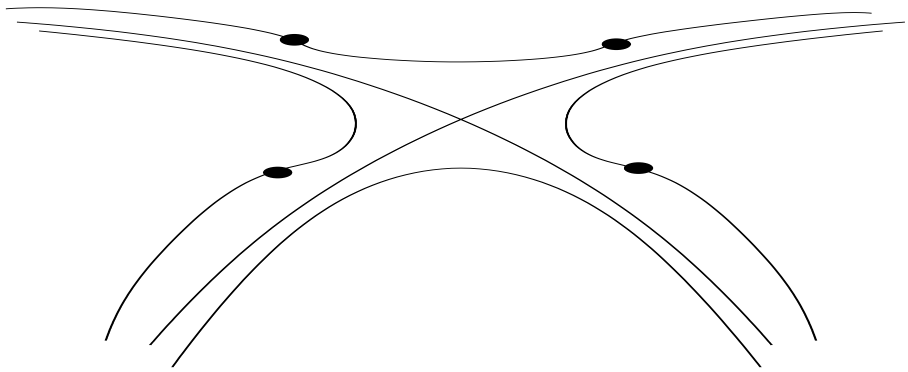

FIGURE 1. *Vanishing inflection points at a generic Morse critical point of a real analytic function f. We have drawn the level curves $f = \varepsilon$ in the affine plane $\mathbb{R}^2$. The complexification of the level curve $f = \varepsilon$ is a Riemann surface with 6 inflection points. Among the six inflection points of this complex holomorphic curve no more than two can reals. The two inflection points of the real curve of equation $f = \varepsilon$ lie either on two distinct components or on the same component (depending on the sign of ε).*

Assume that the family of curves (V_ε) in $\mathbb{C}^n$ is such that:
- for $\varepsilon \neq 0$, V_ε is smooth,
- the germ of V_0 at the origin is triangular.

To each branch C_k of V_0 at the origin, we associate a number η_k as follows. The anomaly sequence of C_k at the origin is say $(a_1^k, \dots, a_s^k)$ or empty. We put:

$$\eta_k = \sum_{j=1}^{s} a_j^k$$

if the anomaly sequence is not empty, $\eta_k = 0$ either.

THEOREM 2.1. *The number $N(f)$ of vanishing flattening points at the origin of the family of curves (V_ε) when $\varepsilon \longrightarrow 0$ is equal to:*

$$N(f) = \frac{n(n+1)}{2}(\mu + r - 1) + \sum_{k=1}^{r} \eta_k.$$

Here r is the number of branches of the curve V_0 and μ is the Milnor number of f.

This theorem is proved in chapter 3, section 2 where the relation with Teissier's polar invariants is given.
Theorem 2.1 implies that the number of vanishing flattening points at the origin of the family of curves (V_ε) depends only on the map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$. Consequently the following definition makes sense.

DEFINITION 2.10. The number $N(f)$ given in theorem 2.1 is called the *number of vanishing flattening points* of f.

Example. Let $f(x,y) = xy + x^3 + y^3$. Put:

$$V_\varepsilon = \left\{(x,y) \in \mathbb{C}^2 : f(x,y) = \varepsilon\right\}.$$

According to Milnor's formula cited above, the Milnor number of $(V_0, 0)$ is $\mu = 1$. The number of branches at the origin of V_0 is equal to $r = 2$.
The germs at the origin of the branches of V_0 are parameterized by:

$$\begin{array}{ccc} (\mathbb{C},0) & \longrightarrow & (\mathbb{C}^2,0) \\ t & \mapsto & (t, -t^2 + (\dots)) \end{array}$$

and:

$$\begin{array}{ccc} (\mathbb{C},0) & \longrightarrow & (\mathbb{C}^2,0) \\ t & \mapsto & (-t^2, t + (\dots)). \end{array}$$

Here the dots stand for higher order terms in the Taylor series. Both anomaly sequences are empty, thus $N = 6$ as found by Plücker.

Remark (for specialists). This theorem can be proved using Chern classes of line bundles and a result of Bassein which generalizes Milnor's formula $2\delta = \mu + r - 1$ ([**Mil**], [**Bas**], [**BuGr**], [**MoSt**]) .
This leads to a more general theorem which is in fact valid for curves which are not necessarily local complete intersections.
The proof we shall give is much more elementary. Vice-versa our proof implies both Milnor's formula $2\delta = \mu + r - 1$ and Bassein's generalization of it for one dimensional complete intersection.

After Plücker, particular cases where obtained by several authors [**EC**], [**Gud**], [**Sh**]. Other kind of generalizations of the so-called Plücker-Poncelet formulas have also been given ([**Kl**], [**Tei1**], [**Vi**], [**Yang**]). In the plane case ($n = 2$), theorem 2.1 is explicitly cited in [**AVGL2**]. A special case of theorem 2.1 was obtained in [**Ga**].

DEFINITION 2.11. Let $V \subset \mathbb{C}P^n$ be a triangular curve. A singular point $p \in V$ is called *minimal* if the number of vanishing flattening points at p is minimal among all the curve-germs having a singularity analytically equivalent to (V, p).

According to Veronese, a smooth curve of degree d and genus g in $\mathbb{C}P^n$ has $(n+1)(d+ng-n)$ flattening points[2] ([**Ve**]).
Assume that there exists a family of curves (V_ε) such that:
- for $\varepsilon \neq 0$, V_ε is a smooth curve of degree d and genus g,
- $V = V_0$.
Then theorem 2.1 implies that V has $(n+1)(d+ng-n)$ flattening points

[2] Such a curve exists only for particular values of g, d ([**GrHa**]).

minus the number of vanishing flattening point at the critical point given by the theorem. This is a generalization of the classical Plücker formula:

$$i = 3d(d-2) - 6m - 8\kappa$$

giving the number i of point of inflection points of a plane curve of degree d with m minimal double points and κ semi-cubical cusps and no other singularities ([**Pl**]).

The most simple non-trivial example is obtained as follows. Consider a curve $V \subset \mathbb{C}P^3$ of degree 6 obtained as the intersection of a quadric with a cubic surface. Assume that V has no other singularities than minimal double points and semi-cubical cusp points.

If V is smooth then according to Veronese it has 60 flattening points ($d = 6,\ g = 4,\ n = 3$).

If V has m minimal double points and κ minimal cusp points then, the number f of flattening points of V is:

$$f = 60 - 12m - 15\kappa.$$

In particular if no branch of V is contained in a plane (i.e.: V is triangular) then V has can have at most:

- 5 double points,
- 3 double points and 1 semi-cubical cusp point,
- 2 double points and 2 semi-cubical cusp points,
- 1 double point and 3 semi-cubical cusp points,
- 4 semi-cubical cusp points.

2.3. The generic number of vanishing flattening points for A, D, E singularities.

Definition 2.12. A holomorphic function-germ $H : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belongs to a *singularity class* A_k, D_k, E_6, E_7, E_8 provided that there exists a biholomorphic map-germ $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that one of the following equalities hold

$$\begin{array}{ll} A_k & H \circ g(x,y) = y^2 + x^{k+1}, \\ D_k & H \circ g(x,y) = x^2 y + y^{k-1}, \\ E_6 & H \circ g(x,y) = x^3 + y^4, \\ E_7 & H \circ g(x,y) = x^3 + xy^3, \\ E_8 & H \circ g(x,y) = x^3 + y^5 \end{array}$$

Choose a singularity class X among the singularity classes A_k, D_k, E_6, E_7, E_8.

Let H be a function-germ belonging to X.

Let $\bar{H}$ be a representative of H. Denote by H_ε the level curves $\bar{H}^{-1}(\varepsilon)$.

Given a holomorphic map $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ denote by N_φ the number of vanishing flattening points at the origin of the family of curves $V_\varepsilon = \varphi(H_\varepsilon)$ when $\varepsilon \longrightarrow 0$.

Theorem 2.1 implies that, for any k big enough, there exists a semi-algebraic variety Σ of codimension at least one in the space $J_0^k(\mathbb{C}^2, \mathbb{C}^n)$ of k-jets at the origin of maps such that the number N_φ is constant for $j_0^k\varphi \notin \Sigma$. In the table below this number is denoted by $N[X]$.

$n=$	2	3	4	n	$n=$	2	3	4	n
$N[A_1]=$	6	12	20	$n(n+1)$	$N[D_4]=$	18	36	60	$3n(n+1)$
$N[A_2]=$	8	15	24	$n(n+2)$	$N[D_5]=$	20	39	64	$n(3n+4)$
$N[A_3]=$	12	24	40	$2n(n+1)$	$N[E_6]=$	22	43	70	$n(3n+6)-2$
$N[A_4]=$	15	29	47	$2n(n+2)-1$	$N[E_7]=$	26	51	84	$n(4n+5)$
$N[A_5]=$	18	36	60	$3n(n+1)$	$N[E_8]=$	29	56	92	$4n(n+2)-4,\ n\geq 3$

Remark that if $H \in X$ is such that the branches of $H^{-1}(0)$ are smooth then theorem 2.1 implies the equality

$$N[X] = (\mu(H)-1)N[A_1].$$

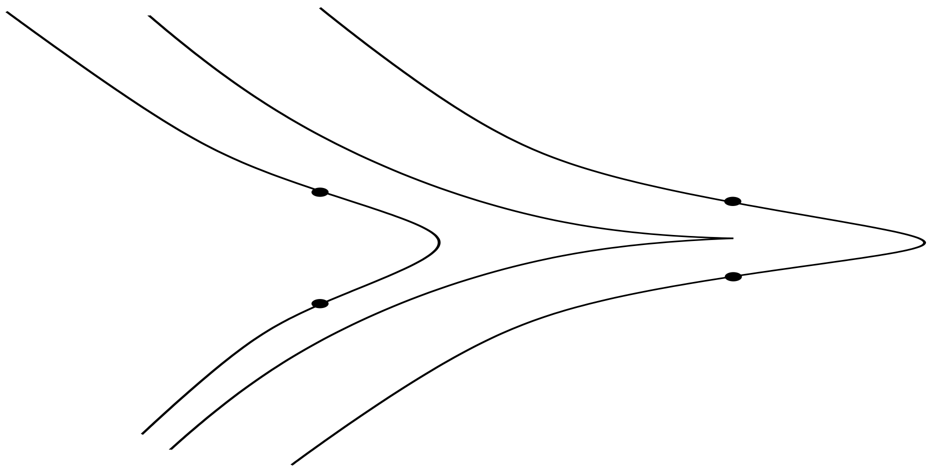

FIGURE 2. *The real part of the family of curves* $V_\varepsilon = \left\{(x,y) \in \mathbb{C}^2 : y^2 + x^3 = \varepsilon\right\}$ *for real values of the parameter* ε. *Among the* 8 *complex inflection points of* V_ε, $\varepsilon \neq 0$ *only* 2 *are real.*

3. The Plücker discriminant.

The results of this section and of the next one can be stated in the real C^∞ case, in the $\mathbb{K}$ analytic case and for $\mathbb{K}$ formal power series with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. However, for simplicity, we give the definitions in the complex holomorphic case. The theorems will be stated in the real C^∞ case and in the complex holomorphic case but they are valid for all the other categories stated above.

3.1. Inflection points of holomorphic curves. The flattening points of plane curves in the affine plane $\mathbb{C}^2$ (or in the projective plane $\mathbb{C}P^2$) are the inflection points.
For the reader's convenience, we state equivalent but simpler definitions for this case ($n = 2$).

Definition 2.13. Let $p \in \mathbb{C}^2$ (or in $\mathbb{C}P^2$) be a smooth point of a holomorphic curve $V \subset \mathbb{C}^2$. Denote by L the tangent complex line of V at p. The point p is called an *inflection point* provided that the multiplicity of intersection at p of V with L is at least equal to 3.

Definition 2.14. The *anomaly* at a point p of a holomorphic curve $V \subset \mathbb{C}^2$ is the maximal multiplicity of intersection at p of a line with the curve V.

Example. At an inflection point the anomaly is at least equal to 1.

Definition 2.15. An inflection point $p \in \mathbb{C}^2$ of a holomorphic curve $V \subset \mathbb{C}^2$ is called a *degenerate inflection point* provided the anomaly of $p \in V$ is at least equal to 2.

Example. The holomorphic curve:

$$V_k = \left\{(x, y) \in \mathbb{C}^2 : y - x^{k+2} = 0\right\}$$

has an inflection point at the origin provided that $k > 0$. For $k > 0$, the anomaly of the origin is equal to k. Thus, the origin is a degenerate inflection point of the curve V_k provided that $k > 1$.

3.2. The Plücker discriminant of a family of curves. In the sequel, by *family of holomorphic curves* (V_λ) in the affine complex plane $\mathbb{C}^2$, we mean the choice of a holomorphic map:

$$G : \Lambda \times U \longrightarrow \mathbb{C}$$

such that the curves of our family are given by

$$V_\lambda = \{p \in U : G(\lambda, p) = 0\}\,.$$

Here $\Lambda \subset \mathbb{C}^k$ is the space of the values of the parameter λ and U denotes an open neighbourhood of the complex affine plane $\mathbb{C}^2$.

Definition 2.16. Let (V_λ) be a family of holomorphic curves in $\mathbb{C}^2$. The set of the values of the parameter λ for which the curve V_λ has either a degenerate inflection point or a singular point is called *the $\mathcal{P}$-discriminant* (read Plücker discriminant) of the family (V_λ).

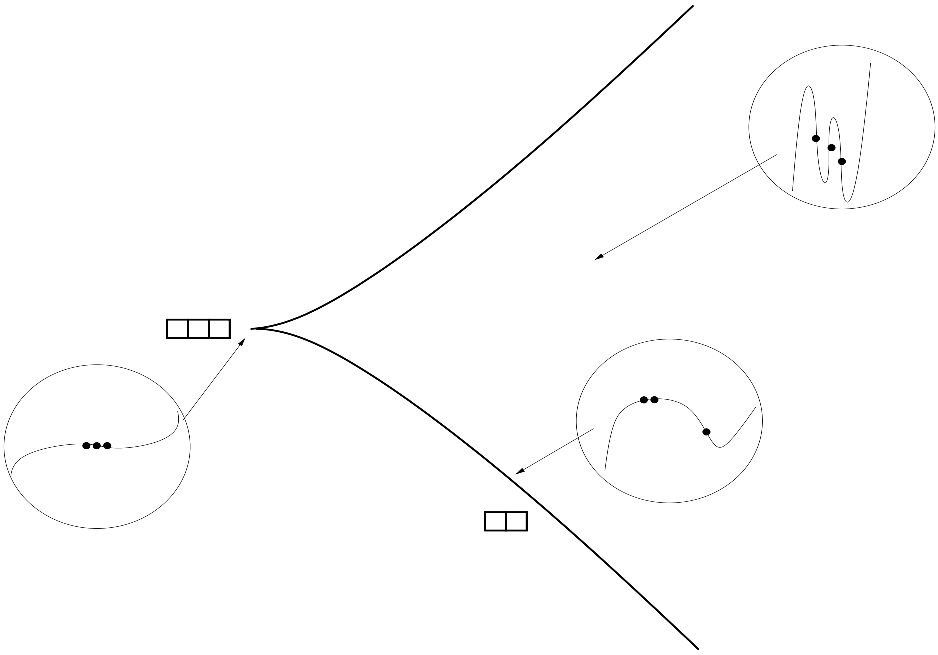

FIGURE 3. *The $\mathcal{P}$-discriminant of the family of curves (V_λ) given by $V_\lambda = \left\{(x,y) \in \mathbb{R}^2 : y + x^5 + \lambda_1 x^3 + \lambda_2 x^2 = 0\right\}$. Inside the disks, we represented the real parts of the curve of the corresponding value of the parameter. The marked points on the curves stand for the inflection points. In case of a degenerate inflection point, we have put the marked points near one to the other.*

Example 1. Consider the two parameter family of mappings

$$\begin{array}{rccc} f_\lambda : & \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ & t & \longrightarrow & (t, t^5 + \lambda_1 t^3 + \lambda_2 t^2), \end{array}$$

depending on the parameter $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$.
Denote by V_λ the curve parameterized by f_λ.
The curve $V_\lambda \subset \mathbb{C}^2$ is smooth for any value of $\lambda \in \mathbb{C}^2$. Thus, the $\mathcal{P}$-discriminant consists only of the values of the parameter $\lambda \in \mathbb{C}^2$ for which the curve V_λ has a degenerate inflection point.
Consider the two parameter family of holomorphic functions $g_\lambda : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$g_\lambda(t) = t^5 + \lambda_1 t^3 + \lambda_2 t^2.$$

The point $f_\lambda(t)$ is an inflection point of the curve V_λ if and only if the second derivative of g_λ vanishes at t, that is $g"_\lambda(t) = 0$.
Consequently, the $\mathcal{P}$-discriminant Σ of the family of curves (V_λ) is given by

$$\Sigma = \left\{\lambda \in \mathbb{C}^2 : \exists t \in \mathbb{C},\ g_\lambda"(t) = g_\lambda^{(3)}(t) = 0\right\}.$$

By direct computations, we get that the $\mathcal{P}$-discriminant of the family of curves (V_λ) is the semi-cubical parabola

$$\Sigma = \left\{\lambda \in \mathbb{C}^2 :\ 2\lambda_1^3 + 5\lambda_2^2 = 0\right\}.$$

We depicted the $\mathcal{P}$-discriminant Σ for real values of the parameter λ in figure 3.

Example 2. Consider the holomorphic function $G : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ defined by the polynomial

$$G(\lambda, x, y) = y^2 + x^3 + \lambda_1 x + \lambda_2.$$

The plane holomorphic curves $V_\lambda = \{(x, y) : G(\lambda, x, y) = 0\}$ are of degree 3. Hence, they have no degenerate inflections.
The $\mathcal{P}$-discriminant of the family (V_λ) consists only of the values of the parameter λ for which the curve V_λ is singular. Consequently, the $\mathcal{P}$-discriminant Σ of the family (V_λ) is given by:

$$\Sigma = \left\{\lambda \in \mathbb{C}^2 : \exists (x, y) \in \mathbb{C}^2,\ \partial_x G(x, y, \lambda) = \partial_y G(x, y, \lambda) = G(x, y, \lambda) = 0\right\}.$$

By direct computations, we get:

$$\Sigma = \left\{\lambda \in \mathbb{C}^2 :\ 4\lambda_1^3 + 27\lambda_2^2 = 0\right\}.$$

The singularities of Plücker discriminants of families of rational holomorphic curves where studied by M.E. Kazarian. He obtained in particular the bifurcation diagrams for the two and three parameters "generic" families of such curves (see [**Ka1**]).

3.3. The genericity notion. In the following theorems, we consider "generic holomorphic functions" of the type $F : \Lambda \times U \longrightarrow \mathbb{C}$ where Λ and U are open neighbourhoods of a in the affine plane $\mathbb{C}^2$ and in the analytic space $\mathbb{C}^k$.
The word generic means that there exists an integer N and a variety $M \subset J^N(\Lambda \times U, \mathbb{C})$ such that F satisfies the theorem provided that the map

$$j^N F : \Lambda \times U \longrightarrow J^N(\Lambda \times U, \mathbb{C})$$

is transversal to M.
In fact in all our cases M will be of codimension strictly higher than $k + 2$. Consequently the transversality condition means simply that the image of $j^N F$, which is of dimension at most $k + 2$, does not intersect M.
Replacing the words complex and holomorphic by real and C^∞ we get the meaning of the genericity notion in the real case.

3.4. The generic Plücker discriminants of foliations by plane curves. The theorems stated in this subsection are corollaries of the general methods developed in section 4 and section 5.
Consider a holomorphic function $F : \Lambda \times U \longrightarrow \mathbb{C}$ depending on the parameter $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda$. Here U denotes an open subset in the complex affine plane $\mathbb{C}^2$ and Λ is an open neighbourhood in the analytic space $\mathbb{C}^k$. The level-curves of $F(\lambda, .)$ define a family of holomorphic curves $(V_{\lambda,\varepsilon})$ in the affine plane $\mathbb{C}^2$. The curve $V_{\lambda,\varepsilon}$ being defined by:

$$V_{\lambda,\varepsilon} = \{p \in U : F(\lambda, p) = \varepsilon\}.$$

The curve $V_{\lambda,\varepsilon}$ is defined as the zero level-set of the function $G(\lambda, \varepsilon, .) = F(\lambda, .) - \varepsilon$ depending on the $k+1$ parameters $\lambda_1, \dots, \lambda_k, \varepsilon$.
Thus, the parameter of the family of curves which was denoted by λ in the preceding subsection is denoted in this subsection by (λ, ε).

Definition 2.17. The $\mathcal{P}$-*discriminant* of the function F is the $\mathcal{P}$-discriminant of the family of curves $(V_{\lambda,\varepsilon})$.

Denote respectively by Λ^1, U neighbourhoods of the origin in $\mathbb{C}$ and in $\mathbb{C}^2$.

Theorem 2.2. *Let $F : \Lambda^1 \times U \longrightarrow \mathbb{C}$ be a generic function. Let $\Sigma \subset \Lambda \times \mathbb{C}$ be the $\mathcal{P}$-discriminant of F. Then, the germ of Σ at an arbitrary point $(\lambda, \varepsilon) \in \Sigma$ is biholomorphically equivalent to the germ at the origin of one of the following curves in $\mathbb{C}^2$, provided that U is small enough:*
$\mathcal{P}A_1, K_1$: A line.
$\mathcal{P}A_1^1$: The curve $\{(\lambda, \varepsilon) \in \mathbb{C}^2 : \varepsilon(\varepsilon - \lambda^4) = 0\}$.
$\mathcal{P}A_2, K_2$: The curve $\{(\lambda, \varepsilon) \in \mathbb{C}^2 : \varepsilon^2 - \lambda^3 = 0\}$.

Remark. In the C^∞ category, for a generic C^∞ function $F : \Lambda \times U \longrightarrow \mathbb{R}$, $\Lambda \subset \mathbb{R}$, $U \subset \mathbb{R}^2$, the germ at the origin of the $\mathcal{P}$-discriminant is diffeomorphic to the real part of one of the complex $\mathcal{P}$-discriminants listed above.

The "typical" deformations giving the bifurcation diagrams $\mathcal{P}A_1$, $\mathcal{P}A_1^1$ and $\mathcal{P}A_2$ are as follows:

$\mathcal{P}A_1$	$\mathcal{P}A_1^1$	$\mathcal{P}A_2$
$xy + x^3 + y^3$	$xy + x^4 + \lambda x^3 + y^3$	$y^2 + x^3 + \lambda x$

The first deformation is constant in the parameter λ.
The $\mathcal{P}$-discriminant of the family $F : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by the polynomial

$$F(\lambda, x, y) = xy + x^4 + \lambda x^3 + y^3$$

is depicted in figure 4.
The level-curves

$$V_{\lambda,\varepsilon} = \{p \in \mathbb{R}^2 : F(\lambda, p) = \varepsilon\}$$

corresponding to different values of the parameter (λ, ε) are depicted in figure 5.

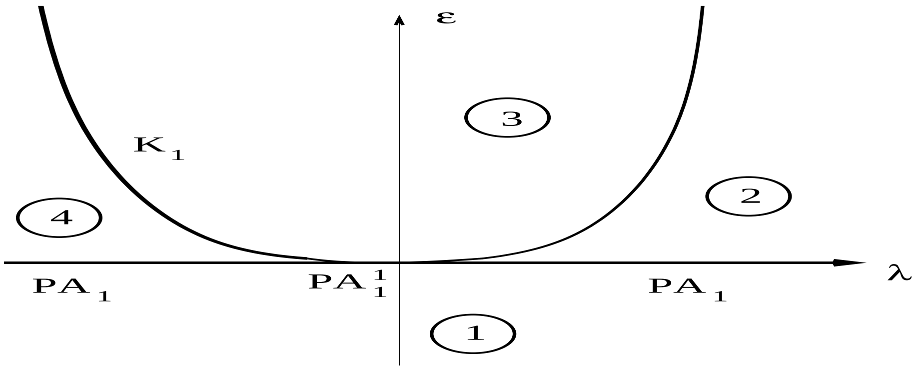

FIGURE 4. *The real $\mathcal{P}$-discriminant of the function $F(\lambda, x, y) = xy + x^4 + \lambda x^3 + y^3$. The curves corresponding to values of the parameter (λ, ε) in the connected components* $1, 2, 3$ *are depicted in figure 5. The stratum K_1 corresponds to the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection point of anomaly* 2. *The values of the parameters (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ is singular has two strata. One stratum of dimension* 1 *denoted $\mathcal{P}A_1$ and a stratum of dimension* 0 *denoted $\mathcal{P}A_1^1$ which lies in the closure of the the strata of dimension* 1. *The stratum $\mathcal{P}A_1$ is not connected (while in the complex case it is connected).*

We turn on to the case where the space of the parameters (λ, ε) is of dimension 3. We recall the following definition.

DEFINITION 2.18. A holomorphic function $f : U \longrightarrow \mathbb{C}$ is called a *Morse function* provided for any critical point $p \in U$ of f, the second differential of f at p is a non-degenerate quadratic form. Here U is an open neighbourhood in $\mathbb{C}^2$.

Denote by Λ^2, U neighbourhoods of the origin in $\mathbb{C}^2$.

THEOREM 2.3. *Let $F : \Lambda^2 \times U \longrightarrow \mathbb{C}$ be a generic function. Let λ_0 be such that $F(\lambda_0, .)$ is a Morse function. Denote by $p_0 \in \mathbb{C}^2$ a critical point of $F(\lambda_0, .)$ of critical value $\varepsilon_0 \in \mathbb{C}$.*
Then, the germ at $(\lambda_0, \varepsilon_0)$ of the $\mathcal{P}$-discriminant of F is biholomorphically equivalent to the germ at the origin of one of the following 4 *surfaces in $(\lambda, \varepsilon)'s$ space $\Lambda^2 \times \mathbb{C} \subset \mathbb{C}^3$, provided that U is a sufficiently small neighbourhood of p_0:*
1) $\mathcal{P}A_1^{1,1}$: One of the polynomials $x^4 + \lambda_1 x^3 + \varepsilon$, $y^4 + \lambda_2 y^3 + \varepsilon$ has at least a double root .
2) $\mathcal{P}A_1^2$: The polynomial $x^5 + \lambda_1 x^4 + \lambda_2 x^3 + \varepsilon$ has at least a double root.
3,4) Cylinders over the $\mathcal{P}$-discriminants denoted $\mathcal{P}A_1^1$ or $\mathcal{P}A_1$ in the list of theorem 2.2.

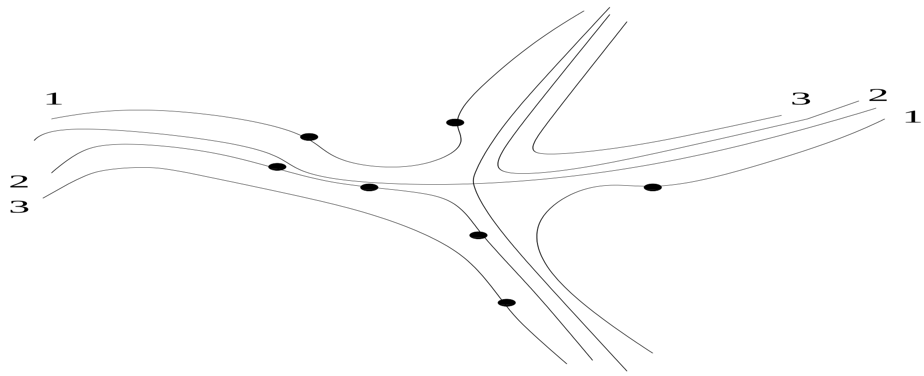

FIGURE 5. *We have numbered the level curves in* $\mathbb{R}^2$ *by* $1, 2, 3$*. The black disks stand for the inflection points of the curves labelled* $1, 2, 3$*. The number* 1 *curve and the number* 2 *curve have* 3 *(real) inflection points. The number* 3 *curve has only one real inflection point. Between the curve* 2 *and the curve* 3 *there is a curve with a degenerate inflection point. Between* 1 *and* 2 *there is a curve with a node singular point. This singular curve has an inflection point.*

The "typical" deformations giving the bifurcation diagrams $\mathcal{P}A_1^{1,1}$ and $\mathcal{P}A_1^2$ of theorem 2.3 are:

$\mathcal{P}A_1^{1,1}$	$\mathcal{P}A_1^2$
$xy + x^4 + y^4 + \lambda_1 x^3 + \lambda_2 y^3$	$xy + x^5 + y^3 + \lambda_1 x^4 + \lambda_2 x^3$

.

The real C^∞ version of this theorem is as follows.
The levels $V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\}$ of a Morse function $f : U \longrightarrow \mathbb{R}$ whose second differential has signature $(+, +)$ or $(-, -)$ are either convex curves or empty, provided that $\varepsilon \neq 0$ is small enough. Thus, they have no inflection points provided that U is small enough. Consequently, we do not consider these cases. Denote by Λ^2, U neighbourhoods of the origin in $\mathbb{R}^2$.

THEOREM 2.4. *Let* $F : \Lambda^2 \times U \longrightarrow \mathbb{R}$ *be a generic function,* $U \subset \mathbb{R}^2$*.*
Let λ_0 *be such that* $F(\lambda_0, .)$ *is a Morse function. Denote by* $p_0 \in \mathbb{R}^2$ *a critical point of* $F(\lambda_0, .)$ *of critical value* $\varepsilon_0 \in \mathbb{R}$*. Assume that the Hessian of* $F(\lambda_0, .)$ *at* p_0 *is neither positive definite nor negative definite.*
Then, the germ at $(\lambda_0, \varepsilon_0)$ *of the* $\mathcal{P}$*-discriminant of* F *is diffeomorphic to the germ at the origin of one of the following* 5 *surfaces in* $(\lambda, \varepsilon)'$*s space* $\Lambda^2 \times \mathbb{R}$*, provided that* U *is a sufficiently small neighbourhood of* p_0*:*
1,2) $\mathcal{P}A_1^{1,1,\pm}$ *: One of the polynomials* $x^4 + \lambda_1 x^3 + \varepsilon$, $\pm y^4 + \lambda_2 y^3 + \varepsilon$ *has at least a (real) double root .*
3) $\mathcal{P}A_1^2$ *: The polynomial* $x^5 + \lambda_1 x^4 + \lambda_2 x^3 + \varepsilon$ *has at least a (real) double root.*

4,5) Cylinders over the $\mathcal{P}$-discriminants denoted $\mathcal{P}A_1^1$ or $\mathcal{P}A_1$ in the list of theorem 2.2.

The "typical" deformations giving the bifurcation diagrams $\mathcal{P}A_1^{1,1,\pm}$ and $\mathcal{P}A_1^2$ of theorem 2.3 and theorem 2.4 are:

$\mathcal{P}A_1^{1,1,\pm}$	$\mathcal{P}A_1^2$
$xy + x^4 \pm y^4 + \lambda_1 x^3 + \lambda_2 y^3$	$xy + x^5 + y^3 + \lambda_1 x^4 + \lambda_2 x^3$

.

The surfaces corresponding to the $\mathcal{P}$-discriminant of theorem 2.4 are depicted in figure 6, figure 8 and figure 9. The complex $\mathcal{P}$-discriminants of theorem 2.3 are obtained by complexifying these real surfaces. Remark that the surface drawn in figure 6 is not the discriminant of the boundary singularity B_3 ([**AVGL1**]). Indeed a transversal slice of this surface gives two curve intersecting with multiplicity 4 at the origin (instead of 2 for the discriminant of the boundary singularity B_3).

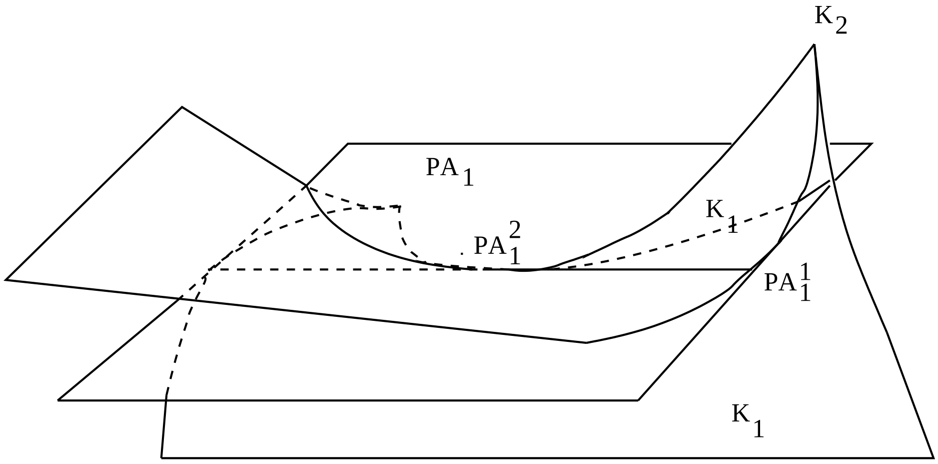

FIGURE 6. *The $\mathcal{P}$-discriminant of the family of curves $\mathcal{P}A_1^2$: $xy+x^5+\lambda_1x^4+\lambda_2x^3+y^3 = \varepsilon$. The stratum K_j corresponds to the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection of anomaly $j + 1$. A transversal slice of the surface gives two curves intersecting with multiplicity 4 at the origin.*

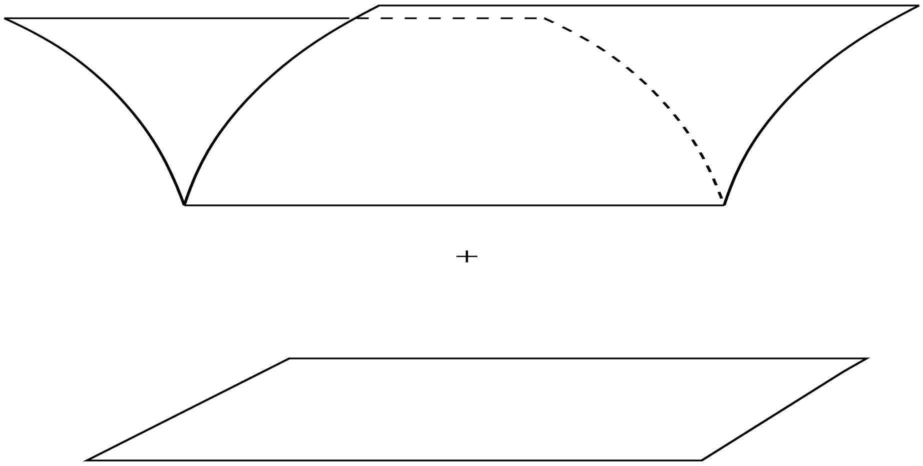

FIGURE 7. *One component of the $\mathcal{P}$-discriminant of $\mathcal{P}A_1^2$ is diffeomorphic to a cylinder over a semi-cubical parabola. The other component is a plane.*

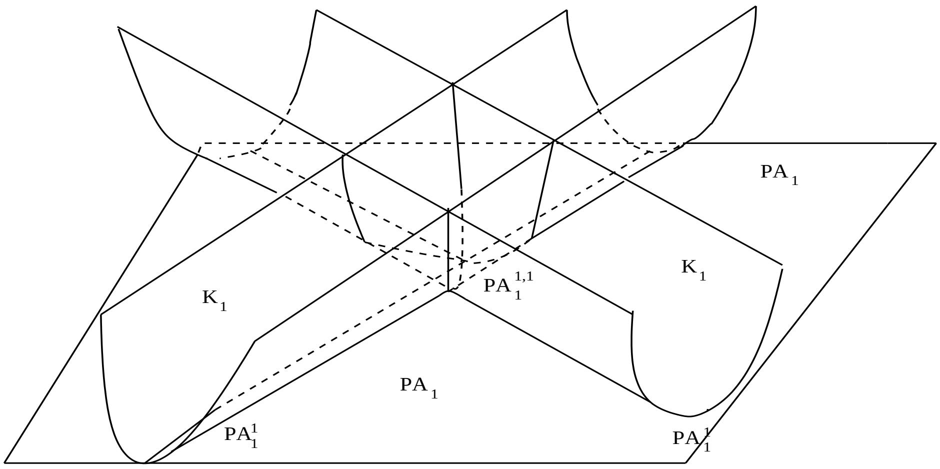

FIGURE 8. *The $\mathcal{P}$-discriminant of the family of curves $\mathcal{P}A_1^{1,1,+}$: $xy + x^4 + y^4 + \lambda_1 x^3 + \lambda_2 y^3 = \varepsilon$. The stratum K_j corresponds to the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection point of anomaly $j + 1$.*

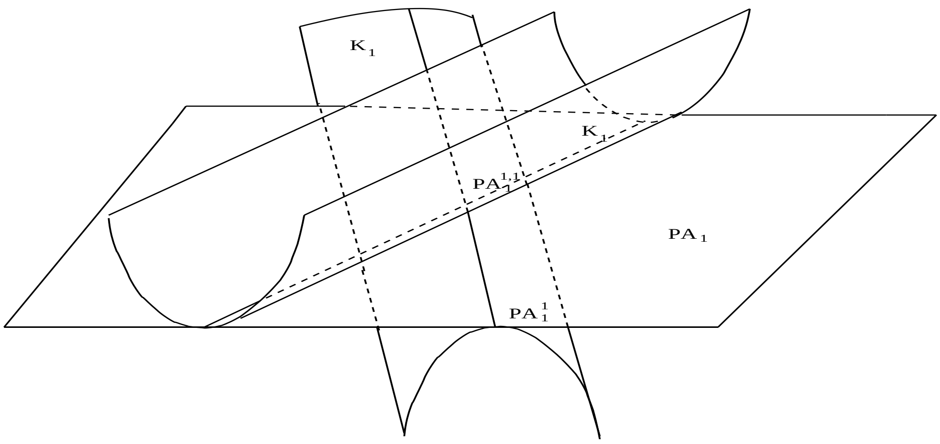

FIGURE 9. *the $\mathcal{P}$-discriminant of the family of curves $\mathcal{P}A_1^{1,1,-} : xy + x^4 - y^4 + \lambda_1 x^3 + \lambda_2 y^3 = \varepsilon$. The stratum K_j corresponds to the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection point of anomaly $j+1$.*

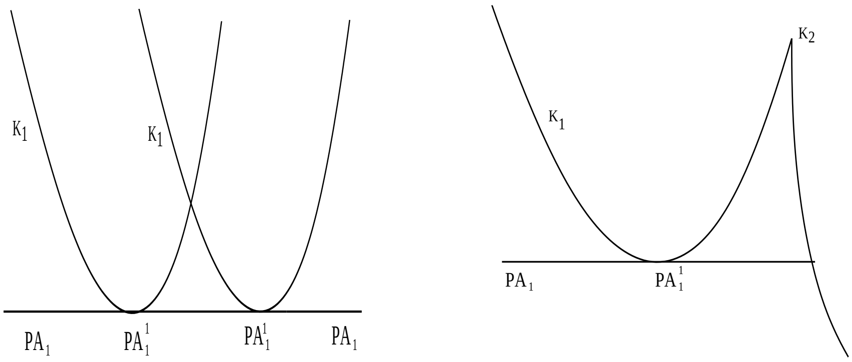

FIGURE 10. *Left-hand side: plane section of the $\mathcal{P}$-discriminant $\mathcal{P}A_1^{1,1,+}$. Right-hand side: plane section of the $\mathcal{P}$-discriminant $\mathcal{P}A_1^2$. The stratum K_j corresponds to the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection point of anomaly $j+1$.*

4. The classification with respect to inflection points.

The results that we have presented in section 3 are by-products of a classification method that we shall explain in this section.
In this section, several uses will be made of the symbol $\mathbb{C}^2$.
It will denote either the affine complex plane, the complex two dimensional vector space, the analytic manifold. If no precision is given then we are just considering the analytic manifold otherwise we shall specify the additional structure which is involved. We made that choice in order to avoid complicated notations.

4.1. The bordered Hessian. Denote by U an open neighbourhood of the affine complex plane $\mathbb{C}^2$.
Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function with isolated critical points. Consider the family of curves (V_ε) defined by:

$$V_\varepsilon = \{p \in U : f(p) = \varepsilon\}\,.$$

For simplicity, assume that:
1. the set of values of $\varepsilon \in f(U)$ for which the curve V_ε has a degenerate inflection point is finite.
2. $\dot{X}_f = \left\{p \in U : \ p \text{ is an inflection point of } V_{f(p)}\right\}$ is a curve.

Definition 2.19. The closure X_f of the curve $\dot{X}_f$ is called the *Hessian curve* of f (see figure 11).

Fix an affine coordinate-system in affine space $\mathbb{C}^2$.

Definition 2.20. The *bordered Hessian* , denoted Δ_f, of a holomorphic function $f : U \longrightarrow \mathbb{C}^2$ is the determinant of the matrix:

$$\begin{pmatrix} f_{xx} & f_{xy} & f_x \\ f_{xy} & f_{yy} & f_y \\ f_x & f_y & 0 \end{pmatrix}.$$

Remark. The bordered Hessian depends on the choice of the affine coordinate system.

According to Klein [**Klein1**], the following proposition is due to Plücker (for simplicity, we assume that f satisfies the conditions 1 and 2 stated above).

Proposition 2.1. *1. The Hessian curve X_f is given by:*

$$X_f = \{(x, y) \in U : \Delta_f(x, y) = 0\}\,.$$

2. The multiplicity of intersection at p of the curve V_ε with the Hessian curve $X_f = \{(x, y) \in U : \Delta_f(x, y) = 0\}$ is equal to the anomaly of p.

The proof of this proposition is elementary, it is given, in a more general form, in chapter 3, page 59.
Example. Consider the function $f(x, y) = y - x^3$. The Hessian curve of f is

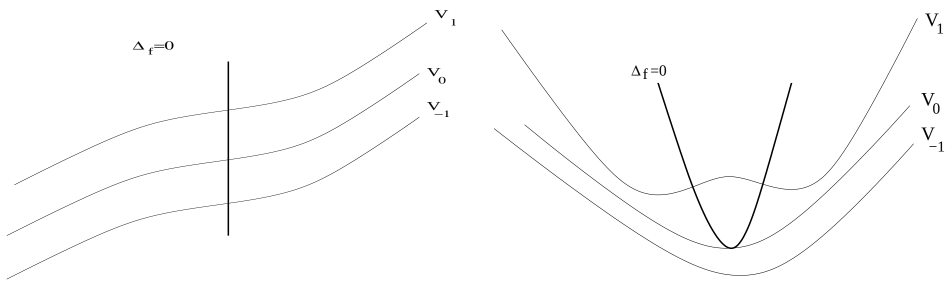

FIGURE 11. *Two examples of Hessian curves. The Hessian curve $X_f = \{(x,y) : \Delta_f(x,y) = 0\}$ intersects the curve $V_\varepsilon = \{(x,y) : f(x,y) = \varepsilon\}$ at an inflection point. If the point p is a non-degenerate inflection point then the intersection of V_ε with X_f at p is transversal (left-hand side).*

the line of equation $x = 0$ in the (x,y)-plane (see left part of figure 11).

The map f together with the function Δ_f contains all the information on the inflection points of the curves V_ε. Once Δ_f is calculated, we forget the affine structure of $U \subset \mathbb{C}^2$ and we study the map:

$$(f, \Delta_f) : U \longrightarrow \mathbb{C} \times \mathbb{C}.$$

This is the main idea for the classification of function-germs with respect to inflection points. In order to study the map (f, Δ_f), we apply $\mathcal{G}$-equivalence. This equivalence relation was introduced by V.V. Goryunov ([**Go**],[**AVGL2**]).

4.2. V-equivalence, $\mathcal{G}$-equivalence and the Plücker space. In this subsection, we recall basic facts of singularity theory ([**Tyu**],[**Math**],[**Mar**]) and introduce the $\mathcal{G}$-equivalence relation ([**Go**]).
A more detailed exposition of the notions introduced in this section is given in chapter 4.
We use the following notations:
1. $Diff(k)$ is the group of biholomorphic map-germs $\varphi : (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0)$ preserving the origin.
2. $\mathcal{O}_{x,y}$ is the ring of holomorphic function-germs $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$.
3. $GL(2, \mathcal{O}_{x,y})$ is the ring of invertible matrices with entries in $\mathcal{O}_{x,y}$.
The space $Diff(2) \times GL(2, \mathcal{O}_{x,y})$ can be endowed with a group structure which makes it a semi-direct product. This semi direct product is defined as follows.
Any matrix $A \in GL(2, \mathcal{O}_{x,y})$ is of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathcal{O}_{x,y}$ and $(\alpha\delta - \beta\gamma)(0) \neq 0$.
Given a holomorphic map germ $\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$, define $A \times \tilde{f}$ by:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tilde{f} = (\alpha f + \beta E_f, \gamma f + \delta E_f),$$

where $\tilde{f} = (f, E_f)$.
The action of $Diff(2) \times GL(2, \mathcal{O}_{x,y})$ on $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$ is defined by:

$$(\varphi, A).\tilde{f} = A \times (\tilde{f} \circ \varphi^{-1}).$$

The composition law in $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$ induces a semi-direct product structure on $Diff(2) \times GL(2, \mathcal{O}_{x,y})$, namely:

$$(\varphi, A).(\varphi', A') = (\varphi \circ \varphi', A \times (A' \circ \varphi)). \tag{5}$$

This group is denoted by $\mathcal{K}$, it is sometimes called the contact-group but we shall not use this terminology.

Definition 2.21. Two holomorphic map-germs $\tilde{f}, \tilde{g} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ are called *V-equivalent* provided that they are in the same orbit under the action of the group $\mathcal{K}$ defined above.

Definition 2.22. The *group* $\mathcal{G}$ is the subgroup of $Diff(2) \times GL(2, \mathcal{O}_{x,y})$ defined by the following condition:
$(\varphi, A) \in \mathcal{G} \iff \exists \alpha, \beta \in \mathcal{O}_{x,y},\ A = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}.$

Remark. With the notations of the definition, since A is invertible we have $\beta(0) \neq 0$.

Definition 2.23. Two holomorphic map-germs $\tilde{f}, \tilde{g} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ are called *$\mathcal{G}$-equivalent* provided that there exists a biholomorphic map-germ $\psi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that $\psi \circ \tilde{f}$ and $\tilde{g}$ are in the same orbit under the action of the group $\mathcal{G}$.

Definition 2.24. The *Plücker space* is the set of holomorphic map-germs $(g, E_g) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ for which there exists a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ and a coordinate system $\mathbb{C}^2$ centered at 0 such that the following $\mathcal{G}$-equivalence relation holds:

$$(f, \Delta_f) \sim (g, E_g).$$

Remark. The holomorphic function-germ $E_g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0) g$ is not necessarily the bordered Hessian of g.

Choose a vector space structure on $\mathbb{C}^2$.

Definition 2.25. Two holomorphic function-germs $f, g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ are called *$\mathcal{P}$-equivalent* if the map-germs $(f, \Delta_f), (g, \Delta_g)$ are $\mathcal{G}$-equivalent.

Remark. The function-germ Δ_f depends on the choice of the linear coordinates in $\mathbb{C}^2$. However, the $\mathcal{G}$-orbit of (f, Δ_f) depends only on the choice of the vector space structure in $\mathbb{C}^2$ and not on the choice of the linear coordinate system introduced for computing the bordered Hessians. Indeed, it is readily verified that the computations of the bordered Hessians in two distinct systems of linear coordinates are equal up to a multiplicative non-zero constant.

4.3. The finite determinacy theorem for $\mathcal{G}$-equivalence. Consider a holomorphic map-germ: $\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$. Put $\tilde{f} = (f, E_f)$. Fix coordinates (x, y) in $\mathbb{C}^2$ and in $\mathbb{C} \times \mathbb{C}$.

DEFINITION 2.26. The $\mathcal{G}$-*tangent space* to $\tilde{f}$, denoted $T\tilde{f}$ is the $\mathcal{O}_{x,y}$-submodule of $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$ generated by the 6 map-germs $x\partial_x \tilde{f}$, $x\partial_y \tilde{f}$, $y\partial_x \tilde{f}$, $y\partial_y \tilde{f}$, $(E_f, 0)$, $(0, E_f)$.

Remark. We have used here a shortened formulation. The $\mathcal{G}$-tangent space to $\tilde{f}$ is in fact the tangent space at the "point" $\tilde{f} \in (\mathcal{O}_{x,y} \times \mathcal{O}_{x,y})$ to the orbit of $\tilde{f}$ under the action of the group $\mathcal{G} \times Diff(1)$ (with the semi-direct product structure induced by the composition law). .

The space $(\mathcal{O}_{x,y} \times \mathcal{O}_{x,y})/T\tilde{f}$ is a $\mathbb{C}$-vector space.

DEFINITION 2.27. The $\mathcal{G}$-*Milnor number* of a holomorphic map-germ $\tilde{f}$: $(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ is defined by the formula

$$\mu_{\mathcal{G}}(\tilde{f}) = dim_{\mathbb{C}}[(\mathcal{O}_{x,y} \times \mathcal{O}_{x,y})/T\tilde{f}].$$

Remark. When no confusion is possible we simply write $\mu_{\mathcal{G}}$ instead of $\mu_{\mathcal{G}}(\tilde{f})$.

Denote by $\mathcal{M}^k$ the k^{th} power of the maximal ideal of $\mathcal{O}_{x,y}$.
Denote by $\widetilde{\mathcal{M}}^k$ the submodule of $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$ of map-germs of the type $g = (g_1, g_2)$ such that $g_1, g_2 \in \mathcal{M}^k$.
The following theorem is the *finite determinacy theorem* for $\mathcal{G}$-equivalence.

THEOREM 2.5. *Let $\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ be a holomorphic map-germ satisfying $\mu_{\mathcal{G}}(\tilde{f}) < +\infty$. Then for any map-germ $\psi \in \widetilde{\mathcal{M}}^{\mu_{\mathcal{G}}+1}$, we have the following $\mathcal{G}$-equivalence*

$$\tilde{f} + \psi \sim \tilde{f}.$$

This theorem is proved in a slightly a more general form in appendix B.
It is proved along the same lines than the standard finite determinacy theorem ([**Math**], [**Tyu**], [**Mar**]). I do not know whether Damon's general theory ([**Da**]) can be applied to the group $\mathcal{G}$ or not.

4.4. The Plücker theorem. Here is the most elementary result that one can obtain using the methods of the preceding subsection (compare chapter 1, section 2).

THEOREM 2.6. *If* $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ *is a generic Morse function-germ*[3] *then the pair* (f, Δ_f) *is* $\mathcal{G}$*-equivalent to* $(x^3 + y^3, xy)$.

Let $\bar{f}$ be a representative of a generic Morse function-germ.
Then, theorem 2.6 implies that:
1) there are 6 inflection points of the curve $V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\}$ that "vanish" at the origin when $\varepsilon \longrightarrow 0$ ([**Pl**]),
2) when ε turns counterclockwise around the origin the 6 inflection points permute of V_ε and one can number the inflection points in such a way that the resulting monodromy is (123)(456).

4.5. The $\mathcal{P}$-simple function germs. Following Arnold, who introduced the modality and the simplicity notions in [**Arn2**] for the case of critical points of functions, we introduce the notion of modality in Plücker space.
Assume that we are given an equivalence relation on a (finite dimensional) manifold M. Then according to Arnold, the *modality* of a point in M is the least number m such that a neighbourhood of the point is covered by a finite number of m-parameter families of equivalence classes[4]. If $m = 0$ then the point is called simple. More details on the modality under various circumstances can be found in [**AVGL1**].
Here and in all this subsection $\mathbb{C}^2$ *denotes the two-dimensional vector space.*
Consider the algebra $\bar{\mathcal{M}}^2$ of the holomorphic function-germs of the type

$$g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

with a critical point at the origin.
The algebra $\bar{\mathcal{M}}^2$ is not the square of the maximal ideal of $\mathcal{O}_{x,y}$ because we are considering germs of function defined on the vector space $\mathbb{C}^2$ and not on the analytic variety $\mathbb{C}^2$.

Let $f \in \bar{\mathcal{M}}^2$ be a holomorphic function-germ such that (f, Δ_f) has finite $\mathcal{G}$-Milnor number, that is $\mu_{\mathcal{G}}(f, \Delta_f) < +\infty$.

DEFINITION 2.28. The holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ has $\mathcal{P}$*-modality* m provided that m is the least number satisfying the following

[3] The Morse function-germs that do not satisfy the property form a semi-algebraic variety of codimension at least one in the space of r-jets of Morse function of two variables for any $r > 3$.

[4] One of the basic examples is the following.
Let $M = GL(2, \mathbb{R})$ be the space of invertible 2×2 matrices. Consider the equivalence relation:

$$A \sim B \iff (\exists P \in GL(2, \mathbb{R}) \text{ such that } A = PBP^{-1})$$

Let $\rho : GL(2, \mathbb{R}) \longrightarrow \mathbb{R}^2$ be the map sending a matrix A to its characteristic polynomial $det(A - Id.X)$.
Assume that A has two distinct eigenvalues. Then, in a neighbourhood of A the equivalence classes are parameterized by the value of ρ. Hence the modality of A is equal to 2.

property. For any $k > \mu_{\mathcal{G}}$, a neighbourhood of $j^k f \in J^k\bar{\mathcal{M}}^2$ is parameterized by a finite number of m-parameter families of k-jets of $\mathcal{P}$-equivalence classes.

Remark. The $\mathcal{G}$-finite determinacy theorem implies that m does not depend on the choice of k.

THEOREM 2.7. *For any $k > 3$, the variety of the non $\mathcal{P}$-simple function-germs is of codimension 2 in the space $J^k\bar{\mathcal{M}}^2$ of k-jets of maps with a critical point at the origin of critical value equal to zero.*

In order to give the complete list of the $\mathcal{P}$-simple function-germs, we need to state some definitions.

DEFINITION 2.29. A *$\mathcal{P}$-singularity class* is a subset of the space of function-germs $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ in $\bar{\mathcal{M}}^2$ which is invariant under $\mathcal{P}$-equivalence.

DEFINITION 2.30. A holomorphic Morse function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, with $p \leq q$, provided that the following property holds. There exists a linear coordinate-system in the vector space $\mathbb{C}^2$ such that the two branches of the zero-level set of f admit holomorphic parameterizations of the type:

$$\begin{array}{ccc} (\mathbb{C}, 0) & \longrightarrow & (\mathbb{C}^2, 0) \\ t & \mapsto & (t, t^{2+p} + o(t^{2+p})) \end{array}$$

and

$$\begin{array}{ccc} (\mathbb{C}, 0) & \longrightarrow & (\mathbb{C}^2, 0) \\ t & \mapsto & (t^{2+q} + o(t^{2+q}), t). \end{array}$$

Here p, q are either integers or equal to zero. If p or q is equal to zero, we omit to write it.

Example. The function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by the polynomial $f(x, y) = xy + x^4 + y^5$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{1,2}$.

DEFINITION 2.31. A holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ with an isolated critical point at the origin belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$ provided that the following property holds. There exists a linear coordinate-system in $\mathbb{C}^2$ such that the zero-level set of f admits a holomorphic parameterization of the type:

$$\begin{array}{ccc} (\mathbb{C}, 0) & \longrightarrow & (\mathbb{C}^2, 0) \\ t & \mapsto & (t^2 + o(t^2), t^3 + o(t^3)). \end{array}$$

THEOREM 2.8. *The (complex holomorphic) classes $\mathcal{P}A_1^{p,q}$ and $\mathcal{P}A_2$ coincide with the list of the $\mathcal{P}$-simple singularities. Moreover any function-germ belonging to one of these $\mathcal{P}$-singularity classes is $\mathcal{P}$-equivalent to one of the following function-germs:*

$\mathcal{P}A_1$	$\mathcal{P}A_1^p$	$\mathcal{P}A_1^{p,q}$	$\mathcal{P}A_2$
$xy + x^3 + y^3$	$xy + x^{p+3} + y^3$	$xy + x^{p+3} + y^{q+3}$	$y^2 + x^3$
$\tau = 0$	$\tau = p$	$\tau = p + q$	$\tau = 1$

where p, q are strictly positive integers. Here τ denotes the codimension of the $\mathcal{P}$-singularity class in the space $\bar{\mathcal{M}}^2$ (see below).

The techniques for proving theorem 2.7 and theorem 2.8 are developed in chapter 4. The details of the computations are given in appendix A.

We explain the meaning of the number τ in the table of theorem 2.8.
Recall that $\bar{\mathcal{M}}^2$ denotes the space of function germs:

$$g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

with a critical point at the origin.
Consider a function-germ $f \in \bar{\mathcal{M}}^2$ of the previous list. Remark that the $\mathcal{G}$-Milnor number of (f, Δ_f) for such a function-germ is finite.
Denote by V the variety of map-germs which are $\mathcal{P}$-equivalent to f.
The finite determinacy theorem for $\mathcal{G}$-equivalence implies that the codimension of $J^kV \subset J^k\bar{\mathcal{M}}^2$ does not depend on k provided that $k > \mu_{\mathcal{G}}(f, \Delta_f)$. This codimension is denoted by τ (because of the analogy with the Tyurina number of singularity theory).
Although, we defined it in the complex holomorphic case, the $\mathcal{P}$-equivalence relation has an immediate variant in the real C^∞ case.
Let $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ be a (real C^∞) Morse function-germ whose second differential at the origin has signature $(+,+)$ or $(-,-)$. Here $\mathbb{R}^2$ denotes the (real) two dimensional vector space.
The curves $V_\varepsilon = \{p \in U : f(p) = \varepsilon\}$ are either convex curves or empty, provided that $\varepsilon \neq 0$ is small enough. Hence we do not consider these cases.

Definition 2.32. A C^∞ function-germ $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ is *of the type* $\mathcal{P}A_1^{p,q,\pm}$ provided that the following property holds. There exists a linear coordinate-system in $\mathbb{R}^2$ such that the two branches of the zero-level set of f admit C^∞ parameterizations of the type:

$$\begin{array}{ccc} (\mathbb{R}, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ t & \mapsto & (t, t^{2+p} + o(t^{2+p})) \end{array}$$

and

$$\begin{array}{ccc} (\mathbb{R}, 0) & \longrightarrow & (\mathbb{R}^2, 0) \\ t & \mapsto & (\pm t^{2+q} + o(t^{2+q}), t). \end{array}$$

Here p, q are either integers or equal to zero. If p or q is equal to zero, we omit to write it.

Remark 1. The real C^∞ function-germs $f, g : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ defined by the polynomials $f(x, y) = xy + x^4 + y^4$ and $g(x, y) = xy - x^4 - y^4$ are $\mathcal{P}$-equivalent since $-f(-x, -y) = g(x, y)$.
Remark 2. For $p = q(mod2)$ the $\mathcal{P}$-singularity classes $\mathcal{P}A_1^{p,q,+}$ and $\mathcal{P}A_1^{p,q,-}$ are equal. We denote this $\mathcal{P}$-singularity by $\mathcal{P}A_1^{p,q}$ omitting the $+, -$ symbols.

The definition of the $\mathcal{P}$-singularity class $\mathcal{P}A_2$ in the real C^∞ case is obtained from the complex one by replacing the words complex holomorphic by real C^∞.

THEOREM 2.9. *Any (real C^∞) function germ belonging to the class $\mathcal{P}A_1^{p,q}$ or $\mathcal{P}A_2$ is $\mathcal{P}$-equivalent to one of the following function-germ:*

$\mathcal{P}A_1^{p,q}$	$\mathcal{P}A_1^{p,q,\pm}$	$\mathcal{P}A_2$
$xy+x^{p+3}+y^{q+3}$	$xy+x^{p+3}\pm y^{q+3}$	y^2+x^3
$\tau=p+q,\ p=q(mod2)$	$\tau=p+q,\ p\neq q(mod2)$	$\tau=1$

,

where $p\geq 0, q\geq 0$. If p or q is equal to 0 then we omit to write it.

4.6. Adjacencies of the $\mathcal{P}$-singularity classes.

DEFINITION 2.33. A $\mathcal{P}$-singularity class L is *adjacent* to a $\mathcal{P}$-singularity class K, denoted $L\longrightarrow K$, if every map $f\in L$ can be deformed to a map of class K by an arbitrary small perturbation.

If L is adjacent to K and K is adjacent to J then L is adjacent to J. We simply write:

$$L\longrightarrow K\longrightarrow J,$$

omitting the arrow between L and J.

Here are the list of all the adjacencies for the $\mathcal{P}$-simple singularities in the space $\bar{\mathcal{M}}^2$ of holomorphic function-germs with a critical point of critical value 0.
The $\mathcal{P}$-singularity class A_3 denotes the function-germs $f:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C},0)$ equal to y^2+x^4 up to a biholomorphic change of variables where $\mathbb{C}^2$ denotes the complex two dimensional vector space. The parenthesis means that this $\mathcal{P}$-singularity class is not $\mathcal{P}$-simple.

$$\begin{array}{cccccccccc}
 & & \mathcal{P}A_2 & \leftarrow & (A_3) & & & & & \\
 & \swarrow & & \swarrow & & & & & & \\
\mathcal{P}A_1 & \leftarrow & \mathcal{P}A_1^1 & \leftarrow & \mathcal{P}A_1^2 & \leftarrow & \mathcal{P}A_1^3 & \leftarrow & \mathcal{P}A_1^4 & \leftarrow\dots \\
 & & & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow \\
 & & & & \mathcal{P}A_1^{1,1} & \leftarrow & \mathcal{P}A_1^{1,2} & \leftarrow & \mathcal{P}A_1^{1,3} & \leftarrow\dots \\
 & & & & & & & \nwarrow & & \nwarrow \\
 & & & & & & & & \mathcal{P}A_1^{2,2} & \leftarrow\dots
\end{array}$$

Remark that there is only one non-simple class A_3 "bounding" the list of $\mathcal{P}$-simple singularities.
The relation with Kazarian's classification (in the plane case only) is as follows. Denote by K_j the set of function germs $g:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C},0)$ such that the origin is an inflection point of the 0 level-curve germ of g with anomaly $j+1$.

DEFINITION 2.34. A $\mathcal{P}$-singularity class L is *adjacent* to K_j, denoted $L\longrightarrow[K_j]$, if every function-germ $f\in L$ can be deformed to a map $g\in K_j$ by an arbitrary small perturbation

Remark. K_j is not a $\mathcal{P}$-singularity class for example both map-germ $f(x, y) = y + x^4$ and $g(x, y) = y + x^4 + yx^2$ are in K_2 and are not $\mathcal{P}$-equivalent. Indeed, denote respectively by Δ_f, Δ_g the Hessian determinants of f and g. Direct computations show that the origin is not an isolated critical point of Δ_f, while it is an isolated critical point of Δ_g.

The adjacencies with the K_j's are as follows:

$$\begin{tikzcd}
{[K_1]} & {[K_2]} \arrow[l] & {[K_3]} \arrow[l] & {[K_4]} \arrow[l] & {[K_5]} \arrow[l] & \dots \arrow[l] \\
\mathcal{P}A_1 \arrow[l, phantom] & \mathcal{P}A_1^1 \arrow[l] \arrow[ul] & \mathcal{P}A_1^2 \arrow[l] \arrow[ul] & \mathcal{P}A_1^3 \arrow[l] \arrow[ul] & \mathcal{P}A_1^4 \arrow[l] \arrow[ul] & \dots \arrow[l] \arrow[ul] \\
 & & \mathcal{P}A_1^{1,1} \arrow[ul] & \mathcal{P}A_1^{1,2} \arrow[l] \arrow[ul] & \mathcal{P}A_1^{1,3} \arrow[l] \arrow[ul] & \dots \arrow[l] \arrow[ul] \\
 & & & & \mathcal{P}A_1^{2,2} \arrow[ul] & \dots \arrow[l] \arrow[ul]
\end{tikzcd}$$

Remark that the adjacencies $\mathcal{P}A_1^1 \longrightarrow K_1$ and $\mathcal{P}A_1^2 \longrightarrow K_2$ can be seen in figure of the corresponding $\mathcal{P}$-discriminants that we depicted in subsection 3.4. The adjacencies listed above are direct corollaries of theorem 2.11 cited below in page 42.

5. $\mathcal{P}$-versal deformation theory.

In this section, we introduce the notion of versality with respect to inflection points of plane curves. As usual the versal deformation is the "largest" possible deformation.
In subsection 5.1, we define $\mathcal{G}$-versality and state the $\mathcal{G}$-versal deformation theorem.
In subsection 5.2, we define $\mathcal{P}$-versality, that is versality with respect to inflection points.
In subsection 5.3, we apply these techniques to the case of Morse functions.
A more detailed exposition of the notions developed in the two first subsections of this section is given in chapter 4.

5.1. Versal deformation theory for $\mathcal{G}$-equivalence. Consider a holomorphic map-germ:

$$\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0).$$

In this subsection, no affine structure is involved, in particular $\mathbb{C}^2$ denotes the analytical manifold and neither the affine complex plane nor the two-dimensional vector space.

Definition 2.35. A holomorphic map-germ $\tilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ such that $\tilde{F}(0, .) = \tilde{f}$ is called a *deformation* of $\tilde{f}$.

Definition 2.36. A deformation $\tilde{G} : (\mathbb{C}^r \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of $\tilde{f}$ is *induced* from a deformation $\tilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of $\tilde{f}$, provided that there exists a holomorphic map-germ $h : (\mathbb{C}^r, 0) \longrightarrow (\mathbb{C}^k, 0)$ such that $\tilde{G}(\lambda, .) = \tilde{F}(h(\lambda), .)$. We use the notation: $\tilde{G} = h^* \tilde{F}$.

Definition 2.37. The *translation* by a vector $u \in \mathbb{C}^2$ is the map-germ:

$$\begin{array}{cccc} \tau & (\mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C}^2, u) \\ & x & \longrightarrow & x + u. \end{array}$$

We denote by $T \approx \mathbb{C}^2$ the group of all translations.

Definition 2.38. Two deformations $\tilde{F}, \tilde{G} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ are called *$\mathcal{G}$-equivalent* provided that there exist holomorphic map-germs $\gamma : (\mathbb{C}^k, 0) \longrightarrow \mathcal{G} \oplus T$ $\psi : (\mathbb{C}^k \times \mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that the following equality holds identically

$$(\gamma(\lambda).(G_1, G_2))(\lambda, p) = (\psi(\lambda, F_1(\lambda, p)), F_2(\lambda, p))$$

and $\psi(0, .)$ is a biholomorphic map-germ.
Here $\tilde{F} = (F_1, F_2)$, $\tilde{G} = (G_1, G_2)$.

Definition 2.39. A deformation $\tilde{F}$ is called *$\mathcal{G}-$versal* if any other deformation of the same germ is $\mathcal{G}$-equivalent to a deformation induced from $\tilde{F}$.

Fix a coordinates $(\lambda_1, \ldots, \lambda_k)$ in $\mathbb{C}^k$ and (x, y) in $\mathbb{C}^2$.

DEFINITION 2.40. Let $\tilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ be a deformation of a holomorphic map-germ $\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$. The *$\mathcal{G}$-tangent space* to $\tilde{F}$ is the sum of the following $\mathbb{C}$-vector subspaces of $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$:
- the $\mathcal{O}_{x,y}$ module generated by the four map-germs $\partial_x \tilde{f}, \partial_y \tilde{f}, (E_f, 0),\ (0, E_f)$,
- the $\mathbb{C}$-vector space generated by the restriction to $\lambda = 0$ of the $\partial_{\lambda_i} \tilde{F}$'s,
- the $\mathbb{C}$-vector space generated by $(1, 0)$.
The tangent space to $\tilde{F}$ is denoted by $T\tilde{F}$.

Remark. We have used a shortened formulation. The tangent space is the restriction to $\lambda = 0$ of the tangent space at F to the orbit of F under the action of the group $Diff(1) \times \mathcal{G} \times T$ (with the obvious semi-direct product structure induced by the composition law).

The following theorem is the *versal deformation theorem for $\mathcal{G}$-equivalence.*

THEOREM 2.10. *A deformation $\tilde{F} : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of a holomorphic map-germ $\tilde{f} = \tilde{F}(0, .)$ is $\mathcal{G}$-versal provided that the equality*

$$T\tilde{F} = \mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$$

holds.

The proof is given in a slightly more general form in appendix B. It is analogous to the proof for the standard versal deformation theorem ([**Math**], [**Tyu**], [**Mar**]). I do not know whether this theorem follows from Damon's general theory or not ([**Da**]).

5.2. Versal deformation theory in Plücker space. In this subsection, we denote by $\mathbb{C}^2$ the (complex) two dimensional vector space.
Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function-germ.

DEFINITION 2.41. A holomorphic map-germ $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ such that $F(0, .) = f$ is called a *deformation of f*.

Fix coordinates $(\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$ and linear coordinates (x, y) in $\mathbb{C}^2$.
Given a deformation $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ of a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$.
We denote by $\Delta_F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ the Hessian determinant of F with respect to the variables (x, y).

DEFINITION 2.42. A deformation $F(\lambda, .)$ of a function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is called *$\mathcal{P}$-versal* if for any other deformation G of f, the germ of the deformation $\tilde{G} = (G, \Delta_G)$ is $\mathcal{G}$-equivalent to the germ of a deformation induced from $\tilde{F} = (F, \Delta_F)$.

5.3. $\mathcal{P}$-versal deformations of germs belonging to $\mathcal{P}A_1^{p,q}$. Denote by $\mathbb{C}^2$ the two-dimensional vector space.
Let $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function-germ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$.

Theorem 2.8 asserts that g is $\mathcal{P}$-equivalent to the holomorphic function-germ f defined by

$$\begin{array}{rccc} f: & (\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C},0) \\ & (x,y) & \mapsto & xy+x^{p+3}+y^{q+3}. \end{array}$$

We keep these notations.
For notational reasons assume that $pq>0$.

THEOREM 2.11. *The $p+q$-parameter deformation $F:(\mathbb{C}^{p+q}\times\mathbb{C}^2,0)\longrightarrow(\mathbb{C},0)$ of f defined by:*

$$F(\alpha,\beta,x,y)=xy+x^{p+3}+y^{q+3}+\sum_{j=1}^{p}\alpha_j x^{j+2}+\sum_{k=1}^{q}\beta_k y^{k+2}$$

is $\mathcal{P}$-versal.
Moreover, the deformation (F,Δ_F) is $\mathcal{G}$-equivalent to the deformation $(\alpha,\beta,x,y)\mapsto(P(\alpha,\beta,x,y),xy)$ defined by:

$$P(\alpha,\beta,x,y)=x^{p+3}+y^{q+3}+\sum_{j=1}^{p}\alpha_j x^{j+2}+\sum_{k=1}^{q}\beta_k y^{k+2}.$$

Here Δ_F denotes the bordered Hessian of F with respect to the variables (x,y), $\alpha=(\alpha_1,\dots,\alpha_p)$ and $\beta=(\beta_1,\dots,\beta_q)$.

*Remark.*If $p\neq 0$ and $q=0$, then the theorem holds if we replace the expressions of F and P by the function-germs defined by the polynomials

$$\begin{cases} F(\alpha,x,y)=xy+x^{p+3}+y^3+\sum_{j=1}^{p}\alpha_j x^{j+2}, \\ \quad P(\alpha,x,y)=x^{p+3}+y^3+\sum_{j=1}^{p}\alpha_j x^{j+2}. \end{cases}$$

The case $p=q=0$ has been considered in theorem 2.6.
The proof of this theorem is given appendix A.
The case $p=1,\ q=0$ is treated in chapter 4 page 95.
The results of section 3 on the bifurcation diagrams associated to Morse functions follow directly from this theorem.

6. The modality of map-germs with respect to flattening points.

In the previous sections, we have concentrated ourselves to the case of plane curves. In this section, we deal with the higher dimensional case. A more detailed exposition of the notions introduced in the first two subsections is given in chapter 3.

6.1. The generalized Wronskian. Denote by U a neighbourhood of the origin in the analytic space $\mathbb{C}^2$.
Consider a holomorphic map $H : U \longrightarrow \mathbb{C}$ with isolated critical points.
Notations. Denote by h the Hamilton vector field of H. Let $h.$ be the Lie derivative along the vector field h. We denote by $[g_1, \dots, g_n]$ the determinant whose columns are $(h.g_i, h.h.g_i, \dots, h^n g_i)$. Fix an affine coordinate system in the affine space $\mathbb{C}^n$.

Definition 2.43. The *(generalized) Wronskian* of a holomorphic map $\varphi : U \longrightarrow \mathbb{C}^n$ with respect to H is the determinant

$$\mathcal{D}[\varphi] = [\varphi_1, \dots, \varphi_n].$$

Remark. The generalized Wronskian depends on the choice of the analytic coordinate-system in $\mathbb{C}^2$ and on the affine coordinate-system in $\mathbb{C}^n$. However, the zero-level set of the generalized Wronskian does not depend on these choices.

In order to simplify the notations, we omit the dependence of the differential operator $\mathcal{D}$ on H and on the coordinate systems in $\mathbb{C}^2$ and $\mathbb{C}^n$.

Let $p \in U$ be a smooth point of the curve:

$$H_\varepsilon = \{p \in U : H(p) = \varepsilon\}.$$

Assume that φ is a holomorphic embedding.
Denote by V_ε the image of the curve H_ε under φ. The proof of the following proposition is analogous to that of proposition 3.3 (see chapter 3 section 1).

Proposition 2.2. *a. The curve H_ε intersects the variety $\{x \in \mathbb{C}^p : \mathcal{D}[\varphi] = 0\}$ at p if and only if $\varphi(p)$ is a flattening point of V_ε.*
b. A flattening point $\varphi(p)$ is a degenerate flattening of the curve V_ε if and only the multiplicity of the solution p of the system of equations

$$\begin{cases} H(p) &= \varepsilon, \\ \mathcal{D}[\varphi](p) &= 0. \end{cases}$$

is strictly higher than one (see figure 12).

6.2. $\mathcal{P}$-equivalence in higher dimensions. The map H together with the function $\mathcal{D}[\varphi]$ contains all the information on the flattening points of the curves V_ε. Once $\mathcal{D}[\varphi]$ is calculated, we forget the affine structure of $\mathbb{C}^n$ and we study the orbit of the map:

$$(H, \mathcal{D}[\varphi]) : U \longrightarrow \mathbb{C} \times \mathbb{C},$$

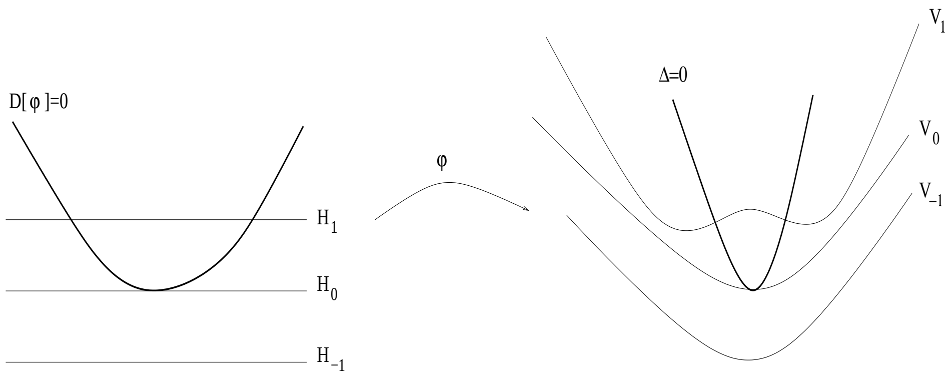

FIGURE 12. *The inflection points of $V_\varepsilon = \varphi(H_\varepsilon)$ are the images under φ of the intersection points of the curve $X = \{(x,y) \in \mathbb{R}^2 : \mathcal{D}[\varphi](x,y) = 0\}$ with the level-curve $H_\varepsilon = \{(x,y) \in \mathbb{R}^2 : H(x,y) = \varepsilon\}$. For $\varepsilon = 1$ the curve V_ε has two (real) inflection points that coalesce for $\varepsilon = 0$. The intersection of the curves H_0 and X at the origin is not transversal.*

under $\mathcal{G}$-equivalence.
We denote by $\mathbb{C}^n$ the n-dimensional complex vector space.

DEFINITION 2.44. Two holomorphic map-germs
$(H,\varphi),(H',\varphi') : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}\times\mathbb{C}^n,0)$ are called *$\mathcal{P}$-equivalent* if the map-germs
$(H,\mathcal{D}[\varphi]),\ (H',\mathcal{D}[\varphi']) : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}\times\mathbb{C},0)$ are $\mathcal{G}$-equivalent.

The $\mathcal{P}$-modality of a map germ $(H,\varphi) : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}\times\mathbb{C}^n,0)$ is the modality with respect to $\mathcal{P}$-equivalence.

6.3. A lower bound for the $\mathcal{P}$-modality. We fix the dimension n of the affine space $\mathbb{C}^n$.
We now state a theorem giving a lower bound for the $\mathcal{P}$-modality. This theorem is related to some combinatorics.
Denote by T_j the triangle of vertices $(0,0),\ (j,0),\ (0,n-j+1)$. Let h_j be the number of integer points lying on the hypotenuse of T_j distinct from the vertices. Let a_j be the number of integer points contained in the interior of T_j.
For $j \in \{1,\ldots,n\}$, we define the integer c_j by:

$$c_j = max\,\{0, j - a_j - h_j\}\,.$$

THEOREM 2.12. *Let $H : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C},0)$ be a holomorphic function-germ with a critical point at the origin. Then, for any holomorphic map-germ $\varphi : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^n,0)$ the $\mathcal{P}$-modality of the map-germ (H,φ) is not less* *than:*

$$m = (\sum_{j=1}^{n} c_j) - (n+1),$$

provided that $n > 2$.

Example 1. Let $n = 3$. We have three triangles depicted in the left part of figure 13. None of these triangles contains integer points in its interior. Consequently, the numbers a_i vanish

$$a_1 = 0,\ a_2 = 0,\ a_3 = 0.$$

Only the second triangle contains an integer point on its hypotenuse distinct from the vertices, thus the values of the h_j's are equal to

$$h_1 = 0,\ h_2 = 1,\ h_3 = 0.$$

We get that the values of the c_j's are equal to

$$c_1 = 1,\ c_2 = 1,\ c_3 = 3.$$

According to the theorem, for $n = 3$ the $\mathcal{P}$-modality of a pair (H, φ), where H has a critical point at the origin is at least equal to $5 - 4 = 1$.
Example 2. Let $n = 4$. We have 4 triangles depicted in the right part of figure 13. the values of the a_j's are equal to:

$$a_1 = 0,\ a_2 = 1,\ a_3 = 1,\ a_4 = 0.$$

None of the triangles contains integer points on its hypotenuse distinct from the vertices, thus the values of the h_i's vanish:

$$h_1 = 0,\ h_2 = 0,\ h_3 = 0,\ h_4 = 0.$$

We get that the values of the c_j's are equal to

$$c_1 = 1,\ c_2 = 1,\ c_3 = 2,\ c_4 = 4.$$

According to the theorem, for $n = 4$ the $\mathcal{P}$-modality of a pair (H, φ), where H has a critical point at the origin is at least equal to $8 - 5 = 3$.
Example 3. Let $n = 5$. We have 5 triangles (see figure 14). The sequences are:

$$\begin{cases} a_1 = 0, & a_2 = 1, \quad a_3 = 1, \quad a_4 = 1, \quad a_5 = 0 \\ h_1 = 0, & h_2 = 1, \quad h_3 = 2, \quad h_4 = 1, \quad h_5 = 0 \\ c_1 = 1, & c_2 = 0, \quad c_3 = 0, \quad c_4 = 2, \quad c_5 = 5 \end{cases}$$

According to the theorem, for $n = 5$ the $\mathcal{P}$-modality of a pair (H, φ), where H has a critical point at the origin is at least equal to $8 - 6 = 2$.

Remark that the first triangle and the last do not contain integer points in their interior, thus $c_1 + c_n = n + 1$.

Moreover the values of c_{n-1} are easily computed.
We have the equalities

$$c_{n-1} = \begin{cases} \frac{n-1}{2} & \text{for } n \text{ odd} \\ \\ \frac{n}{2} & \text{for } n \text{ even} \end{cases}$$

Thus we have the following corollary of theorem 2.12.

COROLLARY 2.1. *Let $H : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a map-germ with a critical point at the origin. Then, for any map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}^n$ the $\mathcal{P}$-modality of the map-germ (H, φ) is at least equal to $\frac{n-1}{2}$ provided that $n > 2$.*

In particular the list of the $\mathcal{P}$-simple map-germs for map-germs of the type

$$(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$$

such that H has a critical point at the origin coincides with the list of the $\mathcal{P}$-simple function germs of section 4.

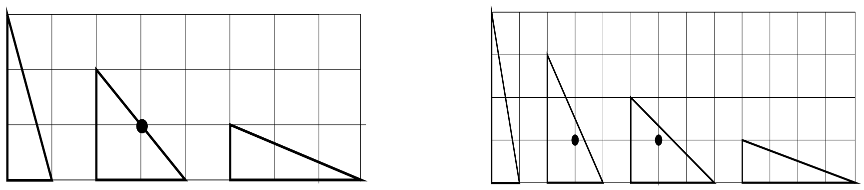

FIGURE 13. *Counting a lower bound of the $\mathcal{P}$-modality by means of combinatorics for the cases $n = 3$ and $n = 4$.*

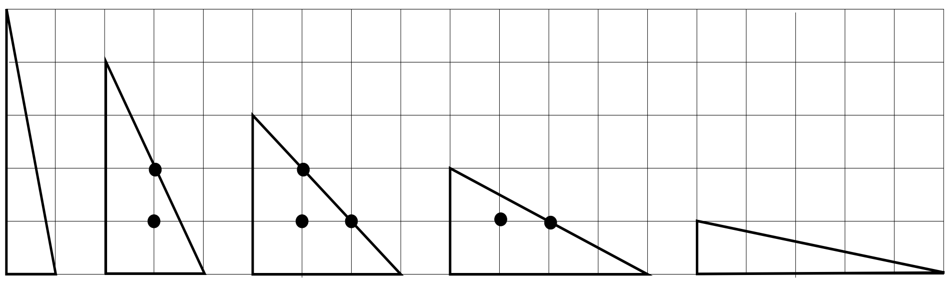

FIGURE 14. *Counting a lower bound of the $\mathcal{P}$-modality by means of combinatorics for the case $n = 5$.*

6.4. Vanishing flattening points at a node singular point. We now state a simple theorem generalizing theorem 2.6. This theorem is related to the lower bound for the $\mathcal{P}$-modality given in theorem 2.12, as we shall see in chapter 6.
In this subsection, $H : U \longrightarrow \mathbb{C}$ denotes the map defined by $H(x, y) = xy$. Here U is a neighbourhood of the origin in the analytic space $\mathbb{C}^2$ with a

fixed coordinate-system.
Consider a non-constant holomorphic map $\varphi : U \longrightarrow \mathbb{C}^n$ where $\mathbb{C}^n$ denotes the n-dimensional complex vector space. For simplicity, assume that φ preserves the origin that is $\varphi(0) = 0$ holds.
Denote by V_ε the image of the curve $H^{-1}(\varepsilon)$ under the holomorphic map φ.

DEFINITION 2.45. A value $x \in \mathbb{C}$ for which there exists y such that $\varphi(x, y)$ is a flattening point of V_ε, with $\varepsilon = H(x, y)$, is called a *label* of a flattening point of V_ε.

We use the old-fashioned language of multi-valued functions. Thereafter, we explain how to avoid it.
We shall prove the following theorem in chapter 6 section 1 (compare chapter 1, section 2).

THEOREM 2.13. *The vanishing flattening points of the curves V_ε when $\varepsilon \mapsto 0$, are given n-labels. These labels are $(n+1)$-valued function $x_1, \dots, x_n$ of the type:*

$$x_i(\varepsilon) = a_i \varepsilon^{\frac{i}{n+1}} + o(\varepsilon^{\frac{i}{n+1}}),$$

provided that $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ is generic[5].

Remark 1. There are n labels each of them is $n+1$-valued. Thus the number of vanishing flattening at a "generic" Morse singular points is $n(n+1)$. This is in accordance with the generalized Plücker formula (theorem 2.1 of page 18).
Remark 2. To avoid using multi-valued function put $\varepsilon = t^{n+1}$. Then, for each $i \in \{1, \dots, n\}$, instead of one $(n+1)$-valued function x_i we get $(n+1)$ holomorphic functions $x_{i,1}, \dots, x_{i,(n+1)}$ of the type:

$$x_{i,k} = a_i \omega^k t^i + o(t^i),$$

where $\omega = e^{\frac{2i\pi}{n+1}}$.

Theorem 2.13 does not give only the number of "vanishing flattening points", it also gives the monodromy of the vanishing flattening points that is how the points permute when ε makes a turn around the origin.
The monodromy is one of the so-called *projective topological invariants* that we shall study. The multi-valued functions giving the vanishing flattening points are the continuous invariants related to the extrinsic projective structure. Whereas the monodromy and the number of vanishing flattening points are discrete invariants related to it.

[5]In the space of k-jets $J_0^k(\mathbb{C}^2, \mathbb{C}^n)$ the set of maps that do not satisfy this theorem form a semi-algebraic variety of codimension at least one, for any $k > n$.

7. Projective-topological invariants.

7.1. Introduction. In [**Arn10**], Arnold introduced the notion of projective topological invariant of a variety in real or complex projective plane. It is a discrete invariant of the variety defined by means of the projective structure. For example the number of inflection points of a plane curve is a projective topological invariant.

Now, a *local projective topological invariant* (abbreviated PT-invariant) of a holomorphic function $f : U \longrightarrow \mathbb{C}$ is a discrete invariant depending on the projective structure $\mathbb{C}P^2$. Here $U \subset \mathbb{C}P^2$ denotes an open neighbourhood.

That an invariant of a holomorphic function $f : U \longrightarrow \mathbb{C}$ depends only on the projective structure means that for any projective transformation $A \in PGL(2, \mathbb{C})$, the invariant take the same value for f and for $f \circ A$.

As a general rule, we consider the open subset U of the affine plane $\mathbb{C}^2$ and not in $\mathbb{C}P^2$. This means that there exists a complex line $L \subset \mathbb{C}P^2$ which does not intersects U. The affine plane $\mathbb{C}^2$ is identified with $\mathbb{C}P^2 \setminus L$. The projective-topological of a holomorphic function $f : U \longrightarrow \mathbb{C}$ should not depend on the choice of the line L.

We already encountered a projective-topological invariant of a function: the sequence giving the number of vanishing flattening points at the critical points of the function.

Assume that f has just one critical point in U. Then, the number of inflection points of the plane holomorphic curves:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\},$$

that vanish at that critical point of f when ε approaches the corresponding critical value is a PT-invariant of f.

For example, we saw that the number of vanishing inflection points at a generic Morse critical point of a holomorphic function is equal to[6] 6 ([**Pl**]). The general formula for the number of vanishing flattening points of a family of curves at a singular point was given in section 2. We now turn on to subtler PT-invariants.

7.2. PT-invariants of a family of curves. Let $(V_\lambda), \lambda \in \Lambda$ be a family of holomorphic curves in $\mathbb{C}^2$. We recall that the values of the parameters $\lambda \in \Lambda$ for which the curve V_λ is either singular or has a degenerate inflection point is called the *Plücker-discriminant* (abbreviated $\mathcal{P}$-discriminant) of the family (V_λ) (see section 3).

Definition 2.46. A family of curves $(V_\lambda), \lambda \in \Lambda$ is called *good* provided that:

- for any $\lambda \in \Lambda$, the number of inflection points of the curve V_λ does not

[6] According to our classification by $\mathcal{P}$-singularity types (see section 4), $f : U \longrightarrow \mathbb{C}$ has a generic critical point at $p \in U$ means that the germ of f at p belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1$.

depend on the choice of λ,
- the $\mathcal{P}$-discriminant of (V_λ) is a variety of codimension one.

Example. Let $B_r = \{\varepsilon \in \mathbb{C} :\mid \varepsilon \mid < r\}$, $\tilde{B}_r = \left\{(x,y) \in \mathbb{C}^2 :\mid x \mid^2 + \mid y \mid^2 < r^2\right\}$. The family of complex holomorphic curves $(V_\lambda), \lambda \in B_3$:

$$V_\lambda = \left\{(x,y) \in \tilde{B}_2 : y = x^3 + \lambda\right\}$$

is not good. For $\mid \lambda \mid < 2$, the curve V_λ has one inflection point. For $\mid \lambda \mid \geq 2$ it has no inflection point. The "subfamily" $(V_\lambda), \lambda \in B_1$ is good.

Definition 2.47. The *PT-covering* (read projective-topological covering) of a good family of curves $(V_\lambda), \lambda \in \Lambda$ is the covering:
1. whose base is $B = \Lambda \setminus \Sigma$, where Σ denotes the $\mathcal{P}$-discriminant of (V_λ) .
2. whose fibre at a point $\lambda \in B$ is the set of inflection points of the curve $V_\lambda \subset \mathbb{C}^n$.

Remark. It is readily seen that the PT-covering is locally trivial (just apply the implicit function theorem).

Definition 2.48. The *PT-fundamental group* (read projective-topological fundamental group) of a good family of curves (V_λ) is the fundamental group of the complement $\Lambda \setminus \Sigma$ of the $\mathcal{P}$-discriminant Σ of (V_λ).

Definition 2.49. The *PT-monodromy group* (read projective-topological monodromy group) of a good family of curves is the monodromy group of the PT-covering.

Example. Consider the family $(V_\lambda), \lambda \in \mathbb{C}$ of holomorphic plane curves defined by:

$$V_\lambda = \left\{(x,y) \in \mathbb{C}^2 : y = x^4 - 6\lambda x^2\right\}.$$

Obviously, the Plücker discriminant $\Sigma \subset \mathbb{C}$ of this family is $\Sigma = \{0\}$.
The curve V_λ has two inflection points namely $p_\pm = (\pm\sqrt{\lambda}, -5\lambda^2)$ provided that $\lambda \neq 0$. The curve V_λ is parameterized by its first coordinate.
Thus, the PT-covering of (V_λ) is biholomorphically equivalent to the covering:

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^* \\ \alpha & \mapsto & \alpha^2. \end{array}$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
Indeed $(\alpha, -5\alpha^4)$ and $(-\alpha, -5\alpha^4)$ are the inflection points of V_λ for $\lambda = \alpha^2$.
Consequently, we get that:
1) the PT-fundamental group of the family (V_λ) is $\pi_1(\mathbb{C}^*) = \mathbb{Z}$,
2) the PT-monodromy group of the family (V_λ) is $\mathbb{Z}_2$.

7.3. *PT*-invariants of holomorphic functions. Let U be an open neighbourhood in the affine complex plane $\mathbb{C}^2$.
Let $f : U \longrightarrow \mathbb{C}$ be a holomorphic function with μ Morse critical points of $\mathcal{P}$-singularity class $\mathcal{P}A_1$ (roughly speaking "generic Morse critical points").

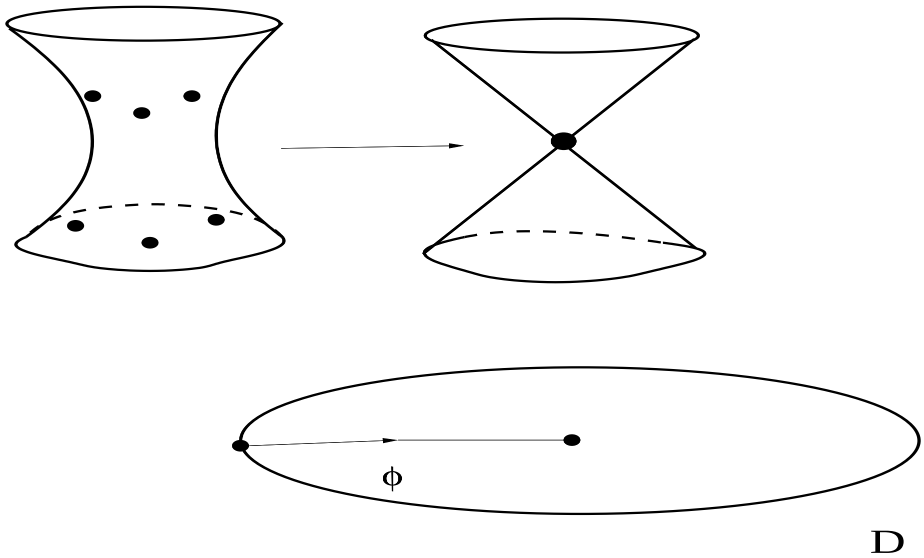

FIGURE 15. *Vanishing inflection points at a generic Morse critical point of a holomorphic function f from the topological view-point (compare with figure 1 of page 18). The complex curve of equation $f = \varepsilon$ is locally homeomorphic to a cylinder provided that $\varepsilon \neq 0$. The inflection points are* 6 *marked points on the cylinder. When ε goes to zero the* 6 *points converge towards the singular point of the cone.*

Denote by $\varepsilon_1, \ldots, \varepsilon_\mu$ the corresponding critical values. Assume that the curves:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\},$$

have non-degenerate inflection points except for a finite number of values $\varepsilon = v_1, \ldots, v_p$.
The holomorphic curve V_ε is a Riemann surface for $\varepsilon \notin \{\varepsilon_1, \ldots, \varepsilon_\mu\}$. When ε approach a critical value of f some of the inflection points vanish at the corresponding critical point.
From the topological view-point the local "behaviour" at a Morse critical point of the type $\mathcal{P}A_1$ is shown schematically in figure 15.
Indeed, assume that f has a Morse critical point at the origin of critical value equal to zero.
Then, the intersection of the surface V_ε with a small ball B centered at the origin is:
- homeomorphic to a cylinder provided that $\varepsilon \neq 0$ is small enough,
- homeomorphic to a cone for $\varepsilon = 0$.
To see it, just apply the Morse lemma stating that there exists (non-affine) coordinates $\tilde{x}, \tilde{y}$ such that in this coordinates f is given by

$$f(\tilde{x}, \tilde{y}) = \tilde{x}\tilde{y}.$$

Now, the curve $V_\varepsilon \subset \mathbb{C}^2$ has 6 inflection points vanishing at the critical point of the cone (Plücker formula again). These considerations "explain" figure 15.

DEFINITION 2.50. A holomorphic function $f : U \longrightarrow \mathbb{C}$ is called *good* provided that the family of curves (V_ε) defined by:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\}$$

for $\varepsilon \in f(U)$ is good.

DEFINITION 2.51. The *PT-covering* of a good function $f : U \longrightarrow \mathbb{C}$ is the PT-covering of the family of curves:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\} .$$

DEFINITION 2.52. The *PT-monodromy group* of a good function $f : U \longrightarrow \mathbb{C}$ is the PT-monodromy group of the family of curves:

$$V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\} .$$

We now define a braid group related to the projective structure. First, we recall some basic definitions from topology.

DEFINITION 2.53. The *configuration space* $B(X, k)$ of a topological manifold X is the topological space of k pairwise distinct unordered points on X:

$$B(X, n) = \{\{x_1, \dots, x_n\} : x_l \in X, x_j \neq x_k \text{ for } j \neq k\} .$$

DEFINITION 2.54. The *(classical) braid group* on k strands, denoted $Br(k)$, is the fundamental group of the configuration space $B(\mathbb{C}, k)$.

We come back to our discussion.
We shall use the notations $D = f(U)$, $D' = D \setminus \{\varepsilon_1, \dots, \varepsilon_\mu, v_1, \dots, v_p\}$.
Choose a base point $\alpha^* \in D'$. Fix a closed loop $\alpha \subset D'$ starting at α^*. The inflection points of V_ε lie on the intersection of V_ε with the smooth part of the Hessian curve X of f.
When t varies, the points of intersection of $V_{\alpha(t)}$ with the Hessian curve X vary continuously with t. Consequently each element of the fundamental group $\pi_1(D')$ gives rise to an element of $\pi_1(B(\dot{X}, k))$ where $\dot{X}$ is the smooth part of the Hessian curve X.

DEFINITION 2.55. The image of the group $\pi_1(D')$ in $\pi_1(B(\dot{X}, k))$ is called the *PT-braid group* of the holomorphic function $f : U \longrightarrow \mathbb{C}$.

DEFINITION 2.56. The PT-invariants defined above are called the *fundamental PT-invariants* of the holomorphic function f.

Example. Let $f : B_\delta \longrightarrow \mathbb{C}$ be a holomorphic function with a Morse critical point at the origin. Here $B_\delta \subset \mathbb{C}^2$ denotes the closed ball centered at the origin of radius δ. Assume that the germ of f at the origin belongs to the $\mathcal{P}$ singularity class $\mathcal{P}A_1$. Roughly speaking "the origin is a generic Morse critical point".

Let Δ_f be the Hessian determinant of f.
Theorem 2.6 asserts that the germ at the origin of (f, Δ_f) is $\mathcal{G}$-equivalent to the map germ

$$\begin{array}{lcl} (\mathbb{C}^2, 0) & \longrightarrow (\mathbb{C} \times \mathbb{C}, 0) & \\ (x, y) & mapsto & (x^3 + y^3, xy). \end{array}$$

This means that there exists $r \leq \delta$ such that for any $\varepsilon \in f(B_r)$, $\varepsilon \neq 0$, the two following facts hold.
1) The inflection points of $V_\varepsilon = \{(x, y) \in B_r : f(x, y) = \varepsilon\}$ are in one-to-one (biholomorphic) correspondence with the solutions of the system of equations

$$\begin{cases} x^3 + y^3 & = \varepsilon, \\ \quad xy & = 0. \end{cases} \tag{6}$$

2) The Hessian curve of f is biholomorphically equivalent to the curve $\{(x, y) \in B_r : xy = 0\}$.
Consequently the Hessian curve is homeomorphic to a cone.
As an immediate corollary of 1) and 2), we get that the PT-invariants of the restriction of f to B_r (under the smallness assumptions of r and ε stated above) are as follows (compare chapter 1 section 2):
- the PT-braid group of f is isomorphic to the group $\mathbb{Z}$,
- the PT-monodromy group of f is isomorphic to the cyclic group $\mathbb{Z}_3$ of order 3.

7.4. Plücker functions and the fundamental PT-invariants of a function-germ. Denote by B_r the closed ball of radius r centered at the origin in the complex two dimensional vector space $\mathbb{C}^2$.
By analogy with Morse functions, we define Plücker functions.

Definition 2.57. A holomorphic function $g : B_\delta \longrightarrow \mathbb{C}$ is called a *Plücker function* if the conditions 1,2,3,4,5 below are satisfied.
Denote by V_ε the curve

$$V_\varepsilon = \{p \in B_\delta : g(p) = \varepsilon\}$$

1) g is a Morse function,
2) there are only a finite number of values of the parameter $\varepsilon \in g(B_\delta)$ for which the curve V_ε has a degenerate inflection point,
3) the anomaly sequence at a degenerate inflection point of a curve V_t is equal to (2),
4) the number of singular points of a curve V_t plus the number of degenerate inflections of it is at most equal to one,
5) the Hessian curve of g is transverse to the boundary of the ball B_δ.

By analogy with the Morsification of a function germ, we define the Plückerization of a function germ.
Recall that for a given holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$, $N(f)$ is the number of vanishing flattening points of f (see page 17).
Denote by D the unit disk of $\mathbb{C}$ centered at the origin.

DEFINITION 2.58. A *Plückerization* of a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ with an isolated critical point at the origin is any Morse function $g : B_\delta \longrightarrow \mathbb{C}$ for which there exists a holomorphic function $G : D \times B_\delta \longrightarrow \mathbb{C}$ satisfying the conditions 1,2,3,4 below.
1) The germ of $G(0, .)$ at the origin is equal to f and $g_1 = g$.
2) for any $t \in]0, 1]$, g_t has exactly $\mu(f)$ distinct Morse critical points of the type $\mathcal{P}A_1$ with distinct critical values in the interior of B_δ,
3) for any $(t, \varepsilon) \notin \Sigma$, $V_{t,\varepsilon}$ has exactly $N(f)$ distinct inflection points in the interior of B_δ,
4) for each $t \in]0, 1]$, the function $G(t, .)$ is a Plücker function.

Remark. Like for the Morsification, a Plückerization of a function germ is not always possible. Take for example the function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by $f(x, y) = xy$.
Note that the zero-level set of f consists of two lines (and consequently is not a triangular curve).
The Hessian of f is equal to $\Delta_f(x, y) = 2xy$. In particular, the multiplicity at the origin of the system of equations

$$\begin{cases} f(x,y) & = & 0, \\ \Delta_f(x,y) & = & 0 \end{cases}$$

is undefined.
However such cases are of infinite codimension in the space of function-germs at the origin. If a function germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ admits a $\mathcal{P}$-versal deformation then one can construct the Plückerization from it (in the same way that the Morsification can be defined using a versal deformation [**AVGL1**]).

DEFINITION 2.59. The *fundamental PT-invariants* of a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ are the fundamental PT-invariants of any Plückerization of f.

Remark. It is readily verified that the fundamental PT-invariants of a holomorphic function-germ do not depend on the choice of the Plückerization.

7.5. Computation of some of the PT-invariants for the $\mathcal{P}$-simple singularities. We first need a definition. The PT-braid group of a holomorphic function is a subgroup of the fundamental group of a configuration space $B(\dot{X}, k)$. Here $\dot{X}$ is a holomorphic curve (the smooth part of a Hessian curve). Assume that the closure X of $\dot{X}$ has only one singular point. Let $\widetilde{X} \longrightarrow X$ be a resolution of the singular point. We get an inclusion map:

$$\mathcal{F} : B(\dot{X}, k) \longrightarrow B(\widetilde{X}, k)$$

which induces a group homomorphism

$$\mathcal{F}_* : \pi_1(B(\dot{X}, k)) \longrightarrow \pi_1(B(\widetilde{X}, k))$$

between the fundamental groups of both space. We call this (injective) group homomorphism, the forgetting homomorphism (because we forget the singular point).

THEOREM 2.14. *The image of the PT-braid groups $Br(Y)$ for $f \in Y$ under the forgetting homomorphism is as follows (in particular it does not depend on the choice of the resolution):*

$$\begin{aligned}\mathcal{F}_*(Br(\mathcal{P}A_1)) &= \mathbb{Z},\\ \mathcal{F}_*(Br(\mathcal{P}A_1^p)) &= Br(3+p) \oplus \mathbb{Z},\\ \mathcal{F}_*(Br(\mathcal{P}A_1^{p,q})) &= Br(3+p) \oplus Br(3+q).\end{aligned}$$

Here p, q denote strictly positive integers.

THEOREM 2.15. *The PT-monodromy group $\mathcal{S}(Y)$ of the $\mathcal{P}$-simple function germs $f \in Y$ are given by the following list:*

$$\begin{aligned}\mathcal{S}(\mathcal{P}A_1) &= \mathbb{Z}_3,\\ \mathcal{S}(\mathcal{P}A_1^p) &= \mathbb{Z}_3 \oplus \mathcal{S}_{p+3},\\ \mathcal{S}(\mathcal{P}A_1^{p,q}) &= \mathcal{S}_{p+3} \oplus \mathcal{S}_{q+3},\\ \mathcal{S}(\mathcal{P}A_2) &= SL(2, \mathbb{Z}_3).\end{aligned}$$

Here:
1. $\mathbb{Z}_3$ is the cyclic group of order 3.
2. $\mathcal{S}_k$ is the permutation group on k elements.
3. $SL(2, \mathbb{Z}_3)$ is the group of invertible linear transformation in the two dimensional vector space $(\mathbb{Z}_3)^2$ over the field $\mathbb{Z}_3$.
And p, q denote strictly positive integers.

Theorem 2.14 and theorem 2.15 are direct corollaries of theorem 2.11.
For the description of the PT-fundamental groups, we will use Dynkin type diagrams.
Notation. We consider the free group G generated by $a_1, \dots, a_k$. Each circle of the diagram represents an a_i. We describe the subgroup R of G giving the relation. A line between two circles corresponding to a_i, a_{i+1} means that $a_i a_{i+1} a_i (a_{i+1} a_i a_{i+1})^{-1} \in R$. A double line means that $(a_i a_{i+1})^4 (a_{i+1} a_i)^{-4} \in R$. The minimal invariant subgroup H containing R is called *the relation subgroup.* That is H is generated by the elements of R and by all their conjugates. The group corresponding to the Dynkin diagram is G/H.

Let $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a $\mathcal{P}$-versal deformation .
Take a representative $\bar{F} : \Lambda \times U \longrightarrow \mathbb{C}$ of the function germ F. Denote respectively by $\tilde{B}_r \subset \mathbb{C}^2$ and $B_\delta \subset \mathbb{C}^k \times \mathbb{C}$ the closed balls of radius r and δ centered at the origin in $\mathbb{C}^2$ and in $\mathbb{C}^k$.
Consider the family of curves:

$$V_{\lambda,\varepsilon} = \left\{ p \in U : \bar{F}(\lambda, p) = \varepsilon \right\}.$$

THEOREM 2.16. *Assume that $F(0,.)$ is a $\mathcal{P}$-simple function-germ. Then, for any r, δ small enough the PT-fundamental group of the family of curves $(V_{\lambda,\varepsilon} \cap \tilde{B}_r)$, $(\lambda, \varepsilon) \in B_\delta$ is given by one of the Dynkin type diagrams of figure 16.*

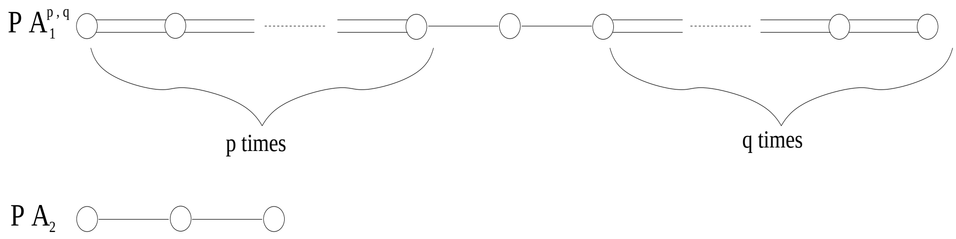

FIGURE 16. *Dynkin type diagrams for the $\mathcal{P}$-fundamental groups of the $\mathcal{P}$-simple singularities.*

The proof of this theorem is given in chapter 5. It requires the construction of a Lyaschko-Loojenga type mapping (see below).

7.6. A $K(\pi, 1)$ theorem for $\mathcal{P}$-simple singularities.

DEFINITION 2.60. A connected topological space X is called a $K(\pi, 1)$ *space* if $\pi_i(X) = 0$ for all $i > 1$.

One of the classical questions in singularity theory is the problem of knowing if the complement of a discriminant or of a bifurcation diagram is a $K(\pi, 1)$ space or not.
Consider a holomorphic function $G : \Lambda \times B_r \longrightarrow \mathbb{C}$ and the family of curves:

$$V_{\lambda,\varepsilon} = \{p \in B_r : G(\lambda, p) = \varepsilon\}\,.$$

Here B_r denotes the closed ball of radius r centered at the origin in the vector space $\mathbb{C}^2$.

DEFINITION 2.61. The *$\mathcal{P}$-bifurcation diagram* of the holomorphic function $G : \Lambda \times B_r \longrightarrow \mathbb{C}$ is the set of values of the parameter $\lambda \in \Lambda$ for which $G(\lambda, .)$ is not a Plücker function.

For the definition of a Plücker function see page 52.
Let $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a $\mathcal{P}$-versal deformation . Take a representative $\bar{F} : \Lambda \times U \longrightarrow \mathbb{C}$ of the function-germ F. Denote respectively by $B_\delta \subset \mathbb{C}^k$ and $\tilde{B}_r \subset \mathbb{C}^2$ the open balls of radius r and δ. Let $F_{\delta,r}$ be the restriction of $\bar{F}$ to $B_\delta \times \tilde{B}_r$.

THEOREM 2.17. *If $F(0,.)$ is a $\mathcal{P}$-simple function germ then for any (δ, r) small enough the complement of the $\mathcal{P}$-bifurcation diagram of $F_{\delta,r}$ is a $K(\pi, 1)$ space.*

Remark. I do not know whether a similar theorem does hold for $\mathcal{P}$-discriminants or not.

The proof of this theorem is given in chapter 5. It is based on the construction of a variant of the Lyaschko-Loojenga mapping, that we shall give in the next subsection.

7.7. A Lyaschko-Loojenga mapping for $\mathcal{P}A_1^{p,q}$. We keep the same notations than those of the preceding subsection.

DEFINITION 2.62. The *PT Lyaschko-Loojenga mapping* is the map sending $\lambda \in B_\delta \setminus \Delta$ to the polynomial whose roots are:
1) the critical values of $F_{\delta,r}(\lambda, .)$ and,
2) the values of ε for which the curve $V_{\lambda,\varepsilon} = \left\{p \in \tilde{B}_r : F(\lambda, p) = \varepsilon\right\}$ has a degenerate inflection point.

Using the covering property of the usual Lyaschko-Loojenga mapping ([**Arn4**], [**Loo**]), theorem 2.11 implies the following result (the details are given in chapter 5).

THEOREM 2.18. *If $f \in \mathcal{P}A_1^{p,q}$ then the PT Lyaschko-Loojenga mapping of $F_{\delta,r}$ defines a covering onto its image provided that r and δ are small enough. Moreover, the image of this PT Lyaschko-Loojenga mapping is diffeomorphic to the complement of the discriminant of the direct product $B_p \times B_q$ of the Coxeter group B_p and B_q.*

CHAPTER 3

The generalized Hessian.

Consider a non-constant holomorphic function $f : U \longrightarrow \mathbb{C}$, where U denotes an open neighbourhood of the affine complex plane $\mathbb{C}^2$.
The inflection points of a plane holomorphic curve:

$$V = \{(x, y) \in U : f(x, y) = 0\}$$

can be computed in terms of f.
Indeed, we saw in chapter 2, section 4 that one can consider a determinant Δ_f called the *bordered Hessian* of f such that the points of inflection of V lie on the curve

$$X = \{(x, y) \in U : \Delta_f(x, y) = 0\}.$$

The curve X_f is called the *Hessian curve.* In this chapter, we generalize this fact to higher dimensions.
Let U be an open neighbourhood in the affine space $\mathbb{C}^n$. Fix a complete intersection map:

$$f : U \longrightarrow \mathbb{C}^{n-1}$$

whose level-curves have isolated flattening points.
Given an affine coordinate-system in $\mathbb{C}^n$, we construct a holomorphic function $\Delta_f : U \longrightarrow \mathbb{C}$ called the *generalized Hessian determinant.*
The function Δ_f satisfies the following property: the flattening points of the level curves of f lie on the hypersurface:

$$X = \{p \in U : \Delta_f(p) = 0\}.$$

The hypersurface $X \subset \mathbb{C}^n$ will be called the *Hessian hypersurface.* We also study an analog of the bordered Hessian for the case of non-affine coordinate-systems.
We apply these techniques in order to generalize Plücker formula giving the number of vanishing inflection points at node and cusp singular points of space curves to higher dimensions.

1. The Hessian hypersurface.

1.1. The Hessian equation in affine coordinates. Let U be an open neighbourhood in the affine space $\mathbb{C}^n$.
In all this subsection we fix an arbitrary complete intersection map:

$$f : U \longrightarrow \mathbb{C}^{n-1}$$

and an affine coordinate system $x = (x_1, \ldots, x_n)$ in $\mathbb{C}^n$.

The calculation of the equation of the Hessian hypersurface (in case it is a hypersurface) is based on the choice of parameterizations of the curves V_ε. These parameterizations are given by the time of the so-called *generalized Hamilton vector field* that we shall now define.
We consider the restriction to $U \subset \mathbb{C}^n$ of the differential n-form:

$$\omega = dx_1 \wedge \ldots \wedge dx_n.$$

We denote abusively this restriction by the same symbol ω.
First, we recall the following definition.

Definition 3.1. The *interior product* $i_X\omega$ of ω with the vector field X is the $n-1$ differential form defined by:
$\forall p \in U$, $i_X\omega$ takes the value $\omega(p)(X(p), Y_1, \ldots, Y_{n-1})$ on arbitrary vectors $Y_1, \ldots, Y_{n-1}$ of the tangent plane to $\mathbb{C}^n$ at p.

The following proposition is straightforward ([**AVGL2**]).

Proposition 3.1. *There exists a unique holomorphic vector field X_f such that $i_{X_f}\omega = df_1 \wedge \ldots \wedge df_{n-1}$.*

Remark. The Hamilton vector-field X_f of f depends on the choice of the affine coordinates in $\mathbb{C}^n$.

Definition 3.2. The vector-field of the previous proposition is called *the (generalized) Hamilton vector field* of f.

Remark 1. The time differential of the Hamilton vector-field is defined by

$$\frac{dx_1 \wedge \ldots \wedge dx_n}{df_1 \wedge \ldots \wedge df_{n-1}}.$$

Remark 2. The Hamilton vector-field depends on the choice of the affine coordinate system.
Example 1. If $n = 2$, then the Hamilton vector field of $f : U \longrightarrow \mathbb{C}$ is given by the formula

$$X_f = (\partial_{x_2} f)\partial_{x_1} - (\partial_{x_1} f)\partial_{x_2}.$$

Definition 3.3. *The (generalized) Hessian determinant* Δ_f of f is the determinant of the matrix whose k^{th} column is formed by the k^{th} derivatives of the coordinates $x = (x_1, \ldots, x_n)$ with respect to the Hamilton vector field of f for $1 \leq k \leq n$.

Definition 3.4. The *Hessian hypersurface* of $f : U \longrightarrow \mathbb{C}^{n-1}$ is the zero level-set of the holomorphic function Δ_f.

Example. Assume that $n = 2$ and that $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is defined by the polynomial

$$f(x_1, x_2) = x_2 - x_1^3.$$

Then, the Hamilton vector-field of f is equal to

$$X_f = \partial_{x_1} + 3x_1^2 \partial_{x_2}.$$

In the old-fashioned notations, the Hamilton vector-field is defined by the Hamilton differential equations

$$\begin{cases} \dot{x}_1 & = & 1, \\ \dot{x}_2 & = & 3x_1^2. \end{cases}$$

The Hessian determinant Δ_f is equal to:

$$\begin{vmatrix} 1 & 3x_1^2 \\ 0 & 6x_1 \end{vmatrix} = 6x_1.$$

Straightforward computations show that we get the same result by computing the bordered Hessian of f (defined at page 31).
The Hessian curve of f is the plane curve of equation equation $x_1 = 0$.

Fix a point called 0 in $\mathbb{C}^n$. The proof of the following proposition is obvious (see also lemma 3.1 of page 60).

Proposition 3.2. *Let $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a linear non-degenerate map then the following equality holds*

$$\Delta_{f \circ A} = k(\Delta_f \circ A)$$

with $k = (detA)^{n(n+1)-1}$.

Thus up to a multiplicative non-zero constant, the function Δ_f does not depend on the choice of the affine coordinate system.
Consider the curve $V = \{x \in U : f(x) = 0\}$.

Proposition 3.3. *a. The curve V intersects the Hessian hypersurface $X = \{x \in U : \Delta_f(x) = 0\}$ at a smooth point p of V if and only if p is a flattening point of V (see figure 1).*
b. A flattening point $p \in V$ having an anomaly sequence equal to $(a_1, \dots, a_k)$ is a solution of the system of equations

$$\begin{cases} \quad f(p) & = \varepsilon, \\ \Delta_f(p) = 0 & \end{cases}$$

of multiplicity $\sum_{i=1}^k a_1$ (see figure 1, page 1).

1.2. Proof of proposition 3.3. Consider a biholomorphic map $\varphi : U \longrightarrow U'$ where U and U' denote open neighbourhoods in affine space $\mathbb{C}^n$. Fix affine coordinates $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$.
Denote respectively by TU and TU' the holomorphic tangent bundles over U and U'.
The map φ sends a holomorphic vector field $X : U' \longrightarrow TU'$ to a holomorphic vector field $Y : U \longrightarrow TU$ by:

$$Y(x) = ((D\varphi)(x))^{-1} X(\varphi(x)).$$

Here D denotes the derivative of φ. The map $(D\varphi)(x)$ is a linear map from T_xU to $T_{\varphi(x)}U'$. Hence $((D\varphi)(x))^{-1}$ is a linear map from $T_{\varphi(x)}U'$ to T_xU. We use the notation $Y = \varphi^* X$.

LEMMA 3.1. *Let $f : U \longrightarrow \mathbb{C}^{n-1}$, $g : U' \longrightarrow \mathbb{C}^{n-1}$ be two holomorphic maps such that $g = f \circ \varphi$. Then the Hamilton vector fields X_f, X_g of f and g satisfy the relation*

$$X_f(x) = det(B)(\varphi^* X_g)(x),$$

where $B : T_xU \longrightarrow T_{\varphi(x)}U'$ denotes the linear map $(D\varphi)(x)$.

Proof.
Let $Y_1, \dots, Y_n$ be arbitrary vectors of the tangent plane at $x \in \mathbb{C}^n$.
The map $B = (D\varphi)(x)$ sends a vector Y_i of tangent plane at $x \in U$ to the vector BY_i of the tangent plane at $\varphi(x) \in U'$. Moreover:

$$(df_1 \wedge \dots \wedge df_{n-1})(x).(Y_1 \dots, Y_{n-1}) = (dg_1 \wedge \dots \wedge dg_{n-1})(\varphi(x)).(BY_1 \dots, BY_{n-1}).$$

The definition of the Hamilton vector field implies:

$$(dg_1 \wedge \dots \wedge g_{n-1})(\varphi(x)).(BY_1 \dots, BY_{n-1}) = \omega(\varphi(x))(X_g(\varphi(x)), BY_1, \dots, BY_{n-1}).$$

Put $\xi(x) = B^{-1}(X_g(\varphi(x)))$. We get:

$$\omega(\varphi(x))(X_g(\varphi(x)), BY_1, \dots, BY_{n-1}) = det(B)\omega(x)(\xi(x), Y_1, \dots, Y_{n-1}).$$

Hence $X_f(x) = det(B)\xi(x)$. Lemma is proved.

Lemma 3.1 implies that in order to prove proposition 3.3, we can use an arbitrary affine coordinate-system in $\mathbb{C}^n$.
Recall that we consider only triangular curves. This means that the germ of V at a smooth point p is parameterized by a triangular holomorphic map-germ of the type

$$\begin{array}{rccc} \gamma : & (\mathbb{C}, 0) & \longrightarrow & (\mathbb{C}^n, 0) \\ & t & \longrightarrow & (t + (\dots), t^{\alpha_2} + (\dots), \dots, t^{\alpha_n} + (\dots)), \end{array}$$

with $0 < \alpha_2 < \dots < \alpha_n$. Here the dots stand for higher order terms in the Taylor series.
Recall from chapter 2 section 1 that by definition the anomaly sequence $(a_1, \dots, a_k)$ of the point $p \in V_\varepsilon$ is defined by

$$a_1 = \alpha_n - n, \dots, a_{n-j+1} = \alpha_j - j$$

where j is the least number such that $\alpha_j - j > 0$.
Take the point p for the origin of the time τ of the Hamilton vector-field.
The time τ of the Hamilton vector-field X_f can be expressed as a holomorphic function h of the parameter t.
The proof of the following lemma is straightforward.

LEMMA 3.2. *The Taylor expansion at the origin of the holomorphic function h is of the type $h(t) = ct + o(t)$ with $c \neq 0$.*

The restriction of the Hessian determinant Δ_f to V can be expressed as a holomorphic function δ of the parameter t.
By definition, the order of δ at the origin is equal to the multiplicity of intersection of the system $f = 0,\ \Delta_f = 0$.
Denote by W the determinant whose k^{th} column is the k^{th}-derivative of γ with respect to t.
Lemma 3.2 implies that the order of W at the origin is equal to the order of δ at the origin.
By straightforward computations, we get:

$$W(t) = ct^k + o(t^k),$$

where $k = \sum_{i=1}^{k} a_i$, $c \neq 0$. Proposition 3.3 is proved.

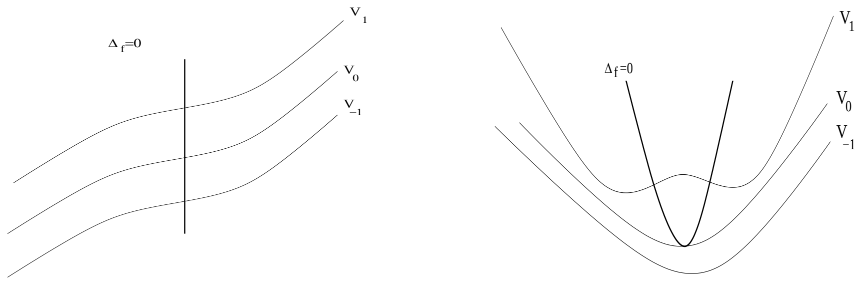

FIGURE 1. *Two examples of Hessian curves. At a non-degenerate inflection point the Hessian curve intersects transversally the level curve (left part).*

1.3. The Hessian equation in non-affine coordinates. Denote by U a neighbourhood of the origin in the analytic space $\mathbb{C}^p$.
Consider a complete intersection map $H : U \longrightarrow \mathbb{C}^{p-1}$ and a holomorphic map $\varphi : U \longrightarrow \mathbb{C}^n$ with $n \geq p$.
We fix a coordinate system in $\mathbb{C}^p$ and denote by h the (generalized) Hamilton vector field of H defined in the preceding subsection. Recall that this vector field depends on the choice of the coordinate system. Let $h.$ be the Lie derivative along the vector field h. We denote by $[g_1, \dots, g_n]$ the determinant whose k^{th} column is given by $(h.g_k, h.h.g_k, \dots, h^n g_k)$.

DEFINITION 3.5. The *generalized Wronskian* of the map φ (with respect to h) is the determinant

$$\mathcal{D}[\varphi] = [\varphi_1, \dots, \varphi_n].$$

Remark. In order to simplify the notations, we omit the dependence of the differential operator $\mathcal{D}$ on H and on the coordinate system.

Let $m \in U$ be a smooth point of the curve:

$$H_\varepsilon = \{z \in U : H(z) = \varepsilon\}.$$

Assume that $\varphi : U \longrightarrow \mathbb{C}^n$ is a holomorphic embedding.
Denote by V_ε the image of the curve H_ε under φ. The proof of the following proposition is analogous to that of proposition 3.3.

PROPOSITION 3.4. *a. The curve H_ε intersects the variety $\{x \in \mathbb{C}^p : \mathcal{D}[\varphi] = 0\}$ at m if and only if $\varphi(m)$ is a flattening point of V_ε.*
b. The multiplicity of the solution m of the system of equations

$$\begin{cases} H(m) &= \varepsilon, \\ \mathcal{D}[\varphi](m) &= 0 \end{cases}$$

is equal to $\sum_{i=1}^k a_i$ where $(a_1, \dots, a_k)$ is the anomaly sequence of the point $\varphi(m) \in V_\varepsilon$ (see figure 2).

Example 1. Consider the pair

$$\begin{array}{rcl} H(x,y) &=& y, \\ \varphi(x,y) &=& (x, x^4 + yx^2 + y). \end{array}$$

The Hamilton vector field of H is $h = \partial_x$.
We get the following values for the derivatives of φ along the Hamilton vector-field h

$$\begin{cases} (h.\varphi)(x,y) &= (1, 4x^3 + 2yx), \\ (h.h.\varphi)(x,y) &= (0, 12x^2 + 2y). \end{cases}$$

Consequently, the function $\mathcal{D}[\varphi]$ evaluated at (x,y) is equal to

$$\mathcal{D}[\varphi] = 12x^2 + 2y.$$

We apply proposition 3.4. The image of the curve $H_\varepsilon = \big\{(x,y) \in \mathbb{C}^2 : y = \varepsilon\big\}$ under φ has two non-degenerate inflection points provided that $\varepsilon \neq 0$. It has a degenerate inflection point at the origin if $\varepsilon = 0$ (see figure 2).
Take $\varepsilon \in \mathbb{R}$. If $\varepsilon > 0$ the two inflection points are real. If $\varepsilon < 0$ the two inflection points are complex conjugate.

Another fundamental example was given in chapter 1, section 2.

1.4. Vanishing flattening points at a Morse double point. In this subsection, we prove proposition 3.5 cited below. This proposition is a particular case of the generalized Plücker formula.
The flattening points of a curve are counted with multiplicities.
Denote by $H : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ the holomorphic function-germ defined by

$$H(x,y) = xy.$$

PROPOSITION 3.5. *For a generic holomorphic map $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ the number N of vanishing flattening points of the curves $\bar{\varphi}(\bar{H}_\varepsilon)$ when $\varepsilon \longrightarrow 0$ is equal to $n(n+1)$ for any representatives $\bar{\varphi}$, $\bar{H}$ of the germs of φ and H.*

Remark. The word generic means that in the space of N-jets at the origin of such map-germs, the set of map-germs that do not satisfy the theorem form a semi-algebraic variety of codimension at least one for any $N > n$.

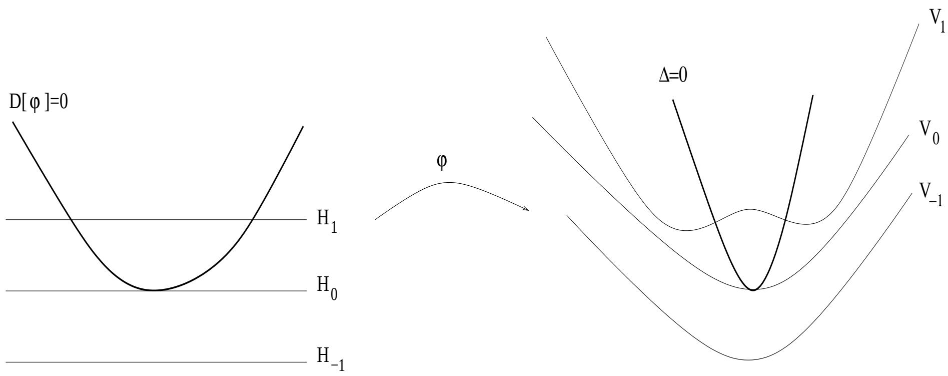

FIGURE 2. *The inflection points of V_ε are the images under φ of the intersection points of the curve $X = \left\{(x,y) \in \mathbb{C}^2 : \mathcal{D}[\varphi](x,y) = 0\right\}$ with $H_\varepsilon = \left\{(x,y) \in \mathbb{C}^2 : H(x,y) = \varepsilon\right\}$. For $\varepsilon = 1$ the curve V_ε has two real inflection points that coalesce for $\varepsilon = 0$. The intersection of the curves H_0 and X at the origin is not transversal.*

Proof.
Proposition 3.4 implies that the number N is equal to the multiplicity of the solution $x = 0,\ y = 0$ of the system of equations

$$\left\{\begin{array}{rcl} H(x,y) & = & 0 \\ \mathcal{D}[\varphi](x,y) & = & 0 \end{array}\right. \tag{7}$$

The plane curve-germ of equation $H = 0$ is the germ at the origin of the two coordinate axis in the plane x, y.
Denote respectively by $\delta_1,\ \delta_2 : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ the holomorphic function-germ obtained by substituting x by 0 and y by 0 in $\mathcal{D}[\varphi]$.
The holomorphic function-germ δ_i is of the form:

$$\delta_i(t) = c_i t^{N_i} + o(t^{N_i}),$$

with $i = 1$ or $i = 2$.
By definition of the multiplicity of a solution of a system of equations, the number N is equal to $N_1 + N_2$.
Let $\psi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, 0)$ be the restriction of the map $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ to the line of equation $y = 0$.
The holomorphic map ψ can be represented in the form

$$\psi(t) = a_1 t + \cdots + a_n t^n + o(t^n)$$

where the a_i's are vectors of $\mathbb{C}^n$.
If the map φ is generic then the a_i's form a basis of $\mathbb{C}^n$.
Denote by D the restriction of the derivation along the Hamilton vector field of H to $y = 0$.

In the basis $a_1, \dots, a_n$, the function-germ δ_1 is the determinant of the matrix whose k^{th} column is

$$(Dt^k, D^{(2)}t^k, \dots, D^{(n)}t^k), \tag{8}$$

where $D^{(m)}$ is the m^{th} derivative with respect to D.

LEMMA 3.3. *We have the equality $Dt^k = kt^k$ for any integer $k > 0$.*

Proof.
The Hamilton differential equations are:

$$\begin{cases} \dot{x}(x,y) &= & x \\ \dot{y}(x,y) &= & -y \end{cases} \tag{9}$$

The first equation of the system can be interpreted as follows.
The derivative of the linear function $(x,y) \mapsto x$ along the Hamilton vector-field is equal to $(x,y) \mapsto x$. Consequently the derivative of the function t with respect to D satisfies the equality $Dt = t$. This proves the lemma.

This lemma implies that the expression (8) is equal to

$$t^k(k, k^2, \dots, k^n).$$

Consequently, the first term appearing in the Taylor series at the origin of δ_1 with a possibly non-zero coefficient is the determinant of the matrix

$$t^{\frac{n(n+1)}{2}} \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^2 & 3^2 & \dots & n^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^n & 3^n & \dots & n^n. \end{pmatrix}$$

Thus, we get the equality

$$\delta_1(t) = c_1 t^{\frac{n(n+1)}{2}} + o(t^{n(n+1)})$$

where c_1 is the Vandermonde determinant of $(1, 2, \dots, n)$ times $n!$, in particular $c_1 \neq 0$.
Similarly, we get the equality

$$\delta_2(t) = c_2 t^{\frac{n(n+1)}{2}} + o(t^{n(n+1)})$$

with $c_2 \neq 0$.
From this two last equalities, we deduce that the multiplicity of the solution $x = 0, y = 0$ of the system (7) is equal to $n(n+1)$. Proposition is proved.

2. Generalized Plücker formulas.

The techniques introduced in the preceding chapter allow us to generalize Plücker's theorem on the 6 vanishing inflections at a "generic" Morse critical point. The proof is related to the classical Poncelet-Plücker formula and to polar varieties. So, we would like to make a digression on this subject and prove a generalization of the Poncelet-Plücker formula. This digression will show the relationship between vanishing flattening and polar varieties. We point out that the generalized Poncelet-Plücker formula giving the degree of the dual curve to a given curve that we will obtain is not new. Indeed, B. Teissier obtained a generalization of the Poncelet-Plücker formula to hypersurfaces ([**Tei2**]). Teissier's formula was completed by Kleiman ([**Kl**]) who obtained a formula much more general than ours. However, our proof is elementary and lies on our way[1]. Hence, we have thought that it was worthwhile to present it and to relate it to the calculation of vanishing flattening points.

2.1. The classical Poncelet-Plücker formula. The *dual projective space* $(\mathbb{C}P^n)^\vee$ to $\mathbb{C}P^n$ is the space of the hyperplanes of $\mathbb{C}P^n$. The space $(\mathbb{C}P^n)^\vee$ can be identified with $\mathbb{C}P^n$. Indeed:
- a point of $\mathbb{C}P^n$ is a complex line in $\mathbb{C}^{n+1}$ passing through the origin,
- a point in $(\mathbb{C}P^n)^\vee$ is a complex hyperplane in $\mathbb{C}^{n+1}$ passing through the origin.
Fix an Hermitian product in $\mathbb{C}^{n+1}$. Then, to a complex hyperplane passing through the origin corresponds the Hermitian orthogonal complex line passing through the origin.
Let $V \subset \mathbb{C}P^n$ be an algebraic manifold.
If V is smooth then the *dual variety* $V^\vee$ to V is the variety formed by the hyperplanes tangent to V in dual projective space $(\mathbb{C}P^n)^\vee$.
If V has isolated singular points then the dual variety is the closure of the dual variety of the smooth part of V.

Consider an algebraic curve V of degree d in projective space $\mathbb{C}P^2$. First assume that V is smooth. According to Poncelet ([**Poncelet**]), the degree $d^\vee$ of the dual curve to V is given by the formula

$$d^\vee = d(d-1).$$

This is essentially Bézout's theorem. Indeed, a complex line $l^\vee$ in dual projective space $(\mathbb{C}P^2)^\vee$ corresponds to a pencil of complex lines (L_t), $t \in \mathbb{C}P^1$ in $\mathbb{C}P^2$ (since each point of the line $l^\vee$ corresponds to a line in $\mathbb{C}P^n$). The statement: "the complex line $l^\vee$ intersects the dual curve $V^\vee$ at a point $p^\vee$" means that the complex line corresponding to the point $p^\vee$ is tangent to the curve V. Consequently the degree of the dual curve is the number of complex lines of the pencil (L_t) tangent to the curve (counted

[1] Our proof uses Picard-Lefschetz theory, hence it is not completely self-contained.

with multiplicities).
Choose homogeneous coordinates $[x : y : z]$ in $\mathbb{C}P^2$. Take the pencil $L_t = \{[x : y : z] \in \mathbb{C}P^2 : x = tz\}$.
Let $f(x, y, z) = 0$ be an equation of the curve V. Assume that the complex line $z = 0$ is not tangent to the curve. Put $g(x, y) = f(x, y, 1)$. Then the points $[x : y : 1]$ where the complex line L_t is tangent to V satisfy:

$$\begin{cases} g(x,y) & = & 0, \\ \partial_y g(x,y) & = & 0. \end{cases}$$

The first equation is of degree d, the second equation is of degree $d-1$. The Bézout theorem implies that the number of solution of this system, counted with multiplicities, is equal to

$$d^{\vee} = d(d-1). \tag{10}$$

This proves Poncelet's formula.
Remark that the points of V for which there exists a complex line of the pencil (L_t) tangent to the curve V lie on the curve:

$$\left\{(x,y) \in \mathbb{C}^2 : \partial_y g(x,y) = 0\right\}.$$

The closure of this curve in $\mathbb{C}P^2$ is called *the polar curve* associated to the pencil (L_t). We denote it by Γ (see figure 3 on page 67).
Denote by $(V.\Gamma)$ the number of points of intersection of V with the polar curve Γ counted with multiplicities and $deg(\Gamma)$ stands for the degree of the curve Γ.
Then the Poncelet formula (10) can be divided into two parts, as follows

$$\begin{cases} d^{\vee} & = & (V.\Gamma), \\ deg(\Gamma) & = & (d-1). \end{cases} \tag{11}$$

Here, we have identified the affine space $\mathbb{C}^2$ with the complex projective plane $\mathbb{C}P^2$ minus the complex line $\{[x : y : z] : z = 0\}$. That is we represent the affine plane as

$$\mathbb{C}^2 \approx \left\{[x : y : z] \in \mathbb{C}P^2 : z \neq 0\right\}.$$

The identification is given by $[x : y : z] \mapsto (x/z, y/z)$. This makes sense since $z \neq 0$.

We come back to Poncelet's formula. Plücker observed that the formula fails if V is not smooth. Geometrically, the reason is as follows (see figure 3 on page 67). Consider a holomorphic function $f : U \longrightarrow \mathbb{C}$ with an only critical point in $U \subset \mathbb{C}^2$. For simplicity, assume that:
- the critical point is the origin,
- the critical point is a Morse critical point,
- the critical value is equal to zero.

Put $V_\varepsilon = \{(x, y) \in U : f(x, y) = \varepsilon\}$. Fix a direction transversal to the branches of V_0, say $x = constant$ for simplicity.
For a real value $\varepsilon \neq 0$, there are two real points of the curve V_ε for which

a line $x = constant$ is tangent to V_ε. When $\varepsilon \longrightarrow 0$ these two real tangent lines "vanish" at the singular point (see figure 3). According to Plücker ([**Pl**]), if $V \subset \mathbb{C}P^2$ is an algebraic curve of degree d whose singular points are Morse double points (biholomorphically equivalent to $xy = 0$) then the degree $d^\vee$ of the curve dual to V is ([**Pl**]):

$$d^\vee = d(d-1) - 2\delta \tag{12}$$

where δ is the number of Morse double points of the curve V.
More generally, the general Poncelet formula (10) for a singular curve is given by:

$$d^\vee = d(d-1) - \sum_{p \in \Sigma} (V.\Gamma)_p, \tag{13}$$

where Σ denotes the set of singular points of the curve Γ and $(V.\Gamma)_p$ is the multiplicity of intersection at p of V with Γ. However,it should be remarked that this formula doesn't tell us anything unless we know the value of $(V.\Gamma)_p$ either explicitly or in terms of the usual invariants of a singular point.
For example, Plücker's result (formula (12)) asserts that $(V.\Gamma)_p = 2$ if p is a Morse double point of V (see figure 3).

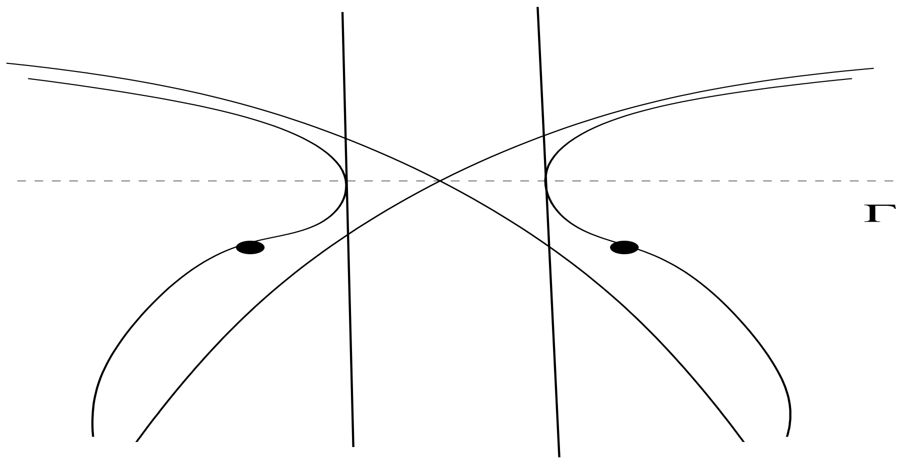

FIGURE 3. *There are two vanishing tangent lines at a node singular point. The polar curve Γ intersects the singular curve with multiplicity equal to two at the singular point.*

2.2. The Poncelet-Plücker formula for space curves. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be an complete intersection map-germ.
Choose coordinates $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ such that the hyperplane of equation

$x_n = 0$ is transverse to the each branch[2] of the curve germ of equation $f = 0$ in $\mathbb{C}^n$.

DEFINITION 3.6. The multiplicity of 0 as a solution of the system of equations

$$\begin{cases} f(x) &= 0, \\ x_n &= 0 \end{cases}$$

is called the *multiplicity* of the map-germ f.

Example. Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be the holomorphic map-germ defined by $f(x, y) = y^3 + x^4 + y^4$. The multiplicity of f is equal to 3. More generally write $f = f_k + \dots$ where f_k is a non-zero homogeneous polynomial of degree k and the dots stand for terms of order higher than k in the Taylor series of f. Then the multiplicity of f is equal to k.

THEOREM 3.1. *Let $\bar{V} \subset \mathbb{C}P^n$ be an algebraic curve given by a system of $n-1$ reduced polynomial equations (i.e. a one-dimensional complete intersection)*

$$g_1 = \dots = \dots = g_{n-1} = 0.$$

Then the degree $d^{\vee}$ of the variety dual to V is equal to

$$d^{\vee} = d \prod_{i=1}^{n-1} (d_i - 1) - \sum_{p \in \Sigma} (m_p + \mu_p - 1),$$

where:
- d_i denotes the degree of the polynomial g_i,
- Σ is the set of singular points of V,
- μ_p and m_p denote respectively the Milnor number and the multiplicity of the germ of $g = (g_1, \dots, g_{n-1})$ at $p \in V$.

Example. Assume that V is a plane curve and that V has no other singular points than Morse double points (biholomorphically equivalent to $xy = 0$) and semi-cubical cusp singular points (biholomorphically equivalent to $y^2 = x^3$).
The multiplicities of both node singular point and cusp singular points are $m = 2$. The Milnor number of the node singular point and of the cusp singular point are respectively $\mu = 1$ and $\mu = 2$. Consequently the formula gives the classical Poncelet-Plücker formula ([**Pl**]):

$$d^{\vee} = d(d - 1) - 2\delta - 3\kappa \tag{14}$$

where δ is the number of Morse double points of the curve and κ is the number of cusps singular points of the curve.

[2]Transverse to a branch means transverse to the limiting tangent line to the branch at the singular point.

2.3. Teissier numbers and the generalized Plücker formula. Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be a complete intersection map-germ with an isolated critical point at the origin.
Let $(V, 0)$ be the curve-germ defined as the zero level-set of f. Choose a linear function $l : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ such that the hyperplane $l = 0$ intersects transversally each branch of the curve germ $(V, 0)$.
Recall that the critical locus of a map is the set of points for which the determinant of derivative of the map vanishes.

DEFINITION 3.7. The critical locus of the map-germ:

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \longrightarrow & (\mathbb{C}^n, 0) \\ x & \mapsto & (f(x), l(x)) \end{array}$$

is called the *polar hypersurface-germ* of f associated to l.

Example. Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be defined by $f(x, y) = y^2 + x^k$ then the polar curve is the germ at the origin of the curve $\Gamma = \{(x, y) : y = 0\}$.

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be a complete intersection-map. Let $(V, 0)$ be the curve-germ defined as the zero level-set of f. We assume that $(V, 0)$ is a triangular curve-germ.
Denote by $(C_1, 0), \dots, (C_r, 0)$ the branches-germ of $(V, 0)$. Let $(\Gamma, 0)$ be a polar hypersurface-germ of f.

DEFINITION 3.8. The *Teissier numbers* of f, denoted $\tau_1(f), \dots, \tau_r(f)$, of a curve-germ $(V, 0)$ in affine space $\mathbb{C}^n$ (or in projective space $\mathbb{C}P^n$) are the intersection multiplicities of the branch-germ $(C_1, 0), \dots, (C_r, 0)$ with the polar curve-germ $(\Gamma, 0)$.

Theorem 3.1 follows directly from the following proposition.

PROPOSITION 3.6. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be a non-degenerate complete intersection map-germ then:*

$$\sum_{k=1}^{r} \tau_k(f) = m(f) + \mu(f) - 1.$$

Here $m(f), \mu(f)$ denote respectively the multiplicity of f and the Milnor number of f. The number $\tau_1(f), \dots, \tau_r(f)$ are the Teissier numbers of f.

Remark. An analogous proposition holds for hypersurfaces ([**Tei1**]). A similar equality holds for arbitrary analytic subvarieties of $\mathbb{C}P^n$ ([**Kl**]).
Example. Fix coordinates x, y in $\mathbb{C}^2$.
Let $f(x, y) = x^2 - y^2$. We have $\mu(f) = 1$. Consider, the polar curve-germ $(\Gamma, 0)$ associated to the linear function $l(x, y) = x$. The germ at the origin of $\{(x, y) \in \mathbb{C}^2 : y = 0\}$ is equal to Γ. Consequently:

$$\tau_1(f) = \tau_2(f) = 1.$$

And the equality $1 + 1 = 2 + 1 - 1$ holds.

With the same notations than those of definition 3.8, we have the following splitting of the generalized Plücker formula (theorem 2.1, page 18).

THEOREM 3.2. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be a complete intersection map-germ such that the curve germ $f^{-1}(0)$ is triangular. Then the multiplicity of intersection of the Hessian hypersurface-germ of f with the branch-germ $(C_k, 0)$ is given by the formula*

$$N_k = \frac{n(n+1)}{2}(\tau_k - m_k + 1) + \eta_k$$

where:
- η_k is the sum of the elements of the anomaly sequence of C_k at the origin ($\eta_k = 0$ if the sequence is empty).,
- m_k is the multiplicity of C_k at the origin.

Remark 1. Combining theorem 3.2 with proposition 3.6 we get the generalized Plücker formula:

$$N(f) = \frac{n(n+1)}{2}(\mu + r - 1) + \sum_{k=1}^{r} \eta_k.$$

Remark 2 (for specialists). We defined the generic number of vanishing flattening points $N[X]$ for a given singularity class (see chapter 2 subsection 2.3). Similarly, we can define the generic number $N_k[X]$ corresponding to the number N_k of theorem 3.2.
Obviously we have the equality

$$N[X] = \sum_{k=1}^{r} N_k[X].$$

It is readily verified that the left hand-side of the equality is a topological invariant while the right hand-side in only an analytical invariant.

2.4. Proof of theorem 3.2. We fix a coordinate system t i $\mathbb{C}$.
Since the branch curve-germ $(C_k, 0)$ of $(V_0, 0)$ is triangular, there exists affine coordinates in $\mathbb{C}^n$ such that the curve germ $(C_k, 0)$ admits a parameterization:

$$\begin{array}{rccc} \gamma_k : (\mathbb{C}, 0) & \longrightarrow & (\mathbb{C}^n, 0) \\ t & \mapsto & (t^{\alpha_1^k} + (\dots), \dots, t^{\alpha_n^k} + (\dots)). \end{array} \tag{15}$$

Here the dots denote higher order terms in the Taylor series.
Denote by:
- $\mathcal{O}_{C_k}$ the ring of holomorphic function-germs on the curve-germ $(C_k, 0)$,
- $\mathcal{O}_t$ the ring of holomorphic function-germ in the variable $t \in \mathbb{C}$.
The parameterization γ_k allows us to identify $\mathcal{O}_{C_k}$ with a subring of $\mathcal{O}_t$.

Let δ_k be the restriction of the Hessian determinant Δ_f to the branch C_k. Via the identification $\mathcal{O}_{C_k} \subset \mathcal{O}_t$, the holomorphic function germ δ_k is identified with an element of $\mathcal{O}_t$.

The number N_k is the order of δ_k that is:

$$\delta_k(t) = ct^{N_k} + o(t^{N_k}),$$

with $c \neq 0$.
Denote by D_k the restriction to C_k of the derivation along the Hamilton vector-field X_f.
Via the identification $\mathcal{O}_{C_k} \subset \mathcal{O}_t$, D_k can be identified with a (holomorphic) derivation on $\mathcal{O}_t$.
We are now going to consider only the branch C_k for a fixed number k. Consequently, in order to avoid too many indices, we drop the index k for the following objects, we write:
- $\alpha_1, \dots, \alpha_n$ instead of $\alpha_1^k, \dots, \alpha_n^k$,
- D for D_k,
- δ for δ_k,
- τ for the Teissier number of the branch C_k instead of τ_k,
- m for the multiplicity of the branch curve-germ $(C_k, 0)$ instead of m_k.
- N for the number N_k.

LEMMA 3.4. *We have:*

$$Dt^j = bjt^{j+\tau-m} + (\dots),$$

where:
- j is an arbitrary positive integer,
- b is a non-zero (complex) multiplicative constant,
- τ is the Teissier number of the branch curve-germ $(C, 0)$,
- m is the multiplicity of the branch curve-germ $(C,0)$
Here the dots stand for higher order terms in the Taylor series.

Proof.
Let l be a linear function such that the hyperplane of equation $l = 0$ is transverse to C. The definition of the multiplicity implies that the function l restricted to C is of the type

$$l_{|C}(t) = at^m + o(t^m),$$

with $a \neq 0$.
By definition of τ, we have

$$Dt = bt^\tau + o(t^\tau),$$

with $b \in \mathbb{C} \setminus \{0\}$. This proves the lemma.

We now prove that:

$$N = \frac{n(n+1)}{2}(\tau - m) + \sum_{j=1}^{n} \alpha_j. \tag{16}$$

Denote by $a_1, \dots, a_s$ the anomaly sequence of C at the origin.
the definition of the anomaly sequence implies that

$$\sum_{j=1}^{n} \alpha_j = \frac{n(n+1)}{2} + \sum_{j=1}^{s} a_j.$$

Consequently, equation (16) concludes the proof of the theorem.
The function-germ $\delta : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ is the determinant of the $n \times n$ matrix whose k^{th} columns is:

$$(Dt^{\alpha_k}, D^{(2)}t^{\alpha_k}, \dots, D^{(n)}t^{\alpha_k}),$$

where $D^{(j)}$ is the j^{th} derivative with respect to the derivation D.
In order to compute the order of δ, we can assume, without loss of generality, that the constant b of lemma 3.4 is equal to one.
Lemma 3.4 implies the equality

$$D^{(j)}t^{\alpha_k} = \alpha_k(\alpha_k + \tau) \dots (\alpha_k + j\tau)t^{\alpha_k + j(\tau - m)} + (\dots),$$

Thus the holomorphic function-germ δ is of the type

$$\delta(t) = ct^{\frac{n(n+1)}{2}(\tau - m) + \sum_{j=1}^{n} \alpha_j} + (\dots).$$

In both formulas the dots stand for higher order terms in the Taylor series.
The number c is the determinant of the $n \times n$ matrix $(a_{k,j})$ where:

$$a_{k,j} = \alpha_k(\alpha_k + \tau) \dots (\alpha_k + j\tau).$$

Consequently c is the Vandermonde determinant of $\alpha_1, \dots, \alpha_n$ times $\alpha_1\alpha_2 \dots \alpha_n$.
Thus $c \neq 0$ and the order of δ is equal to

$$\frac{n(n+1)}{2}(\tau - m) + \sum_{j=1}^{n} \alpha_j.$$

This concludes the proof of theorem 3.2.

2.5. Proof of proposition 3.6. As we pointed out at the beginning of the section, our proof is not completely self-contained because at the end of it, we shall use a fact of Picard-Lefschetz theory which can be proved say using Morse theory ([**Mil**],[**Hamm**]).
Consider a complete intersection map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$. Take a representative $\bar{f} : U \longrightarrow \mathbb{C}^{n-1}$ of f. Consider the curves:

$$V_\varepsilon = \left\{ p \in U : \bar{f}(p) = \varepsilon \right\}.$$

Denote by B_r the closed ball of radius r centered at the origin. Choose r small enough so that V_0 intersects transversally the boundary of the ball B_δ for any $\delta \leq r$.
Consider the curve $X_\varepsilon = V_\varepsilon \cap B_r$, with r small enough.
Let $l : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a linear function such that the hyperplane of equation $l = 0$ is transversal to all the branches of the curve X_0.
Choose base points $\varepsilon^* \in \mathbb{C}^{n-1}$ such that:
- the curve X_{ε^*} is smooth,

- the restriction of l to X_{ε^*} has only Morse critical points of distinct critical values say $\delta_1, \ldots, \delta_k$.

Remark that the critical points of the restriction of l to X_{ε^*} are precisely the points where a hyperplane parallel to $\{x \in \mathbb{C}^n : l = 0\}$ is tangent to X_{ε^*}. Consequently the multiplicity of intersection of X_0 with the polar hypersurface-germ Γ is equal to the number k of critical values of l restricted to X_{ε^*}. Thus, we get the equality

$$k = \sum_{j=1}^{r} \tau_j(f)$$

where the $\tau_k(f)$ are the Teissier numbers of f.
Let $\delta^* \in \mathbb{C}$ be such that the hyperplane $H = \{p \in \mathbb{C}^n : l(p) = \delta^*\}$ intersects X_{ε^*} in m distinct points where m denotes the multiplicity of f. The situation is summarized in figure 4.
We have the exact sequence (we take the homology groups reduced modulo a point):

$$\ldots \longrightarrow \widetilde{H}_j(X_{\varepsilon^*} \cap H) \longrightarrow \widetilde{H}_j(X_{\varepsilon^*}) \longrightarrow \widetilde{H}_j(X_{\varepsilon^*}, (X_{\varepsilon^*} \cap H)) \longrightarrow \widetilde{H}_{j-1}(X_{\varepsilon^*} \cap H) \longrightarrow \ldots$$

But $X_{\varepsilon^*} \cap H$ is a finite set of points. Consequently, the exact sequence reduces to

$$0 \longrightarrow \widetilde{H}_1(X_{\varepsilon^*}) \longrightarrow \widetilde{H}_1(X_{\varepsilon^*}, (X_{\varepsilon^*} \cap H)) \longrightarrow \widetilde{H}_0(X_{\varepsilon^*} \cap H) \longrightarrow 0. \tag{17}$$

By definition:
- $H_1(X_{\varepsilon^*}) = \mathbb{Z}^{\mu}$ where μ is the Milnor number of f,
- $\widetilde{H}_0(X_{\varepsilon^*} \cap H) = \mathbb{Z}^{m-1}$ where m is the multiplicity of f,
- $\widetilde{H}_0(X_{\varepsilon^*}, X_{\varepsilon^*} \cap H) = \mathbb{Z}^k$, where k is the number of critical points of the restriction of l to X_{ε^*}.

The last equality is the "fact from Picard-Lefschetz theory" that we shall not prove. However we shall explain it. But first remark that this exact sequence achieves the proof of proposition 3.6. Indeed the exact sequence (17) implies the equality

$$\mu - (m-1) + k = 0$$

and this concludes the proof of proposition 3.6 since $k = \sum_{j=1}^{r} \tau_j(f)$.
Now, we "explain" why $\widetilde{H}_0(X, X_{\varepsilon^*} \cap H) = \mathbb{Z}^k$.
Consider the function, say $g : X_{\varepsilon^*} \longrightarrow \mathbb{C}$ which is the restriction of l to X_{ε^*}.
Choose an open subset $D \subset \mathbb{C}$ in the image of g containing all the critical values $\delta_1, \ldots, \delta_k$ and containing δ^* on its boundary. Put $D' = D \setminus \{\delta_1, \ldots, \delta_k\}$.
The group $\pi_1(D')$ with base point δ^* is a free group.
Choose a set of path $\phi_1, \ldots, \phi_k : [0,1] \longrightarrow D'$ such that:
- ϕ_j connects δ^* with δ_j,
- for any $t \in [0, 1[$, $\phi_j(t) \in D'$.

Let $\alpha_1, \ldots, \alpha_k$ be a set of loops in D' obtained by following ϕ_j from δ^* to δ_j turning counterclockwise around δ_j and coming back to δ^* along ϕ_j.

We chose the ϕ_j's in such a way that the α_j's generate $\pi_1(D')$ (the path ϕ_j's are called a distinguished basis see page 101).
The homology class of the preimage of a path ϕ_j under $g : X_{\varepsilon^*} \longrightarrow \mathbb{C}$ is a cycle σ_j in $H_1(X_{\varepsilon^*}, X_{\varepsilon^*} \cap H)$ (see figure 4). It can be proved that the homology classes $\sigma_1, \ldots, \sigma_k$ generate $H_1(X_{\varepsilon^*}, X_{\varepsilon^*} \cap H)$ ([**AVGL2**], [**Hamm**]). This implies that $\widetilde{H}_0(X_{\varepsilon^*}, X_{\varepsilon^*} \cap H) = \mathbb{Z}^k$, where k is the number of critical points of the restriction of l to X_{ε^*}. Proposition 3.6 is proved.

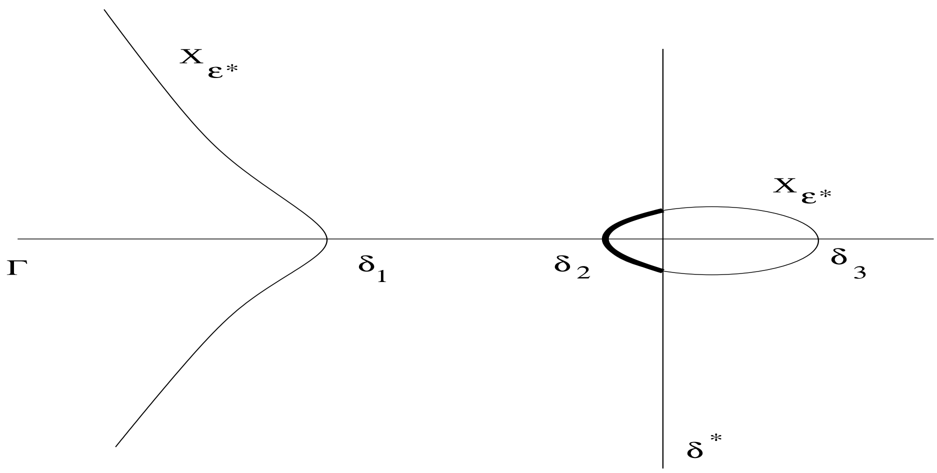

FIGURE 4. *We have drawn the real part of a complex curve (in fact an elliptic curve in this case) X_{ε^*} for $\varepsilon^* \in \mathbb{R}$. The restriction g of the linear function $l : (x, y) \mapsto x$ to X_{ε^*} has three critical points of critical values $\delta_1, \delta_2, \delta_3$. The bolded part of the oval is the pre-image under g of δ_2. It defines a relative one-dimensional cycle associated to δ_2. The other half of the oval is the pre-image under g of δ_3. The relative cycle associated to δ_1 passes through the complex domain.*

CHAPTER 4

The Plücker space.

We come back to our main subject of study that is the classification of families of curves in affine or projective spaces with respect to their flattening points.
Let $f : U \longrightarrow \mathbb{C}^{n-1}$ be a complete intersection map where U denotes an open (connected) subset in affine space $\mathbb{C}^n$.
Define the $n-1$ -parameter family of curves V_ε by

$$V_\varepsilon = \{p \in U : f(p) = \varepsilon\}\,.$$

In the preceding chapter, we defined the Hessian variety (in general a hypersurface) to be the variety of flattening points of the curves V_ε when ε varies. For a given affine coordinate system, we computed an equation of the Hessian variety in case it is a hypersurface (chapter 2, section 4). This equation was denoted by $\Delta_f = 0$, where $\Delta_f : U \longrightarrow \mathbb{C}$ is a holomorphic function called the *Hessian determinant* of the map f.
Instead of considering f with the affine structure on U, we consider (f, Δ_f) with the analytic structure on U and forget the affine structure on U.
Singularity theory for map-germs of the type

$$(g, E_g) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$$

has been settled down by V.V. Goryunov ([**Go**], [**AVGL2**]).
In chapter 2, section 4, we gave an account on $\mathcal{G}$-equivalence for $n = 2$ but there is no essential differential for one-dimensional complete intersections. We make this generalization in this chapter.
Given a map-germ $(f, \Delta_f) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ we search for the "simplest element" (g, E_g) which is $\mathcal{G}$-equivalent to (f, Δ_f).
For example, theorem 2.6 on page 2.6 states that if $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a generic Morse function-germ then (f, Δ_f) is $\mathcal{G}$-equivalent to the map-germ $(x, y) \mapsto (x^3 + y^3, xy)$.
This chapter is divided as follows.
In the first section, we define $\mathcal{G}$-equivalence. This section repeats most of the theory settled in the particular case $n = 2$ in chapter 2, section 4 with more examples and details. For the reader's convenience, they are both independent. We do not assume any background in singularity theory.
The proofs of the $\mathcal{G}$-finite determinacy theorem and of the $\mathcal{G}$-versal deformation theorem for this particular theory are given in appendix B. I do not know whether these theorems follow from J. Damon's theory of geometrical

subgroups or not. Damon's theory is exposed for example in [**Da**]. Also, remark that instead of using V.V. Goryunov's $\mathcal{G}$-equivalence, one can use Zakalyukin's flag equivalence [**Zak2**].

In the second section, we restrict the general theory to the map-germs of the type (g, E_g) for which there exists a map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ and a coordinate system in $\mathbb{C}^n$ such that (g, E_g) is $\mathcal{G}$-equivalent to a map-germ (f, Δ_f) (the map f depends on g). This leads us to the definitions of the Plücker space and of the Plücker equivalence.

In the third section we apply this theory to some simple cases. The complete computations are given in appendix A. We have thought that only the simplest cases are important in order to understand the techniques. The theorems on the generic bifurcation diagrams for Morse functions are direct corollaries of the general theory developed in this chapter (see appendix A, section 4).

Finally remark that we consider the complex holomorphic case for notational reasons. The theory can also be formulated in the real C^∞ case or for $\mathbb{K}$ analytic or formal power series, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ without major differences.

1. Theory of normal forms for $\mathcal{G}$-equivalence.

In this section the notation $\mathbb{C}^n$ stand for the analytical space, no additional structure (affine, vector-space) is involved.
In all this section we fix a coordinate system $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$ and a coordinate system in $\mathbb{C}^{n-1} \times \mathbb{C}$.

1.1. Notations. We shall use the following notations:
1. $Diff(k)$ is the group of biholomorphic map-germs of the type $\varphi : (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0)$ preserving the origin.
2. $diff(k)$ is the group of holomorphic vector fields of the type $v : (\mathbb{C}^k, 0) \longrightarrow (T\mathbb{C}^k, 0)$ vanishing at the origin.
3. $\mathcal{O}_x^k$ is the ring of holomorphic map-germs of the type $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}^k$.
4. $\mathcal{O}_x^*$ is the multiplicative group of holomorphic function-germs of the type $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$ such that $f(0) \neq 0$.

Consider a holomorphic map-germ $v : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}^k$.
Given a $k \times k$ matrix A with elements in $\mathcal{O}_x$. Then $A \times v$ is defined as follows.
For a fixed value of x, $v(x)$ is a vector of $\mathbb{C}^k$ and $A(x)$ is a $k \times k$ matrix. The image of $v(x)$ under $A(x)$ is denoted by $(A \times v)(x)$.
Given a holomorphic function-germ $u : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$. We denote by $u \times v$ the product defined by the formula

$$(u \times v)(x) = u(x)v(x).$$

For a fixed value of x, $v(x)$ is a vector in $\mathbb{C}^k$ and $u(x)$ is a complex number so the product is a vector denoted $(u \times v)(x)$.
When writing a formula, the rules of priority are $\circ, \ \times, \ +$.
For example $f + \alpha \times E \circ \varphi$ means $f + (\alpha \times (E \circ \varphi))$.

1.2. The $\mathcal{G}$-equivalence.

Definition 4.1. Two holomorphic map-germs $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$ are called $(R-L)_0$-*equivalent* if there exist biholomorphic map-germs $\varphi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$, $\psi : (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0)$ such that:

$$f = \psi \circ g \circ \varphi^{-1}.$$

Definition 4.2. Two holomorphic map-germs $E_f, E_g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$ are called V-*equivalent* if there exists an invertible matrix A with coefficients in $\mathcal{O}_x$ and a biholomorphic map-germ $\varphi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that:

$$E_f = A \times (E_g \circ \varphi^{-1}).$$

Example. The function-germs defined by the following 4 polynomials are V-equivalent (here $n = 2$, $k = 1$):

$$x_1x_2, (1 + x_1x_2)x_1x_2, x_1^2 - x_2^2, (x_1 + x_2^2)(x_2 + x_1^2).$$

Remark. If two map-germs are $V-$equivalent then the germs at the origin of their zero level-sets are biholomorphically equivalent varieties.

These definitions are classical, see for instance [**Tyu**], [**Math**], [**AVG**].
Following Goryunov [**Go**], we introduce the analog of $(R-L)_0$-equivalence for the restriction of a map-germ:

$$f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^k, 0)$$

to the hypersurface-germ of equation $E_f = 0$. Here $E_f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ denotes a holomorphic function-germ.
Put $\tilde{f} = (f, E_f), \ \tilde{g} = (g, E_g)$.

Definition 4.3. The map-germs $\tilde{f}, \tilde{g} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ are called $\mathcal{G}$-*equivalent* if there exists $\alpha, \beta, \psi, \varphi$ such that:

$$(\psi \circ (f \circ \varphi + (E_f \circ \varphi) \times \alpha), (E_f \circ \varphi) \times \beta) = (g, E_g),$$

with $\psi \in Diff(n-1)$, $\varphi \in Diff(n)$, $\alpha \in \mathcal{O}_x^{n-1}$ and $\beta \in \mathcal{O}_x^*$.

Remark 1. It is readily verified that $\mathcal{G}$-equivalence is an equivalence relation.
Remark 2. We can forget the product structure on $\mathbb{C}^{n-1} \times \mathbb{C}$. That is we consider $(f, E_f) : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$ as a map from $\mathbb{C}^n$ to $\mathbb{C}^n$. Then if two maps are $\mathcal{G}$-equivalent then they are V-equivalent as maps from $\mathbb{C}^n$ to $\mathbb{C}^n$.

We explain the "meaning" of the formula defining $\mathcal{G}$-equivalence. We use the same notations than the ones of definition 4.3.
We are considering the restriction of the holomorphic map germ:

$$f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$$

to the hypersurface-germ of equation $E_f = 0$ (which is in general a singular hypersurface).
Multiplying the function-germ E_f by a holomorphic function-germ $\beta : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$ such that $\beta(0) \neq 0$ does not change the hypersurface-germ of equation $E_f = 0$ but only the equation.
Algebraically, this means that we can replace E_f by βE_f.
The restriction of f to $E_f = 0$ does not change if we had $E_f \times \alpha$ to f. Thus, we can replace f by $f + E_f \times \alpha$.
We can make change of variables in the source space $\mathbb{C}^n$ and in the target space $\mathbb{C}^{n-1}$.

Examples. Take $n = 2$ then the following $\mathcal{G}$-equivalence relations hold: $(x_1^3 + x_2^3, x_1x_2 + x_1x_2^2) \sim (x_1^3 + x_2^3, x_1x_2 - x_1x_2^3)$ since $x_1x_2 - x_1x_2^3 = (1 - x_2)(x_1x_2 + x_1x_2^2)$.
$(x_1^3 + x_2^3, x_1x_2) \sim ((x_1 + x_1^2)^3 + x_2^3, (x_1 + x_1^2)x_2)$
$(x_1x_2 + x_1^3 + x_2^3, x_1x_2) \sim (x_1^3 + x_2^3, x_1x_2)$ while
$(x_1x_2, x_1x_2 + x_1^3 + x_2^3)$ is not $\mathcal{G}$-equivalent to $(x_1x_2, x_1^3 + x_2^3)$.

1.3. The $\mathcal{G}$-equivalence group. The $\mathcal{G}$-equivalence is given by the action of an "infinite dimensional Lie group[1]".
The groups $Diff(n)$, $Diff(n-1)$, $\mathcal{O}_x^{n-1}$, $\mathcal{O}_x^*$ act on the space $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ as follows.
Let $(f, E) \in \mathcal{O}_x^{n-1} \times \mathcal{O}_x$. We define the actions as follows:

1. $\psi.(f, E) = (\psi \circ f, E)$ for $\psi \in Diff(n-1)$.
2. $\varphi.(f, E) = (f, E) \circ \varphi^{-1}$ for $\varphi \in Diff(n)$.
4. $\alpha.(f, E) = (f + \alpha \times E, E)$ for $\alpha \in \mathcal{O}_x^{n-1}$.
3. $\beta.(f, E) = (f, \beta \times E)$ for[2] $\beta \in \mathcal{O}_x^*$.

For algebraic computations, the action of $Diff(n-1)$ is hard-to-handle. We consider it separately.
The formulas $2, 3, 4$ induce a group structure on the product $Diff(n) \times \mathcal{O}_x^{n-1} \times \mathcal{O}_x^*$, we put:

$$(\varphi, \alpha, \beta).(f, E) = \beta.(\alpha.(\varphi.(f, E))).$$

This group structure is a semi-direct product, where $Diff(n)$ is a distinguished subgroup.

[1] It is not our intention to use infinite dimensional Lie group theory. In the sequel, we shall consider finite dimensional Lie group approximations of the group $\mathcal{G}$.

[2] Recall that $u \times v$ is the function-germ defined by $(u \times v)(x) = u(x)v(x)$

The explicit formula (following from 2, 3, 4 and the composition of maps) is given by:

$$(18)\quad (\varphi,\alpha,\beta).(\varphi',\alpha',\beta') = (\varphi\circ\varphi', (\alpha'\circ\varphi^{-1}) + (\beta'\circ\varphi^{-1})\times\alpha, \beta\times(\beta'\circ\varphi^{-1})),$$

Formula (18) looks rather complicated. It can be written in a more compact form as we shall see in the next subsection.

DEFINITION 4.4. The *$\mathcal{G}$-equivalence group* is the set $\mathcal{G} = Diff(n)\times\mathcal{O}^{n-1}\times\mathcal{O}^*$ endowed with the group structure given by formula (18).

The following proposition is obvious.

PROPOSITION 4.1. *Two holomorphic map-germs $\tilde{f},\tilde{g} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}\times\mathbb{C})$ are $\mathcal{G}$-equivalent provided that there exists a biholomorphic map germ $\psi : (\mathbb{C}^{n-1}, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ such that $\psi\circ\tilde{f}$ and $\tilde{g}$ are in the same orbit under the action of the group $\mathcal{G}$.*

1.4. The V-equivalence group and the $\mathcal{G}$-equivalence group. Denote by $GL(n,\mathcal{O}_x)$ the space of invertible matrices with coefficient in $\mathcal{O}_x$. Remark that a matrix $A = (a_{j,k})$, $a_{j,k}\in\mathcal{O}_x$ is invertible provided that $Det(A(0))\neq 0$.
Consider the group structure on $\mathcal{K} = Diff(n)\times GL(n,\mathcal{O}_x)$ defined by

$$(19)\qquad (\varphi, A).(\varphi', A') = (\varphi\circ\varphi', A\times(A'\circ\varphi)).$$

The group $\mathcal{K}$ acts on $\mathcal{O}_x^{n-1}\times\mathcal{O}_x$ as follows

$$(\varphi, A).\tilde{f} = A\times(\tilde{f}\circ\varphi^{-1})$$

with $\tilde{f}\in\mathcal{O}_x^n$.

The following proposition is obvious.

PROPOSITION 4.2. *Two holomorphic map-germs $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}\times\mathbb{C}, 0)$ are V-equivalent if and only if they are in the same orbit under the action of the group $\mathcal{K}$.*

DEFINITION 4.5. The group $\mathcal{K}$ is called the *V-equivalence* group.

The $\mathcal{G}$-equivalence group can be "naturally' identified with a subgroup of the V-equivalence group $\mathcal{K}$.
Indeed, let $(\varphi,\alpha,\beta)\in Diff(n)\times\mathcal{O}^{n-1}\times\mathcal{O}^*$ be an element of the group $\mathcal{G}$. Put $\alpha = (\alpha_1,\dots,\alpha_{n-1})$. Consider the $n\times n$ invertible matrix $A(\alpha,\beta)$ whose left-upper $(n-1)\times(n-1)$ block is the identity matrix and whose last column is $\alpha_1,\dots,\alpha_{n-1},\beta$. That is:

$$(20)\qquad A(\alpha,\beta) = \begin{pmatrix} 1 & 0 & \dots & 0 & \alpha_1 \\ 0 & 1 & \dots & 0 & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_{n-1} \\ 0 & 0 & \dots & 0 & \beta \end{pmatrix}.$$

since $\beta \in \mathcal{O}_x^*$ the matrix $A(\alpha, \beta)$ is invertible.
The map

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{K} \\ (\varphi, \alpha, \beta) & \mapsto & (\varphi, A(\alpha, \beta)) \end{array}$$

gives the identification of $\mathcal{G}$ with a subgroup of $\mathcal{K}$. The complicated formula (18) is the same than formula (19).

1.5. The Lie algebra of the group $\mathcal{G}$. In this subsection and in the next one e denotes the identity element of the group $\mathcal{G}$.

Definition 4.6. The *tangent space to* $\mathcal{G}$ at a point γ_0, denoted $T_{\gamma_0}\mathcal{G}$, is the vector space of elements m such that there exists a holomorphic map $\gamma : (\mathbb{C}, 0) \longrightarrow (\mathcal{G}, \gamma_0)$ with:

$$\frac{d\gamma}{dt}(0) = m.$$

We denote by $\mathfrak{g}$ the tangent space to $\mathcal{G}$ at e.
Recall that $diff(n)$ denotes the set of germs of vector fields vanishing at the origin of the type

$$\sum_{j=1}^{n} a_j \partial_{x_j},$$

where $a_j : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ denotes a holomorphic function-germ.

Proposition 4.3. *The tangent space $\mathfrak{g}$ to the group $\mathcal{G}$ at e is equal (as a vector space) with the infinite dimensional vector space:*

$$diff(n) \times \mathcal{O}_x^{n-1} \times \mathcal{O}_x.$$

Proof.
Consider a holomorphic map

$$\gamma : (\mathbb{C}, 0) \longrightarrow (\mathcal{G}, e).$$

By definition of $\mathcal{G}$, the element $\gamma(t)$ can be represented as

$$\gamma(t) = (\varphi_t, \alpha_t, \beta_t) \in (Diff(n) \times \mathcal{O}^{n-1} \times \mathcal{O}^*).$$

We represent $\varphi_t : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ in the form

$$\varphi_t = Id + tv + o(t), \tag{21}$$

where $o(t)$ means that all the components are of order at least t and Id stands for the identity map in $\mathbb{C}^n$.
We have $\varphi_t(0) = 0$ hence the vector field-germ v vanishes at 0 ($v(0) = 0$). Consequently v belongs to $diff(n)$.
Similarly write

$$\alpha_t = ta + o(t), \ \beta_t = 1 + tb + o(t) \tag{22}$$

with $a \in \mathcal{O}_x^{n-1}$, $b \in \mathcal{O}_x$ and 1 denotes the constant function equal to 1.

Putting together equation (21) and equation (22) we get the equality

$$\frac{d\gamma}{dt}(0) = (v, a, b)$$

with $(v, a, b) \in (diff(n) \times \mathcal{O}_x^{n-1} \times \mathcal{O}_x)$. This concludes the proof of the proposition.

1.6. The $\mathcal{G}$-tangent space to a germ. The action of the Lie group on the space $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ induces an action of $\mathfrak{g}$ on the space $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$. Let $m \in \mathfrak{g}$. The definition of $\mathfrak{g}$ implies that there exists a holomorphic map:

$$\gamma : (\mathbb{C}, 0) \longrightarrow (\mathcal{G}, e),$$

such that $m = \frac{d\gamma}{dt}(0)$.
We define the action of $m \in \mathfrak{g}$ on a holomorphic map-germ $\tilde{f} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ by the formula

$$m.\tilde{f} = \frac{d}{dt} |_{t=0} (\gamma(t).\tilde{f}).$$

Here $\gamma(t)$ belongs to $\mathcal{G}$ and $\gamma(t).\tilde{f}$ stands for the image of $\tilde{f}$ under $\gamma(t)$.

Definition 4.7. The $\mathcal{G}$-*tangent space* to the holomorphic map-germ $\tilde{f} = (f, E) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$, denoted by $T\tilde{f}$, is the tangent space to the orbit of $\tilde{f}$ under the action of the group $\mathcal{G}$ at the point $\tilde{f}$:

$$T\tilde{f} = \left\{ m.\tilde{f} \in \mathcal{O}_x^{n-1} \times \mathcal{O}_x : \ m \in \mathfrak{g} \right\}.$$

Fix coordinate systems in $\mathbb{C}^n$ and in $\mathbb{C}^{n-1} \times \mathbb{C}$.
Denote by $v^j = (v_1^j, \ldots, v_n^j) \in \mathbb{C}^{n-1} \times \mathbb{C}$ the vector having the coordinates

$$\begin{cases} v_j^j & = \quad 1, \\ v_k^j & = 0 \quad \text{for } j \neq k. \end{cases}$$

Proposition 4.4. *The tangent space to $\tilde{f} = (f, E)$ is the $\mathcal{O}_x$-module generated by the $n^2 + n$ elements of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$:*

$$\begin{cases} x_j \frac{\partial \tilde{f}}{\partial_{x_k}} & \forall j, k \in \{1, \ldots, n\} \\ E \times v^j & \forall j \in \{1, \ldots, n\} \end{cases}$$

Remark. For a fixed value of $x \in \mathbb{C}^n$, $E(x)$ is a complex number, v^j is a vector in $\mathbb{C}^{n-1} \times \mathbb{C}$, $E(x)v^j$ is a vector in $\mathbb{C}^{n-1} \times \mathbb{C}$. Thus $E \times v^j$ is a map-germ of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$.
Example. Consider the case $n = 2$. Let $\tilde{f}(x_1, x_2) = (x_1^2 + x_2^2, x_1 x_2)$. The

tangent space to $\tilde{f}$ is the $\mathcal{O}_x$-module generated by:

$$\begin{cases} x_1\partial_1\tilde{f}(x_1,x_2) & = & (2x_1^2, x_1x_2), \\ x_2\partial_1\tilde{f}(x_1,x_2) & = & (2x_1x_2, x_2^2), \\ x_1\partial_2\tilde{f}(x_1,x_2) & = & (2x_1x_2, x_1^2), \\ x_2\partial_2\tilde{f}(x_1,x_2) & = & (2x_2^2, x_1x_2), \\ E(1,0) & = & (x_1x_2, 0), \\ E(0,1) & = & (0, x_1x_2). \end{cases}$$

The tangent space to $\tilde{f}$ is a $\mathbb{C}$-vector space of codimension 6 in $\mathcal{O}_x \times \mathcal{O}_x$.

1.7. Proof of proposition 4.4. First, we investigate each component of the group $\mathcal{G}$ separately. We assume that the group elements of a one-parameter family depend holomorphically on the parameter t.
Denote by Id the identity map in $\mathbb{C}^n$.
Let φ_t be the germ of a one parameter family of maps in $Diff(n)$ with $\varphi_0 = Id$.
We expand, φ_t with respect to t

$$\varphi_t = Id + tv + o(t).$$

We have $\varphi_t(0) = 0$ hence the vector field-germ v vanishes at 0 ($v(0) = 0$). Consequently v belongs to $diff(n)$.
The action of φ_t on an arbitrary map-germ $\tilde{f} \in \mathcal{O}_x^{n-1} \times \mathcal{O}_x$ is defined by

$$\tilde{f} \circ \varphi_t = \tilde{f} + tv.\tilde{f} + o(t),$$

where $v.\tilde{f}$ denotes the derivative of $\tilde{f}$ along v ($v.\tilde{f} = D\tilde{f}.v$).
Consequently, we get the equality

$$\frac{d}{dt}\,|_{t=0}\,(\varphi_t.\tilde{f}) = v.\tilde{f}. \tag{23}$$

Let α_t be the germ of a one parameter family of maps in $\mathcal{O}_x^{n-1}$ with $\alpha_0 = 0$.
We expand β_t with respect to t:

$$\alpha_t = ta + o(t),\ a \in \mathcal{O}_x^{n-1}$$

The action of $\alpha_t \in \mathcal{O}_x^{n-1}$ on the map-germ $\tilde{f} = (f, E)$ is defined by

$$\alpha_t.(f, E) = (f + E \times \alpha_t, E)$$

Consequently, we get the equality

$$\frac{d}{dt}\,|_{t=0}\,(\alpha_t.\tilde{f}) = (E \times a, 0). \tag{24}$$

Let β_t be the germ of a one parameter family of maps in $\mathcal{O}_x^*$ with $\beta_0 = 1$ (the constant function equal to 1).
We expand β_t with respect to t:

$$\beta_t = 1 + tb + o(t),\ b \in \mathcal{O}_x.$$

The action of β_t on the map-germ $\tilde{f} = (f, E)$ is defined by

$$\beta_t.(f, E) = (f, E \times \beta_t)$$

Consequently, we get the equality

$$\frac{d}{dt}\,|_{t=0}\,(\beta_t.\tilde{f}) = (0, E \times b). \tag{25}$$

Equations 23, 24, 25 imply the equality

$$\frac{d}{dt}\,|_{t=0}\,((\varphi_t, \alpha_t, \beta_t).\tilde{f}) = (v.f + E \times a, E \times b + v.E),$$

with $(v, a, b) \in diff(n) \times \mathcal{O}_x^{n-1} \times \mathcal{O}_x$.
The $\mathcal{O}_x$-module $diff(n)$ is generated (as an $\mathcal{O}_x$-module) by the $x_j \frac{\partial \tilde{f}}{\partial x_k}$'s.
The $\mathcal{O}_x$-module $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ is generated (as an $\mathcal{O}_x$-module) by the vectors v^j of coordinates $v_j^j = 1,\ v_k^j = 0,\ i \neq j$.
This proves the proposition.

1.8. The finite determinacy theorem. Consider a holomorphic map-germ $\tilde{f} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$. The space $(\mathcal{O}_x^{n-1} \times \mathcal{O}_x)/T\tilde{f}$ is a $\mathbb{C}$-vector space.

DEFINITION 4.8. The *$\mathcal{G}$-Milnor number* of the holomorphic map-germ $\tilde{f}$: $(\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ is defined by the formula

$$\mu_{\mathcal{G}}(\tilde{f}) = dim_{\mathbb{C}}[(\mathcal{O}_x^{n-1} \times \mathcal{O}_x)/T\tilde{f}].$$

Example. In subsection 1.6, we found that the tangent space to the germ of the map $\tilde{f} = (x_1^2 + x_2^2, x_1x_2)$ is of codimension 6. In this case $\mu_{\mathcal{G}}(\tilde{f}) = 6$.
Remark. When no confusion is possible we simply write $\mu_{\mathcal{G}}$ instead of $\mu_{\mathcal{G}}(\tilde{f})$.

Fix coordinates $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$. Denote by $\mathcal{M}_x^k$ the k^{th} power of the maximal ideal of $\mathcal{O}_x$.
Denote by $\widetilde{\mathcal{M}_x^k}$ the $\mathcal{O}_x$-submodule of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ which consists of map-germs $g = (g_1, \dots, g_n)$ such that $g_j \in \mathcal{M}_x^k,\ \forall j \in \{1, \dots, n\}$.
The following theorem is the *finite determinacy theorem* for $\mathcal{G}$-equivalence. The proof is given in appendix.

THEOREM 4.1. *Assume that $\mu_{\mathcal{G}}(\tilde{f}) < +\infty$. Then for any holomorphic map-germ ψ such that $\psi \in \widetilde{\mathcal{M}_x}^{\mu_{\mathcal{G}}+1}$, $\tilde{f} + \psi$ is $\mathcal{G}$-equivalent to $\tilde{f}$.*

Remark. This theorem allows us to consider approximations of a function $\tilde{f}$ by Taylor polynomials provided that $\mu_{\mathcal{G}}(\tilde{f}) < +\infty$.

1.9. Goryunov's $C_{2,2}$ singularity class. We give here an example for the reader which is not acquainted with these techniques. In this subsection, we use the notation $(x, y) \in \mathbb{C}^2$ instead of $(x_1, x_2) \in \mathbb{C}^2$.
Denote by $\mathcal{M}^k$ the k^{th} power of the maximal ideal $\mathcal{M}$ of the ring $\mathcal{O}_{x,y}$.
Consider the map-germs $\tilde{f}, \tilde{f}_0 : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ defined by the polynomials

$$\tilde{f}(x, y) = (x^2 + y^2 + r_1(x, y), xy + r_2(x, y))$$

with $r_1, r_2 \in \mathcal{M}^3$, and

$$\tilde{f}_0(x,y) = (x^2+y^2, xy)$$

Denote by $\sim$ the $\mathcal{G}$-equivalence relation.
Assertion (Goryunov): *the map-germ $\tilde{f}$ is $\mathcal{G}$-equivalent to $\tilde{f}_0$.*

In subsection 1.6, we saw that the $\mathcal{G}$-tangent space to $\tilde{f}_0$ contains $\mathcal{M}^2$. This means that there exists $(v,a,b) \in diff(2) \times \mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$ such that the following equality holds

$$(v,a,b).\tilde{f}_0 = (r_1, r_2).$$

Put $\varphi = Id + v$, $\alpha = a$, $\beta = 1 + b$.
It is readily verified that the map-germ φ is biholomorphic and that $\beta(0) \neq 0$.
Thus, $(\varphi, \alpha, \beta) \in \mathcal{G}$.
We have the equality

$$(\varphi, \alpha, \beta).\tilde{f} = \tilde{f}_0 + (r_3, r_4),$$

with $(r_3, r_4) \in \mathcal{M}^4$.
Repeating this process three times, we get that the following $\mathcal{G}$-equivalence holds

$$\tilde{f}(x,y) \sim \tilde{f}_0 + \psi(x,y)$$

where $\psi = (\psi_1, \psi_2)$ and $\psi_1, \psi_2 \in \mathcal{M}^7$.
Consequently, the finite determinacy theorem implies that:

$$\tilde{f}(x,y) \sim (x^2+y^2, xy).$$

This proves the assertion. This is the standard technique in singularity theory for finding normal forms. For the series $\mathcal{P}A_1^{p,q}$, we shall use a more direct method.

1.10. Versal deformation theory for $\mathcal{G}$-equivalence. In this subsection, we fix a holomorphic map-germ:

$$\tilde{f} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0).$$

DEFINITION 4.9. A holomorphic map-germ $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ such that $F(0,.) = \tilde{f}$ is called a *deformation of* $\tilde{f}$.

DEFINITION 4.10. The deformation $G : (\mathbb{C}^r \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ of $\tilde{f}$ is *induced* from the deformation $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ of $\tilde{f}$, if there exists a holomorphic map-germ $h : (\mathbb{C}^r, 0) \longrightarrow (\mathbb{C}^k, 0)$ such that $G(\lambda, .) = F(h(\lambda), .)$. We use the notation: $G = h^*F$.

DEFINITION 4.11. The *translation* by a vector $u \in \mathbb{C}^{n-1} \times \mathbb{C}$ is the map-germ:

$$\begin{array}{cccc} \tau & (\mathbb{C}^n, 0) & \longrightarrow & (\mathbb{C}^{n-1} \times \mathbb{C}, u) \\ & x & \longrightarrow & x + u. \end{array}$$

We denote by $T \approx \mathbb{C}^n$ the group of translations.

Definition 4.12. Two deformations F, G of f are called *$\mathcal{G}$-equivalent* if there exist holomorphic map-germs $\gamma : (\mathbb{C}^k, 0) \longrightarrow \mathcal{G} \oplus T$ and $\psi : (\mathbb{C}^k \times \mathbb{C}^{n-1}, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ such that the following equality of map-germs holds

$$(\gamma(\lambda).G)(\lambda, x) = (\psi(\lambda, F_1(\lambda, x)), F_2(x, \lambda)),$$

with $F = (F_1, F_2)$.

Definition 4.13. A deformation F is called *$\mathcal{G}-$versal* if any other deformation of the same germ is $\mathcal{G}$-equivalent to a deformation induced from F.

Fix coordinate-systems $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$ and $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$. Denote by $v^j = (v_1^j, \dots, v_n^j)$ the constant map-germ of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ defined by

$$\begin{cases} v_k^j & = & 0 \quad \text{for } j \neq k \\ v_j^j & = & 1. \end{cases}$$

Fix a coordinate-systems $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$, $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$ and a coordinate system in $\mathbb{C}^{n-1} \times \mathbb{C}$.

Definition 4.14. The *$\mathcal{G}$-tangent space* to a deformation $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ of $\tilde{f}$ is the $\mathbb{C}$-vector subspace of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$ which is sum of the following $\mathbb{C}$-vector subspaces:
1) the $\mathbb{C}$-vector space generated by the restriction to $\lambda = 0$ of the $\partial_{\lambda_k} \tilde{F}$'s ,
2) the $\mathcal{O}_x$-module generated by the $\partial_{x_k} \tilde{f}$'s,
3) the $\mathbb{C}$-vector space generated by the v^k's for $k \in \{1, \dots, n-1\}$.
The tangent space to F is denoted by TF.

The following theorem is the *versal deformation theorem for $\mathcal{G}$-equivalence.* The proof is given in appendix B.

Theorem 4.2. *A deformation $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ of a holomorphic map-germ $F(0, .)$ is $\mathcal{G}$-versal provided that $TF = \mathcal{O}_x^{n-1} \times \mathcal{O}_x$.*

Example. Let $\tilde{f} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ be the holomorphic map defined by

$$\tilde{f}(x_1, x_2) = (x_1^2 + x_2^2, x_1 x_2).$$

Consider the 3-parameter deformation F of $\tilde{f}$ defined by the formula

$$F(\lambda, x_1, x_2) = (x_1^2 + x_2^2, x_1 x_2 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3).$$

We saw that the tangent space $T\tilde{f}$ to $\tilde{f}$ is $\mathcal{M}^2$. The $\mathcal{G}$-tangent space to $\tilde{f}$ is the sum of $T\tilde{f}$ with the $\mathbb{C}$-vector space generated by the following map-germs

$$(\partial_{\lambda_1} F)_{|\lambda=0} = (0, x_1),\ (\partial_{\lambda_2} F)_{|\lambda=0} = (0, x_2), (\partial_{\lambda_3} F)_{|\lambda=0} = (0, 1),$$

$$(\partial_{x_1} F)_{|\lambda=0} = (2x_1, x_2),\ (\partial_{x_2} F)_{|\lambda=0} = (2x_2, x_1),\ v^1 = (1, 0).$$

Hence we have the equality $TF = \mathcal{O}_x \times \mathcal{O}_x$. Thus, the deformation F of $\tilde{f}$ is $\mathcal{G}$-versal.

2. Theory of normal forms, the Plücker space.

2.1. Summary. Fix an affine coordinate system $x = (x_1, \ldots, x_n)$ in affine space $\mathbb{C}^n$.
Let $f : U \longrightarrow \mathbb{C}^{n-1}$ be a holomorphic map such that, for $\varepsilon \in f(U)$, the varieties:

$$V_\varepsilon = \{x \in U : f(x) = \varepsilon\}$$

are of dimension one. Here U denotes a neighbourhood of the origin in affine space $\mathbb{C}^n$ and $f = \varepsilon$ is assume to be a reduced equation of V_ε.
In chapter 3, we saw that for any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in f(U)$, the flattening points of the curve V_ε are the preimages of $(\varepsilon, 0)$ under the map:

$$(f, \Delta_f) : U \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C},$$

provided that V_ε is smooth. Here Δ_f denotes the generalized Hessian determinant of f with respect to the coordinate system $x = (x_1, \ldots, x_n)$.
By considering (f, Δ_f), we can forget the affine structure on U. This means that we consider the map (f, Δ_f) up to a biholomorphic change of variables in U. In fact, we saw in section 1, that not only biholomorphic change of variables can be made but that when considering the germ of (f, Δ_f) at a point, an equivalence relation, called $\mathcal{G}$-equivalence can be introduced between map-germs of the type:

$$(g, E_g) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0).$$

In this section, we apply $\mathcal{G}$-equivalence to holomorphic map-germs of the type (f, Δ_f). That is we search for the "simplest (g, E_g)" which is $\mathcal{G}$-equivalent to (f, Δ_f). This leads to the construction of the Plücker space, that we shall now describe.

2.2. Preliminary examples. We fix an affine coordinates (x, y) in affine space $\mathbb{C}^2$.
We denote by:
- $B_\delta \subset \mathbb{C}^2$ the closed ball of radius δ centered at the origin,
- $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ a holomorphic function-germ and $\bar{f}$ a representative of f.

Example 1. Assume that the following $\mathcal{G}$-equivalence relation holds

$$(f, \Delta_f) \sim (y, x)$$

The two plane curves $\{(x, y) \in \mathbb{C}^2 : y = 0\}$ and $\{(x, y) \in \mathbb{C}^2 : x = 0\}$ intersect transversally. Consequently, for $\delta \neq 0$ and $\varepsilon \neq 0$ small enough, the $\mathcal{G}$-equivalence relation above implies that the curve:

$$V_\varepsilon = \{(x, y) \in B_\delta : \bar{f}(x, y) = \varepsilon\}$$

has a non-degenerate inflection point in a neighbourhood of the origin (see figure 1).

Example 2. Assume that the following $\mathcal{G}$-equivalence relation holds

$$(f, \Delta_f) \sim (y - x^2, y).$$

The two plane curves $\{(x,y) \in \mathbb{C}^2 : y - x^2 = \varepsilon\}$ and $\{(x,y) \in \mathbb{C}^2 : y = 0\}$ intersect:
- tangentially at the origin for $\varepsilon = 0$,
- transversally in two points for $\varepsilon \neq 0$.
Consequently, for $\delta \neq 0$ and $\varepsilon \neq 0$ small enough, the $\mathcal{G}$-equivalence above implies that curve

$$V_\varepsilon = \{(x,y) \in B_\delta : \bar{f}(x,y) = \varepsilon\}$$

has:
- a degenerate inflection point at the origin (of the type $y = x^4$) for $\varepsilon = 0$,
- two non-degenerate inflection points for $\varepsilon \in \bar{f}(B_\delta)$ and $\varepsilon \neq 0$.
This example is illustrated in figure 2.

Example 3. Assume that $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a Morse function-germ such that none of the branches of the curve-germ of equation $f = 0$ is the germ of a complex line.
We shall prove in section 3 that under these conditions, there exists integers $p, q \geq 0$ such that the following $\mathcal{G}$-equivalence holds.

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy).$$

In particular, for δ small enough, the curve

$$V_\varepsilon = \{(x,y) \in B_\delta : \bar{f}(x,y) = \varepsilon\}$$

has $6 + p + q$ inflection points in B_δ for $\varepsilon \in \bar{f}(B_\delta)$, $\varepsilon \neq 0$.

Example 4. Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function-germ. Denote by $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ a biholomorphic map-germ.
Define the holomorphic function-germ $H : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ by $H = f \circ \varphi$.
With the notation of chapter 3, section 1, we have the $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (H, \mathcal{D}[\varphi]).$$

2.3. The Plücker space (first part). We denote by $\mathbb{C}^n$ the analytic manifold with a marked point, denoted 0, on it.

DEFINITION 4.15. The *Plücker space (for complete intersection map-germs)* is the set of the holomorphic map-germs $(g, E_g) : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}$ for which there exists a holomorphic map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ and a coordinate system centered at the origin in $\mathbb{C}^n$ such that the following $\mathcal{G}$-equivalence relation holds:

$$(f, \Delta_f) \sim (g, E_g),$$

where $\sim$ denotes the $\mathcal{G}$-equivalence[3].

We fix a vector space structure in $\mathbb{C}^n$.

DEFINITION 4.16. Two complete intersection map-germs $f, g : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ are called *$\mathcal{P}$-equivalent* if the map-germs $(f, \Delta_f), (g, \Delta_g)$ are $\mathcal{G}$-equivalent.

Remark. The $\mathcal{P}$-equivalence class of a complete intersection map-germ is well-defined and does not depend on the choice of the linear coordinate system. Indeed, the definition Δ_f requires only a linear coordinate system $(x_1, \dots, x_n)$.
The $\mathcal{G}$-equivalence class of (f, Δ_f) does not depend on this choice. Too see it, consider a non-degenerate linear map $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ (or equivalently a linear change of coordinates). Then we have the equality (chapter 3 page 59)

$$\Delta_{f \circ A} = k^{n(n+1)} \Delta_f \circ A$$

with $k = (detA)^{n(n+1)-1}$.
This equality implies the following $\mathcal{G}$-equivalence

$$(f, \Delta_f) \sim (f \circ A, \Delta_{f \circ A}).$$

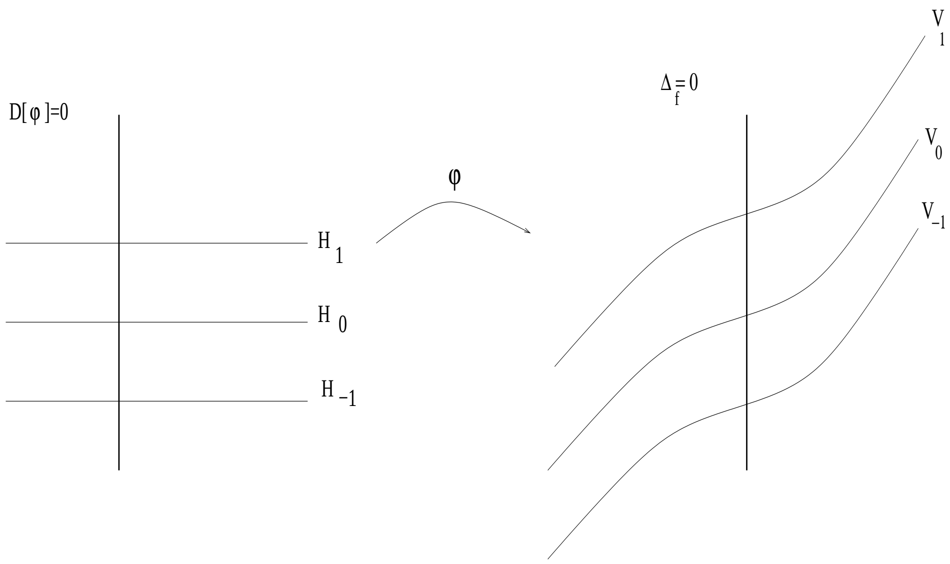

FIGURE 1. *In the left hand-side the intersection points of the curves of equations $y = \varepsilon,\ x = 0$ correspond to inflection points of the curves $V_\varepsilon \subset \mathbb{C}^2$ in the right hand-side.*

[3]The Plücker space depends on the dimension n.

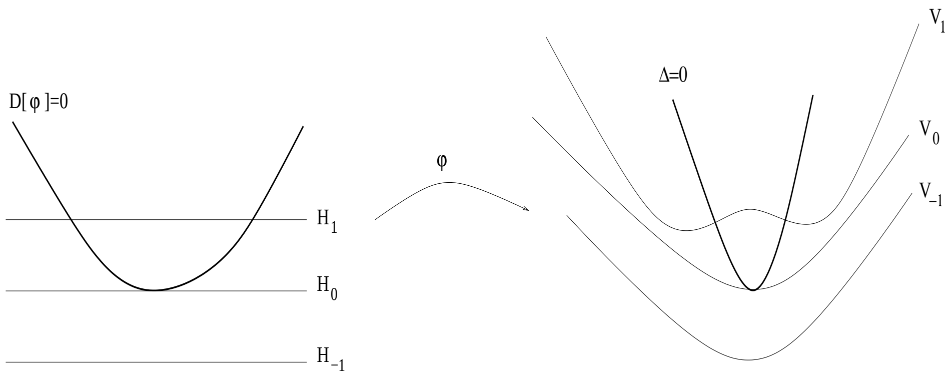

FIGURE 2. *For $\varepsilon = 1$ the curve V_1 has two real inflection points that coalesce when $\varepsilon \longrightarrow 0$.*

2.4. The Plücker space (second part).

DEFINITION 4.17. The *Plücker space*[4] (for map-germs of the type $(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0))$ is the set of holomorphic map-germs $(g, E_g) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$ for which there exist a holomorphic map-germ
$(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$ and coordinate systems in $\mathbb{C}^2$ and in $\mathbb{C}^n$ such that the following $\mathcal{G}$-equivalence relation holds

$$(H, \mathcal{D}[\varphi]) \sim (g, E_g).$$

Denote by $\mathbb{C}^n$ the n-dimensional complex vector space.

DEFINITION 4.18. Two holomorphic map-germs $(H, \varphi), (H', \varphi') : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$ are called *$\mathcal{P}$-equivalent* if the map-germs $(H, \mathcal{D}[\varphi])$ and $(H', \mathcal{D}[\varphi'])$ are $\mathcal{G}$-equivalent.

Remark. The $\mathcal{P}$-equivalence class of (H, φ) does not depend neither on the choice of the coordinates in $\mathbb{C}^2$ nor on the choice of the linear coordinate (see the remark following definition 4.16 on page 88).

2.5. Versal deformation theory in Plücker space. In this subsection $\mathbb{C}^n$ denotes the n-dimensional vector space.

DEFINITION 4.19. A *k-parameter deformation* of a holomorphic map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ is a holomorphic map-germ $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ such that $F(0, .) = f$.

Fix a linear coordinate system $x = (x_1, \ldots, x_n)$ in the vector space $\mathbb{C}^n$. Consider a deformation $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ of a holomorphic map-germ f.
We denote by:

$$\Delta_F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$$

[4] The Plücker space depends on the dimension n.

the Hessian determinant of F with respect to the variables $x = (x_1, \ldots, x_n)$. Like Δ_f the function-germ Δ_F depends on the choice of the coordinate system $x = (x_1, \ldots, x_n)$.

DEFINITION 4.20. A deformation $F(\lambda, .)$ of a holomorphic map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ is $\mathcal{P}$-*versal* if for any other deformation G of f, the germ of the deformation $\tilde{G} = (G, \Delta_G)$ is $\mathcal{G}$-equivalent to the germ of a deformation induced from $\tilde{F} = (F, \Delta_F)$.

Remark. The condition for a deformation to be $\mathcal{P}$-versal does not depend on the choice of the coordinate-systems but only on the vector space structure of a small neighbourhood U of the origin in $\mathbb{C}^n$. This is due to the fact that the orbit of a map-germ (f, Δ_f) under the $\mathcal{G}$-equivalence group depends only on the vector space structure (see the remark following definition 4.16 on page 88).

In the example below, we use the notation (x, y) for an affine coordinate system in affine space $\mathbb{C}^2$ instead of (x_1, x_2).

Example 1. Consider the biholomorphic map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ defined by the polynomials

$$\varphi(x, y) = (x, x^4 + yx^2 + y).$$

Let $H(x, y) = y$. Put $f = H \circ \varphi^{-1}$.
Assertion: the constant deformation $F = f$ is $\mathcal{P}$-versal.
Indeed, with the notations of chapter 3 section 1, we have the following $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (H, \mathcal{D}[\varphi]).$$

We already calculated the function-germ $\mathcal{D}[\varphi]$ on page 62, we found that it is given by the polynomial

$$(\mathcal{D}[\varphi])(x, y) = 12x^2 + 2y.$$

The $\mathcal{G}$-tangent space to $(H, \mathcal{D}[\varphi])$ is the sum of the following $\mathbb{C}$-vector subspaces of $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$:
- the $\mathcal{O}_{x,y}$-module generated by $\partial_x(H, \mathcal{D}[\varphi]),\ \partial_y(H, \mathcal{D}[\varphi]),\ (0, \mathcal{D}[\varphi]),\ (\mathcal{D}[\varphi], 0)$,
- the $\mathbb{C}$-vector space generated by $(1, 0)$.

It is readily verified that this vector space is $\mathcal{O}_{x,y} \times \mathcal{O}_{x,y}$. Consequently the constant deformation $(H, \mathcal{D}[\varphi])$ is $\mathcal{G}$-versal. Thus (f, Δ_f) is $\mathcal{G}$-versal and f is $\mathcal{P}$-versal. Assertion is proved.

Example 2. Consider the holomorphic map-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by the polynomial

$$f(x, y) = xy + x^4 + y^3.$$

We shall prove in the next subsection that the deformation

$$F : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

of f given by the formula

$$F(x, y) = xy + x^4 + \lambda x^3 + y^3$$

is $\mathcal{P}$-versal.

2.6. The modality in Plücker space (first part). Denote by:
- $\mathcal{O}_x$ the ring of holomorphic function-germs in $\mathbb{C}^n$,
- $\mathcal{M}^2$ the square of the maximal ideal of $\mathcal{O}_x$,
- $\widetilde{\mathcal{M}}^2$ the space of holomorphic map germs $f = (f_1, \dots, f_{n-1})$ such that the f_j's are in $\mathcal{M}^2$,
- $J_0^k\mathcal{M}^2$ the space of k-jets at the origin of elements in $\mathcal{M}^2$.

Definition 4.21. A holomorphic map-germ $f \in \widetilde{\mathcal{M}}^2$ has $\mathcal{P}$-*modality* m provided that m is the smallest number satisfying the following property. There exists a number N, such that a neighbourhood of $j_0^k f \in J_0^k\widetilde{\mathcal{M}}^2$ is covered by a finite number of m-parameter families of k-jets of $\mathcal{P}$-equivalence classes for any $k > N$.

Remark. For $n > 2$, it may happen that $\mu(f, \Delta_f) = +\infty$ while f has finite modality.

2.7. The modality in Plücker space (second part). Denote by $\bar{\mathcal{M}}^2$ the space of holomorphic map germs $(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$ such that $H \in \mathcal{M}_2$.
Denote by $J_0^k\bar{\mathcal{M}}^2$ the space of k-jets at the origin of elements in $\bar{\mathcal{M}}^2$.
Let $(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$ be a holomorphic map-germ such that the $\mathcal{G}$-Minor number $\mu_{\mathcal{G}}(H, \mathcal{D}[\varphi])$ of $(H, \mathcal{D}[\varphi])$ is finite.

Definition 4.22. The map (H, φ) has $\mathcal{P}$-modality m provided that for any $k > \mu_{\mathcal{G}}(H, \mathcal{D}[\varphi])$ a neighbourhood of $j_0^k(H, \varphi) \in J_0^k\bar{\mathcal{M}}^2$ is covered by a finite number of m-parameter families.

3. The $\mathcal{P}A_1^{p,q}$ series.

We have now settled the necessary tools for computing the $\mathcal{P}$-normal forms. We treat here the case of $\mathcal{P}A_1^{p,q}$ as an example. The other cases are treated in appendix A.
We shall use some "tricks" in order to avoid long computations. This is of course unnecessary and one can make the computations in the most straightforward manner.
In, the second subsection, we compute the $\mathcal{P}$-versal deformation of $\mathcal{P}A_1^1$.
We have thought that it is more instructive to treat this case rather than the general $\mathcal{P}A_1^{p,q}$ case. The general proof is also given in appendix A. But remark that the difference with the case $\mathcal{P}A_1^1$ lies only in the notations.
Unfortunately, the $\mathcal{P}$-versal deformations are in fact complicated to compute and we shall only do it for the series $\mathcal{P}A_1^{p,q}$.

3.1. Normal form $\mathcal{P}A_1^{p,q}$. We denote by $\sim$ the $\mathcal{G}$-equivalence relation. The aim of this subsection is to prove proposition 4.5 cited below.
In the real C^∞ category the proof of the corresponding statement is analogous.

PROPOSITION 4.5. *For any holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, the following $\mathcal{G}$-equivalence holds:*

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy).$$

Denote by $\mathcal{M}^k$ the k^{th} power of the maximal ideal of $\mathcal{O}_{x,y}$.
A non-degenerate linear map sends the inflection points of a curve to the inflection points of its image.
Consequently, for any non-degenerate linear transformation: $\alpha : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, f is $\mathcal{P}$-equivalent to $f \circ \alpha$.
Hence without loss of generality, we can assume that f is of the form

$$f(x, y) = xy + r_0,$$

with $r_0 \in \mathcal{M}^3$.

LEMMA 4.1. *For $f \in \mathcal{P}A_1^{p,q}$, there exists function germs a, $b : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that the following $\mathcal{G}$-equivalence holds $(f, \Delta_f) \sim (a(x)+b(y), xy)$.*

Proof.
The Hamilton vector-field of f is of the following form

$$X_f = (x + r_1)\partial_x - (y + r_2)\partial_y,$$

where $r_1, r_2 \in \mathcal{M}^2$.
Consequently the Hessian determinant Δ_f of f is given by:

$$\Delta_f(x, y) = 2xy + r_3$$

where $r_3 \in \mathcal{M}^3$.
The Morse lemma implies that there exists a biholomorphic map-germ

$$\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$$

such that we have:

$$(\Delta_f \circ \varphi)(x,y) = xy. \tag{26}$$

By definition of $\mathcal{G}$-equivalence, we have the following $\mathcal{G}$-equivalence relation:

$$(f, \Delta_f) \sim (f \circ \varphi, \Delta_f \circ \varphi).$$

The division theorem implies that one can represent $f \circ \varphi$ in the form

$$(f \circ \varphi)(x,y) = a(x) + b(y) + c(x,y)xy,$$

where a, b, c are holomorphic function germs.
The definition of $\mathcal{G}$-equivalence implies the following $\mathcal{G}$-equivalence relation:

$$(a(x) + b(y) + c(x,y)xy, xy) \sim (a(x) + b(y), xy).$$

Lemma is proved.

Lemma 4.2. *The holomorphic function germs a, b of the preceding lemma are of the form:*

$$\begin{cases} a(x) &= a_0 x^{3+p} + o(x^{3+p}) \\ b(y) &= b_0 y^{3+q} + o(y^{3+q}) \end{cases}$$

with $a_0 b_0 \neq 0$.

Proof.
Put $E(x,y) = xy$. Recall, that we have the following $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (a + b, E).$$

Denote by j (resp. k) the highest number such that $a \in \mathcal{M}^j$ (resp. $b \in \mathcal{M}^k$). That is the first term in the Taylor series of a (resp. b) appearing with a non-zero coefficient is of degree j (resp. k). A priori j or k can be infinite but we shall see that this is not the case.
Denote by:
- C_1 (resp. C_2) the branch of the plane curve-germ of equation $f = 0$ tangent to the x-axis (resp. to the y-axis),
- Δ_1 (resp. Δ_2) the branch of the plane curve-germ of equation $\Delta_f = 0$ tangent to the x-axis (resp. to the y-axis).
- $(C_l.\Delta_m)$ the multiplicity of intersection at the origin of the curve germs C_l and Δ_m.
The definition of the multiplicity of intersection implies the equalities[5]

$$\begin{cases} j &= (C_1.\Delta_1) + (C_2.\Delta_1), \\ k &= (C_1.\Delta_2) + (C_2.\Delta_2). \end{cases}$$

The curve-germ C_1 is tangent to the x-axis while Δ_2 is tangent to the y-axis. Hence, their intersection number is equal to

$$(C_1.\Delta_2) = 1.$$

Similarly, we have the equality

$$(C_2.\Delta_1) = 1.$$

[5]Remark that the equations of the curves that we consider are reduced.

Consequently the numbers j, k are given by the following system of equations

$$\begin{cases} j &= (C_1.\Delta_1) + (C_1.\Delta_2), \\ k &= (C_2.\Delta_1) + (C_2.\Delta_2). \end{cases} \tag{27}$$

The definition of the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ implies that the curve-germ C_1 is parameterized by a holomorphic map-germ of the type

$$\begin{cases} x(t) &= t, \\ y(t) &= -t^{2+p} + o(t^{2+p}). \end{cases} \tag{28}$$

This parameterization allows us to identify the ring of holomorphic function-germs on C_1 is with a subring of $\mathcal{O}_t$.
Denote by $\delta \in \mathcal{O}_t$ the holomorphic function-germ of the parameter t obtained by restricting Δ_f to the curve-germ C_1.
The number $(C_1.\Delta_1) + (C_1.\Delta_2)$ is equal to the degree of the first term in the Taylor series of δ appearing with a non-zero coefficient.
Using the old-fashioned notations, the Hamilton vector field X_f of f is defined by the Hamilton equations

$$\begin{cases} \dot{x}(x,y) &= x + m_1(x,y), \\ \dot{y}(x,y) &= -y + m_2(x,y), \end{cases} \tag{29}$$

with $m_1, m_2 \in \mathcal{M}^2$.
Via the embedding $\mathcal{O}_{C_1} \subset \mathcal{O}_t$, the restriction D of the derivation along the Hamilton vector-field X_f to C_1 is a (holomorphic) derivation of $\mathcal{O}_t$. The first equality of the system of equations (29) implies that:

$$Dt = t + o(t).$$

We get the formula

$$\delta(t) = \begin{vmatrix} t & (2+p)t^{2+p} \\ t & (2+p)^2 t^{2+p} \end{vmatrix} + o(t^{3+p})$$

Thus $\delta(t) = (2+p)(1+p)t^{3+p} + o(t^{3+p})$ and consequently:

$$(C_1.\Delta_1) + (C_1.\Delta_2) = 3 + p.$$

The proof for the branch C_2 is different only in notations. This concludes the proof of the lemma.

Lemma 4.3. *For any holomorphic function germs $a \in \mathcal{O}_x,\ b \in \mathcal{O}_y$ such that $a(x) = a_0x^j + o(x^j),\ b(y) = b_0y^k + o(x^k)$ we have the following $\mathcal{G}$-equivalence relation*

$$(a(x) + b(y), xy) \sim (x^j + y^k, xy)$$

provided that $a_0b_0 \neq 0$.

Proof.
Since $a_0b_0 \neq 0$, we can write:

$$a(x) = (\alpha(x)x)^j$$

and:

$$b(y) = (\beta(y)y)^k$$

where α and β are holomorphic function germs that do not vanish at the origin.
Consider the biholomorphic map-germ $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ defined by

$$g(x, y) = (\alpha(x)x, \beta(y)y).$$

The map-germ g induces the $\mathcal{G}$-equivalence relation

$$(x^j + y^k, xy) \sim ((\alpha(x)x)^j + (\beta(y)y)^k, \alpha(x)\beta(y)xy).$$

By definition of $\mathcal{G}$-equivalence, we have the $\mathcal{G}$-equivalence relation

$$((\alpha(x)x)^j + (\beta(y)y)^k, \alpha(x)\beta(y)xy) \sim ((\alpha(x)x)^j + (\beta(y)y)^k, xy).$$

This concludes the proof of the lemma.
This lemma concludes the proof of proposition 4.5.

3.2. $\mathcal{P}A_1^{p,q}$ is $\mathcal{P}$-simple. Let $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ be a holomorphic function-germ such that $f \in \mathcal{P}A_1^{p,q}$ where $p \geq 0, q \geq 0$ are integers.
The $\mathcal{G}$-Milnor number of f is finite say equal to $k-1$. The finite determinacy theorem for $\mathcal{G}$-equivalence implies that for computing the modality of f is suffices to compute the modality of the k-jet of f in $J_0^k\mathcal{M}^2$.
Here $J_0^k\mathcal{M}^2$ denotes the space of k-jets at the origin of elements of $\mathcal{M}^2$.
Denote by $e_1, \dots, e_s$ a basis of the finite dimensional vector space $J_0^k\mathcal{M}^2$.
Consider the deformation of f defined by:

$$F = f + \sum_{j=1}^{s} \lambda_j e_j$$

for any λ small enough, the germ of $F(\lambda, .)$ at the origin is of the type $\mathcal{P}A_1^{p',q'}$ with $p' \leq p,\ q' \leq q$.
Proposition 4.5 implies that two germs in $\mathcal{P}A_1^{p',q'}$ are in the same $\mathcal{G}$-orbit.
Thus, there is only a finite number of $\mathcal{P}$-equivalence classes, indexed by the integers p', q', in a neighbourhood of the k-jet of f.

3.3. $\mathcal{P}$-versal deformation for the class $\mathcal{P}A_1^1$. In this subsection, we compute the $\mathcal{P}$-versal deformation for the simplest "non-trivial" case, namely $\mathcal{P}A_1^1$.
For the case $\mathcal{P}A_1$ proposition 4.5 implies that the constant deformation is $\mathcal{P}$-versal.
We shall prove the following proposition.

PROPOSITION 4.6. *The deformation $F : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ of $f(x, y) = xy + x^4 + y^3$ defined by:*

$$F(\lambda, x, , y) = xy + x^4 + \lambda x^3 + y^3$$

is $\mathcal{P}$-versal.
Moreover (F, Δ_F) is $\mathcal{G}$-equivalent to the deformation defined by the pair

$$\begin{array}{ccc} (\mathbb{C} \times \mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C} \times \mathbb{C}, 0) \\ (\lambda, x, y) & \mapsto & (x^4 + \lambda x^3 + y^3, xy) \end{array}$$

3.4. Proof of proposition 4.6. Denote by

$$G : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

an arbitrary deformation of the holomorphic function-germ $f \in \mathcal{P}A_1^1$.

LEMMA 4.4. *The deformation (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation induced from the one-parameter deformation*

$$(\lambda, x, y) \mapsto (x^4 + \lambda x^3 + y^3, xy).$$

Proof.
The holomorphic function-germ $f = G(0, .)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^1$. Thus, proposition 4.5 implies that the following $\mathcal{G}$-equivalence holds

$$(f, \Delta_f) \sim (x^4 + y^3, xy)$$

Hence (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation of $(x^4 + y^3, xy)$.
Define the deformation $A : (\mathbb{C}^6 \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of (f, Δ_f) by the polynomials

$$A(\mu, \lambda, x, y) = (x^4 + \lambda x^3 + y^3 + \mu_1 x^2 + \mu_2 y^2 + \mu_3 x + \mu_4 y, xy + \mu_5).$$

Direct calculations show that the $\mathcal{G}$-tangent space to A is $TA = \mathcal{O}_{x,y}^2$.
Thus, the $\mathcal{G}$-versal deformation theorem (see subsection 1.10) implies that the deformation A of (f, Δ_f) is $\mathcal{G}$-versal. Consequently (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation induced from A by a holomorphic map-germ

$$h : (\mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^6, 0).$$

Denote respectively by $\gamma = (\gamma_1, \dots, \gamma_k)$ the parameters of the deformations G.
We use the old-fashioned notation:

$$h(\gamma) = (\mu_1(\gamma), \dots, \mu_5(\gamma), \lambda(\gamma)).$$

Lemma 4.4 is a consequence of the following lemma.

LEMMA 4.5. *The map-germ μ vanishes identically.*

Proof.
Denote by $\bar{G}$ a representative of G.
The function $\bar{G}(\gamma, .)$ has a Morse critical point in a neighbourhood the origin provided that γ is small enough.
A translation sends an inflection point of a curve to an inflection point of its translation. Consequently, we can assume without loss of generality that the Morse critical point of $\bar{G}(\gamma, .)$ is the origin.
Proposition 4.5 implies that for any γ small enough:
1) $\Delta_{\bar{G}}(\gamma, .)$ has a Morse critical point at the origin of critical value 0,
2) the restriction of $\bar{G}(\gamma, .)$ to each branch of the plane curve

$$\left\{(x, y) \in \mathbb{C}^2 : \Delta_{\bar{G}}(\gamma, x, y) = 0\right\}$$

has a critical point of the type A_k with $k \geq 2$.
Condition 1 implies that μ_5 vanishes identically. Then, condition 2 implies that $\mu_1, \ldots, \mu_4$ also vanish identically. Lemma is proved.

LEMMA 4.6. *Assume that the vector $(x^3, 0)$ is contained in the $\mathcal{G}$-tangent space to (G, Δ_G). Then the following $\mathcal{G}$-equivalence holds*
$(G, \Delta_G) \sim (x^4 + y^3 + \lambda x^3, xy)$.

Proof.
By direct calculations, we get that under the conditions of the lemma the $\mathcal{G}$-tangent space to the deformation $B : (\mathbb{C}^{k+5} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of (f, Δ_f) defined by:

$$B(\mu, \gamma, x, y) = (G(\gamma, x, y) + \mu_1 x^2 + \mu_2 y^2 + \mu_3 x + \mu_4 y, \Delta_G(\gamma, x, y) + \mu_5)$$

is $TB = \mathcal{O}^2_{x,y}$.
The $\mathcal{G}$-versal deformation theorem implies that B is $\mathcal{G}$-versal. Hence the deformation

$$\begin{array}{ccc} (\mathbb{C} \times \mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C} \times \mathbb{C}, 0) \\ (\lambda, x, y) & \mapsto & (x^4 + y^3 + \lambda x^3, xy) \end{array}$$

is induced from a deformation $\mathcal{G}$-equivalent to B.
The same argument as the one given in lemma 4.5 implies that $(x^4 + y^3 + \lambda x^3, xy)$ is induced from a deformation $\mathcal{G}$-equivalent to (G, Δ_G).
On the other hand, lemma 4.5 implies that (G, Δ_G) is induced from a deformation $\mathcal{G}$-equivalent to $(x^4 + y^3 + \lambda x^3, xy)$.
We have shown that:
- (G, Δ_G) is induced from a deformation $\mathcal{G}$-equivalent to $(x^4 + y^3 + \lambda x^3, xy)$,
- $(x^4 + y^3 + \lambda x^3, xy)$ is induced from a deformation $\mathcal{G}$-equivalent to (G, Δ_G).

Consequently (G, Δ_G) and $(x^4 + y^3 + \lambda x^3, xy)$ are $\mathcal{G}$-equivalent. Lemma is proved.

By definition of $\mathcal{P}$-versality, these two lemmas imply that a function-germ G satisfying the conditions of lemma 4.6 is a $\mathcal{P}$-versal deformation.

It remains to find a deformation G of f such that $(x^3, 0)$ belongs to the $\mathcal{G}$-tangent space to (G, Δ_G).
Define the deformation $F : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ of $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ by

$$F(\alpha, \beta, x, y) = xy + x^4 + y^3 + \lambda x^3$$

Final assertion. The map germ

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C} \times \mathbb{C}, 0) \\ (x, y) & \mapsto & (x^3, 0) \end{array}$$

is contained in the $\mathcal{G}$-tangent space to the deformation F.

Denote by:
- $\mathcal{M}^k$ the k^{th} power of the maximal ideal $\mathcal{M} \subset \mathcal{O}_{x,y}$,

- M the $\mathcal{O}_{x,y}$-module:

$$M = \left\{(0,m) : m \in \mathcal{M}^2\right\}.$$

LEMMA 4.7. *The module M is contained in the $\mathcal{G}$-tangent space to (F, Δ_F).*

Proof.
The function germ $f = F(0,.)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^1$, thus proposition 4.5 implies the $\mathcal{G}$-equivalence relation

$$(F(0,.), \Delta_F(0,.)) \sim (x^4 + y^3, xy).$$

Hence the $\mathcal{G}$-tangent space to (F, Δ_F) contains the tangent space to the constant deformation equal to $(x^4 + y^3, xy)$. It is readily verified that the $\mathcal{G}$-tangent space to this constant deformation contains the $\mathcal{O}_{x,y}$-module M. Lemma is proved.

We conclude the proof of the assertion.
The Hamilton vector-field of $F(\lambda, .)$ is of the form

$$h_\lambda(x,y) = (x + r_1(\lambda, x, y))\partial_x - (y + r_2(\lambda, x, y))\partial_y.$$

Here $r_1(\lambda, .), r_2(\lambda, .) \in \mathcal{M}^2$. Thus for any value of λ, we have $\Delta_F \in \mathcal{M}^2$. Consequently the restriction of $\partial_\lambda \Delta_F$ to $\lambda = 0$ is contained in $\mathcal{M}_{x,y}^2$.
Lemma 4.7 implies that the restriction of $(0, \partial_\lambda \Delta_F)$ to $\lambda = 0$ is contained in the $\mathcal{G}$-tangent space to (F, Δ_F).
Thus the map germ

$$((\partial_\lambda F, \partial_\lambda \Delta_F) - (0, \partial_\lambda \Delta_F))_{|\lambda=0} = (x^3, 0)$$

is contained in the $\mathcal{G}$-tangent space to (F, Δ_F). This concludes the proof of the final assertion. Proposition 4.6 is proved.

CHAPTER 5

Projective topological invariants and the $K(\pi, 1)$ theorem.

The theory of normal forms that we developed in the preceding chapter allows us to compute most of the projective topological invariants of $\mathcal{P}$-simple function-germs. Indeed, the $\mathcal{P}$-versal deformation theorem for the $\mathcal{P}A_1^{p,q}$ series makes the computation of the corresponding PT-monodromy group a simple exercise. In this chapter, we focus our attention on the computation of the PT-fundamental group for the series $\mathcal{P}A_1^{p,q}$ and on the $K(\pi, 1)$ theorem stated in chapter 2 section 7. This is done by constructing a variant of the Lyaschko-Loojenga mapping. The construction of this mapping is the main tool from which we shall deduce our results.

1. Preliminaries.

1.1. The $\mathcal{P}A_1^{p,q}$ series. For notational reasons assume that $pq > 0$. Let $F : (\mathbb{C}^{p+q} \times \mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ be a $\mathcal{P}$-versal deformation of a function-germ $f \in \mathcal{P}A_1^{p,q}$. Denote by Δ_F the bordered Hessian of F with respect to the variables (x, y). Theorem 2.11 (chapter 2, section 5) asserts that the deformation (F, Δ_F) is $\mathcal{G}$-equivalent to the deformation $(P(\alpha, \beta, x, y), xy)$ with:

$$P(\alpha, \beta, x, y) = x^{3+p} + y^{3+q} + \alpha_1 x^{2+p} + \cdots + \alpha_p x^3 + \beta_1 y^{2+q} + \cdots + \beta_q y^3.$$

The $\mathcal{G}$-equivalence relation preserves all the PT-invariants (in fact $\mathcal{G}$-equivalence was introduced for computing them). This leads us to the following proposition.

Put:

$$\begin{cases} Q(\alpha, x) & = & x^{3+p} + \alpha_1 x^{2+p} + \cdots + \alpha_p x^3, \\ R(\beta, y) & = & y^{3+q} + \beta_1 y^{2+q} + \cdots + \beta_q y^3. \end{cases}$$

Let $\widetilde{\Sigma} \subset \mathbb{C}^{p+q+1}$ be the set of values of the parameters $(\alpha, \beta, \varepsilon)$ for which at least one one of the two polynomials $Q(\alpha, x) - \varepsilon$ or $R(\beta, y) - \varepsilon$ has a double root.

Proposition 5.1. *The PT-covering of f is C^∞-equivalent to the covering whose base is $\mathbb{C}^{p+q+1} \setminus \widetilde{\Sigma}$, whose fibre at $(\alpha, \beta, \varepsilon) \in \mathbb{C}^{p+q+1} \setminus \widetilde{\Sigma}$ is the set of values of x and y such that $Q(\alpha, x) = \varepsilon$, $R(\beta, y) = \varepsilon$.*

This proposition imply the results on the monodromy and on the PT-braid groups for the $\mathcal{P}$-singularity classes $\mathcal{P}A_1^{p,q}$.

All that remains to work out for the $\mathcal{P}A_1^{p,q}$ singularity classes is:
- to calculate the PT-fundamental groups,
- to prove the $K(\pi, 1)$ theorem.

1.2. The $\mathcal{P}A_2$ series. Consider a holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$.
Let $F : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a $\mathcal{P}$-versal deformation of f.
Theorem 2.8 implies that $f = F(0, .)$ is $\mathcal{P}$-equivalent to the function-germ defined by the polynomial $y^2 + x^3$.
The PT-invariants being unchanged under $\mathcal{P}$-equivalence, we can assume without loss of generality that $f(x, y) = y^2 + x^3$.
Take a representative $\bar{F} : \Lambda \times U \longrightarrow \mathbb{C}$ of the function germ F. Put:

$$V_{\lambda,\varepsilon} = \{p \in U : F(\lambda, p) = \varepsilon\}$$

Denote by $B_r \subset \mathbb{C}^2$ the closed ball of radius r.

Lemma 5.1. *The curve $V_{\lambda,\varepsilon} \cap B_r$ does not have degenerate inflection points provided that r, $|| \lambda ||$ and ε are small enough.*

Proof.
First, we make a preliminary remark. Assume that for some value (λ, ε), the curve $V_{\lambda,\varepsilon}$ has a degenerate inflection point p. Let g be the germ at p of the restriction of $\bar{F}$ to the line tangent to $V_{\lambda,\varepsilon}$ at that point. Then, the third degree Taylor polynomial of g at p vanishes.
Let $L \subset \mathbb{C}^2$ be a fixed line. For each value of the parameter (λ, ε), choose a point $p(\lambda, \varepsilon) \in \mathbb{C}^2$ such that:
- the dependence of p on the parameter (λ, ε) is holomorphic,
- $p(0, 0) = 0$.

Denote by k_λ the degree of the first term of the Taylor expansion at $p(\lambda, \varepsilon)$ of the restriction of $\bar{F}(\lambda, .)$ to L.
Since $F(0, x, y) = y^2 + x^3$, we have $k_0 \leq 3$. Thus $k(\lambda) \leq 3$ for λ small enough. According to the preliminary remark this means that $p(\lambda, \varepsilon)$ is not a degenerate inflection point provided that λ is small enough. Lemma is proved.

This lemma implies that the $\mathcal{P}$-discriminant of the family of curves $(V_{\lambda,\varepsilon} \cap B_r)$ consists only of the values of the parameter (λ, ε) for which the curve $V_{\lambda,\varepsilon}$ has a singular point provided that $|| l ||$ and ε are small enough.
Consequently the corresponding $\mathcal{P}$-bifurcation diagram is C^∞ diffeomorphic to the germ of a complex hyperplane H ([**AVGL1**]) in $\mathbb{C}^k$.
Obviously the space $\mathbb{C}^k \setminus H$ is a $K(\pi, 1)$ space. Thus the $K(\pi, 1)$ theorem (theorem (2.17) 55) holds for the $\mathcal{P}A_2$ series.
Another consequence of the lemma is that for any two deformations of the function-germ $f \in \mathcal{P}A_2$ which are $(R-L)$-equivalent, the PT-invariants will be the same.

Thus it suffices to compute the PT-invariants for an $(R-L)$-versal deformation of f say:

$$F(\lambda, x, y) = y^2 + x^3 - 3\lambda x.$$

In this case the curves $V_{\lambda,\varepsilon} \subset \mathbb{C}^2$ are elliptic curves minus one point.
Fix the origin of the group of the elliptic curve $V_{\lambda,\varepsilon}$ to be the "missing" point.
Abel's theorem implies that the inflection points of an elliptic curve are the points of order three of its group ([**Clebsch**]). That makes the computations of the PT-monodromy group trivial[1] .
Indeed, the Plücker discriminant of the family of curves:

$$V_{\lambda,\varepsilon} = \left\{(x, y) \in \mathbb{C}^2 : y^2 + x^3 - 3\lambda x = \varepsilon\right\}$$

is the semi-cubical parabola:

$$\Sigma = \left\{(\lambda, \varepsilon) \in \mathbb{C}^2 : \lambda^3 + \varepsilon^2 = 0\right\}$$

Fix a value of $(\lambda, \varepsilon) \in \mathbb{C}^2 \setminus \Sigma$. Let ω be the restriction of the holomorphic one-form $\frac{dx}{y}$ to $V_{\lambda,\varepsilon}$. The Abel mapping:

$$\begin{array}{ccc} V_{\lambda,\varepsilon} & \longrightarrow & \mathbb{C}/\Gamma \\ p & \mapsto & \int_\infty^p \omega \end{array}$$

sends the inflection points of $V_{\lambda,\varepsilon}$ to the points of order 3 in $\mathbb{C}/\Gamma$ distinct from 0.
We identify these points with the non-zero vectors of the vector space $\mathbb{Z}_3 \times \mathbb{Z}_3$.
Then the PT-monodromy is generated by the matrices ([**BrKn**] for details):

$$\begin{cases} A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{cases}$$

The group generated by these matrices is $SL(2, \mathbb{Z}/3\mathbb{Z})$.

2. Basic singularity theory.

We review various facts that we shall need.

2.1. Distinguished basis and vanishing cycles. In this subsection, we consider no longer holomorphic functions of two variables but of one variable. We follow word for word the textbook [**AVGL1**].
Let $f : B_\delta \longrightarrow \mathbb{C}$ be a holomorphic function with Morse critical points $p_1, \dots, p_\mu$ of corresponding critical values $\varepsilon_1, \dots, \varepsilon_\mu$. Here B_δ denotes the closed disk of radius δ in $\mathbb{C}$ centered at the origin.
We assume that:
- distinct critical points have distinct critical values,
- for any $\varepsilon \in \mathbb{C}$, any point belonging to the set $f^{-1}(\varepsilon)$ lies on the interior of

[1] I do not know how to describe the PT-braid group of the singularity $\mathcal{P}A_2$ in an invariant form although we have this simple normal form.

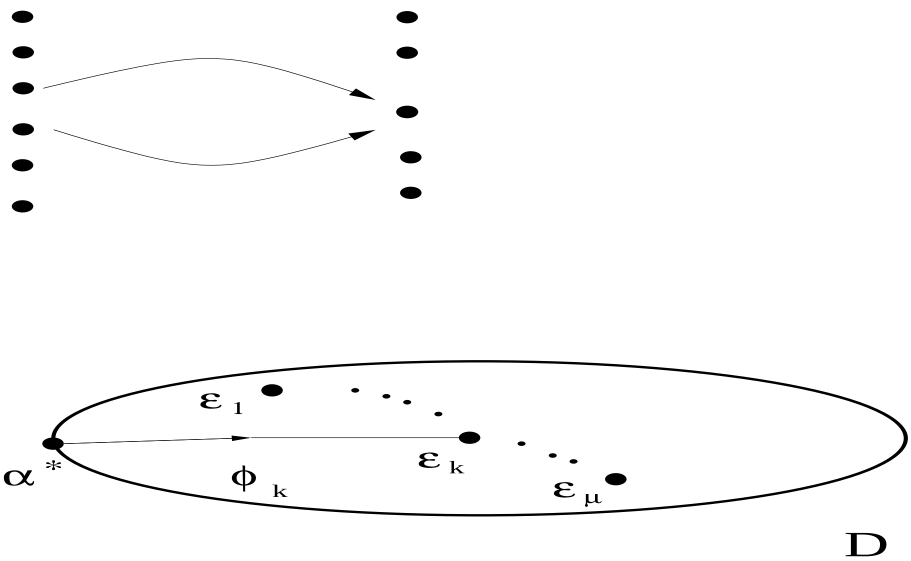

FIGURE 1. *Zero-dimensional vanishing cycle at a Morse critical point. When ε approaches a critical value of a Morse critical point of f two preimages coalesce.*

B_δ.

Put $D = f(B_\delta)$. Choose a point α^* on the boundary of D. Let $D' = D \setminus \{\varepsilon_1, \ldots, \varepsilon_\mu\}$. Fix an arbitrary $j \in \{1, \ldots, \mu\}$. Let $\phi_k : [0,1] \longrightarrow D$ be a path connecting α^* with the critical value $\varepsilon_k \in D$ of f. The word connecting means that:
- for $t \neq 1$, $\phi_k(t) \in D'$,
- $\phi_k(0) = \alpha^*$, $\phi_k(1) = \varepsilon_k$.
When $t \longrightarrow 1$, two preimages of $\phi_k(t)$ under f, say $x_j(t)$ and $x_k(t)$, approach one towards the other continuously with the parameter t (see figure 1).

DEFINITION 5.1. The zero-dimensional reduced homology class
$\Delta_k = \{x_j(0)\} - \{x_k(0)\}$ is called the *vanishing cycle* associated to the path ϕ_k.

Consider a loop β obtained by going along ϕ from α^* then going anti-clockwise around ε_k and returning to α^* along ϕ_k (see figure 2).

DEFINITION 5.2. The loop α described above is called *a loop associated* to the path ϕ_k.

A loop α associated to a path ϕ_k connecting the base point $\alpha^* \in D'$ with a Morse critical value ε_k gives rise to a permutation of two points of the fibre of f at the base point. We say that this permutation is *associated* to the loop α. Classically this permutation is called the monodromy operator associated to α.

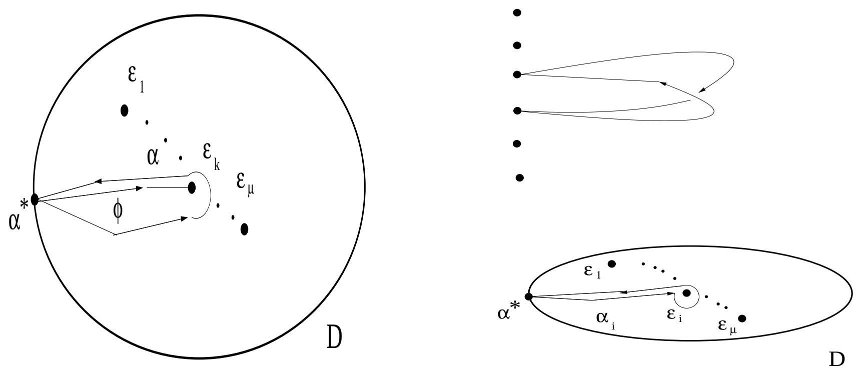

FIGURE 2. *On the left hand side: the closed loop α associated to the path ϕ. On the right hand side: the monodromy associated to the loop α.*

The fundamental group $\pi_1(D')$ is the free group with μ generators. Consider a set $\phi_1, \ldots, \phi_\mu : [0,1] \longrightarrow D$ of paths such that ϕ_k connects the base point α^* to the critical value ε_k.

DEFINITION 5.3. The system of paths $\phi_1, \ldots, \phi_\mu$ is called a *weakly distinguished* basis (of D' with base point α^*) if the set of corresponding loops generates the fundamental group $\pi_1(D')$.

DEFINITION 5.4. A weakly distinguished basis $\phi_1, \ldots, \phi_\mu$ of D' is called *distinguished* provided that:
- the paths ϕ_k's have no self-intersections,
- two different paths ϕ_j, ϕ_k intersect only at $t = 0$ at the base point $\alpha^* = \phi_j(0) = \phi_k(0)$,
- the paths ϕ_k are indexed in the order of increase of $Arg\phi_k'(0)$ (assuming that one reads the picture clockwise, see figure 3).

The following theorem is a particular case of a much more general theorem of Picard-Lefschetz theory ([**Br3**]).

THEOREM 5.1. *The vanishing cycles $\Delta_1, \ldots, \Delta_\mu$ associated to a (weakly) distinguish basis of paths $\phi_1, \ldots, \phi_\mu$ form a basis of the reduced homology group $\widetilde{H}_0(V_*)$ where V_* is the fibre of f at the base point α^*.*

2.2. Zariski's first theorem. In this section we discuss a variant of Zariski's classical theorem ([**Za**], [**VK**]). For details the reader is sent to [**HL**].
The theorem that we shall state depends on a technical "genericity" condition. We first give the theorem and then explain what this genericity condition means.
Let $\Sigma \subset \mathbb{C}^n$ be an algebraic hypersurface.

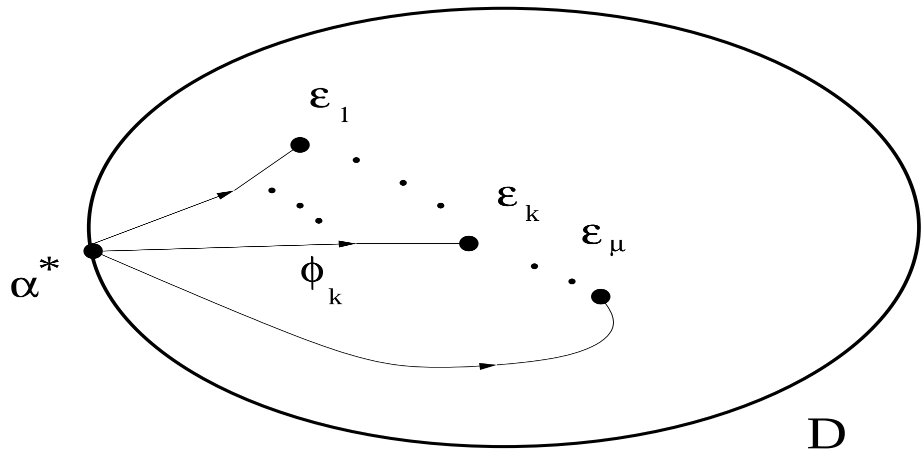

FIGURE 3. *A distinguished basis.*

THEOREM 5.2. *Let $L \subset \mathbb{C}^n$ be a Zariski generic complex line. Then the inclusion:*

$$i : L \setminus (L \cap \Sigma) \longrightarrow \mathbb{C}^n \setminus \Sigma$$

induces a surjective homomorphism

$$i_* : \pi_1(L \setminus (L \cap \Sigma)) \longrightarrow \pi_1(\mathbb{C}^n \setminus \Sigma).$$

We now explain what is a Zariski generic complex line.
First, consider the case $n = 2$.

DEFINITION 5.5. A complex line $L \subset \mathbb{C}^2$ is called *Zariski generic* with respect to a complex algebraic curve $C \subset \mathbb{C}^2$ provided that there exists a pencil of complex lines (L_t), $t \in \mathbb{C}$ such that:
- $L_0 = L$ intersects C transversally at m distinct points,
- for any $t \in \mathbb{C}$ except a finite number of them the complex line L_t intersects Σ transversally at m distinct points,
- for any $t \in \mathbb{C}$, L_t intersects C at a finite number of points.

Remark. We point out that the notion of Zariski genericity does not mean necessarily that the complex line intersects transversally the curve at a maximal number of points. This is made clear by the following example.

Example. Let $\Sigma = \left\{(x, y) \in \mathbb{C}^2 : y = x^2\right\}$. The complex line L of equation $x = 0$ is "Zariski generic". To see it consider the pencil (L_t) defined by $L_t = \{(x, y) : x = t\}$.

We turn on to the general case. To explain this general case, we assume that the reader is acquainted with the definition of a Whitney stratification ([**Wh**],[**Tei3**]).
We fix a hyperplane $H_\infty \approx \mathbb{C}P^{n-1}$ in $\mathbb{C}P^n$ and identify $\mathbb{C}P^n \setminus H_\infty$ with the

affine space $\mathbb{C}^n$.
Given a variety $V \subset \mathbb{C}^n$, we denote by $\bar{V}$ its closure in $\mathbb{C}P^n$.
Let $\Sigma \subset \mathbb{C}^n$ be an an algebraic hypersurface.
Since the surface $\bar{\Sigma} \cup H_\infty$ is algebraic it admits a minimal Whitney stratification ([**Wh**],[**Tei3**]). We consider this minimal stratification.

DEFINITION 5.6. An affine complex line $L \subset \mathbb{C}^n$ is called *Zariski generic* if the following conditions are satisfied:
- there exists an affine complex 2-plane $P \subset \mathbb{C}^n$ containing L such that $\bar{P}$ intersects each stratum of $\Sigma \cup H_\infty$ transversally.
- the complex line L is Zariski generic with respect to $C = P \cap \Sigma$ in $P \approx \mathbb{C}^2$.

2.3. The Zariski, Van-Kampen theorem. We keep the same notations as those of the preceding subsection.
Assume that there exists a linear projection:

$$p : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1},$$

such that:
- the values of the parameter $\varepsilon \in \mathbb{C}^{n-1}$ for which the complex line $p^{-1}(\varepsilon)$ is not Zariski generic form a hypersurface $\Delta \subset \mathbb{C}^{n-1}$,
- there exists a complex line $\widetilde{L} \subset \mathbb{C}^{n-1}$ such that the 2-plane $p^{-1}(\widetilde{L})$ intersects each strata of $\bar{\Sigma} \cup H_\infty$ transversally.

The Zariski Van-Kampen theorem describes the group $\pi_1(\mathbb{C}^n \setminus \Sigma)$ in terms of the projection p.
Fix a base point $\alpha^* \in \mathbb{C}^{n-1} \setminus \Delta$. Then the complex line $L^* = p^{-1}(\alpha^*)$ is Zariski generic.

A loop $\gamma : [0, 1] \longrightarrow \mathbb{C}^{n-1} \setminus \Delta$ acts on the free group $\pi_1(L^* \setminus (L^* \cap \Sigma))$ by monodromy. Indeed,for each value of $t \in [0, 1]$, the set of intersection points of L_t with Σ is identified with a set of points in $\mathbb{C}$ say $\{\varepsilon_1(t), \ldots, \varepsilon_s(t)\}$.
Let $D'_t \subset \mathbb{C}$ be the complement in $\mathbb{C}$ of $\{\varepsilon_1(t), \ldots, \varepsilon_s(t)\}$. Choose the same base point ε^* for the fundamental groups in the complement of $\{\varepsilon_1(t), \ldots, \varepsilon_s(t)\}$ for all values of t.
We get a locally trivial covering:
- the base of the covering is the oriented circle S^1,
- the fibre at t is the fundamental group $\pi_1(D'_t, \varepsilon^*)$ with base point ε^* (this group is a free group).

By definition, the monodromy automorphism of this covering associated to the loop $t \mapsto e^{2i\pi t}$ is the monodromy automorphism associated to γ. We denote it by $\rho[\gamma]$.
The automorphism $\rho[\gamma] : \pi_1(D'_0, \varepsilon^*) \longrightarrow \pi_1(D'_0, \varepsilon^*)$ depends only on the homotopy class of $\gamma : [0, 1] \longrightarrow \mathbb{C}^{n-1} \setminus \Delta$, $\gamma(0) = \gamma(1) = \varepsilon^*$.

Example. Let $\Sigma = \{(x,y) \in \mathbb{C}^2 : x = y^2\}$. Consider the projection $p(x,y) = x$. We have $\Delta = \{0\} \subset \mathbb{C}^*$. Let $\gamma : [0,1] \longrightarrow \mathbb{C}^2$ be defined by $\gamma(t) = (e^{2i\pi t}, 0)$. The line:

$$L_t = \left\{(x,y) \in \mathbb{C}^2 : x = e^{2i\pi t}\right\}$$

is identified with $\mathbb{C}$ via the map:

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}^2 \\ \varepsilon & \mapsto & (e^{2i\pi t}, \varepsilon). \end{array}$$

The points of intersection of L_t with Σ are identified with the points $\varepsilon_1(t),\ \varepsilon_2(t) \in \mathbb{C}$:

$$\begin{cases} \varepsilon_1(t) = e^{i\pi t} \\ \varepsilon_2(t) = -e^{i\pi t} \end{cases}$$

Denote by D'_t the complement of $\{\varepsilon_1(t), \varepsilon_2(t)\}$ in $\mathbb{C}$. We choose the base point ε^* to be any point on the imaginary axis with sufficiently small negative coordinate.

The group $\pi_1(D'_t, \varepsilon^*)$ is the free group with two generators. Consider loops α_1, α_2 of figure 4. Denote respectively by a_1 and a_2 their homotopy classes. They generate $\pi_1(L'_t, \varepsilon^*)$. The monodromy action of γ is as follows:

$$\begin{cases} \rho[\gamma].a_1 = a_2 \\ \rho[\gamma].a_2 = a_2 a_1 a_2^{-1}. \end{cases}$$

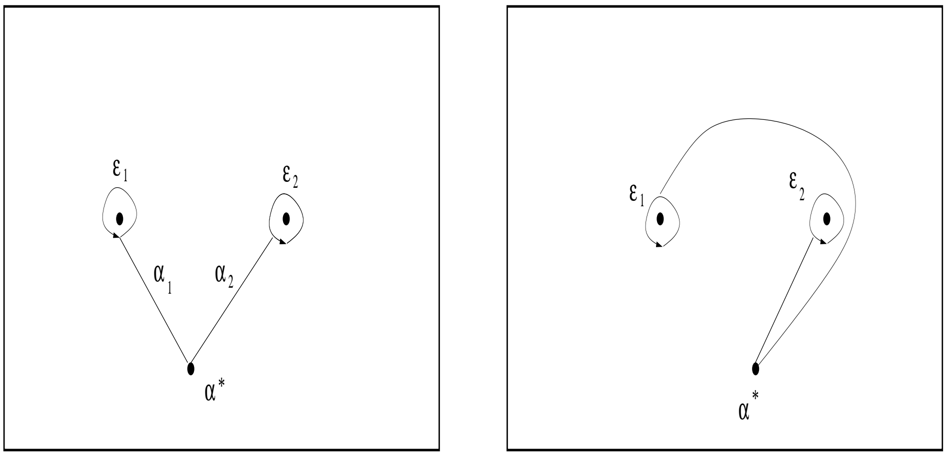

FIGURE 4.

We come back to the general case. We say that a subgroup of a group is *invariant* (sometimes called distinguished) if it is invariant under conjugation by the elements of the group.

Definition 5.7. The *relation subgroup* H of the free group $\pi_1(L^* \setminus (L^* \cap \Sigma))$ is the minimal invariant subgroup containing all the elements of the type $\eta^{-1}(\rho[\gamma].\eta)$ for all $\eta's$ in $\pi_1(L^* \setminus (L^* \cap \Sigma))$ and all γ's in $\pi_1(\mathbb{C}^{n-1} \setminus \Delta)$.

Theorem 5.3. *The quotient of the free group $\pi_1(L^* \setminus (L^* \cap \Sigma))$ by the relation subgroup is isomorphic to the group $\pi_1(\mathbb{C}^n \setminus \Sigma)$.*

Example. We come back to the computations of the preceding example. In this case, the free-group $G = \pi_1(L^* \setminus (L^* \cap \Sigma))$ has two generators a_1, a_2. The relation subgroup H is generated by $a_1 a_2^{-1}$. Thus $\pi_1(\mathbb{C}^2 \setminus \Sigma)$ is isomorphic to $\mathbb{Z}$.
Denote by $\bar{a}_i$ the image of a_i under the canonical projection $G \longrightarrow G/H$. We have $\bar{a}_1 = \bar{a}_2$. This means that the loops α_1 and α_2 are homotopic in $\mathbb{C}^2 \setminus \Sigma$.

3. A variant of the Lyaschko-Loojenga mapping.

3.1. The $\mathcal{P}A_1^p$ Lyaschko-Loojenga mapping. We consider a variant of the usual Lyaschko-Loojenga mapping [**Arn4**], [**Loo**]. This variant is the restriction of the usual mapping to a particular strata. Consequently, it shares the same properties than the usual Lyaschko-Loojenga mapping.

Consider the family of polynomials:

$$Q(\lambda, x) = x^{p+3} + \lambda_1 x^{p+2} + \cdots + \lambda_p x^3. \tag{30}$$

We say that a critical point of a polynomial is *degenerate* if both the first and the second derivative of the polynomial vanish at the given point. Denote by Δ the set of values of the parameter $\lambda \in \mathbb{C}^p$ for which either:
- the polynomial $Q(\lambda, .)$ has a non-zero degenerate critical point or,
- two distinct critical points of $Q(\lambda, .)$ have the same critical value.
Let $\Sigma \subset \mathbb{C}^p$ be the set of values of the parameter $\mu \in \mathbb{C}^p$ for which the polynomial:

$$\tilde{Q}(\mu, x) = x^p + \mu_1 x^{p-1} + \cdots + \mu_{p-1} x + \mu_p$$

either has a double root or vanishes at the origin.

Definition 5.8. The $\mathcal{P}A_1^p$ *Lyaschko-Loojenga mapping*, denoted $\pi[p]$, is the map that sends $\lambda \in \mathbb{C}^p \setminus \Delta$ to the coefficients $\mu = (\mu_1, \dots, \mu_p)$ of the polynomial $Q(\mu, .)$ whose roots are the non-zero critical values of $Q(\lambda, .)$.

Remark. Proposition 5.1 of page 99 implies that this definition is equivalent to the one given in chapter 2 page 56.
Example. Consider the case $p = 1$. Put:

$$Q(\lambda, x) = 3x^4 - 4\lambda x^3.$$

We have changed the coefficients with respect to formula 30 in order to simplify the calculations. The critical points of $Q(\lambda, .)$ are 0 and λ. The critical value corresponding to the critical point $x = \lambda$ of $Q(\lambda, .)$ is $Q(\lambda, \lambda) =$

$-\lambda^4$. Consequently the $\mathcal{P}A_1^1$ Lyaschko-Loojenga mapping $\pi[1]$ is (up to a multiplicative constant) given by the formula

$$\begin{array}{cccc} \pi[1]: & \mathbb{C}^* & \longrightarrow & \mathbb{C}^*, \\ & \lambda & \mapsto & -\lambda^4, \end{array}$$

Here and in the sequel the symbol $\mathbb{C}^*$ stands for $\mathbb{C} \setminus \{0\}$.

The following proposition follows Riemann's theorem on the unicity of the analytic structure of $\mathbb{C}P^1$ (like for the usual Lyaschko-Loojenga mapping).

PROPOSITION 5.2. *The $\mathcal{P}A_1^p$ Lyaschko-Loojenga mapping $\pi[p] : \mathbb{C}^p \setminus \Delta \longrightarrow \mathbb{C}^p \setminus \Sigma$ is a (holomorphic) covering.*

COROLLARY 5.1. *The space $\mathbb{C}^p \setminus \Delta$ is a $K(\pi, 1)$ space.*

We explain why proposition 5.2 implies this corollary.
Recall that given a, say topological, covering the homotopy groups π_k of the base space and of the total space are the same provided that $k > 1$. Indeed, for $k > 1$, the exact homotopy sequence of a fibration $E \longrightarrow B$ of fibre F is:

$$\ldots \longrightarrow \pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \ldots,$$

If F is discrete the $\pi_k(F) = \pi_{k-1}(F) = 0$. Hence $\pi_k(E) \approx \pi_k(B)$ for $k > 1$.
Thus $\mathbb{C}^p \setminus \Delta$ is a $K(\pi, 1)$ space provided that $\mathbb{C}^p \setminus \Sigma$ is a $K(\pi, 1)$ space.
The space $\mathbb{C}^p \setminus \Sigma$ can be identified with the space of polynomials Q with distinct roots that do not vanish at 0. According to Brieskorn, this space is the classifying space of the Coxeter group B_p. In particular it is a $K(\pi, 1)$ space (see [**Br2**] for details).
There is no difficulty in generalizing our discussion to the case $\mathcal{P}A_1^{p,q}$. This will occupy the next subsection.

3.2. The $\mathcal{P}A_1^{p,q}$ Lyaschko-Loojenga mapping. Consider the family of pairs of polynomials (for fixed integers $p > 0,\ q > 0$):

$$\begin{cases} Q(\alpha, x) &= x^{p+3} + \alpha_1 x^{p+2} + \cdots + \alpha_p x^3 \\ R(\beta, y) &= y^{q+3} + \beta_1 y^{p+2} + \cdots + \beta_q y^3 \end{cases} \tag{31}$$

Denote by Δ the set of values of the parameter $(\alpha, \beta) \in \mathbb{C}^{p+q}$ for which either:
- at least one of the polynomials $Q(\alpha, .)$ or $R(\beta, .)$ has a non-zero degenerate critical point or,
- there are two critical points of P or of Q with the same critical values (it can be two critical points of the same polynomials or of the two polynomials).

Let $\Sigma \subset \mathbb{C}^{p+q}$ be the set of values of the parameters $(a, b) \in \mathbb{C}^{p+q}$ for which one of the polynomials:

$$\begin{cases} \tilde{Q}(a, x) &= x^p + a_1 x^{p-1} + \cdots + a_{p-1} x + a_p \\ \tilde{R}(b, y) &= y^q + b_1 y^{p+2} + \cdots + b_{q-1} y + b_q \end{cases}$$

either has a double root or vanishes at the origin.

DEFINITION 5.9. The $\mathcal{P}A_1^{p,q}$ *Lyaschko-Loojenga mapping*, denoted $\pi[p,q]$ is the map which sends the parameter $(\alpha,\beta) \in \mathbb{C}^{p+q}$ to the coefficients $a = (a_1,\dots,a_p)$, $b = (b_1,\dots,b_q)$ of the polynomials $\tilde{Q}(a,.)$, $\tilde{R}(b,.)$ whose roots are the non-zero critical values of Q and of R.

Remark. Proposition 5.1 of section 1 implies that this definition is equivalent to the one given in chapter 2, subsection 7.7 .
Example. Consider the case $p = q = 1$. Take:

$$\begin{cases} Q(\alpha,x) &= 3x^4 - 4\alpha x^3, \\ R(\beta,y) &= 3y^4 - 4\beta y^3. \end{cases}$$

We have multiplied the coefficients of Q, R by constants with respect to formula 31 in order to simplify the calculations.
The variety Δ is the union of 6 complex lines passing through the origin:
- the 4 complex lines $\{(\alpha,\beta) \in \mathbb{C}^2 : \alpha^4 = \beta^4\}$,
- the 2 complex lines $\{(\alpha,\beta) \in \mathbb{C}^2 : \alpha\beta = 0\}$.
The variety Σ is the union of 3 complex lines:
- the coordinate axis of equation $a = 0$ and $b = 0$,
- the complex line $\{(a,b) \in \mathbb{C}^2 : a = b\}$.
The only non-zero critical point of $Q(\alpha,.)$ (resp. $R(\beta,.)$) is α (resp. β). The corresponding critical value is $Q(\alpha,\alpha) = -\alpha^4$ (resp. $R(\beta,\beta) = -\beta^4$). Consequently the $\mathcal{P}A_1^{1,1}$ Lyaschko-Loojenga mapping is (up to a multiplicative constant) given by the formula

$$\begin{array}{cccc} \pi[1,1]: & \mathbb{C}^2 \setminus \Delta & \longrightarrow & \mathbb{C}^2 \setminus \Sigma, \\ & (\alpha,\beta) & \mapsto & (-\alpha^4, -\beta^4). \end{array}$$

Like for the $\mathcal{P}A_1^p$ we have the following proposition.

PROPOSITION 5.3. *The restriction of the $\mathcal{P}A_1^{p,q}$ Lyaschko-Loojenga mapping $\pi[p,q] : \mathbb{C}^{p+q} \setminus \Delta \longrightarrow \mathbb{C}^{p+q} \setminus \Sigma$ is a covering.*

PROPOSITION 5.4. *The space $\mathbb{C}^{p+q} \setminus \Delta$ is a $K(\pi,1)$ space.*

In order to prove proposition 5.4, we need to make a digression.

3.3. Coloured configuration spaces. Consider a topological space X. We have denoted by $B(X,k)$ the configuration space with k elements. That is the space whose points are sets with k pairwise distinct elements in X.
We define the two coloured configuration space of X with p elements of one colour and q elements of another colour denoted $B(X,p,q)$.
We use the notation $\varepsilon = \{\varepsilon_1,\dots,\varepsilon_p\} \in B(X,p)$ and $\varepsilon' = \{\varepsilon_{-1},\dots,\varepsilon_{-q}\} \in B(X,q)$.

Definition 5.10. The topological space

$$B(X,p,q) = \left\{(\varepsilon,\varepsilon') \in B(X,p) \times B(X,q) : \varepsilon_j \neq \varepsilon_k\right\},$$

where j (resp. k) runs over all values in $\{1,\ldots,p\}$(resp. $\{-1,\ldots,-q\}$) is called the *two coloured configuration space of* X with p white elements and q black elements.

Remark. The topological spaces $B(X,p,q)$ and $B(X,q,p)$ are homeomorphic.

With the notations of the previous subsection we have the following proposition.

Proposition 5.5. *The topological space $B(\mathbb{C}^*,p,q)$ is homeomorphic*[2] *to $\mathbb{C}^{p+q} \setminus \Sigma$.*

The proof is obvious. We send the point $(\{\varepsilon_1,\ldots\varepsilon_p\},\{\varepsilon_{-1},\ldots\varepsilon_{-q}\}) \in B(\mathbb{C}^*,p,q)$ to the value of the parameters $(a,b) \in \mathbb{C}^{p+q} \setminus \Sigma$ such that:
- the polynomial $\tilde{Q}(a,.)$ has the roots $\{\varepsilon_1,\ldots\varepsilon_p\}$,
- the polynomial $\tilde{R}(a,.)$ has the roots $\{\varepsilon_{-1},\ldots\varepsilon_{-q}\}$.

3.4. Proof of proposition 5.4. In order to prove that $\mathbb{C}^{p+q} \setminus \Delta$ is a $K(\pi,1)$ space, it is sufficient to prove that $\mathbb{C}^{p+q} \setminus \Sigma$ is a $K(\pi,1)$ space. As usual this is due to the following two facts:
- the $\mathcal{P}A_1^{p,q}$ Lyaschko-Loojenga mapping $\pi[p,q] : \mathbb{C}^{p+q} \setminus \Delta \longrightarrow \mathbb{C}^{p+q} \setminus \Sigma$ is a covering,
- the homotopy groups of order strictly higher than one of the base space of a covering coincide with those of the total space.

As we saw in the preceding subsection, the space $\mathbb{C}^{p+q} \setminus \Sigma$ is homeomorphic to $B(\mathbb{C}^*,p,q)$.
Assertion: *the space $B(\mathbb{C}^*,p,q)$ is a $K(\pi,1)$ space.*

The proof of this assertion will conclude the proof of proposition 5.4.
Consider the map $B(\mathbb{C}^*,p,q) \longrightarrow B(\mathbb{C}^*,p+q)$ obtained by forgetting the colours of the elements.
This map is a covering, hence $B(\mathbb{C}^*,p,q)$ is a $K(\pi,1)$ space provided that $B(\mathbb{C}^*,p+q)$ is a $K(\pi,1)$ space.
The space $B(\mathbb{C}^*,p+q)$ is the $K(\pi,1)$-classifying space of the Coxeter group B_{p+q}(see [**Br2**]). This concludes the proof of proposition 5.4.

[2] We defined the configuration spaces $B(X,k)$ as topological spaces. However this space carries a natural (obvious) analytical structure when X is an analytical manifold. For this structure $B(\mathbb{C}^*,p,q)$ and $\mathbb{C}^{p+q} \setminus \Sigma$ are biholomorphic.

3.5. Description of the fundamental group via the $\mathcal{P}A_1^p$ Lyaschko-Loojenga mapping. We are going to describe the fundamental group of $\mathbb{C}^p \setminus \Delta$ in terms of the covering:

$$\pi[p] : \mathbb{C}^p \setminus \Delta \longrightarrow \mathbb{C}^p \setminus \Sigma.$$

We fix respective base points ε^* and λ^* in $\mathbb{C}^p \setminus \Sigma$ and $\mathbb{C}^p \setminus \Delta$ such that λ^* belongs to the fibre of ε^*, that is $\pi[p](\lambda^*) = \varepsilon^*$.
Any element $\gamma \in \pi_1(\mathbb{C}^p \setminus \Sigma)$ acts on the fibre of the covering $\pi[p]$ at ε^* by monodromy. We denote by $\gamma.$ this action.

Definition 5.11. The *order of* γ (with respect to the base point λ^*) is the smallest number k such that $\gamma^k.\lambda^* = \lambda^*$.

So if we have a loop γ of order k (with respect to the base point λ^*), then starting from λ^*, γ^k can be lifted to a closed loop.
Example. Put $Q(\lambda, x) = 3x^4 - 4\lambda x^3$. We computed the corresponding Lyaschko-Loojenga mapping $\pi[1]$ previously. We found that it is given by the formula

$$\pi[1] : \begin{array}{ccc} \mathbb{C}_1^* & \longrightarrow & \mathbb{C}_2^*, \\ \lambda & \mapsto & -\lambda^4. \end{array}$$

In order to distinguish the source space and the target space (both equal to $\mathbb{C}^*$) of $\pi[1]$, we have put lower indices $1, 2$.
Choose $1 \in \mathbb{C}_1^*$ and $-1 \in \mathbb{C}_2^*$ as the base points.
Consider the loop γ defined by

$$\gamma : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C}_2^* \\ t & \mapsto & -e^{2i\pi t} \end{array}$$

The pre-images of $-1 \in \mathbb{C}_2^*$ under $\pi[1]$ are the four values $\pm 1, \pm i$ corresponding to the four polynomials $Q(\pm 1, .),\ Q(\pm i, .)$.
The action of γ on them is the multiplication by $-i$. Indeed the loop γ can be lifted to the loop

$$\gamma : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C}_1^*, \\ t & \mapsto & e^{\frac{i\pi t}{2}}. \end{array}$$

Thus γ is of order 4. Indeed, the loop γ^4 given by the formula

$$\gamma^4 : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C}_2^*, \\ t & \longrightarrow & e^{8i\pi t}, \end{array}$$

can be lifted to the closed loop:

$$\widetilde{\gamma} : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{C}_1^*, \\ t & \mapsto & e^{2i\pi t}. \end{array}$$

We now describe a set of loops $\{\gamma_{j,k}\}$, $0 \leq j < k \leq p$ in $\mathbb{C}^p \setminus \Sigma$ with base point ε^* and a sequence of numbers $(m_{j,k})$ such that:
- the loop $\gamma_{j,k}$ has order $m_{j,k}$ i.e. the loop $\gamma_{j,k}^{m_{j,k}}$ can be lifted to a closed loop $\tilde{\gamma}_{j,k}$,
- the homotopy classes of the loops $\tilde{\gamma}_{j,k}$'s generate the fundamental group

$\mathbb{C}^p \setminus \Delta$.

Recall that $\mathbb{C}^p \setminus \Sigma$ is homeomorphic to the configuration space $B(\mathbb{C}^*, p)$ of p unordered pairwise distinct points in $\mathbb{C}^*$. In the sequel, we identify both spaces.
Denote by $\varepsilon^* = \{\varepsilon_1, \dots, \varepsilon_p\} \in B(\mathbb{C}^*, p)$, the set of non-zero critical values of $Q(\lambda^*, .)$, $\lambda^* \in \mathbb{C}^p \setminus \Delta$.
The ε_k's are numbered such that $\varepsilon_k < \varepsilon_{k+1}$.
For simplicity, we choose the base point $\lambda^* \in \mathbb{C}^p \setminus \Delta$ so that the critical values of $Q(\lambda^*, .)$ are real and the critical points non-negative.
For $0 < j < k$, the loop $\gamma_{j,k}$ is any loop obtained by exchanging ε_j and ε_k on the lower half-plane ($Im(\varepsilon) < 0$) as indicated in the left part of figure 5).
The loops $\gamma_{0,k}$ are obtained by turning counterclockwise the critical value ε_j around the origin as indicated in the right part of figure 5.
In the description of the loops, we have used the fact $\mathbb{C}^p \setminus \Sigma$ and $B(\mathbb{C}^*, p)$ are homeomorphic. Moreover the corresponding loops are closed in $B(\mathbb{C}^*, p)$ because we start with the configuration $\varepsilon^* \in B(\mathbb{C}^*, p)$ and end with the same configuration.

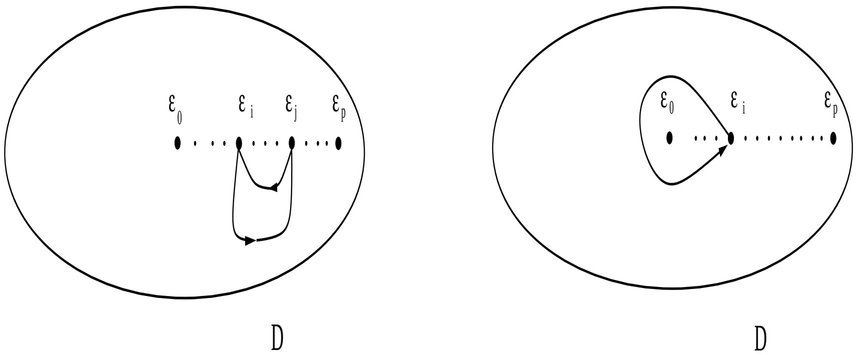

FIGURE 5. *The loops $\gamma_{j,k}$ that generate the fundamental group of $\mathbb{C}^p \setminus \Sigma$.*

PROPOSITION 5.6. *Let $m_{j,k}, j < k$ be the sequence defined by:*

$$m_{j,k} = \begin{cases} 2 & \text{if} \quad k - j > 1, \\ 3 & \text{if} \quad k = j+1,\ i > 0 \\ 4 & \text{if} \quad j = 0,\ k = 1. \end{cases}$$

Then the loops $\gamma_{j,k}^{m_{j,k}}$ can be lifted to a system of generators of $\pi_1(\mathbb{C}^p \setminus \Delta)$ via the map:

$$\pi[p] : \mathbb{C}^p \setminus \Delta \longrightarrow \mathbb{C}^p \setminus \Sigma.$$

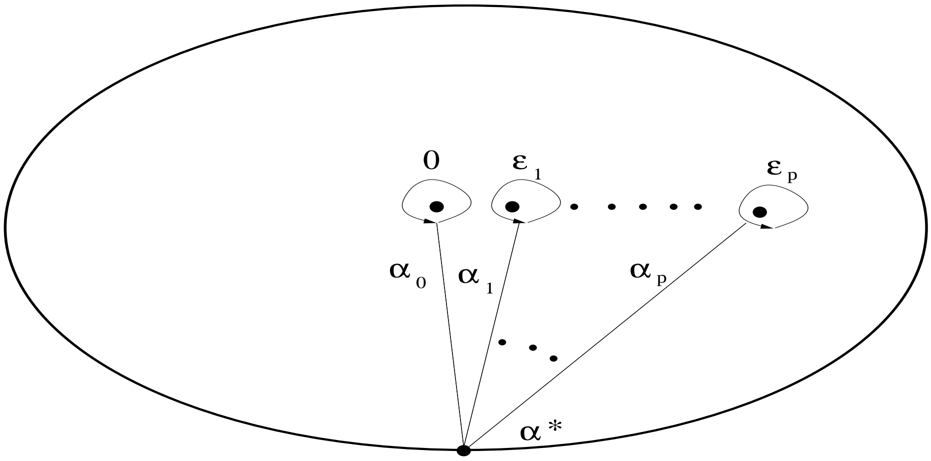

FIGURE 6.

3.6. Proof of proposition 5.6. Choose a closed disk $D \subset \mathbb{C}$ sufficiently big so that it contains all the ε_k's in its interior.
Put $D' = \mathbb{C} \setminus \{\varepsilon_0, \dots, \varepsilon_p\}$. Choose a base point α^* in the boundary of D' and on the lower half-plane. Fix a set of loops $\alpha_0, \dots, \alpha_p$ associated to a distinguished basis of L' like in figure 6.
Consider the map:

$$\begin{array}{cccc} f: & \mathbb{C}_1 & \longrightarrow & \mathbb{C}_2, \\ & x & \mapsto & Q(\lambda^*, x). \end{array}$$

We have used here indices to distinguish the source and the target space of f which are both equal to $\mathbb{C}$.
By definition:
- the critical values of f are $\varepsilon_0, \dots, \varepsilon_p$,
- the restriction of f to $f^{-1}(D')$, $D' \subset \mathbb{C}_2$ is a covering.
Each loop $\alpha_k : [0,1] \longrightarrow D'$ acts on the fibre of f at the base point $\alpha^* \in D'$ by permutation.
We denote formally by $0, 1, \dots, p+3$ the points of the fibre of f at α^*. Let s_k be the permutation associated to the loop a_k. It is readily seen that the numbering can be chosen so that:

$$\begin{cases} s_0 & = & (123) \\ s_k & = & ((2+k)(3+k)) \quad \text{for } k > 0 \end{cases}$$

The $\gamma_{j,k}$'s act on the loops $\alpha_k \subset D'$ by monodromy and consequently they act on the permutations s_k's associated to the α_k's. The action[3] $\tau.$ of the loop $\gamma_{j,k}$ on the permutations is as follows:

[3]The map τ depends on j, k but we want to avoid too many indices.

- for $0 < j < k$,

$$\begin{cases} \tau.s_j &= s_k \\ \tau.s_k &= s_k s_j s_k^{-1} \\ \tau.s_r &= s_r \qquad \text{for } r \neq i,\ r \neq j, \end{cases}$$

- for $j = 0,\ k > 0$,

$$\begin{cases} \tau.s_0 &= s_k s_0 s_k^{-1} \\ \tau.s_k &= s_k s_0 s_k s_0^{-1} s_k^{-1} \\ \tau.s_r &= s_r \qquad \text{for } r \neq 0,\ r \neq j, \end{cases}$$

We denote by:
- $\mathcal{S}_{p+4}$ the group of permutation on $p+4$ elements,
- $\mathcal{S}_{p+4}^{p+1}$ the direct product of $p+1$ copies of the group $\mathcal{S}_{p+4}$.

Definition 5.12. The transformation $\tau : \mathcal{S}_{p+4}^{p+1} \longrightarrow \mathcal{S}_{p+4}^{p+1}$ defined for $0 < j < k$ by

$$\begin{cases} \tau.\sigma_j &= \sigma_k \\ \tau.\sigma_k &= \sigma_k \sigma_j \sigma_k^{-1} \\ \tau.\sigma_r &= \sigma_r \qquad \text{for } r \neq i,\ r \neq j, \end{cases}$$

and for $j = 0,\ k > 0$ by

$$\begin{cases} \tau.\sigma_0 &= \sigma_k \sigma_0 \sigma_k^{-1} \\ \tau.\sigma_k &= \sigma_k \sigma_0 \sigma_k \sigma_0^{-1} \sigma_k^{-1} \\ \tau.\sigma_r &= \sigma_r \qquad \text{for } r \neq 0,\ r \neq j, \end{cases}$$

for $\sigma = (\sigma_1, \ldots, \sigma_p) \in \mathcal{S}_{p+4}^{p+1}$ is called the *transformation associated* to the loop $\gamma_{j,k}$.

We keep the previous notations. The following (classical) lemma is also a consequence of the unicity of analytic structure on the Riemann sphere.

Lemma 5.2. *A loop $\gamma_{j,k}$ has order $m_{j,k}$ if and only if $m_{j,k}$ is the smallest number such that $\tau^{m_{j,k}}.(s_0, \ldots, s_k) = (s_0, \ldots, s_k)$. Here τ is the transformation associated to the loop $\gamma_{j,k}$.*

Thus we can calculate explicitly the $m_{j,k}$'s, we get:

$$m_{j,k} = \begin{cases} 2 & \text{if} \quad j - i > 1, \\ 3 & \text{if} \quad j = i+1,\ i > 0 \\ 4 & \text{if} \quad i = 0,\ j = 1. \end{cases}$$

Let $\tilde{g}_{j,k}$ be a lifting of the loop $\gamma_{j,k}^{m_{j,k}}$. It remains to prove the following lemma.

Lemma 5.3. *The fundamental group of the space $\mathbb{C}^p \setminus \Delta$ is generated by the $\tilde{\gamma}_{j,k}$'s where j,k run over all values such that $0 \leq j < k \leq p$.*

Proof.
Take a line $L \subset \mathbb{C}^p$ Zariski generic with respect to Δ containing λ^*.
The first Zariski's theorem implies that the resulting homomorphism

$$\pi_1(L \setminus (L \cap \Delta)) \longrightarrow \pi_1(\mathbb{C}^p \setminus \Delta)$$

is surjective.
Fix a disk in L containing all the points of $L \cap \Delta$ with base point λ^* on its boundary.
Consider a set of loops $\beta_1, \dots, \beta_s$ associated to a distinguished basis $\phi_1, \dots, \phi_s$ of $L \setminus (L \cap \Delta)$.
The map $\pi[p] : \mathbb{C}^p \setminus \Delta \longrightarrow \mathbb{C}^p \setminus \Sigma$ sends the distinguished basis $\phi_1, \dots, \phi_s$ to a set of paths

$$\phi'_1, \dots, \phi'_s : [0,1] \longrightarrow \mathbb{C}^p$$

such that for $t \neq 1$, $\phi'_k(t) \in \mathbb{C}^p \setminus \Sigma$ for any $k \in \{1, \dots, s\}$.
Denote by β'_j the image of β_j under $\pi[p]$.
Fix $k \in \{1, \dots, s\}$ and consider a path ϕ_k of the distinguished basis.
For $t \neq 1$, we have $\phi(t) \in \mathbb{C}^p \setminus \Delta$. A point $\mathbb{C}^p \setminus \Delta$ can be identified with a set $\{\varepsilon_1, \dots, \varepsilon_p\}$ of unordered points in $\mathbb{C}^*$.
When $t \longrightarrow 1$, either to points say $\varepsilon_j, \varepsilon_k$ coalesce or a point say ε_j approaches 0. Consequently the loop β'_k is conjugated to $\gamma_{j,k}^{m_{j,k}}$.
Now, it is readily seen that a loop conjugated to a $\gamma_{j,k}$ is necessarily of the type $\gamma_{r,s}$. Thus the loop β_k is in the same homotopy class as $\tilde{\gamma}_{r,s}$ where $\tilde{\gamma}_{r,s}$ is the lifting of $\gamma_{r,s}^{m_{r,s}}$. This concludes the proof of the lemma.
Proposition is proved.

3.7. An example. We use the same notations than those of the preceding subsection.
We consider the case $p = 1$. As before, for simplicity, we take:

$$Q(\lambda, x) = 3x^4 - 4\lambda x^3.$$

The $\mathcal{P}A_1^1$ Lyaschko-Loojenga mapping is given by

$$\begin{array}{cccc} \pi[1] & \mathbb{C}_1^* & \longrightarrow & \mathbb{C}_2^*, \\ & \lambda & \mapsto & -\lambda^4. \end{array}$$

Here again we have used small subscripts for the source and the target space in order to distinguish them.
Take the base points $\lambda^* = 1 \in \mathbb{C}_1$, $\varepsilon^* = -1 \in \mathbb{C}_2^*$. In this case $B(\mathbb{C}^*, 1)$ is homeomorphic to $\mathbb{C}^*$. That is there is an unique non-zero critical value ε^*.
Choose a base point α^* in $\mathbb{C}^*$. Take the loops α_0, α_1 as indicated in figure 7. Choose the numbering of the 4 points of the fibre of the function $Q(\lambda^*, .)$ at α^* so that:
- the permutation associated to s_0 is (123),
- the permutation associated to s_1 is (34).
Let a_k be the homotopy class in $\mathbb{C} \setminus \{-1, 1\}$ of the loop α_k. The loop $\gamma_{1,0}$

acts on the homotopy classes a_0, a_1 of the loops α_0 and α_1 as indicated in figure 7 that is:

$$\begin{array}{rcl} \tau.a_0 & = & a_1 a_0 a_1^{-1} \\ \tau.a_1 & = & a_1 a_0 a_1 a_0^{-1} a_1^{-1}. \end{array}$$

Consequently:

$$\begin{cases} \tau(s_0,s_1) & = & ((124),(32)), \\ \tau^2(s_0,s_1) & = & ((134),(12)), \\ \tau^3(s_0,s_1) & = & ((234),(14)), \\ \tau^4(s_0,s_1) & = & ((123),(34)). \end{cases}$$

Hence $\tau^4(s_0,s_1) = (s_0,s_1)$. This implies $\gamma_{1,0} : [0,1] \longrightarrow \mathbb{C}_2^*$ has order 4 with respect to $\lambda^* = 1 \in \mathbb{C}_1^*$. Indeed $\gamma_{1,0}^4 : t \mapsto e^{8i\pi t}$ can be lifted to:

$$\begin{array}{rccc} \tilde{\gamma}_{1,0} : & [0,1] & \longrightarrow & \mathbb{C}_1 \\ & t & \mapsto & e^{2i\pi t} \end{array}$$

and the homotopy class of $\tilde{\gamma}_{1,0}$ generates the fundamental group of $\mathbb{C}_1^*$.

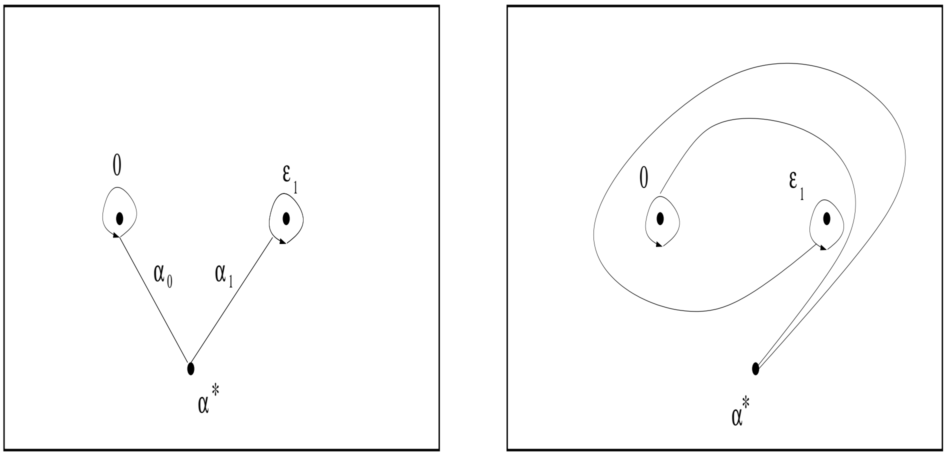

Figure 7.

3.8. Computation of the fundamental group for $\mathcal{P}A_1^p$. We keep the same notations as that of the preceding subsection.
We have the projection:

$$\begin{array}{rccc} \xi : & \mathbb{C}^{p+1} & \longrightarrow & \mathbb{C}^p, \\ & (\alpha_1, \dots, \alpha_p, \varepsilon) & \longrightarrow & (\alpha_1, \dots, \alpha_p). \end{array}$$

Denote by $\widetilde{\Sigma}$ the set of values of the parameter $(\alpha_1, \dots, \alpha_p, \varepsilon)$ for which the polynomial

$$x^{3+p} + \alpha_1 x^{2+p} + \cdots + \alpha_p x^3 - \varepsilon$$

has at least double root.
A point of $\mathbb{C} \setminus \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_p\}$ can be identified with a point of $L \setminus (L \cap \widetilde{\Sigma})$ where L is the complex one-line $\xi^{-1}(\alpha^*)$ via the map:

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}^{p+1}, \\ \varepsilon & \mapsto & (\alpha^*, \varepsilon). \end{array}$$

This map gives an identification of a loops α_k in D' with a loops in $L \setminus (L \cap \widetilde{\Sigma})$. The first Zariski theorem (theorem 5.2) implies that the resulting homomorphism:

$$\pi_1(D') \longrightarrow \pi_1(\mathbb{C}^{p+1} \setminus \Sigma)$$

is surjective. It remains to calculate the relation subgroup.
The group $\mathbb{C}^p \setminus \Delta$ is generated by the $\tilde{\gamma}_{j,k}$'s. The monodromy automorphism associated to $\tilde{\gamma}_{j,k}$ is easily calculated. Let a_r be the homotopy class of α_r. Define $\tau_{j,k}$ by the formula

$$\tau_{j,k} a_r = \begin{cases} a_k & \text{if } r = i \\ a_k a_j a_k^{-1} & \text{if } r = j \\ a_r & \text{if } r \neq i \text{ and } r \neq j \end{cases}$$

Then one finds the action

$$\rho[\tilde{\gamma}_{j,k}] = \tau_{j,k}^{m_{j,k}}$$

provided that $lk \neq 0$ and:

$$\rho[\tilde{\gamma}_{j,k}] = \tau_{j,k}^{2m_{j,k}}$$

if $jk = 0$.
This concludes the computations of the fundamental group for the $\mathcal{P}$-singularity class $\mathcal{P}A_1^p$.

3.9. Description of the fundamental group via the $\mathcal{P}A_1^{p,q}$ Lyaschko-Loojenga mapping. The procedure is analogous to that of $\mathcal{P}A_1^p$. The difference lies essentially in the notations.
We describe the fundamental group of $\mathbb{C}^{p+q} \setminus \Delta$ in terms of the covering:

$$\pi[p,q] : \mathbb{C}^{p+q} \setminus \Delta \longrightarrow \mathbb{C}^{p+q} \setminus \Sigma.$$

We fix respective base points ε^* and λ^* in $\mathbb{C}^{p+q} \setminus \Sigma$ and in $\mathbb{C}^{p+q} \setminus \Delta$ such that λ^* belongs to the fibre of ε^*, that is $\pi[p,q](\lambda^*) = \varepsilon^*$.

We consider a set of loops $\{\gamma_{j,k}\}$, $-q \leq j < k \leq p$ in $\mathbb{C}^p \setminus \Sigma$ with base point ε^* and a sequence of numbers $(m_{j,k})$ such that:
- the loop $\gamma_{j,k}$ has order $m_{j,k}$ i.e. the loop $\gamma_{j,k}^{m_{j,k}}$ can be lifted to a closed loop $\tilde{\gamma}_{j,k}$,
-the $\tilde{\gamma}_{j,k}$'s generate the fundamental group of $\mathbb{C}^{p+q} \setminus \Delta$.

For $-q \leq j < k \leq p$ and $jk \neq 0$, the loop $\gamma_{j,k}$ is any loop obtained by exchanging ε_j and ε_k on the lower half-plane ($Im(\varepsilon) < 0$) like in the case $\mathcal{P}A_1^p$.

The loops $\gamma_{0,j}$ are obtained by turning counterclockwise the critical value ε_j around the origin like in the case $\mathcal{P}A_1^p$.

PROPOSITION 5.7. *Let $m_{j,k}$, $-q \leq j < k \leq p$ be the integers defined by:*

$$m_{j,k} = \begin{cases} 2 & \text{if} \quad \mid k-j \mid > 1, \\ 3 & \text{if} \quad k = j+1,\ ij \neq 0 \\ 4 & \text{if} \quad j = 0,\ k = \pm 1. \end{cases}$$

Then the loops $\gamma_{j,k}^{m_{j,k}}$ can be lifted to a system of generators of $\pi_1(\mathbb{C}^p \setminus \Delta)$ via the map:

$$\pi[p] : \mathbb{C}^p \setminus \Delta \longrightarrow \mathbb{C}^p \setminus \Sigma.$$

The rest of the proof is the same than for $\mathcal{P}A_1^p$.

CHAPTER 6

The modality in Plücker space.

The projective topological invariants of a family of curves (V_ε) are discrete invariants related to the extrinsic projective structure. In this chapter, we study the continuous invariant corresponding to the PT-monodromy group: the asymptotics of the multivalued functions giving the vanishing flattening points of a one-parameter family of curves (V_ε) when $\varepsilon \longrightarrow 0$. The computations of these functions at a node singular point are made in the first section.

In the second section, we apply the results obtained in the first section in order to obtain a lower bound for the $\mathcal{P}$-modality. The answer is given in terms of the geometry of a Newton Polygon. This is how the geometry of integer points contained in a triangle arises in this study.

In this chapter, we shall say that a property holds for a generic holomorphic map $\varphi : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ provided that there exists an integer N and a semi-algebraic variety $\Sigma \subset J^N_O(\mathbb{C}^2, \mathbb{C}^n)$ of codimension at least one such that for $j^N_0\varphi \notin \Sigma$, φ satisfies the given property.

1. Asymptotics of vanishing flattening points.

1.1. Summary and notations. Let $H : U \longrightarrow \mathbb{C}$ be a non-constant holomorphic function and $\varphi : U \longrightarrow \mathbb{C}^n$ holomorphic embedding. Here U denotes a neighbourhood of the origin in $\mathbb{C}^2$ and $\mathbb{C}^n$ denotes the affine n-dimensional space.

Denote by $V_\varepsilon[\varphi]$ the image under φ of the curve $H^{-1}(\varepsilon)$.

We use the notations of chapter 3, subsection 1.3.

We fix a coordinate-system (x, y) in U.

Definition 6.1. The value of the coordinate x at a flattening point of the curve $V_\varepsilon[\varphi]$ is called a *label* of the flattening point.

Remark. This definition is, of course, coordinate dependent. We do not take into account the value of the y-coordinate.

A label of a flattening point of a curve $V_\varepsilon[\varphi]$ depends on the parameter ε as a multi-valued function. These multi-valued function-germs is the object that we are going study.

1.2. Asymptotics of vanishing flattening points (case $n = 3$). For the rest of section 1, we put $H(x, y) = xy$.

We state a refinement of theorem 2.13 in the particular case $n = 3$. We use the old-fashioned language of multi-valued functions. We keep the notations introduced in the preceding subsection.

THEOREM 6.1. *The germs of the labels of the flattening points of $V_\varepsilon[\varphi]$ at the origin are function-germs given by convergent series of the form*

$$\begin{cases} x_1(\varepsilon) &= \alpha_1\varepsilon^{1/4} + o(\varepsilon^{1/4}), \\ x_2(\varepsilon) &= \alpha_2\varepsilon^{1/2} + o(\varepsilon^{1/2}), \\ x_3(\varepsilon) &= i\alpha_2\varepsilon^{1/2} + o(\varepsilon^{1/2}), \\ x_4(\varepsilon) &= \alpha_3\varepsilon^{3/4} + o(\varepsilon^{3/4}) \end{cases}$$

provided that φ is generic.

The function-germs x_1 and x_4 are 4-valued. The function germs x_2 and x_3 are 2-valued. This makes 12 values. That is the curves $V_\varepsilon[\varphi]$ have 12 flattening points that "vanish" at the origin when $\varepsilon \longrightarrow 0$. When ε makes a turn around the origin the 4 values of x_1 and x_4 are permuted as well as the two values of x_2 and x_3.
The language of multi-valued function can be avoided. Put $\varepsilon = t^4$ and consider the curves $V_{t^4}[\varphi]$. Instead of a set of 4 multi-valued function, we get a set of 12 holomorphic functions.
Theorem 6.1 admits a converse. In order to avoid complicated notations, we return to the language of multi-valued functions.

THEOREM 6.2. *Let $g_1, \dots, g_4$ be multi-valued functions given by convergent (or formal) power series*

$$\begin{cases} g_1(\varepsilon) &= \sum_{k>0} a_k\varepsilon^{k/4}, \\ g_2(\varepsilon) &= \sum_{k>0} b_k\varepsilon^{k/2}, \\ g_3(\varepsilon) &= \sum_{k>0} c_k\varepsilon^{k/2}, \\ g_4(\varepsilon) &= \sum_{k>0} d_k\varepsilon^{3k/4}. \end{cases}$$

Assume that $b_1 = \pm ic_1$. Then for any N there exists a holomorphic map $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that the labels $x_1, \dots, x_4$ of the flattening of the curves $V_\varepsilon[\varphi]$ satisfy

$$x_k - g_k = o(\varepsilon^N)$$

provided that the functions $g_1, g_2, \dots, g_4$ are generic[1].

Keeping in mind our general philosophy, once this theorem is proved we study a functional space and forget about the projective nature of our problem.

[1]For any k, the set of coefficients $a_1, \dots, a_N, b_1, \dots, b_N, c_1, \dots, c_N, d_1, \dots, d_N$ such that $b_1 = \pm ic_1$ that do not satisfy the theorem form a semi-algebraic variety of codimension at least one in the space $\mathbb{C}^{4N-1}$ of the values of the a_i, b_i, c_i, d_i's.

1.3. Proof of theorem 6.1 and theorem 6.2. In order to simplify the notations, we denote by the same characters $\mathcal{D}[\varphi]$ the germ at the origin of $\mathcal{D}[\varphi]$ which was previously a function in U.

LEMMA 6.1. *One can represent $\mathcal{D}[\varphi]$ in the form*

$$\mathcal{D}[\varphi](x,y) = a_1x^6 + a_2x^3y + a_3xy^3 + a_4y^6$$

where the a_k's denote some holomorphic function-germs.

Proof.
We start by proving the equality

$$[x^{\alpha_1}y^{\beta_1}, x^{\alpha_2}y^{\beta_2}, x^{\alpha_3}y^{\beta_3}] = cx^{\alpha_1+\alpha_2+\alpha_3}y^{\beta_1+\beta_2+\beta_3}, \tag{32}$$

where the constant c is equal to the Vandermonde determinant of

$$(\alpha_1-\beta_1, \alpha_2-\beta_2, \alpha_3-\beta_3)$$

multiplied by $\prod_{i=1}^3(\alpha_i-\beta_i)$.
The Hamilton vector-field of $H(x,y)=xy$ is $h(x,y)=x\partial_x - y\partial_y$. Thus, the derivative of $x^{\alpha_k}y^{\beta_k}$ along h is given by

$$h.(x^{\alpha_k}y^{\beta_k}) = (\alpha_k-\beta_k)(x^{\alpha_k}y^{\beta_k}.$$

Consequently the k^{th} column of the 3×3 determinant $[x^{\alpha_1}y^{\beta_1}, x^{\alpha_2}y^{\beta_2}, x^{\alpha_3}y^{\beta_3}]$ is given by:

$$x^{\alpha_k}y^{\beta_k}((\alpha_k-\beta_k),(\alpha_k-\beta_k)^2,(\alpha_k-\beta_k)^3).$$

This proves equality 32.
The multilinearity of the determinant implies that $\mathcal{D}[\varphi]$ is the sum of such terms. All these terms belong to the ideal generated by x^6, x^3y, xy^3, y^6. Hence, lemma is proved.

Let I be the ideal generated by x^6, x^3y, xy^3, y^6. Denote by I_k the elements of I of degree not more than k.
For the rest of this chapter, we denote by j^kg denotes the Taylor polynomial of degree k of a holomorphic function germ $g:(\mathbb{C}^r,0)\longrightarrow\mathbb{C}$ at the origin. It is usually denoted by j_0^kg but we drop the subscript 0 to simplify the notations.
The way the coefficients of $\mathcal{D}[\varphi]$ depend on the coefficients of φ is complicated. Nevertheless, one has the following lemma, which is the main lemma of this chapter.

LEMMA 6.2. *For any generic polynomial*[2] *$m \in I_k$ and for any number N there exists a map-germ $\varphi:(\mathbb{C}^2,0)\longrightarrow\mathbb{C}$ (depending in N) such that $j^N(\mathcal{D}[\varphi]) = j^Nm$.*

[2] The subspace of I_k of polynomials that do not satisfy this property is a semi-algebraic variety of codimension at least one in I_k.

Proof.
For any $N \geq 4$, consider the map:

$$\begin{array}{rccc} P: & J^N(\mathbb{C}^2,\mathbb{C}^3) & \longrightarrow & I_N, \\ & j^N\varphi & \mapsto & j^N(\mathcal{D}[\varphi]). \end{array}$$

This map is well-defined since $j^N(\mathcal{D}[\varphi])$ depends only on the jet of order N of φ (in fact even less than N is sufficient).
The map P is a polynomial map. Hence the image of P is a semi-algebraic subvariety V. The lemma asserts that the subvariety V is of codimension 0. We give here a proof which can be readily extended to the case where the dimension of the affine space is greater than 3.
Assume that there exists a polynomial function $F : I_N \longrightarrow \mathbb{C}$ such that:

$$F(a) = 0, \ \forall a \in V.$$

We are going to prove that F is identically 0. This will conclude the proof of the lemma.
Let $e_1, \dots, e_r$ be a basis of monomials of I_N. Denote by $a_1, \dots, a_r : I_N \longrightarrow \mathbb{C}$ the coordinates associated to this basis.
Let $\partial_1, \dots, \partial_r$ be the partial derivatives with respect to $a_1, \dots, a_r$. Fix an arbitrary vector $j = (j_1, \dots, j_s)$ where the j_k's are integers not greater than r. We denote by ∂_j the operator obtained by applying successively the partial derivatives $\partial_{j_1} \dots \partial_{j_s}$.

Assertion: for any integer vector $j = (j_1, \dots, j_s)$, we have $\partial_j F(0) = 0$.

Since F is a polynomial function this assertion implies the lemma. We prove this assertion.
Fix an arbitrary vector $j = (j_1, \dots, j_s)$.
Recall that we have numbered (in an arbitrary way) the monomials of I_N by $e_1, \dots, e_r$.
For any $m \in \{1, \dots, s\}$, define the holomorphic map-germ $f_m : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}^3, 0)$ by the formulas

$$f_m(t_m, x, y) = \begin{cases} \frac{1}{2}t_m^2(0, 0, x^j y^{2+k}) & \text{if} \quad e_{j_m} = x^{1+j}y^{3+k}, \\ \frac{1}{2}t_m^2(0, 0, x^{2+j} y^{k}) & \text{if} \quad e_{j_m} = x^{3+j}y^{1+k}, \\ t_m(0, x^2, x^3) & \text{if} \quad e_{j_m} = x^6, \\ t_m(y^2, 0, y^3)) & \text{if} \quad e_{j_m} = y^6. \end{cases}$$

Let $\varphi_0 : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$ be the map-germ defined by the formula

$$\varphi_0(x, y) = (x, y, 0).$$

Define the s-parameter family of mapping $\varphi_t : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$, with $t = (t_1, \dots, t_s) \in \mathbb{C}^s$ by the equation

$$\varphi_t = \varphi_0 + \sum_{m=1}^{s} f_m(t_m, .).$$

By Leibniz rule, we get the evaluation

$$(\partial_{t_1}^2 \dots \partial_{t_s}^2 (F \circ p))(0) = c(\partial_j F)(0), \tag{33}$$

where c is a non-zero constant.
The constant can be calculated without difficulty, it is equal to$\prod_{l=1}^s c_m$ where:

$$c_m = \begin{cases} [x, y, x^j y^{2+k}] & \text{if} \quad e_{j_m} = x^{1+j} y^{3+k}, \\ [x, y, x^{2+j} y^k] & \text{if} \quad e_{j_m} = x^{3+j} y^{1+k}, \\ [x, x^2, x^3]) & \text{if} \quad e_{j_m} = x^6, \\ [y^2, y, y^3]) & \text{if} \quad e_{j_m} = y^6. \end{cases}$$

We come back to equation 33. The function $F \circ p$ vanishes identically, hence:

$$(\partial_{t_1}^2 \dots \partial_{t_s}^2 (F \circ p))(0) = 0.$$

Thus equation 33 implies the evaluation

$$\partial_j F = 0.$$

Assertion is proved.

We now come to the end of the proof of the theorem. We search the solutions of the system of equations:

$$\begin{cases} xy & = & \varepsilon, \\ a_1 x^6 + a_2 x^3 y + a_3 x y^3 + a_4 y^6 & = & 0, \end{cases}$$

where the $a_k' s$ are arbitrary holomorphic function-germs.
The second equation of this system is just $\mathcal{D}[\varphi] = 0$. Lemma 6.2 implies that for a generic map φ we have $a_1 a_2 a_3 a_4(0) \neq 0$.
Thus, the Newton algorithm for finding the Puiseux series ([**New**]) implies the curve germ of equation $\mathcal{D}[\varphi] = 0$ has four branches. Each branch being given by one of the edges of the Newton diagram of $\mathcal{D}[\varphi]$.
The Puiseux series of the branches are holomorphic functions of the form:

$$\begin{cases} y & = & -\frac{b_1}{b_2} x^3 + o(x^3) \\ y & = & -\sqrt{\frac{-b_3}{b_2}} x + o(x) \\ y & = & \sqrt{\frac{-b_3}{b_2}} x + o(x) \\ x & = & -\frac{b_4}{b_3} y^3 + o(y^3) \end{cases}$$

Substituting these equation in $xy = \varepsilon$ we get the 4 multi-valued functions we were looking for

$$\begin{cases} x_1(\varepsilon) & = & \alpha_1 \varepsilon^{1/4} + o(\varepsilon^{1/4}) \\ x_2(\varepsilon) & = & \alpha_2 \varepsilon^{1/2} + o(\varepsilon^{1/2}) \\ x_3(\varepsilon) & = & -\alpha_2 \varepsilon^{1/2} + o(\varepsilon^{1/2}) \\ x_4(\varepsilon) & = & \alpha_3 \varepsilon^{3/4} + o(\varepsilon^{3/4}) \end{cases} \tag{34}$$

This functions being given by convergent power series.
If we want to avoid the multi-valued function language then we just put

$\varepsilon = t^4$ as we pointed out at the beginning of our discussion.
Conversely let $g_1, \dots, g_4$ be four multi-valued function-germs of the type:

$$\begin{cases} g_1(\varepsilon) &= \sum_{k>0} a_k \varepsilon^{k/4}, \\ g_2(\varepsilon) &= \sum_{k>0} b_k \varepsilon^{k/2}, \\ g_3(\varepsilon) &= \sum_{k>0} c_k \varepsilon^{k/2}, \\ g_4(\varepsilon) &= \sum_{k>0} d_k \varepsilon^{3k/4}, \end{cases} \tag{35}$$

with $b_1 = \pm i c_1$.
The g_k's parameterize a curve-germ of equation say $S(x, y) = 0$. The equality $b_1 = \pm i c_1$ implies that the coefficient of x^2y^2 in $S(x, y)$ vanishes.
Thus lemma 6.2 implies that there exists φ such that $S = \mathcal{D}[\varphi]$ provided that S is generic. Theorem is proved.
From this particular example, we see how the geometry of integer points arises.
Take an arbitrary set of 4 multivalued function-germs $g_1,\ g_2,\ g_3,\ g_4$ like in system 35. These multivalued functions parameterize a curve-germ of equation say $S(x, y) = 0$. A necessary condition for S to be in the image of $\varphi \longrightarrow \mathcal{D}[\varphi]$ is that the coefficient of x^2y^2 in the Taylor series of S vanishes. That is $b_1 = \pm i c_1$. Lemma 6.2 implies that for finite jets, this necessary condition is sufficient except maybe for some exceptional values of the coefficients of S.
Consequently, the number of independent equations that the coefficients of the series g_i's should satisfy is at most equal to the the number of integer points which are contained in the Newton polygon of the polynomial

$$x^6 + x^3y + xy^3 + y^6$$

and which are not contained in D, where D denotes the points (m, n) for which $x^m y^n$ belongs to the ideal I generated by $x^6,\ x^3y,\ xy^3,\ y^6$. This number is the same than the number of integer points contained either in the interior or in the hypotenuse of one of the triangles $T_1,\ T_2,\ T_3$ depicted in figure 1 on page 125.

1.4. Statement of the theorem (general case). Denote by T_k the triangle of vertices $(0, 0),\ (k, 0),\ (0, n-k+1)$. Let h_k be the number of integer points lying on the hypotenuse of T_k distinct from the vertices. Let a_k be the number of integer points contained in T_k or on its boundary distinct from the vertices.

Convention. In this subsection a multi-valued function of the type $\varepsilon^{2/4}$ is a 2-valued function equal to $\varepsilon^{1/2}$.
We write $A(\varepsilon^{2/4})$ for the 4 valued function, the letter A means all values.

THEOREM 6.3. *The labels $x_1, \dots, x_n$ of the flattening points of $V_\varepsilon[\varphi]$ at the origin are $(n+1)$-valued function-germs, given by convergent series of the*

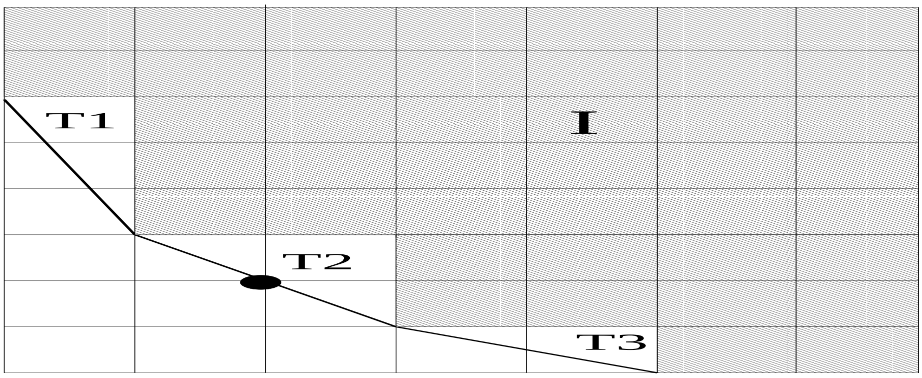

FIGURE 1. *The monomial x^2y^2 is the only monomial belonging to the Newton diagram of $x^6 + x^3y + xy^3 + y^6$ which is not contained in the ideal I. It lies on the hypotenuse of the triangle $T2$.*

type:

$$x_l(\varepsilon) = \alpha_1 A(e^{k_l/(n+1)}) + o(\varepsilon^{k_l/(n+1)}),$$

with $k_l \in \{1, \dots, n\}$ provided that φ is generic.

As in the preceding subsection, this theorem admits a converse.

THEOREM 6.4. *Let $g_{1,1}, \dots, g_{n,h_n+1}$ be a collection of multi-valued functions defined by convergent power series*

$$\begin{cases} g_{1,k}(t) &= \sum_{j>0} b^1_{j,k}\varepsilon^{j/(n+1)} & \text{for } k \in \{1, \dots, h_1+1\}, \\ g_{2,k}(t) &= \sum_{i>0} b^2_{j,k}\varepsilon^{j/(n+1)} & \text{for } k \in \{1, \dots, h_2+1\}, \\ \dots & & \\ g_{n,k}(t) &= \sum_{i>0} b^n_{j,k}\varepsilon^{nj/(n+1)} & \text{for } k \in \{1, \dots, h_n+1\}. \end{cases}$$

Then there exists a set of $h_j + a_j$ polynomials $P_1, \dots, P_s$ in the coefficients $b = (b^l_{j,k})$ of the $g_{j,k}$'s having the following property.
If $P_1(b) = \dots = P_s(b) = 0$, then, for any N there exists a biholomorphic map-germ

$$\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$$

such that the labels $x_{1,1}, \dots, x_{n,h_n+1}$ of the flattening of the curves $V_\varepsilon[\varphi]$ for $\varepsilon = t^{n+1}$ satisfy:

$$x_{j,k} - g_{j,k} = o(t^N),$$

for any $0 < j \leq n$ and $k \in \{1, \dots, h_j+1\}$, provided that the holomorphic functions $g_{1,1}, \dots, g_{n,h_n}$ are generic[3].

Example. For $n = 3$, this theorem is the same as theorem 6.2.

[3]The set of values of b satisfying $P_1(b) = \dots = P_s(b) = 0$ for which the theorem is untrue form a semi-algebraic variety of codimension at least one in the variety of the values of the b satisfying the relations $P_1(b) = \dots = P_s(b) = 0$.

1.5. Proof of theorem 6.3 and theorem 6.4. We use the notation $[k] = 1+2+\cdots+k$, where k is a positive integer.
Let I be the ideal generated by the $x^{[j]}y^{[k]}$'s with $j+k=n$, $j \neq k$. The proofs of the following lemmas are analogous to that for $n=3$.

LEMMA 6.3. *We have the equality*

$$(\mathcal{D}[\varphi])(x,y) = \sum_{i+j=n} a_{i,j} x^{[i]} y^{[j]}$$

where the $a_{i,j}$'s denote some holomorphic function germs.

Denote by I_k the space elements of I of degree at most k.

LEMMA 6.4. *For any generic polynomial*[4] *$m \in I_k$ and for any number N there exists a map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n,)$ such that $j^N \mathcal{D}[\varphi] = j^N m$.*

One can define similar map-germs $f_l : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ than the ones,we define for $n=3$.
For example, in case $n=4$, we take:

$$f_l(t_l, x, y) = \begin{cases} \frac{1}{3!}(0, 0, t_l x^{2+j}, t_l^2 y^{2+k}) & \text{if} \quad e_{j_l} = x^{3+j}y^{3+k}, \\ \frac{1}{3!}(0, 0, t_l x^{2+j}, t_l x^{3+k}) & \text{if} \quad e_{j_l} = x^{6+j}y^{1+k}, \\ \frac{1}{3!}(0, 0, t_l y^{2+j}, t_l y^{3+k}) & \text{if} \quad e_{j_l} = x^{1+j}y^{6+k}, \\ (0, t_l x^2, t_l x^3, t_l x^4) & \text{if} \quad e_{j_l} = x^{10}, \\ (t_l y^2, 0, t_l y^3, t_l y^4) & \text{if} \quad e_{j_l} = y^{10}. \end{cases}$$

The rest of the proof of the lemma is analogous to the proof of lemma 6.2.
Let $\varphi_0 : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ be the map-germ defined by:

$$\varphi_0(x,y) = (x, y, 0).$$

Define the mappings $\varphi_t : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$, depending on the parameter $t = (t_1, \dots, t_s) \in \mathbb{C}^s$ by:

$$\varphi_t = \varphi_0 + \sum_{l=1}^{s} f_l(t_l, .).$$

By Leibniz rule, we get:

$$(\partial_{t_1}^{n-1} \dots \partial_{t_s}^{n-1}(F \circ p))(0) = c(\partial_i F)(0). \tag{36}$$

Where c is a non-zero constant. The conclusion of the proof is the same as that of preceding subsection.

[4]The subspace of I_k of polynomials that do not satisfy this property is a semi-algebraic variety of codimension at least one.

2. The modality representation.

In this section, the notation H stands for the holomorphic function-germ $H : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by

$$H(x, y) = xy.$$

The knowledge of the asymptotics of the flattening points that vanish at Morse double points will give us a lower bound for the $\mathcal{P}$-modality of a map-germ $(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n, 0)$, where $\mathbb{C}^n$ denotes the n-dimensional affine space.
This is of course due to the fact that in the space of function-germs with a critical point, Morse functions form a dense open subset. Consequently the lower bound that we shall find for the modality of map-germs (H, φ) is also a lower bound for the modality of an arbitrary map $(\tilde{H}, \varphi)$ such that $\tilde{H} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a holomorphic function-germ with a critical point at the origin.

2.1. The $\mathcal{H}$-equivalence group. Let $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a generic map-germ.
Denote by $V_t[\varphi]$ the image of the curve $H^{-1}(t^{n+1})$ under φ.
In the preceding section, we saw that there are $n(n+1)$ labels $x_1, \dots, x_{n(n+1)}$ of the curves $V_t[\varphi]$ which are holomorphic functions of the parameter t.
We now search how the holomorphic function-germ $x_1, \dots, x_{n(n+1)}$ are transformed under $\mathcal{G}$-equivalence.
For example, let $\alpha \in \mathcal{O}_{x,y}$ then:

$$\alpha.(H, \mathcal{D}[\varphi]) = (H + \alpha \times \mathcal{D}[\varphi], \mathcal{D}[\varphi]).$$

Here the $\alpha.$ denotes the image under the action of $\alpha \in \mathcal{G}$ (see chapter 4). Let y_k be the multi-valued function-germ such that $x_k(t)y_k(t) = t^{n+1}$. By assumption for any i, $(x_k(t), y_k(t))$ is a solution of;

$$\begin{cases} H(x, y) &= t^{n+1}, \\ \mathcal{D}[\varphi](x, y) &= 0. \end{cases}$$

The solutions of this system are the same than that of:

$$\begin{cases} H(x, y) + \alpha \times \mathcal{D}[\varphi](x, y) &= t^{n+1}, \\ \mathcal{D}[\varphi](x, y) &= 0. \end{cases}$$

Thus the action of $\alpha \in \mathcal{O}$ on the set $(x_1, \dots, x_{n(n+1)})$ is trivial.
Similarly the action of any element $\beta \in \mathcal{O}^*$

$$\beta.(H, \mathcal{D}[\varphi]) = (H, \beta \times (\mathcal{D}[\varphi]))$$

on the holomorphic function-germs $x_1, \dots, x_{n(n+1)}$ is trivial.

- Action of $Diff(1)$.
Let $\psi \in Diff(1)$, it acts as

$$\psi.(H, \mathcal{D}[\varphi]) = (\psi \circ H, \mathcal{D}[\varphi]).$$

The new system to be solved to find the coordinates x, y of the flattening points of the transformed curve is

$$\begin{cases} H(x,y) & = & \psi(t^{n+1}), \\ (\mathcal{D}[\varphi])(x,y) & = & 0. \end{cases}$$

Let $s : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ be any function-germ such that:

$$(s(t))^{n+1} = \psi(t^{n+1}).$$

Then if $(x_k(t), y_k(t))$ is a solution of the old system then $(x_k(s^{-1}(t)), y_k(s^{-1}(t)))$ is a solution of the new one. The map s is not uniquely determined there are $n+1$ possible choices. Denote by $\pi : \widetilde{Diff}(1) \longrightarrow Diff(1)$ the $n+1$ fold-covering of $Diff(1)$ defined by $\pi(s) = \psi$ where ψ is biholomorphic and:

$$(s(t))^{n+1} = \psi(t^{n+1}).$$

The group $\widetilde{Diff}(1)$ acts on the labels $(x_1, \dots, x_{n(n+1)})$ of the flattening points by

$$s.(x_1, \dots, x_{n(n+1)}) = (x_1, \dots, x_{n(n+1)}) \circ s^{-1}.$$

Put $\psi = \pi(s)$ then the $(x_k(s^{-1}(t)), y_k(s^{-1}(t)))$'s are solutions of the equation

$$(\psi \circ H, \mathcal{D}[\varphi]) = (t^{n+1}, 0).$$

Here y_k is the holomorphic function-germ such that $x_k(t)y_k(t) = t^{n+1}$.

- Action of $Diff(2)$.
Let $g \in Diff(2)$, it act as

$$g.(H, \mathcal{D}[\varphi]) = (H \circ g^{-1}, (\mathcal{D}[\varphi]) \circ g^{-1}).$$

The new system is:

$$\begin{cases} (H \circ g^{-1})(x,y)) & = & t^{n+1}, \\ (\mathcal{D}[\varphi] \circ g^{-1})(x,y) & = & 0. \end{cases}$$

Then if (x_k, y_k) is a solution of the old system then $(g_1(x_k, y_k), g_2(x_k, y_k))$ is a solution of the new one. Here $g = (g_1, g_2)$ and y_k is the holomorphic function germ such that $x_k(t)y_k(t) = t^{n+1}$.

The composition of maps induces a direct product group structure on $\widetilde{Diff}(1) \times Diff(2)$:

$$(s, g)(s', g') = (s \circ s', g \circ g').$$

Here $s, s' \in \widetilde{Diff}(1)$, $g, g' \in Diff(2)$.
We shall give the definition of the group $\mathcal{H}$ below. This definition is motivated by a proposition that we state first.
Let $\varphi, \varphi' : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ be two holomorphic map-germs.
Assume that $(H, \mathcal{D}[\varphi])$ is $\mathcal{G}$-equivalent to $(H, \mathcal{D}[\varphi'])$.

This means that there exists $(\psi, g, \alpha, \beta) \in Diff(1) \times Diff(2) \times \mathcal{O}_{x,y} \times \mathcal{O}^*_{x,y}$ such that:

$$\left\{\begin{array}{lcl}(\psi \circ H \circ g^{-1}) + \alpha \times (\mathcal{D}[\varphi] \circ g^{-1}) & = & H \\ \beta \times (\mathcal{D}[\varphi] \circ g^{-1}) & = & \mathcal{D}[\varphi]\end{array}\right. \tag{37}$$

We assume that the dimension n of $\mathbb{C}^n$ is at least equal to 3.

PROPOSITION 6.1. *If (ψ, g, α, β) satisfy equation 37 then:*
a. there exists $a, b \in \mathbb{C}^*$ such that one of the two following equalities hold

$$(Dg)(0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ or } (Dg)(0) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}.$$

b. With the notations of a., we have $(D\psi)(0) = ab$.
c. Put $g = (g_1, g_2)$. Then, the n-degree Taylor polynomial at the origin of the restriction of the function-germ g_1 to $x = 0$ is equal to zero.

DEFINITION 6.2. The $\mathcal{H}$-equivalence group is the subgroup of elements (g, s) of $Diff(2) \times \widetilde{Diff}(1)$ such that:
1. there exists α, β such that $(\pi(s), g, \alpha, \beta)$ satisfy equation 37,
2. there exist $a, b \in \mathbb{C}^*$ such that $Dg(0) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

Recall that a holomorphic function-germ $f : (\mathbb{C}, 0) \longrightarrow \mathbb{C}$ has *order* t^k if:

$$f(t) = ct^k + o(t^k),$$

with $c \neq 0$.

DEFINITION 6.3. The *flattening-space* $\mathcal{F}_n$ is the set of ordered holomorphic function-germs $X = (x_1, \dots, x_{n(n+1)})$ of $\mathcal{O}_t^{n(n+1)}$ such that:
- there exists a map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}^n$ such that X are the labels of the image $V_{t^{n+1}}[\varphi]$ of the curve $H^{-1}(t^{n+1})$ under φ,
- the holomorphic function germs $x_{1+k(n+1)}, \dots, x_{(k+1)(n+1)}$ have order t^{k+1}.

The following proposition is straightforward.

PROPOSITION 6.2. *Let $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^n, 0)$ be a generic map germ. The $\mathcal{P}$-modality of a map-germ $(H, \varphi) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^n)$ is at least equal to the modality of the corresponding holomorphic map $X(\varphi) \in \mathcal{F}_n$ under the action of the $\mathcal{H}$-equivalence group.*

Denote by T_k the triangle of vertices $(0, 0)$, $(k, 0)$, $(0, n - k + 1)$. Let h_k be the number of integer points lying on the hypotenuse of T_k distinct from the vertices.
Let a_k be the number of integer contained in T_k or in its boundary distinct from the vertices. Put:

$$c_k = max\left\{0, k - a_k - h_k\right\}.$$

The results of the preceding section imply the following proposition.

PROPOSITION 6.3. *The dimension of the space $J^n\mathcal{F}_n$ of n-jets of elements in the flattening space $\mathcal{F}_n$ is at least equal to $\sum_{k=1}^{n} c_k$.*

Consequently to find a lower bound for the modality of a map germ $(H,\varphi) : (\mathbb{C}^2,0) \longrightarrow \mathbb{C}^n$ it suffices to find an upper bound for the dimension of the orbit of an element of $J^n\mathcal{F}_n$ under the action of the $\mathcal{H}$-equivalence.
Theorem 2.12 (chapter 2, section6) giving the lower bound for the $\mathcal{P}$-modality is a consequence of the following proposition that we shall prove in the next subsection. Let $\mathcal{H}_0$ be the subgroup of $\mathcal{H}$ containing all the elements that act trivially on $J^n\mathcal{F}_n$.

PROPOSITION 6.4. *The group $\mathcal{H}/\mathcal{H}_0$ is isomorphic to a quotient of a semi-direct product of the Lie groups $\mathbb{C}^{n-1}$ and $(\mathbb{C}^*)^2$. In particular the dimension of the orbit of the n-jet of a function-germ $X = (x_1,\dots,x_{n(n+1)}) \in \mathcal{F}_n$ is not higher than $n+1$.*

Since the dimension of $J^n\mathcal{F}_n$ is at least equal to $\sum_{k=1}^{n} c_k$, the modality of a map germ $(H,\varphi) : (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^n,0)$ is at least $(\sum_{k=1}^{n} c_k) - n - 1$.

2.2. Proof of proposition 6.1. Part a. and part b. of the proposition are easily obtained. Just remark that for $n > 2$, the four-degree Taylor polynomial of $\mathcal{D}[\varphi]$ vanishes. Hence equating the terms of degree 2 in the first equation of the system 37, we get part a. and b. of the proposition. We prove part c. of the proposition.
Define the holomorphic function-germ $m : (\mathbb{C},0) \longrightarrow \mathbb{C}$ by

$$g_1(0,y) = y^k m(y),$$

and $m(0) \neq 0$.
We have to prove that $k > n$.
Restricting the first equation of the system 37 to $x = 0$, we get the equation on the unknown y

$$\psi(y^{k+1}m(y)) + (\alpha \times \mathcal{D}[\varphi] \circ g)(0,y) = 0. \tag{38}$$

The term of least degree in the Taylor series of $\mathcal{D}[\varphi]$ is of degree $2n-2$. Thus $\mathcal{D}[\varphi] \circ g$ is a holomorphic function-germ of order at least $2n-2$:

$$(\mathcal{D}[\varphi] \circ g)(0,y) = O(y^{2n-2}).$$

Coming back to equation 38 and composing by ψ^{-1}, we get an inequality for k given by

$$y^{k+1} = O(y^{2n-2}).$$

That is:

$$k \geq 2n-3 \tag{39}$$

For $n > 3$, we have $2n-3 > n$, this proves the lemma for $n > 3$.
For $n = 3$, inequality 39 implies that there exists $l \in \mathcal{O}_{x,y}$ satisfying the equality

$$g_1(0,y) = y^3 k(y) \tag{40}$$

Lemma 6.1 of the preceding section asserts that the holomorphic function-germ $\mathcal{D}[\varphi]$ belongs to the ring generated by $x^6,\ x^3y,\ xy^3,\ y^6$ oven $\mathcal{O}_{x,y}$. Using equation 40, we get the estimative

$$(\mathcal{D}[\varphi] \circ g)(0, y) = O(y^6),$$

Coming-back to equation 38 we get:

$$y^{k+1} = O(y^6).$$

Thus for $n = 3$, $k \geq 5$, this concludes the proof of the proposition.

2.3. A more precise version of proposition 6.4. To prove proposition 6.4, we make two remarks and set some notations.
Recall that the $\mathcal{H}$-equivalence group is a subgroup of $Diff(2) \times \widetilde{Diff}(1)$. We denote by Id_n the identity map of $\mathbb{C}^n$.

Consider a holomorphic map-germ $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ such that $(g, Id_1) \in \mathcal{H}$ acts on a map $x = (x_1, \dots, x_{n(n+1)}) : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^{n(n+1)}, 0)$ by:

$$g.x_k = g_1(x_k, y_k),$$

where $g = (g_1, g_2)$ and the y_k's are the holomorphic function-germs such that $x_k(t)y_k(t) = t^{n+1}$.
We have used the old fashioned notation $g_1(x_k, y_k)$ instead of $g_1 \circ (x_k, y_k)$.

Denote by $\mathcal{M}_x^k$ the k^{th} power of the maximal ideal of the ring $\mathcal{O}_x$ of holomorphic function in one variable. *Assertion*: *for any holomorphic function-germ* $m : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ *belonging to* $\mathcal{M}_x^2$*, there exists a holomorphic map-germ* $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$*, such that* $(g, Id_1) \in \mathcal{H}$ *and* $g_1(x, y) = x + m(x)$ *where* $g = (g_1, g_2)$.

The map g of the assertion is denoted by $v[m]$.
By definition, the action of $(v[m], Id_1) \in \mathcal{H}$ on a label x_k is given by:

$$(v[m], Id_1).x_k = x_k + m(x_k).$$

We have used here an old-fashioned notation. In modern notations we should write $m \circ x_k$ instead of $m(x_k)$.

We prove the assertion. Since m belongs to $\mathcal{M}_x$ there exists $k \in \mathcal{M}_x$ such that:

$$m(x) = xk(x).$$

Define the map-germ $g = (g_1, g_2)$ by the formula

$$g(x, y) = (x(1 + k(x)), y(1 + k(x))^{-1}).$$

It satisfies the condition $(g, Id_1) \in \mathcal{H}$ and $g_1(x, y) = x + m(x)$.

The $\mathcal{H}$-equivalence group contains the following subgroup L (L for linear) isomorphic to $(\mathbb{C}^*)^2$.

To $(a^{-1}, b^{-1}) \in (\mathbb{C}^*)^2$, we associate the element $(g, s) \in \mathcal{H}$ defined by the linear map

$$g(x, y) = (ax, \frac{b^{n+1}}{a} y), \; s(t) = bt.$$

The equality:

$$\psi \circ H \circ g^{-1} = H,$$

implies that $(g, s) \in \mathcal{H}$. Here $\psi = \pi(s)$ that is $\psi(t^{n+1}) = (s(t))^{n+1}$.

Consider the vector space $V \approx \mathbb{C}^{n-1}$ of biholomorphic map-germs of the type $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ defined by

$$V = \left\{ \sum_{i=2}^{n} \tau_i v[x^i] : (\tau_2, \dots, \tau_n) \in \mathbb{C}^{n-1} \right\}.$$

Denote by $V \oplus L$ the direct sum of V and L as subgroups of $\mathcal{H}$. The subgroup $V \oplus L$ is isomorphic to a semi-direct product of $V \times (\mathbb{C}^*)^2$.

Recall that $\mathcal{H}_0 \subset \mathcal{H}$ denotes the subgroup of elements of $\mathcal{H}$ acting trivially on $J^n\mathcal{F}_n \subset J^n(\mathbb{C}, \mathbb{C}^{n(n+1)})$. Proposition 6.4 is a consequence of the following proposition.

Proposition 6.5. *The restriction of the canonical projection*

$$\mathcal{H} \longrightarrow \mathcal{H}/\mathcal{H}_0$$

to the subgroup $(V \oplus L) \subset \mathcal{H}$ is surjective.

2.4. Proof of proposition 6.5. Let (g, s) be an element of the group $\mathcal{H}$ such that $(Dg)(0) = Id_2$.
The definition of $\mathcal{H}$ and the division theorem imply that there exists holomorphic function-germs p, q, r and a polynomial m of degree n such that:

$$g_1(x, y) = m(x) + x^{n+1}p(x) + y^{n+1}q(y) + r(x, y)xy,$$

where $g = (g_1, g_2)$.
We keep the same notations.

Lemma 6.5. *For any $X = (x_1, \dots, x_{n(n+1)}) \in \mathcal{F}_n$ we have*

$$(g, s).(j^n X) = (v[m], Id_1).(j^n X)$$

provided that $(Dg)(0) = Id_2$.

Proof.
Fix $k \in \{1, \dots, n(n+1)\}$. The action of (g, s) on the label x_k is defined by

$$(g, s).x_k = (x_k + m(x_k) + x_k^{n+1}p(x_k) + y_k^{n+1}q(y_k) + r(x_k, y_k)x_k y_k) \circ s^{-1},$$

where the y_k's are defined by $x_k(t)y_k(t) = t^{n+1}$. We have:

$$\begin{cases} x_k^{n+1} &= o(t^n), \\ y_k^{n+1} &= o(t^n), \\ x_k y_k &= o(t^n). \end{cases}$$

Thus, the following equality holds

$$(g, s).x_k = (v[m], s).x_k + o(t^n).$$

We assert that $s(t) = t + o(t^n)$.
By definition of $Diff(1)$, there exists $\psi \in Diff(1)$ such that the following equality between function-germs holds

$$(s(t))^{n+1} = \psi(t^{n+1}).$$

Thus s is of the type $s(t) = at + o(t^n)$ with $a \in \mathbb{C}^*$.
By definition of $\mathcal{H}$, $Det((Dg)(0)) = a$ consequently $a = 1$.
Thus $(g, s).x_k \circ s = (v[m], Id_1).x_k + o(t^n)$. This proves the lemma.

Let $(\tilde{g}, \tilde{s})$ be an arbitrary element of $\mathcal{H}$. Here $\tilde{g} : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ and $\tilde{s} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ are biholomorphic.
Define $(g, s) \in \mathcal{H}$ by the equality

$$(\tilde{g}, \tilde{s}) = (D(\tilde{g})(0) \circ g, D(\tilde{s})(0) \circ s).$$

with $Dg(0) = Id_2$, $Ds(0) = 1$.
Let $g = (g_1, g_2)$. The definition of $\mathcal{H}$ and the division theorem imply that there exist holomorphic function-germs p, q, r and a polynomial m of degree n such that:

$$g_1(x, y)) = m(x) + x^{n+1}p(x) + y^{n+1}q(y) + r(x, y)xy.$$

Applying lemma 6.5 we get that $(\tilde{g}, \tilde{s}).x_k$ is of the form

$$(\tilde{g}, \tilde{s}).x_k = (D\tilde{g}(0)v[m], D\tilde{s}(0)).x_k + o(t^n).$$

By definition of L and V, we have

$$(D\tilde{g}(0), D\tilde{s}(0)) \in L$$

and

$$(v[m], Id_1) \in V.$$

This concludes the proof of proposition 6.5.

Part II

TOWARDS A LEGENDRIAN THEORY

CHAPTER 7

Legendrian versal deformation theory and its applications.

So far, we have avoided contact and symplectic geometries. In this chapter, we will try to gather the classical projective-geometric approach with contact geometry of Legendre manifolds. We shall do it essentially for families of smooth curves.
There are only a few cases for which both approach coincides. We shall call them the *excellent cases*. We would like first to come back once again to projective geometry. One of the reasons for doing this is to throw the light on the advantages and the difficulties related to this new approach.
In this chapter, unless we mention explicitly the contrary, all the objects that we consider (functions, maps, manifolds) are assumed to be C^∞. In case we consider a function, a map or manifold depending on some parameters, we assume implicitly that the dependence on the parameter is C^∞.

1. The Kazarian folded umbrella.

1.1. Stable Plücker discriminants. For the definitions of the Young diagram of a map germ and of the Plücker discriminant the reader should refer to chapter 2 section 1 and section 3. The definitions were given in the complex holomorphic case. In order to transform the complex holomorphic definitions to the real C^∞ case just replace $\mathbb{C}$ by $\mathbb{R}$ and the words complex holomorphic by real C^∞.

Definition 7.1. The $\mathcal{P}$-discriminant Σ of a family of curves (V_λ) is called $\mathcal{P}$-*stable at a point* λ_0 if the following property holds. Let (V'_λ) be an arbitrary sufficiently small C^∞ perturbation (V'_λ) of the family (V_λ). Denote by Σ' the $\mathcal{P}$-discriminant of (V'_λ). Then there exists a diffeomorphism-germ φ : $(\mathbb{R}^k, \lambda_0) \longrightarrow (\mathbb{R}^k, \varphi(\lambda_0))$ close to identity sending the germ of Σ at λ_0 to the germ of Σ' at some point $\varphi(\lambda_0)$ close to λ_0.

1.2. A preliminary example. The example of this subsection and of the next one are special cases of a theorem of M.E. Kazarian giving the list of the generic $\mathcal{P}$-discriminant for families of smooth curves ([**Ka2**]).

Consider the two-parameter family of curves in $\mathbb{R}^3$ defined by the parameterizations

$$\begin{array}{cccc} f_\lambda : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & t & \mapsto & (t, \frac{1}{6}t^3, \frac{1}{20}t^5 + \frac{1}{12}\lambda_1 t^4 + \frac{1}{2}\lambda_2 t^2). \end{array}$$

Denote by V_λ the curve parameterized by f_λ. The $\mathcal{P}$-discriminant of the family of curves (V_λ) is depicted in figure 1. We shall explain how to compute it but we give first the results of these computations.
In this case the $\mathcal{P}$-discriminant has three strata.
Two of them are of dimension 1, the other component is the point $(0,0)$ of dimension 0.
One of the components of dimension one, denoted $\Sigma(1,1)$ consists of curves having a degenerate flattening of anomaly sequence[1] $(1,1)$. The other component of dimension 1, denoted $\Sigma(2)$ consists of curves having a degenerate flattening of anomaly sequence (2). Both strata consist of two connected components.
The origin $\lambda = 0$ of the space of parameters $\mathbb{R}^2$ corresponds to the curve V_0. This curve V_0 has a degenerate flattening point of anomaly sequence $(2,1)$ at the origin.
The computation of the equation of the $\mathcal{P}$-discriminant is as follows. At a flattening point the first three derivatives of f_λ with respect to t are linearly dependent.
The determinant W_λ whose columns are the first three derivatives of f_λ is called *the Wronskian* . The Wronskian vanishes at a flattening point.
Consequently, the $\mathcal{P}$-discriminant of the family (V_λ) is given by:

$$\Sigma = \left\{ \lambda \in \mathbb{R}^2 : \exists t \in \mathbb{R},\ W_\lambda(t) = \partial_t W_\lambda(t) = 0 \right\}.$$

By explicit calculations we get the equality

$$W_\lambda(t) = 2t^3 + \lambda_1 t^2 - \lambda_2.$$

Consequently:

$$\Sigma(1,1) = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_2 = 0,\ \lambda_1 \neq 0 \right\},$$

and:

$$\Sigma(2) = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_2 = \frac{1}{27}\lambda_1^3,\ \lambda_1 \neq 0 \right\}.$$

M.E. Kazarian proved the stability of this $\mathcal{P}$-discriminant at the origin [**Ka2**]. We shall generalize this result to contact geometry.

1.3. Kazarian folded umbrella and space curves. Consider the three-parameter family of space curves defined by the parameterizations

$$\begin{array}{cccc} f_\lambda : & \mathbb{R} & \longrightarrow & \mathbb{R}^3 \\ & t & \mapsto & (t, t^3, t^6 + \lambda_1 t^5 + \lambda_2 t^4 + \lambda_3 t^2). \end{array}$$

[1] The notation refers to the anomaly sequence and not to the Thom-Mather stratification.

Denote by V_λ the curve parameterized by f_λ .
Direct computations, analogous to the ones of the preceding subsection but longer give the following result.
The $\mathcal{P}$-discriminant has two components of dimension 2. The closure of the component $\Sigma(1,1)$ consisting of the values of the parameter λ for which the curve V_λ has a degenerate flattening of anomaly sequence $(1,1)$ is a plane.
The closure of the component $\Sigma(2)$ consists of of the values of the parameter λ for which the curve V_λ having a degenerate flattening of anomaly sequence 2 is called the *folded umbrella* .
A transversal slice of this $\mathcal{P}$-discriminant by a plane $\lambda_1 =$ constant gives two curves intersecting with multiplicity equal to three at the origin, provided that the constant is distinct from zero.
This $\mathcal{P}$-discriminant is $\mathcal{P}$-stable. This was also proved by M.E. Kazarian. We shall also extend this result to contact geometry.

DEFINITION 7.2. The $\mathcal{P}$-discriminant of example 3 is called *Kazarian's folded umbrella.*

The Legendrian theory that we shall develop will show that the Kazarian folded umbrella is a somehow universal object that may appear in several seemingly unrelated problems. In the next subsection, we shall show another appearance of it.

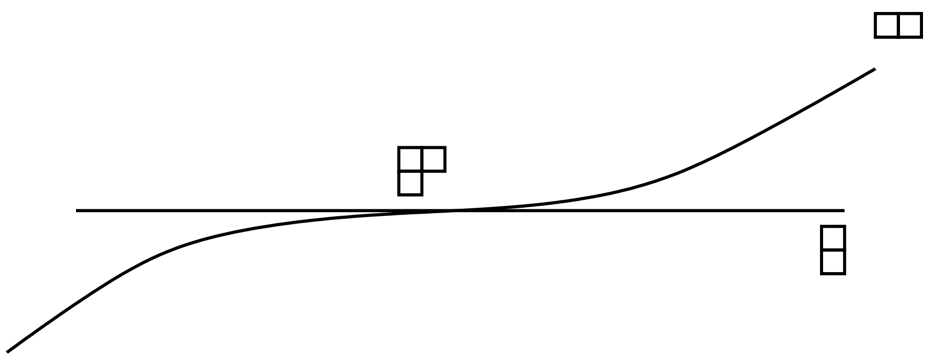

FIGURE 1. *One of Kazarian's "generic" bifurcation diagrams. The curves intersect cubically at the origin (like the plane curves of equation* $y = x^3$ *and* $y = 0$ *at the origin). One of the components corresponds to the curves having a degenerate flattening point of anomaly sequence* 2 *while for the other component the anomaly sequence is* $(1,1)$.

1.4. A Kazarian folded umbrella theorem for surfaces. Consider a surface M embedded in projective space $\mathbb{R}P^3$ or in affine space $\mathbb{R}^3$.

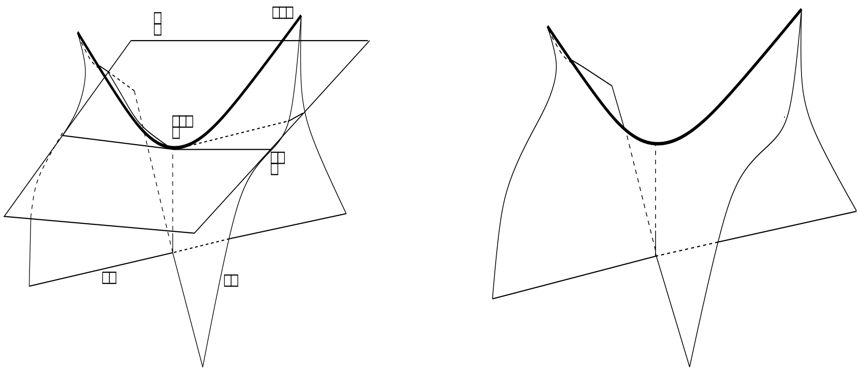

FIGURE 2. *Kazarian's folded umbrella. The cuspidal edge consists of the values of the parameter λ for which the curve V_λ has a degenerate flattening point of anomaly sequence* (3).

DEFINITION 7.3. A line l of the tangent plane to M at a point p is called an *asymptotic direction* of M at p provided that the algebraic multiplicity of intersection of the line with M is at least three.

Remark. Usual textbooks (but not [**Arn6**]) define the asymptotic direction using an Euclidean structure but this is unnecessary.
Example. The two lines $\{(0,t,0)\in\mathbb{R}^3 : t\in\mathbb{R}\}$, and $\{(t,0,0)\in\mathbb{R}^3 : t\in\mathbb{R}\}$, are both asymptotic directions at the origin of the surface defined by

$$M=\left\{(x,y,z)\in\mathbb{R}^3 : z=xy+x^3+y^3\right\}.$$

On a surface $M\subset\mathbb{R}P^3$, one can distinguish (at least) three type of points.
The *elliptic points* are the points of M for which there does not exist an asymptotic direction (for example all the points of a sphere are elliptic).
The *hyperbolic points* are the points of M for which there exists at least two distinct hyperbolic directions (for example on a one-sheeted hyperboloid all the points are hyperbolic).
The *parabolic points* are the points of M for which there is only one asymptotic direction (for example on the affine surface $\{(x,y,z)\in\mathbb{R}^3 : z=x^3+y^2\}$, the set of parabolic points is the line $\{(t,0,0) : t\in\mathbb{R}\}$).

Fix affine coordinates $(x,y,z)\in\mathbb{R}^3$. Assume that a surface $M\subset\mathbb{R}^3$ is the graph of a function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$:

$$M=\left\{(x,y,z)\in\mathbb{R}^3 : z=f(x,y)\right\}.$$

Denote by $Q(p)$ the quadratic form which is the second derivative of f at $p=(x,y,z)\in M$.
Recall that a vector $v\in\mathbb{R}^2$ is an *isotropy vector* of a quadratic form $q:\mathbb{R}^2\longrightarrow\mathbb{R}$ if $q(v)=0$.
We have the following immediate result:

- the point p is elliptic if and only if there are no isotropy vectors of $Q(p)$,
- the point p is hyperbolic there are at least two distinct isotropy vectors of $Q(p)$,
- the point p is parabolic if and only there is only one isotropy vector of $Q(p)$.

Assume that the set of parabolic points of M is a curve.

DEFINITION 7.4. A point of a surface $M \subset \mathbb{R}P^3$ is called a *special parabolic point* if the asymptotic direction of M at this point is tangent to the parabolic curve of M.

Example. The origin is a special parabolic point of the surface parameterized by the mapping

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (s,t) & \mapsto & (t, s+t^2, s^2+t^4). \end{array}$$

We recall some basic facts.
The *dual projective space* to $\mathbb{R}P^3$, denoted by $(\mathbb{R}P^3)^\vee$, is the space of hyperplanes in $\mathbb{R}P^3$.
The space $(\mathbb{R}P^3)^\vee$ can be identified with $\mathbb{R}P^3$. To see it choose a Euclidean scalar product in $\mathbb{R}^4$.
A point in $\mathbb{R}P^3$ is a line in $\mathbb{R}^4$ passing through the origin. A point in $(\mathbb{R}P^3)^\vee$ is a hyperplane in $\mathbb{R}^4$ passing through the origin. To such a hyperplane in $\mathbb{R}^4$, we associate the orthogonal line passing through the origin. This gives an identification of $(\mathbb{R}P^3)^\vee$ with $\mathbb{R}P^3$.
In homogeneous coordinates, the hyperplane $H \in (\mathbb{R}P^3)^\vee$, defined by

$$H = \left\{(x:y:z:t) \in \mathbb{R}P^3 : ax+by+cz+dt = 0\right\},$$

is identified with the point $(a:b:c:d) \in \mathbb{R}P^3$.
The *dual surface* $M^\vee \subset (\mathbb{R}P^3)^\vee$ to a surface $M \subset \mathbb{R}P^3$ is the surface of the tangent planes to M:

$$(H \in M^\vee) \iff (\exists p \in \mathbb{R}P^3,\ H = T_pM).$$

Here T_pM is the plane in $\mathbb{R}P^3$ tangent to M at $p \in \mathbb{R}P^3$.

DEFINITION 7.5. The germ at the origin of the variety:

$$\left\{(q_1,q_2,q_3) \in \mathbb{R}^3 : x^4 + q_1x^2 + q_2x + q_3 \text{ has a real double root}\right\}$$

is called *the swallowtail* .

We depicted this variety in figure 3. To define the degenerate and non-degenerate parabolic points of a surface in $\mathbb{R}P^3$, we fix some conventions.
Consider two surfaces $M, N \subset \mathbb{R}P^3$ (possibly singular) and two points $m \in M$, $n \in N$.
We say that the germ of M at m is *ambientally equivalent* to the germ of N at n provided that the following property holds.
There exists a diffeomorphism map-germ $\varphi : (\mathbb{R}P^3, m) \longrightarrow (\mathbb{R}P^3, n)$ such

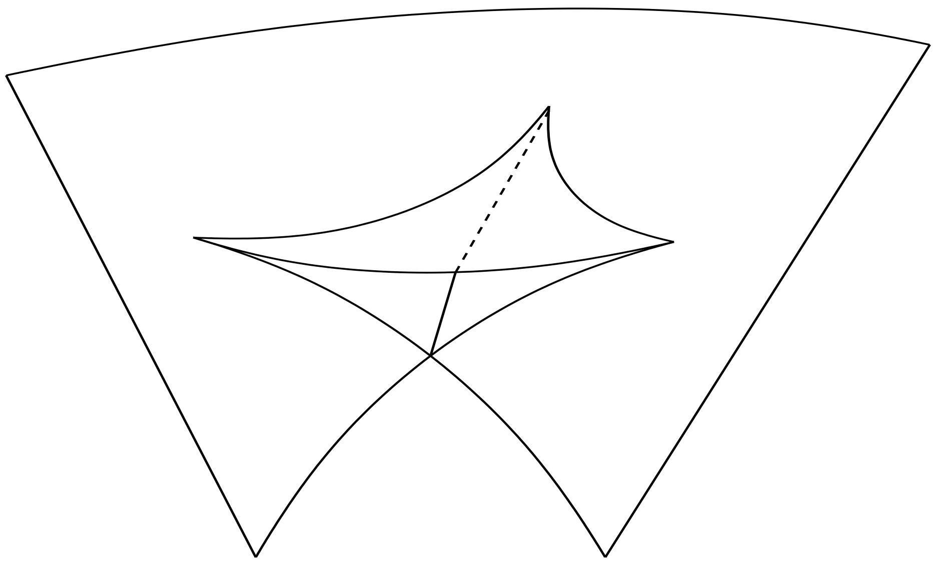

FIGURE 3. *The swallowtail.*

that the restriction $\psi : (M, m) \longrightarrow (N, n)$ of φ to (M, m) is a homeomorphism map-germ.

PROPOSITION 7.1. *If the germ of the surface $M^{\vee}$ dual to a surface $M \subset \mathbb{R}P^3$ at $H = T_pM$ is ambientally equivalent to a swallowtail then p is a special parabolic point of M.*

Remark. We do not prove this (elementary) proposition but we shall come back to it in subsection 4.4 (so the reader will understand why it is an elementary statement).

DEFINITION 7.6. A special parabolic point p of a surface $M \subset \mathbb{R}P^3$ is called a *degenerate special parabolic point* if the germ at T_pM of the dual surface $M^{\vee}$ to M is not diffeomorphic to the swallowtail.

DEFINITION 7.7. The $\mathcal{P}$-*discriminant* of a family (M_λ) of smooth C^∞ surfaces $M_\lambda \subset \mathbb{R}^3$ is the set of values of the parameter λ for which the surface M_λ has a degenerate special parabolic point.

The notion of $\mathcal{P}$-stability extends directly to the case of surfaces in real projective space $\mathbb{R}P^3$.

THEOREM 7.1. *The $\mathcal{P}$-discriminant of the 3-parameter family of surfaces M_λ in $\mathbb{R}^3$ parameterized by the embeddings*

$$\begin{array}{rccc} f_\lambda : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3, \\ & (s,t) & \mapsto & (t, s+t^3, s^2+t^7+t^6+\lambda_1 t^5+\lambda_2 t^4+\lambda_3 t^2) \end{array}$$

is $\mathcal{P}$-stable at the origin. The $\mathcal{P}$-discriminant of the family (M_λ) is the Kazarian folded umbrella.

This theorem is a particular case of a more general theorem (theorem 7.4, page 159).
This $\mathcal{P}$-discriminant is depicted in figure 2. *Remark.* We have restricted ourselves to the most simple case which is not covered neither by Arnold theory of wave fronts nor by Kazarian's theory of flattening points of space curve.

2. Contact geometry.

2.1. Basic definitions. In this subsection, we follow Arnold and Maslov [**Arn3**], [**Mas**].
Consider the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ with standard coordinates $(q, p, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

DEFINITION 7.8. The differential one-form $\omega = du - \sum_{i=1}^n p_i dq_i$ is called *the (standard) contact form* of $J^1(\mathbb{R}^n, \mathbb{R})$. The hyperplane $Ker(\omega(m))$ (contained in the tangent plane to $J^1(\mathbb{R}^n, \mathbb{R})$ at m) is called the *contact hyperplane* at $m \in J^1(\mathbb{R}^n, \mathbb{R})$.

Remark. One can alternatively consider the 1-jet space $J^1(\mathbb{R}^n, \mathbb{R})$ to be the manifold $\mathbb{R}^{2n+1}$ endowed with a product structure so that $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and with the contact form defined above.

DEFINITION 7.9. A diffeomorphism $\varphi : J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R})$ is called a *contactomorphism* provided that $\varphi^*\omega = h\omega$, where $h : J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbb{R}$ is a function such that $h(q, p, u) \neq 0$, $\forall (q, p, u) \in J^1(\mathbb{R}^n, \mathbb{R})$.

Remark. In other words at each point $m \in J^1(\mathbb{R}^n, \mathbb{R})$, the derivative $(D\varphi)(m)$ of a contactomorphism φ sends the contact hyperplane at m to the contact hyperplane at $\varphi(m)$.
Example 1. Take $n = 1$. The map $\varphi : J^1(\mathbb{R}, \mathbb{R}) \longrightarrow J^1(\mathbb{R}, \mathbb{R})$ defined by $\varphi(q, p, u) = (p, q, pq - u)$ is a contactomorphism called Legendre duality map.
Example 2. Take $n = 1$. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. The map $\varphi : J^1(\mathbb{R}, \mathbb{R}) \longrightarrow J^1(\mathbb{R}, \mathbb{R})$ defined by $\varphi(q, p, u) = (q, p + f'(q), u + f(q))$ is a contactomorphism.

DEFINITION 7.10. An integral n-dimensional submanifold $L \subset J^1(\mathbb{R}^n, \mathbb{R})$ of the field of contact hyperplanes is called a *Legendre manifold* .

Remark. An n-dimensional manifold $L \subset J^1(\mathbb{R}^n, \mathbb{R})$ is a Legendre manifold if and only if $\omega_{|L} = 0$, where $\omega_{|L}$ denotes the restriction of the contact form ω to L.
Example 1. A basic example of a Legendre manifold is the 1-graph of a function. Consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then the manifold:

$$L = \left\{ (q, p, f(q)) \in J^1(\mathbb{R}^n, \mathbb{R}) : p_1 = \partial_{q_1} f, \ldots, p_n = \partial_{q_n} f \right\}$$

is a Legendre manifold called the 1-*graph* of f.
Example 2. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Denote by L, the Legendre manifold which is the 1-graph of f. Then, the manifold:

$$L^{\vee} = \left\{ (q, p, u) \in J^1(\mathbb{R}, \mathbb{R}) : q = f'(p),\ u = pf'(p) - f(p) \right\}$$

is the image of L under the Legendre duality map (see example above). Hence $L^{\vee}$ is a Legendre manifold.

Example 3. Another basic example of a Legendre manifold is the fibre at a point (q_0, z_0) of the projection defined by

$$\begin{array}{ccc} J^1(\mathbb{R}^n, \mathbb{R}) & \longrightarrow & J^0(\mathbb{R}^n, \mathbb{R}) \\ (q, p, z) & \mapsto & (q, z). \end{array}$$

That is the manifold

$$L = \left\{ (q_0, p, z_0) \in J^1(\mathbb{R}^n, \mathbb{R}) : \; p \in \mathbb{R}^n \right\}$$

is a Legendre manifold.

DEFINITION 7.11. The map defined by

$$\begin{array}{ccc} J^1(\mathbb{R}^n, \mathbb{R}) & \longrightarrow & J^0(\mathbb{R}^n, \mathbb{R}) \\ (q, p, z) & \mapsto & (q, z), \end{array}$$

is called *the standard projection* from $J^1(\mathbb{R}^n, \mathbb{R})$ to $J^0(\mathbb{R}^n, \mathbb{R})$.

DEFINITION 7.12. The image of a Legendre manifold of the space $J^1(\mathbb{R}^n, \mathbb{R})$ under the standard projection $J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R})$ is called the *wave front* (or simply the front) of the initial Legendre manifold.

Example 1. If the Legendre manifold is the 1-graph of a function (see example above) then the associated front is the graph of the function.
Example 2. If the Legendre manifold is the fibre of the projection $J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R})$, then the front is a point.

2.2. Legendre duality. In this subsection, we fix an Euclidean structure on $\mathbb{R}^n$ and denote by $(,)$ the scalar product.

DEFINITION 7.13. The map :

$$\begin{array}{cccc} l: & J^1(\mathbb{R}^n, \mathbb{R}) & \longrightarrow & J^1(\mathbb{R}^n, \mathbb{R}), \\ & (q, p, z) & \mapsto & (p, q, (p, q) - z). \end{array}$$

is called the *Legendre duality map.* The image under this map of a Legendre submanifold of $J^1(\mathbb{R}^n, \mathbb{R})$ is called the *Legendre dual manifold* to the initial manifold. The front of the dual Legendre manifold is called *the dual front.*

Remark 1. The Legendre duality map depends on the choice of the Euclidean structure in $\mathbb{R}^n$.
Remark 2. The Legendre duality map is an involutive contactomorphism (i.e. $l^2 = Id$).
Remark 3. Consider a wave front $V \subset J^0(\mathbb{R}^n, \mathbb{R}) \approx \mathbb{R}^{n+1}$ which is an affine sub-variety of $\mathbb{R}P^{n+1}$. Then the dual front to V coincides with the projective dual variety to[2] V (see example 1).
Example 1. The front dual to the front γ parameterized by the mapping

$$\begin{array}{cccc} f_1 : & \mathbb{R} & \longrightarrow & \mathbb{R}^2, \\ & t & \mapsto & (t, \frac{1}{3}t^3), \end{array}$$

[2]Via the identification of the dual projective space $(\mathbb{R}P^{n+1})^\vee$ with $\mathbb{R}P^{n+1}$.

is the front $\gamma^{\vee}$ parameterized by the mapping

$$\begin{array}{rccc} f_2: & \mathbb{R} & \longrightarrow & \mathbb{R}^2, \\ & t & \mapsto & (t^2, -\frac{2}{3}t^3). \end{array}$$

From the projective view-point, this is interpreted as follows. Consider the curve $\bar{\gamma}$ parameterized by the mapping

$$\begin{array}{rccc} \bar{f}_1: & \mathbb{R}P^1 & \longrightarrow & \mathbb{R}P^2, \\ & [s:t] & \mapsto & [s:t:\frac{1}{3}t^3]. \end{array}$$

Via the identification $(\mathbb{R}P^2)^{\vee} \sim \mathbb{R}P^2$, the curve projectively dual to $\bar{\gamma}$ is the curve parameterized by the mapping

$$\begin{array}{rccc} \bar{f}_2: & \mathbb{R}P^1 & \longrightarrow & \mathbb{R}P^2 \\ & [s:t] & \mapsto & [s:t:-\frac{2}{3}t^3]. \end{array}$$

With these homogeneous coordinates, denote by L the line defined by

$$L = \left\{[x:y:z] \in \mathbb{R}P^2 : x = 0\right\}.$$

We identify $\mathbb{R}P^2 \setminus L$ with $\mathbb{R}^2$.
We denote by $\infty \in \mathbb{R}P^1$ the point $[1:0]$. We identify $\mathbb{R}P^1 \setminus \{\infty\}$ with $\mathbb{R}$. Then the restriction of the maps $\bar{f}_1$ and $\bar{f}_2$ to $\mathbb{R}$ are the parameterization f_1, f_2 of the curves γ and $\gamma^{\vee}$ given above.
Example 2. Consider a Legendre manifold L which is the one graph of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$:

$$L = \left\{(t, f'(t), f(t)) \in J^1(\mathbb{R}, \mathbb{R}) : t \in \mathbb{R}\right\}.$$

The Legendre dual manifold is given by the parameterization

$$L^{\vee} = \left\{(f'(t), t, tf'(t) - f(t)) \in J^1(\mathbb{R}, \mathbb{R}) : t \in \mathbb{R}\right\}.$$

If f is a convex function then $L^{\vee}$ is the 1-graph of a function g called the *Legendre dual function of f*. That is for f convex, there exists a function g such that the Legendre manifold $L^{\vee}$ is given by

$$L^{\vee} = \left\{(t, g'(t), g(t)) \in J^1(\mathbb{R}, \mathbb{R}) : t \in \mathbb{R}\right\}.$$

2.3. Generating families and singularities of wave fronts. We follow Arnold ([**Arn2**]). Consider a function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}$.
Fix coordinates $q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$, u in $\mathbb{R}$, and $t = (t_1, \ldots, t_k)$ in $\mathbb{R}^k$.
Assume that $\forall (q, u, t) \in \mathbb{R}^{n+k+1}$, $\partial_u F(q, u, t) \neq 0$.
Define the submanifold $S \subset \mathbb{R}^{n+k+1}$, by the equations

$$F(q, u, t) = \partial_{t_1} F(q, u, t) = \cdots = \partial_{t_k} F(q, u, t) = 0.$$

Assume that S is a smooth manifold.
Define the submanifold L of $J^1(\mathbb{R}^n, \mathbb{R})$ with standard coordinates (q, p, u) by the conditions:
1) there exists $t \in \mathbb{R}^k$ such that $(q, u, t) \in S$,
2) $p_i(\partial_u F(q, u, t)) + \partial_{q_i} F(q, u, t)) = 0$ for any $i \in \{1, \ldots, n\}$.

Here p has the coordinates $(p_1, \ldots, p_n)$.
Define the map $v : S \longrightarrow \mathbb{R}^n$ by its components $(v_1, \ldots, v_n)$

$$v_i(q,u,t) = -\frac{\partial_{q_i} F(q,u,t)}{\partial_u F(q,u,t)}.$$

In other words $v_i(q,u,t)$ is the solution p_i of the equation given by condition 2) defined above.
The manifold L is the image of S under the map:

$$\begin{array}{rccc} i_F : & S & \longrightarrow & J^1(\mathbb{R}^n, \mathbb{R}) \\ & (q,u,t) & \longrightarrow & (q, v(q,u,t), u) \end{array}$$

If i_F is an embedding then $L = i_F(S)$ is a Legendre manifold. Under these assumptions (S is smooth and that i_F is an embedding), we give the following definition.

Definition 7.14. The function F is called *a generating family* of the Legendre manifold $L \subset J^1(\mathbb{R}^n, \mathbb{R})$. We say that the Legendre manifold L is *generated* by F.

Theorem 7.2. *Let m be an arbitrary point of a Legendre manifold $L \subset J^1(\mathbb{R}^n, \mathbb{R})$. Then there exists a neighbourhood $U \subset J^1(\mathbb{R}^n, \mathbb{R})$ of m such that $L \cap U$ is generated by a generating family.*

The proof of this theorem is given in [**AVG**] chapter III.
Example 1. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then the function:

$$\begin{array}{rccc} F : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}, \\ & (q,u) & \mapsto & f(q) - u. \end{array}$$

is a generating family of the Legendre manifold:

$$L = \left\{ (q,p,u) \in J^1(\mathbb{R}, \mathbb{R}) : p = \partial_q f, \ u = f(q) \right\}$$

which is the 1-graph of f.
Example 2. The function $F(q,u,t) = \frac{t^3}{3} + qt + u$ is a generating family of the Legendre manifold:

$$L = \left\{ (-t^2, t, \frac{2t^3}{3}) \in J^1(\mathbb{R}, \mathbb{R}) : t \in \mathbb{R} \right\}.$$

The wave-front of L is a semi-cubical parabola.
Example 3. Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. The Legendre manifold:

$$L = \left\{ (f'(t), t, tf'(t) - f(t)) \in J^1(\mathbb{R}, \mathbb{R}) : t \in \mathbb{R} \right\}.$$

is not a 1-graph. However, the dual Legendre manifold $L^\vee$ to L is the 1-graph of the function f. In such a situation, the function $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by:

$$F(q,u,t) = -f(t) + qt - u$$

is a generating family of L.
To conclude with this example, remark that this construction implies theorem 7.2 for $n = 1$.

Indeed take a point z in a Legendre manifold $L \subset J^0(\mathbb{R}^n, \mathbb{R})$.
Then, in a sufficiently small neighbourhood $U \subset J^1(\mathbb{R}^n, \mathbb{R})$ either:
- $L \cap U$ is a 1-graph or,
- the dual Legendre manifold to $L \cap U$ is a 1-graph.

Thus $L \cap U$ admits a generating family.

2.4. Legendre varieties.

DEFINITION 7.15. A variety $L \subset J^1(\mathbb{R}^n, \mathbb{R})$ whose smooth part is a Legendre manifold is called a *Legendre variety*.

We keep the same notations than those of the preceding subsection.
We consider the case, where S is not necessarily smooth.
Denote by Σ the set of singular points of S.
Assume that the restriction of i_F to $S \setminus \Sigma$ is a C^∞ embedding and that i_F is a topological embedding (i.e. a homeomorphism onto its image).
Under such conditions, we have the following definition.

DEFINITION 7.16. The function F is called a *generating family* of the Legendre variety $L \subset J^1(\mathbb{R}^n, \mathbb{R})$.

Example. Consider the function $F : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$F(q, u, t) = 2t^3 - 3qt^2 + u.$$

Then the Legendre subvariety L of $J^1(\mathbb{R}, \mathbb{R})$ generated by F is the union $L = L_1 \cup L_2$ of the two Legendre manifolds L_1, L_2 defined by

$$L_1 = \{(q, p, u) : p = u = 0\}$$

and

$$L_2 = \left\{(q, p, u) : u = q^3,\ p = 3q^2\right\}$$

.

3. A brief review of the theory of Legendre singularities.

We review Arnold's theory of Legendre singularities and singularities of wave-fronts [**Arn3**]. It is necessary for us to give a detailed exposition of it because we shall generalize this theory to the case of "multi-dimensional" wave front propagation.
Following Arnold ([**Arn2**]), we express Legendre equivalence and Legendre stability of singularities of wave fronts in terms of generating family-germs of the corresponding Legendre manifold-germs. This will lead us to an algebraic criteria of the stability of Legendre manifold-germs due to Arnold.
We shall take this theory one step further in section 6.

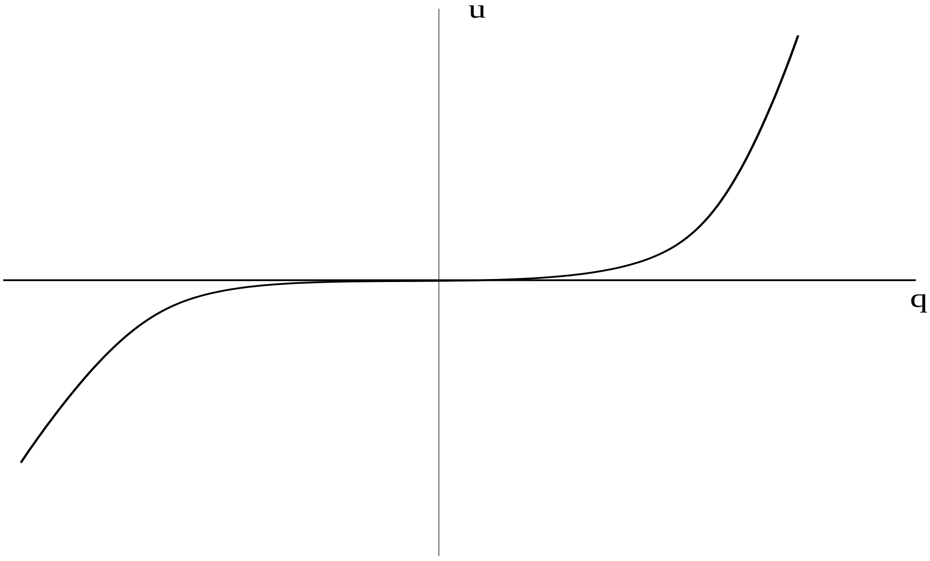

Figure 4. *The wave front of the Legendre variety generated by $F(q,u,t) = 2t^3 - 3qt^2 + u$. It has two branches: the q-axis and a smooth curve cubically tangent to it.*

3.1. Legendrian equivalence (first part).

Definition 7.17. Two Legendre manifolds L, $\tilde{L}$ of $J^1(\mathbb{R}^n,\mathbb{R})$ are called *Legendre-equivalent* provided that there exists a contactomorphism φ and a diffeomorphism ψ such that the following diagram commutes:

$$\begin{array}{ccccc} L & \xrightarrow{i} & J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\pi} & J^0(\mathbb{R}^n,\mathbb{R}) \\ \downarrow & & \varphi\downarrow & & \psi\downarrow \\ \tilde{L} & \xrightarrow{i} & J^1(\mathbb{R}^n,\mathbb{R}) & \xrightarrow{\pi} & J^0(\mathbb{R}^n,\mathbb{R}). \end{array}$$

Here i and π denote respectively the inclusion and the standard projection.

Let L, $\tilde{L}$ be two Legendre manifolds of $J^1(\mathbb{R}^n,\mathbb{R})$. The local version of the previous definition is as follows.

Definition 7.18. The germ of L at a point x is *Legendre-equivalent* to the germ of $\tilde{L}$ at y provided that there exists small neighbourhoods U, V of x and y such that $L \cap U$ is Legendre-equivalent to $\tilde{L} \cap V$.

Definition 7.19. The germ of a Legendre submanifold $L \subset J^1(\mathbb{R}^n,\mathbb{R})$ at a point x is called *Legendre-stable* provided that the following property holds. For any Legendre submanifold $L' \subset J^1(\mathbb{R}^n,\mathbb{R})$ sufficiently C^∞-close to L there exists a point y close to x such that the germ of L at x is Legendre equivalent to the germ of L' at y.

Remark. The study of generating families of Legendre varieties is more general than the study of Legendre manifolds since every Legendre manifold germ admits a generating family germ (see subsection 2.3).

3.2. V-equivalence. In this subsection, we recall basic notions of singularity theory (for details see [**Poin**], [**Tyu**], [**Math**], [**AVG**], [**Da**]).
In order to avoid confusion with contact geometry we use the term V-equivalence and not contact equivalence. The letter V stands for variety.
Notations. Given two function-germs $u, v : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$, we denote by $u \times v$ their product:

$$(u \times v)(x) = u(x)v(x).$$

DEFINITION 7.20. Two function-germs $f, g : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ are called *V-equivalent* if there exists a function-germ $A : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$, with $A(0) \neq 0$, and a diffeomorphism map-germ $\varphi : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ such that:

$$f = A \times (g \circ \varphi^{-1}).$$

Example. The function-germs defined by the germs at the origin of the following polynomials are V-equivalent:

$$t_1 t_2,\ (1 + t_1 t_2) t_1 t_2,\ t_1^2 - t_2^2,\ (t_1 + t_2^2)(t_2 + t_1^2).$$

Remark. If two function-germs are V-equivalent then the germ at the origin of their zero level sets are biholomorphically equivalent.
Denote by $Diff(k)$ the group of diffeomorphism map-germs of the type $\varphi : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ fixing the origin.
The law of composition of functions induces a semi-direct product group structure on $\mathcal{E}_t^* \times Diff(k)$. Here $\mathcal{E}_t^*$ denotes the space of function-germs $A : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ such that $A(0) \neq 0$.
The semi-direct product is given by:

$$(A, \varphi).(A', \varphi') = (A \times (A' \circ \varphi'), \varphi \circ \varphi'),$$

where $A, A' : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ are function-germs such that $A(0) \neq 0$, $A'(0) \neq 0$ and $\varphi, \varphi' : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ are diffeomorphism map-germs.
We denote this group by $\mathcal{K}$. Two function-germs are V-equivalent provided that they are on the same orbit under the action of $\mathcal{K}$.
Notation. Let $\gamma_0 = (A, \varphi)$ be an element of $\mathcal{K}$ and $f : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ a function-germ , we use the notation

$$\gamma_0.f = A \times (f \circ \varphi^{-1}).$$

3.3. Legendrian equivalence of generating families.
Notation. Let $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ. Consider a map-germ $h : (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$. The deformation $\tilde{f}$ defined by $\tilde{F}(q, t) = F(h(q), t)$ is denoted by h^*F.

Definition 7.21. The *translation* by a vector $u \in \mathbb{R}^n$ is the map-germ:

$$\begin{array}{cccc} \tau: & (\mathbb{R}^n, 0) & \longrightarrow & (\mathbb{R}^n, u) \\ & x & \mapsto & x+u. \end{array}$$

We denote by T the space of translations.

Definition 7.22. Two generating family-germs $F, G : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ are called *Legendre-equivalent* provided that there exists a map-germ $\gamma : (\mathbb{R}^k, 0) \longrightarrow \mathcal{K} \oplus T$ and a diffeomorphism map-germ $h : (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$ such that: $h^*(\gamma.G) = F$.

Example. The generating family-germs defined by $F(t, q) = t^2 + q$ and $G(t, q) = t^2 + q + q^2$ are Legendre-equivalent.

Definition 7.23. Two generating family-germs $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$, $G : (\mathbb{R}^{n+m+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ are called *stably Legendre-equivalent* provided that there exists a quadratic form $Q : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}, 0)$ such that the deformation $F \oplus Q : (\mathbb{R}^{n+m+1} \times \mathbb{R}^k, 0)$, defined by:

$$(F \oplus Q)(q, r, t) = F(q, t) + Q(r),$$

is Legendre-equivalent to G.

We have the following proposition ([**AVG**], chapter III).

Proposition 7.2. *Let L, L' be two Legendre submanifold germs of $J^1(\mathbb{R}^n, \mathbb{R})$ generated by two function-germs. Then L and L' are Legendre-equivalent if and only if their generating family-germs are stably Legendre-equivalent.*

3.4. Legendre stability. In this subsection, we give the stability criterion for a Legendre manifold germ due to Arnold ([**Arn2**],[**AVG**]).

Definition 7.24. The *Jacobian ideal* of a function-germ $f : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ is the ideal generated by the partial derivatives $\partial_{t_1} f, \dots, \partial_{t_k} f$, where $t_1, \dots, t_k$ denotes an arbitrary coordinate-system on $\mathbb{R}^k$.

Denote by I_f the ideal which is the sum of the Jacobian ideal of a function germ $f : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ with the ideal generated by f.
Denote by $\mathcal{E}_t = \{g : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}\}$ the ring of germs of functions. Let $\pi : \mathcal{E}_t \longrightarrow \mathcal{E}_t / I_f$ be the canonical projection.

Definition 7.25. *The Tyurina number of a function-germ $f : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$* is the dimension of the $\mathbb{R}$-vector space $\mathcal{E}_t / I_f$.

Example. The Tyurina number of the function germ $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ defined by $f(t) = t^4$ is equal to 3.

Definition 7.26. A *deformation* of a function germ $f : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ is a function germ $F : (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ such that $F(0, .) = f$.

Remark. Let $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ of a Legendre manifold-germ $(L, 0)$. As a function-germ F defines a deformation of[3] $f = F(0, .)$.

Definition 7.27. The deformation $F : (\mathbb{R}^m \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ of $f = F(0, .)$ is a *V-versal deformation* of $f = F(0, .)$ provided that the image under the canonical projection $\mathcal{E}_t \longrightarrow \mathcal{E}_t/I_f$ of the vector space generated by the restrictions of $\frac{\partial F}{\partial \lambda_1}, \dots, \frac{\partial F}{\partial \lambda_m}$ to $\lambda = 0$ is equal to $\mathcal{E}_t/I_f$.

Remark. This definition is not the usual one. However it is equivalent to it because of the V-versal deformation theorem ([**AVG**] chapter I).
Example. Let $f : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be defined by $f(t) = t^4$. The Tyurina number of f is equal to 3. The deformation $F : (\mathbb{R}^3 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ of f defined by $F(\lambda, t) = t^4 + \lambda_1 t^2 + \lambda_2 t + \lambda_3$ is V-versal.

Theorem 7.3. *A Legendre manifold-germ is Legendre stable if and only it is generated by a generating family germ $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ which is a V-versal deformation of $f = F(0, .)$.*

Example 1. Consider the Legendre manifold $L \in J^1(\mathbb{R}, \mathbb{R})$ generated by $F(q, u, t) = -t^3 + qt - u$. That is:

$$L = \left\{(3t^2, t, 2t^3) \in J^1(\mathbb{R}^2, \mathbb{R}) : t \in \mathbb{R}\right\}.$$

The front of this Legendre manifold is the semi-cubical parabola of equation $27u^2 + 4q^3 = 0$. The germ of F at the origin is a V-versal deformation of $f(t) = -t^3$. Hence the germ at the origin of L is Legendre-stable. Roughly speaking: "a generic plane wave front may have semi-cubical cusp points".
Example 2. Consider the Legendre manifold $L \in J^1(\mathbb{R}^2, \mathbb{R})$ generated by the generating family $F(q, u, t) = t^4 + q_1 t^2 + q_2 t - u$. That is:

$$L = \left\{(q_1, q_2, t^2, t, t^4 + q_1 t^2 + q_2 t) \in J^1(\mathbb{R}^2, \mathbb{R}) : 4t^3 + 2q_1 t + q_2 = 0\right\}.$$

The germ at the origin of the front of this Legendre manifold is the swallowtail. The germ of F at the origin is a V-versal deformation of $f(t) = t^4$. Hence the germ at the origin of L is Legendre-stable. Roughly speaking: "a generic wave front in $J^0(\mathbb{R}^2, \mathbb{R}) \approx \mathbb{R}^3$ may have swallowtails singularities."
Our aim is to generalize theorem 7.3 to the case where the Legendre manifold depends on some parameters. Before doing this, we digress in order to give a first result concerning the normal forms of non-stable wave fronts.

4. Geometrical digression: Legendrian special points.

The relation between Legendrian singularities and projective geometry was one of Arnold's motivation for his investigations in symplectic and contact geometries in the late 60's ([**Arn6**]). We give an account on it and then

[3] We point out that in singularity theory of differentiable functions $(q, u) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in \mathbb{R}^k$ are respectively called the parameter and the argument while in contact geometry the convention is the opposite one.

generalize the Young diagram of a point of a curve in projective space to contact geometry.

4.1. Special points of Legendre submanifolds of $J^1(\mathbb{R}^n,\mathbb{R})$. In this subsection, we generalize the notion of flattening points of space curves and of special parabolic points of surfaces.
For defining the special parabolic points and the flattening points an Euclidean structure on $\mathbb{R}^n$ is needed. The definition of Legendrian special points does not involve such a structure but just the standard projection $J^1(\mathbb{R}^n,\mathbb{R}) \longrightarrow J^0(\mathbb{R}^n,\mathbb{R})$.
Consider a Legendre submanifold L of $J^1(\mathbb{R}^n,\mathbb{R})$.
Take a point $m_0 = (q_0, p_0, u_0)$ in L and choose a coordinate-system in $\mathbb{R}^{n+1} \times \mathbb{R}^k$ centered at $(q_0, u_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}^k$.
Let $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ of the germ (L, m_0) of L at the point $m_0 \in J^1(\mathbb{R}^n,\mathbb{R})$.

DEFINITION 7.28. The point $(q_0, p_0, u_0) \in J^1(\mathbb{R}^n,\mathbb{R})$ is called a *Legendrian special point of* L if the Tyurina number of the function-germ of $f = F(0,.) : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ is at least equal to n.

Remark 1. It is readily verified that this definition does not depend on the choice of the generating family-germ nor on the choice of the coordinates in $\mathbb{R}^{n+1} \times \mathbb{R}^k$.
Remark 2. Legendrian equivalence sends a Legendrian special point of a Legendre submanifold of $J^1(\mathbb{R}^n,\mathbb{R})$ to a Legendre special points of its image.

DEFINITION 7.29. A Legendrian special point (q_0, p_0, u_0) of the Legendre submanifold L is called a *degenerate Legendrian special point* provided that either:
- the Tyurina number of f is strictly higher than n or,
- the generating family-germ F is not a V-versal deformation of f.

Example 1. Consider the Legendre manifold L generated by the function $F(q, u, t) = t^4 + q_1t^2 + q_2t - u$. That is:

$$L = \left\{(q_1, q_2, t^2, t, t^4 + q_1t^2 + q_2t) \in J^1(\mathbb{R}^2,\mathbb{R}) : 4t^3 + 2q_1t + q_2 = 0\right\}.$$

The origin is a non-degenerate Legendrian special point of the Legendre manifold L.
Example 2. Let $F(q, u, t) = t^5 + q_1t^2 + q_2t - u$. The origin is a degenerate Legendrian special point of the Legendre manifold generated by F.
Example 3. Let $F(q, u, t) = t^4 + q_1t^3 + q_2t - u$. The origin is a degenerate Legendrian special point of the Legendre manifold generated by F.

4.2. The Young diagram of a point of corank one on a Legendre manifold. Let $F : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ of a Legendre variety.

DEFINITION 7.30. The dimension of the kernel of the second differential of $f = F(0,.)$ at the origin is called the *corank* of F.

Choose a coordinate system $(t, s_1, \dots, s_{k-1})$ in $\mathbb{R}^k$ such that ∂_t is the kernel of the second differential of $f = F(0, .)$ at the origin.
The following definition is a contactification of the notion introduced previously in chapter 2, section 1 for curves in projective spaces.

DEFINITION 7.31. The *Young diagram* (resp. the *anomaly sequence*) of the generating family F is the Young-diagram (resp. the *anomaly sequence*) of the map-germ defined by

$$\begin{array}{ccc} (\mathbb{R}, 0) & \longrightarrow & (\mathbb{R}^{n+1}, 0), \\ t & \mapsto & (F(0,t), (\partial_{q_1} F)(0,t), \dots, (\partial_{q_n} F)(0,t)). \end{array}$$

Remark. Proposition 7.2 implies that the Young diagram is independent on the choice of the coordinate system $(t, s_1, \dots, s_n)$ that we made. It is defined only in terms of the Legendre variety-germ generated by F and of the standard projection $J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R})$.

4.3. Legendrian special points and flattening points of curves. In this subsection, we fix an Euclidean structure on the space $\mathbb{R}^n$ and denote by $(,)$ the scalar product.
Let $L \subset J^1(\mathbb{R}^n, \mathbb{R})$ be a Legendre manifold such that its front $\gamma \subset J^0(\mathbb{R}^n, \mathbb{R})$ is of dimension one. Let π be the projection $J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R})$. Remark that at each point $m \in \gamma$, the intersection of the Legendre submanifold manifold L with the fibre $\pi^{-1}(m)$ is of dimension $n - 1$.
Denote by $L^\vee$ the Legendre manifold dual to L. The following proposition is readily verified.

PROPOSITION 7.3. *The Legendre manifold $L^\vee$ has a Legendrian special point at (q, p, u) with a given Young diagram if and only if the front γ of L has a flattening point at $(p, (p, q) - u)$ with the same Young diagram.*

Example. Consider the front γ parameterized by:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & J^0(\mathbb{R}^2, \mathbb{R}), \\ t & \mapsto & (t, t^2, t^4). \end{array}$$

The front γ has a non-degenerate flattening point at the origin.
Let L be the Legendre manifold whose front is γ. Denote by $L^\vee$ the dual Legendre manifold to L.
A generating family $F : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ for $L^\vee$ is given by the formula

$$F(q, u, t) = -t^4 + q_2 t^2 + q_1 t - u.$$

Indeed, the Legendre manifold corresponding to this generating family is:

$$L^\vee = \left\{ (q_1, q_2, t, t^2, u) \in J^1(\mathbb{R}^2, \mathbb{R}) : \partial_t F(q, u, t) = 0,\ F(q, u, t) = 0 \right\}.$$

And by Legendre duality:

$$L = \left\{ (t, t^2, q_1, q_2, t^4) \in J^1(\mathbb{R}^2, \mathbb{R}) : \partial_t F(q, t) = 0 \right\}.$$

The front of L is:

$$\gamma = \left\{ (t, t^2, t^4) \in J^0(\mathbb{R}^2, \mathbb{R}) : t \in \mathbb{R} \right\}.$$

The origin is a non-degenerate flattening point of γ, while the germ of the function F at the origin is a V-versal deformation of the function-germ

$$\begin{array}{rccc} f: & (\mathbb{R},0) & \longrightarrow & \mathbb{R} \\ & t & \mapsto & t^4. \end{array}$$

We have the following commutative diagram:

$$\begin{array}{ccc} L \subset J^1(\mathbb{R}^2,\mathbb{R}) & \xrightarrow{l} & L^{\vee} \subset J^1(\mathbb{R}^2,\mathbb{R}) \\ \pi_1 \big\downarrow & & \pi_1 \big\downarrow \\ \gamma \subset J^0(\mathbb{R}^2,\mathbb{R}) & & \gamma^{\vee} \subset J^0(\mathbb{R}^2,\mathbb{R}) \\ g \big\downarrow & & \pi_2 \big\downarrow \\ \Sigma \subset \mathbb{R}^2 & \xrightarrow{Id} & \Sigma \subset \mathbb{R}^2 \end{array}$$

Here π_1 is the standard projection. The map π_2 is defined by $\pi_2(q,u) = q$. The 0-jet space $J^0(\mathbb{R}^2,\mathbb{R})$ is diffeomorphic to the product $\mathbb{R}^2 \times \mathbb{R}$. Call a plane in $\mathbb{R}^2 \times \mathbb{R}$ *vertical* if it does not intersects the line $(\{0\} \times \mathbb{R}) \subset \mathbb{R}^2 \times \mathbb{R}$. Then, the intersection of the Legendre manifold L with the fibre $\pi_1^{-1}(m)$, $m \in \gamma$ can be identified with the set of non-vertical planes in $J^0(\mathbb{R}^2,\mathbb{R})$ that are tangent to γ.
The Legendre duality map is denoted by l, Id stands for the identity map. The front $\gamma^{\vee}$ is a swallowtail.
The map g is the Gauss map: if the osculating plane to the curve γ at a point m is $\{(x,y,z) \in J^0(\mathbb{R}^2,\mathbb{R}) : z = ax + by + u\}$ then $g(m) = (a,b) \in \Sigma$. On the other hand, Σ is the image of the critical locus of the restriction of the projection π_2 to the swallowtail $\gamma^{\vee}$.
The restriction of the projection $\pi_2 \circ \pi_1$ to $L^{\vee}$ is the so-called Whitney pleat. The right hand-side of this commutative diagram is depicted in figure 5.

4.4. Legendrian special points and special parabolic points of surfaces. In this subsection, we fix an Euclidean structure on the space $\mathbb{R}^n$ and denote by $(,)$ the scalar product.
Let $L \subset J^1(\mathbb{R}^2,\mathbb{R})$ be a Legendre manifold such that its front $M \subset J^0(\mathbb{R}^2,\mathbb{R})$ is a smooth surface.
Let π be the standard projection $J^1(\mathbb{R}^n,\mathbb{R}) \longrightarrow J^0(\mathbb{R}^n,\mathbb{R})$. Denote by $L^{\vee}$ the Legendre manifold dual to L.

Proposition 7.4. *The Legendre manifold $L^{\vee}$ has a non-degenerate (resp. degenerate) Legendrian special point at (q,p,u) if and only the front M of L has a non-degenerate (resp. degenerate) special parabolic point at $(p,(p,q)-u)$.*

Example. Consider the front M parameterized by

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & J^0(\mathbb{R}^2,\mathbb{R}), \\ (s,t) & \mapsto & (t, s+t^2, s^2+t^4). \end{array}$$

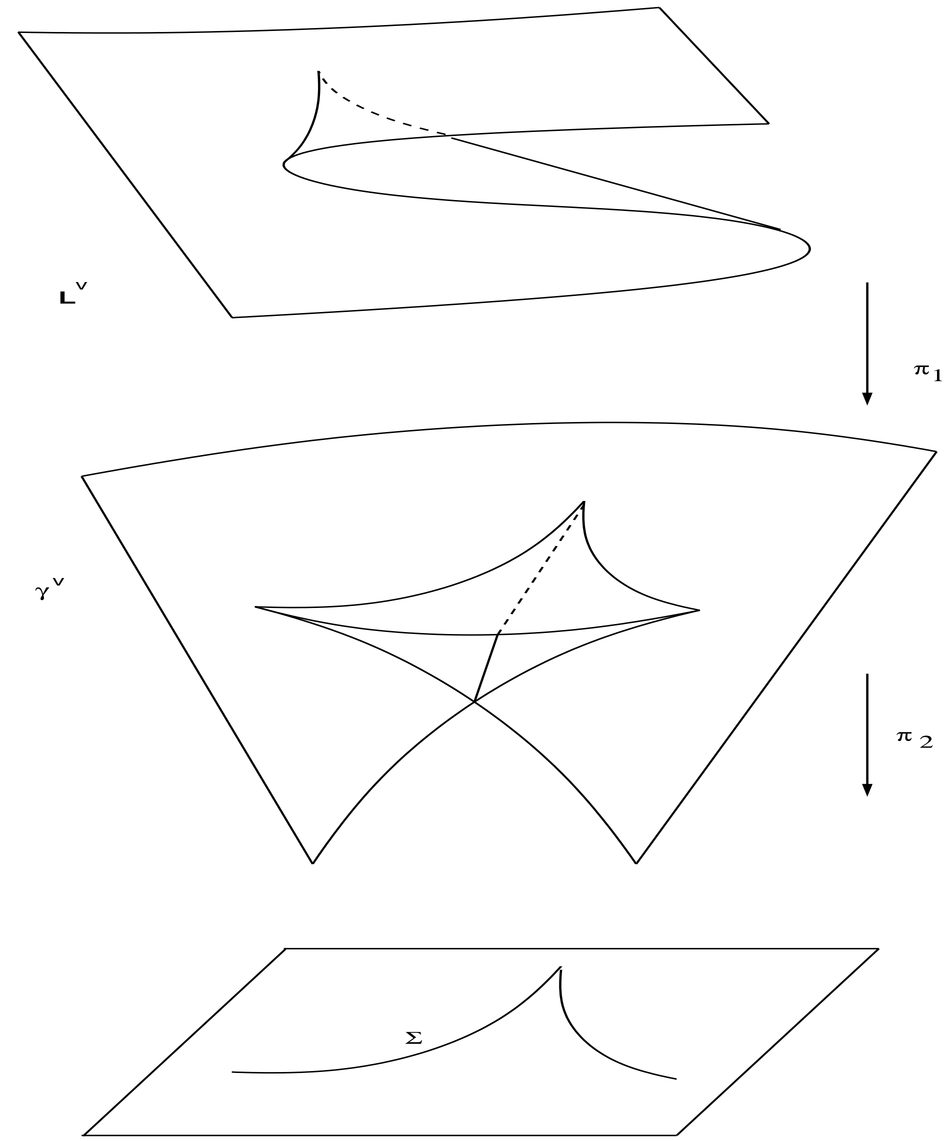

FIGURE 5. *The Legendrian projection corresponding to a non-degenerate flattening point of a space curve.*

Direct calculations show that the front M has a non-degenerate special parabolic point at the origin. We assert that the germ at the origin of the front dual to M is diffeomorphic to a swallowtail.
Let L be the Legendre manifold whose front is M. Denote by $L^{\vee}$ the dual Legendre manifold to L. We assert that a generating family for $L^{\vee}$ is given by:

$$F(q,u,s,t) = -t^4 - s^2 + q_2 s + q_2 t^2 + q_1 t - u.$$

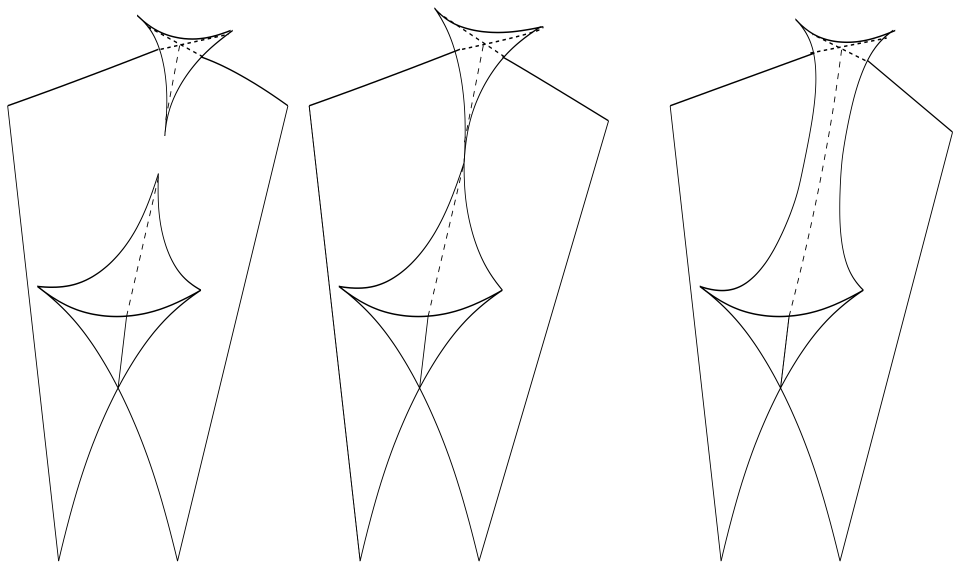

FIGURE 6. *A "typical" bifurcation of wave-fronts.*

Indeed, the Legendre manifold corresponding to this generating family is:
$L^{\vee} = \left\{(q_1, q_2, t, s+t^2, u) \in J^1(\mathbb{R}^2, \mathbb{R}) : \partial_t F(q,u,s,t) = \partial_s F(q,u,s,t) = F(q,u,s,t) = 0\right\}.$
And by Legendre duality, we get that the Legendre manifold L is given by
$L = \left\{(t, s+t^2, q_1, q_2, s^2+t^4) \in J^1(\mathbb{R}^2, \mathbb{R}) : \partial_t F(q,u,s,t) = \partial_s F(q,u,s,t) = 0\right\}.$
The front of L is the surface

$$M = \left\{(t, s+t^2, s^2+t^4) \in J^0(\mathbb{R}^2, \mathbb{R}) : (s,t) \in \mathbb{R}^2\right\}.$$

The germ of F at the origin is a V-versal deformation of the function-germ of:

$$\begin{array}{rccc} f: & (\mathbb{R}^2, 0) & \longrightarrow & \mathbb{R} \\ & (s,t) & \mapsto & s^2+t^4. \end{array}$$

Hence the origin is a non-degenerate Legendrian special point of $L^{\vee}$. We get the following commutative diagram:

$$\begin{array}{ccc} L \subset J^1(\mathbb{R}^2, \mathbb{R}) & \xrightarrow{\ l\ } & L^{\vee} \subset J^1(\mathbb{R}^2, \mathbb{R}) \\ \big\downarrow & & \big\downarrow \\ M \subset J^0(\mathbb{R}^2, \mathbb{R}) & & \gamma^{\vee} \subset J^0(\mathbb{R}^2, \mathbb{R}) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ \Sigma \subset \mathbb{R}^2 & \xrightarrow{\ Id\ } & \Sigma \subset \mathbb{R}^2 \end{array}$$

Assertion. There exists a Legendre manifold $\tilde{L}^{\vee}$ Legendre-equivalent to $\tilde{L}^{\vee}$ whose dual front $\tilde{\gamma}$ is a space curve with a non-degenerate flattening point at the origin.

The Legendre manifold $L^{\vee}$, generated by F, is also generated by the generating family $G : (\mathbb{R}^2 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ defined by

$$G(q, u, t) = t^4 + q_1 t^2 + q_2 t + \frac{1}{4} q_2^2 - u.$$

To see it, remark that:

$$(\partial_s F(q, u, s, t) = 0) \iff (s = \frac{q_2}{2}).$$

Substituting s by $\frac{q_2}{2}$ in F, we get G.
The map:

$$\begin{array}{cccc} \varphi : & J^1(\mathbb{R}^2, \mathbb{R}) & \longrightarrow & J^1(\mathbb{R}^2, \mathbb{R}) \\ & (q_1, q_2, p_1, p_2, u) & \mapsto & (q_1, q_2, p_1 - \frac{1}{2} q_1, p_2, u - \frac{1}{4} q_1^2) \end{array}$$

is a contactomorphism. Moreover φ preserves the standard projection from $J^1(\mathbb{R}^2, \mathbb{R})$ to $J^0(\mathbb{R}^2, \mathbb{R})$. The image of the manifold $L^{\vee}$ under φ is the Legendre manifold $\tilde{L}$ generated by:

$$\widetilde{G}(q, u, t) = t^4 + q_1 t^2 + q_2 t - u,$$

that we were looking for.
Assertion is proved.
The Legendrian equivalence between this example and the example of the preceding subsection explains the analogy between special parabolic points of surfaces and flattening points of space curves: from the Legendrian viewpoint there is no difference between them.

4.5. The $\mathcal{L}$-discriminant. Consider a family of Legendre submanifolds (L_λ) of $J^1(\mathbb{R}^n, \mathbb{R})$.

DEFINITION 7.32. The $\mathcal{L}$-*discriminant* (read Legendrian discriminant) of the family (L_λ) is the set of values of the parameters λ for which the Legendre manifold L_λ has a degenerate Legendrian special point or has a singular point.

Remark. The definition of the $\mathcal{L}$-discriminant depends only on the standard projection $J^1(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^0(\mathbb{R}^n, \mathbb{R})$. In particular, no additional structure (Euclidean, affine, projective) on $\mathbb{R}^n$ is required.
The definition of the stability of the $\mathcal{L}$-discriminant of a given family of Legendre manifolds is analogous to that of the stability of the $\mathcal{P}$-discriminant of a family of curves (see section 1).

4.6. A Kazarian folded umbrella theorem for wave fronts. With the techniques developed below, we shall obtain the following generalization of theorem 7.1 (see section 1).
Fix an Euclidean structure in $\mathbb{R}^2 \times \mathbb{R}$.

Consider the 3-parameter family of surfaces M_λ in $\mathbb{R}^3$ parameterized by the embeddings

$$\begin{array}{cccc} f_\lambda & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R}, \\ & (s,t) & \mapsto & (t, s+t^3, s^2+t^7+t^6+\lambda_1 t^5+\lambda_2 t^4+\lambda_3 t^2). \end{array}$$

Identify $\mathbb{R}^2 \times \mathbb{R}$ with $J^0(\mathbb{R}^2, \mathbb{R})$.
The surfaces are the wave fronts of a family of Legendre manifolds (L_λ), $L_\lambda \subset J^1(\mathbb{R}^2, \mathbb{R})$.
Denote by $L_\lambda^\vee \subset J^1(\mathbb{R}^2, \mathbb{R})$ the Legendre manifold dual to L_λ.

THEOREM 7.4. *The $\mathcal{P}$-discriminant of the* 3*-parameter family of Legendre manifolds* $(L_\lambda^\vee)$ *in* $J^1(\mathbb{R}^2, \mathbb{R})$ *is $\mathcal{L}$-stable at the origin. The $\mathcal{L}$-discriminant of the family* (L_λ) *is the Kazarian folded umbrella.*

Remark. The complete list of $\mathcal{L}$-stable bifurcation diagrams for two and three parameter families is yet unknown.
This theorem is a special case of theorem 7.7 stated in page 162.

5. The excellent Young diagrams.

5.1. New notations. We change slightly our notations. In $J^0(\mathbb{R}^n, \mathbb{R})$, we use to denote the coordinates by $q_1, \dots, q_n, u$. We now use $q_1, \dots, q_n, q_{n+1}$ and $q = (q_1, \dots, q_n, q_{n+1})$ instead of (q, u).
Given a Legendre manifold, we used to denote a generating family of it with capital letters (like F), we now use small letters (like f). The capital letters will be used for another purpose.

5.2. The Legendrian codimension.

DEFINITION 7.33. The *$\mathcal{L}$-tangent space*, denoted Tf, to a generating family-germ $f \in \mathcal{E}_{q,t}$ is the $\mathbb{R}$-vector subspace of $\mathcal{E}_{q,t}$ which is the sum of the following modules:
- the $\mathcal{E}_{q,t}$-module generated by $\partial_{t_1} f, \dots, \partial_{t_k} f, f$,
- the $\mathcal{E}_q$-module generated by $\partial_{q_1} f, \dots, \partial_{q_n} f, \partial_{q_{n+1}} f$.

DEFINITION 7.34. The *Legendrian codimension*, of a Legendre manifold germ $(L, 0)$ generated by a family-germ $f : (\mathbb{R}^k \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ is the codimension in $\mathcal{E}_{q,t}$ of the $\mathcal{L}$-tangent space Tf to f.

5.3. Excellent generating family-germs and excellent Young diagrams. Consider a generating family germ $f : (\mathbb{R}^k \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ of a Legendre manifold-germ. Denote by (a_i) the anomaly sequence of f.

DEFINITION 7.35. A generating family-germ $f : (\mathbb{R}^k \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ of a Legendre manifold-germ is called *excellent* provided that:
- it is of corank one,
- the Legendrian codimension of f is equal to $\sum_{i=1}^s a_i - 1$.

DEFINITION 7.36. A pair (n, Y) where Y denotes a Young diagram is called *excellent* provided that there exists an excellent generating family-germ $f : (\mathbb{R}^k \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ with Young diagram Y.
A Young diagram is called *excellent* provided that there exists an integer $n > 0$ such that (n, Y) is excellent.

THEOREM 7.5. *The excellent generating-family germs and the corresponding Young diagrams are given up to stable Legendrian equivalence by the following list. They form* 4 *infinite series and two sporadic cases.*

$$\begin{array}{ll}
A_n : (\square) & t^{n+2} + q_1 t^n + q_2 t^{n-1} + \dots + q_n t + q_{n+1} \\
W_n : (\square\square) & t^{n+3} + q_1 t^n + q_2 t^{n-1} + \dots + q_n t + q_{n+1} \\
X_n : (\begin{smallmatrix}\square \\ \square\end{smallmatrix}) & t^{n+2} + q_1 t^{n+1} + q_2 t^{n-1} + \dots + q_n t + q_{n+1} \\
Y_n : (\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}) & t^{n+4} + t^{n+3} + q_1 t^{n+1} + q_2 t^{n-1} + \dots + q_n t + q_{n+1} \\
Z_1 : (\square\square\square) & t^7 + t^5 + q_1 t + q_2 \\
Z_2 : (\begin{smallmatrix}\square & \square & \square \\ \square & & \end{smallmatrix}) & t^6 + t^5 + q_1 t^2 + q_2
\end{array}$$

Remark 1. For $n > 1$, the series A_n, W_n, X_n are contained in Arnold's classification [**Arn5**] because their Legendrian codimension is not more than one.
Remark 2. By definition, the Legendrian codimension of an excellent generating family-germ is equal to the number of squares of its Young diagram minus 1.

The proof of this theorem is given in section 2.
The typical families in which the Legendre manifold (or the Legendre variety) of theorem 7.5 arise is given in subsection 6.4.

6. Legendrian versal deformation theory.

In section 4, we saw that the local study of flattening points of space curves as well as the study of special parabolic points of surfaces are included in the study of Legendre manifolds of $J^1(\mathbb{R}^n, \mathbb{R})$ depending on parameters.
In the seventies' Arnold and Zakalyukin studied families of wave front depending on one parameter ([**Arn5**], [**Zak1**]).
This is not sufficient because our families arising from the study of projective geometry might depend on many parameters.
In this section, we introduce the algebraic structures that are needed in order to study multi-parametric families of wave-fronts.

6.1. Legendrian deformations of generating families.

DEFINITION 7.37. A *parametric generating family* of a family (L_λ) of Legendre varieties is a function $F : \mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k \longrightarrow (\mathbb{R}, 0)$ such that $F(\lambda, ., .)$ is a generating family of a Legendre variety L_λ. The family (L_λ) of Legendre varieties is said to be *generated* by F.

Example 1. Consider the parametric generating family $F : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \longrightarrow (\mathbb{R}, 0)$ defined by:

$$F(\lambda, q, t) = t^4 + q_1 t^3 + q_2 t - q_3 + \lambda t^2,$$

with $\lambda \in \mathbb{R}$, $q = (q_1, q_2, q_3) \in \mathbb{R}^3$, $t \in \mathbb{R}$.
The Legendre manifold generated by $F(\lambda, .)$ is:

$$L_\lambda = \left\{ (q_1, q_2, t^3, t, t^4 + q_1 t^3 + \lambda t^2 + q_2 t) \in J^1(\mathbb{R}^n, \mathbb{R}) : \partial_t F(\lambda, q, t) = 0 \right\}.$$

The wave-fronts corresponding to this one-parameter family are depicted in figure 6. The local version of the previous definition is as follows.

Definition 7.38. A *Legendrian deformation* of a generating family-germ $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ of a Legendre variety-germ is the germ at the origin of a parametric generating-family $F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ such that the germ at the origin of $F(0, ., .)$ is equal to f.

The following proposition is a consequence of the construction given in [**Arn2**] and [**AVG**] for the generating family germ of a Legendre manifold germ (see also example 3, subsection 2.3).

Proposition 7.5. *Let (L_λ) be a family of Legendre manifolds. For any point $p \in L_\lambda$ there exists a neighbourhood $U \subset J^1(\mathbb{R}^n, \mathbb{R})$ of p and a parametric generating family F such that the manifold $(L_\lambda \cap U)$ is generated by $F(\lambda, .)$.*

6.2. Legendrian versal deformations. Let $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ of a Legendre of variety-germ.

Definition 7.39. A map-germ $g : (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{l'} \times \mathbb{R}^{n+1}, 0)$ is *fibered* over $\mathbb{R}^l$ and $\mathbb{R}^{l'}$ if the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0) & \xrightarrow{g} & (\mathbb{R}^{l'} \times \mathbb{R}^{n+1}, 0) \\ \downarrow & & \downarrow \\ (\mathbb{R}^l, 0) & \xrightarrow{g_1} & (\mathbb{R}^{l'}, 0) \end{array}$$

In other words $g(\lambda, q) = (g_1(\lambda), g_2(\lambda, q))$.

Let $F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ be of a Legendrian deformation of a generating family-germ $f = F(0, ., .)$.

Definition 7.40. A Legendrian deformation $\tilde{F}$ of the generating family-germ f is *induced* from F provided that there exists the germ of a fibered map $g : (\mathbb{R}^{l'} \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0)$ such that:

$$\tilde{F}(\lambda, q, t) = F(g_1(\lambda), g_2(\lambda, q), t)$$

with $g = (g_1, g_2)$.
We use the notation $\tilde{F} = g^* F$.

Recall that T denotes the space of translations.

DEFINITION 7.41. Two Legendrian deformations $\tilde{F}, F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ of a generating family-germ $f = F(0, ., .)$ are called *$\mathcal{L}$-equivalent* provided that there exists the germ of a map-germ $\gamma : (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow \mathcal{K} \oplus T$ such that:

$$\gamma.\tilde{F} = F.$$

Remark. The formula above can be written in the more explicit form:

$$\tilde{F}(\lambda, q, t) = A(\lambda, q, t)F(\lambda, q, \varphi(\lambda, q, t))$$

where $(A, \varphi) \in \mathcal{K} \oplus T$.

DEFINITION 7.42. The germ of a Legendrian deformation F of a generating family-germ $f = F(0, ., .)$ is called a *$\mathcal{L}$-versal deformation* of f if any other Legendrian deformation of f is induced by a Legendrian deformation $\mathcal{L}$-equivalent to F.

Example. Let $f : (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be defined by:

$$f(q, t) = t^4 + q_1 t^3 + q_2 t - q_3.$$

Consider the Legendrian deformation $F : (\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ of f defined by:

$$F(\lambda, q, t) = f(q, t) + \lambda t^2.$$

We shall prove that F is a $\mathcal{L}$-versal deformation of f.

6.3. The Legendrian versal deformation theorem.

DEFINITION 7.43. The *$\mathcal{L}$-tangent* space to a Legendrian deformation germ $F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ of a generating family-germ $f = F(0, ., .)$, denoted TF, is the sum of the following $\mathbb{R}$-vector subspaces of $\mathcal{E}_{q,t}$:
- Tf,
- the $\mathbb{R}$-vector space generated by the restrictions to $\lambda = 0$ of the function-germs $\partial_{\lambda_1} F, \dots, \partial_{\lambda_l} F$.

THEOREM 7.6. *A Legendrian deformation $F : (\mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ of a generating family-germ is $\mathcal{L}$-versal provided that $TF = \mathcal{E}_{q,t}$.*

Remark. We have considered the C^∞ case. However, a similar theorem holds in the $\mathbb{K}$-analytic category and for $\mathbb{K}$ power series category, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

6.4. The $\mathcal{L}$-versal deformations of the excellent Young diagrams.

THEOREM 7.7. *The Legendrian versal deformations of the excellent generating family germs are given up to stable Legendrian equivalence by the* *following list:*

$$A_n : t^{n+2} + q_1t^n + q_2t^{n-1} + \cdots + q_nt + q_{n+1}$$
$$W_n : t^{n+3} + q_1t^n + q_2t^{n-1} + \cdots + q_nt + q_{n+1} + \lambda_1t^{n+1}$$
$$X_n : t^{n+2} + q_1t^{n+1} + q_2t^{n-1} + \cdots + q_nt + q_{n+1} + \lambda_1t^n$$
$$Y_n : t^{n+4} + t^{n+3} + q_1t^{n+1} + q_2t^{n-1} + \cdots + q_nt + q_{n+1} + \lambda_1t^{n+2} + \lambda_2t^n$$
$$Z_1 : t^7 + t^5 + q_1t + q_2 + \lambda_1t^3 + \lambda_2t^2$$
$$Z_2 : t^6 + t^5 + q_1t^2 + q_2 + \lambda_1t^4 + \lambda_2t^3 + \lambda_3t$$

Remark. The series A_n, W_n, X_n are contained in Arnold's classification [**Arn5**] because their Legendrian codimension is not more than one.
This theorem is proved in section 2.

6.5. Stabilization of $\mathcal{L}$-discriminants.
Let $F : \mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a parametric generating family of Legendre varieties $L_\lambda, L_\lambda \in J^1(\mathbb{R}^n, \mathbb{R})$. Assume that:

$$F(\lambda, q_1, \dots, q_{n+1}, t) = \tilde{F}(\lambda, q_1, \dots, q_{n-1}) + q_nt + q_{n+1}.$$

Then $G = \partial_t F : \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R} \longrightarrow (\mathbb{R}, 0)$ is a parametric generating family of Legendre varieties $\tilde{L}_\lambda, \tilde{L}_\lambda \in J^1(\mathbb{R}^{n-1}, \mathbb{R})$.

Definition 7.44. The parametric generating family G is called a derived generating family from F.
The deformation of a generating family-germ obtained by taking the germ of a derived generating family of an arbitrary representative of the germ is called the *derived deformation*.

The following proposition is straightforward. It generalizes Kazarian's construction of the trace of a curve ([**Ka3**]).

Proposition 7.6. *The $\mathcal{L}$-discriminant of a parametric generating family $F : \mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R} \longrightarrow \mathbb{R}$ coincides with the $\mathcal{L}$-discriminant of a derived parametric generating family of F.*

Remark. This proposition together with the normal forms for the $\mathcal{L}$-versal deformations of perfect pairs allows us to "multiply the theorems". We give an example in the next subsection.

6.6. Theorem 7.4 is a corollary of theorem 7.7. A parametric generating family-germ $F : (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ of the germs at the origin of Legendre manifolds $L^\vee_\lambda$ of subsection 4.6 is given by:

$$F(\lambda, q, t, s) = t^7 + t^6 + q_1t^3 + q_2t + q_3 + \lambda_1t^5 + \lambda_2t^4 + \lambda_3t^2 + s^2 + q_1s.$$

The function F is a deformation of the generating family-germ $f(\mathbb{R}^3 \times \mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ defined by:

$$f(q, t, s) = t^7 + t^6 + q_1t^3 + q_2t + q_3 + s^2 + q_1s.$$

Obviously the deformation F is stably $\mathcal{L}$-equivalent to:

$$\tilde{F}(\lambda, q, t, s) = t^7 + t^6 + q_1 t^3 + q_2 t + q_3 + \lambda_1 t^5 + \lambda_2 t^4 + \lambda_3 t^2$$

To see it, write $s^2 + q_1 s = (s + \frac{1}{2}q_1)^2 - \frac{1}{4}q_1^2$.
The generating family derived from $\tilde{F}$ is a $\mathcal{L}$-versal deformation of a generating family-germ $\mathcal{L}$-equivalent to the element Z_2 of theorem 7.7.
Consequently the $\mathcal{L}$-discriminant of F is stable.

CHAPTER 8

Normal forms of generating families.

1. Normal forms theory.

1.1. The finite determinacy theorem for $\mathcal{L}$-equivalence. Denote by $\mathcal{M}_{q,t}$ the maximal ideal in $\mathcal{E}_{q,t}$ that is the ideal of function-germs of the type

$$g : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0).$$

Let $\mathcal{M}_{q,t}^j$ be the j^{th} power of $\mathcal{M}_{q,t}$.
We denote by Tf the $\mathcal{L}$-tangent space to f (see subsection 5.2).

PROPOSITION 8.1. *Let $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family germ such that there exists a positive integer ν satisfying $\mathcal{M}_{q,t}^{\nu} \subset Tf$.*
Then for any function germ $\psi \in \mathcal{M}_{q,t}^{\nu+2}$ the following $\mathcal{L}$-equivalence holds

$$f + \psi \sim f.$$

The proof of this proposition is given in subsection 3.2.

1.2. Quasi-homogeneous functions. We recall some facts from singularity theory (see [**AVG**] for details). We fix a coordinate system in $\mathbb{R}^m$. The construction depends on the choice of this coordinate-system.

DEFINITION 8.1. A function-germ $f : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}, 0)$ is called *quasi-homogeneous of degree d with exponents $\alpha_1, \ldots, \alpha_m$* provided that for any $\lambda > 0$ and any $(x_1, \ldots, x_m) \in \mathbb{R}^m$ we have:

$$f(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_m} x_m) = \lambda^d f(x_1, \ldots, x_m).$$

Example. The function-germ defined by $f(x_1, x_2) = x_1^p + x_2^q$ is quasi-homogeneous of degree pq with exponents $\alpha_1 = q,\ \alpha_2 = p$. The function germ defined by $f(x_1, x_2) = x_1^3 + x_1 x_2 + x_2^3$ is not quasi-homogeneous.
Fix a vector $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$.

DEFINITION 8.2. A monomial $x_1^{i_1} x_2^{i_2} \ldots x^{i_k}$ has *weight d* provided that $\alpha_1 i_1 + \cdots + \alpha_m i_m = d$.

Remark. Once the weights of the x_i's are fixed, the vector α is fixed and consequently so is the weight of a monomial.

DEFINITION 8.3. A function-germ in $\mathcal{E}_x$ has *order d* if all the monomials appearing with non-zero coefficient in its Taylor series at the origin are of weight at least d.

Notations. The $\mathbb{R}$-vector space of functions-germs of order d is denoted by F_d (F for filtration).
The $\mathbb{R}$-vector space generated by the monomials of weight d is denoted by G_d (G for graduation).

DEFINITION 8.4. The nested sequence $F_0 \supset F_1 \supset F_2 \supset \dots$ of $\mathbb{R}$-vector spaces is called the *quasi-homogeneous filtration* of $\mathcal{E}_x$ associated to α.

1.3. The order of a vector-field. Take coordinates $q = (q_1, \dots, q_{n+1})$ in $\mathbb{R}^{n+1}$ and $t = (t_1, \dots, t_k)$ in $\mathbb{R}^k$.
Fix a quasi-homogeneous filtration $F_0 \supset F_1 \supset F_2 \supset \dots$ of $\mathcal{E}_{q,t}$.
We denote by:
- $L_{q,t}$ the set of vector fields of the type:

$$\sum_{i=1}^{k} a_i \partial_{t_i} + \sum_{i=1}^{n+1} b_i \partial_{q_i}$$

with $a_1, \dots, a_k \in \mathcal{E}_{q,t}$, $b_1, \dots, b_{n+1} \in \mathcal{E}_q$,
- L_q the set of vector fields of the type:

$$\sum_{i=1}^{n+1} c_i \partial_{q_i}$$

with $c_1, \dots, c_{n+1} \in \mathcal{E}_q$.
The letter L stands for Lie.

Any element in $L_{q,t}$ acts on the F_d's. Namely, put:

$$v = \sum_{i=1}^{k} a_i \partial_{t_i} + \sum_{i=1}^{n+1} b_i \partial_{q_i},$$

for $m \in F_d$, the Lie derivative of m along v is defined by the formula

$$v.m = \sum_{i=1}^{k} a_i \partial_{t_i} m + \sum_{i=1}^{n+1} b_i \partial_{q_i} m.$$

We put $v.F_d = \{v.m : m \in F_d\}$.

DEFINITION 8.5. A vector field $v \in L_{q,t}$ (resp. $w \in L_q$) has *order* s provided that for any d and for any $m \in F_d$ we have $v.m \in F_{d+s}$ (resp. $w.m \in F_{d+s}$).

1.4. Normal forms of semi-quasi-homogeneous generating families. Fix a quasi-homogeneous filtration $F_0 \supset F_1 \supset F_2 \supset \dots$ of $\mathcal{E}_{q,t}$.

DEFINITION 8.6. The *reduced tangent space* (with respect to the fixed quasi-homogeneous filtration), denoted $T_r f$, to a generating family germ $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ is the $\mathbb{R}$-vector subspace of $\mathcal{E}_{q,t}$ which is the sum of the following modules:
- the $\mathcal{E}_{q,t}$-module generated by the $v.f$'s where $v \in L_{q,t}$ has order 1,
- the $\mathcal{E}_{q,t}$-module generated by the $h \times f$ where $h \in F_1$,
- the $\mathcal{E}_q$-submodule generated by the $w.f$ where $w \in L_q$ has order 1.

Remark that F_d is an $\mathcal{E}_{q,t}$-submodule of $\mathcal{E}_{q,t}$ while Tf and T_rf are $\mathcal{E}_q$-submodules of $\mathcal{E}_{q,t}$.

PROPOSITION 8.2. *Let $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ such that there exists a positive integer d and pairwise distinct monomials $e_1, \dots, e_s \in \mathcal{E}_{q,t}$ satisfying the following conditions*

$$\begin{cases} F_d \subset (T_r f \oplus < e_1, \dots, e_s >), \\ \forall i \in \{1, \dots, s\}\,,\ e_i \in F_{d+1}. \end{cases}$$

Then for any function-germ $\psi \in F_d$, there exists $a_1, \dots, a_s \in \mathbb{R}$ such that the following $\mathcal{L}$-equivalence relation holds

$$f + \psi \sim f + \sum_{k=1}^{s} a_k e_k.$$

The proof of this proposition is given in subsection 3.3.

1.5. Finite dimensional reduction. We state a proposition which shall be useful for the computations of the normal forms and for the $\mathcal{L}$-versal deformations.
Let $f_0 : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family germ.
We have used here the notation f_0 instead of f to be consistent with the next section.
Fix a quasi-homogeneous filtration of $\mathcal{E}_{q,t}$.
Denote by $T_r f_0$, the reduced $\mathcal{L}$-tangent space to f_0.
Let $\bar{\mathcal{M}}_q \subset \mathcal{E}_{q,t}$ be the $\mathcal{E}_{q,t}$-module of function-germs that vanish at $q_1 = \dots = q_{n+1} = 0$.
Remark that $\bar{\mathcal{M}}_q$ is neither the maximal ideal $\mathcal{M}_{q,t}$ of $\mathcal{E}_{q,t}$ nor the maximal ideal $\mathcal{M}_q$ of $\mathcal{E}_q$.
Denote by V_d the $\mathbb{R}$-vector subspace of $\mathcal{E}_{q,t}$ which is the sum of the following modules:
- the $\mathcal{E}_q$ submodule of $\mathcal{E}_{q,t}$ generated by the $w.f_0$'s where $w \in L_q$ is of order one,
- the $\mathcal{E}_{q,t}$ submodule of $\mathcal{E}_{q,t}$ generated by the $h \times f_0$ where $h \in \mathcal{E}_{q,t}$ is of order one,
- the $\mathcal{E}_q$-submodule $\bar{\mathcal{M}}_q F_d$ of $\mathcal{E}_{q,t}$.

Denote by $\pi_d : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/V_k$ the canonical projection.
The projection π_d is a morphism of $\mathcal{E}_q$-modules and is not a morphism of $\mathcal{E}_{q,t}$-modules.

PROPOSITION 8.3. *Let $f_0 : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family germ.*
Then the inclusion $\pi_d(F_d) \subset \pi_d(T_r f_0)$ implies the inclusion $F_d \subset T_r f_0$.

Example 1. Define a quasi-homogeneous filtration in $\mathcal{E}_{q,t}$ by fixing the following weights:

monomial	t	q_1	q_2	q_3
weight	1	1	3	4

Let $f : (\mathbb{R}^3 \times \mathbb{R}, 0) \longrightarrow \mathbb{R}$ be a generating family germ such that $f = f_0 + f_1$ with

$$f_0(q,t) = t^4 + q_1 t^3 + q_2 t + q_3,$$

$f_1 \in F_5$.

We assert that f is $\mathcal{L}$-equivalent to f_0.

Proposition 8.2 implies that it suffices to prove that F_5 is contained in the reduced tangent space to f_0.

With the notation of subsection 1.5, it suffices to prove that $\pi_5(F_5) \subset \pi_5(T_r f)$.

It is readily seen that the $\mathbb{R}$-vector space $\pi_5(F_4)$ is of dimension three and that it is generated by the images under π_5 of the three monomials:

$$q_1^2 t^2, q_2 t^2, q_3 t^2.$$

Moreover, by straightforward computations we get the equalities

$$\begin{cases} \pi_5(q_1 \partial_t f_0) & = & \pi_5(3q_1^2 t^2), \\ \pi_5(t^2 \partial_t f_0) & = & \pi_5(-3q_2 t^2), \\ \pi_5(t^3 \partial_t f_0) & = & \pi_5(-4q_3 t^2). \end{cases}$$

Consequently we have the inclusion $\pi_5(F_5) \subset \pi_5(T_r f)$. Assertion is proved.

To compute a $\mathcal{L}$-versal deformation of f_0, we proceed as follows.

Recall that G_d denotes the $\mathbb{R}$-vector space generated by the monomials of weight d.

define the $\mathbb{R}$-vector space W by $W = \oplus_{d=0}^{4} G_d$.

It remains to find a basis for the transversal to the $\mathbb{R}$-vector space $W \cap Tf$ in W. We have:

$$\begin{cases} G_0 & = & < 1 >, \\ G_1 & = & < t, q_1 >, \\ G_2 & = & < t^2, q_1 t, q_1^2 >, \\ G_3 & = & < t^3, q_1 t^2, q_1^2 t, q_1^3, q_2 >, \\ G_4 & = & < t^4, q_1 t^3, q_1^2 t^2, q_1^3 t, q_1^4, q_2 t, q_2 q_1, q_3 >, \end{cases}$$

where $< . >$ denotes the $\mathbb{R}$-vector space generated by the monomials inside the brackets.

Straightforward computations show that all the listed monomials except t^2 are contained in Tf.

Thus the Legendrian deformation $F : (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}, 0) \longrightarrow \mathbb{R}$ of f defined by:

$$F(\lambda, q, t) = f(q,t) + \lambda t^2$$

is $\mathcal{L}$-versal.

The corresponding one-parameter family of wave fronts are depicted in figure 6. This is a reformulation of a result due to Arnold [**Arn5**].

If proposition 8.3 applies then the computation of a $\mathcal{L}$-versal deformation is reduced to a simple problem of linear algebra.

2. Proof of the theorem on the excellent generating families.

2.1. Preliminary normal form. Let $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a generating family-germ of anomaly sequence $(a_0, \dots, a_m)$.
By assumption f is of corank one. Consequently, up to $\mathcal{L}$-stable equivalence, we can assume that $k = 1$ without loss of generality.
We fix coordinates $q = (q_1, \dots, q_{n+1})$ in $\mathbb{R}^{n+1}$.
The function germ f admits a representation of the following type

$$f(q,t) = f_0(q,t) + f_1(q,t) + f_2(q,t)$$

where $f_0,\ f_1,\ f_2$ satisfy the 5 conditions cited below.
1) f_0 is of degree at most 1 in $q_1, \dots, q_{n+1}$,
2) $f_1(0,t) = \partial_{q_1} f_1(0,t) = \cdots = \partial_{q_{n+1}} f_1(0,t) = 0$,
3) the anomaly sequences of f_0 and f are equal.
4) see below.
5) see below.
By assumption, one can choose coordinates in $J^0(\mathbb{R}^n, \mathbb{R})$ such that the generating family-germ f is given by:

$$f(q,t) = t^{\alpha_{n+1}} + q_1 t^{\alpha_n} + q_2 t^{\alpha_{n-1}} + \cdots + q_n t^{\alpha_1} + q_{n+1} + f_1(q,t) + f_2(q,t). \tag{41}$$

with $\alpha_{n-s} - (n-s) = a_{s+1}$ if $-1 \le s < m$, and $\alpha_{n-s} = (n-s)$ if $s \ge m$.
We introduce in $\mathcal{E}_{q,t}$ a quasi-homogeneous filtration by specifying the weights of $q_1, \dots, q_n, t$ as follows:
- the weight of t is equal to 1,
- for $i < n+1$, the weight of q_i is equal to $\alpha_n - \alpha_{n-i}$,
- the weight of q_{n+1} is α_n.
Then condition 4 is that $f_2 \in F_{\alpha_{n+1}+1}$ and for any monomial $e \in \mathcal{E}_{q,t}$ appearing with a non-zero coefficient in the Taylor series of f_1, the weight of e is at most equal α_{n+1}.

The function-germ f_2 can be written in the form:

$$f_2(q,t) = c_0(q) + c_1(q)t + c_2(q)t^2 + \cdots + c_{\alpha_n}(q)t^{\alpha_n} + c_{\alpha_{n+1}}(q)t^{\alpha_{n+1}},$$

where the c_i's are arbitrary function germs in $\mathcal{E}_{q,t}$.
By a change of variables in the space $\mathbb{R}^{n+1}$ of the parameter q, we can assume, without loss of generality, that $c_{\alpha_i} = 0$ for all $i \in \{0, \alpha_1, \alpha_2, \dots, \alpha_n\}$.
Moreover, we can also assume that $c_{\alpha_{n+1}} = 0$, just multiply f by $1 - c_{\alpha_{n+1}}(q)$.
Condition 5 is that f_2 can be written in the form described above with $c_{\alpha_1}, \dots, c_{\alpha_{n+1}} = 0$.

2.2. Notations.
For the computations of the normal forms, we use the same notations than those of subsection 1.5.
At the beginning of each computation, we fix a number s and write π instead

of π_s. We also denote by $\widetilde{F}_d$, $\widetilde{G}_d$ the images of the $\mathbb{R}$-vector spaces F_d and G_d under π_s.

2.3. Pairs which are not excellent.

$(n, \square\square\square)$ **is not excellent provided that** $n > 1$

We take $s = n + 5$.

The weights of the quasi-homogeneous filtration are defined by the table

monomial	t	q_1	q_2	$\dots$	q_{n+1}
weight	1	4	5	$\dots$	$n+4$

The elements f_0, f_1 of the prenormal form are given by

$$f_0(q,t) = t^{n+4} + q_1 t^n + q_2 t^{n-1} + \cdots + q_n t + q_{n+1}.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_{n+5}.

By straightforward computations, we get that for $n > 1$,the images of the following 6 monomials under the canonical projection $F_{n+5} \longrightarrow \widetilde{F}_{n+5}/\widetilde{F}_{n+8}$ form a basis of the $\mathbb{R}$-vector space $\widetilde{F}_{n+5}/\widetilde{F}_{n+8}$:

$q_1 t^{n+1}$	$q_1 t^{n+2}$	$q_1 t^{n+3}$
	$q_2 t^{n+1}$	$q_2 t^{n+2}$
		$q_3 t^{n+1}$

Thus the $\mathbb{R}$-vector space $\widetilde{F}_{n+5}/\widetilde{F}_{n+8}$ is of dimension 6.

By elementary linear algebra, we get that the space $\pi(Tf \cap F_{n+5})/\tilde{F}_{n+8}$ is generated by the images of the following 5 vectors:

$t\partial_t f, t^2\partial_t f$	$t^3\partial_t f$	$t^4\partial_t f$
		$q_1\partial_t f$

Thus the $\mathbb{R}$-vector space $\pi(Tf \cap F_{n+5})/\widetilde{F}_{n+8}$ is at most of dimension 5. Consequently the codimension of $Tf \cap F_{n+5}$ in F_{n+5} is at least equal to $6-5=1$. The codimension of $Tf \cap F_{n+5}$ in F_{n+5} is at least equal to 1.

Next, straightforward computations show that the monomials t^{n+2}, t^{n+1} are not contained in Tf.

Thus the codimension of Tf in $\mathcal{E}_{q,t}$ is at least equal to three. Hence $(n, \square\square\square)$ is not excellent provided that $n > 1$.

$(1, \square\square\square\square)$ **is not excellent.**

We take $s = 10$.

The weights of the quasi-homogeneous filtration are defined by the table

monomial	t	q_1	q_2
weight	1	5	6

The elements f_0, f_1 of the prenormal form are given by

$$f_0(q,t) = t^6 + q_1 t + q_2.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_7.

By straightforward computations, we get that the images of the 5 following monomials under the canonical projection $F_7 \longrightarrow \widetilde{F}_7/\widetilde{F}_{10}$ form a basis of the $\mathbb{R}$-vector space $\widetilde{F}_7/\widetilde{F}_{10}$:

q_1t^2	q_1t^3	q_1t^4
	q_2t^2	q_2t^3

Thus the $\mathbb{R}$-vector space $\widetilde{F}_7/\widetilde{F}_{10}$ is of dimension 5.
By elementary linear algebra, we get that the $\mathbb{R}$-vector space $\pi(Tf \cap F_7)/\widetilde{F}_{10}$ is generated by the images of the following 4 function germs:

$t\partial_t f, t^2\partial_t f$	$t^3\partial_t f$	$t^4\partial_t f$

Thus the $\mathbb{R}$-vector space $\pi(Tf \cap F_7)/\widetilde{F}_{10}$ is of dimension at most 4.
Consequently $Tf \cap F_7$ is a least of codimension $5 - 4 = 1$ in F_7.
Moreover, straightforward computations show that none of the monomials t^4, t^3, t^2 is contained in Tf.
Thus Tf is at least of codimension 4 in $\mathcal{E}_{q,t}$. Hence the pair $(1, \square\square\square\square)$ is not excellent.

$(n, \begin{smallmatrix}\square & \square\\ \square & \square\end{smallmatrix})$ is not excellent.

The weights of the quasi-homogeneous filtration are defined by the table

monomial	t	q_1	q_2	q_3	$\dots$	q_{n+1}
weight	1	1	4	5	$\dots$	$n+3$

The elements $f_0,\ f_1,\ f_2$ of the prenormal form are given by

$$\begin{cases} f_0(q,t) &= t^{n+3} + q_1t^{n+2} + q_2t^{n-1} + \cdots + q_nt + q_{n+1}, \\ f_1(q,t) &= \alpha_1 q_1^2 t^{n+1} + \alpha_2 q_1^3 t^n + \alpha_3 q_1^2 t^n, \end{cases}$$

where the α_i's are arbitrary constants and f_2 is an arbitrary function-germ in F_{n+4}.
Denote by W the $\mathbb{R}$ vector subspace of $\mathcal{E}_{q,t}/F_{n+4}$ generated by the images of the following 9 monomials under the canonical projection $\pi : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/F_{n+4}$:

$$t^n,\ q_1t^n,\ q_1^2t^n,\ q_1^3t^n,\ t^{n+1},\ q_1t^{n+1},\ q_1^2t^{n+1}, t^{n+2}, q_1t^{n+2}.$$

By elementary linear algebra, we get that the $\mathbb{R}$-vector space $W \cap \pi(Tf)$ is generated by the images under π of the 5 function-germs:

$$\partial_t f,\ t\partial_t f,\ q_1\partial_t f,\ \partial_{q_1} f,\ q_1\partial_{q_1} f.$$

Thus, the $\mathbb{R}$-vector space Tf is of codimension at least 4 in $\mathcal{E}_{q,t}$. Consequently the pair $(n, \begin{smallmatrix}\square&\square\\ \square&\square\end{smallmatrix})$ is not excellent .

$(n, \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix})$ is not excellent.
The weights of the quasi-homogeneous filtration are defined by the table

monomial	t	q_1	q_2	q_3	$\dots$	q_{n+1}
weight	1	1	2	4	$\dots$	$n+2$

The elements $f_0,\ f_1,\ f_2$ of the prenormal form are given by

$$\begin{cases} f_0(q,t) &= t^{n+2} + q_1t^{n+1} + q_2t^n + q_3t^{n-2} + \cdots + q_nt + q_{n+1}, \\ f_1(q,t) &= \alpha_1 q_1^3 t^{n-1} + \alpha_2 q_1 q_2 t^{n-1}, \end{cases}$$

where the α_i's are arbitrary constants and f_2 is an arbitrary function-germ in F_{n+3}.
Denote by W the vector subspace of $\mathcal{E}_{q,t}/F_{n+3}$ generated by the images of the following 11 monomials under the canonical projection $\pi : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/F_{n+3}$

$$\begin{gathered} t^{n-1},\ q_1t^{n-1},\ q_1^2t^{n-1},\ q_1^3t^{n-1},\ q_1q_2t^{n-1} \\ t^n,\ q_1t^n,\ q_1^2t^n,\ q_2t^n, \\ t^{n+1},\ q_1t^{n+1}. \end{gathered}$$

By elementary linear algebra, we get that $\pi(Tf) \cap V$ is generated by the images under π of the following 8 function-germs:

$$\begin{gathered} \partial_t f, t\partial_t f, q_1\partial_t f, \\ \partial_{q_1} f,\ q_1\partial_{q_1} f,\ \partial_{q_2} f,\ q_1\partial_{q_2} f,\ q_2\partial_{q_2} f. \end{gathered}$$

Thus, the $\mathbb{R}$-vector space Tf is at least of codimension 3 in $\mathcal{E}_{q,t}$. Consequently, the pair $(n, \begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix})$ is not excellent .

2.4. Excellent pairs for $n = 1$.

$(1, \square\square\square)$ is excellent.
We take $s = 8$.
The weights of the quasi-homogeneous filtration are given by:

monomial	t	q_1	q_2
weight	1	4	5

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q,t) = t^5 + q_1t + q_2.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_6.

Lemma 8.1. *We have the inclusion* $F_6 \subset (T_r f_0 \oplus < t^7 >)$.

Proof.
By straightforward computations, we get that the vector space $G_6 \oplus G_7$ is generated by the 6 following monomials

$$q_1 t^2,\ q_1 t^3,\ q_2 t^2,\ t^6,\ t^7,\ q_1 t.$$

The images of the following 6 function-germs under p belong to $G_6 \oplus G_7$

$$t^2 \partial_t f_0,\ t^3 \partial_t f_0,\ t f_0,\ t^2 f_0,\ q_2 \partial_{q_1} f_0,\ t^7$$

Straightforward computations show that these 6 elements of $G_6 \oplus G_7$ are independent. Consequently, the $\mathbb{R}$-vector space $G_6 \oplus G_7$ is contained in $T_r f$.

It remains to prove that $F_8 \subset T_r f_0$.
Proposition 8.3 implies that it suffices to prove the inclusion

$$\widetilde{F}_8 \subset \pi(T_r f_0).$$

Or equivalently that the following property holds:
- any monomial of the type $a(q)t^k$ with $k < 8$ is contained in $T_r f_0$.
We have:
- $a(q)\partial_{q_2} f_0 = a(q)$,
- $a(q)\partial_{q_1} f_0 = a(q)t$,
thus the property holds for $k = 0$ and $k = 1$.
For $k = 2$, we have $a(q)(t^2(\partial_t f_0) - 5tf_0) = -4q_1 t^2 - q_2 t$. Consequently $a(q)t^2 \in T_r f$.
For $k = 3$, we have $a(q)(t^3(\partial_t f_0) - 5t^2 f_0) = -4q_1 t^3 - q_2 t^2$. Consequently $a(q)t^3 \in T_r f_0$.
By induction, we get that $a(q)t^k \in T_r f_0$ for any $k \in \mathbb{N}$. This concludes the proof of the lemma.

This lemma implies the $\mathcal{L}$-equivalence (cf. proposition 8.2):

$$f \sim f_0 + at^7$$

for some value $a \in \mathbb{R}$.

Lemma 8.2. *If $a \neq 0$ then the following $\mathcal{L}$-equivalence holds*

$$f_0 + at^7 \sim f_0 + \epsilon t^7,$$

with $\epsilon = 1$ if $a > 0$ and $\epsilon = -1$ if $a < 0$.

Proof.
Define the function-germ g_a by the formula

$$g_a(q_1, q_2, t) = f_0(q_1, q_2, t) + at^7.$$

For any $\lambda > 0$, we have:

$$\frac{1}{\lambda^5} g_a(\lambda^4 q_1, \lambda^5 q_2, \lambda t) = f_0(q, t) + a\lambda^2 t^7.$$

We take $\lambda = \sqrt{a}$ for $a > 0$ and $\lambda = \sqrt{-a}$ for $a < 0$, we get the $\mathcal{L}$-equivalence

$$f_0 + at^7 \sim f_0 \pm t^7,$$

where the sign is the sign of a. Lemma is proved.

Put $W = \bigoplus_{d=0}^{5} G_d$.
By elementary linear algebra, we get that the monomials t^3 and t^2 form a basis of a transversal to $(T_r f_0 \cap W)$ in W.
Thus, the deformation

$$F(\lambda, q, t) = f_0(q, t) + \lambda_1 t^3 + \lambda_2 t^2 + \lambda_3 t^7$$

is $\mathcal{L}$-versal.
Next, define the function-germs $g_\pm : (\mathbb{R}^2 \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ by the formula

$$g_\pm(q, t) = f(q, t) \pm t^7.$$

The equality

$$t\partial_t g_\pm(q, t) + 3q_1\partial_{q_1} g_\pm(q, t) + 5q_2\partial_{q_2} g_\pm(q, t) - 5g_\pm(q, t) = \pm 5t^7$$

implies that the monomial t^7 belongs to the $\mathcal{L}$-tangent space to $g_{(\mathbb{C}^{n-1},0)}$.
Consequently the deformation $G : (\mathbb{R}^2 \times \mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0)$ of $g_\pm$ defined by the polynomial

$$G(\lambda, q, t) = g_\pm(q, t) + \lambda_1 t^3 + \lambda_2 t^2$$

is $\mathcal{L}$-versal. Thus the pair $(n, \square\square\square)$ is excellent .

$(1, \begin{smallmatrix}\square&\square&\square\\ \square&&\end{smallmatrix})$ is excellent
We take $s = 7$.
The weights of the quasi-homogeneous filtration are given by:

monomial	t	q_1	q_2
weight	1	3	5

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q, t) = t^5 + q_1 t^2 + q_2.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_6.

Lemma 8.3. *We have the inclusion $F_6 \subset (T_r f_0 \oplus < t^6 >)$.*

Proof.
By straightforward computations, we get that the vector space G_6 is generated by the images of the following monomials:

$$t^6,\ q_1 t^3,\ q_2 t,\ q_1^2.$$

By straightforward computations, we get that the 4 function-germs

$$tf_0,\ t^2\partial_t f_0,\ q_1^2\partial_{q_2} f_0,\ t^6$$

are independent and generate G_6.

It remains to prove that $F_7 \subset T_r f_0$.
Proposition 8.3 implies that it suffices to prove the inclusion

$$\widetilde{F}_7 \subset \pi(T_r f_0).$$

Straightforward computations show that the images of the monomials of he following table generate $\widetilde{F}_7$

monomials	$q_1^2 t,\ q_1 t^4$	$q_2 t^3$	$q_1 q_2 t,\ q_1^2 t^3,\ q_2 t^4$	$q_2^2 t$
weight	7	8	9	11

By elementary linear algebra we get that the images under π of the 7 following function-germs

$$t^3 \partial_t f_0, q_1 \partial_t f_0,\ t^4 \partial_t f_0,\ t^5 \partial_t f_0,\ q_1 t^2 \partial_t f_0,\ q_2 \partial_t f_0,\ t^7 \partial_t f_0$$

are independent and consequently generate $\widetilde{F}_7$. This concludes the proof of the lemma.

The lemma implies the $\mathcal{L}$-equivalence:

$$f \sim f_0 + at^6$$

for some value $a \in \mathbb{R}$ (cf proposition 8.2).

The proof of the following lemma is analogous to the proof of lemma 8.2

LEMMA 8.4. *If $a \neq 0$ then the following $\mathcal{L}$-equivalence holds*

$$f_0 + at^6 \sim f_0 + t^6.$$

Like in the preceding case, by elementary linear algebra, we get that the deformations:

$$F(\lambda, q, t) = t^5 + q_1 t^2 + q_2 + \lambda_1 t^6 + \lambda_1 t^4 + \lambda_2 t^3 + \lambda_3 t$$

and

$$G(\lambda, q, t) = t^6 + t^5 + q_1 t^2 + q_2 + \lambda_1 t^4 + \lambda_2 t^3 + \lambda_3 t$$

are $\mathcal{L}$-versal.
Consequently, the pair $(1, \begin{smallmatrix}\square & \square & \square \\ \square & & \end{smallmatrix})$ is excellent.

2.5. Excellent pairs for $n > 1$.
$(n, \square)$ is excellent.
We take $s = n + 3$.
The weights of the quasi-homogeneous filtration are given by the table:

monomial	t	q_1	q_2	$\dots$	q_{n+1}
weight	1	2	3	$\dots$	$n+2$

.

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q, t) = t^{n+2} + q_1 t^n + q_2 t^{n-1} + \cdots + q_n t + q_{n+1}.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_{n+3}.

The monomials generating $\widetilde{F}_{n+3}$ are the n monomials

$$q_1 t^{n+1},\ q_2 t^{n+1}, \dots,\ q_{n+1} t^{n+1}.$$

This n dimensional $\mathbb{R}$-vector space is generated by the images under π of the function-germs

$$q_1 \partial_t f, \dots, q_{n+1} \partial_t f.$$

Consequently f is $\mathcal{L}$-equivalent to f_0.

Obviously, the tangent space to f_0 contains the $\mathbb{R}$-vector space $\bigoplus_{k=1}^{n+2} G_k$. Thus, the constant deformation $F = f$ is $\mathcal{L}$-versal.

$(n, \square\square)$ **is excellent.**
We take $s = n + 4$.
The weights of the quasi-homogeneous filtration are given by the table:

monomial	t	q_1	q_2	$\dots$	q_{n+1}
weight	1	3	4	$\dots$	$n+3$

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q,t) = t^{n+3} + q_1 t^n + q_2 t^{n-1} + \dots + q_n t + q_{n+1}.$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_{n+4}.

By elementary linear algebra, we get that the images under π of the following monomials form a basis $\widetilde{F}_{n+2}$:

$$\begin{cases} q_1 t^{n+1},\ q_2 t^{n+1}, \dots,\ q_n t^{n+1}, \\ q_1 t^{n+2},\ q_2 t^{n+2}, \dots,\ q_{n+1} t^{n+2}. \end{cases}$$

Remark the monomial $q_{n+1} t^{n+1}$ is missing because of the equality:

$$\pi(q_{n+1} t^{n+1}) = -\pi(q_{n+1} t^{n+2}).$$

It is readily verified that the images of the function-germs

$$\begin{cases} t^2 \partial_t f, \dots, t^{n+1} \partial_t f, \\ q_1 \partial_t f, \dots, q_{n+1} \partial_t f \end{cases}$$

under π are independent. Hence they form a basis of $\widetilde{F}_{n+2}$.
By elementary linear algebra, we get that the deformation:

$$F(\lambda, q, t) = f(q,t) + \lambda t^{n+1}$$

is $\mathcal{L}$-versal. Thus the pair $(n, \square\square)$ is excellent.

$(n, \genfrac{}{}{0pt}{}{\square}{\square})$ **is excellent.**
We take $s = n + 3$.
The weights of the quasi-homogeneous filtration are given by the table:

monomial	t	q_1	q_2	$\dots$	q_{n+1}
weight	1	1	3	$\dots$	$n+2$

.

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q,t) = t^{n+2} + q_1 t^{n+1} + q_2 t^{n-1} + \cdots + q_n t + q_{n+1}.$$

$f_1 = \alpha q_1^2 t^n$ with $\alpha \in \mathbb{R}$ and f_2 is an arbitrary function-germ in F_{n+4}.

LEMMA 8.5. *The function-germ f is Legendre equivalent to a function germ of the type $f_0 + f_3$ with $f_3 \in F_{n+4}$.*

Proof.
For any value of λ, the function-germ f is $\mathcal{L}$-equivalent to the function-germ f_λ defined by:

$$f_\lambda(q,t) = f(q, t + \lambda q_1).$$

It is readily verified that f_λ is $\mathcal{L}$-equivalent to a function-germ of the type

$$t^{n+2} + q_1 t^{n+1} + q_2 t^{n-1} + \cdots + q_n t + q_{n+1} + (a(\lambda) + \alpha) q_1^2 t^n + f_3$$

with $f_3 \in F_{n+4}$ and a is a C^∞ function-germ.
Direct computations show that the Taylor series of a is of the type

$$a(\lambda) = 2\lambda + o(\lambda).$$

Thus, the implicit function theorem implies that the equation $a(\lambda) + \alpha = 0$ can be solved in a neighbourhood of $\lambda = -\frac{1}{2}\alpha$.
This concludes the proof of the lemma.

Straightforward computations show that the $\mathbb{R}$-vector space $\widetilde{F}_{n+3}$ is generated by the images under π of the following monomials:

$$q_1^3 t^n,\ q_2 t^n,\ q_3 t^n, \dots,\ q_{n+1} t^n.$$

The images under π of the function-germs

$$q_1^2 \partial_t f, t^2 \partial_t f, \dots, t^n \partial_t f$$

are independent and consequently generate $\widetilde{F}_{n+3}$.
By elementary linear algebra, we get that the deformation:

$$F(\lambda, q, t) = f(q,t) + \lambda_1 t^n$$

is $\mathcal{L}$-versal. Thus the pair $(n, \begin{smallmatrix}\square\\\square\end{smallmatrix})$ is excellent.

$(n, \begin{smallmatrix}\square & \square\\\square & \end{smallmatrix})$ is excellent.
We take $s = n + 5$.
The weights of the quasi-homogeneous filtration are given by the table:

monomial	t	q_1	q_2	$\dots$	q_{n+1}
weight	1	2	4	$\dots$	$n+3$

.

The elements $f_0,\ f_1$ of the prenormal form are given by

$$f_0(q,t) = t^{n+3} + q_1 t^{n+1} + q_2 t^{n-1} + \cdots + q_n t + q_{n+1},$$

$f_1 = 0$ and f_2 is an arbitrary function-germ in F_{n+4}.

LEMMA 8.6. *If f is an excellent generating family-germ then $F_{n+4} \subset Tf$.*

Proof.
Consider the canonical projection $p : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/F_{n+3}$.
By elementary linear algebra, we get that the images under p of the monomials t^{n+2} and t^n generate a transversal to $p(Tf)$.
Consequently, the monomials t^{n+2} and t^n are contained in a basis of a transversal to Tf in $\mathcal{E}_{q,t}$.
Thus, if f is excellent then they should form a basis of a transversal to Tf.
This proves the lemma.

LEMMA 8.7. *The function-germ f is $\mathcal{L}$-equivalent to a function-germ of the type $f_0 + at^{n+4} + f_3$ with $f_3 \in F_{n+5}$.*

Proof.
The function-germ f admits a representation of the form

$$f = f_0 + g_1 + g_2$$

with $g_1 \in G_{n+4}$ and $g_2 \in F_{n+5}$.
By straightforward computations, we get that the $\mathbb{R}$-vector space G_{n+4} is contained in $T_r f_0 \oplus < t^{n+4} >$.
Consequently, the function-germ g_1 admits a representation of the form

$$-g_1 = v.f_0 + w.f_0 + h \times f_0 - at^{n+4}$$

for some $a \in \mathbb{R}$.
Here $v \in L_{q,t}$, $w \in L_q$ and $h \in F_1$ are of order one.
We have

$$(1 + h) \times f(q + w(q), t + v(q,t)) = f + v.f + w.f + h.f + f_3$$

with $f_3 \in F_{n+5}$. This proves the lemma.

The following lemma is straightforward.

LEMMA 8.8. *If the number a of the preceding lemma vanishes then t^{n+4} is not contained in Tf.*

This lemma together with lemma 8.6 implies that if f is excellent then f is $\mathcal{L}$-equivalent to a function-germ of the type $f_0 + at^{n+4} + f_3$ with $a \neq 0$ and $f_3 \in F_{n+5}$.
The proof of the following lemma is analogous to the proof of lemma 8.2.

LEMMA 8.9. *Any function germ of the type $f_0 + at^{n+4} + f_3$ with $a \neq 0$ and $f_3 \in F_{n+5}$ is $\mathcal{L}$-equivalent to $f_0 + t^{n+4} + f_4$ with $f_4 \in F_{n+5}$.*

Put $\tilde{f} = f_0 + t^{n+4}$.

LEMMA 8.10. *The $\mathbb{R}$-vector subspace F_{n+5} is contained in $T_r\tilde{f}$.*

Proof.
Consider the canonical projection of $\mathbb{R}$-vector spaces

$$p_1 : F_{n+5} \longrightarrow (\widetilde{F}_{n+5}/\widetilde{F}_{n+6}).$$

Straightforward computations show that the vector space $\widetilde{F}_{n+5}/\widetilde{F}_{n+6}$ is of dimension one and generated by the image of the monomial q_3t^n under p_1. The image under p_1 of $t^3\partial_t\tilde{f}$ does not vanish. Consequently $\widetilde{F}_{n+5}/\widetilde{F}_{n+6}$ is in the image of $T_r\tilde{f}$ un der p_1.
Consider the canonical projection of $\mathbb{R}$-vector spaces

$$p_2 : F_{n+6} \longrightarrow (\widetilde{F}_{n+6}/\widetilde{F}_{n+7}).$$

Straightforward computations the images under p_2 of the 5 monomials

$$q_4t^n,\ q_1^3t^n,\ q_2t^{n+2},\ q_1q_2t^n,\ q_1t^{n+4}$$

form a basis of the vector space $\widetilde{F}_{n+6}/\widetilde{F}_{n+7}$. Denote this basis by $\mathcal{B}$
Denote by e (e for Euler) the vector field defined by

$$e = t\partial_t + 2q_1\partial_{q_1} + 4q_2\partial_{q_2} + 5q_3\partial_{q_3} + \cdots + (n+2)q_n\partial_{q_n} + (n+3)q_{n+1}\partial_{q_{n+1}}.$$

Remark that the following equality holds:

$$e.\tilde{f} - (n+3)\tilde{f} = t^{n+4}.$$

We assert that the images under p_2 of the following vectors are independent

$$t^4\partial_t\tilde{f},\ q_1^2\partial_t\tilde{f},\ q_2\partial_t\tilde{f},\ q_1^2\partial_t\tilde{f},\ (q_1e).\tilde{f}.$$

To see it just notice that the coordinates of these vectors expressed in the basis $\mathcal{B}$ is triangular.
Consequently $\widetilde{F}_{n+6}/\widetilde{F}_{n+7}$ is in the image of $T_r\tilde{f}$ under p_2.
Consider the canonical projection of $\mathbb{R}$-vector spaces

$$p_3 : F_{n+7} \longrightarrow (\widetilde{F}_{n+7}/\widetilde{F}_{n+8}).$$

Straightforward computations show that:
1) the images under p_3 of the 2 monomials

$$q_5t^n,\ q_3t^{n+2}$$

form a basis of the vector space $\widetilde{F}_{n+7}/\widetilde{F}_{n+8}$,
2) the images under p_3 of the following vectors are independent

$$t^5\partial_t\tilde{f},\ q_3\partial_t\tilde{f}.$$

Consequently $\widetilde{F}_{n+7}/\widetilde{F}_{n+8}$ is in the image of $T_r\tilde{f}$ under p_3.
More generally for any $n-1 \geq s \geq 3$, consider the canonical projection of $\mathbb{R}$-vector spaces

$$p_j : F_{n+4+j} \longrightarrow (\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}).$$

Straightforward computations show that:
1) the images under p_j of the 2 monomials

$$q_{j+2}t^n,\ q_jt^{n+2}$$

form a basis of the vector space $\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}$,
2) the images under p_j of the following vectors generate $\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}$

$$t^{j+2}\partial_t\tilde{f},\ q_j\partial_t\tilde{f}.$$

Consequently $\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}$ is in the image of $T_r\tilde{f}$ under p_j $n-1 \leq j \geq 3$.
For $j = n$ or $j = n+1$, straightforward computations show that:
1) the image under p_j of the monomial q_jt^{n+2} form a basis of the vector space $\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}$, 2) the images under p_j of $q_j\partial_t\tilde{f}$ generates $\widetilde{F}_{n+4+j}/\widetilde{F}_{n+5+j}$. It is readily verified that $F_{n+4+j} = \{0\}$ for $j = n+2$. This concludes the proof of the lemma.

Lemma 8.10 together with proposition 8.2 implies the $\mathcal{L}$-equivalence

$$f \sim \tilde{f}$$

provided that the number a of lemma 8.7 is not equal to zero.
Using the inclusion $F_{n+5} \subset T_rf_0$, we get by elementary linear algebra the equality

$$(T\tilde{f} \oplus < t^{n+2}, t^n >) = \mathcal{E}_{q,t}.$$

Consequently the deformation F of $\tilde{f}$ defined by

$$F(\lambda, q, t) = \tilde{f}(q,t) + \lambda_1 t^{n+2} + \lambda_2 t^n.$$

is $\mathcal{L}$-versal and the pair $(n, {\square \atop \square}\square)$ is excellent.
This concludes the proof of theorem 7.5

3. Proofs of the theorems on normal forms.

3.1. Proof of the $\mathcal{L}$-versal deformation theorem. Following R. Thom, we use the homotopy method. Following Martinet [**Mar**], the fundamental step in the proof of a versal deformation theorem is the following proposition.

PROPOSITION 8.4. *Let $F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be an l-parameter deformation of a generating family-germ $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ such that $TF = \mathcal{E}_{q,t}$.*
Then, for any $(l+1)$-parametric deformation

$$\Phi : (\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$$

of f such that $\Phi(0, .) = F$, the Legendrian deformation Φ of f is $\mathcal{L}$-equivalent to a deformation induced from F.

Proof.
We fix coordinate-systems τ in $\mathbb{R}$, $\lambda = (\lambda_1, \dots, \lambda_l)$ in $\mathbb{R}^l$, $q = (q_1, \dots, q_{n+1})$ in $\mathbb{R}^{n+1}$, $t = (t_1, \dots, t_k)$ in $\mathbb{R}^k$.
Denote by $Diff(k)$ the group of diffeomorphism map germ of the type

$$\varphi : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0),$$

and by $T \approx \mathbb{R}^k$ the space of translations of $\mathbb{R}^k$.
Recall that the group $\mathcal{K}$ is a semi-direct product of $\mathcal{E}_{q,t}$ with $Diff(k)$.
Denote by e the identity element of the direct sum $\mathcal{K} \oplus T$ of the group $\mathcal{K}$ with T.
We search for a map-germ $\gamma : (\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathcal{K} \oplus T, e)$ and for a map-germ $h : (\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0)$ such that:

$$\Phi = h^*(\gamma.F), \tag{42}$$

- $h(\tau, ., .)$ is fibered over $\mathbb{R}^l$,
- $h(0, ., .)$ is the identity mapping.

We shall use the notations

$$\gamma_\tau = \gamma(\tau, .),\ \ h_\tau = h(\tau, .).$$

The equality $\gamma_0 = e$ implies that there exists a map-germ

$$\tilde{\gamma} : (\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathcal{K} \oplus T, e)$$

such that the following equality holds identically

$$\gamma(\tau, \lambda, q).\tilde{\gamma}(\tau, \lambda, q) = e.$$

We denote the map-germ $\tilde{\gamma}(\tau, ., .)$ by γ_τ^{-1}.
Similarly the equality $h_0 = Id$ implies that there exists a map germ

$$\tilde{h} : (\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^l \times \mathbb{R}^{n+1}, 0)$$

such that the following equality holds identically

$$\tilde{h}(\tau, \lambda, q)h(\tau, \lambda, q) = (\lambda, q).$$

We denote the map-germ $\tilde{h}(\tau, ., .)$ by h_τ^{-1}.
Having fixed this notations, equation 42 is then equivalent to:

$$(h_\tau^{-1})^*(\gamma_\tau^{-1}.\Phi) = F, \tag{43}$$

Equation 43 can be written in the less compact but more explicit form:

$$A_\tau(\lambda, q, t)\Phi(h_\tau^{-1}(\lambda, q), \psi_\tau(\lambda, q, t)) = F(\lambda, q, t),$$

where $A_\tau \in \mathcal{E}_{\lambda,q,t}$ is such that $A(0) \neq 0$.
The components of γ_τ^{-1} are given by

$$\gamma_\tau^{-1} = (A_\tau, \psi_\tau)$$

Before differentiating with respect to τ equation 43, we fix some notations:
- $v_\tau.\Phi = \gamma_\tau.\frac{d}{d\tau}(\gamma_\tau^{-1}.\Phi)$,
- $w_\tau.\Phi = h_\tau^*\frac{d}{d\tau}((h_\tau^{-1})^*\Phi)$.

We differentiate with respect to τ equation 43 and multiply on the left the result by $\gamma_\tau h_\tau^*$. We get the *homological equation*:

$$v_\tau.\Phi + w_\tau.\Phi + \partial_\tau\Phi = 0. \tag{44}$$

Equation 44 can be written in the less compact form:

$$B \times \Phi + \sum_{i=1}^{k} a_i \partial_{t_i} \Phi + \sum_{i=1}^{n+1} b_i \partial_{q_i} \Phi + \sum_{j=1}^{l} c_j \partial_{\lambda_j} \Phi = -\partial_\tau \Phi. \tag{45}$$

Here $B \in \mathcal{E}_{\tau,\lambda,q,t}$, $a_1, \dots, a_k \in \mathcal{E}_{\tau,\lambda,q,t}$, $b_1, \dots, b_{n+1} \in \mathcal{E}_{\tau,\lambda,q}$, $c_1, \dots, c_l \in \mathcal{E}_{\tau,\lambda}$.
Define the vector field-germs v and w by

$$v_\tau(\lambda, q, t) = v(\tau, \lambda, q, t), \ w_\tau(\lambda, q) = w(\tau, \lambda, q).$$

The function-germs $a_i(\tau, ., ., .)$, $b_i(\tau, ., .)$, $c_i(\tau, .)$ vanish at the origin. Thus we can integrate v and w along τ.
Consequently, the ordinary theorem for non-autonomous differential equations implies that it is sufficient to find the maps v, w in order to find γ, h.

We interpret equation 45 as follows.
Consider the following $\mathbb{R}$-vector subspaces of $\mathcal{E}_{\tau,\lambda,q,t}$:
- the $\mathcal{E}_{\tau,\lambda,q,t}$-module generated by the $\partial_{t_i}\Phi$'s and by Φ,
- the $\mathcal{E}_{\tau,\lambda,q}$-module generated by the $\partial_{q_i}\Phi$'s,
- the $\mathcal{E}_{\tau,\lambda}$-module generated by the $\partial_{\lambda_i}\Phi$'s.

Main assertion:" the equality $TF = \mathcal{E}_{q,t}$ implies the sum of the three $\mathbb{R}$-vector subspaces of $\mathcal{E}_{\tau,\lambda,q,t}$ described above is equal to $\mathcal{E}_{\tau,\lambda,q,t}$".

Equation 45 can be solved provided that this main assertion is proved.
To prove the main assertion, we use the Weierstrass-Malgrange-Mather ([**Mar**] chapter X) theorem[1].

THEOREM 8.1. *Let M be an $\mathcal{E}_{z,w}$-module of finite type with $z \in \mathbb{R}^u$, $w \in \mathbb{R}^v$. Let $N \subset M$ be an $\mathcal{E}_z$-module of finite type. Denote by $\mathcal{M}_z$ the maximal ideal of $\mathcal{E}_z$. Let $\pi : M \longrightarrow M/(\mathcal{M}_z M)$ be the standard projection. Then $N = M$ provided that $\pi(M) = \pi(N)$.*

Consider the following modules:
- $M_1 = \mathcal{E}_{\tau,\lambda,q,t}/I_1$ where I_1 is the $\mathcal{E}_{\tau,\lambda,q,t}$-module generated by $\partial_{t_1}\Phi, \dots, \partial_{t_k}\Phi, \Phi$,
- $M_2 = M_1/I_2$ where I_2 is the $\mathcal{E}_{\tau,\lambda,q}$-module generated by $\partial_{q_1}\Phi, \dots, \partial_{q_{n+1}}\Phi$.

We shall apply the Weierstrass-Malgrange-Mather theorem twice.
First, in order to prove the following lemma.

LEMMA 8.11. *The equality $TF = \mathcal{E}_{q,t}$ implies that the $\mathcal{E}_{\tau,\lambda,q}$-module M_2 is of finite type.*

Proof.
Assertion: "the $\mathcal{E}_{\tau,\lambda,q}$-module M_2 is generated by the images of the functions germs:

$$\partial_{\lambda_1}\Phi, \dots, \ \partial_{\lambda_l}\Phi$$

[1]We state this theorem in the C^∞ case but it is valid for $\mathbb{K}$ analytic and $\mathbb{K}$ formal power series $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

under the canonical projection $\mathcal{E}_{\tau,\lambda,q,t} \longrightarrow M_2$."
This assertion implies the lemma. To prove the assertion consider the $\mathcal{E}_{\tau,\lambda,q}$-submodule N_1 of M_1 generated by the images under the canonical projection $\mathcal{E}_{\tau,\lambda,q,t} \longrightarrow M_1$ of the functions germs:

$$\partial_{q_1}\Phi, \ldots, \partial_{q_{n+1}}\Phi$$

and by the function-germs

$$\partial_{\lambda_1}\Phi, \ldots,\ \partial_{\lambda_l}\Phi.$$

The assertion is equivalent to the equality $N_1 = M_1$.
To prove this equality, we use the Weierstrass-Malgrange-Mather theorem with $z = (\tau, \lambda, q)$ and $w = t$.
Recall that $F : (\mathbb{R}^l \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ is a deformation of the function-germ $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$.
Denote by I_f the $\mathcal{E}_t$-module (=the ideal) generated by the restrictions of $\partial_{t_1} f, \ldots, \partial_{t_k} f, f$ to $q = 0$.
With the notations of the theorem, we get the following canonical identifications:
- $\pi(M_1) = \mathcal{E}_t / I_f$,
- $\pi(N_1)$ is the vector space generated by the images under the canonical projection $\mathcal{E}_t \longrightarrow \mathcal{E}_t / I_{f_0}$ of the restriction of $\partial_{\lambda_1} F, \ldots \partial_{\lambda_l} F$ to $q = 0, \lambda = 0$.
The hypothesis $TF = \mathcal{E}_{q,t}$ implies that $\pi(M_1) = \pi(N_1)$. Hence the assertion is proved and so is the lemma.

We now prove the main assertion.
Recall that Tf is the sum of the $\mathcal{E}_{q,t}$ module generated by the $\partial_{t_i} f$'s and by f with the $\mathcal{E}_q$-module generated by the $\partial_{q_i} f$'s.
Let N_2 be the $\mathcal{E}_{\tau,\lambda}$-module generated by $\partial_{\lambda_1}\Phi, \ldots, \partial_{\lambda_l}\Phi$. Then the main assertion is equivalent to $M_2 = N_2$. We apply the Weierstrass-Malgrange-Mather theorem with $z = (\tau, \lambda),\ w = q$. We get the natural identifications:
- $\pi(M_2) = \mathcal{E}_{q,t} / Tf$.
- $\pi(N_2)$ is the $\mathbb{R}$-vector space generated by the images under the canonical projection $\mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{t,q,t} / Tf$ of the restriction of the function-germs $\partial_{\lambda_1} F, \ldots, \partial_{\lambda_l} F$ to $\lambda = 0$.
With these identification, the equality $\pi(N_2) = \pi(M_2)$ is equivalent to the equality $TF = \mathcal{E}_{q,t}$. This equality holds and the main assertion is proved. This concludes the proof of the proposition.

The rest of the proof of the $\mathcal{L}$-versal deformation theorem is straightforward.
Let $G : (\mathbb{R}^s \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a deformation of $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ satisfying the condition $TG = \mathcal{E}_{q,t}$.
Let $\tilde{G} : (\mathbb{R}^{s'} \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be an arbitrary deformation of f.
Define the sum

$$G \oplus \tilde{G} : (\mathbb{R}^{s'} \times \mathbb{R}^s \times \mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$$

of the deformations G and $\tilde{G}$ by the formula

$$(G \oplus \tilde{G})(\mu, \lambda, q, t) = G(\lambda, q, t) + \tilde{G}(\mu, q, t) - f(q, t).$$

Fix coordinates $(\lambda_1, \dots, \lambda_s)$ in $\mathbb{R}^s$, $(\mu_1, \dots, \mu_{s'})$ in $\mathbb{R}^{s'}$.
The restriction of $G \oplus \tilde{G}$ to $\lambda = 0$ is equal to $\tilde{G}$. Hence, $\tilde{G}$ is induced from a deformation equivalent to G provided that $\tilde{G} \oplus G$ is induced from a deformation equivalent to G.

LEMMA 8.12. *The deformation $G \oplus \tilde{G}$ is induced from a deformation equivalent to G.*

Proof.
Denote by A_i the restriction of $G \oplus \tilde{G}$ to the vector space $\mu_1, \dots, \mu_i = 0$.
We apply proposition 8.4 with $\Phi = A_{s+s'-1}$, $F = G$. We get that $A_{s+s'-1}$ is induced from a deformation equivalent to G. In particular the $\mathcal{L}$-tangent space to $A_{s+s'-1}$ is equal to $\mathcal{E}_{q,t}$.
Next, we apply Proposition 8.4 with $\Phi = A_{s+s'-2}$ and $F = A_{s+s'-1}$. We get that $A_{s+s'-2}$ is induced from a deformation equivalent to $A_{s+s'-1}$. Hence $A_{s+s'-2}$ is induced from a deformation equivalent to G. By induction we get that $G \oplus \tilde{G}$ is induced from a deformation equivalent to G. Lemma is proved.

This lemma concludes the proof of the $\mathcal{L}$-versal deformation theorem.

3.2. Proof of the finite determinacy theorem for $\mathcal{L}$-equivalence. The proof is along the same lines than that of the standard finite determinacy theorem ([**Math**], [**Mar**], [**AVG**]), see also appendix B of the thesis.
Consider a function-germ $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$.
Fix coordinates $q = (q_1, \dots, q_{n+1})$ in $\mathbb{R}^{n+1}$ and $t = (t_1, \dots, t_k)$ in $\mathbb{R}^k$.
Let $\mathcal{M}_{q,t}$ be the maximal ideal of $\mathcal{E}_{q,t}$ and $\mathcal{M}^d_{q,t}$ the d^{th} power of it.
Following R. Thom, we use the homotopy method.
That is, we search a one parameter family of function-germs

$$A_\tau : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$$

and a one parameter family of diffeomorphism map-germs

$$\varphi_\tau : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0)$$

such that the following equation holds

$$A_\tau \times (f_\tau \circ \varphi_\tau) = f. \tag{46}$$

where $f_\tau = f + \tau\psi$, $\psi \in \widetilde{\mathcal{M}}^{\mu+1}$.
Moreover the maps A_τ should satisfy $A_\tau(0) \neq 0$ and φ_τ is fibered over $\mathbb{R}^{n+1}$.
This means that the components $(\varphi^1_\tau, \varphi^2_\tau)$ of φ_τ are of the type

$$(q, t) \longrightarrow (\varphi^1_\tau(q), \varphi^2_\tau(q, t)).$$

If the functions A_τ and the maps φ_τ are found then:

$$A_1 \times (f \circ \varphi + \psi \circ \varphi) = f,$$

thus f and $f+\psi$ are $\mathcal{L}$-equivalent.

We differentiate equation (46) with respect to τ at $\tau = u$. We obtain:

$$(47)\quad A_u \times \frac{d}{d\tau}_{|\tau=u}(f_u \circ \varphi_\tau) + \frac{d}{d\tau}_{|\tau=u}(A_\tau \times (f_u \circ \varphi_u)) + A_u \times (\frac{d}{d\tau}_{|\tau=u} f_\tau) \circ \varphi_u = 0.$$

Define the time-dependent vector field germ v_u and the one parameter family $h_u : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow \mathbb{R}$ by the formulas

$$\begin{cases} v_u(\varphi_u(q,t)) & = \frac{d}{d\tau}_{|\tau=u}\varphi_\tau(q,t), \\ (A_u^{-1} \times h_u) \circ \varphi_u & = \frac{d}{d\tau}_{|\tau=u} A_\tau. \end{cases}$$

Multiplying equation (47) on the right by φ_u^{-1} and on the left by A_u^{-1}, we get the equation

$$(48)\qquad v_u.f_u + h_u \times f_u + \psi = 0.$$

For any value of u, the vector field-germ v_u vanishes at $q = t = 0$ since $\varphi_u(0,0) = (0,0)$.
The fundamental theorem for non-autonomous differential equations implies that for any vector-field v_u vanishing at the origin there exists a one parameter-family of diffeomorphism φ_u such that

$$v_u(\varphi_u(q,t)) = \frac{d}{d\tau}_{|\tau=u}\varphi_\tau(q,t).$$

Consequently in order to find the one-parameter family of map-germs φ_u it is sufficient to find the one-parameter family of vector field-germs v_u.
Moreover, if the one parameter family of function-germs h_u is known, then by integrating h_u along u we get the one parameter family of function-germs B_u satisfying the equality

$$B_u = log(A_u).$$

Remark that $log(A_u)$ is well-defined since $A_0(0) \neq 0$. From this equality, we get $A_u = e^{B_u}$.
Consequently in order to solve equation (47), it is sufficient to solve equation (48).

Lemma 8.13. *There exists a one-parameter family of vector field-germs v_u and a one-parameter family of function-germs h_u satisfying equation 48 provided that the following inclusion holds*

$$\mathcal{M}_{q,t}^{\nu} \subset Tf.$$

Proof.
The proof is analogous to the proof of Nakayama's lemma.
Denote by $m_1, \dots, m_l$ the monomials generating the $\mathcal{M}_{q,t}$-module $\mathcal{M}_{q,t}^{\nu}$.
Obviously, the following assertion implies the lemma.
Assertion. The monomials $m_1, \dots, m_l$ belong to the $\mathcal{L}$-tangent space to

$$f_u = f + u\psi$$

and the formula expressing the m_i's in terms of the generators of Tf_u has a C^∞ dependence on u.

The inclusion $\mathcal{M}^{\nu}_{q,t} \subset Tf$ implies that for any $j \in \{1, \dots, l\}$, the m_i's can be written in the following form

$$m_i = a_i \times f + v_i.f + w_i.f \tag{49}$$

with $a_i \in \mathcal{E}_{q,t}$, $v_i \in L_{q,t}$, $w_i \in L_q$.
Rewrite the right hand-side of the equality 49, in the following way
(50)
$$a_i\times f+v_i.f+w_i.f = a_i\times(f+u\psi)+v_i.(f+u\psi)+w_i.(f+u\psi)-a_i\times(u\psi)-v_i.(u\psi)-w_i.(u\psi).$$

Since $\psi \in \mathcal{M}^{\nu+2}_{q,t}$, we have that

$$\begin{cases} v_i.\psi \in \mathcal{M}^{\nu+1}_{q,t}, \\ w_i.\psi \in \mathcal{M}^{\nu+1}_{q,t}, \\ a_i \times \psi \in \mathcal{M}^{\nu+2}_{q,t}. \end{cases}$$

Thus there exists function-germs $\alpha_{i,1}, \dots, \alpha_{i,l}$ in $\mathcal{M}_{q,t}$ such that the following equality holds

$$a_i \times \psi + v_i.\psi + w_i.\psi = \sum_{j=1}^{l} \alpha_{i,j} m_j.$$

Coming back to equation (50) and using the notation $f_u = f + u\psi$, we get a system of l equations:

$$\begin{cases} a_1 \times f_u + v_1.f_u + w_1.f_u = m_1 + u\sum_{j=1}^{l} \alpha_{1,j} m_j, \\ \dots \\ a_i \times f_u + v_i.f_u + w_i.f_u = m_i + u\sum_{j=1}^{l} \alpha_{i,j} m_j, \\ \dots \\ a_l \times f_u + v_l.f_u + w_l.f_u = m_j + u\sum_{j=1}^{l} \alpha_{l,j} m_j. \end{cases}$$

Denote by A the matrix whose coefficients are the $\alpha_{i,j}$'s.
Using the matrix formalism, the system of l equations can be written as

$$\vec{a} f_u + \vec{v} f_u + \vec{w} f_u = (Id + uA)\vec{m}$$

where $\vec{a} = (a_1, \dots, a_l)$, $\vec{v} = (v_1, \dots, v_l)$, $\vec{w} = (w_1, \dots, w_l)$, $\vec{m} = (m_1, \dots, m_l)$ and Id is the identity matrix.
The matrix $(Id + uA)$ is invertible indeed $(Id + A)(0) \neq 0$.
By inversion of the matrix $(Id+uA)$ in the last equality, we express the $m_i's$ as elements of the tangent space to $f + u\psi$ with C^∞ dependence on u. This concludes the proof of the assertion.
Lemma is proved.
This lemma achieves the proof of the finite determinacy theorem for $\mathcal{L}$-equivalence.

3.3. Proof of proposition 8.2. The proof is standard ([**Math**],[**Mar**],[**AVG**]). The assumption $F_d \subset T_r f \oplus < e_1, \dots, e_s >$ implies that there exists $h \in \mathcal{E}_{q,t}$, $v \in L_{q,t}$, $w \in L_q$, $b_1, \dots, b_s \in \mathbb{R}$ such that $-\psi$ admits the representation

$$-\psi = v.f + h \times f + w.f + \sum_{j=1} j^s b_j e_j. \tag{51}$$

Define the diffeomorphism map-germ $\varphi : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0)$ by its components:

$$(\varphi_1(q,t), \varphi_2(q)) = (q + w(q), t + v(q,t)).$$

Define the function-germ $A \in \mathcal{E}_{q,t}$ by the formula

$$A(q,t) = 1 + h(q,t).$$

By a straightforward computation, we get that

$$A \times ((f + \psi) \circ \varphi) = f + \psi + h \times f + v.f + w.f + r$$

with $r \in F_{d+1}$.
Using equation (51), we get that

$$A \times ((f + \psi) \circ \varphi) = f + \psi_1 + \sum_{i=1}^{s} b_i e_i$$

with $r \in F_{d+1}$.
By induction, for any $d' > d$ there exists $c_1, \dots, c_s \in \mathbb{R}$ ad $r' \in F_{d'}$ such that the following $\mathcal{L}$-equivalence relation holds

$$f + r \sim f + r' + \sum_{j=1}^{s} c_j e_j. \tag{52}$$

We have the following variant of the Nakayama lemma.

LEMMA 8.14. *There exists a number $\nu \in \mathbb{N}$ such that for any fixed value of $c_1, \dots, c_s$, $\mathcal{M}_{q,t}^{\nu}$ is contained in the $\mathcal{L}$-tangent space to $f + \sum_{j=1}^{s} c_j e_j$.*

Proof.
Denote by $m_1, \dots, m_l$ a set of monomials generating the $\mathcal{M}_{q,t}$-module $\mathcal{M}_{q,t}^{\nu}$. For any $i \in \{1, \dots, l\}$, the m_i's can be written in the following form

$$m_i = a_i \times f + v_i.f + w_i.f \tag{53}$$

where $a_i \times \in \mathcal{E}_{q,t}$, $v_i \in L_{q,t}$, $w_i \in L_q$ are of order one.
Define the function-germ g by

$$g = \sum_{j=1}^{s} c_j e_j$$

Rewrite the right hand-side of the equality 53, in the following way

$$a_i \times f + v_i.f + w_i.f = a_i \times (f+g) + v_i.(f+g) + w_i.(f+g) - a_i \times g - v_i.g - w_i.g.$$

Since a_i, v_i and w_i are of order one, the function germ

$$a_i \times g - v_i.g - w_i.g$$

belongs to $F_{\nu+1}$.
Consequently, there exists $\alpha_{i,1}, \dots, \alpha_{i,l}$ in $\mathcal{M}_{q,t}$ such that the following equality holds

$$a_i \times g + v_i.g + w_i.g = \sum_{j=1}^{l} \alpha_{i,j} m_j.$$

Finally, we get a system of l equations:

$$\begin{cases} a_1(f+g) + v_1(f+g) + w_1(f+g) = m_1 + \sum_{j=1}^{l} \alpha_{1,j} m_j, \\ \dots \\ a_i(f+g) + v_i(f+g) + w_i(f+g) = m_i + \sum_{j=1}^{l} \alpha_{i,j} m_j, \\ \dots \\ a_l(f+g) + v_l(f+g) + w_l(f+g) = m_j + \sum_{j=1}^{l} \alpha_{l,j} m_j. \end{cases}$$

Denote by A the matrix whose coefficients are the $\alpha_{i,j}$'s.
Using the matrix formalism, the system of l equations can be written as

$$\vec{a}(f+g) + \vec{v}(f+g) + \vec{w}(f+g) = (Id + A)\vec{m}$$

where $\vec{a} = (a_1, \dots, a_l)$, $\vec{v} = (v_1, \dots, v_l)$, $\vec{w} = (w_1, \dots, w_l)$, $\vec{m} = (m_1, \dots, m_l)$ and Id is the identity matrix.
The matrix $(Id + A)$ is invertible since $(Id + A)(0) \neq 0$.
By inversion of the matrix $(Id + A)$ in the last equality, we express the $m_i' s$ as elements of the reduced tangent space to $f + g$.
Lemma is proved.

The $\mathcal{L}$-finite determinacy theorem implies that for d' big enough, the following $\mathcal{L}$-equivalence relation holds

$$f + \psi' + \sum_{j=1}^{s} c_j e_j \sim f + \sum_{j=1}^{s} c_j e_j.$$

Using the $\mathcal{L}$-equivalence relation of equation (52), we get that $f + \psi$ is $\mathcal{L}$-equivalent to $f + \sum_{j=1}^{s} c_j e_j$. This concludes the proof of the proposition.

3.4. Proof of proposition 8.3. We fix coordinate-systems $q = (q_1, \dots, q_{n+1})$ in $\mathbb{R}^{n+1}$ and $t = (t_1, \dots, t_k)$ in $\mathbb{R}^k$.
We are given a function germ

$$f_0 : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$$

together with a quasi-homogeneous filtration of $\mathcal{E}_{q,t}$.
To simplify our notations, since we are not going to use the letter f for a special purpose, we write simply f instead of f_0. Denote by M_f denotes the $\mathcal{E}_{q,t}$-submodule of $\mathcal{E}_{q,t}$ which is the sum of the following $\mathcal{E}_{q,t}$-submodules:
- the $\mathcal{E}_{q,t}$-submodule of function-germs of the type $h \times f$ where h has order one,

- the $\mathcal{E}_{q,t}$-submodule of the type $v.f$ where $v \in L_{q,t}$ has order one.

Consider the canonical projections:

$$p_1 : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/(M_f \cap F_d),$$
$$p_2 : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/(\mathcal{M}_q F_d).$$

Put $\bar{F}_d = F_d/(M_f \cap F_d)$, $\bar{\mathcal{E}}_{q,t} = \mathcal{E}_{q,t}/M_f$. We get the commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_{q,t} & \xrightarrow{p_1} & \bar{\mathcal{E}}_{q,t} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \bar{p}_2} \\ \mathcal{E}_{q,t}/(\mathcal{M}_q F_d) & \xrightarrow{\bar{p}_1} & \bar{\mathcal{E}}_{q,t}/(\mathcal{M}_q \bar{\mathcal{E}}_{q,t}) \end{array}$$

The $\mathcal{E}_{q,t}$-module M_f is contained in $T_r f$. Thus we the following equivalence holds

$$(F_d \subset T_r f) \iff (\bar{F}_d \subset p_2(T_r f)) \tag{54}$$

Using the notations of the commutative diagram put $\xi = \bar{p}_2 \circ p_1$.
Our previous discussion implies that proposition 8.3 is equivalent to the following implication

$$(\xi(F_d) \subset \xi(T_r f)) \implies (\bar{F}_d \subset p_2(T_r f)) \tag{55}$$

In order to prove this implication, we apply the Weierstrass-Malgrange-Mather theorem (see theorem 8.1 page 182).
With the notations of theorem 8.1, we put:
- $M = \bar{F}_d$,
- $N = p_1(F_d \cap T_r f)$,
- $z = q$, $w = t$.

The Weierstrass-Malgrange-Mather theorem implies that the implication (55) holds provided that $\bar{F}_d$ is an $\mathcal{E}_q$-module of finite type.
Since $\bar{F}_d$ is a submodule of $\bar{\mathcal{E}}_{q,t}$ proposition 8.3 is a consequence of the following lemma.

Lemma 8.15. *The $\mathcal{E}_q$-module $\bar{\mathcal{E}}_{q,t}$ is of finite type.*

Proof.
The proof of this lemma also requires the use of the Weierstrass-Malgrange-Mather theorem.
Define the function-germ $g : (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ by the formula

$$g(t) = f(0, t).$$

Consider the projection

$$\eta : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_t$$

obtained by restricting a function-germ in $\mathcal{E}_{q,t}$ to $q = 0$. For example $g = \eta(f)$.

This projection can be factorized through the projection (see subsection 1.5 for the notations)

$$\pi_d : \mathcal{E}_{q,t} \longrightarrow \mathcal{E}_{q,t}/V_d.$$

This means that there exists a map

$$\rho : \mathcal{E}_{q,t}/V_d \longrightarrow \mathcal{E}_t$$

such that the following equality holds

$$\rho \circ \pi_d = \eta.$$

The map ρ being also the restriction to $q = 0$.
The assumption $\pi_d(F_d) \subset \pi_d(T_r f)$ implies that $\pi_d(T_r f)$ is a $\mathbb{R}$-vector space of finite codimension in $\mathcal{E}_{q,t}/V_d$. consequently $\eta(T_r f)$ is of finite codimension in $\mathcal{E}_t$.
Let $v_1, \ldots, v_s \in \mathcal{E}_t$ be a basis of a transversal to $\eta(T_r f)$ in $\mathcal{E}_t$.
We assert that the $\mathcal{E}_q$-module, denoted $\widetilde{N}$, generated by $v_1, \ldots, v_s$ is a $\mathbb{R}$-vector space transversal to M_f in $\mathcal{E}_{q,t}$.
In order to prove this assertion, we apply the Weierstrass-Malgrange-Mather theorem (theorem 8.1).
With the notations of the theorem we take:
- $z = q$, $w = t$.
- $M = \mathcal{E}_{q,t}/M_f$.
- $N = \widetilde{N}/M_f$ We have canonical identifications:
- $M/(\mathcal{M}_q M)$ is identified with $\mathcal{E}_t/\eta(T_r f)$.
- $N/(\mathcal{M}_q M)$ is identified with the $\mathbb{R}$-vector space generated by the images of the v_i's under the canonical projection $\mathcal{E}_t \longrightarrow \mathcal{E}_t/\eta(T_r f)$.
The definition of the v_i's implies the equality

$$N/(\mathcal{M}_q M) = M/(\mathcal{M}_q M).$$

Thus the Weierstrass-Malgrange-Mather theorem implies that $M = N$. Lemma is proved and so is the proposition.

Part III

APPENDICES

APPENDIX A

Computations of the normal forms.

1. Quasi-homogeneous filtrations.

In this section, we recall basic facts from singularity theory(see [**AVGL1**] for details). One of the reason for doing this is to fix the notations.
We fix a coordinate system $x = (x_1, \dots, x_m)$ in the analytic space $\mathbb{C}^m$.
The space of germs of holomorphic functions in the variable x is denoted by $\mathcal{O}_x$.
The construction of this subsection depends on the choice of this coordinate-system. We denote by $\mathcal{O}_x$ the ring of germs of holomorphic functions $f : (\mathbb{C}^m, 0) \longrightarrow \mathbb{C}$.

Definition A.1. A holomorphic function-germ $f : (\mathbb{C}^m, 0) \longrightarrow \mathbb{C}$ is called *quasi-homogeneous of degree* d with exponents $\alpha_1, \dots, \alpha_m$ provided that for any $\lambda > 0$ and any $(x_1, \dots, x_m) \in \mathbb{C}^m$ we have:

$$f(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_m} x_m) = \lambda^d f(x_1, \dots, x_m).$$

Example. The function-germ defined by $f(x_1, x_2) = x_1^p + x_2^q$ is quasi-homogeneous of degree pq with exponents $\alpha_1 = q,\ \alpha_2 = p$. The function germ defined by $f(x_1, x_2) = x_1^3 + x_1 x_2 + x_2^3$ is not quasi-homogeneous.

Fix a vector $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$.

Definition A.2. A monomial $x_1^{k_1} x_2^{k_2} \dots x^{k_k}$ has *weight* d provided that $\alpha_1 k_1 + \dots + \alpha_m k_m = d$.

Definition A.3. A function-germ in $\mathcal{O}_x$ has *order* d if all the monomials appearing with non-zero coefficient in its Taylor series at the origin are of weight at least d.

We denote by $F_d \subset \mathcal{O}_x$ the $\mathbb{C}$-vector subspace of function-germs of order d.

Definition A.4. The nested sequence $F_0 \supset F_1 \supset F_2 \supset \dots$ of $\mathbb{C}$-vector spaces is called the quasi-homogeneous filtration of $\mathcal{O}_x$ associated to α.

Definition A.5. A quasi-homogeneous function f is *non-degenerate* if the origin is an isolated critical point of[1] f

[1]The condition for a holomorphic function-germ to have an isolated critical point at the origin is equivalent to $dim[\mathcal{O}_x/J(f)] < +\infty$ where $J(f)$ is the *Jacobian ideal* of f in $\mathcal{O}_x$ generated by the partial derivatives $\partial_{x_k} f$.

Notations. The $\mathbb{C}$-vector space of functions-germs of order d is denoted F_d. The notation:

$$f = f_0 + \tilde{o}(k),$$

means that $f = f_0 + f_1$ where f_0 is a non-degenerate quasi-homogeneous function containing only monomials of weight at most than k and $f_1 \in F_{k+1}$.

2. Normal forms of the $\mathcal{P}$-simple singularities.

We denote by $\mathbb{C}^n$ the n-dimensional complex vector space.
Following Arnold [**Arn2**], we give the following definition.

Definition A.6. A function-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ is *of the type* A_k, denoted $f \in A_k$, provided that there exists a biholomorphic map-germ φ : $(\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ such that

$$(f \circ \varphi)(x) = x_1^{k+1} + \sum_{i=2}^{n} x_i^2.$$

In all this section $\mathbb{C}^2$ denotes the two-dimensional complex vector space.

2.1. Normal form $\mathcal{P}A_1^{p,q}$.

The normal form for a holomorphic function-germ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ has already been calculated in chapter 4 subsection 3.1. In this subsection, the following proposition was proved.

Proposition A.1. *For any holomorphic function-germ f belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, the following $\mathcal{G}$-equivalence holds*

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy).$$

2.2. $\mathcal{P}A_2$ normal form.

Proposition A.2. *For any holomorphic function-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$, the two following statements hold.*
1) The $\mathcal{G}$-Milnor number of (f, Δ_f) is finite,
2) We have the $\mathcal{G}$-equivalence $(f, \Delta_f) \sim (y^2 + x^3, 3x^4 + 4xy^2)$.

We fix a linear coordinate system (x, y) in $\mathbb{C}^2$.
In $\mathcal{O}_{x,y}$ we introduce a quasi-homogeneous filtration.
The weight of the monomial $x^i y^j$ is equal to $2i + 3j$.
Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a function belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$.
A non-degenerate linear map $\alpha : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ sends the inflection points of a curve to the inflection points of its image. Consequently, for any non-degenerate linear map $\alpha : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, the maps f and $f \circ \alpha$ are $\mathcal{P}$-equivalent. Hence, without loss of generality, we can assume that:

$$f(x, y) = y^2 + x^3 + \tilde{o}(6).$$

LEMMA A.1. *The function-germ f is $\mathcal{P}$-equivalent to a function-germ of the type $y^2+x^3+\tilde{o}(7)$.*

Proof.
We have

$$f(x,y)=y^2+x^3+cx^2y+\tilde{o}(7).$$

Moreover

$$y^2+x^3+cx^2y+\tilde{o}(7)=y^2+(x+\frac{c}{3}y)^3+\tilde{o}(7).$$

Put $\alpha(x,y)=(x-\frac{c}{3}y,y)$. Remark that if $g\in F_k$ then $g\circ\alpha\in F_k$ since the weight of y is higher than the weight of x. Hence the function-germ $f\circ\alpha$ is of the type

$$(f\circ\alpha)(x,y)=y^2+x^3+\tilde{o}(7).$$

A non-degenerate linear map sends an inflection point of a curve to an inflection point of its image. Lemma is proved.

Hence, without loss of generality, we can assume that our holomorphic function-germ $f:(\mathbb{C}^2,0)\longrightarrow\mathbb{C}$ is of the type

$$f(x,y)=y^2+x^3+\tilde{o}(7).$$

Put $H(x,y)=y^2+x^3$.
According to Arnold's classification ([**Arn2**]), we have $f\in A_2$. This means that there exists a biholomorphic map-germ $\varphi:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C}^2,0)$ such that:

$$\begin{cases} f\circ\varphi=H. \\ (D\varphi)(0)=Id. \end{cases} \tag{56}$$

Here Id denotes the identity mapping of $\mathbb{C}^2$.
Let $\mathcal{D}[\varphi]$ be the determinant of the 2×2 matrix whose columns are the first and second derivatives of φ along the Hamilton vector field of H.
We have the $\mathcal{G}$-equivalence (see chapter 3, section 1 for the notations)

$$(H,\mathcal{D}[\varphi])\sim(f,\Delta_f).$$

LEMMA A.2. *The function $\mathcal{D}[\varphi]$ is of the type*
$\mathcal{D}[\varphi](x,y)=-18x^4-24xy^2+\tilde{o}(9)$.

Proof.
From the equalities $f=H+\tilde{o}(7)$ and $(D\varphi)(0)=Id$ we get that

$$\varphi(x,y)=(x+\tilde{o}(3),y+\tilde{o}(4)).$$

The Hamilton vector field of H is $h=2y\partial_x-3x^2\partial_y$. The derivation along h increases the weight by 1.
Hence, we find the following expression for $\mathcal{D}[\varphi]$

$$\mathcal{D}[\varphi](x,y)=[x,y]+\tilde{o}(9),$$

where $[x, y]$ denotes the determinant of the matrix

$$\begin{bmatrix} 2y & -3x^2 \\ -6x^2 & -12xy \end{bmatrix}.$$

Lemma is proved.

Define the holomorphic function germ $E \in \mathcal{O}_{x,y}$ by

$$E(x, y) = 3x^4 + 4xy^2.$$

Lemma A.2 implies the following $\mathcal{G}$-equivalence relation

$$(f, \Delta_f) \sim (H, E + r)$$

with $r \in F_{10}$.
We now prove the $\mathcal{G}$-equivalence

$$(H, E + r) \sim (H, E). \tag{57}$$

This will conclude the proof of proposition A.2.
In order to prove the $\mathcal{G}$-equivalence relation (57), it is needed to introduce a subspace of the $\mathcal{G}$-tangent space to (H, E). Recall from section 4 that the $\mathcal{G}$-tangent space to the function-germ (H, E) is the $\mathcal{O}_{x,y}$-module generated by the map-germs:

$$\begin{cases} x\partial_x(H, E),\ y\partial_x(H, E),\ x\partial_y(H, E),\ y\partial_y(H, E), \\ \qquad\qquad (E, 0), (0, E). \end{cases}$$

We denote by $diff(2)$ the set of germs of vector-field f the type $a\partial_x + b\partial_y$ with $a, b \in \mathcal{O}_{x,y}$.
We say that a vector-field $v \in diff(2)$ has *order* d if for any k and any $m \in F_k$, we have $v.m \in F_{k+d}$.
Denote by $\mathcal{T}$ the $\mathcal{O}_{x,y}$-submodule of the tangent space to (H, E) generated by:
- $\{v.(H, E) : v \in diff(2)$ has order $1\}$,
- $(E, 0)$, $(0, xE)$, $(0, yE)$.

Here $v.(H, E)$ denotes the Lie derivative of the map-germ (H, E) along the vector-field v. Thus $v.(H, E) \in \mathcal{O}^2_{x,y}$.
Next, consider the $\mathcal{O}_{x,y}$-module M defined by

$$M = \{(g, h) \in \mathcal{O}_{x,y} \times \mathcal{O}_{x,y} :\ g \in F_8,\ h \in F_{10}\}.$$

LEMMA A.3. *The $\mathcal{G}$-equivalence $(H, E + r) \sim (H, E)$ holds provided that M is contained in $\mathcal{T}$.*

Proof.
The proof is in two steps. First we prove the assertion for formal power series and then use the $\mathcal{G}$-finite determinacy theorem.
Assume that we have proved the following $\mathcal{G}$-equivalence relation:

$$(H, E + r) \sim (H + m_1, E + m_2)$$

with $m_1 \in F_{d_1}$ and $m_2 \in F_{d_2}$, $d_1 \geq 8$, $d_2 \geq 10$.
Assertion. We have a $\mathcal{G}$-relation of the type

$$(H, E + r) \sim (H + \tilde{o}(d_1), E + \tilde{o}(d_2)). \tag{58}$$

The inclusion $M \subset \mathcal{T}$ implies that (m_1, m_2) admits a representation of the type

$$(m_1, m_2) = a(E, 0) + b(0, E) + v.(H, E) \tag{59}$$

with $a, b \in \mathcal{O}_{x,y}$, $v \in diff(2)$.
Put $v(x, y) = v_1(x, y)\partial_x + v_2(x, y)\partial_y$.
Define the biholomorphic map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ by

$$\varphi(x, y) = (x + v_1(x, y), y + v_2(x, y)).$$

Define the matrix $A \in Gl(2, \mathcal{O}_{x,y})$ by

$$A = \begin{pmatrix} 1 & -a \\ 0 & 1-b \end{pmatrix}.$$

With the notations of subsection 4.2, we get the equality

$$(\varphi, A).(H+m_1, E+m_2) = (H, E)-v.(H, E)-(aE, bE)+(m_1, m_2)+(\tilde{o}(d_1), \tilde{o}(d_2)).$$

Coming back to equation (59), we get that equation (58) holds.
This proves the assertion.
We denote by $\mathcal{M}_{x,y}^k$ the k^{th} power of the maximal ideal $\mathcal{M}_{x,y} \subset \mathcal{O}_{x,y}$.
The assertion implies that for any k, there exists a holomorphic map-germ $\psi = (\psi_1, \psi_2) \in \mathcal{O}_{x,y}^2$ with $\psi_1 \in \mathcal{M}^k$, $\psi_2 \in \mathcal{M}^k$ such that the following $\mathcal{G}$-equivalence holds

$$(H, E) \sim (H, E + r) + \psi. \tag{60}$$

We claim that under the assumption of the lemma, this implies that (H, E) and $(H, E + r)$ are $\mathcal{G}$-equivalent.
Indeed, the $\mathcal{O}_{x,y}$ module $\mathcal{T}$ is contained in the $\mathcal{G}$-tangent space to (H, E). Thus, the inclusion $M \subset \mathcal{T}$ implies that the $\mathcal{G}$-Milnor number of (H, E) is finite.
The $\mathcal{G}$-finite determinacy theorem implies that for k big enough, the $\mathcal{G}$-equivalence (60) implies the following $\mathcal{G}$-equivalence

$$(H, E) \sim (H, E + r).$$

This concludes the proof of the lemma.

The following lemma concludes the proof of proposition A.2. We keep the same notations.

Lemma A.4. *The $\mathcal{O}_{x,y}$-module M is contained in $\mathcal{T}$.*

Proof.

The division theorem (or the Nakayama lemma) implies that module M is generated by the 8 following four map-germs:

$$(x^4,0),\ (xy^2,0),(0,x^5),(0,x^2y^2)$$

and

$$(x^3y,0),\ (y^3,0),(0,x^4y),(0,x^3y).$$

Denote by V the $\mathbb{C}$-vector space generated by these 8 map-germs. Direct computations show that the eight following map-germ of $V\cap\mathcal{T}$ are linearly independent:

$$(E,0),(0,xE),x^2\partial_x(H,E),xy\partial_y(H,E)$$

and

$$(0,yE),xy\partial_x(H,E),y^3\partial_y(H,E),x^3y\partial_y(H,E).$$

Thus V is contained in $\mathcal{T}$. Lemma is proved.

2.3. A_3 is not $\mathcal{P}$-simple.

Proposition A.3. *For any holomorphic function-germ $f:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C},0)$ belonging to the singularity class A_3. Then, the holomorphic function-germ $f:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C},0)$ is not $\mathcal{P}$-simple.*

Proof.
Since $f\in A_3$ means that there exists a biholomorphic map-germ $\varphi:(\mathbb{C}^2,0)\longrightarrow(\mathbb{C}^2,0)$ such that:

$$H=f\circ\varphi,$$

with $H(x,y)=y^2+x^4$.
A non-degenerate linear map sends the inflection points of a curve to the inflection points of its image. Hence, without loss of generality we can assume that:

$$(D\varphi)(0)=Id.$$

Consider the one-parameter family f_λ of function-germs f defined as follows. Let $\varphi_\lambda(x,y)=\varphi(x,y)+(0,\lambda x^2)$ and put $f_\lambda=H\circ\varphi_\lambda^{-1}$.
We denote by $\mathcal{D}[\varphi_\lambda]$ the generalized Wronskian of φ.
This means that $\mathcal{D}[\varphi_\lambda]$ is the determinant of the 2×2 matrix whose lines are the first and second derivative of φ_λ along the Hamilton vector-field of H.
We have the following $\mathcal{G}$-equivalence

$$(H,\mathcal{D}[\varphi_\lambda])\sim(f_\lambda,\Delta_{f_\lambda}).$$

The proof of proposition A.3 is based on the three following assertions.
Assertion 1: if f_a is $\mathcal{P}$-equivalent to f_b then $a^2=b^2$.
Assertion 2: if $(H,\mathcal{D}[\varphi_a])$ is V-equivalent to $(H,\mathcal{D}[\varphi_b])$ then $a^2=b^2$.
Assertion 3: assertion 2 implies assertion 1.
Assertion 1 implies that λ is a modulus. Hence the modality of f is at least 1 provided that assertion 1 is proved.
We prove assertion 3.

Denote by $\sim$ the $\mathcal{G}$-equivalence. By definition, f_a is $\mathcal{P}$-equivalent to f_b if and only if the following $\mathcal{G}$-equivalence holds

$$(f_a, \Delta_{f_a}) \sim (f_b, \Delta_{f_b})$$

For any value of λ, we have the $\mathcal{G}$-equivalence:

$$(f_\lambda, \Delta_{f_\lambda}) \sim (H, \mathcal{D}[\varphi_\lambda])$$

Thus the $\mathcal{G}$-equivalence $(f_a, \Delta_{f_a}) \sim (f_b, \Delta_{f_b})$ is equivalent to the $\mathcal{G}$-equivalence $(H, \mathcal{D}[\varphi_a]) \sim (H, \mathcal{D}[\varphi_b])$. If two map-germs are $\mathcal{G}$-equivalent then in particular they are V-equivalent since the group $\mathcal{G}$ is a subgroup of the group $\mathcal{K}$ (see section 4). This proves assertion 3.
It remains to prove assertion 2.
Define the quasi-homogeneous weight of $x^i y^j$ to be $i + 2j$.

Lemma A.5. *The map-germ $(H, \mathcal{D}[\varphi_\lambda])$ is V-equivalent to a holomorphic map-germ of the form $(H, \frac{7}{6}(c+\lambda)y^3 + x^6 + \tilde{o}(6))$ for some $c \in \mathbb{C}$*

Proof.
The holomorphic map-germ $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ is of the form

$$\varphi(x, y) = (x + \tilde{o}(1), y + cx^2 + \tilde{o}(2)).$$

Hence $\varphi_\lambda : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ is of the form:

$$\varphi_\lambda(x, y) = (x + \tilde{o}(1), y + (\lambda + c)x^2 + \tilde{o}(2)).$$

Consequently, we get that

$$\mathcal{D}[\varphi_\lambda] = (\lambda + c)[x, x^2] + [x, y] + \tilde{o}(6),$$

where the brackets are equal to the following 2×2 determinants

$$[x, y] = \begin{vmatrix} 2y & -4x^3 \\ -12x^3 & -24x^2 y \end{vmatrix},$$

$$[x, x^2] = \begin{vmatrix} 2y & 4xy \\ -7x^3 & 8y^2 - 16x^4 \end{vmatrix}.$$

Denote by $\equiv$ the V-equivalence relation.
We substitute y^2 by $-x^4$ in $[x, y]$ and x^4 by $-y^2$ in $[x, x^2]$. We get the V-equivalence

$$[x, y] + (\lambda + c)[x, x^2] \equiv 24x^6 + 28(\lambda + c)y^3.$$

This proves the lemma.

Define the family of function-germs $E_\alpha : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ depending on the parameter $\alpha \in \mathbb{C}$ by

$$E_\alpha(x, y) = \alpha y^3 + x^6.$$

The next lemma concludes the proof of proposition A.3.

Lemma A.6. *If a holomorphic map-germ of the type $(H, E_a + \tilde{o}(6))$ is V-equivalent to a holomorphic map-germ of the type $(H, E_b + \tilde{o}(6))$ then $a^2 = b^2$.*

Proof.
Denote the two holomorphic map-germs by $(H, E_a + r_1)$ and $(H, E_b + r_2)$ with $r_1, r_2 \in F_7$.
Assume that there exist an invertible 2×2 matrix A with elements in $\mathcal{O}_{x,y}$ and biholomorphic map-germs $g : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0), \psi : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$ such that:

$$A \times ((H, E_a + r_1) \circ g) = (\psi \circ H, E_b + r_2). \tag{61}$$

Remark that this equation is in fact a system of two equations, that we shall call the first and second equation of the system (61).
The matrix A is of the type

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta \in \mathcal{O}_{x,y}$.
Write:

$$g(x, y) = (mx + \tilde{o}(1), nx + py + qx^2 + \tilde{o}(2)), \ m, p \in \mathbb{C} \setminus \{0\} .$$

Equating the terms of quasi-homogeneous weight 2 in the first equation of the system (61), we get that $n = 0$.
Equating the terms of quasi-homogeneous weight 4 in the first equation of the system (61), we get that:

$$p^2 y^2 + m^4 x^4 + 2pqx^2 y = c(y^2 + x^4)$$

where c denotes a non-zero constant.
Thus $p = \pm m^2, \ q = 0$. The second equation of the system (61) is of the form:

$$\gamma H + \delta(E_a + r_1) = E_b + r_2.$$

Equating the terms of weight 4 and 5 we get that $\gamma \in F_2$. Equating the terms of weight 6, we get that $\gamma \in F_3$. Consequently:

$$\delta(0)(ap^3 y^3 + m^6 x^6) = by^3 + x^6.$$

As we saw previously $p = \pm m$. Thus identifying the coefficient of x^6 and y^3 in this equality, we get:

$$\delta(0) = m^{-6}$$

and:

$$a^2 = b^2.$$

Lemma is proved.
This lemma concludes the proof of Assertion 2. Proposition A.3 is proved.

2.4. $\mathcal{P}A_2$ is $\mathcal{P}$-simple. First, we consider a one parameter deformation $F : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ of $f \in \mathcal{P}A_2$.
Denote by $\bar{F}$ a representative of the germ F.
Assume that for any λ small enough the origin is a critical point of $\bar{F}(\lambda, .)$ of critical value zero (i.e. $\bar{F}(\lambda, .) \in \mathcal{M}^2$).
According to Arnold's classification [**Arn2**], the germ at the origin of the holomorphic function $\bar{F}(\lambda, .)$ belongs either to the $\mathcal{P}$-singularity class A_1 or A_2 provided that λ is small enough.
Remark that the $\mathcal{P}$-singularity class A_k gives rise to a singularity class in the usual sense by "forgetting" the vector space structure on $\mathbb{C}^2$ ([**AVG**]).
According to our $\mathcal{P}$-classification the germ at the origin of $\bar{F}(\lambda, .)$ belongs either to a $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ for some value of p, q depending on λ or to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$, provided that λ is small enough. By definition of the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, if the germ of $\bar{F}(\lambda, .)$ at the origin belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ then there exists two lines $d_1,\ d_2$ passing through the origin such that:
- the germ at the origin of the restriction of $\bar{F}(\lambda, .)$ to d_1 is a function-germ with a critical point A_{3+p} at the origin,
- the germ at the origin of the restriction of $\bar{F}(\lambda, .)$ to d_2 is a function-germ with a critical point A_{3+q} at the origin.
The following lemma implies that if the germ at the origin of $\bar{F}(\lambda, .)$ belongs either to a $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ then $p = q = 0$.

LEMMA A.7. *Let $G : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be an arbitrary one-parameter deformation of a germ $f \in A_2$. Fix an arbitrary complex line $L \subset \mathbb{C}^2$ passing through the origin. Let $k(\lambda)$ be such that the restriction of $G(\lambda, .)$ to the line L has a singularity $A_{k(\lambda)}$ at the origin.*
Then the inequality $k(\lambda) \leq 2$ holds provided that λ is small enough.

Proof.
Let $L \subset \mathbb{C}^2$ be the line such that the restriction of $G(\lambda, .)$ has a critical point of the type $A_{k(\lambda)}$.
The restriction of $G(0, .)$ to L has a critical point adjacent to $A_{k(\lambda)}$ provided that λ is small enough.
In an appropriate linear coordinate system the function $G(0, .)$ is of the form

$$G(0, x, y) = y^2 + x^3 + yr_1(x, y) + r_2$$

with $r_1 \in \mathcal{M}^2_{x,y}$ and $r_2 \in \mathcal{M}^3_{x,y}$.
Consequently, the germ at the origin of the restriction of $G(0, .)$ to L has either an A_2 critical point or an A_1 critical point. Thus, either A_2 or A_1 is adjacent to $A_{k(\lambda)}$ provided that λ is small enough. Consequently the inequality $k(\lambda) \leq 2$ holds provided that λ is small enough. Lemma is proved.

Remark that at a degenerate inflection point of a curve of equation $f = 0$, the restriction of f to its tangent line has a critical point A_3. Thus the preceding lemma has the following corollary.

COROLLARY A.1. *Let F be a deformation of $f \in A_2$ then the $\mathcal{P}$-discriminant of F consists only of points corresponding to curves with singular points.*

We come back to the simplicity of $\mathcal{P}A_2$. We have proved that:
- all the function-germs in $\mathcal{P}A_1$ are $\mathcal{P}$-equivalent (proposition A.1),
- all the function-germs in $\mathcal{P}A_2$ are $\mathcal{P}$-equivalent (proposition A.2).
Thus lemma A.7 implies that there are only two $\mathcal{P}$-equivalence classes in a small neighbourhood of f namely $\mathcal{P}A_1$ and $\mathcal{P}A_2$. This concludes the proof of the simplicity of $\mathcal{P}A_2$.

2.5. $\mathcal{P}A_1^{p,q}$ and $\mathcal{P}A_2$ are the only $\mathcal{P}$-simple plane singularities.

LEMMA A.8. *If $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a holomorphic function-germ with a critical point at the origin such that $f \notin A_1 \cup A_2$, then f is not $\mathcal{P}$-simple.*

Proof.
Arnold's classification ([**Arn2**]) implies that under the assumptions of the lemma there exists a deformation $F : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ of f satisfying the following property.
There exists a representative $\bar{F}$ of F such that
- for any $\lambda \neq 0$ small enough, the germ at the origin of $\bar{F}(\lambda, .)$ has a critical point of critical value 0;
- for any $\lambda \neq 0$ small enough, the germ at the origin of $F(\lambda, .)$ belongs to the singularity class A_3.
We have proved in subsection 2.3 that any holomorphic function-germ in A_3 satisfying these conditions is not $\mathcal{P}$-simple. Consequently f is not $\mathcal{P}$-simple. Lemma is proved.

LEMMA A.9. *Assume that $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is a Morse function-germ such that for any (p, q), $f \notin \mathcal{P}A_1^{p,q}$ then there are an infinity $\mathcal{P}$-singularity classes in any small neighbourhood of f. In particular f is not $\mathcal{P}$-simple.*

Proof.
By definition the plane curve-germ of equation $f = 0$ contains a line. That is up to a linear change of coordinates f has the form

$$f(x, y) = x(y + ax^{2+p} + o(x^{2+p})).$$

Assume for notational reasons that $a \neq 0$. Consider the deformation:

$$F_q(\lambda, x, y) = f(x, y) + \lambda y^{3+q}.$$

We have $F_q(\lambda, .) \in \mathcal{P}A_1^{p,q}$ provided that $\lambda \neq 0$. Hence for any neighbourhood U of f, for any value of $q \in \mathbb{N}$, there exists a function-germ $F_q(\lambda, .)$ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$. Lemma is proved.

3. Proof of the $\mathcal{P}$-versal deformation theorem for $\mathcal{P}A_1^{p,q}$.

In this section we prove theorem 2.11 of page 42.
For notational reasons, we assume that p and q are strictly positive integers. We fix a linear coordinate-system (x, y) in the two-dimensional vector space

$\mathbb{C}^2$.
Let $P : (\mathbb{C}^{p+q} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be the holomorphic function-germ defined by the polynomial

$$P(\alpha, \beta, x, y) = x^{3+p} + y^{3+q} + \sum_{j=1}^{p} \alpha_j x^{2+j} + \sum_{k=1}^{q} \beta_k y^{2+k},$$

with $\alpha = (\alpha_1, \ldots, \alpha_p)$, $\beta = (\beta_1, \ldots, \beta_q)$.
Denote by

$$G : (\mathbb{C}^k \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$$

an arbitrary deformation of a holomorphic function-germ
$f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$.

LEMMA A.10. *The deformation (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation induced from $(P(\alpha, \beta, x, y), xy)$.*

Proof.
The holomorphic function-germ $f = G(0, .)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$. Thus proposition A.1 implies the following $\mathcal{G}$-equivalence holds

$$(f, \Delta_f) \sim (x^{3+p} + y^{3+q}, xy)$$

Hence (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation of $(x^{3+p} + y^{3+q}, xy)$.
Define the deformation $A : (\mathbb{C}^{p+q+5} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of (f, Δ_f) by the polynomials

$$A(\mu, \alpha, \beta, x, y) = (P(\alpha, \beta, x, y) + \mu_1 x^2 + \mu_2 y^2 + \mu_3 x + \mu_4 y, xy + \mu_5).$$

Direct calculations show that the $\mathcal{G}$-tangent space to A is $TA = \mathcal{O}^2_{x,y}$. Thus, the $\mathcal{G}$-versal deformation theorem (see chapter 4, section 5) implies that the deformation A is $\mathcal{G}$-versal. Consequently (G, Δ_G) is $\mathcal{G}$-equivalent to a deformation induced from A.
Denote respectively by γ, λ the parameters of the deformations G and P. We have $\lambda = (\alpha, \beta) \in \mathbb{C}^{p+q}$. Put $\mu = (\mu_1, \ldots, \mu_5)$. We use the old-fashioned notation:

$$\lambda = \lambda(\gamma)$$
$$\mu = \mu(\gamma)$$

for a map inducing the deformation equivalent to (G, Δ_G) from A. Lemma A.10 is a consequence of the following lemma.

LEMMA A.11. *The map-germ μ vanishes identically.*

Proof.
Denote by $\bar{G}$ a representative of G.
The function $\bar{G}(\gamma, .)$ has a Morse critical point in a neighbourhood the origin provided that γ is small enough.
A translation sends an inflection point of a curve to an inflection point of its translation. Consequently, we can assume without loss of generality that the Morse critical point of $\bar{G}(\gamma, .)$ is the origin.
Proposition A.1 implies that for any γ small enough:

1) $\Delta_{\bar{G}}(\gamma, .)$ has a Morse critical point at the origin of critical value 0,
2) the restriction of $\bar{G}(\gamma, .)$ to each branch of the plane curve

$$\left\{(x, y) \in \mathbb{C}^2 : \Delta_{\bar{G}}(\gamma, x, y) = 0\right\}$$

has a critical point of the type A_k with $k \geq 2$.
Condition 1 implies that μ_5 vanishes identically. Then, condition 2 implies that $\mu_1, \ldots, \mu_4$ also vanish identically. Lemma is proved.

LEMMA A.12. *Assume that $\forall j \in \{1, \ldots, p\}$, $\forall k \in \{1, \ldots, q\}$, the vectors $(x^{2+j}, 0)$, $(y^{2+k}, 0)$ are contained in the $\mathcal{G}$-tangent space to (G, Δ_G). Then the following $\mathcal{G}$-equivalence holds*
$(G, \Delta_G) \sim (P(\alpha, \beta, x, y), xy)$.

Proof.

Consider the deformation $B : (\mathbb{C}^{k+5} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}, 0)$ of (f, Δ_f) defined by the formula

$$B(\mu, \gamma, x, y) = (G(\gamma, x, y) + \mu_1 x^2 + \mu_2 y^2 + \mu_3 x + \mu_4 y, \Delta_G(\gamma, x, y) + \mu_5).$$

By direct computations, we get that under the conditions of the lemma the $\mathcal{G}$-tangent space to B is $TB = \mathcal{O}^2_{x,y}$.
The $\mathcal{G}$-versal deformation theorem implies that B is $\mathcal{G}$-versal.
Consequently the deformation

$$\begin{array}{ccc} (\mathbb{C}^2, 0) & \longrightarrow & (\mathbb{C} \times \mathbb{C}, 0) \\ (\alpha, \beta, x, y) & \mapsto & (P(\alpha, \beta, x, y), xy) \end{array}$$

is induced from a deformation equivalent to B. The same argument as the one given in lemma A.11 implies that $(P(\alpha, \beta, x, y), xy)$ is induced from a deformation equivalent to (G, Δ_G).
On the other hand, lemma A.11 implies that (G, Δ_G) is induced from a deformation equivalent to $(P(\alpha, \beta, x, y), xy)$. We have shown that:
- (G, Δ_G) is induced from a deformation equivalent to $(P(\alpha, \beta, x, y), xy)$,
- $(P(\alpha, \beta, x, y), xy)$ is induced from a deformation equivalent to (G, Δ_G).
Consequently (G, Δ_G) and $(P(\alpha, \beta, x, y), xy)$ are $\mathcal{G}$-equivalent. Lemma is proved.

By definition of $\mathcal{P}$-versality, these two lemmas imply that a function-germ G satisfying the conditions of lemma A.12 is a $\mathcal{P}$-versal deformation.

It remains to find a deformation G such that $(x^{2+j}, 0), (y^{2+k}, 0)$ belong to the $\mathcal{G}$-tangent space to (G, Δ_G) for all[2] $0 < j \leq p,\ 0 < k \leq q$.
We assert that the deformation $F : (\mathbb{C}^{p+q} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ of $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by the formula

$$F(\alpha, \beta, x, y) = xy + x^{3+p} + y^{3+q} + \sum_{j=1}^{p} \alpha_j x^{2+j} + \sum_{k=1}^{q} \beta_k y^{2+k},$$

[2] Recall that for notational convenience, we have assumed that $pq > 0$.

satisfies this property.
We show that the $\mathcal{G}$-tangent space to (F, Δ_F) contains $(x^{2+j}, 0)$, $j \leq p$.
The proof that $(y^{2+k}, 0)$ $k \leq q$ is contained in the $\mathcal{G}$-tangent space to (F, Δ_F) differs only in notations.

Denote by F_j the restriction of F to the complex line $\alpha_k = 0$ for $k \neq j$, $\beta_k = 0$ for all k's. That is F_j is the one-parameter deformation induced from F defined by:

$$F_j(\alpha_j, x, y) = xy + x^{3+p} + y^{3+q} + \alpha_j x^{2+j}.$$

To avoid to many indices we put $\alpha_j = \tau$.
Final assertion: the vector $(x^{2+j}, 0)$ is contained in the $\mathcal{G}$-tangent space to F_j.
The $\mathcal{G}$-tangent space to F contains the $\mathcal{G}$-tangent space to F_j. Thus, this assertion implies that $(x^{2+j}, 0)$ is contained in the $\mathcal{G}$-tangent space to F. Consequently the proof of this assertion will conclude the proof of the theorem.

Denote by $\mathcal{M}^k$ the k^{th} power of the maximal ideal $\mathcal{M} \subset \mathcal{O}_{x,y}$ and by M the $\mathcal{O}_{x,y}$-module:

$$M = \left\{(0, m) : m \in \mathcal{M}^2\right\}.$$

LEMMA A.13. *The module M is contained in the $\mathcal{G}$-tangent space to (F_j, Δ_{F_j}).*

Proof.
The function germ $f = F_j(0, .)$ belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$, thus proposition A.1 implies that:

$$(F_j(0, .), \Delta_{F_j}(0, .)) \sim (x^{3+p} + y^{3+q}, xy).$$

Hence the $\mathcal{G}$-tangent space to (F_j, Δ_{F_j}) contains the tangent space to the constant deformation equal to $(x^{3+p} + y^{3+q}, xy)$. It is readily verified that the $\mathcal{G}$-tangent space to this constant deformation contains the $\mathcal{O}_{x,y}$-module M. Lemma is proved.

We conclude the proof of the final assertion.
The Hamilton vector-field of $F_j(\tau, .)$ is:

$$h_\tau = (x + r_1(\tau, x, y))\partial_x - (y + r_2(\tau, x, y))\partial_y$$

with $r_1(\tau, .), r_2(\tau, .) \in \mathcal{M}^2$.
Thus for any value of τ, we have $\Delta_{F_j} \in \mathcal{M}^2$. Consequently the restriction of $\partial_\tau \Delta_{F_j}$ to $\tau = 0$ is contained in $\mathcal{M}^2$.
Lemma A.13 implies that the restriction of $(0, \partial_\tau \Delta_{F_j})$ to $\tau = 0$ is contained in the $\mathcal{G}$-tangent space to (F_j, Δ_{F_j}). Thus $(\partial_\tau F, 0)_{|\tau=0} = (x^{2+j}, 0)$ is contained in the $\mathcal{G}$-tangent space to (F_j, Δ_{F_j}). This concludes the proof of the final assertion. Theorem is proved.

4. The "generic" bifurcation diagrams.

4.1. A general remark concerning the notion of genericity. Fix respective neighbourhoods Λ, U of the origin in $\mathbb{C}^k$ and $\mathbb{C}^2$. A holomorphic functions $F : \Lambda \times U \longrightarrow \mathbb{C}$ defines a $k+3$-parametric deformation $G : (\mathbb{C}^{k+3} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by

$$G(\lambda, \alpha, \gamma, p) = F(\lambda, p + \alpha) + \gamma$$

where $\alpha \in \mathbb{C}^2$, $\gamma \in \mathbb{C}$ are the additional parameters.
The deformation G defines a $k+3$-dimensional variety in the space $J_0^N(\mathbb{C}^2, \mathbb{C})$ of N-jets at the origin, for N big enough.
Consequently, the property:
- "there exists a point λ_0, p_0 such that the germ of $F(\lambda_0, .)$ at p_0 belongs to the $\mathcal{P}$-singularity class X"
is generic provided that for any N big enough the set of N-jets of function-germs $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ such that $f \in X$ is a variety of codimension at most than $k + 3$ in $J_0^N(\mathbb{C}^2, \mathbb{C})$.
The case where f is a function-germ without a critical point at the origin is already treated by Kazarian's theory [**Ka1**].
Consequently for the rest of the section, we consider only the case where $f \in \mathcal{M}^2$ has critical point at the origin.
The space $\mathcal{M}^2$ is of codimension 3 in $J_0^N(\mathbb{C}^2, \mathbb{C})$.
Consequently, in order to compute the "generic bifurcation diagrams" appearing in k-parameter families (which do not follow from Kazarian's theory), we do as follows:
- first, we compute the R-singularity classes of codimension at most k in the space $\mathcal{M}^2$ of function-germs with a critical point at the origin,
- for a given R-singularity class, we search the $\mathcal{P}$-singularity classes contained in this R-singularity classes of codimension at most k in $\mathcal{M}^2$,
- when possible, we compute the $\mathcal{P}$-versal deformations for each representative of a $\mathcal{P}$-singularity class. This $\mathcal{P}$-versal deformation gives a "generic" bifurcation diagram.

4.2. Proof of theorem 2.2 of page 25. Denote by $\Sigma[X]$ the set of holomorphic function germs in $\mathcal{M}^2$, belonging to a given $\mathcal{P}$-singularity class X.
According to Arnold's classification [**Arn2**] , the complement of $\Sigma[A_1] \cup \Sigma[A_2]$ is of codimension 2 in the space $\mathcal{M}^2$.
Let $G : (\mathbb{C} \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a generic 1-parameter deformation of the holomorphic function-germ $f = G(0, .)$.
The $\mathcal{P}$-versal deformation theorem proved in section 3 of this chapter implies that the codimension of $\Sigma[\mathcal{P}A_1^{p,q}]$ in $\mathcal{M}^2$ is equal to $p + q$.
Consequently, if G is generic such that $f \in A_1$ then f belongs either to the $\mathcal{P}$-singularity class $\mathcal{P}A_1$ or to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^1$.
According to same $\mathcal{P}$-versal deformation theorem, if $f \in \mathcal{P}A_1^1$ then the $\mathcal{P}$-discriminant of G is biholomorphically equivalent to the germ at the origin

of the values (λ, ε) such that ε is a critical value of the restriction of

$$x^4 + \lambda x^3 + y^3$$

to the curve of equation $xy = 0$.
Hence the $\mathcal{P}$-discriminant of G is biholomorphically equivalent to the germ of the curve of equation $\varepsilon(\varepsilon - \lambda^4) = 0$.
For $f \in \mathcal{P}A_1$ the $\mathcal{P}$-discriminant is biholomorphically equivalent to the germ at the origin of the values (λ, ε) such that ε is a critical value of the restriction of

$$x^3 + y^3$$

to the curve $xy = 0$. That is the germ of the line $\varepsilon = 0$.
If f belongs to the singularity class A_2, then f also belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_2$.
Lemma A.7 of page 201, implies that the $\mathcal{P}$-discriminant of G consists only of singular curves.
Hence the $\mathcal{P}$-discriminant coincides with the germ at the origin of the usual discriminant consisting of the values of the parameter (λ, ε) for which the curve

$$V_{\lambda,\varepsilon} = \{p \in U : G(\lambda, p) = \varepsilon\}$$

is singular.
Usual singularity theory ([**AVGL1**]) implies that this discriminant is biholomorphically to the germ at the origin of the semi-cubical parabola given by

$$\left\{(\lambda, \varepsilon) \in \mathbb{C}^2 : \varepsilon^2 = \lambda^3\right\}.$$

4.3. Proof of theorem 2.3 of page 26. Let $G : (\mathbb{C}^2 \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a generic 2-parameter deformation of the function $f = G(0, .)$ such that for each λ, $G(\lambda, .)$ is a Morse function. Then, according to the previous discussion, the holomorphic function germ f belongs to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$ with $p + q \leq 2$.
The case $p + q < 2$ has been treated in the preceding subsection.
If $p + q = 2$ then either

$$f \in \mathcal{P}A_1^{1,1}$$

or

$$f \in \mathcal{P}A_1^2.$$

If $f \in \mathcal{P}A_1^{1,1}$ then the bifurcation diagram is biholomorphically equivalent to the germ at the origin of the variety consisting of the values (λ, ε) for which ε is a critical value of the restriction of the function

$$(x, y) \mapsto x^4 + \lambda_1 x^3 + y^4 + \lambda_2 y^3$$

to the plane curve-germ of equation $xy = 0$.
If $f \in \mathcal{P}A_1^2$ then the bifurcation diagram is biholomorphically equivalent to

the germ at the origin of the variety consisting of the values (λ, ε) for which ε is a critical value of the restriction of the function

$$(x, y) \mapsto x^5 + \lambda_1 x^4 + \lambda_2 x^3 + y^3$$

to the plane curve-germ of equation $xy = 0$.

APPENDIX B

The finite determinacy and versal deformation theorems for $\mathcal{G}$-equivalence.

1. Proof of the finite determinacy theorem for $\mathcal{G}$-equivalence.

In this section, we use the notation $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ instead of $\tilde{f} : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ as we did before.
We denote by e the identity element of the group $\mathcal{G}$.
We used the symbols $\mu_{\mathcal{G}}$ for the $\mathcal{G}$-Minor number in order to make the distinction with the usual Minor number μ of singularity theory. We are not going to use the the usual Minor number any longer. Consequently, we simply denote by μ the $\mathcal{G}$-Milnor number of f instead of $\mu_{\mathcal{G}}$ as we did before. The other notations are those of chapter 4.
So, let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ be a holomorphic map-germ.
Fix coordinates $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}_x$ and $\mathcal{M}^k$ the k^{th} power of it.
consider the $\mathcal{O}_x$-module of map-germs vanishing at order at least k, that is

$$\widetilde{\mathcal{M}}^k = \left\{ (\psi_1, \dots, \psi_n) \in \mathcal{O}_x^n : \forall i, \ \psi_i \in \mathcal{M}^k \right\}.$$

Remark that the $\mathcal{O}_x$-module $\widetilde{\mathcal{M}}^k$ is finitely generated.
Following R. Thom, we use the homotopy method [**AVG**], [**Math**], [**Mar**].

Denote by $D \subset \mathbb{C}$ the unit disk centered at the origin

$$D = \{ t \in \mathbb{C} : \mid t \mid \leq 1 \}.$$

Consider the one-parameter family $f_t, \ t \in D$ defined by

$$f_t = f + t\psi,$$

where ψ is an arbitrary map in $\widetilde{\mathcal{M}}^{\mu+1}$.
The finite determinacy theorem for $\mathcal{G}$-equivalence asserts that f_1 is $\mathcal{G}$-equivalent to f.
To prove this theorem, we search for a holomorphic map $\gamma : D \longrightarrow (\mathcal{G}, e)$ such that:

$$\gamma(t).f_t = f. \tag{62}$$

For any value of t $\gamma(t)$ is an element of the group $\mathcal{G}$ and $\gamma(t).f$ is the image of f under the action of $\gamma(t)$.

If a holomorphic map γ satisfying equation (62) is found then the following equality holds

$$\gamma(1).(f+\psi)=f,$$

and consequently f and $f+\psi$ are $\mathcal{G}$-equivalent.
We differentiate equation (62) with respect to t at t_0.
To do this we identify the group $\mathcal{G}$ with a subgroup of the V-equivalence group $\mathcal{K}$ (see page 79).
Via this identification, we write

$$\gamma(t)=(\varphi_t, A_t)$$

where $\varphi_t:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^n,0)$ is a biholomorphic depending on the parameter $t\in D$ and A_t is a matrix of $GL(n,\mathcal{O}_x)$ also depending on the parameter t.
Equation 62 can be written in the more explicit form

$$A_t\times(f_t\circ\varphi_t)=f. \tag{63}$$

We differentiate equation 63 with respect to t at $t=u$, we get

$$A_u\times\frac{d}{dt}_{|t=u}(f_u\circ\varphi_t)+\frac{d}{dt}_{|t=u}(A_t\times(f_u\circ\varphi_u))+A_u\times(\frac{d}{dt}_{|t=u}f_t)\circ\varphi_u=0. \tag{64}$$

Define the time-dependent holomorphic vector field germ v_u and the one parameter family of matrices h_u with entries in $\mathcal{O}_x$ by the formulas

$$\begin{cases} v_u(\varphi_u(x)) &= \frac{d}{dt}_{|t=u}\varphi_t(x),\\ (A_u^{-1}\times h_u)\circ\varphi_u &= \frac{d}{dt}_{|t=u}A_t.\end{cases}$$

Multiplying equation (47) on the right by φ_u^{-1} and on the left by A_u^{-1}, we get the *homological* equation

$$v_u.f_u+h_u\times f_u+\psi=0. \tag{65}$$

Where $m_u=(v_u,h_u)$ belong to the tangent space $\mathfrak{g}$ to $\mathcal{G}$ at the identity element (see page 80).

LEMMA B.1. *Equation (63) can be solved provided that there exists a holomorphic map*

$$\begin{array}{ccc} D & \longrightarrow & \mathfrak{g} \\ u\longrightarrow(v_u,h_u) & & \end{array}$$

satisfying equation (65).

Proof.
This is a straightforward application of the fundamental theorem on differential equations.
First we search for a one-parameter family of biholomorphic map germ $g_t:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^n,0)$ such that

$$\frac{dg}{dt}\,|_{t=u}\,(t,x)=v_u(g_u(x)) \tag{66}$$

and $t \in D$.
The germ of the time dependent vector field

$$v_t : (\mathbb{C}^n, 0) \longrightarrow (T\mathbb{C}^n, 0)$$

gives rise to the germ of a vector field

$$\tilde{v} : D \times (\mathbb{C}^n, 0) \longrightarrow TD \times T(\mathbb{C}^n, 0).$$

The vector field $\tilde{v}$ being defined by the formula

$$\tilde{v}(t, x) = (1, v_t(x)).$$

We have the equality

$$v_t(0) = 0, \ \forall t \in D.$$

Hence $S = \{(t, x) \in D \times \mathbb{C}^n : x = 0\}$ is everywhere tangent to $\tilde{v}$.
The fundamental theorem on differential equations implies that in a neighbourhood of S, we can integrate the vector field $\tilde{v}$. That is there exists a biholomorphic map

$$\tilde{g} : D \times (\mathbb{C}^n, 0) \longrightarrow D \times (\mathbb{C}^n, 0)$$

such that:

$$\frac{d\tilde{g}}{dt} \,|_{t=u}\, (t, x) = \tilde{v}(\tilde{g}(u, x)),$$

where $\tilde{g}(t, x) = (t, g(t, x))$. Put $g_t = g(t, .)$. The map g_t, satisfies equation (66).
Moreover, if the one parameter family of function-germs h_t is known, then by integrating h_t along t we get the one parameter family of holomorphic function-germs B_t satisfying the equality

$$B_t = log(A_t).$$

Remark that $log(A_t)$ is well-defined since $det(A_0(0)) \neq 0$. From this equality, we get $A_t = e^{B_t}$. Lemma is proved.

Lemma B.2. *There exists a map satisfying equation 65 provided that* $\widetilde{\mathcal{M}}^\mu \subset Tf$.

Proof.
The proof is the same as the proof of Nakayama's lemma.
Fix coordinates in $\mathbb{C}^{n-1} \times \mathbb{C}$.
Denote by $v^1, \dots, v^n$ the vectors of $\mathbb{C}^{n-1} \times \mathbb{C}$ having the components:

$$\begin{cases} v_j^j &= 1 \\ v_k^i &= 0 \quad \text{for } k \neq j \end{cases}$$

We construct a basis $\alpha_1, \dots, \alpha_l$ of the $\mathcal{O}_x$-module $\widetilde{\mathcal{M}}^\mu$ as follows.
Let $e_1, \dots, e_s$ be a basis of monomials of the μ^{th} power $\mathcal{M}_x^\mu \subset \mathcal{O}_x$ of the maximal ideal $\mathcal{M}_x \subset \mathcal{O}_x$. Then the map-germs $e_j.v^k$'s form a basis of $\widetilde{\mathcal{M}}^\mu$.
It remains to prove is that there exists holomorphic maps:

$$v_1, \dots, v_l : D \longrightarrow \mathfrak{g}$$

such that:

$$v_j.f_t = \alpha_j.$$

If this assertion is proved then we write:

$$\psi = \sum_{j=1}^{l} b_j\alpha_j, \ b_j \in \mathcal{M}.$$

This is made possible by the fact that $\psi \in \widetilde{\mathcal{M}}^{\mu+1}$. Then $m_t = \sum b_j v_j \in \mathfrak{g}$ is a holomorphic map from D to $\mathfrak{g}$ satisfying equation (65).

Thus we search for the maps $v_1, \dots, v_l$.
The assumption $\widetilde{\mathcal{M}}^{\mu} \subset Tf$, implies that for all $j \in \{1, \dots, l\}$ there exists $m_j \in \mathfrak{g}$ such that:

$$m_j.f = \alpha_j.$$

Write the left-hand side of this equation in the following way

$$m_j.f = m_j.f_t - tm_j.\psi. \tag{67}$$

Remark that by definition of $\mathcal{G}$, we have the implication

$$(\psi \in \widetilde{\mathcal{M}}^{\mu+1}) \implies (m_j.\psi \in \widetilde{\mathcal{M}}^{\mu+1}).$$

Hence there exists $a_{j,k} \in \widetilde{\mathcal{M}}$ such that for all $j \in \{1, \dots, l\}$:

$$m_j.\psi = \sum_{k=1}^{l} a_{j,k} \times \alpha_k.$$

Here $\alpha_k : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ is a holomorphic map-germ. The $a_{j,k}$'s are holomorphic function germs, hence to multiply α_k by $a_{j,k}$ makes sense. Substituting these equalities in formula 67, we get a system of l-equations

$$\begin{cases} m_1.f_t = \alpha_1 + t\sum_{k=1}^{l} a_{1,k} \times \alpha_1, \\ \quad\quad \dots \\ m_j.f_t = \alpha_j + t\sum_{k=1}^{l} a_{j,k} \times \alpha_k, \\ \quad\quad \dots \\ m_l.f_t = \alpha_l + t\sum_{k=1}^{l} a_{l,k} \times \alpha_k. \end{cases} \tag{68}$$

where the $a_{j,k}$ belong to the maximal ideal $\mathcal{M}_x$.
Define the map-germs $\vec{v}_t$ and $\vec{\alpha}$ by their components

$$\vec{v}_t = (m_1.f_t, \dots, m_k.f_t),$$
$$\vec{\alpha} = (\alpha_1, \dots, \alpha_k).$$

Let A be the matrix whose coefficients are the function-germs $a_{j,k}$'s.
With the matrix formalism, the system of equations (68) is written in the following form

$$\vec{v}_t = (Id + tA)\vec{\alpha}.$$

We assert that for any $t \in D$, the matrix $B(t) = Id + tA$ is invertible and that $B^{-1}(t)$ depends holomorphically on $t \in D$.
Fix the value of $t \in D$. The coefficients of A are function germs in $\mathcal{O}_x$.

For $x = 0$ the matrix $(Id + tA)(0) = Id$ is invertible. Since we are only considering germs at the origin this is a sufficient condition for the matrix to be invertible (in the same way that for a function-germ $f(0) \neq 0$ implies that it is invertible). The formula giving the inverse of a matrix implies that the dependence on t is holomorphic.

We have:

$$\vec{\alpha} = (Id + tA)^{-1}\vec{v}_t.$$

This means that for any $j \in \{1, \ldots, k\}$, we have:

$$\alpha_j = \sum_{k=1}^{k} b_{j,k}(m_k.f_t).$$

Here the $b_{j,k}$'s are the coefficients of the matrix $(Id + tA)^{-1}$. The m_k's are elements of $\mathfrak{g}$.
Define the map-germ v_j by

$$v_j = \sum_{k=1}^{k} b_{j,k} m_k,$$

we get the equality

$$v_j.f_t = \alpha_j.$$

According to the discussion that we made at the beginning of the proof of the lemma, this concludes the proof of the lemma.

It remains to prove that the $\mathcal{O}_x$-module $\widetilde{\mathcal{M}}^\mu$ is contained in the $\mathcal{G}$-tangent space Tf to f.

Lemma B.3. $\widetilde{\mathcal{M}}^\mu \subset Tf$.

Proof.
We use the same notations as the one that we have fixed at the beginning of the proof of the preceding lemma.
The $\mathcal{O}_x$-module $\widetilde{\mathcal{M}}^\mu$ is generated by the map-germs $\alpha_1, \ldots, \alpha_l$.
Each of these monomials is of the form $e_j v^k$. Thus we have to prove that for an arbitrary j, k, $e_j v^k$ belongs to Tf.
Here e_j is a function-germ belonging to the ideal $\mathcal{M}^\mu \subset \mathcal{O}_x$ and v^k is a vector in $\mathbb{C}^{n-1} \times \mathbb{C}$. Hence $e_j v^k$ is an element of $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$.
We fix j, k. Since e_j is a monomial of $\mathcal{M}^\mu$, there exist function-germs $a_1, \ldots, a_\mu \in \mathcal{M}$ such that:

$$e_j = a_1 a_2 \ldots a_\mu.$$

The $\mathbb{C}-$vector space Tf is of codimension μ in $\mathcal{O}_x^{n-1} \times \mathcal{O}_x$. Consequently Tf contains a linear combination of the $\mu + 1$ vectors:

$$a_0 v^j,\ a_1 v^j,\ a_1 a_2 v^j,\ \ldots,\ a_1 a_2 \ldots a_\mu v^j$$

with $a_0 = 1$.
That is there exists complex numbers $c_k, c_{k+1}, \dots, c_\mu$ such that:

$$(c_k a_0 \dots a_k + \dots + c_\mu a_0 \dots a_\mu) v^j \in Tf$$

and $c_k \neq 0$.
We have the identity

$$(c_k a_0 . \dots . a_k + \dots + c_\mu a_0 . \dots . a_\mu) v^j = (c_k + \dots + c_\mu a_{k+1} . \dots . a_\mu) a_0 . \dots . a_k v^j$$

But $c_k + c_{k+1} a_{k+1} + \dots + c_\mu a_{k+1} . \dots . a_\mu$ is invertible, hence the previous identity implies the inclusion

$$a_0 . \dots . a_k v^j \in Tf.$$

Consequently $(a_{k+1} \dots a_\mu)(a_0 \dots a_k v^j) \in Tf$. Lemma is proved.
This lemma achieves the proof of the finite determinacy theorem for $\mathcal{G}$-equivalence.

2. Proof of the versal deformation theorem for $\mathcal{G}$-equivalence.

We first state an easy lemma that we shall need.
Let $C : (\mathbb{C}^r \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ be an arbitrary deformation of a holomorphic map-germ

$$f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$$

Denote by $A : \mathbb{C}^r \times (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$, $B : (\mathbb{C}^r \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ the components of C so that $C = (A, B)$.
Let $g : (\mathbb{C}^r, 0) \longrightarrow (\mathbb{C}^{n-1}, 0)$ be the holomorphic map-germ defined by:

$$g(\lambda) = A(\lambda, 0).$$

LEMMA B.4. *The deformation $C = (A, B)$ is $\mathcal{G}$-equivalent to the deformation $(A - g, B)$.*

The proof is of course obvious. However this lemma implies that we need only to consider deformations (A, B) such that $A(\lambda, .)$ vanishes.
Having made this remark, we start the proof of the theorem. The notations are those of chapter 4.
Following R. Thom, we use the homotopy method. Following Martinet [**Mar**], the fundamental step in the proof of a versal deformation theorem is the following proposition.

PROPOSITION B.1. *Let $F : (\mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ be a k-parameter deformation of a map-germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$ such that $TF = \mathcal{O}_x^{n-1} \times \mathcal{O}_x$. Then, for any holomorphic map-germ*

$$\Phi : (\mathbb{C} \times \mathbb{C}^k \times \mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n-1} \times \mathbb{C}, 0)$$

such that $\Phi(0, .) = F$, the $k+1$ parameter deformation Φ of f is equivalent to a deformation induced from F.

Proof.
We fix coordinate-systems $x = (x_1, \dots, x_n)$ in $\mathbb{C}^n$, $\lambda = (\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$, τ in $\mathbb{C}$ and a coordinate-system in $\mathbb{C}^{n-1} \times \mathbb{C}$.
In the sequel, for simplicity we write $\mathcal{O}_z^n$ instead of $\mathcal{O}_z^{n-1} \times \mathcal{O}_z$ for $z = x$ or $z = (\tau, \lambda, x)$.
Denote by e the identity element of the group $\mathcal{G} \oplus T$ where $T \approx \mathbb{C}^n$ denotes the group of translation in $\mathbb{C}^n$.
We search for a holomorphic map-germ $\gamma : (\mathbb{C} \times \mathbb{C}^k, 0) \longrightarrow (\mathcal{G} \oplus T, e)$ and a holomorphic map-germ $h : (\mathbb{C} \times \mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0)$ such that:

$$\Phi = h^*(\gamma.F), \tag{69}$$

We shall use the notations $\gamma_\tau = \gamma(\tau, .)$ and $h_\tau = h(\tau, .)$. We have the equalities

$$\gamma_0 = e, \ h_0 = Id.$$

Consequently there exists holomorphic map-germs

$$\begin{array}{r}\tilde{\gamma} : (\mathbb{C} \times \mathbb{C}^k, 0) \longrightarrow (\mathcal{G} \oplus T, e), \\ \tilde{h} : (\mathbb{C} \times \mathbb{C}^k, 0) \longrightarrow (\mathbb{C}^k, 0),\end{array}$$

such that the following equalities hold identically

$$\tilde{\gamma}(\tau, \lambda)\gamma(\tau, \lambda) = e, \ \tilde{h}(\tau, \lambda)h(\tau, \lambda) = \lambda.$$

The maps $\tilde{\gamma}(\tau, .)$ and $\tilde{h}(\tau, .)$ will be denoted by γ_τ^{-1} and by h_τ^{-1}.
Equation 69 is equivalent to the equation

$$(h_\tau^{-1})^*(\gamma_\tau^{-1}.\Phi) = F. \tag{70}$$

The map Φ has n components. So, equation 70 is in fact a system of n equations. They can be written in a more explicit form but this is unnecessary.
Before differentiating with respect to τ equation 70, we fix some notations:
- $m_\tau.\Phi = \gamma_\tau.\frac{d}{d\tau}(\gamma_\tau^{-1}.\Phi)$,
- $w_\tau.\Phi = h_\tau^*\frac{d}{d\tau}((h_\tau^{-1})^*\Phi)$.
We have $m_\tau = (\sum_{j=1}^n a_j\partial_{x_j}, \alpha_\tau)$ and $\alpha_\tau \in \mathcal{O}_x^{n-1} \times \mathcal{O}_x$, the a_j's being holomorphic map-germs belonging to $\mathcal{O}_{\tau,\lambda}$.
We differentiate with respect to τ equation 70 and multiply the result on the left by h_τ^* and then by γ_τ.
We get the so-called homological equation:

$$\Phi_n \times \alpha_\tau + v_\tau.\Phi + \partial_\tau\Phi = 0. \tag{71}$$

with $\Phi = (\Phi_1, \dots, \Phi_n)$.

The vector fields $v_\tau, \ w_\tau, \ t_\tau$ vanish at the origin. Consequently, the ordinary theorem on non-autonomous differential equation implies that it is sufficient to find the maps $v_\tau, \ w_\tau, \ t_\tau$ in order to find $\gamma, \ h$.

We interpret equation 71 as follows.
Consider the $\mathbb{C}$-vector subspace of $\mathcal{O}_{\tau,\lambda,x}^n$ which is the sum of the following

modules:
- the $\mathcal{O}_{\tau,\lambda,x}$-module consisting of all maps of the form $\Phi_n \times \alpha_\tau + v_\tau.\Phi$ where (v_τ, α_τ) is the value at τ of an arbitrary holomorphic map-germ $m = (v, \alpha)$: $(\mathbb{C} \times \mathbb{C}^k, 0) \longrightarrow \mathfrak{g}$,
- the $\mathcal{O}_{\tau,\lambda}$-module generated by the $\partial_{\lambda_i}\Phi$'s.
Main assertion:" the $\mathbb{C}$-vector space described above is equal to $\mathcal{O}^n_{\tau,\lambda,x}$ provided that the equality $TF = \mathcal{O}^{n-1}_{\tau,\lambda,x} \times \mathcal{O}_{\tau,\lambda,x}$".
Then equation (71) can be solved provided that this main assertion is proved. To this aim, we use the Weierstrass-Malgrange-Mather ([**Mar**] chapter X) theorem.

THEOREM B.1. *Let M be an $\mathcal{O}_{z,x}$-module of finite type. Let $N \subset M$ be an $\mathcal{O}_z$ module of finite type. Denote by $\mathcal{M}_z$ the maximal ideal of $\mathcal{O}_z$. Let $\pi : M \longrightarrow M/(\mathcal{M}_z M)$ be the standard projection. Then $N = M$ provided that $\pi(M) = \pi(N)$.*

Denote by $e^k \in \mathbb{C}^n$ the vector whose coordinates are all zero except the k^{th} coordinate which is equal to 1.
Consider the following modules:
- $M = \mathcal{O}^n_{\tau,\lambda,x}/I$ where I is the $\mathcal{O}_{\tau,\lambda,x}$-module generated by the $\partial_{x_k}\Phi$'s and by the $e^k\Phi_n$'s,
- N is the $\mathcal{O}_{\tau,\lambda}$-submodule of M generated by the images of the $\partial_{\lambda_i}\Phi$'s under the canonical projection $\mathcal{O}^n_{\tau,\lambda,x} \longrightarrow M$.

The main assertion is equivalent to the equality $M = N$.
We prove that this equality holds by applying the Weierstrass-Malgrange-Mather theorem with $z = (\tau, \lambda)$.
The map π can be identified with the map sending a map-germ of $\mathcal{O}^n_{\tau,\lambda,x}$ to its restriction to $\lambda = 0,\ \tau = 0$ in $\mathcal{O}_x$.
Consequently, we get the natural identifications:
- $\pi(M) = \mathcal{O}^n_x/(Tf \oplus V)$ where Tf is the $\mathcal{G}$-tangent space to f and V is the $\mathbb{C}$-vector space generated by the $\partial_{x_k} f$'s ,
- $\pi(N)$ is the $\mathbb{C}$-vector space generated by the images under the canonical projection $\mathcal{O}_x \longrightarrow \mathcal{O}_x/(Tf \oplus V)$ of the restrictions to $\lambda = 0$ of $\partial_{\lambda_1} F, \ldots, \partial_{\lambda_k} F$.

With these identification, the equality $\pi(N) = \pi(M)$ is equivalent to the equality $TF = \mathcal{O}^{n-1}_x \times \mathcal{O}_x$. This equality holds and the main assertion is proved. This concludes the proof of the proposition.

The rest of the proof of the $\mathcal{G}$-versal deformation theorem is straightforward. Let $G : (\mathbb{C}^s \times \mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a s-parameter deformation of f satisfying the condition $TG = \mathcal{O}^n_x$.
Let $\tilde{G} : (\mathbb{C}^{s'} \times \mathbb{C}^n \times \mathbb{C}^k, 0) \longrightarrow (\mathbb{C}, 0)$ be an arbitrary s'-parameter deformation of f.
Fix coordinates $\lambda = (\lambda_1, \ldots, \lambda_s)$ in $\mathbb{C}^s$, $\mu = (\mu_1, \ldots, \mu_{s'})$ in $\mathbb{C}^{s'}$. Define the

sum
$$G \oplus \tilde{G} : (\mathbb{C}^s \times \mathbb{C}^{s'} \times \mathbb{C}^n \times \mathbb{C}^k, 0) \longrightarrow (\mathbb{C}, 0)$$
of the deformations G and $\tilde{G}$ by:
$$G \oplus \tilde{G}(\lambda, \mu, x) = G(\lambda, x) + \tilde{G}(\mu, x) - f(x).$$
The restriction of $G \oplus \tilde{G}$ to $\lambda = 0$ is equal to $\tilde{G}$. Hence, $\tilde{G}$ is induced from a deformation equivalent to G provided that $\tilde{G} \oplus G$ is induced from a deformation equivalent to G.

LEMMA B.5. *The deformation $G \oplus \tilde{G}$ is induced from a deformation equivalent to G.*

Proof.
For $i \leq s'$, denote by A_i the restriction of $G \oplus \tilde{G}$ to the vector subspace $\mu_1, \dots, \mu_i = 0$.
We apply proposition B.1 with $\Phi = A_{s+s'-1}$, $F = G$. We get that $A_{s+s'-1}$ is induced from a deformation equivalent to G. In particular $TA_{s+s'-1} = \mathcal{O}_x^n$.
Next, we apply proposition B.1 with $\Phi = A_{s+s'-2}$ and $F = A_{s+s'-1}$. We get that $A_{s+s'-2}$ is induced from a deformation equivalent to $A_{s+s'-1}$. Hence $A_{s+s'-2}$ is induced from a deformation equivalent to G. By induction we get that $G \oplus \tilde{G}$ is induced from a deformation equivalent to G. This lemma concludes the proof of the $\mathcal{G}$-versal deformation theorem.

APPENDIX C

Other results concerning PT-monodromy groups

1. Statement of the results

We fix some notations.
Take a linear coordinate system X, Y, Z in the vector space $\mathbb{C}^3$.
Identify $\mathbb{C}P^2$ with the set of lines of $\mathbb{C}^3$ passing through the origin. We denote by $[X : Y : Z]$ the line passing through the point of coordinate (X, Y, Z). The set of lines for which $Z \neq 0$ can be identified with the affine plane $\mathbb{C}^2$, we shall use the notations $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ for the corresponding affine coordinate system.
To a homogeneous polynomial P in three variables, we associate the curve $V[P] \subset \mathbb{C}P^2$ defined by

$$V[P] = \{[X : Y : Z] : P(X, Y, Z) = 0\}.$$

Let B be the topological space of homogeneous polynomials P of degree 4 such that the curve $V[P]$ is smooth and does not have degenerate inflection points.
Applying Bézout's theorem to the curve $V[P]$ and to its Hessian we get that for $P \in B$, the number of inflection points of $V[P]$ is 24.
Denote by $\xi : E \longrightarrow B$ the topological covering whose fibre at the point $P \in B$ is the set formed by the 24 inflection points of $V[P]$.
Fix a polynomial $P_0 \in B$, the fundamental group $\pi_1(B, P_0)$ acts on the fibre $\xi^{-1}(P_0)$ by monodromy. Consequently, we get a group homomorphism:

$$\rho_\xi : \pi_1(B, P_0) \longrightarrow \mathcal{S}_{24}.$$

The image of the homomorphism ρ_ξ defines an abstract group G called the monodromy group of the covering $\xi : E \longrightarrow B$. This is a basic notion of topology.

Theorem C.1. *The monodromy group of the covering $\xi : E \longrightarrow B$ is the full permutation group $\mathcal{S}_{24}$.*

1.1. Scheme of the proof.

This theorem is proved in three steps.
The three steps of the proof of theorem C.1 are summarized by the following three lemmas.
Put $F(\lambda, x, y) = \lambda_1 x^2 y + \lambda_2 x^3 + \lambda_3 x^2 + \lambda_4$.

LEMMA C.1. *The PT-monodromy group of the family of (affine) curves* (V_λ) *defined by*

$$V_\lambda = \left\{(x,y) \in \mathbb{C}^2 : \frac{1}{2}y^2 + \frac{1}{4}x^4 + F(\lambda, x, y) = 0\right\}$$

is equal to $\mathcal{S}_{12}$.

LEMMA C.2. **??** *The PT-monodromy group of the family of (affine) curves* $(W_{\lambda,\alpha})$ *defined by*

$$W_\varepsilon = \left\{(x,y) \in \mathbb{C}^2 : \frac{1}{2}(y-x)^2 - \frac{1}{5}x^3 y - \varepsilon = 0\right\}$$

contains a subgroup isomorphic to $\mathbb{Z}_5$ *which permutes cyclically* 15 *of the inflection points.*

Put $P(\lambda, X, Z) = \lambda_1 X^3 Z + \lambda_3 X^2 Z^2 + \lambda_4 Z^4$.

LEMMA C.3. *The PT-monodromy group of the family of (projective) curves* (S_λ) *defined by*

$$S_{\lambda,\mu} = \left\{[X:Y:Z] \in \mathbb{C}P^2 : Y^2Z^2 + X^4 + X^2YZ + P(\lambda, X, Z) + P(\mu, X, Y) = 0\right\}$$

contains a subgroup equal to $\mathcal{S}_{12} \times \mathcal{S}_{12}$.

Before proving each of the lemmas, we make some straightforward remarks. The only subgroup of $\mathcal{S}_15$ wich contains a cyclic permutation of order 5 permuting ciclically 15 points and a subgroup isomorphic to $\mathcal{S}_{12}$ is $\mathcal{S}_{15}$. Consequently, the first two lemmas imply that the monodromy group of $\xi : E \longrightarrow B$ contains a subgroup isomorphic to $\mathcal{S}_{15}$.
The only subgroup of $\mathcal{S}_{24}$ containing both a subgroup isomomorphic to $\mathcal{S}_{12} \times \mathcal{S}_{12}$ and a subgroup isomorphic to $\mathcal{S}_{15}$ is $\mathcal{S}_{24}$. Consequently, the three lemma imply theorem C.1.
Next, remark that lemma C.3 is a corollary of lemma C.1. Indeed denote by Σ the Plücker discriminant of the family $S_{\lambda,\mu}$. For $(\lambda, \mu) \notin \Sigma$ the curve $S_{\lambda,\mu}$ has 24 inflection point.
Take λ and μ small enough.
When $\lambda \longrightarrow 0$, there are 12 inflection point that vanish at the point $[0:0:1]$ (theorem **??** **??**. Similarly, when $\mu \longrightarrow 0$, there are 12 other inflection point that vanish at $[0:1:0]$. If we restrict the family $(S_{\lambda,\mu}$ to $\mu = 0$ and consider the curve $S_{\lambda,0} \setminus (S_{\lambda,O} \cap \{Z = 0\})$ we get the family V_λ. Thus lemma C.1 implies that the *PT*-monodromy group of $S_{\lambda,\mu}$ contains all the permutation on the 12 inflection points that vanish at $[0:0:1]$. In a similar way, we get that the *PT*-monodromy group of $S_{\lambda,\mu}$ contains all the permutation on the 12 inflection points that vanish at $[0:1:0]$. Thus lemma C.3 is a consequence of lemma C.1.
Consequently it remains to prove lemma C.1 and lemma **??**.

1.2. Proof of lemma ??. The proof of the lemma is done by a direct computation.
Let f be the polynomial defined by

$$f(x,y) = \frac{1}{2}(y-x^2)^2 - \frac{1}{5}x^3y.$$

The curves W_ε are the level-curves $f = \varepsilon$ of f.
The function f has an A_4 critical point at the origin. This means that there exists neighbourhoods U and V of the origin and a biholomorphic map $\varphi : U \longrightarrow V$ such that:

$$f \circ \varphi^{-1}(x,y) = \frac{1}{2}y^2 - \frac{1}{5}x^5.$$

By direct computations, we get that the Taylor expansion of φ at the origin is given by:

$$\varphi(x,y) = (x + \dots, y - x^2 + \dots).$$

We explain the meaning of the dots.
In the space $\mathcal{O}_{x,y}$ of holomorphic function-germ, we introduce a quasi-homogeneous filtration by fixing the weight of x to be equal to 2 and the weight of y to be equal to 5. the dots stand for higher order terms in this quasi-homogeneous filtration.
We compute the generalized Wronskian of φ with respect to $H(x,y) = \frac{1}{2}y^2 - \frac{1}{5}x^5$ (see page **??**).
With the notations of section **??**, we get that

$$\mathcal{D}[\varphi] = -[x,x^2] + [x,y] + \dots$$

where the dots denote higher order terms in the quasi-homogeneous filtration defined above.
We have the equalities

$$[x,x^2] = \left| \begin{matrix} y & -2xy \\ -x^4 & -2y^2 + x^5 \end{matrix} \right|$$

and

$$[x,y] = \left| \begin{matrix} y & -x^4 \\ -x^4 & -4xy^3 \end{matrix} \right|.$$

Consequently, the Taylor expansion at the origin of $\mathcal{D}[\varphi]$ is given by

$$\mathcal{D}[\varphi] = y^3 + x^3y^2 + x^8 + \dots$$

where the dots denote higher order terms i the quasi-homogeneous filtration.
The zero level-set of $\mathcal{D}[\varphi]$ has two branches at the origin are parameterized by two convergent power series

$$\begin{cases} t & \mapsto \quad (t, t^3 + \dots) \\ t & \mapsto \quad (t^2, t^5 + \dots) \end{cases}$$

where the dots denote higher order terms.
Substituting these series in H we get that the parameters corresponding to the inflection points satisfy the equations

$$\begin{cases} t^5 + \cdots = \varepsilon \\ t^{10} + \cdots = \varepsilon \end{cases}$$

Consequently, when ε make a small turn around the origin the 5 points corresponding to the first equation are cyclically permuted while the 10 points corresponding to the second equation are also‘ cyclically permuted. This proves lemma **??**.

1.3. Proof of lemma C.1. Define the polynomial f by

$$f(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2.$$

The germ of f at the origin has a critical point belonging to the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{1,1}$.
Consider the two-parameter deformation $F : (\mathbb{C}^2 \times \mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ defined by the formula

$$F(\lambda, x, y) = f + \lambda_1 x^2 y + \lambda_2 x^3.$$

LEMMA C.4. *The deformation F is a $\mathcal{P}$-versal deformation of f.*

THe proof of this lemma is analoguous to the proof given at page **??**f theorem **??**.
Consequently, theorem **??** implies that:
1) there are the eight inflection points of f_ε that vanish at the origin when $\varepsilon \longrightarrow 0$,
2) the vanishing inflection points are given by two formal power series of the type

$$\begin{cases} x_i(\varepsilon) = a_i \varepsilon^{1/4} + o(\varepsilon^{1/4}) \\ y_i(\varepsilon) = b_i \varepsilon^{1/4} + o(\varepsilon^{1/4}) \end{cases}$$

3) the PT-monodromy group of any representative of F contains all the permutation on the 4 points that vanish along the same branch ($y = x$ or $y = -x$).
The first two facts can easily be proved by explicit computations.
For our computations it is needed to number these eight points. We chose this numbering as follows. For $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$, $p_1(\varepsilon), \dots, p_4(\varepsilon)$ denote the points vanishing along the branch $y = x$ and $p_1(\varepsilon), \dots, p_4(\varepsilon)$ denote the points vanishing along the branch $y = -x$.
The x-coordinate of the points p_1 and p_5 are real positive. The point p_{k+1} is the image of p_k when ε makes a turn couter-clockwise around the origin. In first approximation p_{k+1} is obtain from p_k by multiplying it by $i = \sqrt{-1}$. To avoid complicated notations the point $p_k(\varepsilon)$ will be denoted by the letter k, omiting the dependence on ε.
In figure **??**, we depicted the projections of the inflection points of V_ε on the x-line and on the y-line (the lines are complex lines hence two dimensional).

The notation (k,l) on this figure means that the point k and the point l are projected to the corresponding marked point on the line.
As we pointed out before the PT-monodromy group of W_λ contains all the permutations on $1,2,3,4$ and all the permutations on $5,6,7,8$.
The following assertions are straightforward
1) for any value of ε the curve f_ε has no degenerate inflection point,
2) the curve V_ε is singular only for the values $\varepsilon = 0$ and $\varepsilon = -\frac{1}{4}$, 3) the germs of f at the points $x = 1$ and $x = -1$ belong to the $\mathcal{P}$-singularity class $\mathcal{P}A_1$.
We now search the inflection points that vansih at $x = \pm 1$ when $\varepsilon \longrightarrow -\frac{1}{4}$.
Once these points are known the monodromy is given by theorem **??** at **??**.
Let ε be a small real strictly positive number.
The curve V_ε has 12 inflection points. We have chosen the numbering of 8 of the points. To number the other 4 points, we first need the following lemma.

LEMMA C.5. *The 4 inflection point that do not vanish at the origin when $\varepsilon \longrightarrow 0$ have real and opposite x-coordinates and pure imaginary conjuguate y-coordinates (provided that ε be a small real strictly positive number).*

Proof.
The implicit function theorem implies that it suffices to prove the lemma for the curve V_0.
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So we fix the numbering as indicated by figure **??**.
In the space of the parameter ε, we chose the path as indicated by figure **??**, where A, B are choosen sufficiently close to the origin so that theorem **??** giving the normal form of the PT-covering in a small neighbourhood of the origin can be applied to a neihgbourhood containing A and B.
Consequently, theorem **??** implies that at the point B the points are located as indicated by figure **??**.
By location, we take care of the quadrant(s) to which the point belongs (if it is real or pure imaginary it belongs to 2 quadrants).
The fundamental assertions of this subsection are the two lemma that follow.

LEMMA C.6. *When ε goes from B to C, the x-labels of the inflection points stay in the same quadrants.*

Proof.
The projection of f_e on the x-line namely

$$\begin{array}{ccc} f_\varepsilon & \longrightarrow & \mathbb{C} \\ (x,y) & \longrightarrow & y \end{array}$$

is a ramified covering of degree 2.
Since the equations of the inflection are with real coefficients if x is the x-label of an inflection point then its complex conjuguate $\bar{x}$ is the x-label of

the conjuguate inflection point.
The equality $f(x,y) = f(-x,y)$ implies that if x is the x-label of an inflection point then $-x$ is also the x-label of an inflection point.
Assume that for some value ε_0, the labels (x_k, y_k) are such that the value $x_k(\varepsilon)$ converges to a point of the real axis in the x-line when $\varepsilon \longrightarrow \varepsilon_0$. Then, the same thing occurs for the conjuguate labels $(\bar{x}_k, \bar{y}_k)$. Since, the curve f_ε is invariant under the involution $(x,y) \longrightarrow (-x,y)$, the same thing occur for the inflection points $(x_k, -y_k)$ and $(\bar{x}_k, -\bar{y}_k)$.
The projection p of f_e on the x-line namely

$$p: \begin{array}{ccc} f_{\varepsilon_0} & \longrightarrow & \mathbb{C} \\ (x,y) & \longrightarrow & y \end{array}$$

is a ramified covering of degree 2. Consequently the four points $\pm x_k(\varepsilon_0), \pm\bar{x}_k(\varepsilon_0)$ cannot correspond to four distinct inflection points under p. Consequently the curve f_{ε_0} has a degenerate inflection point. This contradict assertion 2) stated above.

LEMMA C.7. *When ε goes from B to C, the y-labels of the inflection points stay in the same quadrants.*

Proof.
Put $w = y^2$, $z = x^2$. In the variables w, z the system of equations (**??**) whose solutions are the inflection points of f_ε becomes

$$\{$$

Write $w = w_1 + iw_2$, $z = z_1 + iz_2$ with $w_1,\ w_2,\ z_1,\ z_2 \in \mathbb{R}$. Then, direct computations show that if there exists a solution (w,z) of the system of equation (**??**) such that $w_2 = 0$ then either $z_2 = 0$ or $w = 0$.
The solution $z_2 = 0$ is forbidden otherwise we would have a solution of the system (**??** with $z \in \mathbb{R}$. Such a solution corresponds to a solution of the system of equations (**??**) with either $x \in \mathbb{R}$ or $ix \in \mathbb{R}$. This would contradict lemma **??**. Lemma C.7 is proved.

With the help of the two preceeding lemmas, we can compute what are the numbers of the 6 inflection points vanishing at $(x,y) = (1,0)$ and the 6 inflection points vanishing at $(x,y) = (-1,0)$ when $\varepsilon \longrightarrow -\frac{1}{4}$. The two lemmas give also the branches along wich the inflection point vanish.
If three inflection points numbered j,k,l vanish for $\varepsilon \longrightarrow -\frac{1}{4}$ are such that the x-coordinate of j is real, the imaginary part of the x-coordinate of k is positive and the imaginary part of the x-coordinate of l is negative, we say that the *triangle denoted $T(j,k,l)$ vanish* at the given point. As a corollary of the two preceeding lemma, we get the following lemma.

LEMMA C.8. *The 6 inflection points vanishing at $(1,0)$ form two triangles $T(9,5,4)$ and $T(10,1,8)$. The 6 inflection points vanishing at $(-1,0)$ form two triangles $T(12,7,2)$ and $T(11,3,6)$.*

Using theorem **??** of page **??**, we get that the PT-monodromy group of F contains the cyclic permutations:
$c_1 = (9,5,4)(10,1,8)$ and $c_2 = (12,7,2)(11,3,6)$.
The following lemma concludes the roof of proposition **??**.

LEMMA C.9. *The subgroup of* $\mathcal{S}_{12}$ *containing* $c_1,\ c_2$ *anbd the subgroup of all permutations on* $\{1,2,\ldots,8\}$ *is equal to* $\mathcal{S}_{12}$.

Proof.
The proof is elementary. Indeed, denote by G the subgroup of $\mathcal{S}_{12}$ containing $c_1,\ c_2$ anbd the subgroup of all permutations on $\{1,2,\ldots,8\}$. Then G contains the transpositions:

$$c_2(5,8)c_2^{-1} = (9,8),\ c_2^{-1}(9,8)c_2 = (5,10).$$

Thus G contains all the permutations on $\{1,2,\ldots,10\}$.
Next, G contains the transpositions:

$$c_1(3,1)c_1^{-1} = (1,11),\ c_1(7,1)c_1^{-1} = (1,12)$$

and consequently is equal to $\mathcal{S}_{12}$.
Lemma is proved.
This lemma concludes the proof of proposition C.1 part 1.

APPENDIX D

A guide for the reader.

1. Notations that are commonly used (I).

Unless, we mention explicitly the contrary, the following notations are used from chapter 1 to chapter 6.
$\circ$ is the composition law between maps, functions.
$\times$ the multiplication between maps germs or a map with a matrix.
Rules of priority are: first the composition laws, second the multiplications and the additions:

$$\alpha \times f \circ \varphi + g = (\alpha \times (f \circ \varphi)) + g.$$

$\mathbb{C}^k$ the affine space, the vector space or the analytic variety depending on the context.
$\mathbb{C}^*$ the multiplicative group $\mathbb{C} \setminus \{0\}$.
$\mathcal{O}_x$ the ring of holomorphic function germs of the type $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$.
$\mathcal{O}_x^*$ the multiplicative group of holomorphic function germs of the type $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$ with $f(0) \neq 0$.
$\mathcal{O}_x^k$ the ring of holomorphic map-germs of the type $f : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}^k$.
$\mathcal{M}_x$ the maximal ideal of $\mathcal{O}_x$ consisting of function-germs vanishing at the origin.
$\mathcal{M}_x^k$ the k^{th} power of $\mathcal{M}_x$.
$\mathcal{O}_{x,y}$ the ring of holomorphic function germs of the type $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$.
$\mathcal{O}_{x,y}^*$ the multiplicative group of holomorphic function germs of the type $f : (\mathbb{C}^2, 0) \longrightarrow \mathbb{C}$ with $f(0) \neq 0$.
$\mathcal{M}_{x,y}$ the maximal ideal of $\mathcal{O}_{x,y}$.
$\mathcal{M}_{x,y}^k$ the k^{th} power of $\mathcal{M}_{x,y}$.
$\mathcal{M}$ the maximal ideal of $\mathcal{O}_x$ or of $\mathcal{O}_{x,y}$.
F_d the vector space of elements of $\mathcal{O}_x$ of order d for a given quasi-homogeneous filtration.
G_d the vector subspace of $\mathcal{O}_x$ generated by the monomials of weight d for a given quasi-homogeneous filtration.
$GL(n, \mathcal{O}_x)$ the group of invertible $n \times n$ matrix with coefficients in $\mathcal{O}_x$.
$GL(n, \mathcal{O}_{x,y})$ the group of invertible $n \times n$ matrix with coefficients in $\mathcal{O}_{x,y}$.
$Diff(n)$ the group of biholomorphic map germs from $\mathbb{C}^n$ to itself, preserving the origin.
$diff(n)$ the space of germs of holomorphic vector-fields that vanish at the origin.

$\mathcal{G}$ the $\mathcal{G}$-equivalence group.
$\mathcal{H}$ the $\mathcal{H}$-equivalence group.
$\mathcal{K}$ the V-equivalence group.
$\gamma.f$ the image of a map germ f under the action of $\gamma \in \mathcal{G}$.
$m.f$ the image of a map germ f under the action of $m \in \mathfrak{g}$.
$v.f$ the Lie derivative of a map germ f along a vector field v.
$\mu_{\mathcal{G}}$ the $\mathcal{G}$-Minor number.
μ the usual Milnor number.
TF the $\mathcal{G}$-tangent space to a deformation F.
Tf the $\mathcal{G}$-tangent space to a holomorphic map-germ.

$N(f)$ the number of vanishing flattening points of a complete intersection map-germ.
$N[X]$ the generic number of vanishing flattening points for a given singularity class X.
$J_0^k(M,N)$ space of jets of order k at the origin of (holomorphic or C^∞ depending on the context) function-germs $f : M \longrightarrow N$.
$j_0^k f$ jet of order k at the origin of a (holomorphic or C^∞ depending on the context) function-germs $f : M \longrightarrow N$.
$J^k(M,N)$ space of jets of order k of (holomorphic or C^∞ depending on the context) function-germs $f : M \longrightarrow N$.
$j^k f$ jet of order k a (holomorphic or C^∞ depending on the context) function-germs $f : M \longrightarrow N$.
$\mathcal{D}[\varphi]$ the generalized Wronskian.
Δ_f the generalized Hessian.
$\mathcal{F}_n$ the flattening space.
$\pi[p]$, $\pi[p,q]$ the $\mathcal{P}A_1^p$ and the $\mathcal{P}A_1^{p,q}$ Lyaschko-Loojenga mapping.
$Br(k)$ the braid group with k elements.
$B(X,p)$ the configuration of p elements in a topological space X.
$B(X,p,q)$ the configuration of p white elements and q black elements in a topological space X.
$\widetilde{H}_k(X)$ the reduced homology group with coefficients in $\mathbb{Z}$ of the topological space X.

2. Notations which are commonly used (II).

Unless, we mention explicitly the contrary, the following notations are used in chapter 7 (we do not repeat the notations which are common with the other chapters).
$\mathbb{R}^{n+1}$ either the Euclidean space, the affine space or the C^∞ manifold depending on the context.
$\mathbb{R}P^n$ the n-dimensional real projective space.
$J^k(\mathbb{R}^n,\mathbb{R})$ the space of k-jets of C^∞ functions from $\mathbb{R}^n$ to $\mathbb{R}$.
$L^\vee$ the Legendre dual manifold to a Legendre manifold $L \subset J^1(\mathbb{R}^n,\mathbb{R})$.
$V^\vee$ the Legendre dual wave front to a given wave front $V \subset J^0(\mathbb{R}^n,\mathbb{R})$.

$M^\vee$ the dual variety to a variety $M \subset \mathbb{R}P^n$.
$\mathcal{E}_t$ the ring of C^∞ function-germs of the type $f : (\mathbb{R}^k, 0) \longrightarrow \mathbb{R}$.
$\mathcal{E}_q$ the ring of C^∞ function-germs of the type $f : (\mathbb{R}^{n+1}, 0) \longrightarrow \mathbb{R}$.
$\mathcal{E}_{q,t}$ the ring of C^∞ function-germs of the type $f : (\mathbb{R}^{n+1} \times \mathbb{R}^k, 0) \longrightarrow \mathbb{R}$.
$\mathcal{M}_t$ the maximal ideal of $\mathcal{E}_t$.
$\mathcal{M}_q$ the maximal ideal of $\mathcal{E}_q$.
$\mathcal{M}_{q,t}$ the maximal ideal of $\mathcal{E}_{q,t}$.
$\bar{\mathcal{M}}_{q,t}$ the $\mathcal{E}_q$-submodule of $\mathcal{E}_{q,t}$ of function-germs that vanish for $q = 0$.
F_d the vector space of elements of $\mathcal{E}_{q,t}$ of order d for a given quasi-homogeneous filtration.
G_d the vector subspace of $\mathcal{E}_{q,t}$ generated by the monomials of order d for a given quasi-homogeneous filtration.

3. A quick survey of the thesis.

In chapter 1, we discuss a fundamental example (Plücker theorem) from an informal standpoint.
In chapter 2, we state the results that we have obtained concerning the classical theory of vanishing flattening points of curves.
After giving the basic definitions in section 1, we generalize Plücker's theorem in section 2.
In section 3, we define the Plücker discriminant which plays the role of the bifurcation diagram of families of curves with respect to the flattening points. We give most of the generic Plücker discriminant arising in 2 and 3 parameters families of curves.
The results of section 3 are mainly a motivation for the theory developed in section 4.
In section 4, an equivalence relation between function-germs of the type

$$f : (\mathbb{K}^2, 0) \longrightarrow (\mathbb{K}, 0)$$

is introduced. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and the germs that we consider are holomorphic if $\mathbb{K} = \mathbb{C}$ and C^∞ if $\mathbb{K} = \mathbb{R}$. The notation $\mathbb{K}^2$ stands for the affine plane. The equivalence relation, called $\mathcal{P}$-equivalence, preserves the inflection points of the level curves of f.
A notion of modality, called $\mathcal{P}$-modality, analogous to the usual modality notion of singularity theory is introduced. The $\mathcal{P}$-simple elements are those for which the $\mathcal{P}$-modality is equal to zero. The classification of $\mathcal{P}$-simple function-germs is given.
In section 5, we introduce the notion of $\mathcal{P}$-versal deformations and compute the $\mathcal{P}$-versal deformations of almost all the $\mathcal{P}$-simple function-germs.
Most of the results of section 3 follow from the $\mathcal{P}$-versal deformation theorem stated in subsection 5.3.
In section 6, we generalize the $\mathcal{P}$-equivalence relation for curves in higher dimensional spaces. Under some assumptions, a lower bound for the $\mathcal{P}$-modality is given for curves in $\mathbb{C}^n$ for $n > 2$. In particular, the list of the

$\mathcal{P}$-simple function-germs contains all the $\mathcal{P}$-simple elements in all dimensions.
In section 7, we define the discrete invariants related to the extrinsic projective structure. We compute most of these invariants for the $\mathcal{P}$-simple function-germs. The section ends with the statement of a $K(\pi, 1)$ theorem for the $\mathcal{P}$-simple singularities.

In chapter 3, we generalize the classical notion of bordered Hessian of plane curves to one-dimensional complete intersections.
This allows us to generalize the classical Plücker theorem to local one-dimensional complete intersection. We also give a splitting of the formula and show the relation with Teissier polar invariants.
We could have given a more general proof avoiding the hypothesis that the curves are local one-dimensional complete intersection. We did not make this choice because only the elementary proof shows the relation with polar invariants.

Chapter 4 repeats the notions introduced in chapter 2 section 4 and section 5 with more details and in a slightly more general context.
In section 1 and section 2, we give detailed accounts on $\mathcal{G}$-equivalence and on $\mathcal{P}$-equivalence.
In section 3, by means of examples, we show how to use the methods developed in the preceding two sections.
Once the theory is settled, the proofs the theorems of chapter 2 are straightforward. The details are given in appendix A.

In chapter 5, we compute the fundamental group of the complement of the $\mathcal{P}$-discriminant of the $\mathcal{P}$-simple singularities and prove the $K(\pi, 1)$ theorem stated in chapter 2, subsection 7.6.
To do this, we introduce a variant of the Lyaschko-Loojenga mapping.

In chapter 6, we prove the theorem on the $\mathcal{P}$-modality stated in chapter 2 section 6. To do this, we investigate the multi-valued functions giving the coordinates of the flattening points of families of curves.
In section 1, an equivalence relation, called $\mathcal{H}$-equivalence, between these multi-valued functions is introduced. By these means, we get the lower bound for the $\mathcal{P}$-modality in section 2.

In the second part of this thesis, we study the Legendrian counterpart of the classical theory. This leads us to new problems in the theory of wave front propagation.
In chapter 7 we give the results that we have obtained concerning this relationship. The most simple corollary of the methods developed in the sequel is the stability of a certain bifurcation diagram, called the folded umbrella, arising in the study of parabolic curves on surfaces (section 1, theorem 7.1).

In section 2 and section 3, we recall the theory of singularities of wave fronts developed by Arnold and Zakalyukin in the seventies'.
In section 4, we discuss the geometric aspects of this theory.
In section 5, we give the start of a classification of non-generic wave-fronts.
In section 6, a versal deformation theory of non-generic wave fronts is introduced called $\mathcal{L}$-versal deformation theory.
We state a $\mathcal{L}$-versal deformation theorem and compute the $\mathcal{L}$-versal deformations of the start of the classification that we gave previously.
In chapter 8, we prove the theorems stated in chapter 7.
In section 1, we adapt the classical tools of quasi-homogeneous filtrations and normal form theory to the case of germs of generating families of Legendre manifolds.
In section 2 we prove the theorems stated in sections 5 and section 6.
In section 3, we prove the theorems for the reduction to normal forms of section 1 and the $\mathcal{L}$ versal deformation theorem.

In appendix A, using the theory developed in chapter 4, we give the details of the computations that are needed in order to prove the theorems of chapter 2 giving the $\mathcal{P}$-normal forms and $\mathcal{P}$-versal deformation theorem for the $\mathcal{P}$-singularity class $\mathcal{P}A_1^{p,q}$.

In appendix B, we prove the finite determinacy and the versal deformation theorem for $\mathcal{G}$-equivalence.

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