Ties for the integral group ring of the symmetric group

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Conventions

Some of them are repeated in the text.

1. We freely use the notations of maps on the left, on the right, as an upper index etc.
2. For a, $b \in \mathbf{Z}$ we denote the integral interval by
$$[a, b] := \{x \in \mathbf{Z} \mid a \leq x \leq b\}.$$
3. Let $n \geq 1$ be an integer. Let
$$\mathcal{S}_n := \mathrm{Aut}_{\,\mathrm{Sets}}[1, n]$$
be the group of permutations of n elements, which are written in cycle notation, so that e.g. $(124)(35)$ sends 1 to 2, 2 to 4, 4 to 1 and interchanges 3 and 5. We write composition of permutations on the right, so that e.g. $(123)(12) = (23)$. The sign of a permutation $\sigma \in \mathcal{S}_n$ is denoted by ε_σ.
4. Throughout, the letter n is reserved to denote the number of permuted elements, i.e. 'the n as in $\mathcal{S}_n$'. This convention does not extend to variables such as n_λ etc. Also exception is made in case we regard a single symmetric group. p denotes an arbitrarily chosen integral prime number, except stated otherwise.
5. Conjugation in a group G is denoted by $g^h := h^{-1}gh$, ${}^hg := hgh^{-1}$, where $g, h \in G$. Analogously conjugation with units in a ring.
6. ∂ sometimes denotes **Kronecker's delta.** I.e. for elements x, y taken from some set, we let $\partial_{x,y} = 1$ for $x = y$ and $\partial_{x,y} = 0$ for $x \neq y$.
7. Let $1 = \sum_\lambda \varepsilon^\lambda$ be the orthogonal decomposition of the $1 \in \mathbf{Q}\mathcal{S}_n$ into rational central primitive idempotents ε^λ. Here ε^λ acts on the Specht lattice S^λ as the identity, λ being a partition of n (cf. 4.1.1). Let
$$Q^\lambda := \varepsilon^\lambda \mathbf{Z}\mathcal{S}_n$$
be the **quasiblock** associated to λ.
8. More generally, for an arbitrary R-order Λ, R being an integral domain with field of fractions K, and a central primitive idempotent ε of $K \otimes_R \Lambda$, the R-order $\Lambda\varepsilon$ is called a **quasiblock** of Λ. A Λ-lattice X is called **simple** provided $K \otimes_R X$ is a simple $K\Lambda$-module. A simple Λ-lattice is not a simple Λ-module.
9. By $v_p(a)$ we denote the valuation at p of a rational number a (e.g. $v_2(-9/40) = -3$). By a_p we denote the p-part of a rational number a, provided it follows from the context that p is not an ordinary index (e.g. $(-9/40)_2 = 1/8$). Thus $p^{v_p(a)} = a_p$. Analogously prime ideal valuations on the field of fractions of a Dedekind domain.
10. The direct sum of m copies of a single object X is denoted by X^m. In case X is an ideal of a ring, this must not be confused with its mth power as an ideal.
11. The additive group of morphisms between left (or right) modules X and Y over the ring A is sometimes denoted by
$$\mathrm{Hom}_A(X, Y) =: {}_A(X, Y).$$
In case A is commutative, we frequently use the abbreviation $X/a := X/aX$, $a \in A$.
12. The **Circonference Lemma** asserts that for a commutative triangle in an abelian category the induced sequence on the kernels and on the cokernels is 6-term long exact.
13. (C x), (A X), (S x.y) or (S x.y.z) refers to a chapter, an appendix, a section or a subsection respectively. The sole number (x.y.z) refers to a lemma, a definition ... So that e.g. (C.3.5) refers to assertion 3.5 in the appendix C, whereas (C 3) refers to chapter 3, and (A C.3) refers to the third section of appendix C.

Chapter 0

Introduction

0.1 The problem

0.1.1 The situation

WEDDERBURN's Theorem describes the structure of the rational group ring $\mathbf{Q}G$ of a finite group G as follows. Consider a direct sum decomposition of $\mathbf{Q}G$ as a left module over itself into indecomposable summands. By MASCHKE's Lemma, indecomposable left $\mathbf{Q}G$-modules are simple. Therefore, using SCHUR's Lemma, the Pierce decomposition of $\mathbf{Q}G$ corresponding to this direct sum decomposition consists of a direct product of matrix rings over the endomorphism skewfields of these indecomposable summands. The isomorphism from $\mathbf{Q}G$ to this direct product can be described by sending a group element $g \in G$ to the tuple of left multiplications on a system of representatives of isomorphism classes of simple $\mathbf{Q}G$-modules - one column per ring direct factor.

In case of the symmetric group, these endomorphism skewfields coincide with $\mathbf{Q}$. This is the same as to say that the irreducible $\mathbf{Q}\mathcal{S}_n$-modules, that is, the rational Specht modules, are absolutely irreducible [J 78, 4.12]. Choosing simple $\mathbf{Z}\mathcal{S}_n$-lattices, not necessarily the Specht lattices, inside the rational Specht modules, we obtain an inclusion of $\mathbf{Z}$-orders

$$\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_{\lambda} (\mathbf{Z})_{n_\lambda},$$

called **Wedderburn embedding,** where λ runs over the partitions of n. Viewed as an inclusion of abelian groups, it is of finite index $\sqrt{n!^{n!} / \prod_\lambda n_\lambda^{(n_\lambda^2)}}$ (cf. 1.1.4), whence it allows a description by **congruences between matrix entries,** called **ties.** The aim is to gain control of these ties in order to obtain a workable isomorphic copy of the integral group ring $\mathbf{Z}\mathcal{S}_n$ as a subring of the direct product of integral matrix rings $\prod_\lambda (\mathbf{Z})_{n_\lambda}$.

0.1.2 Guiding examples

The complexity of the resulting system of ties strongly depends on the chosen $\mathbf{Z}$-linear bases of the Specht lattices. In the examples directly calculated by computer, we start from the combinatorially given integral representations on the Specht lattices and use 'obvious conjugations' by elementary matrices to simplify (cf. S 0.5). For $n \leq 6$, the complexity of the respective system of ties collapsed at a certain point, and we obtained the systems displayed in (C 2). For $n = 7$, we contended ourselves with the **quasiblocks,** that is, with the images of the projections into the single integral matrix rings.

We regard an embedding $\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda}$, corresponding to a choice of bases, as **satisfactory,** if we can read off a Pierce decomposition of the localized versions $\mathbf{Z}_{(p)}\mathcal{S}_n$ as well as the associated Morita equivalent basic $\mathbf{Z}_{(p)}$-order from its description via ties. In this sense, the examples we calculated are in fact satisfactory, in contrast to the embeddings that use the combinatorially given bases.

But by means of such a vague notion ('we can read off'), we cannot determine the bases uniquely. However, **any** satisfactory embedding allows to search for interpretations of the resulting system of ties. This is what we use our guiding examples for.

0.1.3 Modular morphisms

Suppose given $\mathbf{Z}\mathcal{S}_n$-lattices X and Y, that is, finitely generated $\mathbf{Z}\mathcal{S}_n$-modules free over $\mathbf{Z}$. A modular morphism from X to Y is a $\mathbf{Z}\mathcal{S}_n$-linear map from X/m to Y/m for some integer $m \geq 2$. A necessary condition for an element of the direct product of integral matrix rings to lie in the image of the Wedderburn embedding is, besides that it should act on X and Y, the congruence resulting from the diagram that expresses $\mathbf{Z}\mathcal{S}_n$-linearity. If we denote by ξ the operation of $\mathcal{S}_n$ on X, by η the operation on Y, and by $X \xrightarrow{f} Y$ a $\mathbf{Z}$-linear map, then f yields a $\mathbf{Z}\mathcal{S}_n$-linear map modulo m if and only if the congruence

$$\xi_\sigma f \equiv_m f\eta_\sigma.$$

holds for all $\sigma \in \mathcal{S}_n$. And since the operations ξ and η are pieced together from the tuple of operations on simple lattices in a way independent of σ, the tuples lying in the image of the Wedderburn embedding satisfy certain resulting congruences between matrix entries (usually several ones per morphism).

We shall exhibit several **generic modular morphisms** between simple lattices, by which we understand a family of such modular morphisms given by a formula depending polynomially on combinatorial data (cf. S 4.5).

The specializations of the generic modular morphism given in (4.3.31) already suffice to describe $\mathbf{Z}_{(p)}\mathcal{S}_p$, p prime (cf. 4.2.8).

But e.g. for $\mathbf{Z}_{(3)}\mathcal{S}_6$, which is of index 3^{558} in $\prod_\lambda (\mathbf{Z}_{(3)})_{n_\lambda}$, the specializations of our generic modular morphisms between simple lattices merely describe an intermediate order, which is of index 3^{397} in $\prod_\lambda (\mathbf{Z}_{(3)})_{n_\lambda}$ (cf. 4.4.2).

The reason for this failure is the following. Suppose given a $\mathbf{Z}\mathcal{S}_n$-linear map $X/m \longrightarrow Y/m$, and assume, for sake of simplicity, that $\mathrm{Hom}_{\mathbf{Q}\mathcal{S}_n}(\mathbf{Q}X, \mathbf{Q}Y) = 0$, i.e. that X and Y are rationally disjoint. The long exact Ext-sequence on

$$(*) \qquad 0 \longrightarrow Y \xrightarrow{m} Y \longrightarrow Y/m \longrightarrow 0$$

supplies us with the isomorphism

$$(**) \qquad \mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(X/m, Y/m) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(X, Y)[m],$$

which maps $X/m \longrightarrow Y/m$ to the pullback

$$(***) \qquad 0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

of $(*)$ along $X \longrightarrow X/m \longrightarrow Y/m$. Here $[m]$ denotes the m-torsion part. Thus we obtain a $\mathbf{Z}\mathcal{S}_n$-lattice E, which might now itself be source or target of another modular morphism, say $E/m' \longrightarrow Z/m'$ or $Z/m' \longrightarrow E/m'$. The operation on E being pieced together from the operation on X and the operation on Y by means of the cocycle corresponding to $(***)$, we thus obtain congruences that involve the quasiblocks already involved in X and Y, and in addition those involved in Z. But modular morphisms between simple lattices involve exactly two quasiblocks only.

0.1.4 Extensions

To see that there exists a set of modular morphisms that yields a complete system of ties, by which we understand a system of congruences of matrix entries necessary and, taken together, also sufficient for a tuple of integral matrices to lie in the image of the Wedderburn embedding, we start at the other extreme and writes the regular representation $\mathbf{Z}\mathcal{S}_n$ as an iterated extension of simple lattices. This is possible a priori, since a $\mathbf{Z}\mathcal{S}_n$-lattice is either simple itself, or contains a proper pure sublattice. We thus obtain a finite binary tree of extensions. Now by dint of the correspondence $(**)$, we obtain a corresponding tree of modular morphisms, yielding a complete system of ties. See (S 0.1.5) for more details.

Fortunately, a tree that unscrews the regular representation in combinatorial terms had already been established by JAMES. This tree consists of James lattices, that is, generalized Specht lattices. The unscrewing proceeds by means of James extensions ([J 78, 17.13], cf. 5.1.18), which even ensures that the simple lattices we end up with are actually Specht lattices. The only piece of information that had to be added was to give the inverse of $(**)$ explicitely, that is, to construct a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{m} & Y & \longrightarrow & Y/m & \longrightarrow & 0
\end{array}$$

starting from a James extension in the upper row. As a byproduct, we extract a formula, polynomial in the combinatorial data, for the order of an occurring James extension as an element of the abelian group Ext^1.

The collection of all maps occuring in this procedure furnishes a rather complicated diagram, which can be viewed as a module over a path algebra, and which is called the **truss** (cf. 5.3.8, S 5.4.2). Our initial aim to find a suitable set of bases now boils down to find a normal form for the truss. This is not a well defined problem, but informally, a normal form might be regarded as satisfactory provided its choice of bases yields a satisfactory Wedderburn embedding.

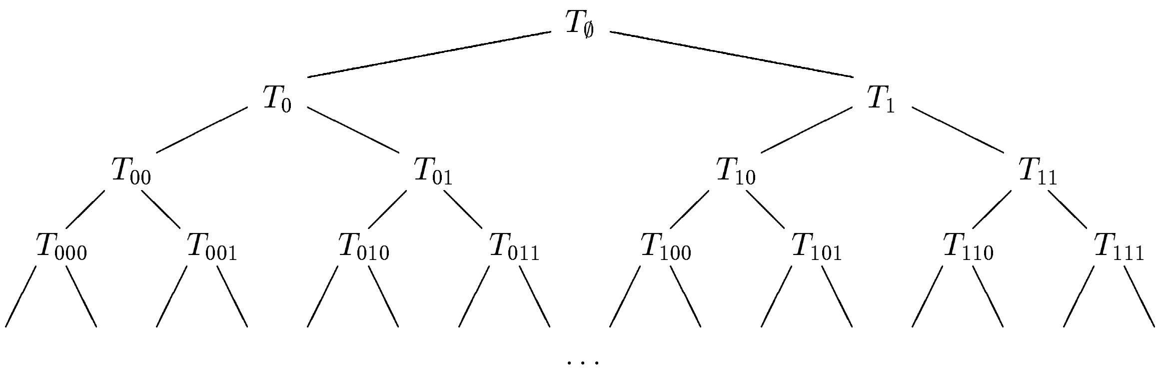
This hypothetical procedure is modelled on the case $\mathbf{Z}_{(p)}\mathcal{S}_p$, in which a long exact sequence of modular morphisms between simple lattices already gives a complete set of ties – a choice of bases adapted to this long exact sequence yields a satisfactory Wedderburn embedding.

A complete solution to this normal form problem seems to be out of reach, and we have tried our hands only on the cases $n = 3, 4$. One might hope that certain parts of the truss will yield partial results, for instance after localization. Or one could try to regard the image of a projection into a certain subdirect product of $\prod_\lambda (\mathbf{Z})_{n_\lambda}$, that is, a generalized quasiblock. But we have not pursued such attempts here. In particular, we do not dispose of a general satisfactory normal form for the truss.

0.1.5 A tree

We shall sketch the general framework employed in (S 0.1.4).

Let R be an integral domain with field of fractions K. Let $\Lambda \subseteq \Gamma$ be a full inclusion of R-orders. Consider a finite binary tree of Λ-lattices



ending at possibly different stages, such that the ends carry Γ-lattices which sum up to Γ, such that $T_\emptyset \simeq \Lambda$ and such that there exist short exact sequences

$$0 \longrightarrow T_{e0} \xrightarrow{e0_*} T_e \xrightarrow{e1^*} T_{e1} \longrightarrow 0,$$

e being a word in 0 and 1's. Moreover, assume given $e0_* e0^* = m_e \in R$, which allows to construct

$$\begin{pmatrix} e0^* & e1^* \end{pmatrix} \begin{pmatrix} e0_* \\ e1_* \end{pmatrix} = m_e.$$

Suppose an element $g \in \Gamma$ acts on T_{e0} via g_{e0} and on T_{e1} via g_{e1}. It acts on T_e if and only if

$$g_e := \frac{1}{m_e} \begin{pmatrix} e0^* & e1^* \end{pmatrix} \begin{pmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{pmatrix} \begin{pmatrix} e0_* \\ e1_* \end{pmatrix}$$

is integral. It acts on $T_\emptyset$ if and only if it is contained in Λ. $e0^*$ induces a modular morphism on the cokernels

$$T_{e1} \longrightarrow T_{e0}/m_e,$$

which is respected by g if and only if g_e is integral.

0.1.6 Quasiblocks

In principle, the ties describing a quasiblock Q^μ, defined as the image of the composition

$$\mathbf{Z}\mathcal{S}_n \longrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda} \longrightarrow (\mathbf{Z})_{n_\mu},$$

can be deduced from the ties that describe a satisfactory embedding, thus explaining its ties as being 'caused' by modular morphisms involving in general several lattices. It would be interesting, however, to have an interpretation in terms of the properties of the Specht lattice S^μ alone, by which it is also determined alone, after all. At least an approximation is given by the Gram matrix G^μ of the $\mathcal{S}_n$-invariant bilinear form on S^μ, which forces

$$Q^\mu \subseteq (\mathbf{Z})_{n_\mu} \cap (G^\mu)^{-1}(\mathbf{Z})_{n_\mu} G^\mu.$$

Though comparably small, the isolated quasiblocks seem to be hard to describe combinatorially. Moreover, an index formula for the inclusion $Q^\mu \subseteq (\mathbf{Z})_{n_\mu}$ is missing (cf. 1.1.3).

0.2 An example

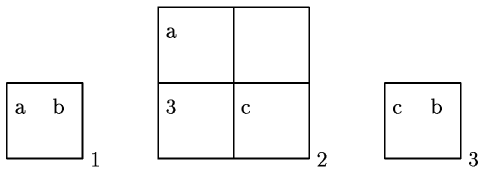
Consider the well known case $\mathbf{Z}\mathcal{S}_3$ [Rog 80, 0.7.2]. The image Λ of the embedding of $\mathbf{Z}$-orders

$$\begin{array}{rcccccc} \mathbf{Z}\mathcal{S}_3 & \longrightarrow & \mathbf{Z} & \times & (\mathbf{Z})_2 & \times & \mathbf{Z} \\ (12) & \longrightarrow & 1 & \times & \left(\begin{smallmatrix} -2 & -1 \\ 3 & 2 \end{smallmatrix}\right) & \times & -1 \\ (123) & \longrightarrow & 1 & \times & \left(\begin{smallmatrix} -2 & -1 \\ 3 & 1 \end{smallmatrix}\right) & \times & 1 \end{array}$$

allows a description as

$$\begin{array}{rl} \Lambda = \{ & x^1_{11} \times \left(\begin{smallmatrix} x^2_{11} & x^2_{12} \\ x^2_{21} & x^2_{22} \end{smallmatrix}\right) \times x^3_{11} \mid x^1_{11} \equiv_2 x^3_{11},\ x^1_{11} \equiv_3 x^2_{11},\ x^2_{22} \equiv_3 x^3_{11},\ x^2_{21} \equiv_3 0\} \\ \subseteq & \mathbf{Z} \times (\mathbf{Z})_2 \times \mathbf{Z} =: \Gamma, \end{array}$$

which we rather depict as



$$\begin{array}{llll} \text{a} & x^1 & \equiv_3 & x^2 \\ \text{b} & x^1 & \equiv_2 & x^3 \\ \text{c} & x^2 & \equiv_3 & x^3. \end{array}$$

This is a satisfactory embedding in the sense of (S 0.1.2). For instance, localized at 3 we obtain the Pierce decomposition

$$1 \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times 1 = \left(1 \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times 0\right) + \left(0 \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times 1\right).$$

Consider the exact sequence of $\mathbf{Z}$-linear maps

$$0 \longrightarrow \mathbf{Z}_1 \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} \begin{array}{c}\mathbf{Z}\\ \mathbf{Z}_2\end{array} \xrightarrow{(0\ 1)} \mathbf{Z}_3 \longrightarrow 0,$$

the indices denoting merely the number of the quasiblock we have taken the column from. This sequence becomes an exact sequence of $\mathbf{Z}\mathcal{S}_3$-linear maps when tensored with $\mathbf{Z}/3 \otimes_{\mathbf{Z}} -$. Conversely, a tuple

$$x^1_{11} \times \begin{pmatrix} x^2_{11} & x^2_{12} \\ x^2_{21} & x^2_{22} \end{pmatrix} \times x^3_{11} \in \Gamma$$

respects both maps modulo 3 if and only if the ties at 3 given above are satisfied, that is, if and only if $x^1_{11} \equiv_3 x^2_{11}$, $x^2_{22} \equiv_3 x^3_{11}$ and $x^2_{21} \equiv_3 0$ hold.

Actually, this exact sequence is a specialization of a generic one (S 4.2.1), and it is not by chance that we have obtained all ties at 3 (cf. 4.2.8).

Let us regard the inclusions of simple lattices over the second quasiblock. There is a $\mathcal{S}_3$-invariant bilinear form defined on its column $\binom{\mathbf{Z}}{\mathbf{Z}}$, given by the Gram matrix $\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$, yielding the inclusion of the Specht lattice $\binom{\mathbf{Z}}{\mathbf{Z}}$ into its dual. This dual lattice $\binom{\mathbf{Z}}{\mathbf{Z}}^*$ is isomorphic to the other column $\binom{\mathbf{Z}}{3}$, where 3 stands for (3), via

$$\begin{array}{ccc} \binom{\mathbf{Z}}{3} & \xrightarrow{\sim} & \binom{\mathbf{Z}}{\mathbf{Z}}^* \\ \binom{1}{0} & \longrightarrow & (2\ 1) \\ \binom{0}{3} & \longrightarrow & (3\ 2), \end{array}$$

as can be derived e.g. from restriction of the map given by the Gram matrix to $\binom{\mathbf{Z}}{3} \subseteq \binom{\mathbf{Z}}{\mathbf{Z}}$ followed by division by 3. Using this isomorphism for isomorphic substitution, we see that the inclusion of the Specht lattice into its dual is isomorphic (as a diagram) to the inclusion $\binom{3}{3} \subseteq \binom{\mathbf{Z}}{3}$. Thus an element of the second quasiblock remains integral by left conjugation with $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, which means that x^2_{21} is divisible by 3.

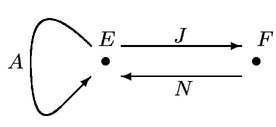
0.3 Motivation

We would like to explain why we consider the problem described in (S 0.1) to be worthwhile. So suppose given a satisfactory description of $\mathbf{Z}_{(p)}\mathcal{S}_n$ as a full suborder of $\prod_\lambda (\mathbf{Z}_{(p)})_{n_\lambda}$ in the sense of (S 0.1.2), so that a Pierce decomposition and the corresponding Morita multiplicities can be read off.

Therefore, the Pierce components of the corresponding basic $\mathbf{Z}_{(p)}$-order as rings resp. as bimodules are given in a very explicit manner, viz. in $\mathbf{Z}_{(p)}$-linear bases, with multiplication derived from matrix multiplication. A path algebra being a universal Pierce decomposition, we may map some convenient path algebra over $\mathbf{Z}_{(p)}$ onto this $\mathbf{Z}_{(p)}$-order to obtain a presentation as a path algebra modulo relations. E.g. the principal block of $\mathbf{Z}_{(2)}\mathcal{S}_5$ turns out to be Morita equivalent to

$$\mathbf{Z}_{(2)}\Xi/(A^2 - 2A, NJ - 2F, (AJN)^2 - (JNA)^2 - 2(AJN - JNA)),$$

where

$\Xi :=$ 

(cf. S 2.2.5). Reformulated as a path algebra modulo relations or not, the Pierce decomposition is the **main reason** to search for a tie form. The path algebra presentation sometimes reveals algebra automorphisms (cf. S 2.3.4).

The calculation of the radical of $\mathbf{Z}_{(p)}\mathcal{S}_n$ may be done as follows. To begin with, we reduce to the calculation of the radicals of the endomorphism rings of the indecomposable projective modules. Moreover, we may reduce to the calculation of the radicals of the quasiblocks of such an endomorphism ring, since its radical can be recovered by intersection (E.1.28). Again, we are reduced to the calculation of the endomorphism rings of the indecomposable projectives of such a quasiblock. Furthermore, if in such an endomorphism ring, embedded into a single matrix ring, either the position ij or its transpose ji carries a single p-tie for each $i \neq j$, its radical is given by imposing single p-ties on the main diagonal entries (E.1.30).

For instance, consider the $\mathbf{Z}_{(3)}$-order $\mathbf{Z}_{(3)}\mathcal{S}_3$, isomorphic to its image Λ under the embedding given in (S 0.2). Its radical is given by

$$\mathfrak{r}\Lambda = 3 \times \left(\begin{smallmatrix} 3 & \mathbf{Z}_{(3)} \\ 3 & 3 \end{smallmatrix}\right) \times 3$$

where 3 stands for (3).

Using the isomorphism

$$\begin{array}{rcccc} \Lambda/\mathfrak{r}\Lambda & \overset{\sim}{\longrightarrow} & \mathbf{F}_3 & \times & \mathbf{F}_3 \\ x_{11}^1 \times \left(\begin{smallmatrix} x_{11}^2 & x_{12}^2 \\ x_{21}^2 & x_{22}^2 \end{smallmatrix}\right) \times x_{11}^3 & \longrightarrow & x_{11}^1 & \times & x_{11}^3 \end{array}$$

we obtain its unit group to be

$$\Lambda^* = \{x_{11}^1 \times \left(\begin{smallmatrix} x_{11}^2 & x_{12}^2 \\ x_{21}^2 & x_{22}^2 \end{smallmatrix}\right) \times x_{11}^3 \in \Lambda \mid x_{11}^1 \not\equiv_3 0,\ x_{11}^3 \not\equiv_3 0\}.$$

One also may pick subsets of the unit group with extra properties, such as torsion units (use characteristic polynomials) or central units (cf. also 1.1.4). We will not pursue this possibility, however.

Units are a prominent example for the usage of matrix multiplication. In general, the tie form allows to think of calculations in the group ring as of calculations in matrices subject to some ties. For instance, the center, or also a maximal commutative suborder (a torus) become visible.

The decomposition matrix may be calculated by counting multiplicities of simple lattices in projective indecomposable lattices. Let S^λ denote the Specht lattice over $\mathbf{Z}_{(p)}\mathcal{S}_n$ to the partition λ of n, p prime. Let μ be a p-regular partition, let $D^\mu = S^\mu/\mathfrak{r}S^\mu$ be the corresponding simple module, let P^μ denote its projective cover over $\mathbf{Z}_{(p)}\mathcal{S}_n$. Consider the semisimple Loewy layers

$$X_i := (\mathfrak{r}^i S^\lambda + pS^\lambda)/(\mathfrak{r}^{i+1} S^\lambda + pS^\lambda).$$

We obtain the decomposition number

$$\begin{array}{rcl} [S^\lambda : D^\mu] & = & \sum_{i \geq 0} \dim_{\mathbf{F}_p} \mathrm{Hom}_{\mathbf{F}_p\mathcal{S}_n}(D^\mu, X_i) \\ & = & \sum_{i \geq 0} \dim_{\mathbf{F}_p} \mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, X_i) \\ & = & \dim_{\mathbf{F}_p} \mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, S^\lambda/pS^\lambda) \\ & = & \dim_{\mathbf{F}_p} \left(\mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, S^\lambda)/\mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, pS^\lambda)\right) \\ & = & \dim_{\mathbf{F}_p} \left(\mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, S^\lambda)/p\,\mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, S^\lambda)\right) \\ & = & \mathrm{rk}_{\mathbf{Z}_{(p)}} \mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, S^\lambda) \\ & = & \dim_{\mathbf{Q}} \mathrm{Hom}_{\mathbf{Q}\mathcal{S}_n}(\mathbf{Q}P^\mu, \mathbf{Q}S^\lambda). \end{array}$$

Moreover, usage of the radical enables us to refine this calculation to the Loewy layers of S^λ/pS^λ. We write $P^\mu = \mathbf{Z}_{(p)}\mathcal{S}_n e^\mu$, e^μ being a primitive idempotent of $\mathbf{Z}_{(p)}\mathcal{S}_n$ and obtain

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{F}_p\mathcal{S}_n}(D^\mu, X_i) & = & \mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, X_i) \\ & = & \mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, \mathfrak{r}^i S^\lambda + pS^\lambda)/\mathrm{Hom}_{\mathbf{Z}_{(p)}\mathcal{S}_n}(P^\mu, \mathfrak{r}^{i+1} S^\lambda + pS^\lambda) \\ & = & e^\mu(\mathfrak{r}^i S^\lambda + pS^\lambda)/e^\mu(\mathfrak{r}^{i+1} S^\lambda + pS^\lambda). \end{array}$$

For instance, in the situation of (S 0.2) we obtain for $n = 3$, $p = 3$, $\lambda = (2,1)$, $\mu = (3)$, $e^{(3)} = 1 \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times 0$ and $i = 1$

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{F}_3\mathcal{S}_3}(D^{(3)}, X_1) & = & e^{(3)}(\mathfrak{r}S^{(2,1)} + 3S^{(2,1)})/e^{(3)}(\mathfrak{r}^2 S^{(2,1)} + 3S^{(2,1)}) \\ & = & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{Z} \\ 3 \end{smallmatrix}\right) / \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right) \\ & = & \mathbf{Z}/3. \end{array}$$

As a disadvantage we mention that the subgroups are hidden. For instance, to calculate vertices of indecomposable lattices over $\mathbf{Z}_{(p)}\mathcal{S}_n$, p prime, one seems to be forced to use the representing matrices, due to the fact that the group ring of a subgroup in general does not allow a convenient description in the same product of matrix rings over $\mathbf{Z}_{(p)}$ into which $\mathbf{Z}_{(p)}\mathcal{S}_n$ is embedded. For some representing matrices, we refer to (C 2).

0.4 Necessity of prime powers

We want to stress by an experiment the necessity, even for ordinary modular representation theory, of the prime powers occurring in the ties resp. in the modular morphisms evoking the ties. In general, changing the prime powers for these modular morphisms changes the objects (such as modules, morphism groups, ...) obtained by reduction modulo that prime. To see this, we shall lower by one an exponent of such a prime power in an example.

Consider the $\mathbf{Z}_{(2)}$-order

$$E := \{x \times y \times z \mid y \equiv_4 z,\ 2x \equiv_8 y + z\} \subseteq \mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)} =: \Gamma,$$

which is the endomorphism ring of an indecomposable projective $\mathbf{Z}_{(2)}\mathcal{S}_4$-module (S 2.1). Note that therefore $E/2$ is the endomorphism ring of an indecomposable projective $\mathbf{F}_2\mathcal{S}_4$-module.

Sending X to $0 \times 2 \times -2$ and Y to $0 \times 0 \times 8$, we obtain the presentation

$$\mathbf{Z}_{(2)}[X,Y]/(X^2 - 2X - Y,\ XY + 2Y,\ Y^2 - 8Y) \xrightarrow{\sim} E,$$

whence the reduction modulo 2 reads

$$\mathbf{F}_2[X]/(X^3) \xrightarrow{\sim} \mathbf{F}_2[X,Y]/(X^2 - Y,\ XY,\ Y^2) \xrightarrow{\sim} E/2.$$

Consider the linear map

$$\begin{array}{rcccl} U := \mathbf{Z}_{(2)} & \xrightarrow{f'} & \mathbf{Z}_{(2)} \times \mathbf{Z}_{(2)} & =: V' \\ 1 & \longrightarrow & 2 \times -2, & \end{array}$$

where the left hand side consists of the first and the right hand side of the second and third ring direct factor of Γ. This linear map factors over the E-sublattice

$$V := \{y \times z \mid y \equiv_4 z\} \subseteq V',$$

and this factorization

$$U \xrightarrow{f} V$$

is E-linear modulo 4. Conversely, E consists of those tuples in Γ that act on V and that respect this map modulo 4.

We pass to the intermediate order $E \subseteq E' \subseteq \Gamma$ consisting of those tuples $x \times y \times z \in \Gamma$ that still act on V but that commute with $U \xrightarrow{f} V$ modulo 2 and not necessarily modulo 4. This amounts to the description

$$E' = \{x \times y \times z \mid y \equiv_4 z,\ 2x \equiv_4 y + z\} \subseteq \Gamma.$$

Sending X to $0 \times 2 \times -2$ and Y to $0 \times 0 \times 4$ yields the presentation

$$\mathbf{Z}_{(2)}[X, Y]/(X^2 - 2X - 2Y,\ XY + 2Y,\ Y^2 - 4Y) \xrightarrow{\sim} E',$$

whence the reduction modulo 2

$$\mathbf{F}_2[X, Y]/(X^2, XY, Y^2) \xrightarrow{\sim} E'/2,$$

which is not isomorphic to $E/2$ since the radical square of $E'/2$ vanishes.

0.5 A washing machine

The problem in formalizing the direct calculational simplification of a given system of ties lies in the choice of a suitable conjugation of a single representation by an elementary matrix, i.e. we have to formalize the meaning of an **obvious step.** In case one chooses the respective step by hand - which is possible, say, for $\mathbf{Z}\mathcal{S}_5$ -, one makes use of the human ability of pattern recognition.

The automatized solution to this problem is rather simple. It is efficient for $n \leq 7$, but it is of **non-algorithmic** nature. It is doubtful whether a practicable algorithm exists. Note that we deal with a finite problem of huge proportions.

We shall now give a sketch of the method employed instead of giving a complete documentation in the main text.

To begin with, we start with the ties given by Serre's Inverse Fourier Transformation Formula (1.1.1) for an arbitrarily chosen embedding, thus avoiding an $n! \times n!$-matrix inversion.

Conjugation of a representation by an elementary matrix is performed by a multiplication from the left and the inverse multiplication from the right. To keep control of the effect on the ties, we **test** either only multiplication from the left or only multiplication from the right, translated to the effect on the tie matrix, i.e. on the matrix which records the ties. For a fixed position of the non main diagonal entry of the conjugating elementary matrix, the entry with **maximal reduction** of the number of involved nonzero positions in the tie matrix is determined. The conjugation itself then is, of course, carried out correctly from both sides. Now we let the position of this non main diagonal entry run through our conjugating elementary matrix. Thus we have two methods at hand, one testing from the left and one testing from the right, which we more or less alternate.

The resulting process is a non-algorithm in the sense that even for the vague aim of a satisfactory embedding (cf. S 0.1.2) there is no theoretical reason to be achieved. For $n \leq 7$, however, the system of ties almost collapses down to a sensible one: the resulting ties regularly involve at most one entry per quasiblock, and, exceptionally, up to three entries per quasiblock.

As a consequence of using a process which is not an algorithm, the tie matrix of $\mathbf{Z}_{(3)}\mathcal{S}_7$ needs two successive treatments by the program until it is cleaned - as one might expect of a washing machine.

The tie matrix being simplified, there remains the extra task of parallelizing the ties occurring in a Pierce component $e\mathbf{Z}_{(p)}\mathcal{S}_n f$. By parallel ties, we understand ties of the same shape for all matrix tuple positions belonging to this Pierce component. Parallelization is necessary for to see Morita multiplicities (cf. e.g. S 2.1.1). But so far, parallel ties are not distiguishable by computer from their non-parallel linear combinations. This is the reason why we haven't obtained satisfactory embeddings for $\mathbf{Z}_{(p)}\mathcal{S}_7$ for $p = 2, 3, 5$ yet, although our washing machine has been able to handle them. For to proceed further in this direct manner, one should attempt to automatize also this parallelization as far as possible, algorithmically or not. For possible theoretical obstacles, cf. (D.1.4).

Trying to deal with ties without computer – not only in this first step for to calculate satisfactory embeddings directly, but also in the subsequent search for generic ties – would be similar to trying to do astronomy without telescope. We should not overestimate our eyes.

0.6 Some results

0.6.1 A one-box-shift morphism

We use the language of JAMES [J 78]. We shall give a formula for a morphism

$$S^\lambda/m \xrightarrow{\bar{f}} S^\mu/m,$$

in case μ arises from λ by **shifting one box** to the left and down, m being the length of the path covered by the shifted box (4.3.31, cf. S 0.6.3). This morphism is induced by a morphism from the free $\mathbf{Z}\mathcal{S}_n$-module on one generator, denoted by F^λ and equipped with the λ-tableaux as $\mathbf{Z}$-linear basis, to the Specht lattice S^μ. The image of a λ-tableau $[t]$ under this morphism is given by a sum of μ-polytabloids according to the following combinatorial rule. Take an entry of the tableau t from the column of the box which is to be shifted and replace an entry in some column further to the left with it. Take this replaced entry and replace an entry in a column further to the left with it. And so on. Put the last replaced entry into the shifted box, but at its position in μ (cf. e.g. S 4.3.5). Form an integral linear combination of the resulting μ-polytabloids with coefficients polynomial in the column lengths of λ. Finally, divide out a redundant and likewise polynomial factor, which occurs because of a linear dependence of the resulting μ-polytabloids that ensues from the GARNIR relations [G 50, p. 56], cf. (4.1.4). The result passes down to $S^\lambda/m \xrightarrow{\bar{f}} S^\mu/m$. $\bar{f} \neq 0$, even modulo primes dividing m, can be seen by a standard polytabloid argument.

Based on [CL 74], Carter and Payne [CP 80] **in particular** show that such a nonzero morphism exists over an infinite field of characteristic p dividing m (cf. 4.3.33, S 4.3.5). This ensues also from (4.3.31).

For the application to integral representation theory, however, we need morphisms modulo prime powers, and moreover, we need to know their behaviour under composition, for instance in order to describe $\mathbf{Z}_{(p)}\mathcal{S}_p$, p prime (cf. e.g. 4.2.4, 4.2.8, S 4.4.2).

Furthermore, Carter and Payne assert the nonvanishing of Hom in the situation of the simultaneous shift of several boxes from a column to another column further down to the left. We could figure out some modular morphisms in simple special cases of this combinatorial situation (cf. 4.4.3, 4.4.5), but we haven't been able to generalize the one-box-shift morphism (4.3.31) accordingly.

A. Kleshchev [Kles 98] has given an argument for the dimension of the Hom-space treated by Carter and Payne to be one-dimensional in case of a one-box-shift in characteristic $\neq 2$ (cf. 4.3.33).

Further exceptional modular morphisms appeared modulo 2, one of them specializing to the nontrivial endomorphism of $S^{(3,1,1)}/2$, causing a tie with two entries in a single quasiblock (4.2.11). This endomorphism is described, up to Morita equivalence, by the matrix

$$\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}.$$

Thus the element $\begin{pmatrix} x_{11}&x_{12}&x_{13} \\ x_{21}&x_{22}&x_{23} \\ x_{31}&x_{32}&x_{33} \end{pmatrix}$ of the matrix ring containing the according quasiblock respects this endomorphism modulo 2 if and only if

$$\begin{pmatrix} x_{13}&0&0 \\ x_{23}&0&0 \\ x_{33}&0&0 \end{pmatrix} = \begin{pmatrix} x_{11}&x_{12}&x_{13} \\ x_{21}&x_{22}&x_{23} \\ x_{31}&x_{32}&x_{33} \end{pmatrix} \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} \equiv_2 \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix} \begin{pmatrix} x_{11}&x_{12}&x_{13} \\ x_{21}&x_{22}&x_{23} \\ x_{31}&x_{32}&x_{33} \end{pmatrix} = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ x_{11}&x_{12}&x_{13} \end{pmatrix},$$

i.e. if and only if $x_{12} \equiv_2 x_{13} \equiv_2 x_{23} \equiv_2 0$ and $x_{11} \equiv_2 x_{33}$. These ties already describe that quasiblock (cf. S 2.2.4).

A table of some modular morphisms is given in (S 4.5).

0.6.2 A retraction up to an integer

In (S 0.1.4) it has been explained that the missing detail in order to describe $\mathbf{Z}\mathcal{S}_n$ via ties arising from modular morphisms, using an unscrewing of the regular lattice by James extensions, is a retraction to the inclusion of an occurring James extension up to its order in Ext^1. The kernel of such an occurring James extension is given by a Specht lattice S^ν, its middle term is given by a James lattice $S^{\lambda\subseteq\nu}$ (cf. 5.1.2). There is an epimorphism from F^ν onto $S^{\lambda\subseteq\nu}$ for which generators of the kernel are known. This enables us to exhibit a retraction up to the Ext^1-order as being induced from a $\mathbf{Z}\mathcal{S}_n$-morphism $F^\nu \xrightarrow{f} S^\nu$ (5.2.25, cf. S 0.6.3)

The formula for f is very similar to that described in (0.6.1), and I do not know why. Suppose given a ν-tableau $[t]$ which is to be mapped. Let y be the entry of t at the unique position outside of λ (modulo 5.3.3). Replace an entry to the right of y by y. Take the replaced entry to replace an entry to the left of it. Take the replaced entry to replace an entry to the left of it. And so on. In the last step, insert the replaced entry at the original position of y. Form an integral linear combination of the resulting ν-polytabloids with coefficients polynomial in the column lengths of ν and divide out a factor of redundancy.

0.6.3 Announcement of results

We shall give an account of (4.3.31, 5.2.25), slightly deviating in notation from our working language of (C 4, C 5).

Let

$$\begin{array}{ccc} \mathbf{N} & \stackrel{\lambda}{\longrightarrow} & \mathbf{N}_0 \\ i & \longrightarrow & \lambda_i \end{array}$$

be a **partition of** n**,** i.e. assume $\sum_i \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for $i \in \mathbf{N}$. Let $P^\lambda := \{i \times j \in \mathbf{N} \times \mathbf{N} \mid j \leq \lambda_i\}$ be the **picture** of λ. A λ**-tableau** is a bijection

$$\begin{array}{ccc} P^\lambda & \stackrel{[a]}{\longrightarrow} & [1,n] \\ i \times j & \longrightarrow & a_{ij}. \end{array}$$

$\sigma \in \mathcal{S}_n$ acts on the set of λ-tableaux T^λ via composition $[a] \stackrel{\sigma}{\longrightarrow} [a]\sigma$. Let F^λ be the free $\mathbf{Z}$-module on T^λ with the induced action of the $\mathcal{S}_n$. Let

$$\begin{array}{ccc} P^\lambda & \stackrel{\rho}{\longrightarrow} & \mathbf{N} \\ i \times j & \longrightarrow & i \end{array} \qquad\qquad \begin{array}{ccc} P^\lambda & \stackrel{\kappa}{\longrightarrow} & \mathbf{N} \\ i \times j & \longrightarrow & j \end{array}$$

denote the projections and let

$$\begin{array}{ccc} T^\lambda & \stackrel{r}{\longrightarrow} & \mathbf{N}^{[1,n]} \\ [a] & \longrightarrow & [a]^{-1}\rho \end{array} \qquad\qquad \begin{array}{ccc} T^\lambda & \stackrel{c}{\longrightarrow} & \mathbf{N}^{[1,n]} \\ [a] & \longrightarrow & [a]^{-1}\kappa \end{array} .$$

The fibers of r are called **tabloids**, and the fiber containing $[a]$ is denoted by $\{a\}$. The free $\mathbf{Z}$-module on the set of tabloids, equipped with the induced action of the $\mathcal{S}_n$, is denoted by M^λ. Let

$$C_{[a]} := \{\sigma \in \mathcal{S}_n \mid [a]c = ([a]\sigma)c\}$$

be the **column stabilizer** of $[a]$. Let the **Specht lattice** S^λ be the $\mathbf{Z}\mathcal{S}_n$-sublattice of M^λ generated over $\mathbf{Z}$ by the λ**-polytabloids**

$$\langle a \rangle := \sum_{\sigma \in C_{[a]}} \{a\}\sigma\varepsilon_\sigma.$$

λ' denotes the **transposed partition** of λ, i.e. $j \leq \lambda_i \iff i \times j \in P^\lambda \iff i \leq \lambda'_j$.

Let $s, t \in \mathbf{N}$, $s < t$, assume $\lambda'_t > \lambda'_{t+1}$, and assume $s = 1$ or $\lambda'_{s-1} > \lambda'_s$. These assumptions ensure

$$\mu'_i := \begin{cases} \lambda'_i + 1 & \text{for } i = s \\ \lambda'_i - 1 & \text{for } i = t \\ \lambda'_i & \text{else} \end{cases}$$

to define a partition μ of n. Let $l \geq 1$. A **path** of length l is a map

$$\begin{array}{ccc} [0,l] & \stackrel{\gamma}{\longrightarrow} & P^\lambda \cup P^\mu \\ k & \longrightarrow & \alpha_k \times \beta_k \end{array}$$

such that $k < k'$ implies $\beta_k < \beta_{k'}$, such that $\alpha_0 \times \beta_0 = \mu'_s \times s$ and such that $\beta_l = t$. For a λ-tableau $[a]$ define the μ-tableau $[a^\gamma]$ by

$$\begin{array}{rcll} a^\gamma_{ij} & := & a_{ij} & \text{for } i \times j \in P^\mu \backslash (\gamma([1,l]) \cup \mathbf{N} \times \{t\}) \\ a^\gamma_{\alpha_k \beta_k} & := & a_{\alpha_{k+1}\beta_{k+1}} & \text{for } k \in [0, l-1] \\ a^\gamma_{it} & := & a_{it} & \text{for } i < \alpha_l \\ a^\gamma_{it} & := & a_{i+1,t} & \text{for } i \geq \alpha_l. \end{array}$$

For $i \in [s+1, t-1]$ we denote $X_i := (t - \lambda'_t) - (i - \lambda'_i)$. Let

$$x_\gamma := (-1)^{\alpha_l} \frac{\prod_{i \in [s+1,t-1],\ \mu'_i > \mu'_{i+1}} X_i}{\prod_{k \in [1,l-1]} X_{\beta_k}}.$$

Let $m := 1 + (t - \lambda'_t) - (s - \lambda'_s)$. Let Γ be the set of paths of some length $l \in [1, t-s]$.

Theorem 0.6.1 (formula for a modular morphism, equivalent to (4.3.31)) *There is a commutative diagram of* $\mathbf{Z}\mathcal{S}_n$*-linear maps*

$$\begin{array}{ccccc} & [a] & \longrightarrow & \sum_{\gamma \in \Gamma} x_\gamma \langle a^\gamma \rangle & \\ [a] & F^\lambda & \xrightarrow{f} & S^\mu & \langle b \rangle \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \langle a \rangle & S^\lambda & \xrightarrow{\bar{f}} & S^\mu/m & \langle b \rangle \end{array}$$

f can be written as an integral matrix such that at least one entry equals ± 1. In particular, $\bar{f}$ does not vanish.

A **prepartition of** n is a map

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{\nu} & \mathbf{N}_0 \\ i & \longrightarrow & \nu_i \end{array}$$

such that $\sum_i \nu_i = n$. The notation in case of a partition carries over verbatim up to the definition of M^ν.

Let λ be a partition of some number such that $P^\lambda \subseteq P^\nu$. Let $[a]$ be a ν-tableau, let $P^\nu \backslash P^\lambda \overset{\iota}{\hookrightarrow} P^\nu$. Let

$$C_{[a],\lambda} := \{\sigma \in \mathcal{S}_n \mid [a]c = [a]\sigma c,\ \iota[a] = \iota[a]\sigma\}$$

be the column stabilizer of $[a]$ inside λ. Let the **James lattice** $S^{\lambda \subseteq \nu}$ be the $\mathbf{Z}\mathcal{S}_n$-sublattice of M^ν generated over $\mathbf{Z}$ by the $\lambda \subseteq \nu$**-semitabloids**

$$\langle a \rangle_\lambda := \sum_{\sigma \in C_{[a],\lambda}} \{a\} \sigma \varepsilon_\sigma.$$

Note that $S^{\lambda \subseteq \nu} = S^{\tilde{\lambda} \subseteq \nu}$, where $\tilde{\lambda}_1 := \nu_1$, $\tilde{\lambda}_i := \lambda_i$ for $i \geq 2$.

Let $z \geq 2$. Assume $\lambda_z < \lambda_{z-1}$ and $\lambda_z < \nu_z$. The assignments

$$(\lambda A_z)_i \;:=\; \begin{cases} \lambda_i + 1 & \text{for } i = z \\ \lambda_i & \text{for } i \neq z \end{cases}$$

$$(\nu R_z)_i \;:=\; \begin{cases} \nu_i + \nu_z - \lambda_z & \text{for } i = z-1 \\ \lambda_z & \text{for } i = z \\ \nu_i & \text{for } i \neq z-1, z \end{cases}$$

define a partition λA_z of some number with $P^{\lambda A_z} \subseteq P^\nu$ and a prepartition μR_z of n with $P^\lambda \subseteq P^{\nu R_z}$. For a ν-tableau $[a]$, we define the νR_z-tableau $[aR_z]$ by

$$\begin{array}{rcll} (aR_z)_{ij} & := & a_{ij} & \text{for } i \times j \in P^{\nu R_z} \backslash \{z-1\} \times [\nu_{z-1}+1, \nu_{z-1}+\nu_z-\lambda_z] \\ (aR_z)_{z-1,\,\nu_{z-1}+j} & := & a_{i,\lambda_z+j} & \text{for } j \in [1, \nu_z - \lambda_z]. \end{array}$$

Theorem 0.6.2 (James, [J 78, 17.13, proof of 17.12], cf. (5.1.18))
The sequence of $\mathbf{Z}\mathcal{S}_n$-lattices

$$\begin{array}{ccccccc} 0 \longrightarrow & S^{\lambda A_z \subseteq \nu} & \longrightarrow & S^{\lambda\subseteq\nu} & \longrightarrow & S^{\lambda\subseteq\nu R_z} & \longrightarrow 0, \\ & \langle a \rangle_{\lambda A_z} & \longrightarrow & \langle a \rangle_{\lambda A_z} & & & \\ & & & \langle a \rangle_\lambda & \longrightarrow & \langle aR_z \rangle_\lambda & \end{array}$$

called the **James extension,** *is short exact.*

Simple case. Let $t := \nu_1$ and suppose $P^\nu \backslash P^\lambda = \{\nu'_s \times s\}$ for some $s \in [1,t]$. Let $z := \nu'_s$. Note that $S^{\lambda A_z \subseteq \nu} = S^\nu$. Let $l \geq 0$. A **cycle** of length l is a map

$$\begin{array}{ccc} [0,l] & \xrightarrow{\gamma} & P^\nu \\ k & \longrightarrow & \alpha_k \times \beta_k \end{array}$$

such that $k < k'$ implies $\beta_k < \beta_{k'}$ and such that $\alpha_0 \times \beta_0 = z \times s$. For a ν-tableau $[a]$ define the ν-tableau $[a^\gamma]$ by multiplication of $[a]$ with the corresponding $(l+1)$-cycle in $\mathcal{S}_n$,

$$[a^\gamma] := [a] \cdot (a_{\alpha_0,\beta_0}, \ldots, a_{\alpha_l,\beta_l}).$$

For $i \in [s+1,t]$ we denote $Y_i := (s - \nu'_s) - (i - \nu'_i)$. Let

$$y_\gamma := \frac{\prod_{i\in[s+1,t],\, \nu'_i > \nu'_{i+1}} Y_i}{\prod_{k\in[1,l]} Y_{\beta_k}}.$$

Let Γ be the set of cycles of some length $l \in [0, t-s]$. Let

$$m := (\nu'_s + t - s) \prod_{j\in[s+1,t],\, \nu'_{j-1} > \nu'_j} (Y_j + 1).$$

Theorem 0.6.3 (formula for a retraction up to m, equivalent to (5.2.25, 5.2.26))
There is a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^\nu & \longrightarrow & S^{\lambda\subseteq\nu} & \longrightarrow & S^{\lambda\subseteq\nu R_z} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow {\scriptstyle \langle a \rangle_\lambda \,\mapsto\, \sum_{\gamma\in\Gamma} y_\gamma \langle a^\gamma \rangle} & & \Big\downarrow & & \\ 0 & \longrightarrow & S^\nu & \xrightarrow{m} & S^\nu & \longrightarrow & S^\nu/m & \longrightarrow & 0 \end{array}$$

in which the upper sequence is the James extension (0.6.2), having order m in Ext^1.

0.7 Acknowledgements

Trying to describe the modular group rings $\mathbf{F}_2\mathcal{S}_3$ and $\mathbf{F}_3\mathcal{S}_3$ when calculating an example, I was advised by PD Dr. S. König to consider $\mathbf{Z}\mathcal{S}_3$ via the inclusion $\mathbf{Z}\mathcal{S}_3 \subseteq \mathbf{Z} \times (\mathbf{Z})_2 \times \mathbf{Z}$ instead (cf. S 0.2). Amazed by this solution, I started to tackle the obvious next question. Moreover, I'd like to thank him for several simplifications, for help in discovering subtle as well as stupid mistakes and for his steady interest, patience and support.

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The representation theory of the symmetric group I've learned from the beautifully written book of James [J 78].

Bielefeld, May 1999

Second corrected version, Bielefeld, April 2001

Chapter 1

Preliminaries

We need a certain amount of theory a priori in order to handle our guiding examples, in particular, to prove that they are correct, independent of computer calculations. For this purpose we give a total index formula, describing the quantity of the system of ties, we recall the modified Coxeter relations, giving a presentation of the $\mathcal{S}_n$ in terms of a transposition and an n-cycle, and finally we recall the point of view of path algebras as universal Pierce decompositions. By rights, also appendix (A D), containing the notions of a homogenus ring, i.e. of a ring which has a Pierce decomposition such that the summands are either isomorphic or lie in different genera, and of a naive localization, i.e. a way to work at a single prime while keeping the global ground ring, should have appeared at this point. However, due to its length, the according section has been shifted to the back.

1.1 Serre's Fourier inversion formula

The reader who is merely interested in our guiding examples may restrict his attention to (S 1.1.1). This section may be regarded as a long corollary to the vertical orthogonality relations of the character table. We do not claim originality, in fact, we have rediscovered several well known assertions (1.1.4, 1.1.5, 1.1.8, 1.1.10, 1.1.13) by this ad hoc method (cf. S 1.1.6).

1.1.1 The tie matrix and the total index

Let G be a finite group. Let $\{\rho^\lambda\}$ be a complete set of complex irreducible representations of G, given by matrix valued functions $\rho^\lambda = (\rho^\lambda_{ij})_{i\times j\in[1,n_\lambda]\times[1,n_\lambda]}$, where $n_\lambda := \dim\rho^\lambda$.

The vertical orthogonality relation applied to the first column of the character table of G, belonging to $1\in G$, and to the column belonging to the conjugacy class of an arbitrary element $g\in G$ reads

$$\sum_\lambda n_\lambda \,\mathrm{Tr}\,\rho^\lambda(g) \overset{(*)}{=} |G|\,\partial_{1,g}.$$

Hence, composing the linear maps

$$\begin{array}{rcl} \mathbf{C}G & \xrightarrow{r} & \prod_\lambda(\mathbf{C})_{n_\lambda} \\ g & \longrightarrow & (\rho^\lambda(g))_\lambda \end{array}$$

and

$$\begin{array}{rcl} \prod_\lambda (\mathbf{C})_{n_\lambda} & \xrightarrow{t} & \mathbf{C}G \\ (\xi^\lambda)_\lambda & \longrightarrow & \sum_g \left(\sum_\lambda n_\lambda \mathrm{Tr}(\xi^\lambda \rho^\lambda(g^{-1})) \right) g, \end{array}$$

whose matrix arises from the matrix of r via transposition, a reordering of the rows and columns and a multiplication of the rows with respective factors n_λ, we obtain the

Lemma 1.1.1 (Serre's Fourier inversion formula [Se 77, 6.2 prop. 11])

$$rt = |G|.$$

Given $h \in G$, we plug in $\xi^\lambda := (h)r = \rho^\lambda(h)$ to get

$$\begin{array}{rcl} \sum_g \left(\sum_{\lambda,i,j} n_\lambda \rho^\lambda_{ij}(h) \rho^\lambda_{ji}(g^{-1}) \right) g & = & \sum_g \left(\sum_{\lambda,i} n_\lambda \rho^\lambda_{ii}(hg^{-1}) \right) g \\ & \overset{(*)}{=} & |G| \sum_g \partial_{h,g} g \\ & = & |G| h. \end{array}$$

Note that both maps r and t depend on the chosen representations.

For $(\xi^\lambda)_\lambda \in \prod_\lambda (\mathbf{C})_{n_\lambda}$ we have, using $rt = |G|$ and $tr = |G|$, the

Remark 1.1.2 (cf. [Klei 96, Prop. 1])

$$(\xi^\lambda)_\lambda \in (\mathbf{Z}G)r \iff ((\xi^\lambda)_\lambda)t \in |G|\mathbf{Z}G.$$

We return to the case $G = \mathcal{S}_n$, in which we may realize the ρ^λ integrally [J 78, 4.2, 4.3, 4.12]. For such a tuple of representations, the restriction of r to $\mathbf{Z}\mathcal{S}_n$ yields an inclusion

$$\mathbf{Z}\mathcal{S}_n \overset{r}{\hookrightarrow} \prod_\lambda (\mathbf{Z})_{n_\lambda},$$

for which we have used the system of ties given by (1.1.2) in order to calculate our guiding examples (C 2) directly by the method described in the introduction (S 0.5).

Corollary 1.1.3 *The elementary divisors of the inclusion* $\mathbf{Z}\mathcal{S}_n \overset{r}{\hookrightarrow} \prod_\lambda (\mathbf{Z})_{n_\lambda}$ *divide* $n!$, *i.e.*

$$n! \prod_\lambda (\mathbf{Z})_{n_\lambda} \subseteq (\mathbf{Z}\mathcal{S}_n)r \subseteq \prod_\lambda (\mathbf{Z})_{n_\lambda}.$$

Since

$$(\det r)(\det t) = n!^{n!},$$

and, considering the matrices as explained above,

$$|\det t| = |\det r| \prod_\lambda n_\lambda^{(n_\lambda^2)},$$

we obtain the

Proposition 1.1.4 (total index formula)

$$|\det r| = \sqrt{\frac{n!^{n!}}{\prod_\lambda n_\lambda^{(n_\lambda^2)}}}$$

is the index of the inclusion of abelian groups

$$\mathbf{Z}S_n \overset{r}{\hookrightarrow} \prod_\lambda (\mathbf{Z})_{n_\lambda}.$$

1.1.2 Total index, more general version

Keep the notation of (S 1.1.1) in the general case. Let $\{\rho^\lambda\}_{\lambda\in L}$ be a complete set of complex irreducible representations of G.

Let R be a principal ideal domain with field of fractions the algebraic number field K. Suppose that the endomorphism rings of the simple KG-modules are commutative and galois over K.

Let $\{r^\mu\}_{\mu\in M}$ be a complete set of K-rational irreducible representations of G, let K^μ be the endomorphism ring of r^μ. Let $d_\mu := |K^\mu/K|$, let $\Gamma^\mu := \mathrm{Gal}(K^\mu/K)$. Note that via

$$\begin{array}{ccccc} \mathbf{C} & \otimes_K & K^\mu & \xrightarrow{\sim} & \prod_{\sigma\in\Gamma^\mu}\mathbf{C} \\ z & \otimes & x & \longrightarrow & (z\sigma(x))_\sigma \end{array}$$

we may further blockwise decompose the tensor product

$$\begin{array}{ccccc} \mathbf{C}G = \mathbf{C}\otimes_K KG & \xrightarrow{\sim} & \prod_\mu \mathbf{C}\otimes_K (K^\mu)_{m_\mu} & \xrightarrow{\sim} & \prod_\mu\prod_{\sigma\in\Gamma^\mu}(\mathbf{C})_{m_\mu} \\ g & \longrightarrow & (r^\mu(g))_\mu & \longrightarrow & (\sigma(r^\mu(g)))_{\sigma,\mu}. \end{array}$$

Therefore we have a surjection $L \longrightarrow M$ corresponding to the Galois orbits of $\{\rho^\lambda\}_{\lambda\in L}$. Let $\{\rho^\mu\}_{\mu\in M}$ be a set of representatives of these orbits, so that in particular $n_\mu = m_\mu$.

Suppose the the complex matrix ρ^μ can be chosen to have entries in the integral closure R^μ of R in K^μ.

In case R^μ is a principal ideal domain, this condition is automatically fulfilled by choosing a R^μ-basis for the RG-lattice $\sum_{g\in G} gR^\mu B$, B being a K^μ-basis of our simple KG-module.

Restricting on the left to $RG\subseteq \mathbf{C}G$ and on the right to $M\subseteq L$ we obtain

$$\begin{array}{ccc} RG & \xrightarrow{r'} & \prod_{\mu\in M}(R^\mu)_{n_\mu} \\ g & \longrightarrow & (\rho^\mu(g))_\mu, \end{array}$$

which becomes an isomorphism under $\mathbf{C}\otimes_R -$ since we may compose with

$$\begin{array}{ccccc} \mathbf{C}\otimes_R R^\mu = \mathbf{C} & \otimes_K & K^\mu & \xrightarrow{\sim} & \prod_{\sigma\in\Gamma^\mu}\mathbf{C} \\ z & \otimes & x & \longrightarrow & (z\sigma(x))_\sigma \end{array}$$

to recover

$$\mathbf{C}G \underset{\sim}{\xrightarrow{r}} \prod_{\lambda\in L}(\mathbf{C})_{n_\lambda}.$$

Consider the linear map

$$\begin{array}{ccl} \prod_{\mu\in M}(R^\mu)_{n_\mu} & \xrightarrow{t'} & \mathbf{C}G \\ (\xi^\mu)_\mu & \longrightarrow & \sum_g \left(\sum_{\mu\in M} n_\mu \sum_{\sigma\in\Gamma^\mu}\mathrm{Tr}(\sigma(\xi^\mu)\sigma(\rho^\mu(g^{-1})))\right) g \\ & = & \sum_g \left(\sum_{\mu\in M} n_\mu \mathrm{Tr}_{K^\mu/K}(\mathrm{Tr}(\xi^\mu\rho^\mu(g^{-1})))\right) g \end{array}$$

The image of t' is contained in RG. Serre's Fourier inversion formula (1.1.1) reads

$$r't' = |G|.$$

Let $\{x_1^\mu,\ldots,x_{d_\mu}^\mu\}$ be an R-linear basis of R^μ and write an element $a\in R^\mu$ as

$$a = \sum_{s\in[1,d_\mu]} a_s x_s^\mu$$

with coefficients $a_s\in R$.

The R-linear matrix attached to r' in the obvious bases has the entry

$$\rho^\mu_{ijs}(g)$$

at the position $(g, \mu ijs)$. The matrix attached to t' has the entry

$$n_\mu \mathrm{Tr}_{K^\mu/K}(x_s^\mu \rho_{ji}^\mu(g^{-1})) \quad = \quad \textstyle\sum_{t\in[1,d_\mu]} n_\mu \mathrm{Tr}_{K^\mu/K}(x_s^\mu x_t^\mu)\rho_{jit}^\mu(g^{-1})$$

at the position $(\mu ijs, g)$. Denoting the discriminant by $\Delta_\mu := \det(\mathrm{Tr}_{K^\mu/K}(x_s^\mu x_t^\mu))_{st}$, we obtain

$$\det t' = \pm \det r' \prod_{\mu\in M} \Delta_\mu^{(n_\mu^2)} n_\mu^{(n_\mu^2 d_\mu)}.$$

The argument from (S 1.1.1) together with an elementary divisor argument in order to pass down to $\mathbf{Z}$ gives the

Proposition 1.1.5 (total index formula II) *The index of*

$$RG \xrightarrow{r'} \prod_{\mu\in M} (R^\mu)_{n_\mu}$$

as abelian groups is

$$\sqrt{\left| \mathrm{N}_{K/\mathbf{Q}} \left(\frac{|G|^{|G|}}{\prod_{\mu\in M} \Delta_\mu^{(n_\mu^2)} n_\mu^{(n_\mu^2 d_\mu)}} \right) \right|}.$$

In particular, in case $R = \mathbf{Z}$ the prime divisors of Δ_μ form a subset of the prime divisors of $|G|$.

1.1.3 Rough estimates for quasiblock indices

Retain the notation from (S 1.1.1), case $G = \mathcal{S}_n$. Let L be a subset of the set of partitions of n, let $\varepsilon^L := \sum_{\lambda\in L} \varepsilon^\lambda$.

Lemma 1.1.6 *The generalized quasiblock index, i.e. the index of the inclusion*

$$\begin{array}{rcl} Q^L := \mathbf{Z}\mathcal{S}_n\varepsilon^L & \xrightarrow{r_L} & \prod_{\lambda\in L}(\mathbf{Z})_{n_\lambda} =: \Gamma_L \\ s\varepsilon^L & \longrightarrow & (\rho^\lambda(s))_{\lambda\in L} \end{array}$$

is independent of the choice of the integral representations ρ^λ.

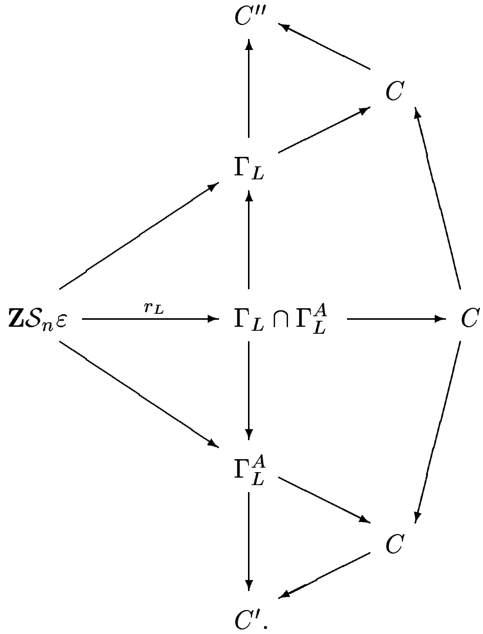
NB in case L consists of all partitions of n this follows from the total index formula (1.1.4). Consider a tuple $A := (\alpha^\lambda)_{\lambda\in L} \in \Gamma_L$ such that the determinant of $\alpha^\lambda \in (\mathbf{Z})_{n_\lambda}$ is nonzero and such that

$$\alpha^\lambda \rho^\lambda(s)(\alpha^\lambda)^{-1}$$

is integral for each $\lambda \in L$ and each $s \in \mathcal{S}_n$, i.e. such that r_L maps into

$$\Gamma_L \cap \Gamma_L^A \subseteq \prod_{\lambda\in L} (\mathbf{Q})_{n_\lambda}.$$

Consider the diagram of abelian groups, where on the right hand side we insert the respective cokernels C,



Using an elementary divisor form of A, we conclude that C' and C'' are isomorphic as abelian groups, so that the assertion results from the Circonference Lemma.

NB the quasiblock $Q^{(3,1)}$

$$\begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ 4 & 4 & \mathbf{Z} \end{pmatrix} \simeq \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & 2 \\ \mathbf{Z} & \mathbf{Z} & 2 \\ 2 & 2 & \mathbf{Z} \end{pmatrix}$$

of $\mathbf{Z}\mathcal{S}_4$ (2.1) shows that the elementary divisors of $Q^L \xrightarrow{r_L} \Gamma_L$ are not well defined.

Now we set out to give estimates for this generalized quasiblock index. Let

$$\Gamma_L \xrightarrow{t_L} \mathbf{Z}\mathcal{S}_n$$

denote the restriction of t to Γ_L, let

$$\mathbf{Z}\mathcal{S}_n \xrightarrow{r'_L} \Gamma_L$$

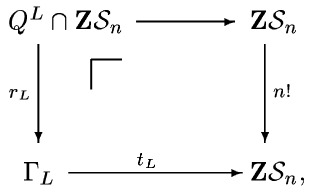
denote the composition of r_L and the projection to Γ_L. By virtue of (1.1.1) we conclude that

$$(\Gamma_L \xrightarrow{t_L r'_L} \Gamma_L) = n!.$$

For a linear map a between $\mathbf{Z}$-lattices, let $|\det a|$ denote the product of the nonnegative elementary divisors of a. Regarding the corresponding matrices we notice that

$$|\det t_L| = |\det r'_L| \prod_{\lambda \in L} n_\lambda^{(n_\lambda^2)}.$$

Denote by $(\bar{-})$ the reduction modulo $n!$. Let M_L be the image of $\bar{t}_L$, let W_L be the cokernel of $\bar{r}'_L$, i.e. the cokernel of r_L, so that $w_L := |W_L|$ is the generalized quasiblock index of Q^L. Note that M_L is the cokernel of $Q^L \cap \mathbf{Z}\mathcal{S}_n \xrightarrow{r_L} \Gamma_L$ (intersection in $\mathbf{Q}\mathcal{S}_n$), for we have a pullback diagram



so that $m_L := |M_L|$ is the index of the inclusion $Q^L \cap \mathbf{Z}\mathcal{S}_n \overset{r_L}{\hookrightarrow} \Gamma_L$. In particular we see, since the greatest common divisor n_L of the n_λ, $\lambda \in L$, divides t_L by construction, that

Lemma 1.1.7

$$n!/n_L \prod_{\lambda\in L} (\mathbf{Z})_{n_\lambda} \subseteq (Q^L \cap \mathbf{Z}\mathcal{S}_n)r_L \subseteq (Q^L)r_L \subseteq \prod_{\lambda\in L} (\mathbf{Z})_{n_\lambda}.$$

I.e. for a generalized quasiblock as well as for its intersection with $\mathbf{Z}\mathcal{S}_n$ all ties can be written modulo $n!/n_L$.

By definition we have

$$m_L = \frac{n!^{\sum_{\lambda\in L} n_\lambda^2}}{|\det t_L|} = \frac{n!^{\sum_{\lambda\in L} n_\lambda^2}}{|\det r'_L| \prod_{\lambda\in L} n_\lambda^{(n_\lambda^2)}}$$

and

$$w_L = |\det r'_L|.$$

Furthermore, the Circonference Lemma applied to $\bar{t}_L \bar{r}'_L = 0$ yields

$$n!^{\sum_{\lambda\in L} n_\lambda^2} \;\Big|\; (n!^{n!}/m_L) w_L.$$

Hence,

Lemma 1.1.8 (cf. [P 80/2, II.3]) ([1]) *The product of the index w_L of $(Q^L)r_L$ in $\prod_{\lambda\in L}(\mathbf{Z})_{n_\lambda}$ with the index m_L of $(Q^L \cap \mathbf{Z}\mathcal{S}_n)r_L$ in $\prod_{\lambda\in L}(\mathbf{Z})_{n_\lambda}$ is*

$$w_L m_L = \frac{n!^{\sum_{\lambda\in L} n_\lambda^2}}{\prod_{\lambda\in L} n_\lambda^{(n_\lambda^2)}}.$$

Since $Q^L \cap \mathbf{Z}\mathcal{S}_n \subseteq Q^L$, we obtain in particular

$$\frac{n!^{2(\sum_{\lambda\in L} n_\lambda^2)-n!}}{\prod_{\lambda\in L} n_\lambda^{(n_\lambda^2)}} \;\Big|\; w_L^2 \;\Big|\; \frac{n!^{\sum_{\lambda\in L} n_\lambda^2}}{\prod_{\lambda\in L} n_\lambda^{(n_\lambda^2)}}$$

In practice, both the upper and the lower bound tend to be far from the actual value. For L being the set of all partitions of n we recover the total index formula (1.1.4).

1.1.4 Ties for the center

The embedding $Z(\mathbf{Z}\mathcal{S}_n) \hookrightarrow \prod_\lambda \mathbf{Z}$ is the bonsai version of the embedding $\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda}$. It might be illuminating to have seen it in advance.

Retain the notation from (S 1.1.1), case $G = \mathcal{S}_n$. By (1.1.2), $(\xi^\lambda)_\lambda \in \prod_\lambda (\mathbf{Z})_{n_\lambda}$ is central in $(\mathbf{Z}\mathcal{S}_n)r$ iff $\xi^\lambda = x^\lambda \cdot 1_\lambda$ is a multiple of the identity matrix for all λ and

$$((\xi^\lambda)_\lambda)t = \sum_g \left(\sum_\lambda n_\lambda x^\lambda \mathrm{Tr}(\rho^\lambda(g^{-1}))\right) g \in n!\ \mathbf{Z}\mathcal{S}_n.$$

Let $\chi^\lambda(g) := \mathrm{Tr}(\rho^\lambda(g))$. Via

$$Z(\mathbf{Z}\mathcal{S}_n) \xrightarrow[\sim]{r} \{(x^\lambda)_\lambda | \sum_\lambda x^\lambda n_\lambda \chi^\lambda(g) \equiv_{n!} 0 \text{ for all } g \in \mathcal{S}_n\} \subseteq \prod_\lambda \mathbf{Z}$$

the center of $\mathbf{Z}\mathcal{S}_n$ resp. of its localizations can be read off the character table.

[1]An earlier version of our proof of the estimate contained a simplification due to S. König, the idea of which also went into the present proof.

Remark 1.1.9
(i). Because of

$$\varepsilon^\lambda = \frac{n_\lambda}{n!}\sum_{s\in\mathcal{S}_n}\chi^\lambda(s^{-1})s$$

we know that $\frac{n!}{n_\lambda}\varepsilon^\lambda \in \mathbf{Z}\mathcal{S}_n$, in accordance with (1.1.7).

(ii). Let p be a prime. We **claim** that $Z((\mathbf{Z}\mathcal{S}_n)_{[p]}) = Z(\mathbf{Z}\mathcal{S}_n)_{[p]}$, the naive localizations taken inside $\prod_\lambda(\mathbf{Z})_{n_\lambda}$ and $\prod_\lambda \mathbf{Z}$ respectively (cf. D.2.10). $Z((\mathbf{Z}\mathcal{S}_n)_{[p]})$ is a p-order (or equal to $\prod_\lambda \mathbf{Z}$, cf. D.2.8) since the diagram comparing the inclusion and the inclusion of the centers is a pullback. This gives $\supseteq$. To see $\subseteq$, we consider an element z that is central in the naive localization $(\mathbf{Z}\mathcal{S}_n)_{[p]}$, and thus central in $\prod_\lambda(\mathbf{Z})_{n_\lambda}$. z is contained in $Z(\mathbf{Z}\mathcal{S}_n)_{[p]}$ iff there is some integer m coprime to p such that $mz \in Z(\mathbf{Z}\mathcal{S}_n)$. But there is such an m multiplying z into $\mathbf{Z}\mathcal{S}_n$, thus also into $Z(\mathbf{Z}\mathcal{S}_n)$.

Lemma 1.1.10 (central index formula, [CPW 87, 4.1]) ([2]) *Let p_n be the number of partitions of n. Let c_λ be the length of the conjugacy class of elements of cycle type λ. The index of the inclusion $Z(\mathbf{Z}\mathcal{S}_n) \xrightarrow{r} \prod_\lambda \mathbf{Z}$ is*

$$\frac{1}{\prod_\lambda n_\lambda}\sqrt{n!^{p_n}\prod_\lambda c_\lambda}.$$

Let γ^λ denote the conjugacy class of elements of cycle type λ, so $c_\lambda = \#\gamma^\lambda$. Let $\sum\gamma^\lambda$ denote the sum of its elements in $\mathbf{Z}\mathcal{S}_n$. The restriction of r to $Z(\mathbf{Z}\mathcal{S}_n)$ maps

$$\begin{array}{rcl}\sum\gamma^\mu & \xrightarrow{r} & (\sum_{g\in\gamma^\mu}\rho^\lambda(g))_\lambda \\ & = & (\frac{1}{n_\lambda}\sum_{g\in\gamma^\mu}\chi^\lambda(g))_\lambda \\ & = & (\frac{c_\mu}{n_\lambda}\chi^\lambda(\gamma^\mu))_\lambda,\end{array}$$

the first equality resulting from knowing the image element to be central, the index of $\mathbf{Z}\mathcal{S}_n$ in $\prod_\lambda(\mathbf{Z})_{n_\lambda}$ being finite.

Now $t/n!$ is a ring isomorphism, in particular, t respects the centers. More precisely, the restriction of t to $\prod_\lambda \mathbf{Z}$ maps

$$\begin{array}{rcl}(\partial_{\mu\lambda})_\lambda & \xrightarrow{t} & \sum_\lambda \partial_{\mu\lambda}n_\lambda\sum_\nu\chi^\lambda((\gamma^\nu)^{-1})\sum\gamma^\nu \\ & = & n_\mu\sum_\nu\chi^\mu((\gamma^\nu)^{-1})\sum\gamma^\nu.\end{array}$$

Thus the matrix for t arises from the matrix for r by transposition, reordering columns and imposing factors n_λ^2 on each row λ and $1/c_\mu$ on each column μ, whence

$$|\det t| = |\det r|\prod_\lambda\frac{n_\lambda^2}{c_\lambda}.$$

Now

$$(\det r)(\det t) = n!^{p_n}$$

yields the required formula.

[2] G. Nebe provided the reference.

Example 1.1.11 According to (1.1.10), the index of $Z(\mathbf{Z}\mathcal{S}_5) \subseteq \prod_\lambda \mathbf{Z}$ is

$$\frac{1}{1^2 \cdot 4^2 \cdot 5^2 \cdot 6}\sqrt{120^7 \cdot 24 \cdot 30 \cdot 20 \cdot 20 \cdot 15 \cdot 10 \cdot 1} = 2^{10}3^4 5^4.$$

We multiply the rows of the character table of the $\mathcal{S}_5$ [J 78, 6.3] with the degree of the respective representation and reduce modulo 8 to obtain, in the notation of loc. cit.,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 3 & -3 & 3 & -3 & -3 & 1 \\ -2 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & -3 & 3 & 3 & -3 & 3 & 1 \\ 4 & 0 & 4 & 4 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Column simplifications yield

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 4 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 4 & 1 \end{bmatrix},$$

whence

$$\begin{array}{rl} Z(\mathbf{Z}\mathcal{S}_5)_{[2]} \xrightarrow{\sim} \{x^2 \times x^4 \times x^6 \times x^7 \times x^5 \times x^3 \times x^1 \mid & x^5 + x^6 \equiv_8 x^1 + x^2 \equiv_8 2x^7,\ x^7 \equiv_2 x^1 \equiv_4 x^6,\ x^3 \equiv_2 x^4\} \\ & \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}, \end{array}$$

the numbering of the quasiblocks chosen as in (2.2.1), viz.

$$\begin{array}{lcl} 1 & : & (1,1,1,1,1) \\ 2 & : & (1,1,1,1,1)' \\ 3 & : & (2,1,1,1) \\ 4 & : & (2,1,1,1)' \\ 5 & : & (2,2,1) \\ 6 & : & (2,2,1)' \\ 7 & : & (3,1,1). \end{array}$$

The following basis, written as consisting of row vectors,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

confirms the 2-part of the central index. Let

$$\begin{array}{rcl} A & := & \{x^2 \times x^6 \times x^7 \times x^5 \times x^1 | x^5 + x^6 \equiv_8 x^1 + x^2 \equiv_8 2x^7,\ x^7 \equiv_2 x^1 \equiv_4 x^6\} \\ B & := & \{x^3 \times x^4 | x^3 \equiv_2 x^4\} \end{array}$$

so that $Z(\mathbf{Z}\mathcal{S}_5)_{[2]} \simeq A \times B$. Let $\mathfrak{a} \subseteq A_{(2)}$ be the ideal having

$$\begin{bmatrix} 0 & 0 & 4 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

as $\mathbf{Z}_{(2)}$-linear basis. Then

$$1 + \mathfrak{a} \xrightarrow{\sim} A^*_{(2)}/\mathbf{Z}^*_{(2)}$$

is a multiplicative isomorphism. Also note that essentially the only nontrivial central involution is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -1 & 1. \end{bmatrix}$$

1.1.5 A refinement of the total index formula

We shall refine the total index formula (1.1.4) to the Pierce components.

Let $R = \mathbf{Z}_{(p)}$, p prime. Retain the notation of (S 1.1.1).

Suppose given a Pierce decomposition $1 = \sum_{i\in[1,s]} e_i$ of the image Λ of the embedding $R\mathcal{S}_n$ into $\Gamma := \prod_\lambda (R)_{n_\lambda}$. Refine it to a Pierce decomposition of $1 = \sum_{i\in[1,s],\ \gamma\in[1,r_i]} e_{i,\gamma}$ of Γ.

Lemma 1.1.12 *Suppose given two orthogonal primitive idempotent decompositions*

$$\begin{array}{rcl} 1 & = & \sum_{i\in[1,t]} f_i \\ 1 & = & \sum_{j\in[1,u]} g_j \end{array}$$

of Γ. Then $t = u$, and there exists a unit $x \in \Gamma^*$ and a permutation σ such that

$$f_i = g_{i\sigma}^x$$

for all i.

Since each f_i is primitive in Γ, we may group the decomposition into the f_i's by multiplication with the central primitive idempotents of Γ into decompositions of these central primitive idempotents. Thus we may assume $\Gamma = (R)_m$. Moreover, we may assume g_i to be the i-th main diagonal primitive idempotents of Γ, and $m = u$.

Since $(R)_m$-proj $\simeq R$-proj has only one indecomposable projective module, up to isomorphism, we have $t = m$ by comparison of ranks, and, moreover, isomorphisms $R^m \xrightarrow[\sim]{b_i} (R)_m f_i$. Since $\begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ and $(\, f_1 \cdots f_m\,)$ are mutually inverse isomorphisms between $\bigoplus_i (R)_m f_i$ and $(R)_m$, we obtain, letting $R^m \xrightarrow{b_i f_i} (R)_m$ be given by right multiplication with a column, the equation

$$\begin{pmatrix} b_1 f_1 \\ \vdots \\ b_m f_m \end{pmatrix} f_i = g_i \begin{pmatrix} b_1 f_1 \\ \vdots \\ b_m f_m \end{pmatrix}$$

in $(R)_m$, $\begin{pmatrix} b_1 f_1 \\ \vdots \\ b_m f_m \end{pmatrix}$ being a unit therein.

Therefore, we may assume the $e_{i,\gamma}$ to be the main diagonal primitive idempotents of Γ.

Let M_{ij} be the matrix describing the embedding

$$e_i \Lambda e_j \subseteq e_i \Gamma e_j,$$

for some R-linear basis of $e_i\Lambda e_j$ and for the canonical basis of $e_i\Gamma e_j$, consisting of matrix tuples with one nonvanishing entry equal to 1. This is, the rows of M_{ij} furnish a basis of $e_i\Lambda e_j$ in terms of the canonical basis of $e_i\Gamma e_j$, and the columns are indexed by matrix tuple positions.

Collect the bases of $e_i\Lambda e_j$ to a basis $\mathcal{B}$ of Λ, and collect the bases of $e_i\Gamma e_j$ to a basis $\mathcal{C}$ of Γ, such that the embedding $\Lambda \subseteq \Gamma$ is given by a main diagonal block matrix M consisting of the M_{ij}'s. Let $\mathcal{G}$ be the basis of Λ consisting of the images of the group elements of the $\mathcal{S}_n$.

Write the matrix of the embedding r with respect to the bases $\mathcal{G}$ and $\mathcal{C}$ as a product of the base change matrix U from $\mathcal{G}$ to $\mathcal{B}$ with M.

The matrix of t with respect to the bases $\mathcal{C}$ and $\mathcal{G}$ arises from the matrix of r by transposition; followed by a permutation of the rows corresponding to transposition of the factors $(R)_{n_\lambda}$, given by a permutation matrix T; followed by left multiplication with n_λ of each row for the respective λ the corresponding matrix tuple position belongs to, given by a diagonal matrix D; followed by a permutation of the columns corresponding to group element inversion, given by a permutation matrix V. Hence Serre's Fourier inversion formula $rt = n!$ (1.1.1) reads

$$(UM)(DTM^tU^tV) = n!.$$

Lemma 1.1.13 (basis-ties duality) *We abbreviate*

$$w_{ij} := \mathrm{rk}\; e_i\Lambda e_j = \mathrm{rk}\; e_j\Lambda e_i$$

and obtain

$$M_{ij}D_{ij}T_{ij}M_{ji}^t \in n!\,\mathrm{GL}_{w_{ij}}(R),$$

where T_{ij} permutes the rows of M_{ji}^t such that the matrix tuple position corresponding to the k-th row of M_{ji} and to the k-th column of M_{ij} are mutually transposed. D_{ij} is a main diagonal matrix consisting of n_λ's according to the matrix tuple position corresponding to the respective column of M_{ij}, or, equivalently, the respective row of $T_{ij}M_{ji}^t$.

In particular, a **basis** *of*

$$e_i\Lambda e_j$$

furnishes a complete set of **ties** *for*

$$e_j\Lambda e_i,$$

in the sense that an element of $e_j\Gamma e_i$ is in $e_j\Lambda e_i$ iff, written as a column in the same ordering as the columns of M_{ij} are written, its product with $M_{ij}D_{ij}$ is in $n!\,R^{w_{ij}}$.

Suppose given an element x with

$$M_{ij}D_{ij}x = n!\,y,$$

y being an integral vector. We have to show that there exists an integral vector z such that $x = T_{ij}M_{ji}^t z$. Letting

$$z = (\frac{1}{n!}M_{ij}D_{ij}T_{ij}M_{ji}^t)^{-1}y,$$

we obtain

$$T_{ij}M_{ji}^t z = T_{ij}M_{ji}^t(\frac{1}{n!}M_{ij}D_{ij}T_{ij}M_{ji}^t)^{-1}y = x.$$

Let

$$w_{ij}^\lambda := \mathrm{rk}(e_i(R)_{n_\lambda}e_j)$$

and note that

$$\det D_{ij} = \prod_\lambda n_\lambda^{w_{ij}^\lambda}.$$

Taking p-parts of determinants, (1.1.13) admits the

Corollary 1.1.14 (refined index formula)

$$(\det M_{ij})_p(\det M_{ji})_p = \left(\frac{n!^{\,w_{ij}}}{\prod_\lambda n_\lambda^{w_{ij}^\lambda}}\right)_p.$$

In particular,

$$(\det M_{ii})_p = \left(\sqrt{\frac{n!^{\,w_{ii}}}{\prod_\lambda n_\lambda^{w_{ii}^\lambda}}}\right)_p.$$

NB already $\mathbf{Z}_{(3)}\mathcal{S}_3$ shows that in general $(\det M_{ji})_p$ depends on the chosen embedding $R\mathcal{S}_n \overset{r}{\hookrightarrow} \Gamma$.

We recover the total index formula (1.1.4) by remarking that it is of local nature and that the product over the indices of the inclusions

$$e_i\Lambda e_j \subseteq e_i\Gamma e_j$$

yields the local total index, since $\sum_{ij} w_{ij}^\lambda = n_\lambda^2$ and $\sum_{ij} w_{ij} = n!$.

Question 1.1.15 Consider the index of the inclusion

$$e_i \Lambda e_j \subseteq \bigoplus_{\gamma,\delta} e_{i,\gamma} \Lambda e_{j,\delta}$$

resulting from 'dropping those ties which involve more than one position'. Is it independent of the choices made? Is it invariant under exchange of i and j?

Example 1.1.16 Let $R := \mathbf{Z}_{(2)}$, let the inclusion $\Lambda \subseteq \Gamma$ be as given further down in (S 2.1), localized at (2). Let

$$\begin{array}{rcl} e_1 & := & 0 \times 0 \times \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \\ e_2 & := & 0 \times 0 \times \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \\ e_3 & := & 1 \times 1 \times \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \end{array}$$

We obtain

$$M_{11} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{bmatrix}, \; M_{13} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \; M_{31} = \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}, \; M_{33} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{bmatrix},$$

and, moreover, $M_{11} = M_{12} = M_{21} = M_{22}$, $M_{13} = M_{23}$, $M_{31} = M_{32}$.
We index the factors of Γ from left to right with $\lambda = 1, 2, 3, 4, 5$. For M_{11}, the columns are indexed by matrix tuple positions $(\lambda, i, j) = (5,1,1), (3,1,1), (4,1,1)$, for M_{13} by $(3,1,3), (4,1,3)$, for M_{31} by $(3,3,1), (4,3,1)$, for M_{33} by $(3,3,3), (4,3,3), (1,1,1), (2,1,1)$. The positions for the remaining matrices arise from these by parallel shift. Furthermore, we have $D_{11} = \mathrm{diag}(2,3,3)$, $D_{13} = \mathrm{diag}(3,3)$, $D_{31} = \mathrm{diag}(3,3)$, $D_{33} = \mathrm{diag}(3,3,1,1)$. Finally, we note that all permutation matrices T_{ij} equal the identity and obtain

$$\begin{array}{rcl} M_{11} D_{11} M_{11}^t & = & 24 \cdot \begin{bmatrix} 1/3 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 8 \end{bmatrix} \\ M_{13} D_{13} M_{31}^t & = & 24 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ M_{31} D_{31} M_{13}^t & = & 24 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ M_{33} D_{33} M_{33}^t & = & 8 \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 4 & 4 \\ 1 & 2 & 4 & 8 \end{bmatrix}. \end{array}$$

The formulas of (1.1.14) yield

$$\begin{array}{rcl} (\det M_{11})_2 & = & 2^{(3\cdot 3-1)/2} \\ (\det M_{33})_2 & = & 2^{(4\cdot 3-0)/2} \\ (\det M_{13})_2 \cdot (\det M_{31})_2 & = & 2^{(2\cdot 3-0)}, \end{array}$$

returning the local total index at 2 as

$$2^{4\cdot(3\cdot 3-1)/2+(4\cdot 3-0)/2+2\cdot(2\cdot 3-0)} = 2^{34}$$

(cf. S 2.1.1).

1.1.6 det exp

G. Nebe pointed out that (1.1.4, 1.1.5, 1.1.8, 1.1.10, 1.1.13) may be derived more conceptionally using the associative bilinear form on $\mathbf{Z}S_n$ (cf. 1.1.18). Here, we have left our arguments unchanged, to gain some variety [3].

We keep the notation of (S 1.1.1), for a general finite group G. exp **denotes the formal exponential function.**

Lemma 1.1.17 *For $g \in G$ the equalities*

$$\prod_{\lambda}(\det\exp(t\cdot\rho^{\lambda}(g)))^{n_\lambda} = \det\exp(t\cdot g(-)) = \begin{cases} \exp(|G|t) & \text{for } g=1 \\ 1 & \text{for } g\neq 1 \end{cases}.$$

hold in $\mathbf{C}[[t]]$. $g(-)$ is to be read as the linear endomorphism of $\mathbf{C}G$ given by left multiplication with g.

The first equality follows by Wedderburn's isomorphism and by $\det(A(-)) = (\det A)^k$ for $A \in (\mathbf{C}[[t]])_k$.

We **claim** the second equality. Choosing G as basis of $\mathbf{C}G$ and sorting it into cosets modulo the cyclic subgroup $\langle g\rangle \leq G$ we are reduced to the case $G = C_m$, $m \geq 1$. The case $m = 1$ corresponds to $g = 1$, so we may assume $m \geq 2$ and claim the result to be 1.

Let the generalized hyperbolic sine be defined by

$$s_{m,i}(t) := \sum_{j\in\mathbf{Z},\ i+jm\geq 0} t^{i+jm}/(i+jm)!,$$

depending on i only modulo m. Note that

$$\frac{d}{dt}s_{m,i}(t) = s_{m,i-1}(t)$$

With respect to the basis $(1, g, g^2, \ldots, g^{m-1})$, we obtain

$$\exp(t\cdot g(-)) = (s_{m,i-j}(t))_{ij}.$$

On the one hand we have

$$\begin{aligned} \frac{d}{dt}\det(s_{m,i-j}(t))_{ij} &= \sum_{u\in[1,m]} \det\left(\begin{cases} s_{m,i-j-1}(t) & \text{for } j=u \\ s_{m,i-j}(t) & \text{for } j\neq u \end{cases}\right)_{ij} \\ &= \sum_{u\in[1,m]} 0, \end{aligned}$$

on the other hand we see

$$\det(s_{m,i-j})_{ij}(0) = 1.$$

Corollary 1.1.18 *For $g \in G$ the equalities*

$$\sum_{\lambda} n_\lambda \operatorname{tr}\rho^{\lambda}(g) = \operatorname{tr}(g(-)) = \begin{cases} |G| & \text{for } g=1 \\ 0 & \text{for } g\neq 1 \end{cases}$$

hold. In particular, the associative bilinear form $\langle g,h\rangle := \partial_{g,h^{-1}}$, $g,h\in G$, reads

$$\langle g,h\rangle = \frac{1}{|G|}\sum_{\lambda} n_\lambda \operatorname{tr}(\rho^{\lambda}(g)\rho^{\lambda}(h)).$$

Consider the **first** derivative of the identities in (1.1.17) and evaluate at zero. Note that for $A \in (\mathbf{C}[[t]])_k$ we have

$$\begin{aligned} (\frac{d}{dt}\det(1+tA))(0) &= \sum_{u\in[1,k]} \det(1+tA)_u(0) \\ &= \operatorname{tr}(\frac{d}{dt}A)(0), \end{aligned}$$

where B_u denotes the matrix B with uth column replaced by its derivative. Or use a modification of the argument of (1.1.17) to argue directly.

[3] At the time this subsection was written, I was unaware of Jacobi's formula det exp = exp trace. See for instance I. P. GOULDEN, D. M. JACKSON, *Combinatorial Enumeration,* Wiley 1983, page 11, formula 7. For the conceptional approach mentioned in the text, cf. e.g. `http://xxx.lanl.gov/abs/math.NT/0102048`.

1.1.7 Strong horizontal orthogonality relations

Two asides.

Retain the notation from (S 1.1.1), case $G = \mathcal{S}_n$. The horizontal orthogonality relations of the character table are equivalent to its vertical orthogonality relations. These have as a corollary Serre's Fourier inversion formula $rt = n!$ (1.1.1), which in turn is equivalent to $tr = n!$. This formula, applied to a tuple of matrices having only one nonzero entry 1 at position ij in the factor μ, written $(\partial_{\mu ij,\lambda st})_{\lambda st}$, yields

$$\begin{array}{rcl} (\partial_{\mu ij,\lambda st})_{\lambda st} & \stackrel{t}{\longrightarrow} & \sum_g \left(\sum_{s,t,\lambda} n_\lambda \rho^\lambda_{ts}(g^{-1})\partial_{\mu ij,\lambda st}\right) g \\ & = & \sum_g n_\mu \rho^\mu_{ji}(g^{-1}) g \\ & \stackrel{r}{\longrightarrow} & (\sum_g n_\mu \rho^\mu_{ji}(g^{-1})\rho^\lambda_{st}(g))_{\lambda st}, \end{array}$$

hence

Lemma 1.1.19 (strong horizonal orthogonality relations [Se 77, 2.2 cor. 2, cor. 3])

$$\sum_g \rho^\mu_{ji}(g^{-1})\rho^\lambda_{st}(g) = \frac{n!}{n_\mu}\partial_{\mu ij,\lambda st}.$$

In particular, suppose the entry at the position of μij of any element of $(R\mathcal{S}_n)r$ to be divisible by p^α and the entry at the position of μji of any element of $(R\mathcal{S}_n)r$ to be divisible by p^β. I.e. suppose we have a p^α-tie at the single position μij and a p^β-tie at the single position μji. Then

$$p^{\alpha+\beta} \;\Big|\; \frac{n!}{n_\lambda}.$$

Cf. e.g. $\mathbf{Z}_{(3)}\mathcal{S}_6$ (S 2.3.3).

In particular, letting $\chi^\lambda(g) := \sum_i \rho^\lambda_{ii}(g)$, we conclude that the horizontal orthogonality relations

$$\sum_g \chi^\mu(g^{-1})\chi^\lambda(g) = n!\, \partial_{\mu,\lambda}$$

hold, so that the circle of implications is closed again.

Now consider the matrix $G = (\gamma_{ij})_{ij} := \sum_g \rho^\lambda(g)^t\rho^\lambda(g)$, which is invertible, since $x^tGx > 0$ for any nonzero rational vector x, and which has the property that $\rho^\lambda(h)^tG\rho^\lambda(h) = G$ for any $h \in \mathcal{S}_n$, i.e. $\rho^\lambda(h)^tG\rho^\lambda(h^{-1}) = G\rho^\lambda(h^{-2})$. Thus

$$\begin{array}{rcl} \sum_m \sum_g \gamma_{im}\rho_{mj}(g^{-2}) & = & \sum_{kl}\sum_g \rho^\lambda_{ki}(g)\gamma_{kl}\rho^\lambda_{lj}(g^{-1}) \\ & \stackrel{(1.1.19)}{=} & \frac{n!}{n_\lambda}\sum_{kl}\partial_{ik,lj}\gamma_{kl} \\ & = & \frac{n!}{n_\lambda}\gamma_{ij}, \end{array}$$

i.e. $G\sum_g \rho^\lambda(g^{-2}) = \frac{n!}{n_\lambda}G$, whence the

Lemma 1.1.20 ([Se 77, 13.2 prop. 39])

$$\sum_g \rho^\lambda(g^2) = \frac{n!}{n_\lambda}.$$

In particular, the tuple of scalar matrices $(n!/n_\lambda)_\lambda$ is contained in the image of r, in accordance with (1.1.7).

1.2 Modified Coxeter relations

In order to be able to check the correctness of a representation of the $\mathcal{S}_n$ given by the operation matrices of a transposition w and an n-cycle z, we recall the relations that generate as a normal subgroup the kernel of the map from the free group on two elements W and Z to the $\mathcal{S}_n$ which maps W to w and Z to z.

First we regard the ordinary Coxeter relations. We **claim** that the group epimorphism from the

$$\tilde{\mathcal{S}}_n := \left\langle W_i \;\middle|\; i \in [1, n-1], \left\{ \begin{array}{rll} W_i^2 & \text{for} & i \in [1, n-1] \\ [W_i, W_j] & \text{for} & i - j \geq 2, \\ W_{i+1}W_iW_{i+1} = W_iW_{i+1}W_i & \text{for} & i \in [1, n-2] \end{array} \right. \right\rangle$$

onto the $\mathcal{S}_n$ via

$$W_i \longrightarrow (i, i+1)$$

is injective. We have a morphism

$$\begin{array}{ccc} \tilde{\mathcal{S}}_{n-1} & \longrightarrow & \tilde{\mathcal{S}}_n \\ W_i & \longrightarrow & W_i \end{array}$$

and **claim** that its image has index $\leq n$ in $\tilde{\mathcal{S}}_n$, thus proving $\tilde{\mathcal{S}}_n \xrightarrow{\sim} \mathcal{S}_n$. Let

$$X_k := \left\{ \begin{array}{ll} W_{n-1}W_{n-2}\cdots W_k & \text{for } k \in [1, n-1] \\ 1 & \text{for } k = n. \end{array} \right.$$

We **claim** more precisely that each right coset of the image of $\tilde{\mathcal{S}}_{n-1}$ in $\tilde{\mathcal{S}}_n$ contains an X_k. In case W_{n-1} appears in a word representing an element of $\tilde{\mathcal{S}}_n$, we claim that we can find a representing word of the same element with indices decreasing with step 1 from some W_{n-1} to the right and also no further W_{n-1} to the left of it. Regard the first W_{n-1} from the left, then regard the first letter to the right of this W_{n-1} (possibly $= W_{n-1}$) not having a successor with index decreasing by 1 (called 'the successor'). For example, for $n = 6$, in $W_3W_5W_4W_2W_5W_1$ we regard W_4, the successor being W_2. In the occurring cases we will give the method pars pro toto, letting $n = 6$, $W_{n-1} = W_5$.

Case 1. The index of the successor increases by 1. Then

$$\begin{array}{rcl} W_5W_4W_3W_2W_3 & = & W_5W_4W_2W_3W_2 \\ & = & W_2W_5W_4W_3W_2. \end{array}$$

Case 2. The index of the successor increases by ≥ 2. Then

$$\begin{array}{rcl} W_5W_4W_3W_2W_4 & = & W_5W_4W_3W_4W_2 \\ & = & W_5W_3W_4W_3W_2 \\ & = & W_3W_5W_4W_3W_2. \end{array}$$

Case 3. The index of the successor decreases by ≥ 2. Then

$$W_5W_4W_3W_1 = W_1W_5W_4W_3.$$

Since none of the reduction steps produces a new letter W_{n-1} to the left of the picked letter W_{n-1}, this proves the claim.

We define

$$\hat{\mathcal{S}}_n := \left\langle Z, W \;\middle|\; \left\{ \begin{array}{rll} W^2 & & \\ Z^n & & \\ (ZW)^{n-1} & & \\ W(W^Z)W = (W^Z)W(W^Z) & & \\ [W, W^{Z^i}] & \text{for} & i \in [2, n-2] \end{array} \right. \right\rangle,$$

the relations of which we shall call **modified Coxeter relations**, and obtain mutually inverse group morphisms

$$\begin{array}{rcl} \tilde{\mathcal{S}}_n & \longleftrightarrow & \hat{\mathcal{S}}_n \\ W_i & \longrightarrow & W^{Z^{i-1}} \\ W_1 & \longleftarrow & W \\ W_{n-1}W_{n-2}\cdots W_1 & \longleftarrow & Z \end{array}$$

which is verified in direction $\longleftarrow$ using the isomorphism $\tilde{\mathcal{S}}_n \xrightarrow{\sim} \mathcal{S}_n$.

1.3 The universal Pierce decomposition

We formalize the point of view of path algebras as universal Pierce decompositions in a fairly obvious and well-known manner.

Let k be a commutative ring.

Definition 1.3.1 *A* **quiver** *is a quadruple $Q = (V, \Lambda, s, t)$, consisting of a* **finite** *set of* **vertices** *V and a set of* **arrows** *Λ, together with two maps $\Lambda \xrightarrow{s} V$ - the* **start** *- and $\Lambda \xrightarrow{t} V$ - the* **target.** *A morphism of quivers is a pair of maps, one on the vertices, one on the arrows, which is compatible with s and t. The quivers as objects and the morphisms of quivers as morphisms furnish the category* quiv.

*The category of k-***algebras with Pierce decompositions,** *k-*algpierce*, is defined as follows. Objects are pairs*

$$(A, (e_i)_{i\in I})$$

(more formally, $(A, I \xrightarrow{e_{(-)}} A)$), consisting of a k-algebra A and a tuple $(e_i)_{i\in I}$ which gives a finite orthogonal decomposition $\sum_{i\in I} e_i = 1$ into idempotents. A morphism

$$(A, (e_i)_{i\in I}) \xrightarrow{(u,v)} (B, (f_j)_{j\in J})$$

is a pair, consisting of a morphism $A \xrightarrow{u} B$ of k-algebras and a map $I \xrightarrow{v} J$ of indexing sets such that

$$e_i u = f_{iv}.$$

For short, we also denote $(A, (e_i)_{i\in I})$ by A and a morphism (u, v) by u.

NB we do not require primitivity for the occurring idempotents.

Lemma 1.3.2 *The forgetful functor*

$$\begin{array}{ccc} k\text{-algpierce} & \xrightarrow{(\vec{-})} & \text{quiv} \\ A & \longrightarrow & \vec{A} \end{array}$$

which associates with $(A, (e_i)_{i\in I})$ the quiver having the set I as set of vertices and the set e_iAe_j as set of arrows with start i and target j, has a left adjoint

$$\begin{array}{ccc} \text{quiv} & \xrightarrow{k(-)} & k\text{-algpierce} \\ Q & \longrightarrow & kQ, \end{array}$$

where kQ has, as a module over k, a basis consisting of these words in the arrows of Q in which the target of each letter coincides with the start of the subsequent letter, if existent. In particular, to each vertex i we associate an empty word e_i, which has by convention i as start and target. The product is defined on this basis as the concatenation of these words if possible, and as zero otherwise. The tuple of idempotents is given by the empty words, indexed by the associated vertices.

Both functors are to be read as operating on the morphisms in the expected manner.

The unit of this adjunction

$$Q \longrightarrow (kQ)^{\rightarrow}$$

is given by sending the vertices and the arrows to themselves.

The counit of this adjunction

$$k\vec{A} \longrightarrow A$$

is given on the basis of the algebra by sending a word, i.e. a formal product of letters being elements in various Pierce components, to its actual product in A, and on the indexing set by sending an index to itself.

The adjunction triangles have to be verified.

Corollary 1.3.3 (the morphism construction principle) *Suppose given an algebra A and an orthogonal decomposition $1_A = \sum_{i \in I} e_i$ into idempotents. Let Q be a quiver having I as set of vertices. Each map which sends an arrow of Q with start i and target j to an arbitrarily chosen element of $e_i A e_j$ can be prolonged uniquely to an algebra morphism from kQ to A, sending i to e_i.*

Apply the universal property of $Q \longrightarrow (kQ)^{\to}$ expressed as adjunction in (1.3.2) as follows. We have a bijection

$$k\text{-algpierce}(kQ, A) \xrightarrow{\sim} \text{quiv}(Q, \vec{A})$$

given by an application of $(\vec{-})$ followed by composition with the unit $Q \longrightarrow (kQ)^{\to}$. Giving a map which sends an arrow of Q with start i and target j to an element of $e_i A e_j$ amounts to give a morphism of quivers

$$Q \xrightarrow{q} \vec{A}$$

By adjunction there is a unique morphism in k-algpierce

$$kQ \xrightarrow{u} A$$

such that

$$(Q \longrightarrow (kQ)^{\to} \xrightarrow{\vec{u}} \vec{A}) = (Q \longrightarrow \vec{A}),$$

i.e. such that u restricts to q in this sense.

Chapter 2

Guiding examples

Our guiding examples consist of embeddings of $\mathbf{Z}\mathcal{S}_4$, $\mathbf{Z}\mathcal{S}_5$, $\mathbf{Z}\mathcal{S}_6$ and the quasiblocks of $\mathbf{Z}\mathcal{S}_7$ into products of integral matrix rings.

We discuss the case $\mathbf{Z}\mathcal{S}_4$ in full detail and abbreviate later on. From $\mathbf{Z}\mathcal{S}_5$ on, we make use of the possibility of giving different embeddings at the various primes together with a constructive argument that this suffices to give a simultaneous embedding. Which we refrain from calculating, since it would contain no new information.

In case of the quasiblocks of $\mathbf{Z}\mathcal{S}_7$ the reader is asked to trust the computer calculations insofar that we won't give a way to check the details of its correctness by theoretical means. However, it is of course still possible to check its correctness via computer. The reason for this inconvenience is that we do **not dispose of an index formula for the quasiblocks** (cf. S 1.1.3). Because of its preliminary nature and because of its length, the section on these quasiblocks is presented as an appendix (A F).

2.1 $\mathbf{Z}\mathcal{S}_4$

2.1.1 Description of $\mathbf{Z}\mathcal{S}_4$

Definition 2.1.1 *A* **tie** *is a congruence of matrix entries.*

This notion is used in the context of embeddings of suborders of a product of integral matrix rings, where a set of ties describes the suborder as an abelian subgroup. I.e. writing the set of ties as a linear map to a torsion module, the suborder is given as the kernel of this map.

The index of $\mathbf{Z}\mathcal{S}_4$ in $\Gamma := \prod_\lambda (\mathbf{Z})_{n_\lambda}$ is

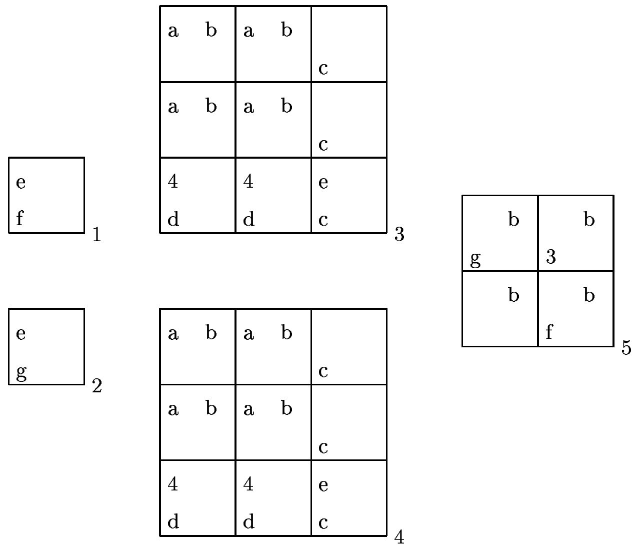
$$\sqrt{\frac{24^{24}}{1^1 1^1 3^9 3^9 2^4}} = 2^{34} 3^3,$$

cf. (1.1.4).

We claim that $\mathbf{Z}\mathcal{S}_4$ can be embedded into

$$\Gamma = \mathbf{Z} \times \mathbf{Z} \times (\mathbf{Z})_3 \times (\mathbf{Z})_3 \times (\mathbf{Z})_2$$

such that the image allows the following description. The respective lower right number indicates the position of the factor in the product of matrix rings, counted from left to right.



$$\begin{array}{lrll}
\mathrm{a} & x^3 & \equiv_4 & x^4 \\
\mathrm{b} & x^3+x^4 & \equiv_8 & 2x^5 \\
\mathrm{c} & x^3 & \equiv_2 & x^4 \\
\mathrm{d} & x^3 & \equiv_8 & x^4 \\
\mathrm{e} & x^1-x^3 & \equiv_8 & x^2-x^4 \equiv_4 0 \\
\mathrm{f} & x^1 & \equiv_3 & x^5 \\
\mathrm{g} & x^2 & \equiv_3 & x^5
\end{array}$$

This is to be read as the subset of the product of the matrix rings, consisting of elements satisfying the ties a to g as given in the table as well as the one-entry-ties as given by number in the picture, indicating the entry to be divisible by this number. The ties given in the table have to be read **parallel** for the matrix entries, so that e.g. the tie labelled by a reads as, x^k_{ij} being the entry in the i-th row and the j-th column of the quasiblock k,

$$\begin{array}{lll}
x^3_{11} & \equiv_4 & x^4_{11} \\
x^3_{12} & \equiv_4 & x^4_{12} \\
x^3_{21} & \equiv_4 & x^4_{21} \\
x^3_{22} & \equiv_4 & x^4_{22}
\end{array}$$

The embedding is given by

$$\begin{array}{ccccccccccc}
 & & 1 & & 2 & & 3 & & 4 & & 5 \\
(12) & \longrightarrow & -1 & \times & 1 & \times & \begin{pmatrix} -11 & -24 & 2 \\ 5 & 11 & -1 \\ 0 & 0 & -1 \end{pmatrix} & \times & \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} & \times & \begin{pmatrix} -5 & 24 \\ -1 & 5 \end{pmatrix} \\
(1234) & \longrightarrow & -1 & \times & 1 & \times & \begin{pmatrix} 26 & 57 & 2 \\ -11 & -24 & -1 \\ -4 & -8 & -1 \end{pmatrix} & \times & \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 0 & 1 \end{pmatrix} & \times & \begin{pmatrix} 4 & -15 \\ 1 & -4 \end{pmatrix}
\end{array}$$

For this map to give a morphism we check the modified Coxeter relations (S 1.2).

The correspondence of the quasiblocks to the partitions (cf. 4.1.1) is given by, the dash indicating transposition,

$$\begin{array}{rcl} 1 & : & (1,1,1,1) \\ 2 & : & (1,1,1,1)' \\ 3 & : & (2,1,1) \\ 4 & : & (2,1,1)' \\ 5 & : & (2,2), \end{array}$$

as we check using a character table. This comparison also shows that rationally we have obtained all simple modules and that therefore the morphism $\mathbf{Z}\mathcal{S}_4 \longrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda}$ as given above is injective.

For a non dashed partition λ of this list we use the Specht lattice S^λ (4.1.1), for its transpose λ' we use the lattice $S^{\lambda,-} \simeq S^{\lambda',*}$ (cf. 6.2.5) for our embedding.

Denote by A the abelian subgroup A described by the ties given above inside Γ.

At this stage we do not know yet that A equals the subring Λ generated by the images of (12) and (1234), being an isomorphic copy of $\mathbf{Z}\mathcal{S}_4$. We proceed in three steps.

(i) Let $A_{[p]}$ be the kernel of the map from Γ to $(\Gamma/A)_{(p)}$, p prime. A equals Λ iff $A_{[p]}$ equals the naive localization $\Lambda_{[p]}$ (D.2.10) for each prime p dividing 4!, as follows by intersection.

(ii) We show that $A_{[p]}$ is in fact a subring, from which we conclude that $\Lambda_{[p]} \subseteq A_{[p]}$ after having checked that the ties are satisfied by the images of the generators (12) and (1234).

(iii) We show that the index of $A_{[p]}$ in Γ equals the index of $\Lambda_{[p]}$ in Γ, viz. 2^{34} for $p = 2$ and 3^3 for $p = 3$. By (ii) we may now conclude that $\Lambda_{[p]} = A_{[p]}$.

To carry out (i-iii), we exhibit a Pierce decomposition $A_{[p]} = \bigoplus_{ij} e_i A_{[p]} e_j$, e_i and e_j being idempotents of Γ contained in $A_{[p]}$. We shouldn't call them 'idempotents of $A_{[p]}$' until we know that $A_{[p]}$ is a ring, but such a direct sum decomposition exists regardless whether $A_{[p]}$ is a subring or not. Then we exhibit $\mathbf{Z}$-linear bases for the Pierce components $e_i A_{[p]} e_j$, so that we are reduced to showing that the products of basis elements of Pierce components which multiply nontrivially are contained in $A_{[p]}$, which is a calculation.

Moreover, for (ii) we may regard the candidate ring direct factors of A separately. I.e. in case the projection of $A_{[p]}$ to a product of a subset of the factors of Γ is contained in $A_{[p]}$, we may consider such projections instead of $A_{[p]}$.

Furthermore, in case the ties describing $A_{[p]}$ are ordered blockwise in a parallel manner, we may as well choose parallel bases for the Pierce components. Therefore, concerning the question whether $A_{[p]}$ is a subring we may shrink these blocks to the size of 1×1 - which then becomes Morita reduction for the homogenus ring $A_{[p]}$ (D.1.1) as soon as (ii) is proven.

Choosing the basis elements of the Pierce components $e_i A e_j$ a priori in an upper triangular manner allows to check (iii) easily. Note that the Morita multiplicities enter when multiplying the separate indices of the Pierce components of the Morita reduction together again.

The case $p = 3$.

Instead of $A_{[3]}$ we consider its projection A' to the product of the factors 1, 2, and 5 of Γ. For the idempotents

$$\begin{array}{rcccccc} & & 1 & & 2 & & 5 \\ e & := & 1 & \times & 0 & \times & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ f & := & 0 & \times & 1 & \times & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

we exhibit bases of the corresponding Pierce components

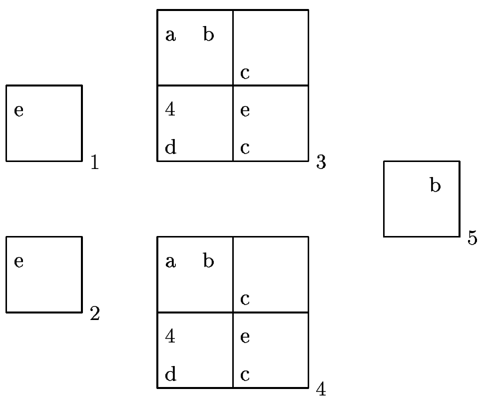
$$\begin{array}{rclrcccccl} eA'e & = & \mathbf{Z}\langle & e & = & 1 & \times & 0 & \times & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ & & & x & := & 0 & \times & 0 & \times & \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}\rangle \\ eA'f & = & \mathbf{Z}\langle & g & := & 0 & \times & 0 & \times & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\rangle \\ fA'e & = & \mathbf{Z}\langle & h & := & 0 & \times & 0 & \times & \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\rangle \\ fA'f & = & \mathbf{Z}\langle & f & = & 0 & \times & 1 & \times & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ & & & y & := & 0 & \times & 0 & \times & \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\rangle. \end{array}$$

Thus $A_{[3]}$ is a subring of Γ with index 3^3, so $A_{[3]} = \Lambda_{[3]}$.

For $A_{[3]}$ to be homogenus (D.1.1) it suffices to show that $(eA'e)/3$ and $(fA'f)/3$ are indecomposable as left modules over themselves, since Ae and Af lie in different genera because of different annihilators (D.1.5, D.2.21). But now both rings are isomorphic to the local ring $\mathbf{F}_3[X]/X^2$.

The case $p = 2$.

We shrink $A_{[2]}$ blockwise to obtain the following abelian subgroup B in $\Gamma' := \mathbf{Z} \times \mathbf{Z} \times (\mathbf{Z})_2 \times (\mathbf{Z})_2 \times \mathbf{Z}$.



$$\begin{array}{clcl} \text{a} & x^3 & \equiv_4 & x^4 \\ \text{b} & x^3 + x^4 & \equiv_8 & 2x^5 \\ \text{c} & x^3 & \equiv_2 & x^4 \\ \text{d} & x^3 & \equiv_8 & x^4 \\ \text{e} & x^1 - x^3 & \equiv_8 & x^2 - x^4 \equiv_4 0 \end{array}$$

We choose the idempotents

$$\begin{array}{rcl} e & := & 0 \times 0 \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times 1 \\ f & := & 1 \times 1 \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \times 0 \end{array}$$

and exhibit bases for the corresponding Pierce components

$$
\begin{array}{rclcccccccccl}
eBe & = & \mathbf{Z}\langle\, e & = & 0 & \times & 0 & \times & \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) & \times & 1\,, \\
 & & x & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}-2&0\\0&0\end{smallmatrix}\right) & \times & 0\,, \\
 & & y & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right) & \times & 0 \quad \rangle \\
eBf & = & \mathbf{Z}\langle\, g & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) & \times & 0\,, \\
 & & h & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right) & \times & 0 \quad \rangle \\
fBe & = & \mathbf{Z}\langle\, i & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\-4&0\end{smallmatrix}\right) & \times & 0\,, \\
 & & j & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\8&0\end{smallmatrix}\right) & \times & 0 \quad \rangle \\
fBf & = & \mathbf{Z}\langle\, f & = & 1 & \times & 1 & \times & \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) & \times & 0\,, \\
 & & u & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\0&-4\end{smallmatrix}\right) & \times & 0\,, \\
 & & v & := & 0 & \times & 2 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\0&2\end{smallmatrix}\right) & \times & 0\,, \\
 & & w & := & 0 & \times & 0 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right) & \times & 0 \quad \rangle.
\end{array}
$$

Hence, checking products of these basis elements, B is a subring in Γ'. Therefore also $A_{[2]}$ is a subring of Γ. Moreover $A_{[2]}$ has index $2^{(4\cdot 4+2\cdot 1+2\cdot 5+1\cdot 6)} = 2^{34}$ in Γ, the Morita factors taken into account. Thus $A_{[2]} = \Lambda_{[2]}$.

For $A_{[2]}$ to be homogenus (D.1.1) it suffices to show that $(eBe)/2$ and $(fBf)/2$ are local rings, since Be and Bf lie in different genera because of different annihilators (D.1.5, D.2.21) and since parallel ties yield isomorphic corresponding indecomposable projectives.

But, for $\alpha, \beta, \gamma \in \mathbf{Z}/2$ the equation

$$\alpha e + \beta x + \gamma y = (\alpha e + \beta x + \gamma y)^2 = \alpha^2 e + \beta^2 y$$

has only trivial solutions.

And also, for $\alpha, \beta, \gamma, \delta \in \mathbf{Z}/2$ the equation

$$\alpha f + \beta u + \gamma v + \delta w = (\alpha f + \beta u + \gamma v + \delta w)^2 = \alpha^2 f$$

has only trivial solutions.

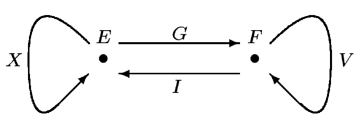
Altogether, A equals Λ, which is isomorphic to $\mathbf{Z}\mathcal{S}_4$. $A_{[2]}$ and $A_{[3]}$ are homogenus.

2.1.2 $\mathbf{F}_2\mathcal{S}_4$ as path algebra modulo relations

We shall write, up to Morita equivalence, $\mathbf{Z}_{(2)}\mathcal{S}_4$ and, derived from this, $\mathbf{F}_2\mathcal{S}_4$, as path algebra modulo relations. This is of course possible in a trivial manner - take one point and one arrow for each element of a generating subset of the group, modulo the relations defining the group (as a semigroup). Therefore, we require the points to correspond to **primitive** idempotents.

Maintain the notation from (S 2.1.1). Consider the quiver

$\Xi :=$



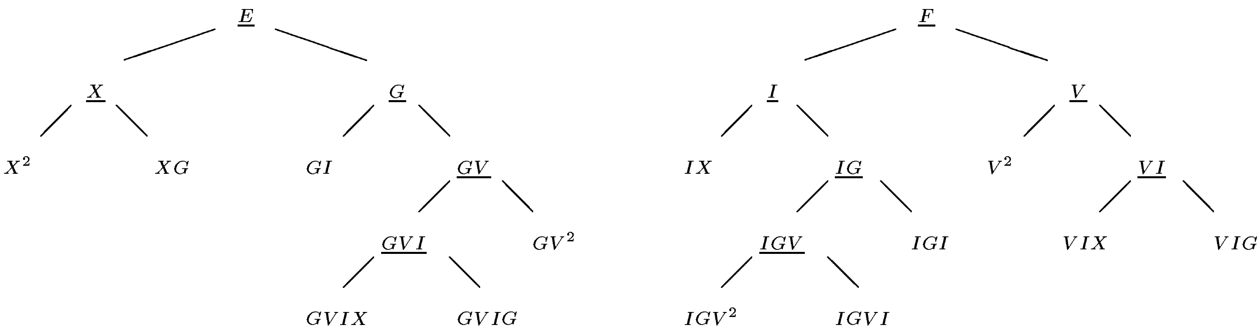
A ring morphism

$$\mathbf{Z}\Xi \longrightarrow B$$

is defined by sending the capital to the small letter elements (1.3.3). Since

$$\begin{array}{rcl} y & = & -gvi \\ h & = & gv \\ j & = & -vi \\ u & = & ig \\ w & = & -igv, \end{array}$$

this morphism is surjective. In order to calculate its kernel, we regard the multiplication trees of Ξ, consisting of concatenable words in the arrows. We shall give ideal generators which on the one hand lie in the kernel and which on the other hand allow to express the nonunderlined elements as linear combinations of the underlined ones modulo this ideal. This shows that this ideal coincides with the kernel, since the rank of $\mathbf{Z}\Xi$ modulo this ideal is less or equal than the rank of B.



The kernel is generated as an ideal by

$$\begin{array}{rcl} X^2 & - & (2X - GVI) \\ XG & - & (2G - 2GV) \\ GI & - & 2X \\ V^2 & - & 2V \\ IX & - & (2I - 2VI) \\ VIG & - & IGV. \end{array}$$

The kernel K of $\mathbf{F}_2\Xi \longrightarrow B/2$ now is generated as an ideal by

$$\begin{array}{rcl} X^2 & - & GVI \\ XG & & \\ GI & & \\ V^2 & & \\ IX & & \\ VIG & - & IGV, \end{array}$$

thus giving a Morita equivalence between $\mathbf{F}_2\Xi/K$ and $\mathbf{F}_2\mathcal{S}_4$. In K. ERDMANN's notation [Er 90, Tables, p. 295] the algebra $\mathbf{F}_2\Xi/K$ is called $D(2\mathcal{B})_{k=1,c=0,s=2}$. It differs from the algebra given in [GR 92, p. 74] already by the sizes of the Pierce components ([1]).

[1] M. KAUER provided the reference.

2.2 $\mathbf{Z}\mathcal{S}_5$

2.2.1 Setup

The index of $\mathbf{Z}\mathcal{S}_5$ in $\Gamma := \prod_\lambda (\mathbf{Z})_{n_\lambda}$ is

$$\sqrt{\frac{120^{120}}{1^1 1^1 4^{16} 4^{16} 5^{25} 5^{25} 6^{36}}} = 2^{130} 3^{42} 5^{35},$$

cf. (1.1.4).

A complete set of integrally realized ordinary irreducible representations gives an embedding

$$\begin{array}{ccccccccc}
\mathbf{Z}\mathcal{S}_5 & \longrightarrow & \mathbf{Z} & \times & \mathbf{Z} & \times & (\mathbf{Z})_4 & \times & (\mathbf{Z})_4 \\
 & & & & & & (\mathbf{Z})_5 & \times & (\mathbf{Z})_5 \\
 & & & & & & & \times & (\mathbf{Z})_6 \\
(12) & \longrightarrow & -1 & \times & 1 & \times & \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \times & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \times & \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
(12345) & \longrightarrow & 1 & \times & 1 & \times & \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} & \times & \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \times & \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},
\end{array}$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters, yielding the correspondence of the quasiblocks to the partitions of 5 as

$$\begin{array}{rcl}
1 & : & (1,1,1,1,1) \\
2 & : & (1,1,1,1,1)' \\
3 & : & (2,1,1,1) \\
4 & : & (2,1,1,1)' \\
5 & : & (2,2,1) \\
6 & : & (2,2,1)' \\
7 & : & (3,1,1),
\end{array}$$

where we have numbered the factors of Γ from left to right.

We conjugate this embedding separately at the primes 2, 3 and 5 via tuples of SL-elements. Since

$$\mathrm{SL}_{n_\lambda}(\mathbf{Z}) \longrightarrow \mathrm{SL}_{n_\lambda}(\mathbf{Z}/8) \times \mathrm{SL}_{n_\lambda}(\mathbf{Z}/3) \times \mathrm{SL}_{n_\lambda}(\mathbf{Z}/5)$$

is surjective (A.2.1), we may map the conjugators C_p needed at a prime divisor p of 5! to $\mathrm{SL}_{n_\lambda}(\mathbf{Z}/p^{v_p(n!)})$ and choose an inverse image C of this tuple in $\mathrm{SL}_{n_\lambda}(\mathbf{Z})$. Conjugation with the C yields the same ties at p as conjugation with C_p, since the matrices $C_p C^{-1}$ map to $1 \in \mathrm{SL}_{n_\lambda}(\mathbf{Z}/p^{v_p(n!)})$, having no effect on the ties (1.1.2). Hence we obtain an embedding

$$\mathbf{Z}\mathcal{S}_5 \xrightarrow{\sim} (\mathbf{Z}\mathcal{S}_5)_{[2]} \cap (\mathbf{Z}\mathcal{S}_5)_{[3]} \cap (\mathbf{Z}\mathcal{S}_5)_{[5]} \subseteq \Gamma,$$

where the occurring naive localizations are realized as subrings of Γ by the ties given below (S 2.2.2, S 2.2.3, S 2.2.4).

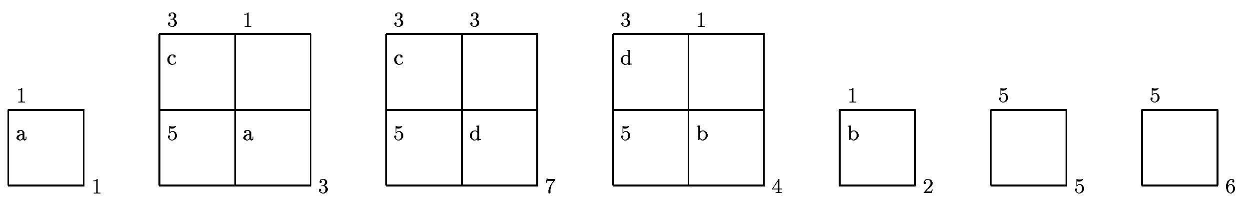
The proof of (A.2.1) is constructive in the case of the $\mathrm{SL}_m(\mathbf{Z})$, but applied to our present argument this construction would yield absolutely large matrix entries not carrying any extra information, so it does not make sense to compute them.

Moreover, we use the language of **Morita multiplicities.** At each prime we first claim the naive localization (D.2.10) to be homogenus (D.1.1). Then the associated Morita equivalent basic ring is displayed, together with the multiplicities of the indecomposable projective left lattices in the naive localization, displayed on top of each involved quasiblock column, where the sets of isomorphism classes of indecomposable projectives of both rings are identified via Morita equivalence.

E.g. to describe $A_{[2]}$ in the case $p = 2$ of (S 2.1.1) in terms of the picture describing B, we would place the Morita multiplicity 2 on top of the left columns of the quasiblocks 3 and 4 and on top of the quasiblock 5, and the Morita multiplicity 1 on top of the remaining columns.

2.2.2 $(\mathbf{Z}\mathcal{S}_5)_{[5]}$

We claim that $(\mathbf{Z}\mathcal{S}_5)_{[5]}$ is homogenus and takes the following form.



a	x^1	$\equiv_5$	x^3
b	x^2	$\equiv_5$	x^4
c	x^3	$\equiv_5$	x^7
d	x^4	$\equiv_5$	x^7

First proof (theoretical). (4.2.8).

Second proof (pedestrian). Needed to illustrate the results of (S 4.2).

We conjugate the embedding given in (S 2.2.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$1 \times 1 \times \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 1&0&0&0\\ -1&1&-1&1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ -1&1&-1&1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1&0&0&-1&1&0\\ 0&1&0&-1&0&1\\ 1&-1&1&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&2&1\\ 0&0&0&1&1&2 \end{bmatrix}$$

to obtain the embedding

$$
\begin{array}{rcccccccl}
\mathbf{Z}\mathcal{S}_5 & \longrightarrow & \mathbf{Z} & \times & \mathbf{Z} & \times & \begin{array}{c}(\mathbf{Z})_4 \\ (\mathbf{Z})_5\end{array} & \begin{array}{c}\times \\ \times \\ \times\end{array} & \begin{array}{l}(\mathbf{Z})_4 \\ (\mathbf{Z})_5 \\ (\mathbf{Z})_6\end{array} \\
(12) & \longrightarrow & -1 & \times & 1 & \times & \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & -1 & -1 \\ 1 & -1 & -2 & -1 \\ -5 & 5 & 5 & 4 \end{bmatrix} & \times & \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -5 & -5 & -5 & -4 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \times & \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} -2 & 1 & 1 & -5 & 5 & 0 \\ 1 & -2 & -1 & -5 & 0 & 5 \\ -4 & 4 & 3 & 0 & 5 & -5 \\ 0 & 0 & 0 & -3 & 1 & 1 \\ 0 & 0 & 0 & -4 & 2 & 1 \\ 0 & 0 & 0 & -4 & 1 & 2 \end{bmatrix} \\
(12345) & \longrightarrow & 1 & \times & 1 & \times & \begin{bmatrix} 1 & -1 & -2 & -1 \\ -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 5 & -5 & -5 & -4 \end{bmatrix} & \times & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & -5 & -5 & -4 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} & \times & \begin{bmatrix} 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} -4 & 4 & 3 & 0 & 5 & -5 \\ -2 & 1 & 1 & -5 & 5 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 11 & -4 & -4 \\ 0 & 3 & 1 & 16 & -4 & -8 \\ 1 & 1 & 0 & 12 & -4 & -4 \end{bmatrix} .
\end{array}
$$

The ties are satisfied on these generators.

In order to prove that the abelian subgroup A described by the ties given above coincides with the image of the embedding given above, we shrink A to the overall Morita multiplicity 1, drop the quasiblocks 5 and 6 and call the resulting subgroup B (cf. 2.1.1). Writing the factors ordered 1, 3, 7, 4, 2 - as in the picture -, we obtain a Pierce decomposition

$$
\begin{array}{rclcccccccc}
e & := & 1 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & 0 \\
f & := & 0 & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & 0 \\
g & := & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & 0 \\
h & := & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \times & 1
\end{array}
$$

into idempotents.

Bases for the Pierce components are given by, dropping zero matrices,

$$
\begin{array}{rclccccccl}
eBe & = & \mathbf{Z}\langle 1 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \times & & \times & \times & , \\
 & & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 5 \end{smallmatrix}\right) & \times & & \times & \times & \rangle \\
eBf & = & \mathbf{Z}\langle & \times & \left(\begin{smallmatrix} 0 & 0 \\ 5 & 0 \end{smallmatrix}\right) & \times & & \times & \times & \rangle \\
fBe & = & \mathbf{Z}\langle & \times & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \times & & \times & \times & \rangle \\
fBf & = & \mathbf{Z}\langle & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \times & , \\
 & & & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 5 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \times & \rangle \\
fBg & = & \mathbf{Z}\langle & \times & & \times & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \times & \times & \rangle
\end{array}
$$

$$\begin{array}{rclcccccccc}
gBf & = & \mathbf{Z}\langle & \times & & \times & \left(\begin{smallmatrix}0&0\\5&0\end{smallmatrix}\right) & \times & & \times & \rangle \\
gBg & = & \mathbf{Z}\langle & \times & & \times & \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) & \times & , \\
 & & & \times & & \times & \left(\begin{smallmatrix}0&0\\0&5\end{smallmatrix}\right) & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) & \times & \rangle \\
gBh & = & \mathbf{Z}\langle & \times & & \times & & \times & \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) & \times & \rangle \\
hBg & = & \mathbf{Z}\langle & \times & & \times & & \times & \left(\begin{smallmatrix}0&0\\5&0\end{smallmatrix}\right) & \times & \rangle \\
hBh & = & \mathbf{Z}\langle & \times & & \times & & \times & \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) & \times & 1, \\
 & & & \times & & \times & & \times & \left(\begin{smallmatrix}0&0\\0&5\end{smallmatrix}\right) & \times & 0\rangle
\end{array}$$

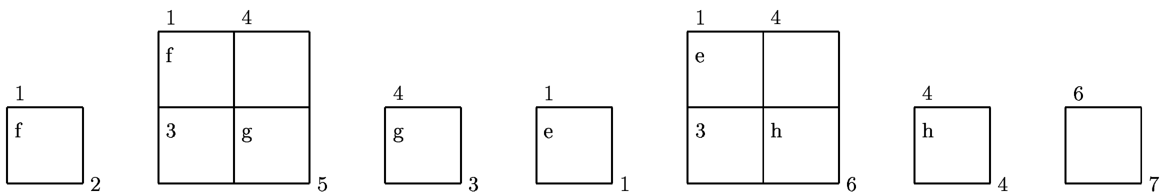
Thus the index of A in Γ is

$$5^{1\cdot1+3\cdot1+3\cdot0+9\cdot1+9\cdot0+9\cdot1+9\cdot1+3\cdot0+3\cdot1+1\cdot1} = 5^{35}.$$

This idempotent decomposition remains primitive modulo 5, since $eBe/5$, $fBf/5$, $gBg/5$ and $hBh/5$ are isomorphic to $\mathbf{F}_5[X]/X^2$. The indecomposable projectives Be, Bf, Bg and Bh lie in different genera because of different annihilators (D.1.5, D.2.21). Thus A is homogenus.

2.2.3 $(\mathbf{Z}\mathcal{S}_5)_{[3]}$

We claim that $(\mathbf{Z}\mathcal{S}_5)_{[3]}$ is homogenus and takes the following form.



$$\begin{array}{cccc}
\text{e} & x^1 & \equiv_3 & x^6 \\
\text{f} & x^2 & \equiv_3 & x^5 \\
\text{g} & x^3 & \equiv_3 & x^5 \\
\text{h} & x^4 & \equiv_3 & x^6
\end{array}$$

We conjugate the embedding given in (S 2.2.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$\begin{array}{cccccc}
1 & \times & 1 & \times & \left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right] & \times \left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{smallmatrix}\right] \\
 & & & \times & \left[\begin{smallmatrix}0&1&0&0&0\\-1&1&2&0&-3\\1&0&0&1&-1\\0&-2&0&-1&3\\0&0&1&0&-1\end{smallmatrix}\right] & \times \left[\begin{smallmatrix}0&1&0&0&0\\-1&1&2&0&-3\\1&0&0&1&-1\\0&-2&0&-1&3\\0&0&1&0&-1\end{smallmatrix}\right] \\
 & & & & & \times \left[\begin{smallmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{smallmatrix}\right]
\end{array}$$

to obtain the embedding

$$
\begin{array}{rcccccccl}
\mathbf{Z}\mathcal{S}_5 & \longrightarrow & \mathbf{Z} & \times & \mathbf{Z} & \times & (\mathbf{Z})_4 & \times & (\mathbf{Z})_4 \\
 & & & & & & (\mathbf{Z})_5 & \times & (\mathbf{Z})_5 \\
 & & & & & & & \times & (\mathbf{Z})_6 \\
(12) & \longrightarrow & -1 & \times & 1 & \times & \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \times & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} -2 & -1 & -1 & -1 & 1 \\ -3 & -4 & -3 & -3 & 8 \\ -3 & -3 & -4 & -3 & 7 \\ 9 & 9 & 9 & 8 & -16 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \times & \begin{bmatrix} 2 & 1 & 1 & 1 & -1 \\ 3 & 4 & 3 & 3 & -8 \\ 3 & 3 & 4 & 3 & -7 \\ -9 & -9 & -9 & -8 & 16 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
(12345) & \longrightarrow & 1 & \times & 1 & \times & \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} & \times & \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \\
 & & & & & \times & \begin{bmatrix} 1 & 3 & 3 & 2 & -6 \\ -12 & -15 & -15 & -12 & 28 \\ -9 & -16 & -15 & -12 & 29 \\ 12 & 15 & 14 & 12 & -26 \\ -6 & -9 & -9 & -7 & 17 \end{bmatrix} & \times & \begin{bmatrix} 1 & 3 & 3 & 2 & -6 \\ -12 & -15 & -15 & -12 & 28 \\ -9 & -16 & -15 & -12 & 29 \\ 12 & 15 & 14 & 12 & -26 \\ -6 & -9 & -9 & -7 & 17 \end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.
\end{array}
$$

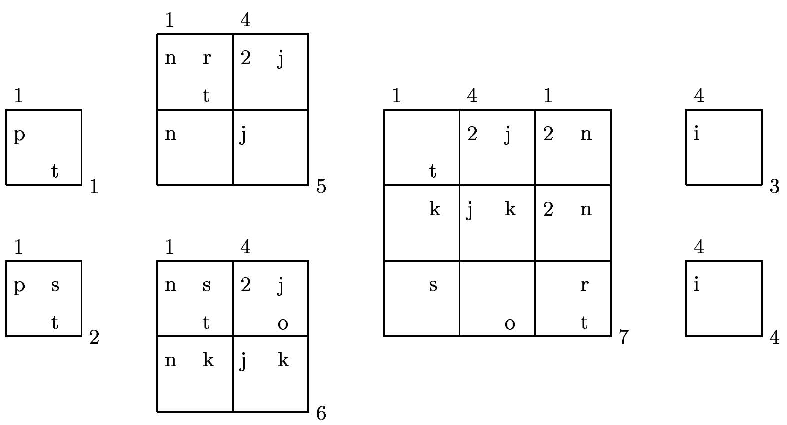
The ties are satisfied by these generators.

For Pierce decompositions of the ring direct factors in spe, viz. 2, 5, 3 and 1, 6, 4, see Case $p = 3$ of (S 2.1.1). The index of the subring described by these ties in Γ is

$$3^{2(1\cdot 1+4\cdot 1+4\cdot 0+16\cdot 1)} = 3^{42}.$$

2.2.4 $(\mathbf{Z}\mathcal{S}_5)_{[2]}$

We claim that $(\mathbf{Z}\mathcal{S}_5)_{[2]}$ is homogenus and takes the following form.



$$\begin{array}{lrll}
\mathrm{i} & x^3 & \equiv_2 & x^4 \\
\mathrm{j} & x^5+x^6 & \equiv_8 & 2x^7 \\
\mathrm{k} & x^6 & \equiv_2 & x^7 \\
\mathrm{n} & x^5-x^6 & \equiv_4 & x^7 \\
\mathrm{o} & x^6 & \equiv_4 & 2x^7 \\
\mathrm{p} & x^1 & \equiv_2 & x^2 \\
\mathrm{r} & x^5 & \equiv_2 & x^7 \\
\mathrm{s} & x^2-x^6 & \equiv_4 & 2x^7 \\
\mathrm{t} & x^1+x^2+x^5+x^6 & \equiv_8 & 2x^7_{11}+2x^7_{33} \equiv_4 0
\end{array}$$

We conjugate the embedding given in (S 2.2.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$\begin{array}{ccccccc}
1 & \times & 1 & \times & \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} & \times & \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \\
 & & & \times & \begin{bmatrix}-1&-1&-1&-1&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\-1&0&0&0&0\end{bmatrix} & \times & \begin{bmatrix}-1&-1&-1&-1&1\\0&1&-2&0&0\\0&2&-5&0&0\\-2&2&-2&5&0\\1&0&-2&-2&0\end{bmatrix} \\
 & & & & & \times & \begin{bmatrix}3&-5&-3&-3&-5&1\\0&1&-2&-2&0&9\\2&0&5&-2&-4&-11\\0&-2&4&5&0&-21\\-1&0&-2&0&2&7\\0&-1&0&0&0&4\end{bmatrix}
\end{array}$$

to obtain the embedding

$$\begin{array}{ccccccccc}
\mathbf{Z}\mathcal{S}_5 & \longrightarrow & \mathbf{Z} & \times & \mathbf{Z} & \times & (\mathbf{Z})_4 & \times & (\mathbf{Z})_4 \\
 & & & & & & (\mathbf{Z})_5 & \times & (\mathbf{Z})_5 \\
 & & & & & & & \times & (\mathbf{Z})_6 \\
(12) & \longrightarrow & -1 & \times & 1 & \times & \begin{bmatrix}-1&0&0&-1\\0&-1&0&1\\0&0&-1&-1\\0&0&0&1\end{bmatrix} & \times & \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&-1\end{bmatrix} \\
 & & & & & \times & \begin{bmatrix}3&4&2&4&-4\\0&-1&-1&0&0\\0&0&1&0&0\\-1&-1&-1&-2&1\\1&1&0&1&-2\end{bmatrix} & \times & \begin{bmatrix}-3&-64&42&-12&-28\\0&11&-5&0&0\\0&24&-11&0&0\\3&67&-41&10&21\\-1&-11&8&-3&-6\end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix}-5&-1850&-294&-860&-600&-110\\2&1025&161&476&328&64\\-4&-1680&-265&-780&-540&-100\\-5&-2627&-413&-1220&-841&-164\\3&1419&224&659&456&86\\0&134&21&62&42&9\end{bmatrix} \\
(12345) & \longrightarrow & 1 & \times & 1 & \times & \begin{bmatrix}0&0&0&1\\-1&0&0&-1\\0&-1&0&1\\0&0&-1&-1\end{bmatrix} & \times & \begin{bmatrix}0&0&0&1\\-1&0&0&-1\\0&-1&0&1\\0&0&-1&-1\end{bmatrix} \\
 & & & & & \times & \begin{bmatrix}3&4&6&6&-2\\0&0&0&1&0\\-1&-1&-1&-2&1\\0&0&-1&-1&-1\\1&1&2&2&-1\end{bmatrix} & \times & \begin{bmatrix}3&60&-38&10&22\\2&40&-28&9&20\\5&99&-69&22&49\\4&104&-73&23&55\\1&9&-6&2&3\end{bmatrix} \\
 & & & & & & & \times & \begin{bmatrix}-7&-3540&-560&-1644&-1138&-212\\8&4408&698&2049&1422&270\\-13&-6987&-1103&-3246&-2243&-426\\-18&-9984&-1581&-4641&-3221&-612\\7&3861&610&1794&1241&236\\3&1668&263&775&535&103\end{bmatrix} .
\end{array}$$

The ties are satisfied by these generators.

In order to prove that the abelian subgroup A described by the ties given above coincides with the image of the embedding given above, we shrink A to the overall Morita multiplicity 1, drop the quasiblocks 3 and 4 and call the resulting subgroup B (cf. 2.1.1). Writing the factors ordered 1, 2, 5, 6, 7, we obtain a Pierce decomposition of B

$$\begin{array}{lclllllllll}
e & := & 1 & \times & 1 & \times & \begin{pmatrix}1&0\\0&0\end{pmatrix} & \times & \begin{pmatrix}1&0\\0&0\end{pmatrix} & \times & \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix} \\
f & := & 0 & \times & 0 & \times & \begin{pmatrix}0&0\\0&1\end{pmatrix} & \times & \begin{pmatrix}0&0\\0&1\end{pmatrix} & \times & \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},
\end{array}$$

into idempotents.

Bases for the Pierce components are given by

$$\begin{array}{rcrcccccccccc}
eBe & = & \mathbf{Z}\langle e & = & 1 & \times & 1 & \times & \begin{pmatrix} 1&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 1&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 1&0&0\\0&0&0\\0&0&1 \end{pmatrix}, \\
& & a & := & 0 & \times & 2 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0&2\\0&0&0\\0&0&0 \end{pmatrix}, \\
& & b & := & 0 & \times & 0 & \times & \begin{pmatrix} 2&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0&0\\0&0&0\\-1&0&0 \end{pmatrix}, \\
& & c & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 4&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \\
& & d & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 2&0&0\\0&0&0\\0&0&2 \end{pmatrix}, \\
& & g & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&0&0\\0&0&4 \end{pmatrix}, \\
& & h & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&4\\0&0&0\\0&0&0 \end{pmatrix}, \\
& & i & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&0&0\\2&0&0 \end{pmatrix}\rangle \\
eBf & = & \mathbf{Z}\langle j & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&2\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&2\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&2&0\\0&0&0\\0&-1&0 \end{pmatrix}, \\
& & k & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&4\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&2&0\\0&0&0\\0&0&0 \end{pmatrix}, \\
& & l & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&4&0\\0&0&0\\0&0&0 \end{pmatrix}, \\
& & m & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&0&0\\0&2&0 \end{pmatrix}\rangle \\
fBe & = & \mathbf{Z}\langle n & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\1&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\1&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \\
& & p & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\2&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&0&2\\0&0&0 \end{pmatrix}, \\
& & q & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\2&0&0\\0&0&0 \end{pmatrix}, \\
& & r & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&0&4\\0&0&0 \end{pmatrix}\rangle \\
fBf & = & \mathbf{Z}\langle f & = & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&1 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&1 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \\
& & s & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&4 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&2&0\\0&0&0 \end{pmatrix}, \\
& & t & := & 0 & \times & 0 & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0\\0&0 \end{pmatrix} & \times & \begin{pmatrix} 0&0&0\\0&4&0\\0&0&0 \end{pmatrix}\rangle
\end{array}$$

For to see that e (resp. f) is primitive, we check that $eBe/2$ (resp. $fBf/2$) does not contain nontrivial idempotents. Regard

$$\begin{array}{rcl}
(\varepsilon e + \alpha a + \beta b + \zeta c + \delta d + \gamma g + \vartheta h + \iota i)^2 & = & \varepsilon^2 e + \zeta^2 g \\
(\varphi f + \sigma s + \tau t)^2 & = & \varphi^2 f + \sigma^2 t.
\end{array}$$

The indecomposable projectives Be and Bf lie in different genera because of different annihilators (D.1.5, D.2.21). Thus A is homogenus. The index of A in Γ is calculated to be

$$2^{(1\cdot 10+4\cdot 6+4\cdot 4+16\cdot 4)+16\cdot 1} = 2^{130}.$$

Remark 2.2.1 (sketch) Since Krull-Schmidt holds in $\mathbf{Z}_{(2)}\mathcal{S}_5$-lat (C.2.15), each lattice has a vertex, i.e. a $\mathcal{S}_n$-conjugacy class of subgroups of $D_8 = \langle(12),(1324)\rangle \leq \mathcal{S}_5$ minimal w.r.t. the lattice being projective relative to it. By abuse of notation in the sequel we talk about subgroups representing conjugacy classes as vertices.

One may define relative projectivity in the following manner. A $\mathbf{Z}_{(2)}\mathcal{S}_5$-lattice X is projective relative to $H \le D_8$ iff there exists a $\mathbf{Z}_{(2)}$-linear endomorphism $X \xrightarrow{f} X$ such that

(i) $|H| = \sum_{g\in\mathcal{S}_5} {}^g f$

(ii) $|H| \mid \sum_{h\in H} {}^h f$,

where ${}^g f(x) := gf(g^{-1}x)$. This is the same as to require the multiplication map $\mathbf{Z}_{(2)}\mathcal{S}_5 \otimes_H X \longrightarrow X$ be split.

The lattice X is called **quasiprojective** iff there exists an idempotent $e \in \mathbf{Q}\mathcal{S}_5$ such that $X \simeq \mathbf{Z}_{(2)}\mathcal{S}_5 e$.

For a natural number a we call X **a-projective** iff there exists a $\mathbf{Z}_{(2)}$-linear endomorphism $X \xrightarrow{f} X$ such that

$$\sum_{g\in\mathcal{S}_5} {}^g f = a,$$

i.e. if we can satisfy (i) for a instead of $|H|$. Thus an H-projective $\mathbf{Z}_{(2)}\mathcal{S}_5$-lattice is $|H|$-projective.

X is a-projective iff every pure epimorphism $Y \xrightarrow{f} X$ of $\mathbf{Z}_{(2)}\mathcal{S}_5$-lattices with has a coretraction g up to the scalar factor a, i.e. such that $gf = a \cdot 1_X$. A quasiprojective $\mathbf{Z}_{(2)}\mathcal{S}_n$-lattice $\mathbf{Z}_{(2)}\mathcal{S}_5 e$ with rational idempotent e therefore is a-projective iff $ae \in \mathbf{Z}_{(2)}\mathcal{S}_5$.

Now taking

$$e_0 := 1_1 \times 1_2 \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)_5 \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)_6 \in B$$

and e a corresponding idempotent in $\mathbf{Z}_{(2)}\mathcal{S}_5$, a direct computer calculation ([2]) has shown the vertex of $\mathbf{Z}_{(2)}\mathcal{S}_5 e$ to be $V_4 = \langle (12)(34), (13)(24)\rangle$, whereas $2e \in \mathbf{Z}_{(2)}\mathcal{S}_5$, i.e. $2e_0 \in B$. I.e. **not** every a-projective quasiprojective lattice is H-projective for some $H \le D_8$ with $|H| = a$.

Note that calculations are simplified by the following observation [D 70]. Let G be a finite group, let R be a discrete valuation ring, let H be a subgroup of G. The RG-lattice X is projective relative to H iff the module $X/|G|$ is projective relative to H. This hinges on the fact that an epimorphism $Y \xrightarrow{f} X$ is split iff $Y/|G| \xrightarrow{f} X/|G|$ is split. In fact, consider the corresponding short exact sequence as an element of $H^1(G, {}_R(X,Z))$, where Z is the kernel of f. The exact sequence

$$H^1(G, {}_R(X,Z)) \xrightarrow{|G|} H^1(G, {}_R(X,Z)) \longrightarrow H^1(G, {}_R(X/|G|, Z/|G|))$$

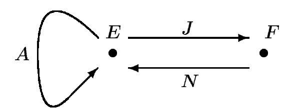
contains a monomorphism on the right hand side.

I don't know whether it is possible to formalize the impression that glueing lattices reduces the vertex in some other way (cf. S 5.3). I do not know how the vertices of the three terms of a pure short exact sequences of lattices are related, except for an estimate which follows from the long exact relative Ext-sequence. Note that in the short exact sequence arising from the inclusion of the sublattice into $\mathbf{Z}_{(2)}\mathcal{S}_5 e$ which is given by intersection with the product of the quasiblocks 5 and 6 all terms have vertex V_4. I do not know how the relative Ext-groups for varying subgroups relate.

2.2.5 $\mathrm{F}_2\mathcal{S}_5$ as path algebra modulo relations

Maintain the notation from (S 2.2.4). Consider the quiver

$\Xi :=$



[2] for which J. Künzer wrote an essential part of the program

We have a ring morphism

$$\mathbf{Z}\Xi \longrightarrow B$$

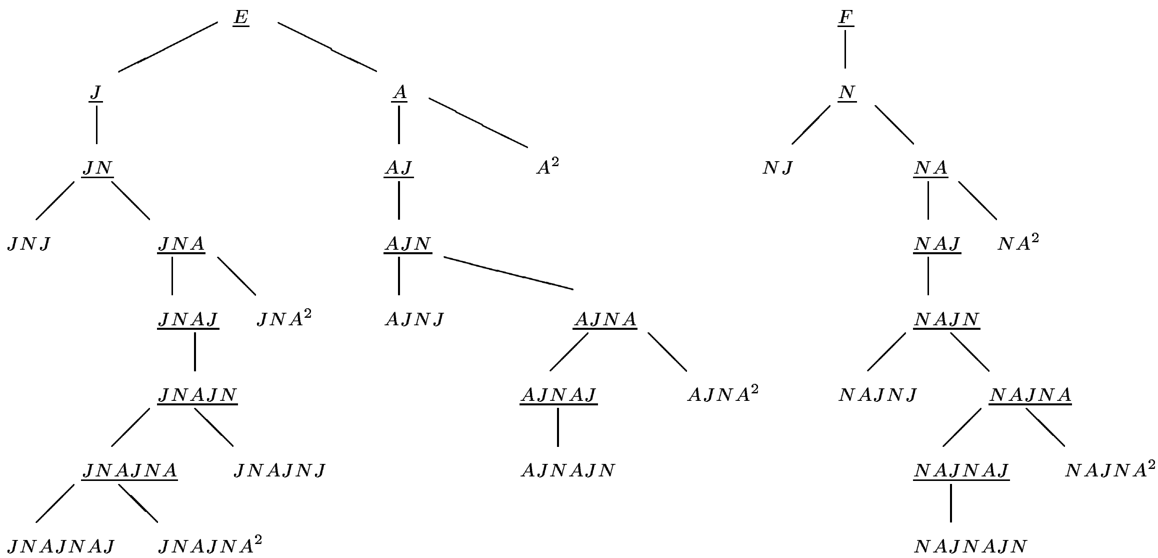
by sending the capital to the small letter elements (1.3.3). Since

$$\begin{array}{rcl} e & = & e \\ a & = & a \\ b & = & jn \\ c & = & ajn \\ d & = & jna - ajn + jnajn + ajna - jnajna \\ g & = & -jnajna + 2jnajn + 2ajna - 4ajn \\ h & = & ajna - 2ajn \\ i & = & -jnajn + 2ajn \\ j & = & j \\ k & = & aj \\ l & = & -ajnaj + 4aj \\ m & = & -jnaj + 2aj \\ n & = & n \\ p & = & -najn - na + najna \\ q & = & 2na + najn - najna \\ r & = & najna - 2najn \\ f & = & f \\ s & = & naj \\ t & = & 4naj - najnaj, \end{array}$$

this morphism is surjective. We **claim** that its kernel is generated as an ideal by

$$\begin{array}{rcl} A^2 & - & 2A \\ NJ & - & 2F \\ (AJN)^2 - (JNA)^2 & - & 2(AJN - JNA). \end{array}$$

Regard the multiplication trees of Ξ (cf. S 2.1.2).



The kernel K of $\mathbf{F}_2\Xi \longrightarrow B/2$ now is generated an an ideal by

$$\begin{array}{l} A^2 \\ NJ \\ (AJN)^2 - (JNA)^2, \end{array}$$

thus giving a Morita equivalence between $\mathbf{F}_2\Xi/K \times \mathbf{F}_2[X]/X^2$ and $\mathbf{F}_2 S_5$. In K. Erdmann's notation [Er 90, Tables, p. 294] this algebra $\mathbf{F}_2\Xi/K$ is called $D(2\mathcal{A})_{k=1}$ [Er 90, Tables, p. 294].

Question 2.2.2 Is there a concept of graph orders analoguous to the concept of graph algebras, sufficient, say, to describe blocks of dihedral defect of group rings over $\mathbf{Z}_{(2)}$? Cf. [Ka 98, prop. 4.1].

2.3 $\mathbf{Z}\mathcal{S}_6$

2.3.1 Setup

The index of $\mathbf{Z}\mathcal{S}_6$ in $\Gamma := \prod_\lambda (\mathbf{Z})_{n_\lambda}$ is

$$\sqrt{\frac{720^{720}}{1^1 1^1 5^{25} 5^{25} 9^{81} 9^{81} 5^{25} 5^{25} 10^{100} 10^{100} 16^{256}}} = 2^{828} 3^{558} 5^{210}.$$

A complete set of rationally irreducible integral representations gives an embedding

$$\begin{array}{lcl} \mathbf{Z}S_6 & \longrightarrow & \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \\ & \times & \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \times (\mathbf{Z})_{16} \end{array}$$

$$(12) \longrightarrow -1 \times \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \; 1 \times \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\times \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
(123456) \longrightarrow -1
\times \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&0&-1\\ 0&1&0&0&1\\ 0&0&1&0&-1\\ 0&0&0&1&1 \end{bmatrix}
\times \begin{bmatrix} 0&0&0&-1&0&0&0&1&1\\ 0&0&0&0&0&0&-1&-1&-1\\ 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&-1&-1\\ -1&0&0&-1&0&0&1&1&1\\ 0&-1&0&1&0&0&0&-1&0\\ 0&0&-1&-1&0&0&0&1&1\\ 0&0&0&0&-1&0&1&1&0\\ 0&0&0&0&0&-1&-1&-1&-1 \end{bmatrix}
\times \begin{bmatrix} 1&-1&-1&-1&0\\ -1&0&1&0&0\\ 0&-1&-1&0&0\\ 1&0&-1&-1&0\\ 0&1&1&1&1 \end{bmatrix}
\times \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1\\ -1&0&0&-1&0&0&-1&0&0&0\\ 0&-1&0&1&0&0&0&0&-1&0\\ 0&0&-1&-1&0&0&0&0&0&-1\\ 0&0&0&0&-1&0&1&0&1&0\\ 0&0&0&0&0&-1&-1&0&0&1\\ 0&0&0&0&0&0&0&-1&-1&-1 \end{bmatrix}
$$

$$
\times 1
\times \begin{bmatrix} 0&0&0&0&-1\\ -1&0&0&0&1\\ 0&-1&0&0&-1\\ 0&0&-1&0&1\\ 0&0&0&-1&-1 \end{bmatrix}
\times \begin{bmatrix} 0&0&0&1&0&0&0&-1&-1\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&1&1\\ 1&0&0&1&0&0&-1&-1&-1\\ 0&1&0&-1&0&0&0&1&0\\ 0&0&1&1&0&0&0&-1&-1\\ 0&0&0&0&1&0&-1&-1&0\\ 0&0&0&0&0&1&1&1&1 \end{bmatrix}
\times \begin{bmatrix} -1&1&1&1&0\\ 1&0&-1&0&0\\ 0&1&1&0&0\\ -1&0&1&1&0\\ 0&-1&-1&-1&-1 \end{bmatrix}
\times \begin{bmatrix} 0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0&-1\\ 1&0&0&1&0&0&1&0&0&0\\ 0&1&0&-1&0&0&0&0&1&0\\ 0&0&1&1&0&0&0&0&0&1\\ 0&0&0&0&1&0&-1&0&-1&0\\ 0&0&0&0&0&1&1&0&0&-1\\ 0&0&0&0&0&0&0&1&1&1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&-1\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 0&0&-1&-1&-1&0&0&-1&0&0&1&-1&-1&0&0&1\\ 0&0&0&1&0&0&0&0&0&-1&-1&0&0&-1&-1&-1\\ 0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\ 1&0&-1&-1&0&0&0&0&0&1&1&0&0&1&0&1\\ 0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&1&0&-1&0&-1&0&-1&0&-1&0&0\\ 0&0&0&0&0&0&1&1&0&0&-1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1 \end{bmatrix},
$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters. The correspondence of the quasiblocks to the partitions of 5 parametrizing the irreducible characters is given by

$$
\begin{array}{rcl}
1 & : & (1,1,1,1,1,1)\\
2 & : & (1,1,1,1,1,1)'\\
3 & : & (2,1,1,1,1)\\
4 & : & (2,1,1,1,1)'\\
5 & : & (2,2,1,1)\\
6 & : & (2,2,1,1)'\\
7 & : & (2,2,2)\\
8 & : & (2,2,2)'\\
9 & : & (3,1,1,1)\\
10 & : & (3,1,1,1)'\\
11 & : & (3,2,1),
\end{array}
$$

where we number the factors of Γ from left to right.

We shall make use of the possibility to give separate embeddings at the prime divisors of $n!$, yielding a global embedding

$$\mathbf{Z}\mathcal{S}_6 \xrightarrow{\sim} (\mathbf{Z}\mathcal{S}_6)_{[2]} \cap (\mathbf{Z}\mathcal{S}_6)_{[3]} \cap (\mathbf{Z}\mathcal{S}_6)_{[5]} \subseteq \Gamma,$$

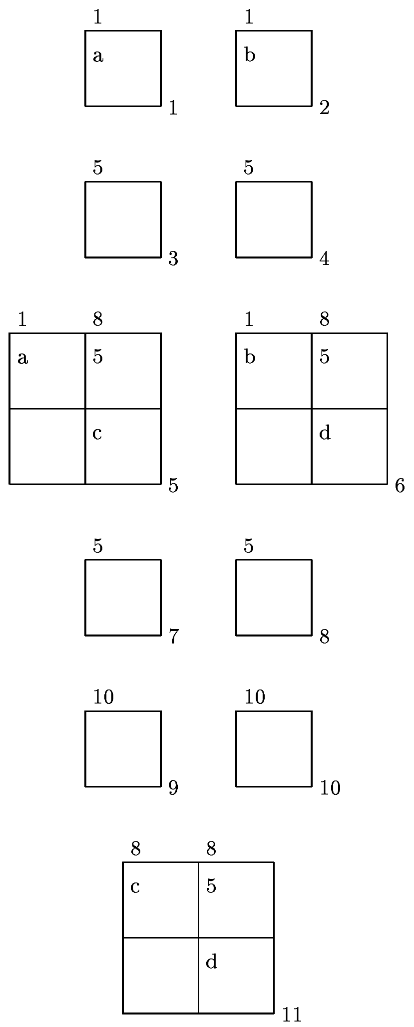
in a constructive, but not explicitely given manner (cf. S 2.2.1).

Furthermore, we employ the language of Morita multiplicities (cf. S 2.2.1).

Moreover, in writing down bases for the Pierce components of the naive localizations resp. for their appropriately reduced versions we drop the redundant information of the position of the matrix entries, already being encoded in the idempotents.

2.3.2 $(\mathbf{Z}\mathcal{S}_6)_{[5]}$

We claim that $(\mathbf{Z}\mathcal{S}_6)_{[5]}$ is homogenus and takes the following form.



a	x^1	$\equiv_5$	x^5
b	x^2	$\equiv_5$	x^6
c	x^5	$\equiv_5$	x^{11}
d	x^6	$\equiv_5$	x^{11}

We conjugate the embedding given in (S 2.3.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$1 \times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 2&3&2&-2&2&-2&2&2&-2\\ 0&0&5&0&-2&0&0&0&0\\ 0&0&0&5&0&-2&0&0&0\\ 0&5&0&0&0&0&3&0&0\\ 0&0&5&0&0&0&0&3&0\\ 0&0&0&5&0&0&0&0&3\\ -5&9&-5&5&-5&5&5&-5&5\\ 0&0&4&0&5&0&0&10&0\\ 0&0&0&4&0&5&0&0&10 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\times\ 1 \times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 38&-438&738&1012&-1182&1432&273&958&1482\\ 0&-228&418&684&30&40&-145&-68&107\\ 0&-38&76&152&-440&545&80&368&508\\ -25&289&-487&-668&780&-945&-180&-632&-978\\ 0&150&-275&-450&-21&-25&96&46&-69\\ 0&25&-50&-100&293&-363&-53&-244&-338\\ -15&175&-295&-405&473&-573&-109&-383&-593\\ 0&90&-165&-270&-12&-16&58&28&-42\\ 0&15&-30&-60&176&-218&-32&-147&-203 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1&0&0&0&0&-2&0&0&0&0&2&-2&2&2&-2&-2\\ 0&1&0&0&0&0&-2&0&0&0&-2&2&0&-6&0&0\\ 0&0&1&0&0&0&0&-2&0&0&-3&0&2&0&-6&4\\ 0&0&0&1&0&0&0&0&-2&0&2&-1&0&2&0&-2\\ 0&0&0&0&1&0&0&0&0&-2&-2&0&-1&0&2&2\\ -1&0&0&-1&1&3&0&0&1&-1&-6&5&-5&-5&5&6\\ 0&-1&0&1&0&0&3&0&-3&0&5&-5&0&10&0&-1\\ 0&0&-1&0&1&0&0&3&0&-3&2&0&-5&0&10&-4\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

to obtain the embedding

$$\mathbf{Z}\mathcal{S}_6 \longrightarrow \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \times \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \times (\mathbf{Z})_{16}$$

$$(12) \longrightarrow -1 \times \begin{bmatrix} -1&0&0&0&1\\ 0&-1&0&0&-1\\ 0&0&-1&0&1\\ 0&0&0&-1&-1\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} 249&0&-50&-330&0&65&100&0&-20\\ -50&-1&125&66&0&-166&-20&0&50\\ 50&0&-151&-66&0&200&20&0&-60\\ 150&0&50&-199&0&-65&60&0&20\\ 0&0&0&0&-1&-1&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ -125&0&290&165&0&-377&-51&0&116\\ 125&0&-315&-165&0&415&50&-1&-126\\ -125&0&380&165&0&-500&-50&0&151 \end{bmatrix} \times \begin{bmatrix} -1&0&1&0&-1\\ 0&-1&-1&0&0\\ 0&0&1&0&0\\ 0&0&0&-1&-1\\ 0&0&0&0&1 \end{bmatrix} \times \begin{bmatrix} -1&0&0&-1&0&0&-1&0&0&0\\ 0&-1&0&1&0&0&0&0&-1&0\\ 0&0&-1&-1&0&0&0&0&0&-1\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&-1&0&1&0&1&0\\ 0&0&0&0&0&-1&-1&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-1&-1&-1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$\times\ 1 \times \begin{bmatrix} 1&0&0&0&-1\\ 0&1&0&0&1\\ 0&0&1&0&-1\\ 0&0&0&1&1\\ 0&0&0&0&-1 \end{bmatrix} \times \begin{bmatrix} -126299&223090&106270&-255030&550640&280970&105090&-352680&-198850\\ -26600&41689&10074&-55366&93020&39410&24890&-49520&-39970\\ -50275&86310&42931&-101120&214540&110620&41170&-138940&-75560\\ 83375&-147265&-70145&168356&-363480&-185465&-69375&232800&131265\\ 17425&-27275&-6545&36275&-60811&-25721&-16315&32320&26161\\ 33500&-57500&-28625&67375&-142950&-73724&-27425&92600&50325\\ 50590&-89355&-42555&102155&-220540&-112525&-42096&141244&79649\\ 10600&-16590&-4045&22055&-37050&-15725&-9905&19761&15885\\ 20100&-34500&-17175&40425&-85770&-44235&-16455&55560&30196 \end{bmatrix}$$

$$\times \begin{bmatrix} 1&0&-1&0&1\\ 0&1&1&0&0\\ 0&0&-1&0&0\\ 0&0&0&1&1\\ 0&0&0&0&-1 \end{bmatrix} \times \begin{bmatrix} 1&0&0&1&0&0&1&0&0&0\\ 0&1&0&-1&0&0&0&0&1&0\\ 0&0&1&1&0&0&0&0&0&1\\ 0&0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&1&0&-1&0&-1&0\\ 0&0&0&0&0&1&1&0&0&-1\\ 0&0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&0&-1 \end{bmatrix}$$

$$\times \begin{bmatrix} -1&0&1&0&-1&0&0&0&0&-5&5&0&5&0&5&-10\\ 0&-1&-1&0&0&0&0&0&0&0&-10&0&0&0&0&15\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&-1&-1&0&0&0&0&0&5&0&0&0&0&-5\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&3&-1&0&1&0&10&-10&0&-10&0&-5&20\\ 0&0&0&0&0&0&-1&-1&0&0&20&0&0&0&0&-25\\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$(123456) \longrightarrow$$

$$-1 \times \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 49 & 375 & 275 & -65 & -495 & -365 & 20 & 150 & 110 \\ -54 & 131 & -136 & 72 & -173 & 180 & -22 & 52 & -54 \\ -30 & -75 & -95 & 40 & 99 & 125 & -12 & -30 & -38 \\ -125 & -5 & -5 & 165 & 6 & 6 & -50 & -2 & -2 \\ 75 & 314 & 0 & -99 & -414 & 0 & 30 & 125 & 0 \\ -75 & -200 & -86 & 99 & 264 & 114 & -30 & -80 & -35 \\ -535 & -950 & -700 & 707 & 1252 & 927 & -215 & -380 & -280 \\ 385 & 715 & 340 & -510 & -942 & -450 & 155 & 285 & 135 \\ -175 & -475 & -45 & 230 & 627 & 62 & -70 & -190 & -20 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\times 1 \times \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix} \times \begin{bmatrix} -540114 & 788575 & 205415 & -1118145 & 1762115 & 748590 & 495365 & -940900 & -724240 \\ 13315 & -36990 & -15490 & 27605 & -90004 & -44415 & -12279 & 56345 & 34705 \\ -201455 & 296135 & 77785 & -417065 & 662544 & 282339 & 184785 & -354690 & -272380 \\ 356405 & -520345 & -135540 & 737830 & -1162735 & -493955 & -326876 & 620849 & 477894 \\ -9352 & 25210 & 10432 & -19385 & 61182 & 30070 & 8618 & -38137 & -23634 \\ 134061 & -197131 & -51776 & 277546 & -441042 & -187947 & -122975 & 236110 & 181327 \\ 216096 & -315479 & -82173 & 447361 & -704948 & -299475 & -198190 & 376406 & 289743 \\ -5409 & 14796 & 6186 & -11208 & 35983 & 17743 & 4978 & -22507 & -13869 \\ 80557 & -118449 & -31114 & 166776 & -265010 & -112935 & -73894 & 141876 & 108951 \end{bmatrix} \times \begin{bmatrix} -1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} -4 & 4 & 17 & 2 & -5 & -2 & 2 & 11 & 10 & 15 & 5 & 5 & 15 & -10 & 0 & -15 \\ 0 & 0 & 0 & -6 & 0 & -2 & 0 & 2 & -10 & 15 & 0 & 10 & 0 & 15 & -5 & 15 \\ 0 & 0 & 0 & 6 & 6 & 0 & -2 & -2 & 10 & 10 & 25 & -10 & -10 & 15 & 15 & -5 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & -5 & 0 & -10 & 0 & -10 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -5 & 5 & 5 & 0 & 0 & 10 \\ 10 & -10 & -33 & -3 & 7 & 5 & -5 & -20 & -20 & -25 & -15 & -5 & -20 & 25 & -5 & 45 \\ -5 & 0 & 5 & 8 & 0 & 0 & 0 & 0 & 15 & -20 & 0 & -15 & 0 & -35 & 10 & -30 \\ 0 & -5 & -5 & -8 & -8 & 0 & 0 & 0 & -20 & -20 & -40 & 15 & 15 & -20 & -20 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 3 & 0 & -3 & 1 & 0 & 2 & 0 & -2 & 0 & -3 & 0 & -3 & 0 & 3 & -2 & 1 \\ 0 & 3 & 3 & -1 & -1 & 0 & 2 & 2 & 4 & 4 & 1 & 3 & 3 & 0 & 0 & -3 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 1 & -3 & 1 & -3 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 & -3 & 3 & 3 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

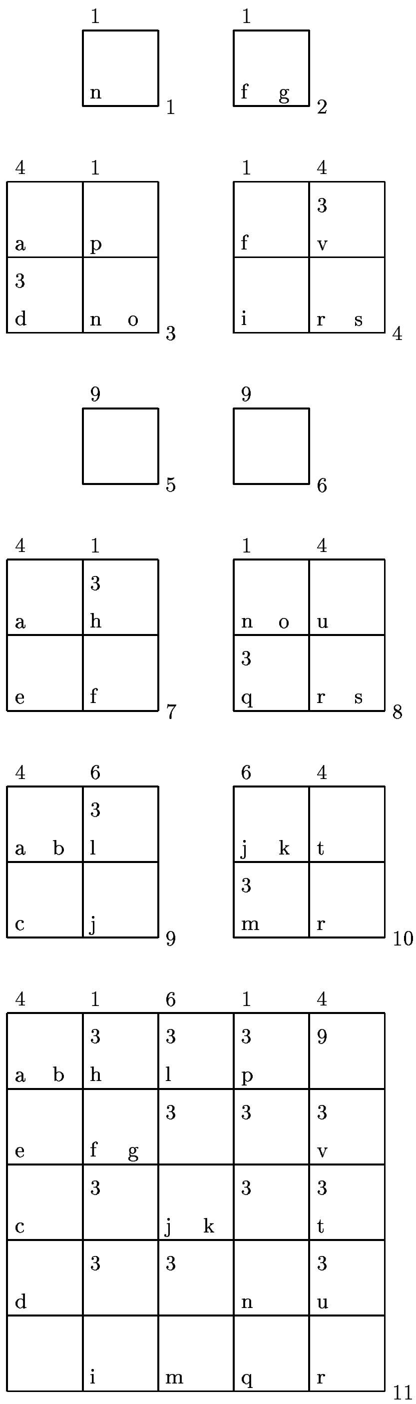
A Pierce decomposition, bases of the Pierce components, the irreducibility of the chosen idempotents and the homogenity of the ring described by the the ties given above is deduced as in (S 2.2.2).

The index in Γ of the subring described by the ties given above is calculated to be, walking along a, c, d, b,

$$5^{1+8+64+64+64+8+1} = 5^{210}.$$

2.3.3 $(\mathbf{Z}\mathcal{S}_6)_{[3]}$

We claim that $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ is homogenus and takes the following form.



$$
\begin{array}{crcl}
\mathrm{a} & x^3 - x^{11} & \equiv_9 & x^9 - x^7 \equiv_3 0 \\
\mathrm{b} & x^9 & \equiv_3 & x^{11} \\
\mathrm{c} & x^9 + x^{11} & \equiv_3 & 0 \\
\mathrm{d} & x^3 & \equiv_9 & 3x^{11} \\
\mathrm{e} & x^7 & \equiv_3 & x^{11} \\
 & & & \\
\mathrm{f} & x^7 - x^{11} & \equiv_9 & x^2 - x^4 \equiv_3 0 \\
\mathrm{g} & x^{11} & \equiv_3 & x^2 \\
\mathrm{h} & x^7 & \equiv_9 & x^{11} \\
\mathrm{i} & x^{11} & \equiv_3 & x^4 \\
 & & & \\
\mathrm{j} & x^9 + x^{11} & \equiv_9 & 2x^{10} \\
\mathrm{k} & x^{10} & \equiv_3 & x^{11} \\
\mathrm{l} & x^9 & \equiv_9 & x^{11} \\
\mathrm{m} & x^{10} & \equiv_9 & 3x^{11} \\
 & & & \\
\mathrm{n} & x^8 - x^{11} & \equiv_9 & x^1 - x^3 \equiv_3 0 \\
\mathrm{o} & x^3 & \equiv_3 & x^8 \\
\mathrm{p} & 3x^3 & \equiv_9 & x^{11} \\
\mathrm{q} & 3x^{11} & \equiv_9 & x^8 \\
 & & & \\
\mathrm{r} & x^{10} - x^4 & \equiv_9 & x^8 - x^{11} \equiv_3 0 \\
\mathrm{s} & x^8 & \equiv_3 & x^4 \\
\mathrm{t} & 3x^{10} + x^{11} & \equiv_9 & 0 \\
\mathrm{u} & 3x^8 & \equiv_9 & x^{11} \\
\mathrm{v} & x^{11} & \equiv_9 & x^4
\end{array}
$$

We conjugate the embedding given in (S 2.3.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$
1 \qquad\qquad \times\ 1
$$

$$
\times \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & -1 & -2 & -1 & -2 \end{bmatrix}
\qquad
\times \begin{bmatrix} 1 & 2 & 1 & -1 & -5 \\ 0 & -1 & 0 & 3 & 1 \\ 0 & -183 & 61 & 635 & 246 \\ 0 & -3 & 1 & 12 & 1 \\ 0 & -81 & 27 & 281 & 109 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 13 & -31 & -8 & 9 & 30 \\ -58 & 138 & 27 & -40 & -125 \\ 261 & -621 & -122 & 180 & 563 \\ 0 & 2 & 0 & -2 & -9 \\ 0 & -9 & 0 & 9 & 40 \end{bmatrix}
\qquad
\times \begin{bmatrix} 4 & -5 & -4 & -4 & 14 \\ -10 & 13 & 10 & 9 & -35 \\ -10 & 13 & 9 & 10 & -35 \\ 11 & -12 & -13 & -13 & 41 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
\times \begin{bmatrix} 0 & 25 & -1 & -49 & -22 & -3 & 99 & -11 & 95 & 29 \\ 2 & -12 & 0 & 23 & 26 & 5 & -50 & 5 & -64 & -18 \\ -13 & -3870 & 27 & 7726 & 2342 & 918 & -15453 & 2010 & -15694 & -3006 \\ 0 & 1294 & 17 & -2631 & -779 & -91 & 5289 & -687 & 4944 & 1197 \\ 3 & 883 & -6 & -1763 & -533 & -210 & 3526 & -459 & 3582 & 684 \\ 0 & -288 & -4 & 586 & 174 & 20 & -1178 & 153 & -1101 & -267 \\ 1 & 0 & 0 & -1 & 2 & -2 & 0 & 0 & 2 & -2 \\ 0 & -1 & 1 & 0 & -2 & 2 & 0 & 0 & -2 & 2 \\ 0 & -37 & 0 & 74 & 16 & 0 & -149 & 20 & -132 & -32 \\ -3 & -627 & 1 & 1256 & 351 & 161 & -2507 & 334 & -2569 & -447 \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & -1 & -2 & 0 & 3 & 2 & 4 & 5 & 0 & 6 & -3 & -1 & 1 & -2 & -4 & 1 \\ 8 & -6 & -9 & -10 & 25 & 10 & 24 & 27 & 40 & 50 & -21 & 13 & -19 & -7 & -20 & -36 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & -5 & 2 & 0 & -2 & 0 & -1 & -8 \\ 0 & -4 & 9 & -5 & 0 & -9 & -5 & -18 & 20 & 0 & -7 & 17 & -9 & 40 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -2 & 1 & -2 & 3 & -1 & 2 & -1 & 8 & 6 & -4 & 5 & -3 & 10 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

to obtain the embedding

$$
\begin{array}{ll}
\mathbf{Z}S_6 \longrightarrow & \\
\mathbf{Z} & \times\,\mathbf{Z} \\
\times\,(\mathbf{Z})_5 & \times\,(\mathbf{Z})_5 \\
\times\,(\mathbf{Z})_9 & \times\,(\mathbf{Z})_9 \\
\times\,(\mathbf{Z})_5 & \times\,(\mathbf{Z})_5 \\
\times\,(\mathbf{Z})_{10} & \times\,(\mathbf{Z})_{10} \\
 & \times\,(\mathbf{Z})_{16}
\end{array}
$$

$$(12) \longrightarrow$$

$$
-1 \qquad\qquad \times\, 1
$$

$$
\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 & -1 \end{bmatrix}
\qquad
\times\begin{bmatrix} 1 & 0 & 387 & -36 & -873 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4342 & 404 & 9797 \\ 0 & 0 & 258 & -23 & -582 \\ 0 & 0 & -1935 & 180 & 4366 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\qquad
\times\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -1 & -252 & -56 & -594 & -132 \\ 0 & 1277 & 284 & 2646 & 588 \\ 0 & -5742 & -1277 & -11907 & -2646 \\ 0 & 18 & 4 & 161 & 36 \\ 0 & -81 & -18 & -720 & -161 \end{bmatrix}
\qquad
\times\begin{bmatrix} 143 & 29 & 30 & 2 & -33 \\ -351 & -71 & -75 & -6 & 84 \\ -360 & -75 & -79 & -10 & 95 \\ 414 & 81 & 78 & -5 & -69 \\ 9 & 0 & -3 & -6 & 13 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 & -3 & 0 & -3 & -3 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\times\begin{bmatrix} 12905 & -2934 & 41878 & -9333 & 186670 & -41985 & 5863 & -12258 & -3025 & 5197 \\ -3852 & 836 & -11980 & 3231 & -53397 & 14455 & -1635 & 3591 & 1188 & -1472 \\ -2256534 & 507078 & -7291241 & 1523745 & -32492268 & 6872733 & -1014543 & 2137674 & 448791 & -897015 \\ 764271 & -174132 & 2487141 & -535456 & 11085623 & -2411983 & 348744 & -728133 & -166491 & 308172 \\ 515196 & -115767 & 1664658 & -347778 & 7418278 & -1568646 & 231624 & -488055 & -102381 & 204789 \\ -170154 & 38781 & -553806 & 119340 & -2468421 & 537553 & -77670 & 162141 & 37155 & -68631 \\ 657 & -180 & 2349 & -594 & 10485 & -2664 & 370 & -723 & -222 & 309 \\ -324 & 9 & -648 & -396 & -2835 & -1692 & -12 & 152 & -361 & -25 \\ -22311 & 5049 & -72477 & 15093 & -323010 & 68067 & -10116 & 21144 & 4507 & -8949 \\ -372645 & 83628 & -1203525 & 249300 & -5363145 & 1124874 & -167355 & 352959 & 72372 & -147902 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -1 & 0 & -2 & 0 & 3 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 9 & 0 & 0 & -9 \\ 0 & -1 & -4 & 0 & 48 & 0 & 0 & 36 & 0 & 90 & -6 & 0 & 63 & 0 & 99 & -72 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 22 & -1 & 27 & 0 & 0 & -18 & 0 & 108 & -57 & 0 & 54 & 0 & 126 & 126 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -3 & 0 & -6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 12 & 0 & 0 & 0 & 0 & 39 & -12 & -1 & 24 & 0 & 42 & 27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$(123456) \longrightarrow$$

$$
-1 \qquad\qquad \times\, 1
$$

$$
\times\begin{bmatrix} 1 & 2 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & -3 & 3 & -1 \end{bmatrix}
\qquad
\times\begin{bmatrix} -2 & -12 & 375 & -30 & -846 \\ 1 & 14 & 320 & -30 & -722 \\ 183 & 2881 & 47891 & -4576 & -108059 \\ 3 & 52 & 1487 & -137 & -3355 \\ 81 & 1275 & 21170 & -2023 & -47767 \end{bmatrix}
$$

$$
\times\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}
\qquad
\times\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -3554 & 61433 & 13829 & 3328 & 750 \\ 15073 & -260682 & -58681 & -13902 & -3132 \\ -67872 & 1173813 & 264232 & 62612 & 14106 \\ 660 & -11363 & -2558 & -810 & -183 \\ -2941 & 50636 & 11399 & 3603 & 814 \end{bmatrix}
\times\begin{bmatrix} 35 & 93 & 92 & 155 & -384 \\ -60 & -232 & -229 & -396 & 976 \\ -78 & -230 & -226 & -385 & 952 \\ 186 & 277 & 278 & 436 & -1096 \\ 45 & 9 & 11 & 2 & -13 \end{bmatrix}
$$

$$
\times\begin{bmatrix} 1 & 2 & -1 & 1 & 3 & 3 & 3 & 6 & 6 & 3 \\ 1 & 0 & -1 & 0 & 3 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -3 & -3 & -3 & -3 & -3 \\ 0 & 1 & 1 & 0 & 3 & 0 & -3 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & -2 & -3 & 0 & -2 & -1 \\ 0 & -1 & -1 & 0 & -1 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 & -1 & -2 & -2 & -3 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}
\qquad
\times\begin{bmatrix} 35948 & -10688 & 136443 & -42852 & 610078 & -190645 & 21924 & -40557 & -19869 & 19015 \\ -13510 & 3915 & -51083 & 15397 & -228384 & 68542 & -8013 & 14890 & 7029 & -7094 \\ -5368761 & 1665179 & -20782828 & 7019050 & -92971526 & 31173684 & -3420408 & 6237274 & 3388927 & -2943402 \\ 1966814 & -596088 & 7522390 & -2447901 & 33641731 & -10881765 & 1223465 & -2250181 & -1156666 & 1055397 \\ 1224738 & -379954 & 4741559 & -1602000 & 21211317 & -7114895 & 780461 & -1423097 & -773632 & 671590 \\ -438093 & 132750 & -1675413 & 545048 & -7492789 & 2422943 & -272466 & 501148 & 257500 & -235046 \\ -765 & 84 & -2013 & -321 & -8907 & -1323 & -160 & 491 & -417 & -182 \\ 570 & -27 & 1284 & 531 & 5649 & 2259 & 43 & -277 & 512 & 82 \\ -57735 & 17490 & -220569 & 71739 & -986409 & 318921 & -35900 & 66068 & 33856 & -30922 \\ -864477 & 270036 & -3357666 & 1147350 & -15021672 & 5094372 & -554799 & 1009316 & 557354 & -476812 \end{bmatrix}
$$

$$
\times\begin{bmatrix} -323 & -37 & -40 & -173 & -6 & 75 & 75 & 75 & -3 & -129 & -411 & 618 & -36 & -18 & -54 & -36 \\ -4457 & -510 & -577 & -2391 & -162 & 1038 & 993 & 975 & -216 & -2151 & -5652 & 8550 & -639 & -459 & -1161 & -711 \\ -102 & -12 & -17 & -55 & -6 & 27 & 24 & 24 & -12 & -75 & -129 & 198 & -18 & -36 & -54 & -54 \\ -1527 & -176 & -155 & -828 & -12 & 354 & 288 & 249 & -15 & -558 & -1944 & 2952 & -135 & 18 & -72 & 108 \\ -268 & -31 & -37 & -144 & -8 & 66 & 63 & 63 & -9 & -132 & -333 & 516 & -30 & -30 & -57 & -45 \\ -341 & -39 & -43 & -183 & -9 & 77 & 73 & 72 & -15 & -152 & -427 & 654 & -45 & -36 & -81 & -48 \\ 90 & 10 & 10 & 48 & 3 & -19 & -18 & -17 & 7 & 36 & 112 & -171 & 9 & 6 & 18 & -3 \\ -183 & -21 & -25 & -98 & -6 & 45 & 44 & 44 & -8 & -89 & -228 & 351 & -18 & -21 & -39 & -27 \\ 78 & 9 & 9 & 42 & 0 & -19 & -18 & -17 & 0 & 28 & 99 & -150 & 3 & 3 & 3 & 0 \\ -78 & -9 & -9 & -42 & 0 & 18 & 17 & 17 & 0 & -27 & -98 & 150 & -6 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -3 & -3 & -3 \\ -827 & -95 & -94 & -446 & -15 & 192 & 171 & 159 & -18 & -336 & -1053 & 1592 & -90 & -30 & -111 & -27 \\ 121 & 14 & 17 & 65 & 4 & -30 & -29 & -29 & 4 & 61 & 150 & -233 & 14 & 14 & 27 & 22 \\ -52 & -6 & -6 & -28 & 0 & 13 & 12 & 11 & 0 & -19 & -66 & 100 & -2 & -1 & -1 & 1 \\ 52 & 6 & 6 & 28 & 0 & -12 & -11 & -11 & 0 & 18 & 65 & -100 & 4 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 2 & 2 & 2 \end{bmatrix}.
$$

The ties are satisfied by these generators.

In order to prove that the abelian subgroup A described by the ties given above coincides with the image of this embedding, we shrink A to the overall Morita multiplicity 1, drop the quasiblocks 5 and 6 and call the resulting subgroup B (cf. 2.1.1). Writing the factors ordered 1, 2, 3, 4, 7, 8, 9, 10, 11, we obtain a Pierce decomposition of B via

$$e := 0 \times 0 \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \begin{pmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$$f := 0 \times 1 \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \begin{pmatrix}0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$$g := 0 \times 0 \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

$$h := 1 \times 0 \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{pmatrix}$$

$$i := 0 \times 0 \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

In the sequel, by index we mean the index of A in Γ. The indices on the entries indicate the quasiblocks they belong to. As announced, we only denote the relevant matrix entries, their position ensueing from the idempotents.

eBe has the following **Z**-basis.

$$\begin{array}{rcl} e &=& 1_3 \times 1_7 \times 1_9 \times 1_{11} \\ j &:=& 0 \times 3 \times 3 \times 0 \\ k &:=& 0 \times 0 \times 3 \times -3 \\ l &:=& 0 \times 0 \times 0 \times 9 \end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $3^{16\cdot 4}$.

fBe has the following **Z**-basis.

$$\begin{array}{r} a' := 1_7 \times 1_{11} \\ 0 \times 3 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 1}$.

gBe has the following **Z**-basis.

$$\begin{array}{r} b' = -1_9 \times 1_{11} \\ 0 \times 3 \end{array}$$

The Morita factor 24 taken under consideration, its contribution to the index is $3^{24\cdot 1}$.

hBe has the following **Z**-basis.

$$\begin{array}{r} c' := 3_3 \times 1_{11} \\ 0 \times 3 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 2}$.

iBe has the following **Z**-basis.

$$1_{11}$$

It does not contribute to the index.

eBf has the following **Z**-basis.

$$\begin{array}{r} a := 3_7 \times 3_{11} \\ 0 \times 9 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 3}$.

fBf has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcccccc} f = 1_2 & \times & 1_4 & \times & 1_7 & \times & 1_{11} \\ 0 & \times & 3 & \times & -3 & \times & 0 \\ 0 & \times & 0 & \times & 3 & \times & 3 \\ 0 & \times & 0 & \times & 0 & \times & 9 \end{array}$$

The Morita factor 1 taken under consideration, its contribution to the index is $3^{1 \cdot 4}$.

gBf has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 6 taken under consideration, its contribution to the index is $3^{6 \cdot 1}$.

hBf has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 1 taken under consideration, its contribution to the index is $3^{1 \cdot 1}$.

iBf has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcc} x := 1_4 & \times & 1_{11} \\ 0 & \times & 3 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4 \cdot 1}$.

eBg has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcc} b := 3_9 & \times & 3_{11} \\ 0 & \times & 9 \end{array}$$

The Morita factor 24 taken under consideration, its contribution to the index is $3^{24 \cdot 3}$.

fBg has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 6 taken under consideration, its contribution to the index is $3^{6 \cdot 1}$.

gBg has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcccccc} g & = & 1_9 & \times & 1_{10} & \times & 1_{11} \\ m & := & 0 & \times & 3 & \times & -3 \\ n & := & 0 & \times & 0 & \times & 9 \end{array}$$

The Morita factor 36 taken under consideration, its contribution to the index is $3^{36 \cdot 3}$.

hBg has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 6 taken under consideration, its contribution to the index is $3^{6 \cdot 1}$.

iBg has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcc} y := 3_{10} & \times & 1_{11} \\ 0 & \times & 3 \end{array}$$

The Morita factor 24 taken under consideration, its contribution to the index is $3^{24 \cdot 2}$.

eBh has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcc} c := 1_3 & \times & 3_{11} \\ 0 & \times & 9 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4 \cdot 2}$.

fBh has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 1 taken under consideration, its contribution to the index is $3^{1 \cdot 1}$.

gBh has the following $\mathbf{Z}$-basis.

$$3_{11}$$

The Morita factor 6 taken under consideration, its contribution to the index is $3^{6 \cdot 1}$.

hBh has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcccccc} h = 1_1 & \times & 1_3 & \times & 1_8 & \times & 1_{11} \\ 0 & \times & 3 & \times & 0 & \times & 3 \\ 0 & \times & 0 & \times & 3 & \times & 3 \\ 0 & \times & 0 & \times & 0 & \times & 9 \end{array}$$

The Morita factor 1 taken under consideration, its contribution to the index is $3^{1\cdot 4}$.
iBh has the following $\mathbf{Z}$-basis.

$$\begin{array}{l} z := 3_8 \times 1_{11} \\ \quad\;\; 0 \times 3 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 2}$.
eBi has the following $\mathbf{Z}$-basis.

$$9_{11}$$

The Morita factor 16 taken under consideration, its contribution to the index is $3^{16\cdot 2}$.
fBi has the following $\mathbf{Z}$-basis.

$$\begin{array}{l} x' := 3_4 \times 3_{11} \\ \quad\;\;\; 0 \times 9 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 3}$.
gBi has the following $\mathbf{Z}$-basis.

$$\begin{array}{l} y' := -1_{10} \times 3_{11} \\ \quad\;\;\;\;\; 0 \times 9 \end{array}$$

The Morita factor 24 taken under consideration, its contribution to the index is $3^{24\cdot 2}$.
hBi has the following $\mathbf{Z}$-basis.

$$\begin{array}{l} z' := 1_8 \times 3_{11} \\ \quad\;\;\; 0 \times 9 \end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $3^{4\cdot 2}$.
iBi has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcccccc} i = 1_4 & \times & 1_8 & \times & 1_{10} & \times & 1_{11} \\ 0 & \times & 3 & \times & 0 & \times & 3 \\ 0 & \times & 0 & \times & 3 & \times & -3 \\ 0 & \times & 0 & \times & 0 & \times & 9 \end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $3^{16\cdot 4}$.
Altogether, the index of A in Γ is

$$3^{558}.$$

For to see that $eBe/3$ and thus eBe do not contain nontrivial idempotents, we regard, for $\alpha, \beta, \gamma, \delta \in \mathbf{Z}/3$,

$$\begin{array}{rcl} \alpha e + \beta j + \gamma k + \delta l & = & (\alpha e + \beta j + \gamma k + \delta l)^2 \\ & = & \alpha^2 e + 2\alpha\beta j + 2\alpha\gamma k + (2\gamma^2 + 2\alpha\delta + 2\beta\gamma) l, \end{array}$$

yielding $\alpha = 0, 1$. In both cases we obtain $\beta = 0$ and $\gamma = 0$, whence, in both cases, $\delta = 0$.
Now eBe, fBf, hBh and iBi are isomorphic to

$$\{a \times b \times c \times d \mid a - d \equiv_9 c - b, a \equiv_3 b \equiv_3 c \equiv_3 d\} \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}.$$

For to see that $gBg/3$ and thus gBg do not contain nontrivial idempotents, we regard, for $\alpha, \beta, \gamma \in \mathbf{Z}/3$,

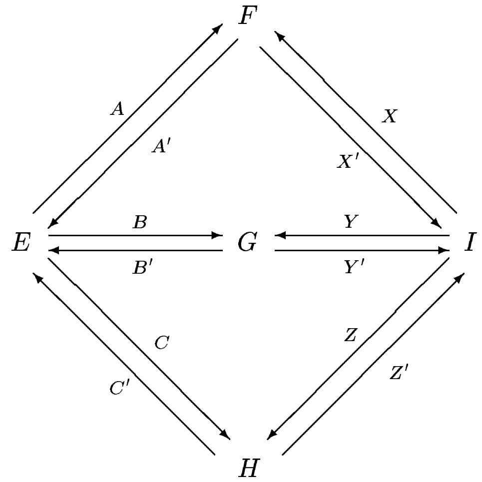
$$\begin{array}{rcl} \alpha g + \beta m + \gamma n & = & (\alpha g + \beta m + \gamma n)^2 \\ & = & \alpha^2 g + 2\alpha\beta m + (2\alpha\gamma + 2\beta^2) n, \end{array}$$

yielding $\alpha = 0, 1$. In both cases we obtain $\beta = 0$, whence, in both cases, $\gamma = 0$.
The indecomposable projectives Be, Bf, Bg, Bh, Bi lie in different genera because of different annihilators (D.1.5, D.2.21). Thus A is homogenus.
For an equivalent description of the quasiblock 11, i.e. of $Q^{(3,2,1)}_{(3)}$, cf. [P 80/1, (III.9)].

2.3.4 $\mathbf{F}_3\mathcal{S}_6$ as path algebra modulo relations

We maintain the notation of (2.3.3). Consider the quiver

$\Xi :=$



We have a ring morphism

$$\mathbf{Z}\Xi \longrightarrow B$$

by sending the capital to the small letter elements (1.3.3), which we shall list again. NB their matrix positions are determined by the chosen idempotents.

$$\begin{array}{lcllll}
e & = & 1_3 & \times\ 1_7 & \times\ 1_9 & \times\ 1_{11} \\
f & = & 1_2 & \times\ 1_4 & \times\ 1_7 & \times\ 1_{11} \\
g & = & 1_9 & \times\ 1_{10} & \times\ 1_{11} & \\
h & = & 1_1 & \times\ 1_3 & \times\ 1_8 & \times\ 1_{11} \\
i & = & 1_4 & \times\ 1_8 & \times\ 1_{10} & \times\ 1_{11} \\
a & = & 3_7 & \times\ 3_{11} & & \\
a' & = & 1_7 & \times\ 1_{11} & & \\
b & = & 3_9 & \times\ 3_{11} & & \\
b' & = & -1_9 & \times\ 1_{11} & & \\
c & = & 1_3 & \times\ 3_{11} & & \\
c' & = & 3_3 & \times\ 1_{11} & & \\
x & = & 1_4 & \times\ 1_{11} & & \\
x' & = & 3_4 & \times\ 3_{11} & & \\
y & = & 3_{10} & \times\ 1_{11} & & \\
y' & = & -1_{10} & \times\ 3_{11} & & \\
z & = & 3_8 & \times\ 1_{11} & & \\
z' & = & 1_8 & \times\ 3_{11} & &
\end{array}$$

This morphism is surjective, as to be seen by direct verification.

We **claim** that its kernel is generated as an ideal by

$$\begin{array}{rclcl}
A'B & = & X'Y & & \\
A'C & = & X'Z & & \\
B'A & = & Y'X & & \\
B'C & = & Y'Z & & \\
C'A & = & Z'X & & \\
C'B & = & Z'Y & & \\
XA' & = & YB' & = & ZC' \\
AX' & = & BY' & = & CZ'
\end{array}$$

$$\begin{aligned}
AA'A &= 3A\\
A'AA' &= 3A'\\
BB'B &= -3B+2BY'Y\\
B'BB' &= -3B'+2Y'YB'\\
CC'C &= 3C\\
C'CC' &= 3C'\\
XX'X &= 3X\\
X'XX' &= 3X'\\
YY'Y &= -3Y+2YB'B\\
Y'YY' &= -3Y'+2B'BY'\\
ZZ'Z &= 3Z\\
Z'ZZ' &= 3Z'
\end{aligned}$$

$$\begin{aligned}
AA'-BB'+CC' &= 3E\\
XX'-YY'+ZZ' &= 3I\\
B'B+Y'Y+3G &= B'BY'Y.
\end{aligned}$$

In order to check that modulo the ideal generated by these elements the nonunderlined elements in the trees below are in fact **Z**-linear combinations of the underlined ones, it is convenient to note that modulo these elements we obtain

$$\begin{aligned}
B'BB'C &= B'BY'Z\\
&= B'AX'Z\\
&= B'AA'C\\
&= Y'XA'C\\
&= Y'YB'C\\
&= Y'YY'Z,
\end{aligned}$$

as well as

$$B'BB'C = -3B'C + 2Y'YB'C,$$

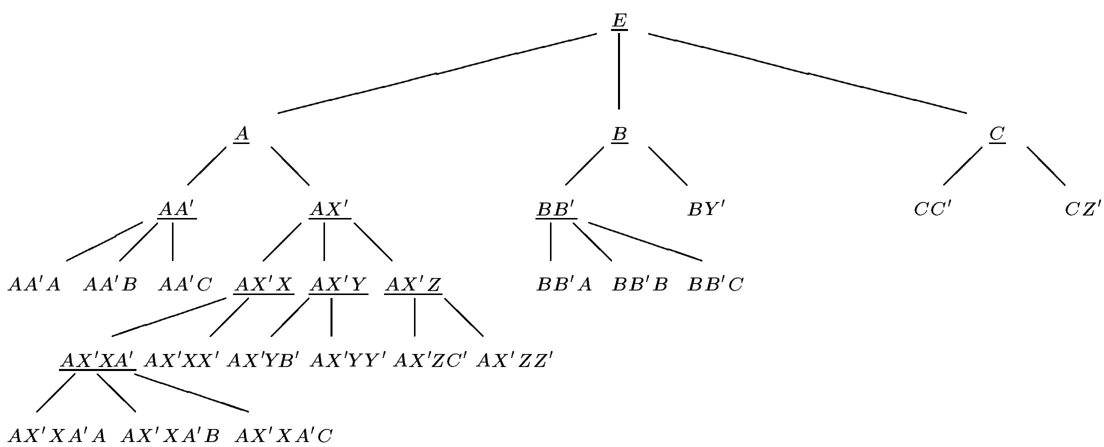
which implies

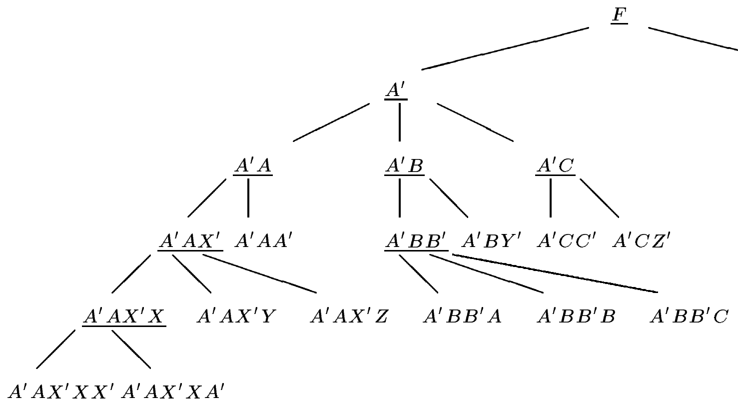
$$B'BB'C = 3B'C,$$

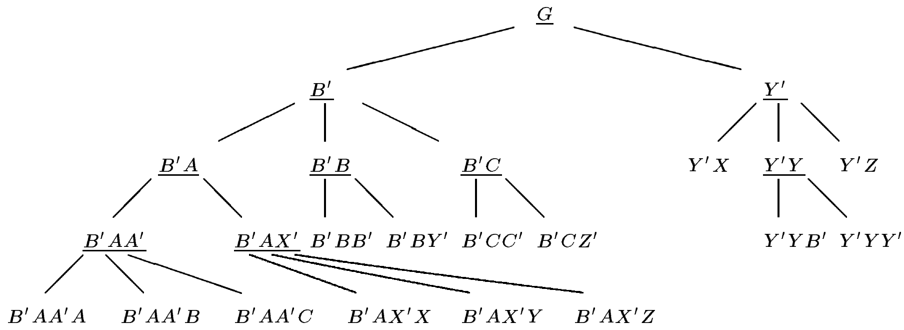
also to be read modulo these elements.

Regard the multiplication trees of Ξ (cf. S 2.1.2).

It suffices by the symmetries $A \longleftrightarrow X$ etc. and $A \longleftrightarrow C$ etc. - expressable a posteriori as automorphisms, cf. the generators for the claimed kernel below - to draw trees for E, F and G. NB there are several possible equivalent ways to end the underlined part of the tree.







The kernel K of $\mathbf{F}_3\Xi \longrightarrow B/3$ now is generated by

$$\begin{aligned}
A'B &= X'Y \\
A'C &= X'Z \\
B'A &= Y'X \\
B'C &= Y'Z \\
C'A &= Z'X \\
C'B &= Z'Y \\
XA' &= YB' = ZC' \\
AX' &= BY' = CZ'
\end{aligned}$$

$$\begin{aligned}
AA'A &= 0 \\
A'AA' &= 0 \\
BB'B + BY'Y &= 0 \\
B'BB' + Y'YB' &= 0 \\
CC'C &= 0 \\
C'CC' &= 0 \\
XX'X &= 0 \\
X'XX' &= 0 \\
YY'Y + YB'B &= 0 \\
Y'YY' + B'BY' &= 0 \\
ZZ'Z &= 0 \\
Z'ZZ' &= 0
\end{aligned}$$

$$\begin{aligned}
AA' - BB' + CC' &= 0 \\
XX' - YY' + ZZ' &= 0 \\
B'B + Y'Y &= B'BY'Y,
\end{aligned}$$

thus giving a Morita equivalence between $\mathbf{F}_3\Xi/K \times \mathbf{F}_3 \times \mathbf{F}_3$ and $\mathbf{F}_3\mathcal{S}_6$.

K. Erdmann and S. Martin also give a complete description, up to the determination of four parameters, in terms of a quiver with relations [EM 94, Th. 7.1]. We couldn't establish full accordance between both presentations. ([3]).

[3]Cf. also G. Nebe, *The principal block of* $\mathbf{Z}_p\mathcal{S}_{2p}$, available under `http://www.mathematik.uni-ulm.de/ReineM/nebe/pl.html`.

2.3.5 $(\mathbf{Z}\mathcal{S}_6)_{[2]}$

Again, we change the notation slightly. We drop the - too many - letters parametrizing the ties and endow our variables with extra lower indices instead, which indicate their position in case it is not uniquely determined by the position of the Pierce component. These Pierce components have to be read cum grano salis: actually, they furnish a decomposition of the Morita reduced ring which we blow up again via the Morita multiplicities, placed on top of the columns. The idempotents e, f, g, h then are the obvious ones on the main diagonals, at this stage used only for grouping and placing the ties.

We claim that $(\mathbf{Z}\mathcal{S}_6)_{[2]}$ is homogenus and takes the following form.

1
eAe

1

1
eAe

2

1	4
eAe	eAf
fAe	fAf

3

1	4
eAe	eAf
fAe	fAf

4

4	1	4
fAf	fAe	fAg
eAf	eAe	eAg
gAf	gAe	gAg

5

4	1	4
fAf	fAe	fAg
eAf	eAe	eAg
gAf	gAe	gAg

6

1	4
eAe	eAg
gAe	gAg

7

1	4
eAe	eAg
gAe	gAg

8

4	1	4	1
fAf	fAe	fAg	fAe
eAf	eAe	eAg	eAe
gAf	gAe	gAg	gAe
eAf	eAe	eAg	eAe

9

4	1	4	1
fAf	fAe	fAg	fAe
eAf	eAe	eAg	eAe
gAf	gAe	gAg	gAe
eAf	eAe	eAg	eAe

10

16
hAh

11

eAe

$$\begin{aligned}
x^8 - x^6 &\equiv_4 x^4 - x^2 \equiv_2 0 \\
x_{22}^{10} &\equiv_2 x^6 \\
x_{24}^9 &\equiv_4 x_{24}^{10} \\
x_{22}^9 - x_{44}^9 &\equiv_4 x_{22}^{10} - x_{44}^{10} \equiv_2 0 \\
x^5 + x^7 + x_{24}^9 &\equiv_8 x^6 + x^8 + x_{24}^{10} \\
x_{22}^9 + x_{22}^{10} &\equiv_4 x_{24}^{10} + x^4 + x^2 \equiv_2 0 \\
2x_{42}^{10} &\equiv_4 x^6 - x^4 \\
x^7 &\equiv_4 x^6 - x_{24}^{10} \\
x^3 - 3x^7 - x_{24}^9 + 2x_{42}^9 &\equiv_8 x^4 - 3x^8 - x_{24}^{10} + 2x_{42}^{10}
\end{aligned}$$

$$\begin{aligned}
&-3x^1 + x^3 - x^5 - x^7 + 2x_{22}^9 - 2x_{24}^9 - 2x_{42}^9 + 2x_{44}^9 + 2x_{42}^{10} \\
\equiv_{16} &-3x^2 + x^4 - x^6 - x^8 + 2x_{22}^{10} - 2x_{24}^{10} - 2x_{42}^{10} + 2x_{44}^{10} + 2x_{42}^9 \equiv_8 0
\end{aligned}$$

eAf

$$\begin{aligned}
x^5 - x^6 &\equiv_4 x_{21}^9 - x_{21}^{10} \equiv_2 0 \\
x_{21}^{10} &\equiv_2 x^6 \\
x_{41}^9 &\equiv_2 x_{41}^{10} \\
x^3 + 2x^5 - 4x_{41}^9 &\equiv_{16} x^4 + 2x^6 - 4x_{41}^{10} \equiv_8 0
\end{aligned}$$

eAg

$$\begin{aligned}
x_{23}^9 &\equiv_4 x_{23}^{10} \equiv_2 0 \\
x^7 - x^5 &\equiv_8 x^8 - x^6 \equiv_4 0 \\
2x_{43}^9 - x^5 &\equiv_8 2x_{43}^{10} - x^6 \equiv_4 0 \\
x^8 - x^5 - 2x_{23}^9 &\equiv_{16} -3x^7 - x^6 - 2x_{23}^{10} \equiv_8 0
\end{aligned}$$

fAe

$$\begin{aligned}
x_{12}^9 &\equiv_4 x_{12}^{10} \equiv_2 0 \\
x_{14}^{10} &\equiv_4 x^6 \\
x^6 &\equiv_4 2x^4 \\
x^5 - x^6 &\equiv_8 x_{14}^9 - x_{14}^{10} \equiv_4 0 \\
x^5 - 2x^3 + 2x_{12}^9 &\equiv_{16} x^6 - 2x^4 + 2x_{12}^{10} \equiv_8 0
\end{aligned}$$

fAf

$$\begin{aligned}
x^3 &\equiv_2 x^4 \\
x^9 &\equiv_2 x^{10} \\
x^3 + x^5 - 2x^9 &\equiv_{16} x^4 + x^6 - 2x^{10} \equiv_8 0 \\
x^3 - x^5 &\equiv_8 x^4 - x^6 \equiv_4 0
\end{aligned}$$

fAg

$$\begin{aligned}
x^5 &\equiv_8 x^6 \equiv_4 0 \\
x^5 - 2x^9 &\equiv_{16} x^6 - 2x^{10} \equiv_8 0
\end{aligned}$$

gAe

$$\begin{aligned}
x_{34}^9 &\equiv_4 x_{34}^{10} \equiv_2 0 \\
x_{32}^9 &\equiv_2 x^5 \\
x^7 - x^8 &\equiv_4 x_{32}^9 - x_{32}^{10} \equiv_2 0 \\
x^8 + x_{34}^9 + x^5 &\equiv_8 -3x^7 + x_{34}^{10} + x^6 \equiv_4 0
\end{aligned}$$

gAf

$$\begin{aligned}
x^9 &\equiv_2 x^{10} \\
x^5 - x^9 &\equiv_4 x^6 - x^{10} \equiv_2 0
\end{aligned}$$

gAg

$$\begin{aligned}
x^5 &\equiv_2 x^9 \\
x^5 + x^8 - 2x^{10} &\equiv_{16} x^6 + x^7 - 2x^9 \equiv_8 0 \\
x^9 - x^{10} &\equiv_4 x^7 - x^8 \equiv_2 0
\end{aligned}$$

We conjugate the embedding given in (S 2.3.1) from the left with the $\prod_\lambda \mathrm{SL}_{n_\lambda}(\mathbf{Z})$-element

$$1 \times \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 & 0 & -5 & -7 & -1 & -4 & -4 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -3 \\ -1 & 0 & -1 & 0 & 2 & 3 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 85 & -85 & -69 & -85 & -85 \\ 0 & 1 & 0 & 0 & 0 \\ 16 & -16 & -13 & -16 & -16 \\ 0 & 0 & 0 & -85 & 16 \\ 0 & 0 & 0 & 16 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & -4 & 0 & 0 & 3 & -1 & 1 & -6 & -6 & 6 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & -1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 1 & 5 \end{bmatrix}$$

$$\times\ 1 \times \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -2 & 0 & 0 & -3 & -7 & -1 & -2 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -3 \\ -617 & 0 & -617 & 0 & 1234 & 1115 & 1852 & 2016 & 1933 \\ -26 & 0 & -26 & 0 & 52 & 47 & 78 & 88 & 74 \\ 6694 & 0 & 6694 & 0 & -13388 & -12088 & -20115 & -22180 & -20314 \\ 6904 & 0 & 6904 & 0 & -13808 & -12468 & -20744 & -22887 & -20918 \\ -109608 & 0 & -109608 & 0 & 219216 & 197936 & 329348 & 363266 & 332357 \end{bmatrix} \times \begin{bmatrix} 1354357 & -2075437 & 131703 & -124353 & -4459921 \\ -2 & 3 & 0 & 2 & 2 \\ 7195022 & -11025760 & 699673 & -660626 & -23693330 \\ 12344 & -18916 & 1200 & -1133 & -40650 \\ 344 & -528 & 36 & -38 & -1119 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & -1 & 1 & -4 & -4 & 6 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & -1 & -2 & 2 & 1 & 0 \\ 2 & -4 & 2 & 0 & 0 & 3 & 2 & -4 & -2 & 1 \\ 2 & -4 & 2 & 0 & 0 & 0 & 9 & -4 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 5 & -3 \\ 4 & -9 & 4 & -1 & 0 & 9 & -1 & -7 & -9 & 15 \end{bmatrix} \times \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

to obtain the embedding

$$\mathbf{Z}S_6 \longrightarrow \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \times \mathbf{Z} \times (\mathbf{Z})_5 \times (\mathbf{Z})_9 \times (\mathbf{Z})_5 \times (\mathbf{Z})_{10} \times (\mathbf{Z})_{16}$$

$$(12) \longrightarrow -1 \times \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -8 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 511 & 0 & -2720 & -2720 & -14450 \\ -16 & -1 & 85 & 0 & 0 \\ 96 & 0 & -511 & -512 & -2720 \\ 0 & 0 & 0 & 1871 & 9945 \\ 0 & 0 & 0 & -352 & -1871 \end{bmatrix} \times \begin{bmatrix} 7 & 24 & 8 & 3 & 8 & 16 & 8 & 24 & 12 & 8 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -12 & -5 & -5 & -4 & -8 & -4 & -12 & -8 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 18 & 6 & 10 & 5 & 12 & 8 & 18 & 14 & 6 \\ -2 & -6 & -2 & -2 & -2 & -5 & -3 & -6 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -4 & -12 & -4 & -4 & -4 & -8 & -4 & -13 & -9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 8 & 24 & 8 & 5 & 8 & 16 & 6 & 24 & 17 & 7 \end{bmatrix}$$

$$\times\ 1 \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 5 & 0 & 2496 & 38752 & 38768 & 4808 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1234 & -44399 & -38314656 & -570592832 & -571112068 & -70811278 \\ 0 & 0 & 0 & -52 & -1870 & -1608129 & -23945203 & -23967036 & -2971632 \\ 0 & 0 & 0 & 13388 & 482240 & 415622912 & 6189244481 & 6194880688 & 768093160 \\ 0 & 0 & 0 & 13808 & 497320 & 428596912 & 6382432192 & 6388244503 & 792068013 \\ 0 & 0 & 0 & -219216 & -7895840 & -6804918944 & -101335267504 & -101427549424 & -12575836455 \end{bmatrix}$$

$$\times \begin{bmatrix} 15507928570849 & -709146032 & -2907736145384 & -6643578612916 & -109727370242 \\ -22886142 & 1039 & 4291151 & 9804364 & 161928 \\ 82385884900032 & -3767338944 & -15447350966239 & -35294017536064 & -582926756064 \\ 141339060648 & -6463152 & -26501069664 & -60549489665 & -1000053837 \\ 3973676064 & -181664 & -745064144 & -1702318048 & -28115983 \end{bmatrix}$$

$$\times \begin{bmatrix} -7 & -8 & -8 & -5 & -344 & -96 & -56 & -160 & -76 & -8 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 5 & 5 & 172 & 48 & 28 & 84 & 40 & 4 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -6 & -10 & -269 & -76 & -44 & -130 & -62 & -6 \\ 14 & 14 & 14 & 22 & 656 & 187 & 107 & 302 & 144 & 14 \\ 8 & 8 & 8 & 16 & 296 & 80 & 49 & 176 & 84 & 8 \\ 4 & 4 & 4 & 4 & 172 & 48 & 28 & 95 & 45 & 4 \\ -8 & -8 & -8 & -8 & -344 & -96 & -56 & -192 & -91 & -8 \\ 32 & 32 & 32 & 51 & 1580 & 452 & 258 & 714 & 341 & 33 \end{bmatrix}$$

$$\times \begin{bmatrix} -1&0&1&0&-1&0&0&0&0&-1&0&0&0&0&1&0 \\ 0&-1&-1&0&0&0&0&0&0&0&-1&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&-1&-1&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-1&0&1&0&1&0&0&-1&0&-1&0 \\ 0&0&0&0&0&0&-1&-1&0&0&1&0&0&0&0&-1 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$(123456) \longrightarrow$

$$-1 \times \begin{bmatrix} -1 & 2 & -2 & -2 & 0 \\ -1 & 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \times 1 \times \begin{bmatrix} 1 & -2 & 14 & 2 & 0 \\ 1 & 5 & -29 & 1 & -6 \\ 0 & 1 & -6 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\times \begin{bmatrix} 21 & 7 & 59 & -4 & 58 & 36 & 12 & 40 & 36 \\ 3 & 0 & 8 & 1 & 8 & 8 & 4 & 4 & 8 \\ 4 & 1 & 12 & -1 & 12 & 4 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 & 0 & 4 & 4 & 4 & 4 \\ -11 & -3 & -31 & 2 & -31 & -16 & -4 & -16 & -16 \\ -1 & -1 & -3 & 1 & -3 & -1 & 0 & -3 & -1 \\ 0 & 0 & -1 & -1 & 0 & -1 & -1 & -2 & -2 \\ -1 & 0 & -3 & 0 & -3 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix} \times \begin{bmatrix} -15 & -37 & -43 & 6 & -602 & -692108 & -10429284 & -10437152 & -1294188 \\ -3 & -6 & -8 & -1 & -256 & -259056 & -3885396 & -3888564 & -482160 \\ -4 & -9 & -12 & 1 & -116 & -142628 & -2154408 & -2155964 & -267340 \\ 0 & 0 & -1 & -1 & -144 & -133596 & -1996188 & -1997916 & -247724 \\ 5599 & 12313 & 17415 & 1354 & 380075 & 400210268 & 6011423128 & 6016209728 & 745983308 \\ 239 & 525 & 743 & 57 & 15835 & 16721019 & 251188548 & 251388189 & 31171043 \\ -61044 & -134176 & -189859 & -14721 & -4109328 & -4331198847 & -65059934017 & -65111705416 & -8073564576 \\ -62971 & -138410 & -195849 & -15180 & -4237319 & -4466312209 & -67089627504 & -67143012380 & -8325437793 \\ 999634 & 2197204 & 3109034 & 241020 & 67279022 & 70913243353 & 1065205133003 & 1066052756095 & 132185845337 \end{bmatrix}$$

$$\times \begin{bmatrix} -511 & -16 & 2720 & 0 & 0 \\ 3 & -1 & -16 & -19 & -101 \\ -96 & -3 & 511 & 0 & 0 \\ 511 & -69 & -2720 & -1056 & -5609 \\ -96 & 13 & 511 & 199 & 1057 \end{bmatrix} \qquad \times \begin{bmatrix} -52328776266333 & 2361986060 & 9811644192372 & 22417470831376 & 370239044628 \\ 16110061 & -733 & -3020636 & -6901507 & -113985 \\ -277996627456520 & 12548051107 & 52124360436031 & 119092815308918 & 1966894949646 \\ -476947504285 & 21528179 & 89427644680 & 204322698292 & 3374521647 \\ -13163176494 & 594171 & 2468095251 & 5639060383 & 93132743 \end{bmatrix}$$

$$\times \begin{bmatrix} -21 & -69 & -19 & -26 & -18 & -30 & -26 & -68 & -54 & -22 \\ 5 & 16 & 4 & 5 & 4 & 8 & 6 & 14 & 12 & 4 \\ 10 & 29 & 10 & 7 & 10 & 22 & 8 & 30 & 22 & 8 \\ 12 & 36 & 13 & 13 & 12 & 26 & 14 & 38 & 26 & 12 \\ 5 & 17 & 7 & 6 & 5 & 10 & 8 & 16 & 10 & 6 \\ 1 & 3 & 1 & 3 & 1 & 1 & 2 & 5 & 3 & 2 \\ -4 & -12 & -5 & -5 & -4 & -9 & -5 & -12 & -8 & -4 \\ -1 & -2 & -1 & 0 & -1 & -3 & 0 & -2 & -1 & 0 \\ 2 & 6 & 2 & 2 & 2 & 3 & 1 & 7 & 5 & 2 \\ -23 & -70 & -24 & -23 & -23 & -48 & -26 & -74 & -52 & -23 \end{bmatrix} \times \begin{bmatrix} 11 & 15 & 11 & 16 & 402 & 98 & 70 & 288 & 138 & 14 \\ -5 & -6 & -4 & -5 & -184 & -52 & -30 & -102 & -48 & -4 \\ -10 & -9 & -10 & -7 & -430 & -126 & -68 & -210 & -98 & -8 \\ -12 & -12 & -13 & -13 & -556 & -158 & -90 & -266 & -126 & -12 \\ -5 & -7 & -7 & -6 & -241 & -66 & -40 & -112 & -54 & -6 \\ 1 & 5 & 3 & -1 & 55 & 13 & 10 & 21 & 11 & 2 \\ 32 & 36 & 41 & 43 & 1528 & 425 & 251 & 714 & 342 & 36 \\ 5 & 4 & 5 & 4 & 175 & 49 & 28 & 110 & 51 & 4 \\ -12 & -10 & -12 & -10 & -416 & -115 & -67 & -269 & -125 & -10 \\ 13 & 20 & 12 & -3 & 389 & 102 & 64 & 306 & 142 & 11 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The ties are satisfied by these generators.

In order to prove that the abelian subgroup A described by the ties given above coincides with the image of this embedding, we shrink A to the overall Morita multiplicity 1, drop the quasiblock 11 and call the resulting subgroup B (cf. 2.1.1). Writing the factors ordered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, we obtain a Pierce decomposition of B via

$$e := 1 \times 1 \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$$

$$f := 0 \times 0 \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$$

$$g := 0 \times 0 \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right).$$

In the sequel, by index we mean the index of A in Γ. The indices on the entries indicate the quasiblocks they belong to. We only denote the relevant matrix entries, their position ensueing from the idempotents.

To give a basis for a Pierce component now becomes a problem of integral linear algebra by its sheer size. In order to solve it, we write the ties as a matrix to be annihilated from the right modulo 16, i.e. we display the ties as rows. Assume the first $s-1$ basis elements of the Pierce component to be found in a lower triangular manner. We drop the first $s-1$ columns of this matrix and perform an elementary divisor simplification on the remaining matrix R to obtain a main diagonal matrix D. In the SL-element acting from the right on R we pick the k-th column with minimal value of $v_2(\text{top entry}) - v_2(k\text{-th main diagonal entry of the } D)$. We multiply this k-th column with $16/(k\text{-th main diagonal entry of the } D)$ and then, regarding it in $\mathbf{Z}/16$, with an element in $(\mathbf{Z}/16)^*$ such that the top entry becomes a power of 2, which we also choose as inverse image in $\mathbf{Z}$ again. The resulting column annihilates R modulo 16 and, furthermore, can be used to kill the top entry of an arbitrary column annihilating R modulo 16. Hence iteration of this process furnishes a basis of our Pierce component in a lower triangular manner.

We have chosen a different order on the quasiblocks to display the factors, viz. $1, 2, 3, 4, 7, 8, 5, 6, 9, 10$.

eBe has the following **Z**-basis.

$$
\begin{array}{rcccccccccccccccccccc}
e & := & 1_1 & \times & 1_2 & \times & 1_3 & \times & 1_4 & \times & 1_7 & \times & 1_8 & \times & 1_5 & \times & 1_6 & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)_{10} \\
x & := & 0 & \times & 2 & \times & 2 & \times & 0 & \times & 0 & \times & 2 & \times & 2 & \times & 0 & \times & \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\
y & := & 0 & \times & 0 & \times & 2 & \times & 2 & \times & 0 & \times & 0 & \times & 2 & \times & 2 & \times & \left(\begin{smallmatrix} 0 & -6 \\ 0 & 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right) \\
xy & = & 0 & \times & 0 & \times & 4 & \times & 0 & \times & 0 & \times & 0 & \times & 4 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & -12 \\ 0 & 4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\
z & := & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 2 & \times & 2 & \times & 2 & \times & 2 & \times & \left(\begin{smallmatrix} 4 & 8 \\ -1 & -2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ -1 & 2 \end{smallmatrix}\right) \\
xz & = & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 4 & \times & 4 & \times & 0 & \times & \left(\begin{smallmatrix} 8 & 16 \\ -2 & -4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\
yz & = & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 4 & \times & 4 & \times & \left(\begin{smallmatrix} 6 & 12 \\ -2 & -4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} -2 & 4 \\ -2 & 4 \end{smallmatrix}\right) \\
xyz & = & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 8 & \times & 0 & \times & \left(\begin{smallmatrix} 12 & 24 \\ -4 & -8 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right)
\end{array}
$$

The Morita factor 1 taken under consideration, its contribution to the index is $2^{1\cdot 28}$.
eBf has the following **Z**-basis.

$$
\begin{array}{rcccccccccccc}
a & := & 2_3 & \times & 2_4 & \times & 1_5 & \times & 1_6 & \times & \left(\begin{smallmatrix} -3 \\ 1 \end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)_{10} \\
xa & = & 4 & \times & 0 & \times & 2 & \times & 0 & \times & \left(\begin{smallmatrix} -6 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\
za & = & 0 & \times & 0 & \times & 2 & \times & 2 & \times & \left(\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \\
xza & = & 0 & \times & 0 & \times & 4 & \times & 0 & \times & \left(\begin{smallmatrix} -8 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right)
\end{array}
$$

The Morita factor 4 taken under consideration, its contribution to the index is $2^{4\cdot 12}$.
eBg has the following **Z**-basis.

$$
\begin{array}{rcccccccccccc}
b & := & 2_7 & \times & 2_8 & \times & 2_5 & \times & 2_6 & \times & \left(\begin{smallmatrix} -4 \\ 1 \end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)_{10} \\
xb & = & 0 & \times & 4 & \times & 4 & \times & 0 & \times & \left(\begin{smallmatrix} -8 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\
yb & = & 0 & \times & 0 & \times & 4 & \times & 4 & \times & \left(\begin{smallmatrix} -6 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right) \\
xyb & = & 0 & \times & 0 & \times & 8 & \times & 0 & \times & \left(\begin{smallmatrix} -12 \\ 4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 8 \\ 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right)
\end{array}
$$

The Morita factor 4 taken under consideration, its contribution to the index is $2^{4\cdot 16}$.
fBe has the following **Z**-basis.

$$
\begin{array}{rcccccccccccc}
a' & := & 1_3 & \times & 1_4 & \times & 2_5 & \times & 2_6 & \times & \left(\begin{smallmatrix} 0 & 2 \end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix} 0 & 2 \end{smallmatrix}\right)_{10} \\
a'x & = & 2 & \times & 0 & \times & 4 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) \\
a'z & = & 0 & \times & 0 & \times & 4 & \times & 4 & \times & \left(\begin{smallmatrix} -2 & -4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} -2 & -4 \end{smallmatrix}\right) \\
a'xz & = & 0 & \times & 0 & \times & 8 & \times & 0 & \times & \left(\begin{smallmatrix} -4 & -8 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 4 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 4 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 8 & 0 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 4 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 4 \end{smallmatrix}\right) \\
 & & 0 & \times & 0 & \times & 0 & \times & 0 & \times & \left(\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right) & \times & \left(\begin{smallmatrix} 0 & 8 \end{smallmatrix}\right)
\end{array}
$$

The Morita factor 4 taken under consideration, its contribution to the index is $2^{4\cdot 16}$.

fBf has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcllllll}
f & := & 1_3 & \times\ 1_4 & \times\ 1_5 & \times\ 1_6 & \times\ 1_9 & \times\ 1_{10} \\
u & := & 2 & \times\ 0 & \times\ 2 & \times\ 0 & \times\ 2 & \times\ 0 \\
a'za & = & 0 & \times\ 0 & \times\ 4 & \times\ 4 & \times\ 2 & \times\ -6 \\
a'xza & = & 0 & \times\ 0 & \times\ 8 & \times\ 0 & \times\ 4 & \times\ 0 \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 4 & \times\ 4 \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 8
\end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $2^{16\cdot 11}$.
fBg has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcllll}
a'b & = & 4_5 & \times\ 4_6 & \times\ 2_9 & \times\ 2_{10} \\
a'xb & = & 8 & \times\ 0 & \times\ 4 & \times\ 0 \\
 & & 0 & \times\ 0 & \times\ 4 & \times\ 4 \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 8
\end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $2^{16\cdot 10}$.
gBe has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcllllll}
b' & := & 1_7 & \times\ 1_8 & \times\ 1_5 & \times\ 1_6 & \times\ (\,-1\ -2\,)_9 & \times\ (\,-1\ 2\,)_{10} \\
b'x & = & 0 & \times\ 2 & \times\ 2 & \times\ 0 & \times\ (\,-2\ -4\,) & \times\ (\,0\ 0\,) \\
b'y & = & 0 & \times\ 0 & \times\ 2 & \times\ 2 & \times\ (\,0\ 2\,) & \times\ (\,0\ 2\,) \\
b'xy & = & 0 & \times\ 0 & \times\ 4 & \times\ 0 & \times\ (\,0\ 4\,) & \times\ (\,0\ 0\,) \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ (\,2\ 0\,) & \times\ (\,2\ 0\,) \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ (\,0\ 0\,) & \times\ (\,4\ 0\,) \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ (\,0\ 4\,) & \times\ (\,0\ 4\,) \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ (\,0\ 0\,) & \times\ (\,0\ 8\,)
\end{array}$$

The Morita factor 4 taken under consideration, its contribution to the index is $2^{4\cdot 12}$.
gBf has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcllll}
b'a & := & 1_5 & \times\ 1_6 & \times\ 1_9 & \times\ 1_{10} \\
b'xa & = & 2 & \times\ 0 & \times\ 2 & \times\ 0 \\
 & & 0 & \times\ 0 & \times\ 2 & \times\ 2 \\
 & & 0 & \times\ 0 & \times\ 0 & \times\ 4
\end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $2^{16\cdot 4}$.
gBg has the following $\mathbf{Z}$-basis.

$$\begin{array}{rcllllll}
g & := & 1_7 & \times\ 1_8 & \times\ 1_5 & \times\ 1_6 & \times\ 1_9 & \times\ 1_{10} \\
v & := & 0 & \times\ 2 & \times\ 2 & \times\ 0 & \times\ 0 & \times\ 2 \\
b'yb & = & 0 & \times\ 0 & \times\ 4 & \times\ 4 & \times\ 2 & \times\ 2 \\
b'xyb & = & 0 & \times\ 0 & \times\ 8 & \times\ 0 & \times\ 4 & \times\ 0 \\
c & := & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 4 & \times\ 4 \\
d & := & 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 0 & \times\ 8
\end{array}$$

The Morita factor 16 taken under consideration, its contribution to the index is $2^{16\cdot 11}$.
In particular, gBg and fBf are isomorphic as R-orders via the quasiblock bijection $3, 4, 5, 6, 9, 10 \longrightarrow 7, 8, 6, 5, 9, 10$.

Since the obvious symmetric form one would like to obtain for the generators is hurt at several places, I am not quite content with the description just given. I suspect that there is a conjugation by $\left(\begin{smallmatrix}1 & 4\\ 0 & 1\end{smallmatrix}\right)$ from the left missing on the eBe-part of quasiblock 9. It shouldn't be too difficult to insert it still.

Altogether, the index of A in Γ is

$$2^{1\cdot 28+4\cdot 12+4\cdot 16+4\cdot 16+16\cdot 11+16\cdot 10+4\cdot 12+16\cdot 4+16\cdot 11} = 2^{828}.$$

For to see that e is primitive, we anticipate a description of $eBe/2$ which ensues from (2.3.6), viz.

$$eBe \simeq \mathbf{F}_2\langle X, Y, Z\rangle/(X^2, XY - YX, XZ - ZX, X^2, Y^2, Z^2, (YZ)^2 - (ZY)^2).$$

Lemma 2.3.1 *Let k be a field. Let $L = \bigoplus_{i\geq 0} L_i$ be a graded finite dimensional k-algebra such that $k \xrightarrow{\sim} L_0$ as rings. Its radical is obtained as*

$$\mathfrak{r}L = \bigoplus_{i\geq 1} L_i,$$

in particular, L is local.

$\bigoplus_{i\geq 1} L_i$ is the maximal nilpotent ideal in L (E.1.8).

For to see that g is primitive we note that we have for $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbf{F}_2$

$$(\alpha g + \beta v + \gamma b' y b + \delta b' x y b + \varepsilon c + \zeta d)^2 = \alpha^2 g + \gamma^2 c.$$

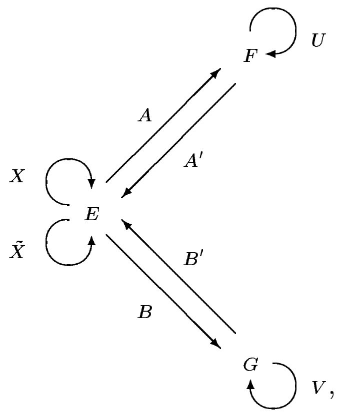
Now $fBf \simeq gBg$ shows that also f is primitive.

The indecomposable projectives Be, Bf and Bg lie in different genera because of different annihilators (D.1.5, D.2.21). Thus A is homogenus.

2.3.6 $\mathrm{F}_2\mathcal{S}_6$ as path algebra modulo relations

We maintain the notation of (2.3.5). Consider the quiver

$\Xi :=$



whose arrow B is not to be confused with the ring of the same name.

We have a ring morphism

$$\mathbf{Z}\Xi \longrightarrow B$$

by sending the capital to the small letter elements (1.3.3), which we shall list again resp. define now.

$$\begin{array}{rcccccccccccccccccccccc}
e & = & 1_1 & \times & 1_2 & \times & 1_3 & \times & 1_4 & \times & 1_7 & \times & 1_8 & \times & 1_5 & \times & 1_6 & \times & \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)_{10} \\
x & = & 0_1 & \times & 2_2 & \times & 2_3 & \times & 0_4 & \times & 0_7 & \times & 2_8 & \times & 2_5 & \times & 0_6 & \times & \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)_{10} \\
\tilde{x} & := & 0_1 & \times & 2_2 & \times & 2_3 & \times & 0_4 & \times & 0_7 & \times & 2_8 & \times & 2_5 & \times & 0_6 & \times & \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)_{10} \\
f & = & & & & & 1_3 & \times & 1_4 & \times & & & & & 1_5 & \times & 1_6 & \times & 1_9 & \times & 1_{10} \\
g & = & & & & & & & & & 1_7 & \times & 1_8 & \times & 1_5 & \times & 1_6 & \times & 1_9 & \times & 1_{10} \\
a & = & & & & & 2_3 & \times & 2_4 & \times & & & & & 1_5 & \times & 1_6 & \times & \left(\begin{smallmatrix}-3\\1\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)_{10} \\
a' & = & & & & & 1_3 & \times & 1_4 & \times & & & & & 2_5 & \times & 2_6 & \times & \left(\begin{smallmatrix}0&2\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}0&2\end{smallmatrix}\right)_{10} \\
b & = & & & & & & & & & 2_7 & \times & 2_8 & \times & 2_5 & \times & 2_6 & \times & \left(\begin{smallmatrix}-4\\1\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)_{10} \\
b' & = & & & & & & & & & 1_7 & \times & 1_8 & \times & 1_5 & \times & 1_6 & \times & \left(\begin{smallmatrix}-1&-2\end{smallmatrix}\right)_9 & \times & \left(\begin{smallmatrix}-1&2\end{smallmatrix}\right)_{10} \\
u & = & & & & & 2_3 & \times & 0_4 & \times & & & & \times & 2_5 & \times & 0_6 & \times & 2_9 & \times & 0_{10} \\
v & = & & & & & & & & & 0_7 & \times & 2_8 & \times & 2_5 & \times & 0_6 & \times & 0_9 & \times & 2_{10}
\end{array}$$

We abbreviate

$$\begin{array}{rcl}
Y & := & AA' \\
Z & := & BB' \\
P & := & YZY + ZYZ - YZYZ - ZY + YZ
\end{array}$$

and note that Y maps to y and Z maps to z.

For to see that this morphism is surjective we consider in particular the following elements in its image. Non mentioned entries are to be read as zero.

Elements in eBe. Let the twiddle denote right conjugation by $\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)$.

$$\begin{array}{rcl} yz - zy &=& \left(\begin{smallmatrix}-2&4\\-2&2\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}-2&4\\-2&2\end{smallmatrix}\right)_{10} \\ yzy - zyz &=& \left(\begin{smallmatrix}0&4\\2&0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}0&4\\2&0\end{smallmatrix}\right)_{10} \\ zyz - 2zy &=& \left(\begin{smallmatrix}0&0\\-2&0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}0&0\\-2&0\end{smallmatrix}\right)_{10} \\ zyzy - 4zy &=& \left(\begin{smallmatrix}0&0\\0&-4\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}0&0\\0&-4\end{smallmatrix}\right)_{10} \end{array}$$

Elements in eBf. Let the twiddle denote left multiplication by $\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)^{-1}$.

$$\begin{array}{rcl} (yz - zy)a &=& \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)_{10} \\ (yzy - zyz)a &=& \left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)_{10} \end{array}$$

Elements in eBg. Let the twiddle denote left multiplication by $\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)^{-1}$.

$$\begin{array}{rcl} (yz - zy)b &=& \left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)_{10} \\ (yzy - zyz)b &=& \left(\begin{smallmatrix}4\\0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}4\\0\end{smallmatrix}\right)_{10} \end{array}$$

Elements in fBe. Let the twiddle denote right multiplication by $\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)$.

$$\begin{array}{rcl} a'(yz - zy) &=& \left(\begin{smallmatrix}-4&4\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}-4&4\end{smallmatrix}\right)_{10} \\ a'(yzy - zyz) &=& \left(\begin{smallmatrix}4&0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}4&0\end{smallmatrix}\right)_{10} \end{array}$$

Elements in gBe. Let the twiddle denote right multiplication by $\left(\begin{smallmatrix}1&4\\0&1\end{smallmatrix}\right)$.

$$\begin{array}{rcl} b'(yz - zy) &=& \left(\begin{smallmatrix}-2&0\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}-2&0\end{smallmatrix}\right)_{10} \\ b'(yzy - zyz) &=& \left(\begin{smallmatrix}4&-4\end{smallmatrix}\right)_9^{\sim} \times \left(\begin{smallmatrix}4&-4\end{smallmatrix}\right)_{10} \end{array}$$

An element in fBf.

$$a'(yzy - zyz)a = 4_9 \times 4_{10}$$

An element in fBg.

$$a'(yz - zy)b = 4_9 \times 4_{10}$$

An element in gBf.

$$b'(yz - zy)a = -2_9 \times -2_{10}$$

An element in gBg.

$$b'(yzy - zyz)b = -4_9 \times -4_{10}$$

We **claim** that its kernel is generated as an ideal by

$$\begin{array}{rcl} A'A &=& 2F \\ B'B &=& 2G \\ (YZ)^2 - (ZY)^2 &=& 2(YZ - ZY) \\ \\ X^2 &=& 2X \\ XY &=& YX \\ XZ &=& ZX \\ \\ \tilde{X}^2 &=& 2\tilde{X} \\ \tilde{X}Y &=& Y\tilde{X} \\ \tilde{X}Z &=& Z\tilde{X} \\ \\ \tilde{X} - X &=& P - XP \\ \\ U^2 &=& 2U \\ UA' &=& A'X \\ AU &=& XA \\ \\ V^2 &=& 2V \\ VB' &=& B'\tilde{X} \\ BV &=& \tilde{X}B \end{array}$$

We note that $P^2 = 2P$ modulo these relations except for the $\tilde{X} - X$-relation, as we check by a direct calculation.

We **claim** that these relations are symmetric in the sense that

$$\begin{array}{ccc} F & \longrightarrow & G \\ U & \longrightarrow & V \\ A & \longrightarrow & B \\ A' & \longrightarrow & B' \\ X & \longrightarrow & \tilde{X} \end{array}$$

induces an automorphism of $\mathbf{Z}\Xi$ modulo these relations. First we remark that

$$\begin{aligned} YZY + ZYZ - YZYZ - ZY + YZ &= YZY + ZYZ - ZYZY - 2YZ + 2ZY - ZY + YZ \\ &= ZYZ + YZY - ZYZY - YZ + ZY. \end{aligned}$$

modulo these relations except for the $\tilde{X} - X$-relation, which means that P is invariant under the automorphism induced on $\mathbf{Z}\Xi$ modulo this smaller ideal. Now

$$\begin{aligned} \tilde{X} - X &= -P + XP \\ &= P + XP - P^2 \\ &= P - (XP + P^2 - XP^2) \\ &= P - \tilde{X}P \end{aligned}$$

modulo the whole ideal, so that the automorphism exists modulo this whole ideal, too.

Note moreover that we have

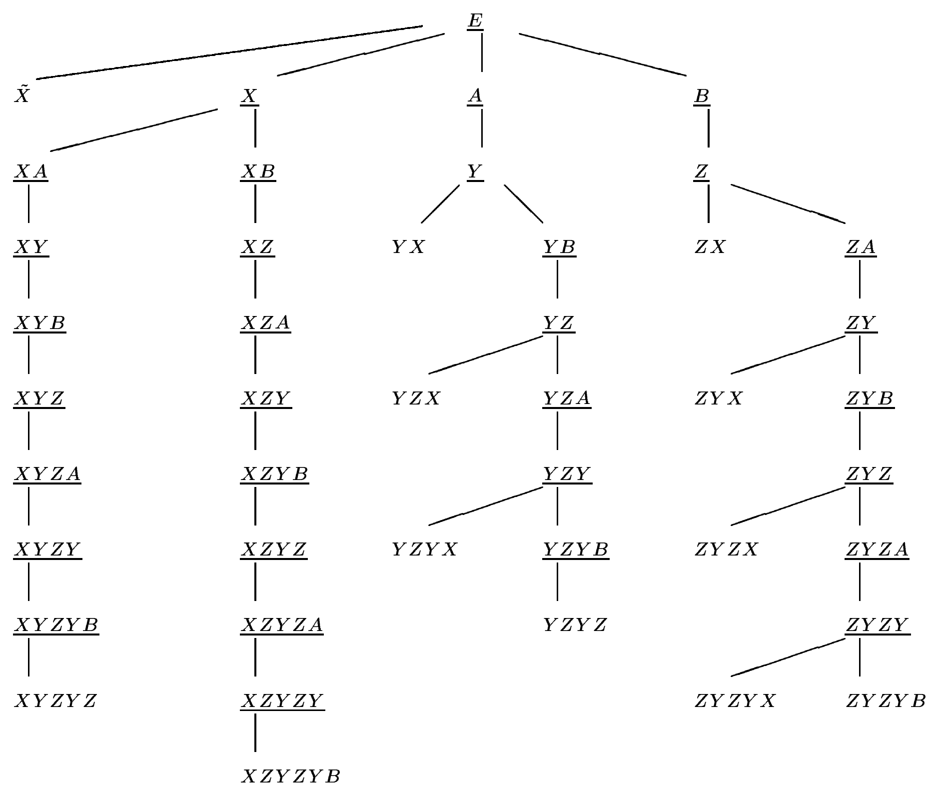
$$\begin{aligned} (YZ - ZY)^2 &= YZYZ - YZZY - ZYYZ + ZYZY \\ &= YZYZ - 2YZY - 2ZYZ + YZYZ - 2YZ + 2ZY \\ &= -2P, \end{aligned}$$

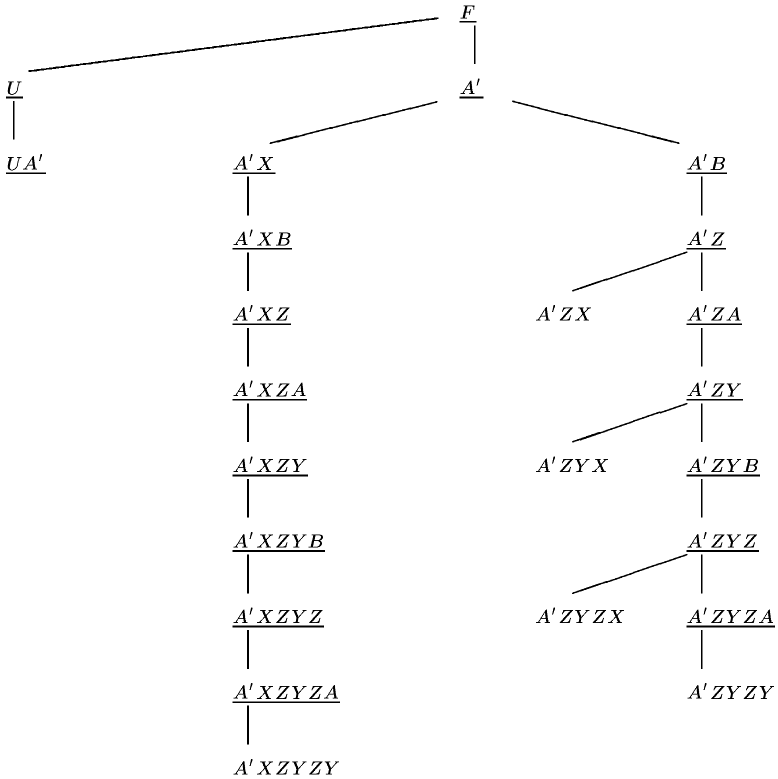
being a more natural expression, but unfortunately only for $2P$.

Regard the multiplication trees of Ξ. We drop the branches which end non underlined because of X^2, U^2, V^2, $A'A$ or $B'B$ in it, possibly after commuting factors according to the relations. Also we drop the branches with $\tilde{X}X$ in it, which we can do, since we may rewrite $WX\tilde{X}$, W being a word - commute X to the right - by the $(\tilde{X} - X)$-relation above as a linear combination

$$WX\tilde{X} = -WXP + 2WX$$

all summands of which however in fact can be written as linear combination of underlined elements by the part of the tree which we do not drop. By symmetry we may disregard the multiplication tree for G.





The kernel K of $\mathbf{F}_2\Xi \longrightarrow B/2$ now is generated by

$$\begin{aligned}
A'A &= 0\\
B'B &= 0\\
(YZ)^2 &= (ZY)^2\\[1ex]
X^2 &= 0\\
XY &= YX\\
XZ &= ZX\\[1ex]
\tilde{X}^2 &= 0\\
\tilde{X}Y &= Y\tilde{X}\\
\tilde{X}Z &= Z\tilde{X}\\[1ex]
\tilde{X} - X &= P - XP\\[1ex]
U^2 &= 0\\
UA' &= A'X\\
AU &= XA\\[1ex]
V^2 &= 0\\
VB' &= B'\tilde{X}\\
BV &= \tilde{X}B,
\end{aligned}$$

thus giving a Morita equivalence between $\mathbf{F}_2\Xi/K \times \mathbf{F}_2$ and $\mathbf{F}_2\mathcal{S}_6$.

Chapter 3

Ties arising from modular morphisms

Our approach to a description of $\mathbf{Z}\mathcal{S}_n$ is based on the philosophy that ties are given by modular morphisms, and, possibly, by inclusions of simple lattices (in case the simple lattices we end up with are not the ones used for the embedding). Since the technicalities arising from that philosophy which are necessary to establish (5.3.15) are contained in (5.3.13), this chapter may be regarded as a digression, intended to motivate (C 4, C 5). The presentation here deviates slightly from that in (5.3.13).

Conventions for (S 3.1,S 3.2).

Let R be a Dedekind domain (to which we refer as 'integral') with field of fractions K (to which we refer as 'rational'). Let Λ be a full (i.e. rationally equal) R-suborder of a direct product of matrix rings over R, $\Lambda \subseteq \Gamma := \prod_{i\in[1,s]}(R)_{m_i}$. Let $\mathfrak{a} \subseteq R$ be the annihilator of Γ/Λ.

By a module we understand a left module, except if stated otherwise. A Λ-lattice is a Λ-module which is finitely generated projective over R. We abbreviate $K\otimes_R -$ by $K(-)$. A simple lattice is a lattice X such that KX is a simple $K\Lambda$-module. A pure monomorphism of Λ-lattices has a torsionfree quotient, a full monomorphism has a torsion quotient, a pure epimorphism is surjective.

3.1 One-step filtrations

Suppose given an extension of Λ-lattices X and Y. Since it is split as an extension of R-lattices, we may write it in the following form, where $X \lhd Y$ stands for the direct sum of R-lattices, equipped with a certain Λ-operation.

$$0 \longrightarrow X \xrightarrow{(1\ 0)} X \lhd Y \xrightarrow{\binom{0}{1}} Y \longrightarrow 0.$$

Let ξ_λ be the operation of $\lambda \in \Lambda$ on X, written as R-linear endomorphism of X. Let η_λ be the operation of λ on Y. The operation of λ on $X \lhd Y$ therefore is given by a matrix of the form $\begin{pmatrix} \xi_\lambda & 0 \\ \partial_\lambda & \eta_\lambda \end{pmatrix}$ such that

$$\partial_{\mu\lambda} = \partial_\lambda \xi_\mu + \eta_\lambda \partial_\mu .$$

I.e.

$$\Lambda \xrightarrow{\partial} {}_R(Y, X)$$

is a Hochschild 1-cocycle over R with values in the $\Lambda \otimes_R \Lambda^o$-module ${}_R(Y, X)$, written

$$\partial \in Z^1_R(\Lambda,\ {}_R(Y, X)).$$

Two such extensions are equivalent iff there exists a diagram of Λ-lattices

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \lhd Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle \left(\begin{smallmatrix}1 & 0\\ f & 1\end{smallmatrix}\right)} & & \Vert & & \\
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \lhd' Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0,
\end{array}$$

i.e. iff there exist $f \in {}_R(Y, X)$ such that

$$\partial_\lambda - \partial'_\lambda = f\xi_\lambda - \eta_\lambda f$$

for all $\lambda \in \Lambda$, i.e. iff

$$\partial - \partial' \in B^1_R(\Lambda,\ {}_R(Y, X))$$

is a coboundary.

Thus we have identified, as sets,

$$\operatorname{Ext}^1_\Lambda(Y, X) = H^1_R(\Lambda,\ {}_R(Y, X)),$$

an identification which is also seen to be R-linear.

Since $\mathfrak{a}$ annihilates $\operatorname{Ext}^1_\Lambda(Y, X)$ (B.2.3), the pullback

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \oplus Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0 \\
 & & \Vert & & {\scriptstyle \left(\begin{smallmatrix}1 & 0\\ g & a\end{smallmatrix}\right)}\downarrow & \ulcorner & \downarrow{\scriptstyle a} & & \\
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \lhd Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0,
\end{array}$$

splits our extension for any $a \in \mathfrak{a} \backslash 0$. Therefore

$$\left(\begin{smallmatrix}\xi_\lambda & 0\\ 0 & \eta_\lambda\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 0\\ g & a\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 0\\ g & a\end{smallmatrix}\right)\left(\begin{smallmatrix}\xi_\lambda & 0\\ \partial_\lambda & \eta_\lambda\end{smallmatrix}\right),$$

whence

$$\partial_\lambda = a^{-1}(\eta_\lambda g - g\xi_\lambda)$$

for all $\lambda \in \Lambda$. This, however, is just a restatement of the annihilator property just used in view of the identification $H^1 = \operatorname{Ext}^1$ above.

In other words, we have explicitely constructed an inverse image of our extension under the connector of the long exact sequence

$$\begin{array}{cccccc}
H^0(\Lambda,\ {}_R(Y, X/a)) & \xrightarrow{\delta} & H^1(\Lambda,\ {}_R(Y, X)) & & \xrightarrow{a} & H^1(\Lambda,\ {}_R(Y, X)) \\
g & \longrightarrow & (\lambda \longrightarrow a^{-1}(\eta_\lambda g - g\xi_\lambda)) & & &
\end{array}$$

induced by the short exact sequence

$$0 \longrightarrow X \xrightarrow{a} X \longrightarrow X/a \longrightarrow 0.$$

This can also be expressed by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \lhd Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \binom{a}{-g}} & & \downarrow{\scriptstyle -g} & & \\
0 & \longrightarrow & X & \xrightarrow{a} & X & \longrightarrow & X/a & \longrightarrow & 0.
\end{array}$$

Note that $\binom{a}{-g}$ is Λ-linear because of the annihilating pushout

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \oplus Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle a} & & \uparrow{\scriptstyle \left(\begin{smallmatrix} a & 0 \\ -g & 1 \end{smallmatrix}\right)} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow{(1\ 0)} & X \lhd Y & \xrightarrow{\binom{0}{1}} & Y & \longrightarrow & 0,
\end{array}$$

$\left(\begin{smallmatrix} a & 0 \\ -g & 1 \end{smallmatrix}\right)$ being Λ-linear since

$$\left(\begin{smallmatrix} a & 0 \\ -g & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ g & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right).$$

Suppose given $\gamma \in \Gamma$. We say γ acts on a Λ-lattice X if its operation ξ_γ on KX restricts to X.

Lemma 3.1.1 *Suppose $\gamma \in \Gamma$ acts on X and Y. γ acts on $X \lhd Y$ iff γ respects $Y \xrightarrow{g} X$ modulo a, this is, iff*

$$\eta_\gamma g \equiv_a g\xi_\gamma.$$

Thus the existence of the extension $X \lhd Y$ imposes a necessary condition on an element $\gamma \in \Gamma$ for to be in Λ.

γ acts on $X \oplus Y$. γ acts on $X \lhd Y$ iff

$$\left(\begin{smallmatrix} a & 0 \\ -g & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi_\gamma & 0 \\ 0 & \eta_\gamma \end{smallmatrix}\right) = \begin{pmatrix} \xi_\gamma & 0 \\ \partial_\gamma & \eta_\gamma \end{pmatrix}\left(\begin{smallmatrix} a & 0 \\ -g & 1 \end{smallmatrix}\right)$$

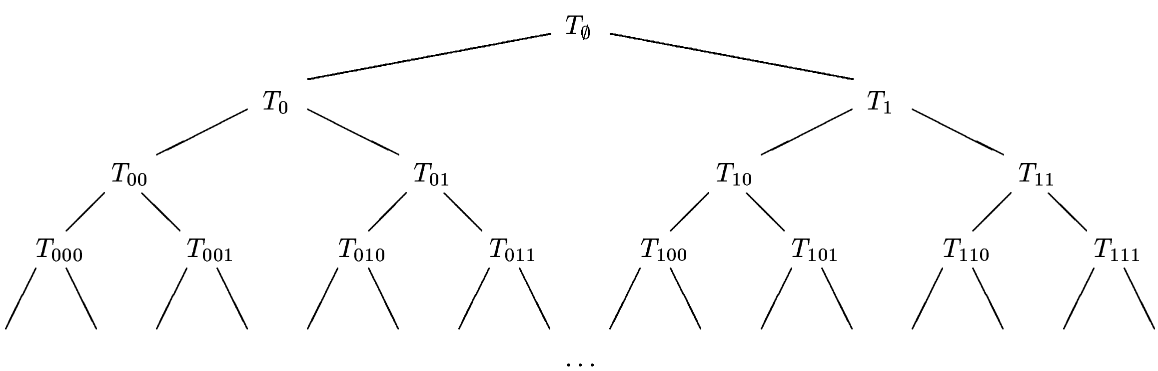
yields an integral map

$$\partial_\gamma = a^{-1}(\eta_\gamma g - g\xi_\gamma).$$

Remark 3.1.2 *Note that $\binom{a}{-g}$ is a Λ-linear retraction to the inclusion $X \xrightarrow{(1\ 0)} X \lhd Y$ of the extension up to the scalar a.*

3.2 Filtrations

Suppose given a finite binary tree of Λ-lattices



ending at possibly different stages, such that the ends carry Γ-lattices, and such that there exists a short exact sequence

$$0 \longrightarrow T_{e0} \xrightarrow{e0_*} T_e \xrightarrow{e1^*} T_{e1} \longrightarrow 0,$$

for each non-end $[1,h] \xrightarrow{e} \{0,1\}$, $h \geq 0$, where $e0$ resp. $e1$ denote concatenation with 0 resp. 1.

Let

$$T_{e1} \xrightarrow{g_e} T_{e0}$$

be an R-linear map giving a Λ-linear map

$$T_{e1} \xrightarrow{g_e} T_{e0}/a_e$$

as in (3.1). Let $\tau_{e,\gamma}$ denote the operation of $\gamma \in \Gamma$ on KT_e. Note that, for e not being an end,

$$\tau_{e,\gamma} = \begin{pmatrix} \tau_{e0,\gamma} & 0 \\ a_e^{-1}(\tau_{e1,\gamma}g_e - g_e\tau_{e0,\gamma}) & \tau_{e1,\gamma} \end{pmatrix}.$$

Lemma 3.2.1 *γ acts on $T_\emptyset$ iff*

$$\tau_{e1,\gamma}g_e \equiv_{a_e} g_e\tau_{e0,\gamma}$$

for all non-ends e.

This follows from (3.1.1).

Remark 3.2.2 Pulling back along the pure epimorphisms of that tree, one obtains a filtration of $T_\emptyset$ by Γ-lattices. Conversely, it is always possible to filter $T_\emptyset$ by simple Λ-lattices, which, however, need not be Γ-lattices. In that case, one obtains the same assertion as in (3.2.1), except that one has to add that γ should act on those simple lattices as well.

Remark 3.2.3 *$\gamma \in \Gamma$ acts on Λ iff it is contained in Λ.*

Consider $1 \in \Lambda$.

Remark 3.2.4 *Let X and Y be isomorphic Λ-lattices. γ acts on X iff it acts on Y.*

Thus we may summarize to the

Observation 3.2.5 *In case $T_\emptyset$ is isomorphic to Λ as Λ-lattices, $\gamma \in \Gamma$ is contained in Λ iff*

$$\tau_{e1,\gamma}g_e \equiv_{a_e} g_e\tau_{e0,\gamma}$$

for all non-ends e. For short, ties are given by modular morphisms.

This follows from (3.2.1, 3.2.3, 3.2.4).

Remark 3.2.6 *Let X and Y be Λ-lattices. $\gamma \in \Gamma$ acts on X and on Y iff it acts on $X \oplus Y$.*

Remark 3.2.7 *Let*

$$\Lambda = \bigoplus_{i \in I} P_i$$

be a decomposition into projective Λ-lattices, I being some indexing set. Let $J \subseteq I$ be a subset such that for each $i \in I$ there is a $j \in J$ with $P_i \simeq P_j$. $\gamma \in \Gamma$ is in Λ iff it acts on P_j for each j.

This ensues from (3.2.3, 3.2.6, 3.2.4). Therefore we could as well use $T_\emptyset$'s isomorphic to the P_j's, $j \in J$.

However, since we do not know of a nontrivial decomposition of $\mathbf{Z}_{(p)}\mathcal{S}_n$ into projectives given in combinatorial terms, we won't make use of such a reduction.

Question 3.2.8 (speculative) Is it possible to use a chain of full inclusions of R-orders

$$\Lambda = \Lambda_0 \subseteq \cdots \subseteq \Lambda_i \subseteq \Lambda_{i+1} \subseteq \cdots \subseteq \Lambda_m = \Gamma$$

for which the indecomposable projective Λ_i-lattices are either projective over Λ_{i+1} or extensions of two indecomposable projective Λ_{i+1}-lattices to 'iterate the theory of cyclic defect'?

3.3 Cocycles

We shall have a look at the matrices giving the operation on a filtered Λ-lattice in terms of the operation on the graduation plus some extra information and encode this extra information in a 'generalized Ext-set', which is neither (known to be) a group nor a functor, except in case of a one-step-filtration. We proceed using cocycles, for lack of a proper formalism.

The aim should be to encode all ties as something like 'modular morphisms between several simple lattices'. Whereas the analogue of Ext^1 for two lattices is easy to see, we don't have an idea what the analogue of the connector from the modular Hom-group to it should be (cf. S 3.1).

Let R be an integral domain of characteristic 0 with field of fractions K, let G be a finite group. Let $X = (X_1, \dots, X_m)$ be a tuple of right RG-lattices, let $\xi_{i,g}$ denote the operation of $g \in G$ on X_i.

Definition 3.3.1 *A* **cocycle** ∂ *of G over R with coefficients in X is a tuple ∂ of R-linear maps*

$$(G \xrightarrow{\partial_{ij}} {}_R(X_i, X_j))_{i,j\in[1,m],\, i<j}$$

such that for $g, h \in G$, we have

$$\partial_{ij}(gh) = \xi_{i,g}\partial_{ij}(h) + \partial_{ij}(g)\xi_{j,h} + \sum_{k\in[i+1,j-1]} \partial_{ik}(g)\partial_{kj}(h)$$

for all $i < j$. The set of cocycles is denoted by $Z(G, X)$.

Example 3.3.2 *Let*

$$(KX_i \xrightarrow{f_{ij}} KX_j)_{i,j\in[1,m],\, i<j}$$

be a tuple of K-linear maps. For $i < j$, let $X_i \xrightarrow{\partial_{ij}(g)} X_j$ be defined by

$$\begin{bmatrix} 1 & & & & \\ f_{m-1,m} & 1 & & & \\ f_{m-2,m} & f_{m-2,m-1} & 1 & & \\ \vdots & \vdots & & \ddots & \\ f_{1,m} & f_{1,m-1} & \cdots & f_{12} & 1 \end{bmatrix} \begin{bmatrix} \xi_{m,g} & & & & \\ & \xi_{m-1,g} & & & \\ & & \xi_{m-2,g} & & \\ & & & \ddots & \\ & & & & \xi_{1,g} \end{bmatrix}$$
$$= \begin{bmatrix} \xi_{m,g} & & & & \\ \partial_{m-1,m}(g) & \xi_{m-1,g} & & & \\ \partial_{m-2,m}(g) & \partial_{m-2,m-1}(g) & \xi_{m-2,g} & & \\ \vdots & \vdots & & \ddots & \\ \partial_{1,m}(g) & \partial_{1,m-1}(g) & \cdots & \partial_{12}(g) & \xi_{1,g} \end{bmatrix} \begin{bmatrix} 1 & & & & \\ f_{m-1,m} & 1 & & & \\ f_{m-2,m} & f_{m-2,m-1} & 1 & & \\ \vdots & \vdots & & \ddots & \\ f_{1,m} & f_{1,m-1} & \cdots & f_{12} & 1 \end{bmatrix}$$

Assume that each ∂_{ij} factors over

$$G \xrightarrow{\partial_{ij}} {}_R(X_i, X_j) \hookrightarrow {}_K(KX_i, KX_j),$$

by slight abuse of notation. Then ∂ is a cocycle of G over R with coefficients in X.

From

$$\begin{bmatrix} \xi_{m,g} & & & & \\ \partial_{m-1,m}(g) & \xi_{m-1,g} & & & \\ \partial_{m-2,m}(g) & \partial_{m-2,m-1}(g) & \xi_{m-2,g} & & \\ \vdots & \vdots & & \ddots & \\ \partial_{1,m}(g) & \partial_{1,m-1}(g) & \cdots & \partial_{12}(g) & \xi_{1,g} \end{bmatrix} \begin{bmatrix} \xi_{m,h} & & & & \\ \partial_{m-1,m}(h) & \xi_{m-1,h} & & & \\ \partial_{m-2,m}(h) & \partial_{m-2,m-1}(h) & \xi_{m-2,h} & & \\ \vdots & \vdots & & \ddots & \\ \partial_{1,m}(h) & \partial_{1,m-1}(h) & \cdots & \partial_{12}(h) & \xi_{1,h} \end{bmatrix}$$
$$= \begin{bmatrix} \xi_{m,gh} & & & & \\ \partial_{m-1,m}(gh) & \xi_{m-1,gh} & & & \\ \partial_{m-2,m}(gh) & \partial_{m-2,m-1}(gh) & \xi_{m-2,gh} & & \\ \vdots & \vdots & & \ddots & \\ \partial_{1,m}(gh) & \partial_{1,m-1}(gh) & \cdots & \partial_{12}(gh) & \xi_{1,gh} \end{bmatrix}$$

we take the required functional equation (3.3.1).

Definition 3.3.3 *The set of cocycles arising from an* **integral** *tuple* $(f_{ij})_{i,j\in[1,m],\, i<j}$ *in the way described in (3.3.2) is denoted by* $B(G,X)$. *An element of* $B(G,X)$ *is called a* **coboundary.** *Via matrix multiplication of the corresponding unipotent lower triangular matrices*

$$\begin{bmatrix} 1 & & & & \\ f_{m-1,m} & 1 & & & \\ f_{m-2,m} & f_{m-2,m-1} & 1 & & \\ \vdots & \vdots & & \ddots & \\ f_{1,m} & f_{1,m-1} & \cdots & f_{12} & 1 \end{bmatrix}$$

$B(G,X)$ *becomes a group which acts via conjugation on* $Z(G,X)$, *a cocycle written as an operating matrix as above. The quotient set is denoted by* $H(G,X) := Z(G,X)/B(G,X)$.

In case $m = 2$ we recover $H(G,X)$ to be the first Hochschild cohomology group $H^1(G, {}_R(X_1, X_2))$ with coefficients in the bimodule ${}_R(X_1, X_2)$.

Remark 3.3.4 *There is an operation of* R *on the set* $Z(G,X)$ *given by*

$$(r\partial)_{ij}(g) := r^{j-i}\partial_{ij}(g)$$

for $r \in R$, $\partial \in Z(G,X)$. *We have*

$$|G|Z(G,X) \subseteq B(G,X).$$

Let

$$|G|f_{ij} := \sum_{g\in G} \partial_{ij}(g).$$

We obtain

$$\begin{aligned} \textstyle\sum_{h\in G} \partial_{ij}(gh)\xi_{j,h^{-1}} &\overset{1.}{=} \textstyle\sum_{h\in G} \partial_{ij}(gh)\xi_{j,(gh)^{-1}}\xi_{j,g} \\ &= |G|f_{ij}\xi_{j,g} \\ &\overset{2.}{=} \textstyle\sum_{h\in G} \left(\xi_{i,g}\partial_{ij}(h) + \partial_{ij}(g)\xi_{j,h} + \displaystyle\sum_{k\in[i+1,j-1]} \partial_{ik}(g)\partial_{kj}(h) \right) \xi_{j,h^{-1}} \\ &= \xi_{i,g}|G|f_{ij} + |G|\partial_{ij}(g) + \sum_{k\in[i+1,j-1]} \partial_{ik}(g)|G|f_{kj}, \end{aligned}$$

whence

$$\partial_{ij}(g) = f_{ij}\xi_{j,g} - \xi_{i,g}f_{ij} - \sum_{k\in[i+1,j-1]} \partial_{ik}(g)f_{kj},$$

which, one the one hand, shows by induction on $j-i$ that f determines ∂ and which, on the other hand, holds in the first matrix equation of (3.3.2). Thus ∂ arises from f in the way described in (3.3.2).

Hence $|G|\partial$ corresponds to $(|G|^{j-i}f_{ij})_{i,j\in[1,m],\, i<j}$, which is integral by construction.

Example 3.3.5 Since also the rational (f_{ij})'s carry a group structure given by matrix multiplication of the corresponding unipotent matrices, one might be tempted to believe that it carries over to the cocycles via the correspondence described in (3.3.2, 3.3.4). In particular, this works in case $m = 2$. However, let $G := \mathcal{S}_4$, $R := \mathbf{Z}$ and consider the operations

$$\begin{array}{lll} \xi_{1,(12)} = \begin{bmatrix} -5 & 24 \\ -1 & 5 \end{bmatrix} & & \xi_{1,(1234)} = \begin{bmatrix} 4 & -15 \\ 1 & -4 \end{bmatrix} \\ \xi_{2,(12)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} & & \xi_{2,(1234)} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 2 \\ -2 & 0 & 1 \end{bmatrix} \\ \xi_{3,(12)} = \begin{bmatrix} -11 & -24 & 8 \\ 5 & 11 & -4 \\ 0 & 0 & -1 \end{bmatrix} & & \xi_{3,(1234)} = \begin{bmatrix} 26 & 57 & 8 \\ -11 & -24 & -4 \\ -1 & -2 & -1 \end{bmatrix}. \end{array}$$

The 'unipotent f-matrix'

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1/4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1/4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 1 & 0 & 0 \\ 1/8 & 0 & 0 & 1/2 & 0 & 0 & 1 & 0 \\ 0 & 1/8 & 0 & 0 & 1/2 & 0 & 0 & 1 \end{bmatrix}$$

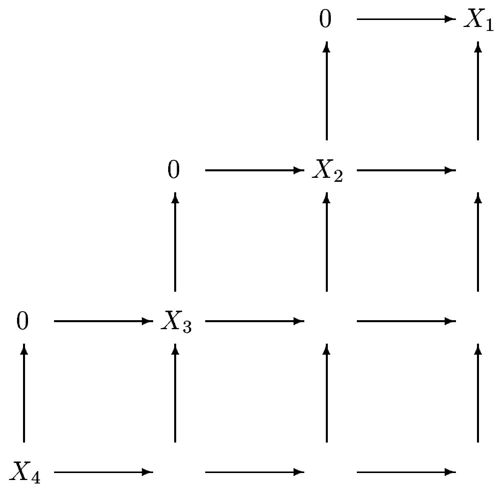
conjugates the direct sum operation to the operation displayed as the 'operating ∂-matrices'

$$\eta_{(12)} = \begin{bmatrix} -11 & -24 & 8 & 0 & 0 & 0 & 0 & 0 \\ 5 & 11 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -3 & 2 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -9 & 1 & 3 & -12 & 0 & -5 & 24 \\ 1 & 0 & 0 & 1 & -3 & 1 & -1 & 5 \end{bmatrix}, \quad \eta_{(1234)} = \begin{bmatrix} 26 & 57 & 8 & 0 & 0 & 0 & 0 & 0 \\ -11 & -24 & -4 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -7 & -14 & -2 & -2 & 1 & 0 & 0 & 0 \\ 2 & 6 & 2 & -3 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 11 & 1 & -3 & 8 & 0 & 4 & -15 \\ -2 & -2 & 0 & -2 & 2 & 1 & 1 & -4 \end{bmatrix},$$

wheras the **square** of that lower triangular f-matrix conjugates the direct sum operation to

$$\zeta_{(12)} = \begin{bmatrix} -11 & -24 & 8 & 0 & 0 & 0 & 0 & 0 \\ 5 & 11 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 12 & -4 & 1 & 0 & 0 & 0 & 0 \\ -2 & -6 & 4 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 9/4 & -18 & 1 & 6 & -24 & 0 & -5 & 24 \\ 7/4 & -9/4 & 3/2 & 2 & -6 & 2 & -1 & 5 \end{bmatrix}, \quad \zeta_{(1234)} = \begin{bmatrix} 26 & 57 & 8 & 0 & 0 & 0 & 0 & 0 \\ -11 & -24 & -4 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -14 & -28 & -4 & -2 & 1 & 0 & 0 & 0 \\ 4 & 12 & 4 & -3 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & -2 & 0 & 1 & 0 & 0 \\ -1/4 & 17 & 1 & -6 & 16 & 0 & 4 & -15 \\ -7/2 & -1/2 & 3/2 & -4 & 4 & 2 & 1 & -4 \end{bmatrix}.$$

Definition 3.3.6 *Let* $\mathrm{Ext}_{RG}(X)$ *be the set of diagrams of RG-lattices of the following shape (e.g.* $m = 4$*)*



with horizontal pure monomorphisms, vertical pure epimorphisms and all squares exact, modulo isomorphisms of diagrams carrying the identity at X_i, $i \in [1, m]$.

For $X = (X_1, X_2)$ we recover $\mathrm{Ext}_{RG}(X) = \mathrm{Ext}^1_{RG}(X_1, X_2)$.

Lemma 3.3.7 *In case $G = 1$, $\mathrm{Ext}_{RG}(X)$ consists of one element.*

Given a diagram $(X_{j/i})_{i,j\in[0,m],\, i<j}$ with $X_{j/j-1} = X_{m+1-j}$ for $j \in [1,m]$, we exhibit a diagram morphism constant on the X_i's onto it, starting from the split diagram $(\bigoplus_{k\in[i,j]} X_{k/k-1})_{i,j\in[0,m],\, i<j}$, equipped with the canonical inclusions and projections.

Choose coretractions

$$(X_{j/j-1} \longrightarrow X_{j/0} \longrightarrow X_{j/j-1}) = 1$$

for $j \in [1,m]$. Let

$$\bigoplus_{k\in[i,j]} X_{k/k-1} \longrightarrow X_{j/i}$$

have the composition

$$X_{k/k-1} \longrightarrow X_{k/0} \longrightarrow X_{j/i}$$

as its component at k.

Let $i' < j'$ be given with $i \leq i'$, $j \leq j'$. Let $k \in [i,j]$. First going through the morphism, then through the diagram yields

$$X_{k/k-1} \longrightarrow X_{k/0} \longrightarrow X_{j/i} \longrightarrow X_{j'/i'}$$

as its component at k. First going through the diagram, then through the morphism yields

$$\begin{cases} X_{k/k-1} = X_{k/k-1} \longrightarrow X_{k/0} \longrightarrow X_{j'/i'} & \text{for } k \geq i' \\ 0 & \text{for } k < i' \end{cases}$$

as its component at k.

Lemma 3.3.8 *There is a bijection*

$$H(G,X) \xrightarrow{\sim} \mathrm{Ext}_{RG}(X).$$

We give a surjection

$$Z(G,X) \longrightarrow \mathrm{Ext}_{RG}(X)$$

with fibers being the orbits under the operation of $B(G,X)$. A cocycle ∂ corresponds to an operation of G on the R-linear direct sum $\bigoplus_{i\in[1,m]} X_i$, given by

$$\begin{bmatrix} \xi_{m,g} & & & & \\ \partial_{m-1,m}(g) & \xi_{m-1,g} & & & \\ \partial_{m-2,m}(g) & \partial_{m-2,m-1}(g) & \xi_{m-2,g} & & \\ \vdots & \vdots & & \ddots & \\ \partial_{1,m}(g) & \partial_{1,m-1}(g) & \cdots & \partial_{12}(g) & \xi_{1,g} \end{bmatrix}$$

Since the subcocycle $(\partial_{ij})_{i,j\in[k,l],\, i<j}$ for $k,l \in [1,m]$, $k < l$, is a cocycle of G over R with coefficients in $(X_k,\ldots,X_l)$, it corresponds to an operation on $\bigoplus_{i\in[k,l]} X_i$. These RG-lattices, together with their canonical inclusions and projections form a diagram as in (3.3.6), to which we map ∂.

We **claim** this map to be surjective. Given a diagram of RG-lattices as in (3.3.6), we may replace it by an isomorphism of diagrams constant on each X_i by a diagram which consists of R-linear direct sums of type $\bigoplus_{i\in[k,l]} X_i$, equipped with the canonical inclusions and projections (3.3.7). This yields an operation matrix on the lower right object which is of lower triangular shape and which contains the desired cocycle.

If two diagrams consisting of R-linear direct sums are isomorphic, constant on each X_i, then the operation matrices on the lower right objects are conjugate by an integral matrix of the form

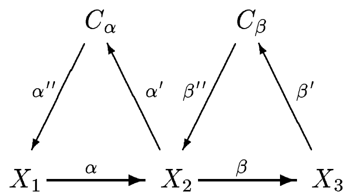
$$\begin{bmatrix} 1 & & & & \\ f_{m-1,m} & 1 & & & \\ f_{m-2,m} & f_{m-2,m-1} & 1 & & \\ \vdots & \vdots & & \ddots & \\ f_{1,m} & f_{1,m-1} & \cdots & f_{12} & 1 \end{bmatrix},$$

containing the components of that isomorphism. And conversely.

Remark 3.3.9 *Consider the case* $m = 3$, $X = (X_1, X_2, X_3)$. $\mathrm{Ext}_{RG}(X_1, X_2, X_3)$ *surjects onto*

$$\{\alpha \times \beta \in \mathrm{Ext}_{RG}(X_1, X_2) \times \mathrm{Ext}_{RG}(X_2, X_3) \mid \alpha \cdot \beta = 0 \ \textit{(Yoneda)}\}$$

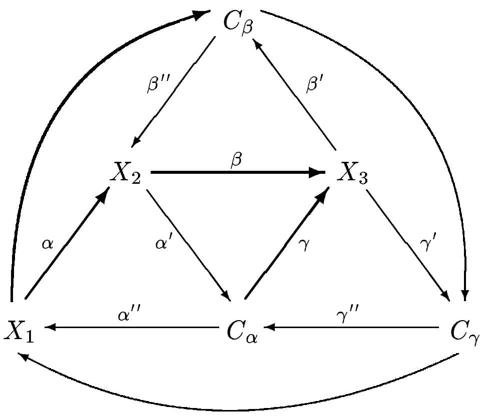
via projection onto the respective parts of the diagram (3.3.6).
Each fiber is in bijection with the cokernel of the map ${}_{RG}(X_2, X_3) \longrightarrow \mathrm{Ext}_{RG}(X_1, X_3)$ *induced by the respective* $\alpha \in \mathrm{Ext}_{RG}(X_1, X_2)$.
Suppose given elements $\alpha \in D^b(RG)(X_1, X_2[1])$ and $\beta \in D^b(RG)(X_2, X_3[1])$ such that $\alpha\beta[1] = 0$. Consider the cones on α and β



where the thick arrows display graded morphisms. Since $\alpha\beta[1] = 0$, there is a $C_\alpha \xrightarrow{\gamma} X_3[1]$ such that $\alpha'\gamma = \beta$, the cone of which is also in the canonical heart, forming a short exact sequence

$$0 \longrightarrow X_3 \xrightarrow{\gamma'} C_\gamma \xrightarrow{\gamma''} C_\alpha \longrightarrow 0$$

there. The octahedron



furnishes the required diagram when dropping all graded morphisms.
Conversely, given a diagram representing an element of $\mathrm{Ext}_{RG}(X_1, X_2, X_3)$, we obtain such an octahedron, forcing $\alpha\beta[1] = 0$.
Suppose given α and β with $\alpha\beta[1] = 0$. The corresponding fiber is in bijection with the set of morphisms $C_\alpha \xrightarrow{\gamma} X_3[1]$ such that $\alpha'\gamma = \beta$, which is, since nonempty, in bijection with the set of morphisms $C_\alpha \xrightarrow{\gamma} X_3[1]$ such that $\alpha'\gamma = 0$. Consider the exact sequence

$$(X_2, X_3[1]) \xleftarrow{\alpha'(-)} (C_\alpha, X_3[1]) \xleftarrow{\alpha''(-)} (X_1, X_3[1]) \xleftarrow{\alpha(-)} (X_2[1], X_3[1]).$$

Chapter 4

Generic modular morphisms

We shall search for modular morphisms between Specht lattices, i.e. for $\mathbf{Z}\mathcal{S}_n$-linear maps $S^\lambda \longrightarrow S^\mu/m$ for some $m \geq 2$. Some easy and some exceptional cases are given in (S 4.2). In (S 4.3) a formula for a one-box-shift morphism is derived. As an example, we treat the consequences for $\mathbf{Z}\mathcal{S}_6$ at 3 in (S 4.4).

The morphisms we found are **generic** in the sense that their formulas depend (at most) polynomially on combinatorial data (cf. S 4.5).

A Specht lattice has a presentation as a regular lattice on a $\mathbf{Z}$-linear tableaux basis modulo the Garnir relations, roughly speaking. The structure and the sheer amount of these relations cause lenghty case-by-case analyses (cf. 4.3.9, 5.2.7).

A Specht lattice also has a $\mathbf{Z}$-linear basis given by standard polytabloids. However, since standard polytabloids are not stable under the occurring combinatorial operations, it is hardly possible to work with this basis (thus avoiding Garnir relations), except for the case of hooks. In particular, we are not able to write down matrices for the resulting morphisms in a combinatorial way. Even the question whether the morphism (4.3.31) is nonzero is not clear a priori, and has to be dealt with by standard polytabloid methods. A formula for the rank of that modular morphism remains to be found, its behaviour under composition is unknown in general (cf. 4.2.4), and, moreover, we were not able to dualize it (cf. 6.2.6).

A more conceptual way to derive (4.3.31, 5.2.25) would be desirable, since our ad hoc method does not explain the structure of the resulting formulas.

Let n be a natural number, let $\lambda \in \Lambda_n$ be a partition of n, i.e. $[1,n] \xrightarrow{\lambda} [1,n]$, $\sum \lambda_i = n$, $\lambda_i \geq \lambda_{i+1}$ for $i \in [1, n-1]$. Let λ' denote the transpose of λ, i.e. $j \leq \lambda_i \iff i \leq \lambda'_j$ for $i, j \in [1,n]$. In this chapter, lattices are right lattices, since we write the composition in the $\mathcal{S}_n$ on the right. ε_σ denotes the signature of a permutation $\sigma \in \mathcal{S}_n$.

We think of a partition as being a diagram of type

$$(5,2,1,1) = \begin{array}{ccccc} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & & & \\ \times . & & & & \end{array}$$

In particular, we talk of rows and columns. We also denote $(5,2,1,1) =: (5,2,1^2)$ **etc.**

4.1 Garnir relations

We recall a part of the basic machinery from [J 78].

Definition 4.1.1 A **λ-tableau** is a tuple

$$[a] = (a_{ij})_{i\in[1,n],j\in[1,\lambda_i]} = (a_{ij})_{j\in[1,n],i\in[1,\lambda'_j]}$$

such that

$$\begin{array}{rcl} \{(i,j)\in[1,n]\times[1,n] \mid j\leq\lambda_i\} & \xrightarrow{a} & [1,n] \\ (i,j) & \longrightarrow & a_{ij} \end{array}$$

is a bijection. i being the row index, j being the column index, we think of a tableau $[a]$ as of a distribution of the set $[1,n]$ onto the diagram associated to the partition λ. The brackets [] are used only to distinguish a tableau from a tabloid resp. from a polytabloid and may be dropped.

The $\mathcal{S}_n$ operates from the right on the set of λ-tableaux via composition of bijections. The $\mathbf{Z}\mathcal{S}_n$-module having as $\mathbf{Z}$-basis the λ-tableaux is denoted by F^λ. It is isomorphic to $\mathbf{Z}\mathcal{S}_n$ as a right lattice.

Let a_{*j} denote the j-th column of $[a]$, let a_{i*} denote the i-th row of $[a]$.

Denote by R_a the Young subgroup of $\mathcal{S}_n$ stabilizing the rows of a, i.e.

$$R_a := \{\sigma\in\mathcal{S}_n \mid a_{ij}\sigma\in a_{i*} \text{ for all } i,j\}.$$

Denote by C_a the Young subgroup of $\mathcal{S}_n$ stabilizing the columns of a, i.e.

$$C_a := \{\sigma\in\mathcal{S}_n \mid a_{ij}\sigma\in a_{*j} \text{ for all } i,j\}.$$

Note that $R_{a\tau} = (R_a)^\tau$, $C_{a\tau} = (C_a)^\tau$.

On the set of λ-tableaux we define an equivalence relation by

$$[a]\sim[b] :\Longleftrightarrow \text{ there is a } \sigma\in R_a \text{ such that } [a]\sigma=[b].$$

Note that $[a]\sim[b]$ implies $R_a = R_b$, so that this in fact is an equivalence relation. The equivalence class represented by the λ-tableau $[a]$ is called a **λ-tabloid** and is denoted by $\{a\}$. Informally, it is a 'tableau with unordered rows'.

This equivalence relation is compatible with the action of the $\mathcal{S}_n$, for, given a λ-tableau $[a]$ and elements $\sigma\in R_a$ and $\rho\in\mathcal{S}_n$, we have $[a]\sigma\rho = [a\rho]\sigma^\rho \sim [a\rho]$. This operation turns the free $\mathbf{Z}$-module on the set of λ-tabloids into a $\mathbf{Z}\mathcal{S}_n$-lattice, denoted by M^λ.

Let the **polytabloid** $\langle a\rangle$ attached to the λ-tableau $[a]$ (NB not to the λ-tabloid $\{a\}$) be defined as

$$\langle a\rangle := \sum_{\sigma\in C_a}\varepsilon_\sigma\{a\}\sigma\in M^\lambda$$

Note that for $\rho\in\mathcal{S}_n$ we obtain

$$\begin{array}{rcl} \langle a\rangle\rho & = & \sum_{\sigma\in C_a}\varepsilon_\sigma\{a\}\sigma\rho \\ & = & \sum_{\sigma'\in C_{a\rho}}\varepsilon_{\sigma'}\{a\rho\}\sigma' \\ & = & \langle a\rho\rangle \end{array}$$

so that the abelian subgroup of M^λ generated by the λ-polytabloids is a $\mathbf{Z}\mathcal{S}_n$-sublattice of M^λ, called the **Specht lattice** S^λ**.** In particular, we have a $\mathbf{Z}\mathcal{S}_n$-morphism

$$\begin{array}{rcl} F^\lambda & \longrightarrow & S^\lambda \\ [a] & \longrightarrow & \langle a\rangle. \end{array}$$

We denote $n_\lambda := \mathrm{rk}\, S^\lambda$.

The rational Specht modules $\mathbf{Q}S^\lambda$, obtained by tensoring the Specht lattices with $\mathbf{Q}$ over $\mathbf{Z}$, form a complete set of absolutely simple $\mathbf{Q}\mathcal{S}_n$-modules, where λ runs over the set of partitions of n [J 78, 4.12].

A λ-tableau $[a]$ is called **standard** if $a_{ij} \leq a_{i'j'}$ provided $i \leq i'$ and $j \leq j'$. A λ-polytabloid $\langle a \rangle$ is called **standard** if there exists a standard λ-tableau $[a']$ such that $\langle a \rangle = \langle a' \rangle$.

The standard polytabloids form a basis of S^λ [J 78, 8.4] (cf. 4.3.2).

Let λ be a partition of n. Let the hook length at the position (i,j) of λ, $i \leq \lambda'_j$, be given by

$$h_{ij} := (\lambda_i - j) + (\lambda'_j - i) + 1.$$

The rank of S^λ is given by the **hook formula** [J 78, 20.1]

$$n_\lambda = \frac{n!}{\prod_{i \leq \lambda'_j} h_{ij}}.$$

Lemma 4.1.2 *Let $[a]$ be a λ-tableau, let $\rho \in C_a$. Then*

$$\begin{array}{rcl} \langle a \rangle \rho & = & \sum_{\sigma \in C_a} \varepsilon_\sigma \{a\} \sigma \rho \\ & = & \sum_{\sigma' \in C_a} \varepsilon_{\sigma'} \varepsilon_\rho \{a\} \sigma' \\ & = & \varepsilon_\rho \langle a \rangle. \end{array}$$

Definition 4.1.3 *Let F_0^λ be the quotient of F^λ modulo signed column permutations. I.e. use the $\mathbf{Z}\mathcal{S}_n$-sublattice*

$$F_1^\lambda := \mathbf{Z}\langle [a] + [a](st) \mid [a] \text{ a tableau, } s,t \text{ in the same column of } [a] \rangle$$

to define

$$F_0^\lambda := F^\lambda / F_1^\lambda,$$

which is a lattice since it has the set of tableaux with ordered columns as a basis over $\mathbf{Z}$.

Note that by (4.1.2) the canonical map $F^\lambda \longrightarrow S^\lambda$ factors as

$$\begin{array}{ccccc} F^\lambda & \longrightarrow & F_0^\lambda & \longrightarrow & S^\lambda \\ [a] & \longrightarrow & [a] & \longrightarrow & \langle a \rangle. \end{array}$$

Proposition 4.1.4 (Garnir relations, [G 50, p. 56], cf. [J 78, 7.2]) *Let $[a]$ be a λ-tableau. Let a_{*j} denote the j-th column of $[a]$, viewed as a tuple. Let $j < k$. Let $\xi \subseteq a_{*j}$ and $\eta \subseteq a_{*k}$ such that*

$$\#\eta + \#\xi \geq \lambda'_j + 1.$$

For a subset $\zeta \subseteq [1,n]$, let $\mathcal{S}_\zeta$ denote the subgroup of $\mathcal{S}_n$ which moves only ζ, i.e. $\mathcal{S}_\zeta := C_{\mathcal{S}_n}([1,n] \backslash \zeta)$.

The element

$$\frac{1}{\#\xi! \#\eta!} \sum_{\rho \in \mathcal{S}_{\xi \cup \eta}} \varepsilon_\rho [a] \rho \in F_0^\lambda$$

is well defined and goes to zero under

$$\begin{array}{ccc} F_0^\lambda & \longrightarrow & S^\lambda \\ [a] & \longrightarrow & \langle a \rangle. \end{array}$$

Welldefinedness follows from (4.1.2) and from taking right cosets with respect to $\mathcal{S}_\xi \times \mathcal{S}_\eta \leq \mathcal{S}_{\xi \cup \eta}$. To prove vanishing under $F_0^\lambda \longrightarrow S^\lambda$ we may drop the scalar factor.

$$\begin{array}{rcl} \sum_{\rho \in \mathcal{S}_{\xi \cup \eta}} \varepsilon_\rho \langle a \rangle \rho & = & \sum_{\rho \in \mathcal{S}_{\xi \cup \eta}} \sum_{\sigma \in C_a} \varepsilon_\rho \varepsilon_\sigma \{a\} \sigma \rho \\ & = & \sum_{\sigma \in C_a} \varepsilon_\sigma \sum_{\rho \in \mathcal{S}_{\xi \cup \eta}} \varepsilon_\rho \{a\} \sigma \rho. \end{array}$$

Assume that the m-th row of the tableau a intersects $\xi\sigma^{-1}$ and $\eta\sigma^{-1}$ nontrivially, i.e. assume that $\iota := (a\sigma)_{mj} = a_{mj}\sigma \in \xi$ and $\kappa := (a\sigma)_{mk} = a_{mk}\sigma \in \eta$. Such an m exists by the assumption on the sum of the sizes of ξ and η. Note that the transposition $(\iota\ \kappa)$ is in $S_{\xi \cup \eta}$. We continue to treat the inner sum.

$$\begin{array}{rcl} \sum_{\rho \in \mathcal{S}_{\xi \cup \eta}} \varepsilon_\rho \{a\} \sigma \rho & = & \sum_{\rho \in \mathcal{S}_{\xi \cup \eta},\ \iota\rho < \kappa\rho} \left(\varepsilon_\rho \{a\sigma\} \rho + \varepsilon_{(\iota\ \kappa)\rho} \{a\sigma\} (\iota\ \kappa) \rho \right) \\ & = & \sum_{\rho \in \mathcal{S}_{\xi \cup \eta},\ \iota\rho < \kappa\rho} \left(\varepsilon_\rho \{a\sigma\} \rho + \varepsilon_{(\iota\ \kappa)\rho} \{a\sigma\} \rho \right) \\ & = & 0. \end{array}$$

4.2 Hooks

The case of hook partitions, corresponding to the easiest nontrivial Specht lattices, plays the role of a test case, in which, however, it is still possible to work by direct standard-polytabloid-methods.

Let $\lambda^k := (n-k+1, 1^{k-1})$ be a hook, $k \in [1,n]$.

Note that a λ^k-polytabloid is determined by its first column (4.1.4). Since the standard λ^k-polytabloids, which form a $\mathbf{Z}$-basis of S^λ, have an entry $a_{11} = 1$ in the (upper left) corner, we restrict to the consideration of polytabloids with $a_{11} = 1$, arbitrary otherwise, being standard polytabloids up to sign.

By abuse of notation, we denote for a tuple $b = (b_2, \ldots, b_{\lambda_i})$ with pairwise different entries out of $[2,n]$ by $\langle b \rangle$ the λ^k-polytabloid determined by $a_{11} = 1$ and by $a_{i1} = b_i$ for $i \in [2,k]$. I.e. the tuple b is the first column except for the corner. Writing down a λ^k-polytabloid $\langle b \rangle$ in this section implies that b is a tuple of the form just mentioned.

For a tuple c, $x \in c$, $y \notin c$, we let $c^{x,y}$ denote the tuple arising from c by substitution of x by y. We denote the tuple concatenated from c and the single element y by (c,y). We abbreviate $\langle (b,y) \rangle =: \langle b, y \rangle$.

4.2.1 A long exact sequence

Lemma 4.2.1 *Let $k \in [1, n-1]$. The rank of S^{λ^k} is given by*

$$n_{\lambda^k} = \binom{n-1}{k-1}.$$

Lemma 4.2.2 *Let $k \in [1, n-1]$. Let $\langle b \rangle$ be a λ^k-polytabloid, let $u \in [2,n] \backslash b$. We have*

$$\langle b \rangle (1\ u) = \langle b \rangle - \sum_{t \in b} \langle b^{t,u} \rangle.$$

The Garnir relation (4.1.4) reads

$$\begin{aligned} 0 &= \frac{1}{k!} \textstyle\sum_{\rho \in \mathcal{S}_{b \cup u}} \varepsilon_\rho \langle b \rangle \rho \\ &= \langle b \rangle - \langle b \rangle (1\ u) - \textstyle\sum_{t \in b} \langle b^{t,u} \rangle. \end{aligned}$$

This allows to permute the 1 back to the corner at the cost of some other summands with the 1 in the corner.

The following proposition is a special case of (4.3.31).

Proposition 4.2.3 (the box shift morphism for hooks) *Let*

$$\begin{array}{ccc} S^{\lambda^k} & \xrightarrow{f_k} & S^{\lambda^{k+1}} \\ \langle b \rangle & \longrightarrow & \sum_{s \in [2,n] \backslash b} \langle b, s \rangle. \end{array}$$

This is a well defined $\mathbf{Z}$*-linear map, which induces a* $\mathbf{Z}\mathcal{S}_n$*-linear map*

$$S^{\lambda^k}/n \xrightarrow{\bar{f}_k} S^{\lambda^{k+1}}/n.$$

Welldefinedness merely means that for an arbitrary polytabloid $\langle b\rangle$, its image as given above and its image as derived from its presentation as a linear combination of standard polytabloids coincide. But the necessary permutation of the entries of b yields the same sign on both sides.

We **claim** that for $u \in [2,n]$ we have

$$(\langle b\rangle(1\ u))f_k - (\langle b\rangle f_k)(1\ u) \in nS^{\lambda^{k+1}}.$$

Case $u \in b$**.**

$$\begin{array}{rcl}(\langle b\rangle(1\ u))f_k - (\langle b\rangle f_k)(1\ u) & = & -\langle b\rangle f_k - \sum_{s\in[2,n]\backslash b}\langle b,s\rangle(1\ u) \\ & = & -\sum_{s\in[2,n]\backslash b}\langle b,s\rangle + \sum_{s\in[2,n]\backslash b}\langle b,s\rangle \\ & = & 0.\end{array}$$

Case $u \in [2,n]\backslash b$**.**

$$\begin{array}{rcl}(\langle b\rangle(1\ u))f_k - (\langle b\rangle f_k)(1\ u) & \overset{(4.2.2)}{=} & (\langle b\rangle - \sum_{t\in b}\langle b^{t,u}\rangle)f_k - (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle)(1\ u) \\ & \overset{(4.2.2)}{=} & (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle) - (\sum_{t\in b}\sum_{s\in[2,n]\backslash b^{t,u}}\langle b^{t,u},s\rangle) \\ & & -(\sum_{s\in[2,n]\backslash(b\cup u)}(\langle b,s\rangle - \sum_{t\in b}\langle b^{t,u},s\rangle - \langle b,u\rangle)) + \langle b,u\rangle \\ & = & (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle) - (\sum_{t\in b}\sum_{s\in[2,n]\backslash(b\cup u)}\langle b^{t,u},s\rangle) - (\sum_{t\in b}\langle b^{t,u},t\rangle) \\ & & -(\sum_{s\in[2,n]\backslash(b\cup u)}(\langle b,s\rangle - \sum_{t\in b}\langle b^{t,u},s\rangle - \langle b,u\rangle)) + \langle b,u\rangle \\ & = & (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle) + (\sum_{t\in b}\langle b,u\rangle) \\ & & -(\sum_{s\in[2,n]\backslash(b\cup u)}(\langle b,s\rangle - \langle b,u\rangle)) + \langle b,u\rangle \\ & = & (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle) \\ & & -(\sum_{s\in[2,n]\backslash(b\cup u)}\langle b,s\rangle) + (1 + (k-1) + ((n-1)-k))\langle b,u\rangle \\ & = & (1 + 1 + (k-1) + ((n-1)-k))\langle b,u\rangle \\ & = & n\langle b,u\rangle.\end{array}$$

Proposition 4.2.4 (long exact hook sequence, cf. [P 71, Lemma 2])
The sequence of maps

$$0 \longrightarrow S^{\lambda^1} \xrightarrow{f_1} S^{\lambda^2} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} S^{\lambda^n} \longrightarrow 0$$

is exact.

We **claim** that $f_k f_{k+1} = 0$ for $k \in [1, n-2]$. Let $\langle b\rangle \in S^{\lambda^k}$.

$$\begin{array}{rcl}\langle b\rangle f_k f_{k+1} & = & (\sum_{s\in[2,n]\backslash b}\langle b,s\rangle)f_{k+1} \\ & = & \sum_{s\in[2,n]\backslash b}\sum_{t\in[2,n]\backslash(b\cup s)}\langle b,s,t\rangle \\ & = & \sum_{s,t\in[2,n]\backslash b,\ s\neq t}\langle b,s,t\rangle \\ & = & \sum_{s,t\in[2,n]\backslash b,\ s>t}(\langle b,s,t\rangle + \langle b,t,s\rangle) \\ & = & 0.\end{array}$$

We claim that the image of f_k is a pure submodule of $S^{\lambda^{k+1}}$ of rank $\binom{n-2}{k-1}$. More precisely, we **claim** that the set of polytabloids $\langle b\rangle$ with b being a strictly increasing tuple with entries only in $[2, n-1]$ is sent to a basis of the image.

For to see that the image of this set generates the image we consider a tuple $c = (c_2, \dots, c_{k-1})$ with entries taken from $[2, n-1]$ and use the equation already shown in the last step, viz.

$$\sum_{s \in [2,n] \setminus c} (\langle c, s \rangle f_k) = 0.$$

For to see that the image of this set is linearly independent and spans a pure sublattice we write the image of such a polytabloid $\langle b \rangle$ as

$$\langle b \rangle f_k = \langle b, n \rangle + \sum_{s \in [2,n] \setminus (b \cup n)} \langle b, s \rangle.$$

Now the rank of the image of f_k, $\binom{n-2}{k-1}$, equals the rank of the kernel of f_{k+1}, $\binom{n-1}{k} - \binom{n-2}{k}$ (4.2.1).

In particular, the rank of the image of f_1, $\binom{n-2}{0}$, equals the rank of S^{λ^1} and the rank of the image of f_{n-1}, $\binom{n-2}{n-2}$, equals the rank of S^{λ^n}.

Remark 4.2.5 I do not know whether there is there a 'homological reason' for the occurrence of the long exact sequence in (4.2.4). Cf. also [J 78, 24.1].

Corollary 4.2.6 *The subring*

$$\Lambda := \{\rho \mid \rho^{\lambda^k} f_k \equiv_n f_k \rho^{\lambda^{k+1}} \text{ for } k \in [1, n-1]\} \subseteq \prod_\lambda (\mathbf{Z})_{n_\lambda} =: \Gamma$$

described by the generic modular morphism exhibited in (4.2.3) has index

$$|\Gamma / \Lambda| = n^{1/2 \sum_{k \in [1,n]} \binom{n-1}{k-1}^2}$$

as abelian groups and contains the image of the embedding of $\mathbf{Z}S_n$ into Γ via the operations on the Specht lattices.

Writing down a $\mathbf{Z}$-linear basis for Λ respecting the kernels of the long exact hook sequence (4.2.4) shows that the index of the quasiblock of Λ which is contained in $(\mathbf{Z})_{n_{\lambda^k}}$ is

$$n^{\binom{n-2}{k-2}\binom{n-2}{k-1}}.$$

Moreover, the contribution to the index caused by ties between the quasiblock of Λ in $(\mathbf{Z})_{n_{\lambda^k}}$ and the quasiblock of Λ in $(\mathbf{Z})_{n_{\lambda^{k+1}}}$ is

$$n^{\binom{n-2}{k-1}^2}.$$

We calculate the exponent of n for the index of Λ in Γ to be

$$\begin{aligned}
\textstyle\sum_{k\in[1,n]} \binom{n-2}{k-2}\binom{n-2}{k-1} + \sum_{k\in[1,n-1]} \binom{n-2}{k-1}^2 &= \textstyle\sum_{k\in[1,n]} \binom{n-2}{k-2}\binom{n-2}{k-1} + \sum_{k\in[1,n]} \binom{n-2}{k-1}^2 \\
&= \textstyle 1/2\sum_{k\in[1,n]} \binom{n-2}{k-2}\binom{n-2}{k-1} + 1/2\sum_{k\in[1,n]} \binom{n-2}{k-1}^2 \\
&\quad \textstyle + 1/2\sum_{k\in[1,n]} \binom{n-2}{k-2}\binom{n-2}{k-1} + 1/2\sum_{k\in[1,n]} \binom{n-2}{k-2}^2 \\
&= \textstyle 1/2\sum_{k\in[1,n]} (\binom{n-2}{k-2} + \binom{n-2}{k-1}) \binom{n-2}{k-1} \\
&\quad \textstyle + 1/2\sum_{k\in[1,n]} \binom{n-2}{k-2} (\binom{n-2}{k-1} + \binom{n-2}{k-2}) \\
&= \textstyle 1/2\sum_{k\in[1,n]} \binom{n-1}{k-1}\binom{n-2}{k-1} \\
&\quad \textstyle + 1/2\sum_{k\in[1,n]} \binom{n-2}{k-2}\binom{n-1}{k-1} \\
&= \textstyle 1/2\sum_{k\in[1,n]} \binom{n-1}{k-1}^2.
\end{aligned}$$

Cf. (S 2.2.2).

Lemma 4.2.7 *For $n =: p$ prime, we obtain*

$$(n_\lambda)_p = \begin{cases} 1 & \textit{if } \lambda \textit{ is a hook} \\ p & \textit{else.} \end{cases}$$

This follows by the hook formula (4.1.1).

Corollary 4.2.8 (an easy case) *Keep the notation of (4.2.6). Let p be a prime. Then*

$$(\mathbf{Z}\mathcal{S}_p)_{[p]} = \Lambda.$$

Since Λ is a (p)-order (D.2.8), it suffices to show that the indices coincide. The index of $(\mathbf{Z}\mathcal{S}_p)_{[p]}$ in Γ is

$$\begin{aligned} \left(\sqrt{\frac{p!^{p!}}{\prod_\lambda n_\lambda^{n_\lambda^2}}}\right)_p &= \sqrt{\frac{p^{p!}}{\prod_{\lambda \text{ nonhook}} p^{n_\lambda^2}}} \\ &= p^{1/2 \sum_{k\in[1,n]} n_{\lambda k}^2} \\ &= p^{1/2 \sum_{k\in[1,n]} \binom{n-1}{k-1}^2}. \end{aligned}$$

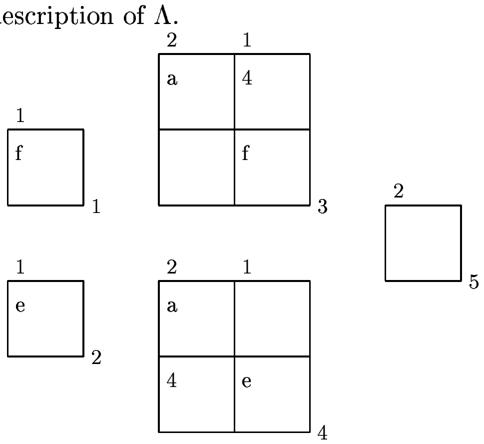
(1.1.4, 4.2.1, 4.2.7), cf. (4.2.6).

Remark 4.2.9 (4.2.8) implies in particular the well known fact that the principal block of the $\mathbf{F}_p\mathcal{S}_p$ is a Brauer tree algebra whose Brauer tree is a line.

Example 4.2.10 Consider $(\mathbf{Z}\mathcal{S}_4)_{[2]}$. Keep the notation of (S 2.1.1, 4.2.6). Note that we worked with right lattices, i.e. with rows in Γ. Choosing bases such that the long exact sequence of f_k's reads, shrunk blockwise, as

$$0 \longrightarrow S^{(4)} \xrightarrow{(0\,0\,1)} S^{(3,1)} \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}} S^{(2,1,1)} \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix}} S^{(1,1,1,1)} \longrightarrow 0,$$

we obtain the following description of Λ.



e	x^2	$\equiv_4$	x^4
a	x^4	$\equiv_4$	x^3
f	x^3	$\equiv_4$	x^1

This is not quite in accordance with (S 2.1.1) since we used rationally isomorphic lattices instead of the Specht modules there. To remedy, one could conjugate the quasiblock 3 here with, shrunk blockwise, $\begin{pmatrix}1 & \\ & 4\end{pmatrix}$ from the right. For full accordance one should check that the morphisms implicitely appearing there conincide with the morphisms used here, which we omit, since this would require an examination of the representing matrices in our guiding examples.

4.2.2 At the prime 2

The propositions in this subsection are **not** special cases of (4.3.31). The technical reason why they work only at the prime 2 is the Garnir relation (4.2.2), used to evaluate $\langle b\rangle(1\ u) = +\langle b\rangle - \sum_{v\in b}\langle b^{v,u}\rangle$ in case $u \notin b$ in contrast to $\langle b\rangle(1\ u) = -\langle b\rangle$ in case $u \in b$, cases which may be collected together again modulo 2.

The recipe to find these somehow exceptional morphisms is to 'genericalize' examples. I doubt that there is a uniform method, let alone a single generic morphism covering all the series exhibited at the prime 2 so far.

'Exceptional' refers to the fact that for $p \geq 3$ prime, a nonzero morphism from S^λ/p to S^μ/p implies that λ dominates μ, i.e. $\sum_{i\in[1,j]}\lambda_i \geq \sum_{i\in[1,j]}\mu_i$ for all $j \in [1,n]$ [J 78, 13.17]. This does not hold for 'one half' of the specializations of the following generic morphisms.

Proposition 4.2.11 (the constant sum morphism) *Let $k, l \in [1,n]$ such that $n-l$ and $k-1$ are even. Let the following* **Z***-linear map of rank* 1 *be defined on the* **standard** *polytabloids $\langle b\rangle$.*

$$\begin{array}{ccc} S^{\lambda^k} & \xrightarrow{s} & S^{\lambda^l} \\ \langle b\rangle & \longrightarrow & \sum_{\langle c\rangle \text{ standard}}\langle c\rangle. \end{array}$$

This map factors over a $\mathbf{Z}S_n$-linear morphism

$$S^{\lambda^k}/2 \xrightarrow{\bar{s}} S^{\lambda^l}/2.$$

Note that modulo 2 the formula for the image $\langle b\rangle s$ holds for all admissible tuples b.

We **claim** that for $u \in [2,n]$ and for $\langle b\rangle$ standard we have

$$(\langle b\rangle(1\ u))s - (\langle b\rangle s)(1\ u) \in 2S^{\lambda^l}.$$

Case $u \in b$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))s - (\langle b\rangle s)(1\ u) & = & -\sum_{\langle c\rangle \text{ st.}}\langle c\rangle - (\sum_{\langle c\rangle \text{ st.}}\langle c\rangle)(1\ u) \\ & = & -\sum_{\langle c\rangle \text{ st.}}\langle c\rangle + \sum_{\langle c\rangle \text{ st.},\, u\in c}\langle c\rangle - \sum_{\langle c\rangle \text{ st.},\, u\in[2,n]\backslash c}(\langle c\rangle - \sum_{t\in c}\langle c^{t,u}\rangle) \\ & \equiv_2 & \sum_{\langle c\rangle \text{ st.},\, u\in[2,n]\backslash c}\sum_{t\in c}\langle c^{t,u}\rangle. \end{array}$$

The number of times a standard polytabloid with an entry u in the first column appears in this sum (coefficient ± 1) equals the number of possible replaced entries, which is given by the even number $n-l$.

Case $u \in [2,n]\backslash b$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))s - (\langle b\rangle s)(1\ u) & = & (\langle b\rangle - \sum_{t\in b}\langle b^{t,u}\rangle)s - \sum_{\langle c\rangle \text{ st.}}\langle c\rangle(1\ u) \\ & \equiv_2 & (1-(k-1))\sum_{\langle c\rangle \text{ st.}}\langle c\rangle + \sum_{\langle c\rangle \text{ st.},\, u\in c}\langle c\rangle \\ & & -\sum_{\langle c\rangle \text{ st.},\, u\in[2,n]\backslash c}(\langle c\rangle - \sum_{t\in c}\langle c^{t,u}\rangle) \\ & \equiv_2 & (k-1)\sum_{\langle c\rangle \text{ st.}}\langle c\rangle - \sum_{\langle c\rangle \text{ st.},\, u\in[2,n]\backslash c}\sum_{t\in c}\langle c^{t,u}\rangle, \end{array}$$

which vanishes modulo 2 since $k-1$ as well as $n-l$ are even (cf. the case $u \in b$).

Example 4.2.12 For $n = 5$ we may let $k = 3$ and $l = 3$ in (4.2.11). This leads to $x^7_{11} \equiv_2 x^7_{33}$ implied by tie t in (S 2.2.4) as well as to the single quasiblock ties modulo 2 in quasiblock 7, since, shrunken blockwise, we may write the constant sum morphism as $\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \end{pmatrix}$. Cf. also (S 0.6). This example, calculated directly by computer first, convinced me of the relevance of modular morphisms.

Proposition 4.2.13 (the transposition morphism) *Let $k \in [1,n]$ be even. Let the following* **Z***-linear map of rank* 1 *be defined on the* **standard** *polytabloids $\langle b\rangle$, where we denote c to be the increasingly ordered tuple in the first row of $\langle b\rangle$, corner excepted, i.e. $c = [2,n]\backslash b$ as sets.*

$$\begin{array}{ccc} S^{\lambda^k} & \xrightarrow{z} & S^{(\lambda^k)'} \\ \langle b\rangle & \longrightarrow & \sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle . \end{array}$$

This map factors over a $\mathbf{Z}\mathcal{S}_n$*-linear morphism*

$$S^{\lambda^k}/2 \xrightarrow{\bar{z}} S^{(\lambda^k)'}/2.$$

Note that modulo 2 the formula for the image $\langle b\rangle z$ holds for all polytabloids consisting of a 1 in the corner, remaining first column b and remaining first row c, regardless whether b or c is ordered increasingly.

We **claim** that for $u \in [2,n]$ and for $\langle b\rangle$ standard we have

$$(\langle b\rangle(1\ u))z - (\langle b\rangle z)(1\ u) \in 2S^{\lambda^l}.$$

Case $u \in b$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))z - (\langle b\rangle z)(1\ u) & = & -(\sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle) - (\sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle(1\ u)) \\ & = & -\sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle \\ & & -\sum_{s\in c}\sum_{t\in b\backslash u}\Big(\langle c^{s,t}\rangle - (\sum_{v\in c\backslash s}\langle (c^{s,t})^{v,u}\rangle) - \langle (c^{s,t})^{t,u}\rangle\Big) \\ & & +\sum_{s\in c}\langle c^{s,u}\rangle \\ & \equiv_2 & \sum_{s\in c}\sum_{t\in b\backslash u}\Big((\sum_{v\in c\backslash s}\langle (c^{s,t})^{v,u}\rangle) + \langle (c^{s,t})^{t,u}\rangle\Big) \\ & = & (\sum_{s,v\in c,\, s\neq v}\sum_{t\in b\backslash u}\langle (c^{s,t})^{v,u}\rangle) + (k-2)(\sum_{s\in c}\langle c^{s,u}\rangle) \\ & = & (\sum_{s,v\in c,\, s>v}\sum_{t\in b\backslash u}(\langle (c^{s,t})^{v,u}\rangle + \langle (c^{v,t})^{s,u}\rangle)) + (k-2)(\sum_{s\in c}\langle c^{s,u}\rangle) \\ & = & (k-2)(\sum_{s\in c}\langle c^{s,u}\rangle). \end{array}$$

Case $u \in c$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))z - (\langle b\rangle z)(1\ u) & = & (\langle b\rangle - \sum_{v\in b}\langle b^{v,u}\rangle)z - (\sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle(1\ u)) \\ & \equiv_2 & \sum_{s\in c}\sum_{t\in b}\langle c^{s,t}\rangle \\ & & -\sum_{v\in b}\sum_{s\in c^{u,v}}\sum_{t\in b^{v,u}}\langle (c^{u,v})^{s,t}\rangle \\ & & +\sum_{s\in c\backslash u}\sum_{t\in b}\langle c^{s,t}\rangle \\ & & -\sum_{t\in b}\Big(\langle c^{u,t}\rangle - (\sum_{v\in c\backslash u}\langle (c^{u,t})^{v,u}\rangle) - \langle (c^{u,t})^{t,u}\rangle\Big) \\ & \equiv_2 & \sum_{v\in b}\sum_{s\in c^{u,v}}\sum_{t\in b^{v,u}}\langle (c^{u,v})^{s,t}\rangle \\ & & +\sum_{t\in b}((\sum_{v\in c\backslash u}\langle c^{v,t}\rangle) + \langle c\rangle) \\ & = & \sum_{v\in b}\sum_{s\in c\backslash u}\sum_{t\in b\backslash v}\langle (c^{u,v})^{s,t}\rangle \\ & & +\sum_{v\in b}\sum_{t\in b\backslash v}\langle (c^{u,v})^{v,t}\rangle \\ & & +\sum_{v\in b}\sum_{s\in c\backslash u}\langle (c^{u,v})^{s,u}\rangle \\ & & +\sum_{v\in b}\langle (c^{u,v})^{v,u}\rangle \\ & & +\sum_{t\in b}\sum_{v\in c\backslash u}\langle c^{v,t}\rangle \\ & & +(k-1)\langle c\rangle \end{array}$$

$$\begin{array}{rl}
= & \sum_{v,t\in b,\, v\neq t}\sum_{s\in c\backslash u}\langle (c^{u,v})^{s,t}\rangle \\
& +\sum_{v,t\in b,\, v\neq t}\langle c^{u,t}\rangle \\
& -\sum_{v\in b}\sum_{s\in c\backslash u}\langle c^{s,v}\rangle \\
& +(k-1)\langle c\rangle \\
& +\sum_{t\in b}\sum_{v\in c\backslash u}\langle c^{v,t}\rangle \\
& +(k-1)\langle c\rangle \\
\equiv_2 & \sum_{v,t\in b,\, v>t}\sum_{s\in c\backslash u}(\langle (c^{u,v})^{s,t}\rangle + \langle (c^{u,t})^{s,v}\rangle) \\
& +(k-2)\sum_{t\in b}\langle c^{u,t}\rangle \\
= & (k-2)\sum_{t\in b}\langle c^{u,t}\rangle .
\end{array}$$

Proposition 4.2.14 (the two box shift morphism) *Let $k\in[1,n-2]$. Assume n to be odd. Let the following* $\mathbf{Z}$*-linear map be defined on the* **standard** *polytabloids $\langle b\rangle$.*

$$\begin{array}{ccl}
S^{\lambda^k} & \xrightarrow{g} & S^{\lambda^{k+2}} \\
\langle b\rangle & \longrightarrow & \sum_{s,t\in[2,n]\backslash b,\, s<t}\langle b,s,t\rangle .
\end{array}$$

This map induces a morphism

$$S^{\lambda^k}/2 \xrightarrow{\bar g} S^{\lambda^{k+2}}/2.$$

Note that the formula for the image $\langle b\rangle g$ holds for all for all admissible tuples b.

We **claim** that for $u\in[2,n]$ and for $\langle b\rangle$ standard we have

$$(\langle b\rangle(1\ u))g - (\langle b\rangle g)(1\ u) \in 2S^{\lambda^{k+2}}.$$

Case $u\in b$.

$$\begin{array}{rcl}
(\langle b\rangle(1\ u))g - (\langle b\rangle g)(1\ u) & = & -\sum_{s,t\in[2,n]\backslash b,\, s<t}\langle b,s,t\rangle + \sum_{s,t\in[2,n]\backslash b,\, s<t}\langle b,s,t\rangle \\
& = & 0.
\end{array}$$

Case $u\notin b$.

$$\begin{array}{rl}
(\langle b\rangle(1\ u))g - (\langle b\rangle g)(1\ u) \;= & \sum_{s,t\in[2,n]\backslash b,\, s<t}\langle b,s,t\rangle \\
& -\sum_{w\in b}\sum_{s,t\in[2,n]\backslash b^{w,u},\, s<t}\langle b^{w,u},s,t\rangle \\
& -\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}(\langle b,s,t\rangle - (\sum_{v\in b}\langle b^{v,u},s,t\rangle) - \langle b,u,t\rangle - \langle b,s,u\rangle) \\
& +\sum_{t\in[u+1,n]\backslash b}\langle b,u,t\rangle \\
& +\sum_{s\in[2,u-1]\backslash b}\langle b,s,u\rangle \\
\equiv_2 & \sum_{w\in b}\sum_{s,t\in[2,n]\backslash b^{w,u},\, s<t}\langle b^{w,u},s,t\rangle \\
& +\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}((\sum_{v\in b}\langle b^{v,u},s,t\rangle) + \langle b,u,t\rangle + \langle b,s,u\rangle) \\
= & \sum_{w\in b}\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}\langle b^{w,u},s,t\rangle \\
& +\sum_{w\in b}\sum_{t\in[w+1,n]\backslash(b\cup u)}\langle b^{w,u},w,t\rangle \\
& +\sum_{w\in b}\sum_{s\in[2,w-1]\backslash(b\cup u)}\langle b^{w,u},s,w\rangle \\
& +\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}\sum_{v\in b}\langle b^{v,u},s,t\rangle \\
& +\sum_{t\in[2,n]\backslash(b\cup u)}\#([2,t-1]\backslash(b\cup u))\langle b,u,t\rangle \\
& +\sum_{s\in[2,n]\backslash(b\cup u)}\#([s+1,n]\backslash(b\cup u))\langle b,s,u\rangle \\
\equiv_2 & \sum_{w\in b}\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}\langle b^{w,u},s,t\rangle \\
& +\sum_{w\in b}\sum_{t\in[w+1,n]\backslash(b\cup u)}\langle b,u,t\rangle \\
& +\sum_{w\in b}\sum_{s\in[2,w-1]\backslash(b\cup u)}\langle b,s,u\rangle \\
& +\sum_{v\in b}\sum_{s,t\in[2,n]\backslash(b\cup u),\, s<t}\langle b^{v,u},s,t\rangle \\
& +\sum_{t\in[2,n]\backslash(b\cup u)}(\#([2,t-1]\backslash(b\cup u)) + \#([t+1,n]\backslash(b\cup u)))\langle b,u,t\rangle \\
\equiv_2 & (k-1)\sum_{t\in[2,n]\backslash(b\cup u)}\langle b,u,t\rangle \\
& +\sum_{t\in[2,n]\backslash(b\cup u)}(n-2-k)\langle b,u,t\rangle \\
= & (n-3)\sum_{t\in[2,n]\backslash(b\cup u)}\langle b,u,t\rangle .
\end{array}$$

Proposition 4.2.15 (the two box cancellation morphism) *Let $k \in [3, n]$. Let the following $\mathbf{Z}$-linear map be defined on the* **standard** *polytabloids $\langle b\rangle$.*

$$\begin{array}{ccl} S^{\lambda^k} & \xrightarrow{h} & S^{\lambda^{k-2}} \\ \langle b\rangle & \longrightarrow & \sum_{s,t\in b,\, s<t}\langle b\backslash\{s,t\}\rangle, \end{array}$$

where the latter expression is to be read as e.g. $(2,3,4,5)\backslash\{2,4\} = (3,5)$.

This map induces a morphism

$$S^{\lambda^k}/2 \xrightarrow{\bar{h}} S^{\lambda^{k-2}}/2.$$

Note that the formula for the image $\langle b\rangle g$ holds modulo 2 for all admissible tuples b.

We **claim** that for $u \in [2, n]$ and for $\langle b\rangle$ standard we have

$$(\langle b\rangle(1\ u))h - (\langle b\rangle h)(1\ u) \in 2S^{\lambda^{k-2}}.$$

Case $u \in b$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))h - (\langle b\rangle h)(1\ u) & = & -\sum_{s,t\in b,\, s<t}\langle b\backslash\{s,t\}\rangle \\ & & +\sum_{s,t\in b\backslash u,\, s<t}\langle b\backslash\{s,t\}\rangle \\ & & +\sum_{s\in[2,u-1]\cap b}\Big(\langle b\backslash\{s,u\}\rangle - \sum_{v\in b\backslash\{s,u\}}\langle(b\backslash\{s,u\})^{v,u}\rangle\Big) \\ & & +\sum_{t\in[u+1,n]\cap b}\Big(\langle b\backslash\{u,t\}\rangle - \sum_{v\in b\backslash\{u,t\}}\langle(b\backslash\{u,t\})^{v,u}\rangle\Big) \\ & \equiv_2 & \sum_{s\in[2,u-1]\cap b}\sum_{v\in b\backslash\{s,u\}}\langle(b\backslash\{s,u\})^{v,u}\rangle \\ & & +\sum_{t\in[u+1,n]\cap b}\sum_{v\in b\backslash\{u,t\}}\langle(b\backslash\{u,t\})^{v,u}\rangle \\ & \equiv_2 & \sum_{s\in[2,u-1]\cap b}\sum_{v\in b\backslash\{s,u\}}\langle b\backslash\{s,v\}\rangle \\ & & +\sum_{t\in[u+1,n]\cap b}\sum_{v\in b\backslash\{u,t\}}\langle b\backslash\{v,t\}\rangle \\ & = & \sum_{s,v\in b\backslash u,\, s\neq v}\langle b\backslash\{s,v\}\rangle \\ & = & 2\sum_{s,v\in b\backslash u,\, s<v}\langle b\backslash\{s,v\}\rangle. \end{array}$$

Case $u \notin b$**.**

$$\begin{array}{rcl} (\langle b\rangle(1\ u))h - (\langle b\rangle h)(1\ u) & = & \sum_{s,t\in b,\, s<t}\langle b\backslash\{s,t\}\rangle \\ & & -\sum_{v\in b}\sum_{s,t\in b^{v,u},\, s<t}\langle b^{v,u}\backslash\{s,t\}\rangle \\ & & -\sum_{s,t\in b,\, s<t}\Big(\langle b\backslash\{s,t\}\rangle - \sum_{w\in b\backslash\{s,t\}}\langle(b\backslash\{s,t\})^{w,u}\rangle\Big) \\ & = & -\sum_{v\in b}\sum_{s,t\in b^{v,u},\, s<t}\langle b^{v,u}\backslash\{s,t\}\rangle \\ & & +\sum_{s,t\in b,\, s<t}\sum_{w\in b\backslash\{s,t\}}\langle(b\backslash\{s,t\})^{w,u}\rangle \\ & = & -\sum_{v\in b}\sum_{s,t\in b\backslash v,\, s<t}\langle b^{v,u}\backslash\{s,t\}\rangle \\ & & -\sum_{v\in b}\sum_{s\in([2,u-1]\backslash v)\cap b}\langle b^{v,u}\backslash\{s,u\}\rangle \\ & & -\sum_{v\in b}\sum_{t\in([u+1,n]\backslash v)\cap b}\langle b^{v,u}\backslash\{u,t\}\rangle \\ & & +\sum_{s,t\in b,\, s<t}\sum_{w\in b\backslash\{s,t\}}\langle b^{w,u}\backslash\{s,t\}\rangle \\ & = & -\sum_{s,t\in b,\, s<t}\sum_{v\in b\backslash\{s,t\}}\langle b^{v,u}\backslash\{s,t\}\rangle \\ & & -\sum_{v\in b}\sum_{s\in([2,u-1]\backslash v)\cap b}\langle b\backslash\{s,v\}\rangle \\ & & -\sum_{v\in b}\sum_{t\in([u+1,n]\backslash v)\cap b}\langle b\backslash\{v,t\}\rangle \\ & & +\sum_{s,t\in b,\, s<t}\sum_{w\in b\backslash\{s,t\}}\langle b^{w,u}\backslash\{s,t\}\rangle \\ & = & -\sum_{v\in b}\sum_{s\in b\backslash v}\langle b\backslash\{s,v\}\rangle \\ & = & -2\sum_{v,s\in b, v<s}\langle b\backslash\{s,v\}\rangle. \end{array}$$

Remark 4.2.16 In the notation of (4.2.11, 4.2.13, 4.2.14, 4.2.15), neither the constant sum morphism $S^{\lambda^k}/2 \xrightarrow{\bar{s}} S^{\lambda^l}/2$ nor the transposition morphism $S^{\lambda^k}/2 \xrightarrow{\bar{z}} S^{(\lambda^k)'}/2$ nor the two box shift morphism $S^{\lambda^k}/2 \xrightarrow{\bar{g}} S^{\lambda^{k+2}}/2$ nor the two box cancellation morphism $S^{\lambda^k}/2 \xrightarrow{\bar{h}} S^{\lambda^{k-2}}/2$ vanish, since in each case there exists an element in the image with a nonvanishing coefficient when displayed as linear combination of standard polytabloids.

In the following examples (4.2.17, 4.2.18, 4.2.19) we regard some ties resulting from specializations of the generic morphisms exhibited so far.

Example 4.2.17 (the Kronecker quiver appears) Let $n = 6$. We have the box shift morphism (4.2.3)

$$\begin{array}{ccc} S^{(4,1,1)}/2 & \xrightarrow{\bar{f}} & S^{(3,1,1,1)}/2 \\ \left\langle\begin{smallmatrix}2\\3\end{smallmatrix}\right\rangle & \longrightarrow & \left\langle\begin{smallmatrix}2\\3\\4\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\3\\5\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\3\\6\end{smallmatrix}\right\rangle \end{array}$$

given by the matrix

$$\begin{bmatrix} 1&1&1&0&0&0&0&0&0&0\\ 1&0&0&1&1&0&0&0&0&0\\ 0&1&0&1&0&1&0&0&0&0\\ 0&0&1&0&1&1&0&0&0&0\\ 1&0&0&0&0&0&1&1&0&0\\ 0&1&0&0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0&1&1&0\\ 0&0&0&1&0&0&1&0&0&1\\ 0&0&0&0&1&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&1&1 \end{bmatrix}$$

with entries in $\mathbf{F}_2$, where the standard polytabloids are ordered by their first column lexicographically and the matrix operates on the right on the row vectors representing elements of the Specht modules.

We dispose of the constant sum morphism (4.2.11)

$$\begin{array}{ccl} S^{(4,1,1)}/2 & \xrightarrow{\bar{s}} & S^{(3,1,1,1)}/2 \\ \left\langle\begin{smallmatrix}2\\3\end{smallmatrix}\right\rangle & \longrightarrow & \left\langle\begin{smallmatrix}2\\3\\4\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\3\\5\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\3\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\4\\5\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\4\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}2\\5\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}3\\4\\5\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}3\\4\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}3\\5\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}4\\5\\6\end{smallmatrix}\right\rangle \end{array}$$

given by the matrix

$$\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1 \end{bmatrix}.$$

Searching simultaneous bases amounts to search for a normal form for the corresponding module over the Kronecker quiver. Whereas the situation generalizes to a generic one, I do not know the generic answer.

In the other direction, we dispose of the transposition morphism (4.2.13)

$$\begin{array}{ccl} S^{(3,1,1,1)}/2 & \xrightarrow{\bar{z}} & S^{(4,1,1)}/2 \\ \left\langle\begin{smallmatrix}2\\3\\4\end{smallmatrix}\right\rangle & \longrightarrow & \left\langle\begin{smallmatrix}2\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}3\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}4\\6\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}5\\2\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}5\\3\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}5\\4\end{smallmatrix}\right\rangle \end{array}$$

given by the matrix

$$\begin{bmatrix} 0&0&1&1&0&1&1&1&1&0\\ 0&1&0&1&1&0&1&1&0&1\\ 0&1&1&0&1&1&0&0&1&1\\ 1&0&0&1&1&1&0&0&1&1\\ 1&0&1&0&1&0&1&1&0&1\\ 1&1&0&0&0&1&1&1&1&0\\ 1&1&1&0&0&0&1&0&1&1\\ 1&1&0&1&0&1&0&1&0&1\\ 1&0&1&1&1&0&0&1&1&0\\ 0&1&1&1&1&1&1&0&0&0 \end{bmatrix}.$$

Denote $X := S^{(4,1,1)}/2$ and $Y := S^{(3,1,1,1)}/2$. We note that

$$\cdots \longrightarrow X \xrightarrow{\bar{f}} Y \xrightarrow{\bar{z}} X \xrightarrow{\bar{f}} Y \longrightarrow \cdots$$

is a periodic exact sequence. Let

$$X' := \mathbf{F}_2\langle \left\langle\begin{smallmatrix}3\\4\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}3\\5\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}3\\6\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}4\\5\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}4\\6\end{smallmatrix}\right\rangle, \left\langle\begin{smallmatrix}5\\6\end{smallmatrix}\right\rangle \rangle \subseteq X$$

i.e. let X' correspond to the last six rows of the matrix of $\bar{f}$ given above. X' is a complement to the kernel of $\bar{f}$. Let Y' be a complement of $X'\bar{f}$ in Y. We obtain, identifying X' and $X'\bar{f}$ via $\bar{f}$ and Y' and the kernel of $\bar{f}$ via $\bar{z}$, the block matrices

$$
\begin{aligned}
(X \xrightarrow{\bar{f}} Y) &= (Y' \oplus X' \xrightarrow{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)} Y' \oplus X') \\
(X \xrightarrow{\bar{s}} Y) &= (Y' \oplus X' \xrightarrow{\left(\begin{smallmatrix}0&0\\0&\tilde{s}\end{smallmatrix}\right)} Y' \oplus X') \\
(Y \xrightarrow{\bar{z}} X) &= (Y' \oplus X' \xrightarrow{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)} Y' \oplus X').
\end{aligned}
$$

Note that in the chosen basis for X' above and the image of this chosen basis - the last six rows of the matrix for $\bar{f}$ given above - $\tilde{s}$ has the matrix

$$
\begin{bmatrix}
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1\\
1&1&1&1&1&1
\end{bmatrix}.
$$

Conjugation by

$$
\begin{bmatrix}
1&0&0&0&0&0\\
1&1&0&0&0&0\\
1&0&1&0&0&0\\
1&0&0&1&0&0\\
1&0&0&0&1&0\\
1&1&1&1&1&1
\end{bmatrix}
$$

from the left turns this matrix into

$$
\begin{bmatrix}
0&0&0&0&0&1\\
0&0&0&0&0&0\\
0&0&0&0&0&0\\
0&0&0&0&0&0\\
0&0&0&0&0&0\\
0&0&0&0&0&0
\end{bmatrix}.
$$

Thus, shrinking blockwise and using the basis for X' corresponding to this conjugation, we obtain

$$
\begin{aligned}
\bar{f} &= \begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \\
\bar{s} &= \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \\
\bar{z} &= \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}
\end{aligned}
$$

in which the blocks have sizes 4, 1, 4 and 1 from left to right resp. from top to bottom. We choose a corresponding integral basis of the Specht lattices (A.2.1). Hence $\bar{f}$, $\bar{s}$ and $\bar{z}$ yield the two-quasiblock-ties, where, as in (S 2.3.5), the quasiblock number 9 resp. 10 belongs to $(3, 1, 1, 1)$ resp. $(4, 1, 1)$, and which we give in the redundant manner in which they result from the morphisms,

4	1	4	1
c			
2	a	a	a
2	a	a	a
2	a 2	a 2	a b

9

4	1	4	1
c	2	2	2
	a b	a	a
	a 2	a	a
	a 2	a	a

10

$$
\begin{aligned}
&\mathrm{a} \quad x^9 \equiv_2 x^{10} \\
&\mathrm{b} \quad x^9 \equiv_2 x^{10} \\
&\mathrm{c} \quad x^9 \equiv_2 x^{10},
\end{aligned}
$$

which one may compare to (S 2.3.5). NB this shows the possibility of the existence of quasiblock ties that involve more than one entry and that are **not** induced by a modular **endo**morphism of the corresponding Specht lattice – we check directly via a computer calculation that there are no nontrivial endomorphisms modulo 2.

A generic extension of this example is unknown so far.

Example 4.2.18 We use matrices as in (4.2.17).

In case $n = 2k - 1$, we regard the transposition endomorphism

$$S^{\lambda_k}/2 \xrightarrow{\bar{z}} S^{\lambda_k}/2$$

of (4.2.13).

For $k = 3$, $n = 5$, we obtain $1 = \bar{z} + \bar{s}$, so that the arising quasiblock ties are known from $\bar{s}$ already (4.2.12).

For $k = 4$, $n = 7$, we let $X := S^{(4,1,1,1)}/2$. $\bar{z}$ has as matrix

$$\begin{bmatrix}
0&1&1&1&1&1&1&0&0&0&1&1&1&0&0&0&0&0&0&0\\
1&0&1&1&1&0&0&1&1&0&1&0&0&1&1&0&0&0&0&0\\
1&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0\\
1&1&1&0&0&0&1&0&1&1&0&0&1&0&1&1&0&0&0&0\\
1&1&0&0&0&1&1&1&1&0&1&0&0&0&0&0&1&1&0&0\\
1&0&1&0&1&0&1&1&0&1&0&1&0&0&0&0&1&0&1&0\\
1&0&0&1&1&1&0&0&1&1&0&0&1&0&0&0&0&1&1&0\\
0&1&1&0&1&1&0&0&1&1&0&0&0&1&0&0&1&0&0&1\\
0&1&0&1&1&0&1&1&0&1&0&0&0&0&1&0&0&1&0&1\\
0&0&1&1&0&1&1&1&1&0&0&0&0&0&0&1&0&0&1&1\\
1&1&0&0&1&0&0&0&0&0&0&1&1&1&1&0&1&1&0&0\\
1&0&1&0&0&1&0&0&0&0&1&0&1&1&0&1&1&0&1&0\\
1&0&0&1&0&0&1&0&0&0&1&1&0&0&1&1&0&1&1&0\\
0&1&1&0&0&0&0&1&0&0&1&1&0&0&1&1&1&0&0&1\\
0&1&0&1&0&0&0&0&1&0&1&0&1&1&0&1&0&1&0&1\\
0&0&1&1&0&0&0&0&0&1&0&1&1&1&1&0&0&0&1&1\\
0&0&0&0&1&1&0&1&0&0&1&1&0&1&0&0&0&1&1&1\\
0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&1\\
0&0&0&0&0&1&1&0&0&1&0&1&1&0&0&1&1&1&0&1\\
0&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&1&1&1&0
\end{bmatrix} .$$

We obtain $\bar{z}^2 = 1$, thus $(1 + \bar{z})^2 = 0$. Let K be the kernel of $(1 + \bar{z})$, which has dimension 14, let I be its image, which thus has dimension 6 and is contained in K. Let C be a complement of K in X, let D be a complement of I in K, so that, identifying C with I via $(1 + \bar{z})$, we obtain

$$(X \xrightarrow{1+\bar{z}} X) = (C \oplus D \oplus C \xrightarrow{\left(\begin{smallmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{smallmatrix}\right)} C \oplus D \oplus C).$$

Cf. (S F.7).

A generic extension of this example is unknown so far.

Example 4.2.19 (a decomposition) We use matrices as in (4.2.17).

Let $n = 7$. We have the constant sum morphisms (4.2.11)

$$\begin{array}{rcl}
S^{(3,1,1,1,1)}/2 & \xrightarrow{\bar{s}_1} & S^{(5,1,1)}/2 \\
S^{(5,1,1)}/2 & \xrightarrow{\bar{s}_2} & S^{(3,1,1,1,1)}/2 \\
S^{(3,1,1,1,1)}/2 & \xrightarrow{\bar{s}_3} & S^{(3,1,1,1,1)}/2 \\
S^{(5,1,1)}/2 & \xrightarrow{\bar{s}_4} & S^{(5,1,1)}/2,
\end{array}$$

all of them given by matrices whose entries all equal 1. In particular, $\bar{s}_1\bar{s}_2 = \bar{s}_3$, $\bar{s}_2\bar{s}_1 = \bar{s}_4$.

We dispose of the two box cancellation morphism (4.2.15)

$$\begin{array}{ccl}
S^{(3,1,1,1,1)}/2 & \xrightarrow{\bar{h}} & S^{(5,1,1)}/2 \\
\left\langle\begin{smallmatrix} 2\\3\\4\\5 \end{smallmatrix}\right\rangle & \longrightarrow & \left\langle\begin{smallmatrix} 2\\3 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 2\\4 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 2\\5 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 3\\4 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 3\\5 \end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix} 4\\5 \end{smallmatrix}\right\rangle
\end{array}$$

given by the matrix

$$\begin{bmatrix} 1&1&1&0&0&1&1&0&0&1&0&0&0&0&0\\ 1&1&0&1&0&1&0&1&0&0&1&0&0&0&0\\ 1&1&0&0&1&1&0&0&1&0&0&1&0&0&0\\ 1&0&1&1&0&0&1&1&0&0&0&0&1&0&0\\ 1&0&1&0&1&0&1&0&1&0&0&0&0&1&0\\ 1&0&0&1&1&0&0&1&1&0&0&0&0&0&1\\ 0&1&1&1&0&0&0&0&0&1&1&0&1&0&0\\ 0&1&1&0&1&0&0&0&0&1&0&1&0&1&0\\ 0&1&0&1&1&0&0&0&0&0&1&1&0&0&1\\ 0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1&0&1&1&0&1&0&0\\ 0&0&0&0&0&1&1&0&1&1&0&1&0&1&0\\ 0&0&0&0&0&1&0&1&1&0&1&1&0&0&1\\ 0&0&0&0&0&0&1&1&1&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1 \end{bmatrix}.$$

We dispose of the two box shift morphism (4.2.14)

$$\begin{array}{ccc} S^{(5,1,1)}/2 & \xrightarrow{\bar{g}} & S^{(3,1,1,1,1)}/2 \\ \left\langle \begin{smallmatrix} 2\\3 \end{smallmatrix} \right\rangle & \longrightarrow & \left\langle \begin{smallmatrix} 2\\3\\4\\5 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} 2\\3\\4\\6 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} 2\\3\\4\\7 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} 2\\3\\5\\6 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} 2\\3\\5\\7 \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} 2\\3\\6\\7 \end{smallmatrix} \right\rangle \end{array}$$

given by the transpose of the matrix of $\bar{h}$.

Denote $X := S^{(3,1,1,1,1)}/2$ and $Y := S^{(5,1,1)}/2$. $\bar{s}_3$ is an idempotent of X, $\bar{s}_4$ is an idempotent of Y. We decompose $X = I_3 \oplus K_3$ into the the image I_3 and kernel K_3 of $\bar{s}_3 = \left(\begin{smallmatrix} 1&0\\0&0 \end{smallmatrix}\right)$ and $Y = I_4 \oplus K_4$ into the image I_4 and the kernel K_4 of $\bar{s}_4 = \left(\begin{smallmatrix} 1&0\\0&0 \end{smallmatrix}\right)$.

Note that

$$\begin{array}{rcl} (X \xrightarrow{\bar{h}} Y) & = & (I_3 \oplus K_3 \xrightarrow{\left(\begin{smallmatrix} 0&0\\0&\alpha \end{smallmatrix}\right)} I_4 \oplus K_4) \\ (Y \xrightarrow{\bar{g}} X) & = & (I_4 \oplus K_4 \xrightarrow{\left(\begin{smallmatrix} 0&0\\0&\beta \end{smallmatrix}\right)} I_3 \oplus K_3) \end{array}$$

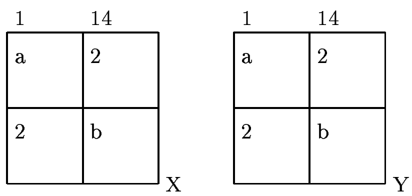
since $\bar{s}_3\bar{h} = 0$, $\bar{h}\bar{s}_4 = 0$, $\bar{s}_4\bar{g} = 0$ and $\bar{g}\bar{s}_3 = 0$. Now $\bar{h}\bar{g} = 1 - \bar{s}_3$ and $\bar{g}\bar{h} = 1 - \bar{s}_4$ yield $\alpha\beta = 1$ and $\beta\alpha = 1$.

Moreover,

$$\begin{array}{rcl} (X \xrightarrow{\bar{s}_1} Y) & = & (I_3 \oplus K_3 \xrightarrow{\left(\begin{smallmatrix} \gamma&0\\0&0 \end{smallmatrix}\right)} I_4 \oplus K_4) \\ (Y \xrightarrow{\bar{s}_2} X) & = & (I_4 \oplus K_4 \xrightarrow{\left(\begin{smallmatrix} \delta&0\\0&0 \end{smallmatrix}\right)} I_3 \oplus K_3) \end{array}$$

since $(1 - \bar{s}_3)\bar{s}_1 = 0$, $\bar{s}_1(1 - \bar{s}_4) = 0$, $(1 - \bar{s}_4)\bar{s}_2 = 0$ and $\bar{s}_2(1 - \bar{s}_3) = 0$. Now $\bar{s}_1\bar{s}_2 = \bar{s}_3$ and $\bar{s}_2\bar{s}_1 = \bar{s}_4$ yield $\gamma\delta = 1$ and $\delta\gamma = 1$.

Identifying K_3 and K_4 via α and identifying I_3 and I_4 via γ we therefore obtain the ties



$$\begin{array}{llll} \text{a} & x & \equiv_2 & y \\ \text{b} & x & \equiv_2 & y, \end{array}$$

the notation being selfexplanatory.

Cf. [J 78, 23.10.iii], (S F.4). A generic extension of this example is unknown so far.

4.3 The one-box-shift morphism

We exhibit a generic modular morphism between Specht lattices attached to a combinatorially related pair of partitions (4.3.31). More specifically, the second partition arises from the first by a downwards shift of one box. The formula for the map is given by a linear combination of polytabloids with coefficients polynomial in the combinatorial data, the occurring polytabloids arising from the one which is to be mapped by shifting an entry stepwise from the column of the first box position to the second.

The reader might wish to regard the 'sufficiently large' example in (S 4.3.5) beforehand.

4.3.1 Preparation

Notation 4.3.1 Let λ be a partition of the natural number n. Let $z := \lambda_1$. Let $g, k \in [1, z]$, $g \leq k$, such that ($g = 1$ or $\lambda'_{g-1} > \lambda'_g$) and ($\lambda'_{k+1} > \lambda'_{k+2}$), in particular, $k + 1 \leq z$. In other words, we require

$$\mu'_i := \begin{cases} \lambda'_i + 1 & \text{for} \quad i = g \\ \lambda'_i - 1 & \text{for} \quad i = k+1 \\ \lambda'_i & \text{else} \end{cases}$$

to define a partition μ. Pictorially, we move a box from the bottom of the $(k+1)$-th column to the bottom of the g-th column in order to pass from λ to μ ([1]).

Note that the free $\mathbf{Z}S_n$-module on one generator can be realized as having as basis the λ-tableaux, equipped with the natural operation, thus called F^λ (cf. 4.1.1). We will define $\mathbf{Z}S_n$-morphisms from F^λ to S^μ as follows. Let

$$\langle \; a_1 \; \ldots \; a_z \; \rangle$$

denote the μ-polytabloid generated by the tableau

$$[\; a_1 \; \ldots \; a_z \;],$$

where the a_i denote μ'_i-tuples giving the columns of the μ-tableau a. The entry in the i-th column and the j-th row is thus denoted by $a_{i,j}$. Note that this is a **change of notation** compared to (4.1.1), necessary in order to handle columns.

Let e denote a function

$$\begin{array}{rcl} [g+1, k] & \stackrel{e}{\longrightarrow} & \{0, 1\} \\ j & \longrightarrow & e_j . \end{array}$$

For example, in case $g = 2$, $k = 5$ we write $e = 101$ for $e_3 = 1, e_4 = 0, e_5 = 1$. Let, accordingly, i denote the strictly monotone function

$$\begin{array}{rcl} [1, l] & \stackrel{i}{\longrightarrow} & [g+1, k] \\ j & \longrightarrow & i_j \end{array}$$

such that $l = \#e^{-1}(1)$, $j \in i_{[1,l]} \iff e_j = 1$, i.e. such that e is the characteristic function of i. Note that l may be zero. Let $i_0 := g$, $i_{l+1} := k + 1$, and, consequently, $e_g := 1$,

[1] C. Ringel suggested to generalize from $\lambda'_{k+1} = 1$ to λ'_{k+1} arbitrary, as presented here.

$e_{k+1} := 1$. Furthermore, let $e_j = 0$ for $j \notin [g, k+1]$. To continue our example, we obtain $e_1 = 0$, $e_2 = 1$, $e_3 = 1, e_4 = 0, e_5 = 1$, $e_6 = 1$ and $e_j = 0$ for $j \geq 7$.

Occasionally, we allow ourselves to treat single elements and tuples of elements as sets, as long as the notation remains unambiguous.

Recall that we write $c^{s,t}$ for the tuple c with its entry s replaced by t. Denote by $c^{s,}$ the tuple c with its entry s removed and the following entries shifted by one to close the gap. (This shift 'causes' the sign in the formula below.) Let, for the tableau a and for $x \in a$, more precisely, $x \in a_j$, $\pi(x)$ be its position in a_j minus one, i.e. $a_{j,\pi(x)+1} = x$.

Let f_e be the $\mathbf{Z}S_n$-morphism defined by

$$\begin{array}{ccc} F^\lambda & \xrightarrow{f_e} & S^\mu \\ [a_1 \dots a_z] & \longrightarrow & \displaystyle\sum_{x_1 \in a_{i_1}, \dots, x_{l+1} \in a_{i_{l+1}}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \dots a_g \dots a_{i_1}^{x_1,x_2} \dots a_{i_2}^{x_2,x_3} \dots a_{i_l}^{x_l,x_{l+1}} \dots a_{k+1}^{x_{l+1},} \dots \\ x_1 \end{matrix} \right\rangle \end{array}$$

We also write $x_j \in a_{i_j}$ as short for $x_1 \in a_{i_1}, \dots, x_{l+1} \in a_{i_{l+1}}$ in such formulas to indicate this multiple sum. $F^\lambda \xrightarrow{f_e} S^\mu$ in fact is a well defined $\mathbf{Z}\mathcal{S}_n$-linear map since its defining operations on the tableau entries depend only on their positions.

For a λ-tableau a, we denote

$$\sum_{\sigma \in S_\zeta} \langle\ a_1\ \dots\ a_z\ \rangle\, \sigma\, \varepsilon_\sigma =: \langle\ a_1\ \dots\ a_z\ \rangle \circ \zeta$$

where $\zeta \subseteq a$, and where S_ζ denotes the symmetric group on the elements of ζ, i.e. $S_\zeta := C_{S_n}([1,n] \backslash \zeta)$. '$\circ$' is to be read 'moved in'. Analoguously for tableaux, the sum being formed inside F^λ.

For a tuple $(s_1, s_2, \dots)$, we denote

$$(s_1, \dots, \hat{s}_{i_1}, \dots, \hat{s}_{i_2}, \dots) := (s_1, \dots, s_{i_1-1}, s_{i_1+1}, \dots, s_{i_2-1}, s_{i_2+1}, \dots).$$

In this section, the variable p is used as an index, which is not a prime number in general.

Lemma 4.3.2 *The kernel of*

$$\begin{array}{ccc} F^\lambda & \longrightarrow & S^\lambda \\ [a_1\ \dots\ a_z] & \longrightarrow & \langle a_1\ \dots\ a_z \rangle \end{array}$$

is generated over $\mathbf{Z}S_n$ *by the* **one-step Garnir relations** *and by the* **signed column transpositions**.

One-step Garnir relations *are elements of the form*

$$G_{a,\xi,\eta} := \sum_{\sigma \in S_\xi \times S_\eta \backslash S_{\xi \cup \eta}} [a_1\ \dots\ a_j\ a_{j+1}\ \dots\ a_z]\, \sigma \varepsilon_\sigma,$$

where $j \in [1,k]$, $\xi \subseteq a_j$, $\eta \subseteq a_{j+1}$, $\xi + \eta = \#a_j + 1$.

Signed column transpositions *are elements of the form*

$$[a_1\ \dots\ a_j\ \dots\ a_z] + [a_1\ \dots\ a_j\ \dots\ a_z](s\ t),$$

where $s, t \in a_j$, $s \neq t$.

Modulo signed column transpositions a one-step Garnir relation can be written more conveniently as

$$G_{a,\xi,\eta} = \frac{1}{\#\xi!\#\eta!}\,[a_1\ \ldots\ a_j\ a_{j+1}\ \ldots\ a_z] \circ (\xi \cup \eta).$$

The usual proof - usually phrased in terms of S^λ - that the standard tableaux generate $\mathbf{Z}$-linearly the module (F^λ modulo one-step Garnir relations and modulo signed column transpositions) suffices to prove the lemma, since therefore (F^λ modulo these relations), mapping onto S^λ (4.1.4), has the same rank over $\mathbf{Z}$. We will give this usual proof, working already in $F_0^\lambda :=$ (F^λ modulo signed column transpositions, i.e. modulo signed column permutations).

We order the tableaux such that for two tableaux the largest entry which lies in different columns decides their ordering in such a way that in the smaller tableau this element lies in a larger numbered column, i.e. further to the right. This leaves us with a total order in the canonical $\mathbf{Z}$-basis of F_0^λ, over which we perform an induction. The smallest tableau is standard (up to column permutation).

Assume, after ordering the columns of the tableau a increasingly from top to bottom, that a is not standard because the entry c in the j-th column a_j is larger than the entry d in the $(j+1)$-th column a_{j+1}, where both c and d lie in the t-th row. Let $\xi := (a_{j,t}, \ldots, a_{j,\lambda'_j})$, let $\eta := (a_{j+1,1}, \ldots, a_{j+1,t})$. Note $a_{j+1,1} < \cdots < a_{j+1,t} = d < c = a_{j,t} < \cdots < a_{j,\lambda'_j}$. All summands occurring in $G_{a,\xi,\eta}$ not equal to a modulo column permutation are smaller than a, since the largest entry being moved out of its column is moved to the right, for it cannot be a member of η. a itself occurs in $G_{a,\xi,\eta}$ with coefficient 1.

Lemma 4.3.3 *The signed column transpositions vanish under f_e (4.3.1).*

Let $s, t \in a_p$, $s \neq t$. We have to consider the case $e_p = 1$ only.

Case $p \in [g+1, k]$, i.e. $p = i_u$, $u \in [1, l]$ (the calculation is valid also for $u = 1$).

$$\begin{aligned}
&\sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \cdots & a_g & \cdots & a_{i_{u-1}}^{x_{u-1},x_u} & \cdots & a_{i_u}^{x_u,x_{u+1}} & \cdots \\ & x_1 & & & & & \end{matrix} \right\rangle (s\ t) \\
= &- \sum_{x_j \in a_{i_j},\ s \neq x_u,\ t \neq x_u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},x_u}\ \cdots\ a_{i_u}^{x_u,x_{u+1}}\ \cdots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},t}\ \cdots\ (a_{i_u}^{s,x_{u+1}})^{t,s}\ \cdots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},s}\ \cdots\ (a_{i_u}^{t,x_{u+1}})^{s,t}\ \cdots \right\rangle \\
= &- \sum_{x_j \in a_{i_j},\ s \neq x_u,\ t \neq x_u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},x_u}\ \cdots\ a_{i_u}^{x_u,x_{u+1}}\ \cdots \right\rangle \\
&- \sum_{x_j \in a_{i_j},\ j \neq u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},t}\ \cdots\ a_{i_u}^{t,x_{u+1}}\ \cdots \right\rangle \\
&- \sum_{x_j \in a_{i_j},\ j \neq u} (-1)^{\pi(x_{l+1})} \left\langle \cdots\ a_{i_{u-1}}^{x_{u-1},s}\ \cdots\ a_{i_u}^{s,x_{u+1}}\ \cdots \right\rangle \\
= &- \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \cdots & a_g & \cdots & a_{i_{u-1}}^{x_{u-1},x_u} & \cdots & a_{i_u}^{x_u,x_{u+1}} & \cdots \\ & x_1 & & & & & \end{matrix} \right\rangle .
\end{aligned}$$

Case $p = k + 1 = i_{l+1}$ (the calculation is valid also for $l = 0$).

$$\begin{aligned}
&\sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \dots & a_g & \dots & a_{i_l}^{x_l,x_{l+1}} & \dots & a_{i_{l+1}}^{x_{l+1},} & \dots \\ & x_1 & & & & & \end{matrix} \right\rangle (st) \\
= &- \sum_{x_j \in a_{i_j},\ s \neq x_{l+1},\ t \neq x_{l+1}} (-1)^{\pi(x_{l+1})} \left\langle \dots \ a_{i_l}^{x_l,x_{l+1}} \ \dots \ a_{i_{l+1}}^{x_{l+1},} \ \dots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq l+1} (-1)^{\pi(s)} \left\langle \dots \ a_{i_l}^{x_l,t} \ \dots \ (a_{i_{l+1}}^{s,})^{t,s} \ \dots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq l+1} (-1)^{\pi(t)} \left\langle \dots \ a_{i_l}^{x_l,s} \ \dots \ (a_{i_{l+1}}^{t,})^{s,t} \ \dots \right\rangle \\
= &- \sum_{x_j \in a_{i_j},\ s \neq x_{l+1},\ t \neq x_{l+1}} (-1)^{\pi(x_{l+1})} \left\langle \dots \ a_{i_l}^{x_l,x_{l+1}} \ \dots \ a_{i_{l+1}}^{x_{l+1},} \ \dots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq l+1} (-1)^{\pi(s)+(\pi(t)-\pi(s)+1)} \left\langle \dots \ a_{i_l}^{x_l,t} \ \dots \ a_{i_{l+1}}^{t,} \ \dots \right\rangle \\
&+ \sum_{x_j \in a_{i_j},\ j \neq l+1} (-1)^{\pi(t)+(\pi(s)-\pi(t)+1)} \left\langle \dots \ a_{i_l}^{x_l,s} \ \dots \ a_{i_{l+1}}^{s,} \ \dots \right\rangle \\
= &- \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \dots & a_g & \dots & a_{i_l}^{x_l,x_{l+1}} & \dots & a_{i_{l+1}}^{x_{l+1},} & \dots \\ & x_1 & & & & & \end{matrix} \right\rangle .
\end{aligned}$$

Case $p = g$. Nothing to do.

If the moved subset is too small for Garnir by one box, we still have the

Lemma 4.3.4 *Consider the μ-polytabloid*

$$\langle \ a_1 \ \dots \ a_z \ \rangle.$$

Let $\xi \subset a_p$, $\emptyset \neq \eta \subseteq a_q$, $p < q$, such that $\#\xi + \#\eta = \#a_p$. Choosing $x \in a_p \backslash \xi$ and $y \in \eta$ we obtain

$$\begin{aligned}
&\langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \circ (\xi \cup \eta) \\
= \frac{\#\eta}{\#\xi + 1} &\langle \ \dots \ a_p^{x,y} \ \dots \ a_q^{y,x} \ \dots \ \rangle \circ (\xi \cup \eta).
\end{aligned}$$

The Garnir relations in S^μ (4.1.4) give

$$\begin{aligned}
0 &= \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \circ (\xi \cup x \cup \eta) \\
&= \sum_{w \in \xi \cup x \cup \eta} \ \sum_{\sigma \in S_{\xi \cup x \cup \eta},\ w\sigma = x} \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \, \sigma \, \varepsilon_\sigma \\
&= \sum_{w \in \xi} \ \sum_{\sigma \in S_{\xi \cup x \cup \eta},\ w\sigma = x} \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \, \sigma \, \varepsilon_\sigma \\
&+ \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \circ (\xi \cup \eta) \\
&+ \sum_{w \in \eta \backslash y} \ \sum_{\sigma \in S_{\xi \cup x \cup \eta},\ w\sigma = x} \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \, \sigma \, \varepsilon_\sigma \\
&+ \sum_{\sigma \in S_{\xi \cup x \cup \eta},\ y\sigma = x} \langle \ \dots \ a_p \ \dots \ a_q \ \dots \ \rangle \, \sigma \, \varepsilon_\sigma
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{w\in\xi}\sum_{\sigma'\in S_{\xi\cup\eta}} \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle\,\sigma'\,\varepsilon_{\sigma'} && \text{(via } \sigma' = (wx)\sigma) \\
&+\ \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta) \\
&+ \sum_{w\in\eta\backslash y}\ \sum_{\sigma'\in S_{\xi\cup x\cup\eta},\ y\sigma'=x} \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle\,\sigma'\,\varepsilon_{\sigma'} && \text{(via } \sigma' = (wy)\sigma) \\
&+ \sum_{\sigma\in S_{\xi\cup x\cup\eta},\ y\sigma=x} \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle\,\sigma\,\varepsilon_{\sigma} \\
&= (\#\xi+1)\,\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta) \\
&+\ \#\eta \sum_{\sigma\in S_{\xi\cup x\cup\eta},\ y\sigma=x} \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle\,\sigma\,\varepsilon_{\sigma} \\
&= (\#\xi+1)\,\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta) \\
&-\ \#\eta \sum_{\sigma'\in S_{\xi\cup\eta}} \langle\ \dots\ a_p^{x,y}\ \dots\ a_q^{y,x}\ \dots\ \rangle\,\sigma'\,\varepsilon_{\sigma'} && \text{(via } \sigma' = (xy)\sigma)
\end{aligned}
$$

Blowing this up a bit, we obtain the

Lemma 4.3.5 *Consider the μ-polytabloid*

$$\langle\ a_1\ \dots\ a_z\ \rangle.$$

Let $\xi \subseteq a_p$, $\eta \subseteq a_q$, $p < q$, such that $\#\xi + \#\eta = \#a_p$. Denote $\bar\xi := a_p\backslash\xi$. Then

$$
\begin{aligned}
&\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta) \\
&= \#\xi!\#\eta!\,\langle\ \dots\ a_p^{\bar\xi,\eta}\ \dots\ a_q^{\eta,\bar\xi}\ \dots\ \rangle.
\end{aligned}
$$

where the tuple substition in the latter expression is to be understood as having fixed a bijection $\bar\xi \xrightarrow{\sim} \eta$ responsible for both substitutions. We will use this notation whenever using this lemma directly or indirectly.

$$
\begin{aligned}
&\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta) \\
\overset{(4.3.4)}{=}\ & \tfrac{\#\eta}{\#\xi+1}\,\langle\ \dots\ a_p^{x,y}\ \dots\ a_q^{y,x}\ \dots\ \rangle \circ (\xi\cup\eta) \\
\overset{(4.3.4)}{=}\ & \tfrac{\#\eta(\#\eta-1)}{(\#\xi+1)(\#\xi+2)}\,\langle\ \dots\ (a_p^{x,y})^{x',y'}\ \dots\ (a_q^{y,x})^{y',x'}\ \dots\ \rangle \circ (\xi\cup\eta) \\
\overset{(4.3.4)}{=}\ & \dots \\
\overset{(4.3.4)}{=}\ & \tfrac{\#\eta!\#\xi!}{\#a_p!}\,\langle\ \dots\ a_p^{\bar\xi,\eta}\ \dots\ a_q^{\eta,\bar\xi}\ \dots\ \rangle \circ (\xi\cup\eta) \\
=\ & \#\eta!\#\xi!\,\langle\ \dots\ a_p^{\bar\xi,\eta}\ \dots\ a_q^{\eta,\bar\xi}\ \dots\ \rangle.
\end{aligned}
$$

So that in case the moved subset would be large enough for Garnir, but is deformed by one box, we obtain the

Lemma 4.3.6 *Consider the μ-polytabloid*

$$\langle\ a_1\ \dots\ a_z\ \rangle.$$

Let $w \in a_r$, $\xi \subseteq a_p$, $\eta \subseteq a_q$, $r < p < q$, such that $\#\xi + \#\eta = \#a_p$. Denote $\bar\xi := a_p\backslash\xi$. Then

$$
\begin{aligned}
&\langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (w\cup\xi\cup\eta) \\
&= \binom{\#a_p}{\#\xi}^{-1}\,\langle\ \dots\ a_r\ \dots\ a_p^{\bar\xi,\eta}\ \dots\ a_q^{\eta,\bar\xi}\ \dots\ \rangle \circ (w\cup\xi\cup\eta).
\end{aligned}
$$

$$
\begin{array}{cl}
 & \langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (w \cup \xi \cup \eta) \\
= & \langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi \cup \eta) \\
- & \sum_{x\in\xi} \langle\ \dots\ a_r^{w,x}\ \dots\ a_p^{x,w}\ \dots\ a_q\ \dots\ \rangle \circ (w \cup (\xi\backslash x) \cup \eta) \\
- & \sum_{y\in\eta} \langle\ \dots\ a_r^{w,y}\ \dots\ a_p\ \dots\ a_q^{y,w}\ \dots\ \rangle \circ (w \cup \xi \cup (\eta\backslash y)) \\
\overset{(4.3.5)}{=} & \#\xi!\#\eta!\Big(\ \langle\ \dots\ a_r\ \dots\ a_p^{\bar{\xi},\eta}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle \\
- & \sum_{x\in\xi} \langle\ \dots\ a_r^{w,x}\ \dots\ (a_p^{x,w})^{\bar{\xi},\eta}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle \\
- & \sum_{y\in\eta} \langle\ \dots\ a_r^{w,y}\ \dots\ a_p^{\bar{\xi},\eta^{y,w}}\ \dots\ (a_q^{y,w})^{\eta^{y,w},\bar{\xi}}\ \dots\ \rangle\Big) \\
= & \#\xi!\#\eta!\Big(\ \langle\ \dots\ a_r\ \dots\ a_p^{\bar{\xi},\eta}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle \\
- & \sum_{x\in\xi} \langle\ \dots\ a_r^{w,x}\ \dots\ (a_p^{\bar{\xi},\eta})^{x,w}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle \\
- & \sum_{y\in\eta} \langle\ \dots\ a_r^{w,y}\ \dots\ (a_p^{\bar{\xi},\eta})^{y,w}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle\Big) \\
= & \dfrac{\#\xi!\#\eta!}{\#a_p!} \langle\ \dots\ a_r\ \dots\ a_p^{\bar{\xi},\eta}\ \dots\ a_q^{\eta,\bar{\xi}}\ \dots\ \rangle \circ (w \cup a_p^{\bar{\xi},\eta})
\end{array}
$$

In case we augment this subset by one box, Garnir applies again, so that we dispose of the

Lemma 4.3.7 *Consider the μ-polytabloid*

$$\langle\ a_1\ \dots\ a_z\ \rangle.$$

Let $w \in a_r$, $\xi \subseteq a_p$, $\eta \subseteq a_q$, $r < p < q$, such that $\#\xi + \#\eta = \#a_p + 1$. Denote $\bar{\xi} := a_p\backslash\xi$. Then

$$\langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (w \cup \xi \cup \eta) = 0.$$

$$
\begin{array}{cl}
 & \langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (w \cup \xi \cup \eta) \\
= & \langle\ \dots\ a_r\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi \cup \eta) \\
- & \sum_{x\in\xi} \langle\ \dots\ a_r^{w,x}\ \dots\ a_p^{x,w}\ \dots\ a_q\ \dots\ \rangle \circ (w \cup (\xi\backslash x) \cup \eta) \\
- & \sum_{y\in\eta} \langle\ \dots\ a_r^{w,y}\ \dots\ a_p\ \dots\ a_q^{y,w}\ \dots\ \rangle \circ (w \cup \xi \cup (\eta\backslash y)) \\
\overset{\text{Garnir, (4.1.4)}}{=} & 0 - \sum_{x\in\xi} 0 - \sum_{y\in\eta} 0.
\end{array}
$$

Finally, a reindexing of a signed sum yields the

Remark 4.3.8 *Consider the μ-polytabloid*

$$\langle\ a_1\ \dots\ a_z\ \rangle.$$

Let $\zeta \subseteq a$, $\sigma \in S_\zeta$. Then

$$
\begin{array}{l}
\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ \zeta \\
= \varepsilon_\sigma\ \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle\, \sigma \circ \zeta.
\end{array}
$$

We will make use of this fact in composition with column permutations **not** necessarily within ζ.

4.3.2 Strategy

Orientation 4.3.9
Our aim is to find a linear combination of the maps f_e as defined in (4.3.1)

$$F^\lambda \xrightarrow{f:=\sum_e u_e f_e} S^\mu,$$

$u_e \in \mathbf{Z}$, e running over the maps $[g+1,k] \xrightarrow{e} \{0,1\}$, which factors as

$$\begin{array}{ccc} F^\lambda & \xrightarrow{f} & S^\mu \\ \downarrow & & \downarrow \\ S^\lambda & \longrightarrow & S^\mu/m, \end{array}$$

where m denotes the **length of the path** covered by the shifted box, i.e.

$$m := 1 + (k+1-g) + (\lambda'_g - \lambda'_{k+1}).$$

This integer m, unfortunately, comes out of the calculation only, a priori there is no reason for it to occur.

Suppose given some f_e (4.3.1) and some one-step Garnir relation $G_{a,\xi,\eta}$ for a λ-tableau a and for $\xi \subseteq a_p$, $\eta \subseteq a_{p+1}$, $p \in [1, z-1]$, $\#\xi + \#\eta = \#a_p + 1$ (4.3.2).

We denote

$$\#_j := \begin{cases} \lambda'_j & = \#a_j \quad \text{for } j \neq g \\ \lambda'_j + 1 & = \#a_j + 1 \quad \text{for } j = g. \end{cases}$$

We have to calculate the images of the one-step Garnir relations under f_e (4.3.1) in case-by-case analysis, i.e. we have to evaluate the expression

$$G_{a,\xi,\eta} f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} \left\langle \begin{array}{ccccccccc} \cdots & a_g & \cdots & a_{i_1}^{x_1,x_2} & \cdots & a_{i_l}^{x_l,x_{l+1}} & \cdots & a_{k+1}^{x_{l+1},} & \cdots \\ & x_1 & & & & & & & \end{array} \right\rangle \circ (\xi \cup \eta)$$

in order to be able to exhibit coefficients u_e such that $\sum_e u_e G_{a,\xi,\eta} f_e \in mS^\mu$ (4.3.2).

We have to distinguish eleven cases.

(I) $e_p = 1$, $e_{p+1} = 1$. Let $i_\nu := p$.

- (i) $\nu = 0 \neq l$.
- (ia) $\nu = 0 = l$, i.e. $g = i_0$, $g+1 = k+1 = i_1$.
- (ii) $\nu \in [1, l-1]$.
- (iii) $\nu = l \neq 0$.

(II) $e_p = 1$, $e_{p+1} = 0$. Let $i_\nu := p$.

(i) $\nu = 0$.

(ii) $\nu \in [1, l]$.

(iii) $\nu = l + 1$.

(III) $e_p = 0$, $e_{p+1} = 1$. Let $i_\nu := p + 1$.

(i) $g \geq 2$, $\nu = 0$.

(ii) $\nu \in [1, l]$.

(iii) $\nu = l + 1$.

(IV) $e_p = 0$, $e_{p+1} = 0$.

We continue to denote $\bar{\xi} = a_p \backslash \xi$, $\bar{\eta} = a_{p+1} \backslash \eta$.

Notation 4.3.10
Suppose given $\xi \subseteq a_g$, $\eta \subseteq a_{g+1}$, $y \in \eta$, such that $\#\xi + \#\eta = \#_g$, and a strictly increasingly ordered tuple of t elements $(s_1, \ldots, s_t) \subseteq [g+2, k]$, possibly empty. Let $s_{t+1} := k + 1$. Let

$$A(s_1, \ldots, s_t)^y_{\xi,\eta} := \sum_{x_j \in a_{s_j},\, j \in [1,t+1]} (-1)^{\pi(x_{t+1})} \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{g+1}^{\eta\backslash y,\bar{\xi}})^{y,x_1} & \cdots & a_{s_1}^{x_1,x_2} & \cdots & a_{s_{t+1}}^{x_{t+1},} & \cdots \\ & y & & & & & & \end{matrix} \right\rangle .$$

Alternatively, we admit, with $[g+1, k] \xrightarrow{e} \{0, 1\}$ such that $e_j = 1 \iff j \in s_{[1,t]}$, i.e. with e being the characteristic function of s, the notation

$$A^y_{e,\xi,\eta} := A(s_1, \ldots, s_t)^y_{\xi,\eta}.$$

Furthermore, we let

$$A'^y_{\xi,\eta} := (-1)^{\pi(y)} \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{g+1}^{\eta\backslash y,\bar{\xi}})^{y,} & \cdots \\ & y & & \end{matrix} \right\rangle (\neq A()^y_{\xi,\eta}).$$

Notation 4.3.11
Suppose given $\xi \subseteq a_p$, $\eta \subseteq a_{p+1}$, $p \in [g+1, k]$, $y \in \eta$, such that $\#\xi + \#\eta = \#_p + 1$.

First, assume $p \in [g+1, k-1]$. Suppose given a strictly increasingly ordered tuple of t elements $(s_1, \ldots, s_t) \subseteq [g+1, k] \backslash \{p, p+1\}$, possibly empty. Let $s_{t+1} := k + 1$. Let

$$\begin{aligned} & B(s_1, \ldots, s_t)^y_{\xi,\eta} \\ := \ & \frac{1}{\#_p!} \sum_{x_j \in a_{s_j},\, j \in [1,t+1]} (-1)^{\pi(x_{t+1})} \left\langle \begin{matrix} \ldots a_g \ldots a_{s_{u-1}}^{x_{u-1},y} \ldots a_p^{\bar{\xi},\eta\backslash y} \, (a_{p+1}^{\eta\backslash y,\bar{\xi}})^{y,x_u} \ldots a_{s_u}^{x_u,x_{u+1}} \ldots a_{s_{t+1}}^{x_{t+1},} \ldots \\ x_1 \end{matrix} \right\rangle \circ (\xi \cup \eta). \end{aligned}$$

Second, assume $p = k$. Suppose given a strictly increasingly ordered tuple of t elements $(s_1, \ldots, s_t) \subseteq [g+1, k-1]$, possibly empty. Let

$$\begin{aligned} & B'(s_1, \ldots, s_t)^y_{\xi,\eta} \\ := \ & \frac{1}{\#_k!} \sum_{x_j \in a_{s_j},\, j \in [1,t]} (-1)^{\pi(y)} \left\langle \begin{matrix} \ldots a_g \ldots a_{s_1}^{x_1,x_2} \ldots a_{s_t}^{x_t,y} \ldots a_k^{\bar{\xi},\eta\backslash y} \, (a_{k+1}^{\eta\backslash y,\bar{\xi}})^{y,} \ldots \\ x_1 \end{matrix} \right\rangle \circ (\xi \cup \eta). \end{aligned}$$

Alternatively, we admit, with $[g+1, k] \xrightarrow{e} \{0, 1\}$ such that $e_j = 1 \iff j \in s_{[1,t]}$, i.e. with e being the characteristic function of s, the notation

$$\begin{aligned} B^y_{e,\xi,\eta} &:= B(s_1, \ldots, s_t)^y_{\xi,\eta} \\ B'^y_{e,\xi,\eta} &:= B'(s_1, \ldots, s_t)^y_{\xi,\eta}. \end{aligned}$$

Remark 4.3.12 The expressions in (4.3.10, 4.3.11) are in fact independent of the choice of $y \in \eta$. However, this will not play a role.

4.3.3 Calculations

We keep all previous notation, e.g. $\bar{\xi} = a_p \backslash \xi$, $\bar{\eta} = a_{p+1} \backslash \eta$.

Calculation 4.3.13 *We treat the* **case** (I.i), *i.e.* $\xi \subseteq a_g = a_{i_0}$, $\eta \subseteq a_{g+1} = a_{i_1}$, $l \neq 0$, $\#\xi + \#\eta = \#_g$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = (1 + \#_g - \#_{g+1}) A(i_2, \dots, i_l)^y_{\xi,\eta}.$$

In order to evaluate

$$G_{a,\xi,\eta} f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{x_1,x_2} & \cdots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta)$$

we distinguish two subcases for the occurring summands.

Subcase $x_1 \in \eta$.

$$\begin{aligned} & \sum_{x_1 \in \eta} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{x_1,x_2} & \cdots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\ \overset{(4.3.8),\, \sigma=(x_1\, y\, x_2)}{=} & \sum_{x_1 \in \eta} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{y,x_2} & \cdots \\ & y & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\ \overset{(4.3.5)}{=} & (\#\xi + 1)!(\#\eta - 1)!\#\eta \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{i_1}^{y,x_2})^{\eta\backslash y,\bar{\xi}} & \cdots \\ & y & & \end{matrix} \right\rangle \end{aligned}$$

Subcase $x_1 \in \bar{\eta}$.

$$\begin{aligned} & \sum_{x_1 \in \bar{\eta}} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{x_1,x_2} & \cdots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\ \overset{(4.3.5)}{=} & \#\xi!\#\eta! \sum_{x_1 \in \bar{\eta}} \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & ((a_{i_1}^{x_1,x_2})^{\eta\backslash y,\bar{\xi}})^{y,x_1} & \cdots \\ & y & & \end{matrix} \right\rangle \\ = & -\#\xi!\#\eta! \sum_{x_1 \in \bar{\eta}} \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{i_1}^{y,x_2})^{\eta\backslash y,\bar{\xi}} & \cdots \\ & y & & \end{matrix} \right\rangle \\ = & -\#\xi!\#\eta!(\#_{i_1} - \#_g + \#\xi) \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{i_1}^{y,x_2})^{\eta\backslash y,\bar{\xi}} & \cdots \\ & y & & \end{matrix} \right\rangle \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} & \frac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{x_1,x_2} & \cdots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\ = & (1 + \#_g - \#_{i_1}) \sum_{x_j \in a_{i_j},\, j \neq 1} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \cdots & a_g^{\bar{\xi},\eta\backslash y} & (a_{i_1}^{y,x_2})^{\eta\backslash y,\bar{\xi}} & \cdots \\ & y & & \end{matrix} \right\rangle \\ \overset{(4.3.10)}{=} & (1 + \#_g - \#_{i_1}) A(i_2, \dots, i_l)^y_{\xi,\eta}. \end{aligned}$$

Calculation 4.3.14 *We treat the* **case** (I.ia), *i.e.* $g = k$, $\xi \subseteq a_g = a_{i_0}$, $\eta \subseteq a_{g+1} = a_{i_1} = a_{k+1}$, $\#\xi + \#\eta = \#_g$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = (1 + \#_g - \#_{g+1}) A'^y_{\xi,\eta}.$$

In order to evaluate

$$G_{a,\xi,\eta} f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_1 \in a_{i_1}} (-1)^{\pi(x_1)} \left\langle \begin{matrix} \cdots & a_g & a_{i_1}^{x_1,} & \cdots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta)$$

we distinguish two subcases for the occurring summands.

Subcase $x_1 \in \eta$.

$$\begin{aligned}
&\sum_{x_1 \in \eta} (-1)^{\pi(x_1)} \left\langle \begin{matrix} \dots & a_g & a_{i_1}^{x_1,} & \dots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\,\sigma=(x_1\,y)}{=} &\sum_{x_1 \in \eta} (-1)^{\pi(y)} \left\langle \begin{matrix} \dots & a_g & a_{i_1}^{y,} & \dots \\ & y & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\
\overset{(4.3.5)}{=} &(-1)^{\pi(y)} (\#\xi + 1)!(\#\eta - 1)!\#\eta \left\langle \begin{matrix} \dots & a_g^{\bar{\xi},\eta\setminus y} & (a_{i_1}^{y,})^{\eta\setminus y,\bar{\xi}} & \dots \\ & y & & \end{matrix} \right\rangle
\end{aligned}$$

Subcase $x_1 \in \bar{\eta}$.

$$\begin{aligned}
&\sum_{x_1 \in \bar{\eta}} (-1)^{\pi(x_1)} \left\langle \begin{matrix} \dots & a_g & a_{i_1}^{x_1,} & \dots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\
\overset{(4.3.5)}{=} &\#\xi!\#\eta! \sum_{x_1 \in \bar{\eta}} (-1)^{\pi(x_1)} \left\langle \begin{matrix} \dots & a_g^{\bar{\xi},\eta\setminus y} & ((a_{i_1}^{x_1,})^{\eta\setminus y,\bar{\xi}})^{y,x_1} & \dots \\ & y & & \end{matrix} \right\rangle \\
= &-\#\xi!\#\eta! \sum_{x_1 \in \bar{\eta}} (-1)^{\pi(y)} \left\langle \begin{matrix} \dots & a_g^{\bar{\xi},\eta\setminus y} & (a_{i_1}^{y,})^{\eta\setminus y,\bar{\xi}} & \dots \\ & y & & \end{matrix} \right\rangle \\
= &-(-1)^{\pi(y)} \#\xi!\#\eta!(\#_{i_1} - \#_g + \#\xi) \left\langle \begin{matrix} \dots & a_g^{\bar{\xi},\eta\setminus y} & (a_{i_1}^{y,})^{\eta\setminus y,\bar{\xi}} & \dots \\ & y & & \end{matrix} \right\rangle
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
&\frac{1}{\#\xi!\#\eta!} \sum_{x_1 \in a_{i_1}} (-1)^{\pi(x_1)} \left\langle \begin{matrix} \dots & a_g & a_{i_1}^{x_1,} & \dots \\ & x_1 & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\
= &(1 + \#_g - \#_{i_1})(-1)^{\pi(y)} \left\langle \begin{matrix} \dots & a_g^{\bar{\xi},\eta\setminus y} & (a_{i_1}^{y,})^{\eta\setminus y,\bar{\xi}} & \dots \\ & y & & \end{matrix} \right\rangle \\
\overset{(4.3.10)}{=} &(1 + \#_g - \#_{i_1}) A'^{y}_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.15 *We treat the* **case** (I.ii), *i.e.* $\xi \subseteq a_p = a_{i_\nu}$, $\eta \subseteq a_{p+1} = a_{i_{\nu+1}}$, $\nu \in [1, l-1]$, $\#\xi + \#\eta = \#_{i_\nu} + 1$. *Choose* $x \in \xi$, $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = -(1 + \#_p - \#_{p+1}) B(i_1, \dots, \hat{i}_\nu, \hat{i}_{\nu+1}, \dots, i_l)^{y}_{\xi,\eta}.$$

In order to evaluate

$$G_{a,\xi,\eta} f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \ \dots \ a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \ \dots \ a_{i_\nu}^{x_\nu,x_{\nu+1}} \ a_{i_{\nu+1}}^{x_{\nu+1},x_{\nu+2}} \ \dots \ \rangle \circ (\xi \cup \eta)$$

we distinguish four subcases for the occurring summands.

The **subcases** $x_\nu \in \bar{\xi}$, $x_{\nu+1} \in \eta$ as well as $x_\nu \in \bar{\xi}$, $x_{\nu+1} \in \bar{\eta}$ yield zero summands by the Garnir relations (4.1.4).

Subcase $x_\nu \in \xi$, $x_{\nu+1} \in \eta$.

$$\begin{aligned}
&\sum_{x_\nu \in \xi} \sum_{x_{\nu+1} \in \eta} \langle \ \dots \ a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \ \dots \ a_{i_\nu}^{x_\nu,x_{\nu+1}} \ a_{i_{\nu+1}}^{x_{\nu+1},x_{\nu+2}} \ \dots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\,\sigma=(x_{\nu+1}\,y\,x_{\nu+2})(x_\nu\,x\,y)}{=} &\sum_{x_\nu \in \xi} \sum_{x_{\nu+1} \in \eta} \langle \ \dots \ a_{i_{\nu-1}}^{x_{\nu-1},x} \ \dots \ a_{i_\nu}^{x,y} \ a_{i_{\nu+1}}^{y,x_{\nu+2}} \ \dots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.6)}{=} &\#\xi\#\eta \binom{\#_{i_\nu}}{\#\xi}^{-1} \langle \ \dots \ a_{i_{\nu-1}}^{x_{\nu-1},x} \ \dots \ (a_{i_\nu}^{x,y})^{\bar{\xi},\eta\setminus y} \ (a_{i_{\nu+1}}^{y,x_{\nu+2}})^{\eta\setminus y,\bar{\xi}} \ \dots \ \rangle \circ (\xi \cup \eta),
\end{aligned}$$

where the (4.3.8)-step remains true, mutatis mutandis, for $x_\nu = x$ or $x_{\nu+1} = y$ or both. Similarly further down, without being explicitely mentioned.

Subcase $x_\nu \in \xi$, $x_{\nu+1} \in \bar\eta$.

$$\begin{aligned}
&\sum_{x_\nu\in\xi}\sum_{x_{\nu+1}\in\bar\eta} \langle \;\dots\; a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \;\dots\; a_{i_\nu}^{x_\nu,x_{\nu+1}} \; a_{i_{\nu+1}}^{x_{\nu+1},x_{\nu+2}} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.8),\,\sigma=(x_\nu\; x\; x_{\nu+1})}{=} &\sum_{x_\nu\in\xi}\sum_{x_{\nu+1}\in\bar\eta} \langle \;\dots\; a_{i_{\nu-1}}^{x_{\nu-1},x} \;\dots\; a_{i_\nu}^{x,x_{\nu+1}} \; a_{i_{\nu+1}}^{x_{\nu+1},x_{\nu+2}} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.6)}{=} &\binom{\#_{i_\nu}}{\#\xi-1}^{-1} \sum_{x_\nu\in\xi}\sum_{x_{\nu+1}\in\bar\eta} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},x} \dots ((a_{i_\nu}^{x,x_{\nu+1}})^{\bar\xi,\eta\backslash y})^{x_{\nu+1},y}\, ((a_{i_{\nu+1}}^{x_{\nu+1},x_{\nu+2}})^{\eta\backslash y,\bar\xi})^{y,x_{\nu+1}} \dots\rangle \circ (\xi\cup\eta) \\
= &-\#\xi\#\bar\eta \binom{\#_{i_\nu}}{\#\xi-1}^{-1} \langle \;\dots\; a_{i_{\nu-1}}^{x_{\nu-1},x} \;\dots\; (a_{i_\nu}^{x,y})^{\bar\xi,\eta\backslash y} \; (a_{i_{\nu+1}}^{y,x_{\nu+2}})^{\eta\backslash y,\bar\xi} \;\dots\;\rangle \circ (\xi\cup\eta)
\end{aligned}$$

Altogether, we obtain thus

$$\begin{aligned}
&G_{a,\xi,\eta}f_e \\
= &\frac{\#\xi-\#\bar\eta}{\#_{i_\nu}!} \sum_{x_j\in a_{i_j},\, j\neq\nu,\nu+1} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},x} \dots (a_{i_\nu}^{x,y})^{\bar\xi,\eta\backslash y}\, (a_{i_{\nu+1}}^{y,x_{\nu+2}})^{\eta\backslash y,\bar\xi} \dots\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.8),\,\sigma=(x\;y)}{=} &-\frac{1+\#_{i_\nu}-\#_{i_{\nu+1}}}{\#_{i_\nu}!} \sum_{x_j\in a_{i_j},\, j\neq\nu,\nu+1} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},y} \dots a_{i_\nu}^{\bar\xi,\eta\backslash y}\, (a_{i_{\nu+1}}^{y,x_{\nu+2}})^{\eta\backslash y,\bar\xi} \dots\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.11)}{=} &-(1+\#_{i_\nu}-\#_{i_{\nu+1}}) B(i_1,\dots,\hat{i}_\nu,\hat{i}_{\nu+1},\dots,i_l)^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.16 *We treat the* **case** (I.iii), *i.e.* $\xi \subseteq a_k = a_{i_l}$, $\eta \subseteq a_{k+1} = a_{i_{l+1}}$, $l \neq 0$, $\#\xi + \#\eta = \#_k + 1$. *Choose* $x \in \xi$, $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta}f_e = -(1+\#_k-\#_{k+1})B'(i_1,\dots,i_{l-1})^y_{\xi,\eta}.$$

In order to evaluate

$$G_{a,\xi,\eta}f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_j\in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x_l} \;\dots\; a_{i_l}^{x_l,x_{l+1}} \; a_{i_{l+1}}^{x_{l+1},} \;\dots\;\rangle \circ (\xi\cup\eta)$$

we distinguish four subcases for the occurring summands.

The **subcases** $x_l \in \bar\xi$, $x_{l+1} \in \eta$ as well as $x_l \in \bar\xi$, $x_{l+1} \in \bar\eta$ yield zero summands by the Garnir relations (4.1.4).

Subcase $x_l \in \xi$, $x_{l+1} \in \eta$.

$$\begin{aligned}
&\sum_{x_l\in\xi}\sum_{x_{l+1}\in\eta} (-1)^{\pi(x_{l+1})} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x_l} \;\dots\; a_{i_l}^{x_l,x_{l+1}} \; a_{i_{l+1}}^{x_{l+1},} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.8),\,\sigma=(x_{l+1}\;y)(x_l\;x\;y)}{=} &\sum_{x_l\in\xi}\sum_{x_{l+1}\in\eta} (-1)^{\pi(y)} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x} \;\dots\; a_{i_l}^{x,y} \; a_{i_{l+1}}^{y,} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.6)}{=} &(-1)^{\pi(y)}\#\xi\#\eta \binom{\#_{i_l}}{\#\xi}^{-1} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x} \;\dots\; (a_{i_l}^{x,y})^{\bar\xi,\eta\backslash y} \; (a_{i_{l+1}}^{y,})^{\eta\backslash y,\bar\xi} \;\dots\;\rangle \circ (\xi\cup\eta)
\end{aligned}$$

Subcase $x_l \in \xi$, $x_{l+1} \in \bar\eta$.

$$\begin{aligned}
&\sum_{x_l\in\xi}\sum_{x_{l+1}\in\bar\eta} (-1)^{\pi(x_{l+1})} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x_l} \;\dots\; a_{i_l}^{x_l,x_{l+1}} \; a_{i_{l+1}}^{x_{l+1},} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.8),\,\sigma=(x_l\;x\;x_{l+1})}{=} &\sum_{x_l\in\xi}\sum_{x_{l+1}\in\bar\eta} (-1)^{\pi(x_{l+1})} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x} \;\dots\; a_{i_l}^{x,x_{l+1}} \; a_{i_{l+1}}^{x_{l+1},} \;\dots\;\rangle \circ (\xi\cup\eta) \\
\overset{(4.3.6)}{=} &\binom{\#_{i_l}}{\#\xi-1}^{-1} \sum_{x_l\in\xi}\sum_{x_{l+1}\in\bar\eta} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{l-1}}^{x_{l-1},x} \dots ((a_{i_l}^{x,x_{l+1}})^{\bar\xi,\eta\backslash y})^{x_{l+1},y}\, ((a_{i_{l+1}}^{x_{l+1},})^{\eta\backslash y,\bar\xi})^{y,x_{l+1}} \dots\rangle \circ (\xi\cup\eta) \\
= &-(-1)^{\pi(y)}\#\xi\#\bar\eta \binom{\#_{i_l}}{\#\xi-1}^{-1} \langle \;\dots\; a_{i_{l-1}}^{x_{l-1},x} \;\dots\; (a_{i_l}^{x,y})^{\bar\xi,\eta\backslash y} \; (a_{i_{l+1}}^{y,})^{\eta\backslash y,\bar\xi} \;\dots\;\rangle \circ (\xi\cup\eta)
\end{aligned}$$

Altogether, we obtain thus

$$\begin{aligned}
& G_{a,\xi,\eta} f_e \\
= \quad & \tfrac{\#\xi - \#\bar\eta}{\#_{i_l}!} \sum_{x_j \in a_{i_j},\, j \le l-1} (-1)^{\pi(y)} \langle \dots a_{i_{l-1}}^{x_{l-1},x} \dots (a_{i_l}^{x,y})^{\bar\xi,\eta\backslash y} (a_{i_{l+1}}^{y,})^{\eta\backslash y,\bar\xi} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\, \sigma=(x\ y)}{=} \quad & -\tfrac{1+\#_{i_l} - \#_{i_{l+1}}}{\#_{i_l}!} \sum_{x_j \in a_{i_j},\, j \le l-1} (-1)^{\pi(y)} \langle \dots a_{i_{l-1}}^{x_{l-1},y} \dots a_{i_l}^{\bar\xi,\eta\backslash y} (a_{i_{l+1}}^{y,})^{\eta\backslash y,\bar\xi} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.11)}{=} \quad & -(1 + \#_{i_l} - \#_{i_{l+1}}) B'(i_1, \dots, i_{l-1})^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.17 *We treat the* **case** (II.i), *i.e.* $\xi \subseteq a_g = a_{i_0}$, $\eta \subseteq a_{g+1}$, $e_{g+1} = 0$, $\#\xi + \#\eta = \#_g$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = A(i_1, \dots, i_l)^y_{\xi,\eta}.$$

$$\begin{aligned}
& \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \dots & a_g & a_{g+1} & \dots & a_{i_1}^{x_1,x_2} & \dots \\ & x_1 & & & & \end{matrix} \right\rangle \circ (\xi \cup \eta) \\
\overset{(4.3.5)}{=} \; & \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \dots & a_g^{\bar\xi,\eta\backslash y} & (a_{g+1}^{\eta\backslash y,\bar\xi})^{y,x_1} & \dots & a_{i_1}^{x_1,x_2} & \dots \\ & y & & & & \end{matrix} \right\rangle \\
\overset{(4.3.10)}{=} \; & A(i_1, \dots, i_l)^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.18 *We treat the* **case** (II.ii), *i.e.* $\xi \subseteq a_p = a_{i_\nu}$, $\nu \in [1, l]$, $\eta \subseteq a_{p+1}$, $e_{p+1} = 0$, $\#\xi + \#\eta = \#_{i_\nu} + 1$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = -B(i_1, \dots, \hat{i}_\nu, \dots, i_l)^y_{\xi,\eta}.$$

$$\begin{aligned}
& \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \dots a_{i_\nu}^{x_\nu,x_{\nu+1}} \; a_{p+1} \dots \rangle \circ (\xi \cup \eta) \\
\overset{\text{Garnir, } (4.1.4)}{=} \; & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\, j \ne \nu} \sum_{x_\nu \in \xi} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \dots a_{i_\nu}^{x_\nu,x_{\nu+1}} \; a_{p+1} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.6)}{=} \; & \tfrac{1}{\#\xi!\#\eta!} \left(\begin{matrix} \#_{i_\nu} \\ \#\xi - 1 \end{matrix} \right)^{-1} \sum_{x_j \in a_{i_j},\, j \ne \nu} \sum_{x_\nu \in \xi} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},x_\nu} \dots ((a_{i_\nu}^{x_\nu,x_{\nu+1}})^{\bar\xi,\eta\backslash y})^{x_{\nu+1},y} (a_{p+1}^{\eta\backslash y,\bar\xi})^{y,x_{\nu+1}} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\, \sigma=(x_\nu\ y)}{=} \; & -\tfrac{1}{\#\xi\#_{i_\nu}!} \sum_{x_j \in a_{i_j},\, j \ne \nu} \sum_{x_\nu \in \xi} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},y} \dots a_{i_\nu}^{\bar\xi,\eta\backslash y} \; (a_{p+1}^{\eta\backslash y,\bar\xi})^{y,x_{\nu+1}} \dots \rangle \circ (\xi \cup \eta) \\
= \; & -\tfrac{1}{\#_{i_\nu}!} \sum_{x_j \in a_{i_j},\, j \ne \nu} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_{\nu-1}}^{x_{\nu-1},y} \dots a_{i_\nu}^{\bar\xi,\eta\backslash y} \; (a_{p+1}^{\eta\backslash y,\bar\xi})^{y,x_{\nu+1}} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.11)}{=} \; & -B(i_1, \dots, \hat{i}_\nu, \dots, i_l)^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.19 *We treat the* **case** (II.iii), *i.e.* $\xi \subseteq a_p = a_{i_{l+1}} = a_{k+1}$, $\eta \subseteq a_{p+1} = a_{k+2}$, $\#\xi + \#\eta = \#_{k+1} + 1$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = 0.$$

$$\begin{aligned}
& \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_l}^{x_l,x_{l+1}} \dots a_{i_{l+1}}^{x_{l+1},} \; a_{p+1} \dots \rangle \circ (\xi \cup \eta) \\
\overset{\text{Garnir, } (4.1.4)}{=} \; & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\, j \ne l+1} \sum_{x_{l+1} \in \xi} (-1)^{\pi(x_{l+1})} \langle \dots a_{i_l}^{x_l,x_{l+1}} \dots a_{i_{l+1}}^{x_{l+1},} \; a_{p+1} \dots \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.7)}{=} \; & 0.
\end{aligned}$$

Calculation 4.3.20 *We treat the* **case** (III.i), *i.e.* $\xi \subseteq a_{g-1}$, $\eta \subseteq a_g$, $\#\xi + \#\eta = \#_{g-1} + 1$, *to obtain*

$$G_{a,\xi,\eta} f_e \overset{\text{Garnir, (4.1.4)}}{=} 0.$$

Calculation 4.3.21 *We treat the* **case** (III.ii), *i.e.* $\xi \subseteq a_p$, $e_p = 0$, $\eta \subseteq a_{p+1} = a_{i_\nu}$, $\nu \in [1, l]$, $\#\xi + \#\eta = \#_p + 1$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = B(i_1, \ldots, \hat{i}_\nu, \ldots, i_l)^y_{\xi,\eta}.$$

$$\begin{aligned}
& \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_{\nu-1},x_\nu}_{i_{\nu-1}} \ \ldots \ a_p \ a^{x_\nu,x_{\nu+1}}_{i_\nu} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{\text{Garnir, (4.1.4)}}{=} & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\ j \neq \nu} \ \sum_{x_\nu \in \eta} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_{\nu-1},x_\nu}_{i_{\nu-1}} \ \ldots \ a_p \ a^{x_\nu,x_{\nu+1}}_{i_\nu} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\, \sigma=(x_\nu\ y\ x_{\nu+1})}{=} & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\ j \neq \nu} \ \sum_{x_\nu \in \eta} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_{\nu-1},y}_{i_{\nu-1}} \ \ldots \ a_p \ a^{y,x_{\nu+1}}_{i_\nu} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.6)}{=} & \tfrac{1}{\#\xi!\#\eta!} \#\eta \binom{\#_p}{\#\xi}^{-1} \sum_{x_j \in a_{i_j},\ j \neq \nu} (-1)^{\pi(x_{l+1})} \langle \ldots a^{x_{\nu-1},y}_{i_{\nu-1}} \ldots a_p^{\bar{\xi},\eta\backslash y} \ (a^{y,x_{\nu+1}}_{i_\nu})^{\eta\backslash y,\bar{\xi}} \ldots \rangle \circ (\xi \cup \eta) \\
= & \tfrac{1}{\#_p!} \sum_{x_j \in a_{i_j},\ j \neq \nu} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_{\nu-1},y}_{i_{\nu-1}} \ \ldots \ a_p^{\bar{\xi},\eta\backslash y} \ (a^{y,x_{\nu+1}}_{i_\nu})^{\eta\backslash y,\bar{\xi}} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.11)}{=} & B(i_1, \ldots, \hat{i}_\nu, \ldots, i_l)^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.22 *We treat the* **case** (III.iii), *i.e.* $\xi \subseteq a_k$, $e_k = 0$, $\eta \subseteq a_{k+1} = a_{i_{l+1}}$, $\#\xi + \#\eta = \#_k + 1$. *Choose* $y \in \eta$. *We* **claim** *that*

$$G_{a,\xi,\eta} f_e = B'(i_1, \ldots, i_l)^y_{\xi,\eta}.$$

$$\begin{aligned}
& \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_l,x_{l+1}}_{i_l} \ \ldots \ a_k \ a^{x_{l+1},}_{i_{l+1}} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{\text{Garnir, (4.1.4)}}{=} & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\ j \neq l+1} \ \sum_{x_{l+1} \in \eta} (-1)^{\pi(x_{l+1})} \langle \ \ldots \ a^{x_l,x_{l+1}}_{i_l} \ \ldots \ a_k \ a^{x_{l+1},}_{i_{l+1}} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.8),\, \sigma=(x_{l+1}\ y)}{=} & \tfrac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j},\ j \neq l+1} \ \sum_{x_{l+1} \in \eta} (-1)^{\pi(y)} \langle \ \ldots \ a^{x_l,y}_{i_l} \ \ldots \ a_k \ a^{y,}_{i_{l+1}} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.6)}{=} & \tfrac{1}{\#\xi!\#\eta!} \#\eta \binom{\#_k}{\#\xi}^{-1} \sum_{x_j \in a_{i_j},\ j \neq l+1} (-1)^{\pi(y)} \langle \ldots a^{x_l,y}_{i_l} \ldots a_k^{\bar{\xi},\eta\backslash y} \ (a^{y,}_{i_{l+1}})^{\eta\backslash y,\bar{\xi}} \ldots \rangle \circ (\xi \cup \eta) \\
= & \tfrac{1}{\#_k!} \sum_{x_j \in a_{i_j}, j \neq l+1} (-1)^{\pi(y)} \langle \ \ldots \ a^{x_l,y}_{i_l} \ \ldots \ a_k^{\bar{\xi},\eta\backslash y} \ (a^{y,}_{i_{l+1}})^{\eta\backslash y,\bar{\xi}} \ \ldots \ \rangle \circ (\xi \cup \eta) \\
\overset{(4.3.11)}{=} & B'(i_1, \ldots, i_l)^y_{\xi,\eta}.
\end{aligned}$$

Calculation 4.3.23 *We treat the* **case** (IV), *i.e.* $e_p = 0$, $e_{p+1} = 0$, *to obtain*

$$G_{a,\xi,\eta} f_e \overset{\text{Garnir, (4.1.4)}}{=} 0.$$

4.3.4 Polynomial coefficients

Example 4.3.24
Let $g = 1$ and $k = 4$. We list the images of $G_{a,\xi,\eta} f_e$ (cf. 4.3.9).

Case $\xi \subseteq a_1$, $\eta \subseteq a_2$.

e	Case	Factor	A, B
111	(I.i)	$(1 + \#_1 - \#_2)$	$A(3,4)^y_{\xi,\eta}$
011	(II.i)	1	$A(3,4)^y_{\xi,\eta}$
101	(I.i)	$(1 + \#_1 - \#_2)$	$A(4)^y_{\xi,\eta}$
001	(II.i)	1	$A(4)^y_{\xi,\eta}$
110	(I.i)	$(1 + \#_1 - \#_2)$	$A(3)^y_{\xi,\eta}$
010	(II.i)	1	$A(3)^y_{\xi,\eta}$
100	(I.i)	$(1 + \#_1 - \#_2)$	$A()^y_{\xi,\eta}$
000	(II.i)	1	$A()^y_{\xi,\eta}$

Case $\xi \subseteq a_2$, $\eta \subseteq a_3$.

e	Case	Factor	A, B
111	(I.ii)	$-(1 + \#_2 - \#_3)$	$B(4)^y_{\xi,\eta}$
011	(III.ii)	1	$B(4)^y_{\xi,\eta}$
101	(II.ii)	-1	$B(4)^y_{\xi,\eta}$
001	(IV)	0	
110	(I.ii)	$-(1 + \#_2 - \#_3)$	$B()^y_{\xi,\eta}$
010	(III.ii)	1	$B()^y_{\xi,\eta}$
100	(II.ii)	-1	$B()^y_{\xi,\eta}$
000	(IV)	0	

Case $\xi \subseteq a_3$, $\eta \subseteq a_4$.

e	Case	Factor	A, B
111	(I.ii)	$-(1 + \#_3 - \#_4)$	$B(2)^y_{\xi,\eta}$
011	(I.ii)	$-(1 + \#_3 - \#_4)$	$B()^y_{\xi,\eta}$
101	(III.ii)	1	$B(2)^y_{\xi,\eta}$
001	(III.ii)	1	$B()^y_{\xi,\eta}$
110	(II.ii)	-1	$B(2)^y_{\xi,\eta}$
010	(II.ii)	-1	$B()^y_{\xi,\eta}$
100	(IV)	0	
000	(IV)	0	

Case $\xi \subseteq a_4$, $\eta \subseteq a_5$.

e	Case	Factor	A, B
111	(I.iii)	$-(1 + \#_4 - \#_5)$	$B'(2,3)^y_{\xi,\eta}$
011	(I.iii)	$-(1 + \#_4 - \#_5)$	$B'(3)^y_{\xi,\eta}$
101	(I.iii)	$-(1 + \#_4 - \#_5)$	$B'(2)^y_{\xi,\eta}$
001	(I.iii)	$-(1 + \#_4 - \#_5)$	$B'()^y_{\xi,\eta}$
110	(III.iii)	1	$B'(2,3)^y_{\xi,\eta}$
010	(III.iii)	1	$B'(3)^y_{\xi,\eta}$
100	(III.iii)	1	$B'(2)^y_{\xi,\eta}$
000	(III.iii)	1	$B'()^y_{\xi,\eta}$

Thus we want to (but not necesarily have to) find a column vector containing the coefficients

u_e, ordered according to the e's as in the tables above, which annihilates

$$\begin{bmatrix}
1+\#_1-\#_2 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1+\#_1-\#_2 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1+\#_1-\#_2 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1+\#_1-\#_2 & +1 \\
-(1+\#_2-\#_3) & +1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(1+\#_2-\#_3) & +1 & -1 & 0 \\
-(1+\#_3-\#_4) & 0 & +1 & 0 & -1 & 0 & 0 & 0 \\
0 & -(1+\#_3-\#_4) & 0 & +1 & 0 & -1 & 0 & 0 \\
-(1+\#_4-\#_5) & 0 & 0 & 0 & +1 & 0 & 0 & 0 \\
0 & -(1+\#_4-\#_5) & 0 & 0 & 0 & +1 & 0 & 0 \\
0 & 0 & -(1+\#_4-\#_5) & 0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & -(1+\#_4-\#_5) & 0 & 0 & 0 & +1
\end{bmatrix}$$

from the right modulo the path length $m = (k+1-g) + (\#_g - \#_{k+1}) = 4 + \#_1 - \#_5$. We substitute

$$\begin{aligned}
X_j &:= 5 - \#_5 + \#_j - j \\
1 + \#_4 - \#_5 &= X_4 \\
1 + \#_j - \#_{j+1} &= X_j - X_{j+1} \\
m &= X_1 \,,
\end{aligned}$$

yielding

$$\begin{bmatrix}
X_1-X_2 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & X_1-X_2 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_1-X_2 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & X_1-X_2 & +1 \\
X_2-X_3 & -1 & +1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & X_2-X_3 & -1 & +1 & 0 \\
X_3-X_4 & 0 & -1 & 0 & +1 & 0 & 0 & 0 \\
0 & X_3-X_4 & 0 & -1 & 0 & +1 & 0 & 0 \\
X_4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & X_4 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & X_4 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & X_4 & 0 & 0 & 0 & -1
\end{bmatrix} .$$

We choose the coefficient vector u to be

$$\begin{bmatrix}
1 \\ X_2 \\ X_3 \\ X_2X_3 \\ X_4 \\ X_2X_4 \\ X_3X_4 \\ X_2X_3X_4
\end{bmatrix} ,$$

thus annihilating this matrix from the right modulo m, and obtain

$$f = f_{111} + X_2 f_{011} + X_3 f_{101} + X_2X_3 f_{001} + X_4 f_{110} + X_2X_4 f_{010} + X_3X_4 f_{100} + X_2X_3X_4 f_{000}.$$

Notation 4.3.25

Let $J \subseteq [g+1,k]$. Let $[g+1,k]\backslash J \xrightarrow{e'} \{0,1\}$, $J \xrightarrow{e''} \{0,1\}$. Denote by $[g+1,k] \xrightarrow{[e''e']} \{0,1\}$ the 'concatenated' map defined by $[e''e']|_{[g+1,k]\backslash J} = e'$ and $[e''e']|_J = e''$. Similarly multi-concatenations.

Furthermore, we make use of multiindices in the sense that for a map $[g+1,k] \supseteq J \xrightarrow{e} \{0,1\}$ we denote

$$X^{1-e} := \prod_{j\in J} X_j^{1-e_j} .$$

We give a precursor of the result (4.3.31), the first subcase occurring in its proof we shall also need later on.

Proposition 4.3.26 *Keep the situation from (4.3.9). Let*

$$\boxed{X_j := k + 1 - \#_{k+1} + \#_j - j}$$

for $j \in [g, k]$. *In particular,* $m = X_g$. *Let*

$$f^0 := \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} X^{1-e} f_e.$$

For a λ*-tableau* a, *for* $p \in [1, z-1]$ *and for* $\xi \subseteq a_p$, $\eta \subseteq a_{p+1}$ *such that* $\#\xi + \#\eta = \#a_p + 1$, *we have*

$$G_{a,\xi,\eta} f^0 \in mS^\lambda.$$

Hence f^0 *induces a morphism of* $\mathbf{Z}S_n$*-modules from* S^λ/m *to* S^μ/m *(4.3.2, 4.3.3, 4.3.9).*

Choose $y \in \eta$.

Case $g < k$**.**

Subcase $p = g$, i.e. (I.i) or (II.i). Let $[g+2, k] \xrightarrow{e'} \{0, 1\}$ be given. From (4.3.13, 4.3.17) we take

$$\begin{array}{rcl} G_{a,\xi,\eta}(X^{1-[1e']} f_{[1e']} + X^{1-[0e']} f_{[0e']}) & = & (1 \cdot (1 + \#_g - \#_{g+1}) + X_{g+1} \cdot 1) X^{1-e'} A^y_{[0e'],\xi,\eta} \\ & = & (X_g - X_{g+1} + X_{g+1}) X^{1-e'} A^y_{[0e'],\xi,\eta} \\ & = & m X^{1-e'} A^y_{[0e'],\xi,\eta}. \end{array}$$

Subcase $p \in [g+1, k-1]$, i.e. (I.ii), (II.ii), (III.ii) or (IV). Let $[g+1, k] \backslash \{p, p+1\} \xrightarrow{e'} \{0, 1\}$ be given. From (4.3.15, 4.3.18, 4.3.21, 4.3.23) we take

$$\begin{array}{rl} & G_{a,\xi,\eta}(X^{1-[11e']} f_{[11e']} + X^{1-[10e']} f_{[10e']} + X^{1-[01e']} f_{[01e']} + X^{1-[00e']} f_{[00e']}) \\ = & (1 \cdot (-(1 + \#_p - \#_{p+1})) + X_{p+1} \cdot (-1) + X_p \cdot 1 + X_p X_{p+1} \cdot 0) X^{1-e'} B^y_{[00e'],\xi,\eta} \\ = & (-X_p + X_{p+1} - X_{p+1} + X_p) X^{1-e'} B^y_{[00e'],\xi,\eta} \\ = & 0. \end{array}$$

Subcase $p = k$, i.e. (I.iii) or (III.iii). Let $[g+1, k-1] \xrightarrow{e'} \{0, 1\}$ be given. From (4.3.16, 4.3.22) we take

$$\begin{array}{rcl} G_{a,\xi,\eta}(X^{1-[1e']} f_{[1e']} + X^{1-[0e']} f_{[0e']}) & = & (1 \cdot (-(1 + \#_k - \#_{k+1})) + X_k \cdot 1) X^{1-e'} B'^y_{[0e']} \\ & = & (-X_k + X_k) X^{1-e'} B'^y_{[0e']} \\ & = & 0. \end{array}$$

Subcase $p \notin [g, k]$, i.e. (II.iii), (III.i) or (IV). Let $[g+1, k] \xrightarrow{e} \{0, 1\}$ be given. From (4.3.19, 4.3.20, 4.3.23) we take

$$G_{a,\xi,\eta} f_e = 0.$$

Case $g = k = p$**.**

The sum furnishing f^0 has only one summand associated to the index $\emptyset \xrightarrow{e} \{0,1\}$ so that $f^0 = f_e$. Hence, by (I.ia) (4.3.14) we obtain

$$\begin{aligned} G_{a,\xi,\eta} f^0 &= (1 + \#_g - \#_{g+1}) A'^{y}_{\xi,\eta} \\ &= m A'^{y}_{\xi,\eta}. \end{aligned}$$

We shall detect a redundant scalar factor in this provisional version.

Lemma 4.3.27 *For* $g+1 \leq \alpha \leq \beta \leq k$ *we have*

$$\sum_{[\alpha,\beta] \xrightarrow{e} \{0,1\}} X^{1-e} = \prod_{j \in [\alpha,\beta]} (X_j + 1).$$

The induction step is given by

$$\sum_{[\alpha,\beta] \xrightarrow{e} \{0,1\}} X^{1-e} = (X_\alpha + 1) \sum_{[\alpha+1,\beta] \xrightarrow{e} \{0,1\}} X^{1-e}.$$

Lemma 4.3.28 *Consider an element* $p \in [g+1, k-1]$ *with* $\#_p = \#_{p+1}$*. Furthermore, let* $[g+1, p-1] \xrightarrow{e} \{0,1\}$*,* $[p+2, k] \xrightarrow{e'} \{0,1\}$ *be given. Then*

$$f_{[e10e']} = f_{[e11e']}.$$

Similarly, for $p \in [g+2, k-1]$*,* $e_{g+1} = 0$*,* $\xi \subseteq a_g$*,* $\eta \subseteq a_{g+1}$*,* $y \in \eta$ *such that* $\#\xi + \#\eta = \#_g$*, we assert that*

$$A^y_{[e10e'],\xi,\eta} = A^y_{[e11e'],\xi,\eta}.$$

Moreover, in case $p = g+1$ *we have*

$$A^y_{[00e'],\xi,\eta} = A^y_{[01e'],\xi,\eta}.$$

The Garnir relation (4.1.4) gives

$$\langle \dots\ a_p^{x_\nu, x_{\nu+1}} a_{p+1}\ \dots \rangle = \sum_{z \in a_{p+1}} \langle \dots\ a_p^{x_\nu, z} a_{p+1}^{z, x_{\nu+1}}\ \dots \rangle.$$

Mutatis mutandis in case $p = g+1$ for $A^y_{[00e'],\xi,\eta}$.

Lemma 4.3.29 *Let*

$$\boxed{r := \prod_{i \in [g+1,k-1],\ \#_i = \#_{i+1}} X_i},$$

where the empty product equals 1*.*

Suppose given $\xi \subseteq a_g$*,* $\eta \subseteq a_{g+1}$*,* $y \in \eta$ *such that* $\#\xi + \#\eta = \#_g$*. The elements*

$$\begin{aligned} [a] f^0 &= \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} X^{1-e} [a] f_e \\ m^{-1} G_{a,\xi,\eta} f^0 &= \sum_{[g+2,k] \xrightarrow{e} \{0,1\}} X^{1-e} A^y_{[0e],\xi,\eta} \end{aligned}$$

of S^μ are divisible by r.

Suppose given $g+1 \le \alpha \le \beta \le k$ such that $\#_i = \#_j$ for $i, j \in [\alpha,\beta]$. For $\tau \in [\alpha, \beta+1]$, we denote

$$\begin{array}{rcl} [\alpha,\beta] & \xrightarrow{e^\tau} & \{0,1\} \\ i & \longrightarrow & e_i^\tau := \begin{cases} 0 & \text{for } i \in [\alpha, \tau-1] \\ 1 & \text{for } i \in [\tau, \beta]. \end{cases} \end{array}$$

We fix maps $[g+1, \alpha-1] \xrightarrow{e'} \{0,1\}$ and $[\beta+1, k] \xrightarrow{e''} \{0,1\}$ and obtain

$$\begin{aligned} \sum_{[\alpha,\beta] \xrightarrow{e} \{0,1\}} X^{1-e} f_{[e'ee'']} &\overset{(4.3.27,\, 4.3.28)}{=} \sum_{\tau\in[\alpha,\beta+1]} f_{[e'e^\tau e'']} \left(\prod_{j\in[\alpha,\tau-1]} X_j\right)\left(\prod_{j\in[\tau+1,\beta]} (X_j+1)\right) \\ &= \left(\sum_{\tau\in[\alpha,\beta]} f_{[e'e^\tau e'']}\right)\left(\prod_{j\in[\alpha,\beta-1]} X_j\right) \\ &\quad + f_{[e'e^{\beta+1}e'']}\left(\prod_{j\in[\alpha,\beta]} X_j\right). \end{aligned}$$

Similarly for A. In case $g+2 \le \alpha$, we fix maps $[g+2, \alpha-1] \xrightarrow{e'} \{0,1\}$, $[\beta+1, k] \xrightarrow{e''} \{0,1\}$ and get

$$\begin{aligned} \sum_{[\alpha,\beta] \xrightarrow{e} \{0,1\}} X^{1-e} A^y_{[0e'ee''],\xi,\eta} &\overset{(4.3.27,\, 4.3.28)}{=} \sum_{\tau\in[\alpha,\beta+1]} A^y_{[0e'e^\tau e''],\xi,\eta} \left(\prod_{j\in[\alpha,\tau-1]} X_j\right)\left(\prod_{j\in[\tau+1,\beta]} (X_j+1)\right) \\ &= \left(\sum_{\tau\in[\alpha,\beta]} A^y_{[0e'e^\tau e''],\xi,\eta}\right)\left(\prod_{j\in[\alpha,\beta-1]} X_j\right) \\ &\quad + A^y_{[0e'e^{\beta+1}e''],\xi,\eta}\left(\prod_{j\in[\alpha,\beta]} X_j\right). \end{aligned}$$

In case $g+1=\alpha$, we fix a map $[\beta+1, k] \xrightarrow{e''} \{0,1\}$ to obtain

$$\begin{aligned} \sum_{[g+2,\beta] \xrightarrow{e} \{0,1\}} X^{1-e} A^y_{[0ee''],\xi,\eta} &\overset{(4.3.27,\, 4.3.28)}{=} A^y_{[e^{g+2}e''],\xi,\eta} \left(\prod_{j\in[g+2,\beta]} (X_j+1)\right) \\ &= A^y_{[e^{g+2}e''],\xi,\eta} \left(\prod_{j\in[g+1,\beta-1]} X_j\right). \end{aligned}$$

Lemma 4.3.30 *f can be written as a matrix with at least one entry equal to ± 1.*

Let $[\check{a}]$ be the standard λ-tableau for which $i < i'$ implies $\check{a}_{i,j} < \check{a}_{i',j'}$, i.e. the smallest one in the sense of the proof of (4.3.2). Suppose given $[g+1,k] \xrightarrow{e} \{0,1\}$ such that for $p, q \in [g+1,k]$, $p < q$ and $\#_p = \#_q$ we have $e_p \le e_q$. The summands of

$$[\check{a}]f_e = \sum_{x_j \in \check{a}_{i_j}} (-1)^{\pi(x_{l+1})} \left\langle \begin{matrix} \ldots \check{a}_g \ldots \check{a}_{i_1}^{x_1,x_2} \ldots \check{a}_{i_2}^{x_2,x_3} \ldots \check{a}_{i_l}^{x_l,x_{l+1}} \ldots \check{a}_{k+1}^{x_{l+1},} \ldots \\ x_1 \end{matrix} \right\rangle$$

are standard λ-polytabloids up to sign. Since we may write the image of $[\check{a}]$ under f as an integral linear combination of such elements (4.3.28), and since the occurring standard polytabloids are pairwise different because of different fillings of the columns, we are reduced to consider a chosen such e and to regard the corresponding summand

$$\frac{1}{r}\left(\sum_{e'\in E(e)} X^{1-e'}\right)[\breve{a}]f_e,$$

where

$$E(e) := \{[g{+}1,k] \xrightarrow{e'} \{0,1\} \mid \forall i \in [g{+}1,k]\ (e'_i = e_i \vee \exists j \in [g{+}1,i{-}1]\ (\#_j = \#_i \wedge e'_j = e_j = 1))\}.$$

However, for $e = 11\dots1$ we obtain

$$\sum_{e'\in E(e)} X^{1-e'} \overset{(4.3.27)}{=} \prod_{i\in[g+2,k],\ \#_{i-1}=\#_i} (X_i+1) \overset{(4.3.29)}{=} r.$$

We summarize to the

Theorem 4.3.31 *Keep the notation of (4.3.1, 4.3.25, 4.3.26, 4.3.29). The* $\mathbf{Z}\mathcal{S}_n$*-linear map*

$$\boxed{f := \frac{1}{r}f^0 = \frac{1}{r}\sum_{[g+1,k]\xrightarrow{e}\{0,1\}} X^{1-e}f_e : F^\lambda \longrightarrow S^\mu}$$

factors over

$$(S^\lambda/m \xrightarrow{\bar{f}} S^\mu/m) \neq 0.$$

More precisely, f can be written as a matrix with at least one entry equal to ± 1.

From the proof of (4.3.26) it follows that (4.3.29) suffices to prove the factorization. The second assertion follows from (4.3.30).

Remark 4.3.32 Based on [CL 74], Carter and Payne [CP 80] have obtained a closely related non-vanishing result ([2]). It asserts **in particular** that for λ and μ as above we have

$$\mathrm{Hom}_{K\mathcal{S}_n}(K\otimes_{\mathbf{Z}} S^\lambda, K\otimes_{\mathbf{Z}} S^\mu) \neq 0,$$

K being an infinite field of characteristic dividing m. This particular case of their result now also ensues from (4.3.31).

For the application to integral representation theory, we need such a morphism in the concrete form as given in (4.3.31) for the following two reasons. First, we need to calculate modulo prime powers, not merely modulo primes (cf. 3.2.1 or e.g. 4.2.10). Second, we need to know the elementary divisors as well as the behaviour under composition of the of the various specializations (cf. e.g. 4.2.4, 4.2.8, S 4.4.2).

The result of [CP 80] also comprises the case of a simultaneous shift of several boxes from a column to a column further to the left. A first step in the direction of a concretization of this result is undertaken in (4.4.3). Some further examples have been calculated directly but are not yet understood (cf. 4.4.5).

Cf. also [J 78, 24.6 (ii), 24.10].

[2] as G. James pointed out to me

Remark 4.3.33 For $n \leq 7$ the following assertions hold (cf. S 4.3.5).

(i) Let $m_{2'}$ denote the $2'$-part of m, i.e. $m_{2'} := m/2^{v_2(m)}$. $\bar{f}$ (4.3.31) generates

$$\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda/m_{2'}, S^\mu/m_{2'}).$$

(ii) Let m' be a natural number divisible by m. The map

$$\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda/m, S^\mu/m) \longrightarrow \mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda/m', S^\mu/m'),$$

induced by multiplication by m'/m, is an isomorphism.

A. Kleshchev [Kles 98] has given an argument for the dimension of the Hom-space treated by Carter and Payne (4.3.32) to be one-dimensional in case of a one-box-shift over a field of characteristic $\neq 2$. Note that in case of characteristic 2, the partitions $\lambda = (4,1,1)$, $\mu = (3,1,1,1)$ furnish an example in which this Hom-space is two-dimensional (S 4.3.5).

Remark 4.3.34 *In case λ and μ are hooks, $\bar{f}$ coincides with the map given in (4.2.3).*

We have $g = 1$, $k = \lambda_1 - 1$ and $m = n$. Moreover, $X_i = k + 1 - i$ for $i \in [2,k]$, $r = (k-1)!$. The proof of (4.3.29) yields

$$f = \sum_{\tau \in [g+1, k+1]} f_{e^\tau},$$

using the notation introduced there.

We record a composition property of certain specializations of our morphism (which originally has been a failed attempt to prove their nonvanishing).

Lemma 4.3.35 *Let $\nu \geq 2$ be a natural number. Let the partition μ be such that the* **binomial condition at** ν *is satisfied which says that*

$$\binom{\mu'_i + 1}{u} \equiv_\nu 0 \text{ for all } u \in [1, \mu'_{i+1}] \text{ for all } i \in [1, z].$$

Fix a μ-tableau $[a]$. The **alternating augmentation**

$$\begin{array}{ccc} S^\mu & \stackrel{d}{\longrightarrow} & (\mathbf{Z}/\nu)^- \\ \langle as \rangle & \longrightarrow & \varepsilon_s \end{array}$$

is a well defined nonzero $\mathbf{Z}\mathcal{S}_n$-morphism, where $s \in \mathcal{S}_n$, and where $(\mathbf{Z}/\nu)^-$ denotes the alternating module structure on $\mathbf{Z}/\nu$. NB the sign of d depends on the choice of $[a]$.

We construct d as the factorization of

$$\begin{array}{ccc} F^\mu & \longrightarrow & (\mathbf{Z}/\nu)^- \\ [as] & \longrightarrow & \varepsilon_s \end{array}$$

over $F^\mu \longrightarrow S^\mu$. Signed column transpositions (4.3.2) vanish under this map. It remains to be shown that the same holds for one-step-Garnir relations (4.3.2). But

$$G_{as,\xi,\eta} = \frac{1}{\#\xi!\#\eta!}[as] \circ (\xi \cup \eta) \in F_0^\lambda$$

where $j \in [1, z-1]$, $\xi \subseteq a_j$, $\emptyset \neq \eta \subseteq a_{j+1}$, $\xi + \eta = \mu'_j + 1$, is mapped to $\binom{\mu'_j+1}{\#\eta}$, which is divisible by ν by the binomial condition at ν for μ.

Lemma 4.3.36 *Keep the situation of (4.3.35). Moreover, assume that* $m \equiv_\nu 0$ *and that* $r = 1$. *Then the composition*

$$S^\lambda/m \xrightarrow{\bar{f}} S^\mu/m \xrightarrow{d} (\mathbf{Z}/\nu)^-$$

(4.3.31) vanishes, except in case $\nu = 2$, $g+1 = \kappa$, $k \equiv_2 g$, *consisting of hooks, in which it is nonzero.*

(The case $r \neq 1$ remains to be investigated.)

Case $\mu'_{k+1} \geq 1$. We **claim** that the image of $[b]f_e$ under the alternating augmentation (4.3.35), $[b]$ being a λ-tableau, is given by

$$\pm(\mu'_{k+1}+1) \prod_{i\in[g+1,k],\ e_i=1} \mu'_i \equiv_\nu \pm(\mu'_{k+1}+1) \prod_{i\in[g+1,k],\ e_i=1} (-1).$$

NB in case $\mu'_{k+2} \neq 0$ we are already done with the whole case.

Replacement of x_{l+1} by x'_{l+1} in the summand occurring in the expression for the image of a under f_e (4.3.1) amounts to an operation of $(x_{l+1}\ x'_{l+1})$ followed by an operation of a cycle of length $|\pi(x_{l+1}) - \pi(x'_{l+1})| - 1$. Replacement of x_j by x'_j in this expression, $j \in [1,l]$, amounts to an operation of $(x_{j+1}\ x_j\ x'_j)$. Therefore all summands of this expression are sent to $+1$ or all summands are sent to -1.

To determine the sign more precisely, we note that the summand of this expression with each x_j being the top entry of its column, changes by an operation $(x_j\ x_{j+1})$ if we drop the column i_j for some $j \in [1,l]$. Thus the image of $[b]f_e$ under the alternating augmentation is given by

$$\pm(\mu'_{k+1}+1)(-1)^{\sum_{i\in[g+1,k]} e_i} \prod_{i\in[g+1,k],\ e_i=1} (-1) = \pm(\mu'_{k+1}+1),$$

where the sign $\pm$ now is independent of e.

Note that by our assumption we have $0 \equiv_\nu m = k - \mu'_{k+1} + \mu'_g - g \equiv_\nu k - g - \mu'_{k+1} - 1$, whence $X_i \equiv_\nu g - i$ for $i \in [g+1,k]$ since $\mu'_i \equiv_\nu -1$. Thus the composition $\bar{f}d$ maps $[b]$ to

$$\begin{array}{rcl} \pm(\mu'_{k+1}+1)\sum_e X^{1-e} & \overset{(4.3.27)}{=} & \pm(\mu'_{k+1}+1)\prod_{i\in[g+1,k]}(X_i+1) \\ & \equiv_\nu & 0 \end{array}$$

because of the factor $X_{g+1} + 1 \equiv_\nu 0$.

Case $\mu'_{k+1} = 0$, $g \leq k-1$, $r = 1$. Note that still $\mu'_k \geq 1$. As in the first case, the image of $[b]f_e$ under the alternating augmentation is given by

$$\pm(-1)^{\sum_{i\in[g+1,k]} e_i} \prod_{i\in[g+1,k],\ e_i=1} \mu'_i \equiv_\nu \pm \begin{cases} (-1)\mu'_k & \text{for } e_k = 1 \\ 1 & \text{for } e_k = 0 \end{cases}.$$

The composition $\bar{f}d$ maps $[b]$ to

$$\pm\left(\sum_{[g+1,k-1] \xrightarrow{e} \{0,1\}} X^{1-e}\right)(1\cdot(-1)\mu'_k + X_k \cdot 1) = 0.$$

Example 4.3.37 Let $\lambda = (4,1)$, $\mu = (3,2)$, $\nu = 3$. $\bar{f}$ has rank 4, so that $\bar{f}$ and d cause ties just as those numbered e and h in (S 2.2.3). As usual, for accordance one should compare this morphism with the one implicitely given there.

4.3.5 Illustration

We perform some direct computer calculations in order to see to what extent our generic result (4.3.31) is relevant when specialized to small cases. To a large extent, these specializations do not look exciting, but we are as well interested in an illustration as in an exhaustive list for $n \leq 7$.

We drop the brackets indicating polytabloids in our notation. Furthermore, by **generate** we mean that the respective maps generate $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/m)$ **Z**-linearly **and** that for $m \mid m'$ the map induced by multiplication by m'/m

$$\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/m) \longrightarrow \mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^\lambda, S^\mu/m')$$

is an isomorphism. Cf. (4.3.33).

To begin with, we give a **'sufficiently large' example.**

Let $n = 9$, $\lambda = (4,3,2)$, $\mu = (3,3,2,1)$, so that $m = 6$, $g = 1$, $k = 3$, $X_2 = k+3-2 = 4$ and $X_3 = k+2-3 = 2$. The specialization takes the form

$$\begin{array}{llll} 1&4&7&9\\ 2&5&8&\\ 3&6&& \end{array} \xrightarrow{\bar{f}} X_2^0X_3^0 \left(\begin{array}{lll} 1&7&9\\ 2&5&8\\ 3&6&\\ 4&& \end{array} + \begin{array}{lll} 1&4&9\\ 2&7&8\\ 3&6&\\ 5&& \end{array} + \begin{array}{lll} 1&4&9\\ 2&5&8\\ 3&7&\\ 6&& \end{array} + \begin{array}{lll} 1&8&7\\ 2&5&9\\ 3&6&\\ 4&& \end{array} + \begin{array}{lll} 1&4&7\\ 2&8&9\\ 3&6&\\ 5&& \end{array} + \begin{array}{lll} 1&4&7\\ 2&5&9\\ 3&8&\\ 6&& \end{array} \right)$$
$$+X_2^1X_3^0 \left(\begin{array}{lll} 1&4&9\\ 2&5&8\\ 3&6&\\ 7&& \end{array} + \begin{array}{lll} 1&4&7\\ 2&5&9\\ 3&6&\\ 8&& \end{array} \right)$$
$$+X_2^0X_3^1 \left(\begin{array}{lll} 1&9&7\\ 2&5&8\\ 3&6&\\ 4&& \end{array} + \begin{array}{lll} 1&4&7\\ 2&9&8\\ 3&6&\\ 5&& \end{array} + \begin{array}{lll} 1&4&7\\ 2&5&8\\ 3&9&\\ 6&& \end{array} \right)$$
$$+X_2^1X_3^1 \left(\begin{array}{lll} 1&4&7\\ 2&5&8\\ 3&6&\\ 9&& \end{array} \right).$$

Case $n = 2$.

$\lambda = (2)$, $\mu = (1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{ll} 1&2 \end{array} \xrightarrow{\bar{f}} \begin{array}{l} 1\\ 2 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_2}(S^{(2)}, S^{(1,1)}/2)$.

Case $n = 3$.

$\lambda = (3)$, $\mu = (2,1)$, $m = 3$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll} 1&2&3 \end{array} \xrightarrow{\bar{f}} \begin{array}{ll} 1&3\\ 2& \end{array} + \begin{array}{ll} 1&2\\ 3& \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_3}(S^{(3)}, S^{(2,1)}/3)$.

$\lambda = (2,1)$, $\mu = (1,1,1)$, $m = 3$, $g = 1$, $k = 1$.

$$\begin{array}{ll} 1&3\\ 2& \end{array} \xrightarrow{\bar{f}} \begin{array}{l} 1\\ 2\\ 3 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_3}(S^{(2,1)}, S^{(1,1,1)}/3)$.

Case $n = 4$.

$\lambda = (4)$, $\mu = (3,1)$, $m = 4$, $g = 1$, $k = 3$, $X_2 = 2$, $X_3 = 1$, $r = 2$.

$$\begin{array}{llll} 1&2&3&4 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1&3&4\\ 2&& \end{array} + \begin{array}{lll} 1&2&4\\ 3&& \end{array} + \begin{array}{lll} 1&2&3\\ 4&& \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_4}(S^{(4)}, S^{(3,1)}/4)$.

$\lambda=(3,1)$, $\mu=(2,2)$, $m=2$, $g=2$, $k=2$.

$$\begin{array}{lll}1&3&4\\2\end{array}\ \xrightarrow{\bar f}\ \begin{array}{ll}1&3\\2&4\end{array}.$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_4}(S^{(3,1)},S^{(2,2)}/2)$.

$\lambda=(3,1)$, $\mu=(2,1,1)$, $m=4$, $g=1$, $k=2$, $X_2=1$.

$$\begin{array}{lll}1&3&4\\2\end{array}\ \xrightarrow{\bar f}\ \begin{array}{ll}1&4\\2\\3\end{array}+\begin{array}{ll}1&3\\2\\4\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_4}(S^{(3,1)},S^{(2,1,1)}/4)$.

$\lambda=(2,2)$, $\mu=(2,1,1)$, $m=2$, $g=1$, $k=1$.

$$\begin{array}{ll}1&3\\2&4\end{array}\ \xrightarrow{\bar f}\ \begin{array}{ll}1&3\\2\\4\end{array}-\begin{array}{ll}1&4\\2\\3\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_4}(S^{(2,2)},S^{(2,1,1)}/2)$.

$\lambda=(2,1,1)$, $\mu=(1,1,1,1)$, $m=4$, $g=1$, $k=1$.

$$\begin{array}{ll}1&4\\2\\3\end{array}\ \xrightarrow{\bar f}\ \begin{array}{l}1\\2\\3\\4\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_4}(S^{(2,1,1)},S^{(1,1,1,1)}/4)$.

Case $n=5$.

$\lambda=(5)$, $\mu=(4,1)$, $m=5$, $g=1$, $k=4$, $X_2=3$, $X_3=2$, $X_4=1$, $r=6$.

$$\begin{array}{lllll}1&2&3&4&5\end{array}\ \xrightarrow{\bar f}\ \begin{array}{llll}1&3&4&5\\2\end{array}+\begin{array}{llll}1&2&4&5\\3\end{array}+\begin{array}{llll}1&2&3&5\\4\end{array}+\begin{array}{llll}1&2&3&4\\5\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(5)},S^{(4,1)}/5)$.

$\lambda=(4,1)$, $\mu=(3,2)$, $m=3$, $g=2$, $k=3$, $X_3=1$.

$$\begin{array}{llll}1&3&4&5\\2\end{array}\ \xrightarrow{\bar f}\ \begin{array}{lll}1&3&5\\2&4\end{array}+\begin{array}{lll}1&3&4\\2&5\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(4,1)},S^{(3,2)}/3)$.

$\lambda=(4,1)$, $\mu=(3,1,1)$, $m=5$, $g=1$, $k=3$, $X_2=2$, $X_3=1$, $r=2$.

$$\begin{array}{llll}1&3&4&5\\2\end{array}\ \xrightarrow{\bar f}\ \begin{array}{lll}1&4&5\\2\\3\end{array}+\begin{array}{lll}1&3&5\\2\\4\end{array}+\begin{array}{lll}1&3&4\\2\\5\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(4,1)},S^{(3,1,1)}/5)$.

$\lambda=(3,2)$, $\mu=(2,2,1)$, $m=4$, $g=1$, $k=2$, $X_2=2$.

$$\begin{array}{lll}1&3&5\\2&4\end{array}\ \xrightarrow{\bar f}\ \left(\begin{array}{ll}1&5\\2&4\\3\end{array}+\begin{array}{ll}1&3\\2&5\\4\end{array}\right)+2\cdot\begin{array}{ll}1&3\\2&4\\5\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(3,2)},S^{(2,2,1)}/4)$.

$\lambda = (3,2)$, $\mu = (3,1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{lll}1&3&5\\2&4\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&3&5\\2&&\\4&&\end{array} - \begin{array}{lll}1&4&5\\2&&\\3&&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(3,2)}, S^{(3,1,1)}/2)$.

$\lambda = (3,1,1)$, $\mu = (2,2,1)$, $m = 2$, $g = 2$, $k = 2$.

$$\begin{array}{lll}1&4&5\\2&&\\3&&\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&4\\2&5\\3&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(3,1,1)}, S^{(2,2,1)}/2)$.

$\lambda = (3,1,1)$, $\mu = (2,1,1,1)$, $m = 5$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll}1&4&5\\2&&\\3&&\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&5\\2&\\3&\\4&\end{array} + \begin{array}{ll}1&4\\2&\\3&\\5&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(3,1,1)}, S^{(2,1,1,1)}/5)$. $\lambda = (2,2,1)$, $\mu = (2,1,1,1)$, $m = 3$, $g = 1$, $k = 1$.

$$\begin{array}{ll}1&4\\2&5\\3&\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&4\\2&\\3&\\5&\end{array} - \begin{array}{ll}1&5\\2&\\3&\\4&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(2,2,1)}, S^{(2,1,1,1)}/3)$.

$\lambda = (2,1,1,1)$, $\mu = (1,1,1,1,1)$, $m = 5$, $g = 1$, $k = 1$.

$$\begin{array}{ll}1&5\\2&\\3&\\4&\end{array} \xrightarrow{\bar{f}} \begin{array}{l}1\\2\\3\\4\\5\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_5}(S^{(2,1,1,1)}, S^{(1,1,1,1,1)}/5$.

Case $n = 6$.

$\lambda = (6)$, $\mu = (5,1)$, $m = 6$, $g = 1$, $k = 5$, $X_2 = 4$, $X_3 = 3$, $X_4 = 2$, $X_5 = 1$, $r = 24$.

$$\begin{array}{llllll}1&2&3&4&5&6\end{array} \xrightarrow{\bar{f}} \begin{array}{lllll}1&3&4&5&6\\2&&&&\end{array} + \begin{array}{lllll}1&2&4&5&6\\3&&&&\end{array} + \begin{array}{lllll}1&2&3&5&6\\4&&&&\end{array}$$
$$+ \begin{array}{lllll}1&2&3&4&6\\5&&&&\end{array} + \begin{array}{lllll}1&2&3&4&5\\6&&&&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(6)}, S^{(5,1)}/6)$.

$\lambda = (5,1)$, $\mu = (4,2)$, $m = 4$, $g = 2$, $k = 4$, $X_3 = 2$, $X_4 = 1$, $r = 2$.

$$\begin{array}{lllll}1&3&4&5&6\\2&&&&\end{array} \xrightarrow{\bar{f}} \begin{array}{llll}1&3&5&6\\2&4&&\end{array} + \begin{array}{llll}1&3&4&6\\2&5&&\end{array} + \begin{array}{llll}1&3&4&5\\2&6&&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(5,1)}, S^{(4,2)}/4)$.

$\lambda = (5,1)$, $\mu = (4,1,1)$, $m = 6$, $g = 1$, $k = 4$, $X_2 = 3$, $X_3 = 2$, $X_4 = 1$, $r = 6$.

$$\begin{array}{lllll}1&3&4&5&6\\2&&&&\end{array} \xrightarrow{\bar{f}} \begin{array}{llll}1&4&5&6\\2&&&\\3&&&\end{array} + \begin{array}{llll}1&3&5&6\\2&&&\\4&&&\end{array} + \begin{array}{llll}1&3&4&6\\2&&&\\5&&&\end{array} + \begin{array}{llll}1&3&4&5\\2&&&\\6&&&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(5,1)}, S^{(4,1,1)}/6)$.

$\lambda = (4,2)$, $\mu = (3,3)$, $m = 2$, $g = 3$, $k = 3$.

$$\begin{array}{llll}1&3&5&6\\2&4\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&3&5\\2&4&6\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(4,2)}, S^{(3,3)}/2)$.

$\lambda = (4,2)$, $\mu = (3,2,1)$, $m = 5$, $g = 1$, $k = 3$, $X_2 = 3$, $X_3 = 1$.

$$\begin{array}{llll}1&3&5&6\\2&4\end{array} \xrightarrow{\bar{f}} \left(\begin{array}{lll}1&5&6\\2&4\\3\end{array} + \begin{array}{lll}1&3&6\\2&5\\4\end{array}\right) + \left(\begin{array}{lll}1&6&5\\2&4\\3\end{array} + \begin{array}{lll}1&3&5\\2&6\\4\end{array}\right)$$
$$+3\cdot\begin{array}{lll}1&3&6\\2&4\\5\end{array} + 3\cdot\begin{array}{lll}1&3&5\\2&4\\6\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(4,2)}, S^{(3,2,1)}/5)$.

$\lambda = (4,2)$, $\mu = (4,1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{llll}1&3&5&6\\2&4\end{array} \xrightarrow{\bar{f}} \begin{array}{llll}1&3&5&6\\2\\4\end{array} - \begin{array}{llll}1&4&5&6\\2\\3\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(4,2)}, S^{(4,1,1)}/2)$.

$\lambda = (4,1,1)$, $\mu = (3,2,1)$, $m = 3$, $g = 2$, $k = 3$, $X_3 = 1$.

$$\begin{array}{llll}1&4&5&6\\2\\3\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&4&6\\2&5\\3\end{array} + \begin{array}{lll}1&4&5\\2&6\\3\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(4,1,1)}, S^{(3,2,1)}/3)$.

$\lambda = (4,1,1)$, $\mu = (3,1,1,1)$, $m = 6$, $g = 1$, $k = 3$, $X_2 = 2$, $X_3 = 1$, $r = 2$.

$$\begin{array}{llll}1&4&5&6\\2\\3\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&5&6\\2\\3\\4\end{array} + \begin{array}{lll}1&4&6\\2\\3\\5\end{array} + \begin{array}{lll}1&4&5\\2\\3\\6\end{array}$$

and

$$\begin{array}{llll}1&4&5&6\\2\\3\end{array} \longrightarrow 3\cdot \sum_{2\le i_1<i_2<i_3\le 6} \begin{array}{lll}1&*&*\\i_1\\i_2\\i_3\end{array}$$

generate $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(4,1,1)}, S^{(3,1,1,1)}/6)$. Cf. (4.2.11, 4.2.17).

$\lambda = (3,3)$, $\mu = (3,2,1)$, $m = 3$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll}1&3&5\\2&4&6\end{array} \xrightarrow{\bar{f}} \left(\begin{array}{lll}1&6&5\\2&4\\3\end{array} + \begin{array}{lll}1&3&5\\2&6\\4\end{array} - \begin{array}{lll}1&5&6\\2&4\\3\end{array} - \begin{array}{lll}1&3&6\\2&5\\4\end{array}\right) + \left(\begin{array}{lll}1&3&5\\2&4\\6\end{array} - \begin{array}{lll}1&3&6\\2&4\\5\end{array}\right)$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,3)}, S^{(3,2,1)}/3)$.

$\lambda = (3,2,1)$, $\mu = (2,2,2)$, $m = 3$, $g = 2$, $k = 2$.

$$\begin{array}{lll}1&4&6\\2&5\\3\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&4\\2&5\\3&6\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,2,1)}, S^{(2,2,2)}/3)$.

$\lambda = (3,2,1)$, $\mu = (2,2,1,1)$, $m = 5$, $g = 1$, $k = 2$, $X_2 = 2$.

$$\begin{array}{lll}1&4&6\\2&5&\\3&&\end{array} \xrightarrow{\bar f} \left(\begin{array}{ll}1&6\\2&5\\3&\\4&\end{array} + \begin{array}{ll}1&4\\2&6\\3&\\5&\end{array}\right) + 2\cdot \begin{array}{ll}1&4\\2&5\\3&\\6&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,2,1)}, S^{(2,2,1,1)}/5)$.

$\lambda = (3,2,1)$, $\mu = (3,1,1,1)$, $m = 3$, $g = 1$, $k = 1$.

$$\begin{array}{lll}1&4&6\\2&5&\\3&&\end{array} \xrightarrow{\bar f} \begin{array}{lll}1&4&6\\2&&\\3&&\\5&&\end{array} - \begin{array}{lll}1&5&6\\2&&\\3&&\\4&&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,2,1)}, S^{(3,1,1,1)}/3)$.

$\lambda = (3,1,1,1)$, $\mu = (2,2,1,1)$, $m = 2$, $g = 2$, $k = 2$.

$$\begin{array}{lll}1&5&6\\2&&\\3&&\\4&&\end{array} \xrightarrow{\bar f} \begin{array}{ll}1&5\\2&6\\3&\\4&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,1,1,1)}, S^{(2,2,1,1)}/2)$.

$\lambda = (3,1,1,1)$, $\mu = (2,1,1,1,1)$, $m = 6$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll}1&5&6\\2&&\\3&&\\4&&\end{array} \xrightarrow{\bar f} \begin{array}{ll}1&6\\2&\\3&\\4&\\5&\end{array} + \begin{array}{ll}1&5\\2&\\3&\\4&\\6&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(3,1,1,1)}, S^{(2,1,1,1,1)}/6)$.

$\lambda = (2,2,2)$, $\mu = (2,2,1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{ll}1&4\\2&5\\3&6\end{array} \xrightarrow{\bar f} -\begin{array}{ll}1&5\\2&6\\3&\\4&\end{array} + \begin{array}{ll}1&4\\2&6\\3&\\5&\end{array} - \begin{array}{ll}1&4\\2&5\\3&\\6&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(2,2,2)}, S^{(2,2,1,1)}/2)$.

$\lambda = (2,2,1,1)$, $\mu = (2,1,1,1,1)$, $m = 4$, $g = 1$, $k = 1$.

$$\begin{array}{ll}1&5\\2&6\\3&\\4&\end{array} \xrightarrow{\bar f} -\begin{array}{ll}1&6\\2&\\3&\\4&\\5&\end{array} + \begin{array}{ll}1&5\\2&\\3&\\4&\\6&\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(2,2,1,1)}, S^{(2,1,1,1,1)}/4)$.

$\lambda = (2,1,1,1,1)$, $\mu = (1,1,1,1,1,1)$, $m = 6$, $g = 1$, $k = 1$.

$$\begin{array}{ll}1&6\\2&\\3&\\4&\\5&\end{array} \xrightarrow{\bar f} \begin{array}{l}1\\2\\3\\4\\5\\6\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_6}(S^{(2,1,1,1,1)}, S^{(1,1,1,1,1,1)}/6)$.

Case $n = 7$.

$\lambda = (7)$, $\mu = (6,1)$, $m = 7$, $g = 1$, $k = 6$, $X_2 = 5$, $X_3 = 4$, $X_4 = 3$, $X_5 = 2$, $X_6 = 1$, $r = 120$.

$$1\;2\;3\;4\;5\;6\;7 \xrightarrow{\bar f} \begin{array}{l}1\;3\;4\;5\;6\;7\\2\end{array} + \begin{array}{l}1\;2\;4\;5\;6\;7\\3\end{array} + \begin{array}{l}1\;2\;3\;5\;6\;7\\4\end{array} + \begin{array}{l}1\;2\;3\;4\;6\;7\\5\end{array} + \begin{array}{l}1\;2\;3\;4\;5\;7\\6\end{array} + \begin{array}{l}1\;2\;3\;4\;5\;6\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(6)}, S^{(5,1)}/7)$.

$\lambda = (6,1)$, $\mu = (5,2)$, $m = 5$, $g = 2$, $k = 5$, $X_3 = 3$, $X_4 = 2$, $X_5 = 1$, $r = 6$.

$$\begin{array}{l}1\;3\;4\;5\;6\;7\\2\end{array} \xrightarrow{\bar f} \begin{array}{l}1\;3\;7\;5\;6\\2\;4\end{array} + \begin{array}{l}1\;3\;4\;7\;6\\2\;5\end{array} + \begin{array}{l}1\;3\;4\;5\;7\\2\;6\end{array} + \begin{array}{l}1\;3\;4\;5\;6\\2\;7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(6,1)}, S^{(5,2)}/5)$.

$\lambda = (6,1)$, $\mu = (5,1,1)$, $m = 7$, $g = 1$, $k = 5$, $X_2 = 4$, $X_3 = 3$, $X_4 = 2$, $X_5 = 1$, $r = 24$.

$$\begin{array}{l}1\;3\;4\;5\;6\;7\\2\end{array} \xrightarrow{\bar f} \begin{array}{l}1\;7\;4\;5\;6\\2\\3\end{array} + \begin{array}{l}1\;3\;7\;5\;6\\2\\4\end{array} + \begin{array}{l}1\;3\;4\;7\;6\\2\\5\end{array} + \begin{array}{l}1\;3\;4\;5\;7\\2\\6\end{array} + \begin{array}{l}1\;3\;4\;5\;6\\2\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(6,1)}, S^{(5,1,1)}/7)$.

$\lambda = (5,2)$, $\mu = (4,3)$, $m = 3$, $g = 3$, $k = 4$, $X_4 = 1$.

$$\begin{array}{l}1\;3\;5\;6\;7\\2\;4\end{array} \xrightarrow{\bar f} \begin{array}{l}1\;3\;5\;7\\2\;4\;6\end{array} + \begin{array}{l}1\;3\;5\;6\\2\;4\;7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(5,2)}, S^{(4,3)}/3)$.

$\lambda = (5,2)$, $\mu = (4,2,1)$, $m = 6$, $g = 1$, $k = 4$, $X_2 = 4$, $X_3 = 2$, $X_4 = 1$, $r = 2$.

$$\begin{array}{l}1\;3\;5\;6\;7\\2\;4\end{array} \xrightarrow{\bar f} \left(\begin{array}{l}1\;5\;7\;6\\2\;4\\3\end{array} + \begin{array}{l}1\;3\;7\;6\\2\;5\\4\end{array} + \begin{array}{l}1\;6\;5\;7\\2\;4\\3\end{array} + \begin{array}{l}1\;3\;5\;7\\2\;6\\4\end{array} + \begin{array}{l}1\;7\;5\;6\\2\;4\\3\end{array} + \begin{array}{l}1\;3\;5\;6\\2\;7\\4\end{array}\right) + 4\cdot\left(\begin{array}{l}1\;3\;7\;6\\2\;4\\5\end{array} + \begin{array}{l}1\;3\;5\;7\\2\;4\\6\end{array} + \begin{array}{l}1\;3\;5\;6\\2\;4\\7\end{array}\right)$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(5,2)}, S^{(4,2,1)}/6)$.

$\lambda = (5,2)$, $\mu = (5,1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{l}1\;3\;5\;6\;7\\2\;4\end{array} \xrightarrow{\bar f} -\begin{array}{l}1\;4\;5\;6\;7\\2\\3\end{array} + \begin{array}{l}1\;3\;5\;6\;7\\2\\4\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(5,2)}, S^{(5,1,1)}/2)$.

$\lambda = (5,1,1)$, $\mu = (4,2,1)$, $m = 4$, $g = 2$, $k = 4$, $X_3 = 2$, $X_4 = 1$, $r = 2$.

$$\begin{array}{l}1\;4\;5\;6\;7\\2\\3\end{array} \xrightarrow{\bar f} \begin{array}{l}1\;4\;7\;6\\2\;5\\3\end{array} + \begin{array}{l}1\;4\;5\;7\\2\;6\\3\end{array} + \begin{array}{l}1\;4\;5\;6\\2\;7\\3\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(5,1,1)}, S^{(4,2,1)}/4)$.

$\lambda = (5,1,1)$, $\mu = (4,1,1,1)$, $m = 7$, $g = 1$, $k = 4$, $X_2 = 3$, $X_3 = 2$, $X_4 = 1$, $r = 6$.

$$\begin{array}{lllll} 1 & 4 & 5 & 6 & 7 \\ 2 \\ 3 \end{array} \xrightarrow{\bar{f}} \begin{array}{llll} 1 & 7 & 5 & 6 \\ 2 \\ 3 \\ 4 \end{array} + \begin{array}{llll} 1 & 4 & 7 & 6 \\ 2 \\ 3 \\ 5 \end{array} + \begin{array}{llll} 1 & 4 & 5 & 7 \\ 2 \\ 3 \\ 6 \end{array} + \begin{array}{llll} 1 & 4 & 5 & 6 \\ 2 \\ 3 \\ 7 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(5,1,1)}, S^{(4,1,1,1)}/7)$.

$\lambda = (4,3)$, $\mu = (3,3,1)$, $m = 5$, $g = 1$, $k = 3$, $X_2 = 3$, $X_3 = 2$, $r = 3$.

$$\begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 7 & 5 \\ 2 & 4 & 6 \\ 3 \end{array} + \begin{array}{lll} 1 & 3 & 5 \\ 2 & 7 & 6 \\ 4 \end{array} + \begin{array}{lll} 1 & 3 & 7 \\ 2 & 4 & 6 \\ 5 \end{array} + \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & 7 \\ 6 \end{array} + 2 \cdot \begin{array}{lll} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 7 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,3)}, S^{(3,3,1)}/5)$.

$\lambda = (4,3)$, $\mu = (4,2,1)$, $m = 3$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 \end{array} \xrightarrow{\bar{f}} \begin{array}{llll} 1 & 5 & 6 & 7 \\ 2 & 4 \\ 3 \end{array} - \begin{array}{llll} 1 & 6 & 5 & 7 \\ 2 & 4 \\ 3 \end{array} + \begin{array}{llll} 1 & 3 & 6 & 7 \\ 2 & 5 \\ 4 \end{array} - \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 6 \\ 4 \end{array} + \begin{array}{llll} 1 & 3 & 6 & 7 \\ 2 & 4 \\ 5 \end{array} - \begin{array}{llll} 1 & 3 & 5 & 7 \\ 2 & 4 \\ 6 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,3)}, S^{(4,2,1)}/3)$.

$\lambda = (4,2,1)$, $\mu = (3,3,1)$, $m = 2$, $g = 3$, $k = 3$.

$$\begin{array}{llll} 1 & 4 & 6 & 7 \\ 2 & 5 \\ 3 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 4 & 6 \\ 2 & 5 & 7 \\ 3 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,2,1)}, S^{(3,3,1)}/2)$.

$\lambda = (4,2,1)$, $\mu = (3,2,2)$, $m = 4$, $g = 2$, $k = 3$, $X_3 = 1$.

$$\begin{array}{llll} 1 & 4 & 6 & 7 \\ 2 & 5 \\ 3 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 4 & 7 \\ 2 & 5 \\ 3 & 6 \end{array} + \begin{array}{lll} 1 & 4 & 6 \\ 2 & 5 \\ 3 & 7 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,2,1)}, S^{(3,2,2)}/4)$.

$\lambda = (4,2,1)$, $\mu = (3,2,1,1)$, $m = 6$, $g = 1$, $k = 3$, $X_2 = 3$, $X_3 = 1$.

$$\begin{array}{llll} 1 & 4 & 6 & 7 \\ 2 & 5 \\ 3 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 6 & 7 \\ 2 & 5 \\ 3 \\ 4 \end{array} + \begin{array}{lll} 1 & 4 & 7 \\ 2 & 6 \\ 3 \\ 5 \end{array} + \begin{array}{lll} 1 & 7 & 6 \\ 2 & 5 \\ 3 \\ 4 \end{array} + \begin{array}{lll} 1 & 4 & 6 \\ 2 & 7 \\ 3 \\ 5 \end{array} + 3 \cdot \left(\begin{array}{lll} 1 & 4 & 7 \\ 2 & 5 \\ 3 \\ 6 \end{array} + \begin{array}{lll} 1 & 4 & 6 \\ 2 & 5 \\ 3 \\ 7 \end{array} \right)$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,2,1)}, S^{(3,2,1,1)}/6)$.

$\lambda = (4,2,1)$, $\mu = (4,1,1,1)$, $m = 3$, $g = 1$, $k = 1$.

$$\begin{array}{llll} 1 & 4 & 6 & 7 \\ 2 & 5 \\ 3 \end{array} \xrightarrow{\bar{f}} \begin{array}{llll} 1 & 5 & 6 & 7 \\ 2 \\ 3 \\ 4 \end{array} - \begin{array}{llll} 1 & 4 & 6 & 7 \\ 2 \\ 3 \\ 5 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,2,1)}, S^{(4,1,1,1)}/3)$.

$\lambda = (4,1,1,1)$, $\mu = (3,2,1,1)$, $m = 3$, $g = 2$, $k = 3$, $X_3 = 1$.

$$\begin{array}{llll} 1 & 5 & 6 & 7 \\ 2 \\ 3 \\ 4 \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 5 & 7 \\ 2 & 6 \\ 3 \\ 4 \end{array} + \begin{array}{lll} 1 & 5 & 6 \\ 2 & 7 \\ 3 \\ 4 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,1,1,1)}, S^{(3,2,1,1)}/3)$.

$\lambda = (4,1,1,1)$, $\mu = (3,1,1,1,1)$, $m = 7$, $g = 1$, $k = 3$, $X_2 = 2$, $X_3 = 1$, $r = 2$.

$$\begin{array}{llll}1&5&6&7\\2\\3\\4\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&7&6\\2\\3\\4\\5\end{array} + \begin{array}{lll}1&5&7\\2\\3\\4\\6\end{array} + \begin{array}{lll}1&5&6\\2\\3\\4\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(4,1,1,1)}, S^{(3,1,1,1,1)}/7)$.

$\lambda = (3,3,1)$, $\mu = (3,2,2)$, $m = 2$, $g = 2$, $k = 2$.

$$\begin{array}{lll}1&4&6\\2&5&7\\3\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&4&7\\2&5\\3&6\end{array} - \begin{array}{lll}1&4&6\\2&5\\3&7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,3,1)}, S^{(3,2,2)}/2)$.

$\lambda = (3,3,1)$, $\mu = (3,2,1,1)$, $m = 4$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll}1&4&6\\2&5&7\\3\end{array} \xrightarrow{\bar{f}} -\begin{array}{lll}1&6&7\\2&5\\3\\4\end{array} - \begin{array}{lll}1&4&7\\2&6\\3\\5\end{array} + \begin{array}{lll}1&7&6\\2&5\\3\\4\end{array} + \begin{array}{lll}1&4&6\\2&7\\3\\5\end{array} - \begin{array}{lll}1&4&7\\2&5\\3\\6\end{array} + \begin{array}{lll}1&4&6\\2&5\\3\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,3,1)}, S^{(3,2,1,1)}/4)$.

$\lambda = (3,2,2)$, $\mu = (2,2,2,1)$, $m = 5$, $g = 1$, $k = 2$, $X_2 = 3$.

$$\begin{array}{lll}1&4&7\\2&5\\3&6\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&7\\2&5\\3&6\\4\end{array} + \begin{array}{ll}1&4\\2&7\\3&6\\5\end{array} + \begin{array}{ll}1&4\\2&5\\3&7\\6\end{array} + 3\cdot\begin{array}{ll}1&4\\2&5\\3&6\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,2,2)}, S^{(2,2,2,1)}/5)$.

$\lambda = (3,2,2)$, $\mu = (3,2,1,1)$, $m = 2$, $g = 1$, $k = 1$.

$$\begin{array}{lll}1&4&7\\2&5\\3&6\end{array} \xrightarrow{\bar{f}} \begin{array}{lll}1&5&7\\2&6\\3\\4\end{array} - \begin{array}{lll}1&4&7\\2&6\\3\\5\end{array} + \begin{array}{lll}1&4&7\\2&5\\3\\6\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,2,2)}, S^{(3,2,1,1)}/2)$.

$\lambda = (3,2,1,1)$, $\mu = (2,2,2,1)$, $m = 3$, $g = 2$, $k = 2$.

$$\begin{array}{lll}1&5&7\\2&6\\3\\4\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&5\\2&6\\3&7\\4\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,2,1,1)}, S^{(2,2,2,1)}/3)$.

$\lambda = (3,2,1,1)$, $\mu = (2,2,1,1,1)$, $m = 6$, $g = 1$, $k = 2$, $X_2 = 2$.

$$\begin{array}{lll}1&5&7\\2&6\\3\\4\end{array} \xrightarrow{\bar{f}} \begin{array}{ll}1&7\\2&6\\3\\4\\5\end{array} + \begin{array}{ll}1&5\\2&7\\3\\4\\6\end{array} + 2\cdot\begin{array}{ll}1&5\\2&6\\3\\4\\7\end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,2,1,1)}, S^{(2,2,1,1,1)}/6)$.

$\lambda = (3,2,1,1)$, $\mu = (3,1,1,1,1)$, $m = 4$, $g = 1$, $k = 1$.

$$\begin{array}{lll} 1 & 5 & 7 \\ 2 & 6 & \\ 3 & & \\ 4 & & \end{array} \xrightarrow{\bar{f}} \begin{array}{lll} 1 & 6 & 7 \\ 2 & & \\ 3 & & \\ 4 & & \\ 5 & & \end{array} - \begin{array}{lll} 1 & 5 & 7 \\ 2 & & \\ 3 & & \\ 4 & & \\ 6 & & \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,2,1,1)}, S^{(3,1,1,1,1)}/4)$.

$\lambda = (3,1,1,1,1)$, $\mu = (2,2,1,1,1)$, $m = 2$, $g = 2$, $k = 2$.

$$\begin{array}{lll} 1 & 6 & 7 \\ 2 & & \\ 3 & & \\ 4 & & \\ 5 & & \end{array} \xrightarrow{\bar{f}} \begin{array}{ll} 1 & 6 \\ 2 & 7 \\ 3 & \\ 4 & \\ 5 & \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,1,1,1,1)}, S^{(2,2,1,1,1)}/2)$.

$\lambda = (3,1,1,1,1)$, $\mu = (2,1,1,1,1,1)$, $m = 7$, $g = 1$, $k = 2$, $X_2 = 1$.

$$\begin{array}{lll} 1 & 6 & 7 \\ 2 & & \\ 3 & & \\ 4 & & \\ 5 & & \end{array} \xrightarrow{\bar{f}} \begin{array}{ll} 1 & 7 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \end{array} + \begin{array}{ll} 1 & 6 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 7 & \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(3,1,1,1,1)}, S^{(2,1,1,1,1,1)}/7)$.

$\lambda = (2,2,2,1)$, $\mu = (2,2,1,1,1)$, $m = 3$, $g = 1$, $k = 1$.

$$\begin{array}{ll} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & \end{array} \xrightarrow{\bar{f}} \begin{array}{ll} 1 & 6 \\ 2 & 7 \\ 3 & \\ 4 & \\ 5 & \end{array} - \begin{array}{ll} 1 & 5 \\ 2 & 7 \\ 3 & \\ 4 & \\ 6 & \end{array} + \begin{array}{ll} 1 & 5 \\ 2 & 6 \\ 3 & \\ 4 & \\ 7 & \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(2,2,2,1)}, S^{(2,2,1,1,1)}/3)$.

$\lambda = (2,2,1,1,1)$, $\mu = (2,1,1,1,1,1)$, $m = 5$, $g = 1$, $k = 1$.

$$\begin{array}{ll} 1 & 6 \\ 2 & 7 \\ 3 & \\ 4 & \\ 5 & \end{array} \xrightarrow{\bar{f}} \begin{array}{ll} 1 & 7 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \end{array} - \begin{array}{ll} 1 & 6 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 7 & \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(2,2,1,1,1)}, S^{(2,1,1,1,1,1)}/5)$.

$\lambda = (2,1,1,1,1,1)$, $\mu = (1,1,1,1,1,1,1)$, $m = 7$, $g = 1$, $k = 1$.

$$\begin{array}{ll} 1 & 7 \\ 2 & \\ 3 & \\ 4 & \\ 5 & \\ 6 & \end{array} \xrightarrow{\bar{f}} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

generates $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_7}(S^{(2,1,1,1,1,1)}, S^{(1,1,1,1,1,1,1)}/7)$.

4.4 Approximating $(\mathbf{Z}\mathcal{S}_6)_{[3]}$

We investigate in (S 4.4.2) the intermediate order between $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ and the direct product of integral matrix rings described by specializations of the generic morphism (4.3.31) and, moreover, by two morphisms which are not covered by this generic morphism. We undertake the first two steps into the direction of a generic morphism in the situation of the simultaneous shift of several boxes in (S 4.4.1) so that, in particular, these two morphisms are obtained as specializations thereof.

4.4.1 Two-box-shift, easy cases

First we exhibit a generic morphism for a horizontal simultaneous two-box-shift in case there are essentially only two rows (4.4.1). Then we exhibit a generic morphism for a vertical simultaneous two-box-shift in case there are only two columns (4.4.3). These morphisms then cover the cases of the nonzero morphisms $S^{(5,1)}/3 \longrightarrow S^{(3,3)}/3$ and $S^{(2,2,2)}/3 \longrightarrow S^{(2,1,1,1,1)}/3$ needed for the approximation of $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ in (S 4.4.2).

Proposition 4.4.1 (a fixed point, cf. [J 78, 24.4]) *Let $n \geq 0$, let $g, l \in [0, n]$ such that $g + 2 \leq l$. Let λ be a partition of n with $\lambda_1 = l + 2$ and $\lambda_2 = g$. Let μ be the partition of n defined by*

$$\mu_i := \begin{cases} l & \text{for } i = 1 \\ g+2 & \text{for } i = 2 \\ \lambda_i & \text{else.} \end{cases}$$

In other words, μ arises from λ by a simultaneous shift of the rightmost two boxes from the first into the second row.

In the sequel we shall restrict ourselves to the consideration of the case

$$g = 0,$$

in which λ is just a row of length $l + 2 = n$. The formula in case $g = 0$ generalizes to the case of $g \geq 0$ by letting the polytabloid entries in columns $[1, g]$ constant under the map, and by performing the place operations on the remaining entries just as in case $g = 0$, only shifted by g columns to the right. The modulus in the general case is obtained by replacement of l by $l - g$.

We denote a μ-polytabloid by recording only the first two columns, i.e. in the form $\left\langle \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right\rangle$. This is to say, we drop the $[3, l]$-part of the first row without loss of information. The $\mathbf{Z}\mathcal{S}_n$-linear map

$$\begin{array}{ccc} F^\lambda & \xrightarrow{v} & S^\mu \\ [12\ldots n] & \longrightarrow & \sum_{i,j \in [3,n],\, i<j} \left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle - \sum_{k \in [4,n]} (k-2) \left\langle \begin{smallmatrix} 1 & 3 \\ 2 & k \end{smallmatrix} \right\rangle \end{array}$$

induces a $\mathbf{Z}\mathcal{S}_n$-linear map

$$S^\lambda/\nu \xrightarrow{\bar{v}} S^\mu/\nu,$$

where ν stands for $l + 1$ in case $l + 1$ is odd, and for $(l + 1)/2$ in case $l + 1$ is even.

We justify the reduction to the case $g = 0$. The Garnir relations involving pairs of subsequent columns in the range $[1, g]$ will map to the according Garnir relations in the target lattice, and thus vanish under the map extended by constant columns. The Garnir relation involving the columns g and $g + 1$ also will do so, because, as the formula shows, the entry in the upper left corner is kept fix in the reduced case. The remaining Garnir relation express, translated to the reduced case $g = 0$, that the image is a fixpoint modulo ν under the operation of the $\mathcal{S}_n$.

We need to show that the right hand side element is invariant modulo ν under the operation of the $\mathcal{S}_n$. So we **claim** that for $d \in [2, n]$ we have

$$\left(\sum_{i,j \in [3,n], i<j} \left\langle \begin{smallmatrix} 1 & 2 \\ i & j \end{smallmatrix} \right\rangle - \sum_{k \in [4,n]} (k-2) \left\langle \begin{smallmatrix} 1 & 3 \\ 2 & k \end{smallmatrix} \right\rangle \right) ((1\ d) - 1) \in \nu S^\mu.$$

Case $d = 2$**.**

$$\begin{aligned}
&\left(\sum_{i,j\in[3,n],\, i<j} \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle - \sum_{k\in[4,n]} (k-2)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\right)((1\ 2)-1)\\
={}& \textstyle\sum_{i,j\in[3,n],\, i<j}(\left\langle\begin{smallmatrix}2&1\\i&j\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle)\\
&\textstyle -\sum_{k\in[4,n]}(k-2)(\left\langle\begin{smallmatrix}2&3\\1&k\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle)\\
={}& \textstyle\sum_{i,j\in[4,n],\, i<j}(-\left\langle\begin{smallmatrix}1&3\\2&j\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}1&3\\2&i\end{smallmatrix}\right\rangle)\\
&\textstyle -\sum_{j\in[4,n]}\left\langle\begin{smallmatrix}1&3\\2&j\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{k\in[4,n]} 2(k-2)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\\
={}& \textstyle\sum_{k\in[4,n]}(-(k-4)+(n-k))\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{k\in[4,n]}(2k-5)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\\
={}& \textstyle\sum_{k\in[4,n]}(n-1)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle.
\end{aligned}$$

Case $d = 3$**.**

$$\begin{aligned}
&\left(\sum_{i,j\in[3,n],\, i<j} \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle - \sum_{k\in[4,n]} (k-2)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\right)((1\ 3)-1)\\
={}& \textstyle\sum_{i,j\in[4,n],\, i<j}(\left\langle\begin{smallmatrix}3&2\\i&j\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle)\\
&\textstyle +\sum_{j\in[4,n]}(\left\langle\begin{smallmatrix}3&2\\1&j\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\3&j\end{smallmatrix}\right\rangle)\\
&\textstyle -\sum_{k\in[4,n]}(k-2)(\left\langle\begin{smallmatrix}3&1\\2&k\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle)\\
={}& \textstyle -\sum_{i,j\in[4,n],\, i<j}\left\langle\begin{smallmatrix}1&2\\3&j\end{smallmatrix}\right\rangle\\
&\textstyle -2\sum_{j\in[4,n]}\left\langle\begin{smallmatrix}1&2\\3&j\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{k\in[4,n]}(k-2)\left\langle\begin{smallmatrix}1&2\\3&k\end{smallmatrix}\right\rangle\\
={}& \textstyle -\sum_{j\in[4,n]}(j-2)\left\langle\begin{smallmatrix}1&2\\3&j\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{k\in[4,n]}(k-2)\left\langle\begin{smallmatrix}1&2\\3&k\end{smallmatrix}\right\rangle\\
={}& 0.
\end{aligned}$$

Case $d \in [4, n]$**.**

$$\begin{aligned}
&\left(\sum_{i,j\in[3,n],\, i<j} \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle - \sum_{k\in[4,n]} (k-2)\left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle\right)((1\ d)-1)\\
={}& \textstyle\sum_{i,j\in[3,n]\backslash d,\, i<j}(\left\langle\begin{smallmatrix}d&2\\i&j\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle)\\
&\textstyle +\sum_{j\in[d+1,n]}(\left\langle\begin{smallmatrix}d&2\\1&j\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\d&j\end{smallmatrix}\right\rangle)\\
&\textstyle +\sum_{i\in[3,d-1]}(\left\langle\begin{smallmatrix}d&2\\i&1\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&d\end{smallmatrix}\right\rangle)\\
&\textstyle -\sum_{k\in[4,n]\backslash d}(k-2)(\left\langle\begin{smallmatrix}d&3\\2&k\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle)\\
&\textstyle -(d-2)(\left\langle\begin{smallmatrix}d&3\\2&1\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle)\\
={}& \textstyle -\sum_{i,j\in[3,n]\backslash d,\, i<j}\left\langle\begin{smallmatrix}1&2\\d&j\end{smallmatrix}\right\rangle\\
&\textstyle -2\sum_{j\in[d+1,n]}\left\langle\begin{smallmatrix}1&2\\d&j\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{i\in[3,d-1]}(\left\langle\begin{smallmatrix}1&i\\2&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&d\end{smallmatrix}\right\rangle)\\
&\textstyle +\sum_{k\in[4,n]\backslash d}(k-2)\left\langle\begin{smallmatrix}1&3\\d&k\end{smallmatrix}\right\rangle\\
&\textstyle -(d-2)(\left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle)\\
={}& \textstyle -\sum_{j\in[3,d-1]}(j-3)(\left\langle\begin{smallmatrix}1&2\\j&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}1&3\\2&j\end{smallmatrix}\right\rangle)\\
&\textstyle -\sum_{j\in[d+1,n]}(j-4)\left\langle\begin{smallmatrix}1&2\\d&j\end{smallmatrix}\right\rangle\\
&\textstyle -2\sum_{j\in[d+1,n]}\left\langle\begin{smallmatrix}1&2\\d&j\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{i\in[4,d-1]}(\left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&i\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\i&d\end{smallmatrix}\right\rangle)\\
&\textstyle +\left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle\\
&\textstyle +\sum_{k\in[4,d-1]}(k-2)(\left\langle\begin{smallmatrix}1&2\\k&d\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}1&3\\2&k\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle)\\
&\textstyle +\sum_{k\in[d+1,n]}(k-2)(\left\langle\begin{smallmatrix}1&2\\d&k\end{smallmatrix}\right\rangle + \left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle)\\
&\textstyle -(d-2)(\left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle)\\
={}& \textstyle\sum_{i\in[4,d-1]}\left\langle\begin{smallmatrix}1&2\\i&d\end{smallmatrix}\right\rangle(-(i-3)-1+(i-2))\\
&\textstyle +\sum_{i\in[d+1,n]}\left\langle\begin{smallmatrix}1&2\\d&i\end{smallmatrix}\right\rangle(-(i-4)-2+(i-2))\\
&\textstyle +\sum_{i\in[4,d-1]}\left\langle\begin{smallmatrix}1&3\\2&i\end{smallmatrix}\right\rangle(-(i-3)-1+(i-2))\\
&\textstyle +\left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle((d-3)(d-4)/2+(d-4)+1+((n-2)(n-1)/2-(d-2)(d-1)/2))+(d-2))\\
&\textstyle -\left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle(n-2)(n-1)/2
\end{aligned}$$

$$= (\left\langle\begin{smallmatrix}1&3\\2&d\end{smallmatrix}\right\rangle - \left\langle\begin{smallmatrix}1&2\\3&d\end{smallmatrix}\right\rangle)(n-2)(n-1)/2.$$

Now we set out to exhibit a somehow 'dual version' (?, cf. 6.2.6) of (4.4.1) by different means. Throughout this enterprise we freely use the language of (S 4.3).

Lemma 4.4.2 *Let n be a natural number, let μ be a partition of n and let $\langle a\rangle$ be a μ-polytabloid. Let $\xi \subset a_p$, $\emptyset \neq \eta \subseteq a_q$, $p < q$, such that $\#\xi + \#\eta = \#a_p - 1$. We obtain*

$$\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi \cup \eta) = \#\xi!\#\eta! \sum_{x\in\bar\xi} \langle\ \dots\ a_p^{\bar\xi\setminus x,\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle.$$

First we do a single step. Choose $x \in \bar\xi$.

$$\begin{aligned}
&\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi \cup x \cup \eta)\\
\overset{1., (4.3.5)}{=}\ &(\#\xi+1)!\#\eta!\ \langle\ \dots\ a_p^{\bar\xi\setminus x,\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle\\
\overset{2., \text{ proof of } (4.3.4)}{=}\ &(\#\xi+1)\ \langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta)\\
&-\#\eta\ \langle\ \dots\ a_p^{x,y}\ \dots\ a_q^{y,x}\ \dots\ \rangle \circ (\xi\cup\eta)
\end{aligned}$$

Iterating this step we obtain, choosing a sequence $x, x', \dots$ resp. $y, y', \dots$ of pairwise different elements of $\bar\xi$ resp. of η such that $x_0 \in \bar\xi$ is not contained in the former,

$$\begin{aligned}
&\langle\ \dots\ a_p\ \dots\ a_q\ \dots\ \rangle \circ (\xi\cup\eta)\\
=\ &\#\xi!\#\eta!\ \langle\ \dots\ a_p^{\bar\xi\setminus x}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle\\
&+\frac{\#\eta}{\#\xi+1}\ \langle\ \dots\ a_p^{x,y}\ \dots\ a_q^{y,x}\ \dots\ \rangle \circ (\xi\cup\eta)\\
=\ &\#\xi!\#\eta!\ \langle\ \dots\ a_p^{\bar\xi\setminus x}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle\\
&+\frac{\#\eta}{\#\xi+1}\Bigg((\#\xi+1)!(\#\eta-1)!\ \langle\ \dots\ a_p^{\bar\xi\setminus x',\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x'}\ \dots\ \rangle\\
&\qquad+\frac{\#\eta-1}{\#\xi+2}\ \langle\ \dots\ (a_p^{x,y})^{x',y'}\ \dots\ (a_q^{y,x})^{y',x'}\ \dots\ \rangle \circ (\xi\cup\eta)\Bigg)\\
=\ &\dots\\
=\ &\#\xi!\#\eta! \sum_{x\in\bar\xi\setminus x_0} \langle\ \dots\ a_p^{\bar\xi\setminus x,\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle\\
&+\frac{\#\eta!\#\xi!}{(\#\xi+\#\eta)!}\ \langle\ \dots\ a_p^{\bar\xi\setminus x_0,\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x_0}\ \dots\ \rangle \circ (\xi\cup\eta)\\
=\ &\#\xi!\#\eta!\sum_{x\in\bar\xi} \langle\ \dots\ a_p^{\bar\xi\setminus x,\eta}\ \dots\ a_q^{\eta,\bar\xi\setminus x}\ \dots\ \rangle\ .
\end{aligned}$$

Proposition 4.4.3 (two columns, two boxes) *Let n be a natural number. Let $k \in [0, n/2-2]$. Let λ be the partition of n having $\lambda'_1 = n-k-2$, $\lambda'_2 = k+2$. Let μ be the partition of n with $\mu'_1 = n-k$, $\mu'_2 = k$.*

In other words, μ arises from λ by the simultaneous shift of two boxes from the second column.

We denote the first column of a λ-tableau by a, the second by b. For elements $i \neq j$ in b we denote

$$\varepsilon_{ij} := (-1)^{\pi i + \pi j},$$

where πi denotes the tuple position of i in b (from top to bottom). ε_{ij} is not to be confused with the signature of a permutation. Given a tuple b and two elements i, j in b, we denote by $b^{ij,}$ the tuple b with i and j dropped and the remaining part shifted accordingly.

The $\mathbf{Z}\mathcal{S}_n$-linear map

$$\begin{array}{ccc}
F^\lambda & \xrightarrow{w} & S^\mu\\
[a\ b] & \longrightarrow & \displaystyle\sum_{i,j\in b,\ \pi i<\pi j} \varepsilon_{ij} \left\langle \begin{matrix} a & b^{ij,}\\ i & \\ j & \end{matrix} \right\rangle
\end{array}$$

factors over

$$S^\lambda/\nu \xrightarrow{\bar{w}} S^\mu/\nu,$$

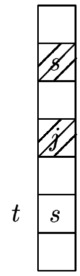
where ν *stands for* $n-2k-1$ *in case* n *is even, and for* $(n-2k-1)/2$ *in case* n *is odd.*

Note that we might still attach further columns to the left of λ *in such a way that* μ *becomes a partition and extend our morphism accordingly since such columns do not affect the following calculations.*

Step 1. We **claim** that the signed column transpositions (4.3.2) vanish under w. It suffices to consider transpositions in b. So, let $s,t\in b$, $\pi s<\pi t$. Let the symbol $\{u<v\}$ take the value $+1$ in case $u<v$ and the value -1 in case $u>v$.

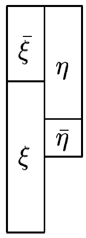
$$\begin{aligned}
&\sum_{i,j\in b,\,\pi i<\pi j}\varepsilon_{ij}\left(\left\langle\begin{matrix} a & b^{ij,}\\ i & \\ j & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & b^{ij,}\\ i & \\ j & \end{matrix}\right\rangle(s\;t)\right)\\
=\;&\textstyle\sum_{i,j\in b\setminus\{s,t\},\,\pi i<\pi j}\varepsilon_{ij}\left(\left\langle\begin{matrix} a & b^{ij,}\\ i & \\ j & \end{matrix}\right\rangle-\left\langle\begin{matrix} a & b^{ij,}\\ i & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{j\in b\setminus t,\,\pi s<\pi j}\varepsilon_{sj}\left(\left\langle\begin{matrix} a & b^{sj,}\\ s & \\ j & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & (b^{sj,})^{t,s}\\ t & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{j\in b,\,\pi t<\pi j}\varepsilon_{tj}\left(\left\langle\begin{matrix} a & b^{tj,}\\ t & \\ j & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & (b^{tj,})^{s,t}\\ s & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{i\in b,\,\pi i<\pi s}\varepsilon_{is}\left(\left\langle\begin{matrix} a & b^{is,}\\ i & \\ s & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & (b^{is,})^{t,s}\\ i & \\ t & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{i\in b\setminus s,\,\pi i<\pi t}\varepsilon_{it}\left(\left\langle\begin{matrix} a & b^{it,}\\ i & \\ t & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & (b^{it,})^{s,t}\\ i & \\ s & \end{matrix}\right\rangle\right)\\
&+\varepsilon_{st}\left(\left\langle\begin{matrix} a & b^{st,}\\ s & \\ t & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & b^{st,}\\ t & \\ s & \end{matrix}\right\rangle\right)\\
=\;&\textstyle\sum_{j\in b\setminus t,\,\pi s<\pi j}\left(\varepsilon_{sj}\left\langle\begin{matrix} a & b^{sj,}\\ s & \\ j & \end{matrix}\right\rangle-\{\pi t<\pi j\}\varepsilon_{jt}\left\langle\begin{matrix} a & b^{tj,}\\ t & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{j\in b,\,\pi t<\pi j}\left(\varepsilon_{tj}\left\langle\begin{matrix} a & b^{tj,}\\ t & \\ j & \end{matrix}\right\rangle-\varepsilon_{sj}\left\langle\begin{matrix} a & b^{sj,}\\ s & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{i\in b,\,\pi i<\pi s}\left(\varepsilon_{is}\left\langle\begin{matrix} a & b^{is,}\\ i & \\ s & \end{matrix}\right\rangle-\varepsilon_{it}\left\langle\begin{matrix} a & b^{it,}\\ i & \\ t & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{i\in b\setminus s,\,\pi i<\pi t}\left(\varepsilon_{it}\left\langle\begin{matrix} a & b^{it,}\\ i & \\ t & \end{matrix}\right\rangle-\{\pi i<\pi s\}\varepsilon_{is}\left\langle\begin{matrix} a & b^{is,}\\ i & \\ s & \end{matrix}\right\rangle\right)\\
=\;&\textstyle\sum_{j\in b,\,\pi s<\pi j<\pi t}\left(\varepsilon_{sj}\left\langle\begin{matrix} a & b^{sj,}\\ s & \\ j & \end{matrix}\right\rangle+\varepsilon_{jt}\left\langle\begin{matrix} a & b^{tj,}\\ t & \\ j & \end{matrix}\right\rangle\right)\\
&+\textstyle\sum_{i\in b,\,\pi s<\pi i<\pi t}\left(\varepsilon_{is}\left\langle\begin{matrix} a & b^{is,}\\ i & \\ s & \end{matrix}\right\rangle+\varepsilon_{it}\left\langle\begin{matrix} a & b^{it,}\\ i & \\ t & \end{matrix}\right\rangle\right)\\
=\;&0.
\end{aligned}$$

It is helpful to draw the modifications of the column b as little diagrams. E.g. for the first step the (abbreviated) equation $(b^{sj,})^{t,s}=-\{\pi t<\pi j\}\varepsilon_{st}b^{tj,}$ can be deduced using the diagram



We leave it to the reader to sketch the necessary diagrams for the various steps, here and below.

Step 2. We **claim** that the one-step Garnir relations $G_{[a\ b],\xi,\eta}$ (4.3.2) such that ξ is the (numerically) upper interval of a and η the (numerically) lower interval of b, and such that a and b are ordered increasingly from top to bottom, vanish under w modulo ν. In particular, we may drop the π's from the formula giving their image of w. Recall that $\#\xi + \#\eta = \#a + 1$. Pictorially, ξ and η are situated as follows.



An inspection of the proof of (4.3.2) shows that this claim suffices to prove the proposition. We have to calculate the expression

$$G_{[a\ b],\xi,\eta}w = \frac{1}{\#\xi!\#\eta!}\left(\sum_{i,j\in b,\, i<j}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta).$$

Case $\#\eta \geq 2$.

Let $y < y'$ be the largest two elements of η, i.e. those sitting at the bottom of η. Note $\varepsilon_{yy'} = -1$. Let

$$\begin{aligned} \tilde{\eta} &:= \eta\backslash y' \\ \eta' &:= \eta\backslash(y,y'). \end{aligned}$$

All occurring bijections in the tuple substitutions are meant to respect the order of the elements they set in correspondence and are thus determined uniquely once given two sets of numbers.

Step 2a. We calculate the following partial sum.

$$\begin{aligned}
&\frac{1}{\#\xi!\#\eta!}\left(\sum_{i,j\in b\backslash\eta,\, i<j}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) \\
\overset{(4.4.2)}{=}\ & \sum_{i,j\in b\backslash\eta,\, i<j}\varepsilon_{ij}\left(\left(\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & ((b^{ij,})^{\eta',\bar{\xi}\backslash x})^{(y,y'),(i,j)} \\ y & \\ y' & \end{matrix}\right\rangle\right) + \left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & ((b^{ij,})^{\tilde{\eta},\bar{\xi}})^{y',j} \\ i & \\ y' & \end{matrix}\right\rangle + \left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & ((b^{ij,})^{\tilde{\eta},\bar{\xi}})^{y',i} \\ y' & \\ j & \end{matrix}\right\rangle\right) \\
=\ & \sum_{i,j\in b\backslash\eta,\, i<j}\left(\varepsilon_{yy'}\left(\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & (b^{yy',})^{\eta',\bar{\xi}\backslash x} \\ y & \\ y' & \end{matrix}\right\rangle\right) + \varepsilon_{iy'}\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{iy',})^{\tilde{\eta},\bar{\xi}} \\ i & \\ y' & \end{matrix}\right\rangle - \varepsilon_{y'j}\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{y'j,})^{\tilde{\eta},\bar{\xi}} \\ y' & \\ j & \end{matrix}\right\rangle\right) \\
=\ & -\binom{\#b\backslash\eta}{2}\left(\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & (b^{yy',})^{\eta',\bar{\xi}\backslash x} \\ y & \\ y' & \end{matrix}\right\rangle\right) + (\#b\backslash\eta - 1)\left(\sum_{i\in b\backslash\eta}\varepsilon_{iy'}\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{iy',})^{\tilde{\eta},\bar{\xi}} \\ i & \\ y' & \end{matrix}\right\rangle\right).
\end{aligned}$$

Step 2b. We calculate the following partial sum.

$$\frac{1}{\#\xi!\#\eta!}\left(\sum_{i\in\eta,\, j\in b\backslash\eta}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta)$$

$$
\begin{aligned}
&\overset{(4.3.8),\,(i\ y')}{=} && \frac{1}{\#\xi!\#\eta!}\left(\sum_{i\in\eta,\, j\in b\backslash\eta}(-\varepsilon_{ij})\left\langle\begin{matrix} a & (b^{ij,})^{y',i} \\ y' & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) \\
&= && \frac{\#\eta}{\#\xi!\#\eta!}\left(\sum_{j\in b\backslash\eta}\varepsilon_{y'j}\left\langle\begin{matrix} a & b^{y'j,} \\ y' & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) \\
&\overset{(4.4.2)}{=} && (\#\xi+1)\sum_{j\in b\backslash\eta}\varepsilon_{y'j}\left(\left(\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & ((b^{y'j,})^{\eta',\bar{\xi}\backslash x})^{y,j} \\ y' & \\ y & \end{matrix}\right\rangle\right)+\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{y'j,})^{\tilde{\eta},\bar{\xi}} \\ y' & \\ j & \end{matrix}\right\rangle\right) \\
&= && (\#\xi+1)\#b\backslash\eta\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & (b^{yy',})^{\eta',\bar{\xi}\backslash x} \\ y & \\ y' & \end{matrix}\right\rangle-(\#\xi+1)\sum_{j\in b\backslash\eta}\varepsilon_{y'j}\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{y'j,})^{\tilde{\eta},\bar{\xi}} \\ j & \\ y' & \end{matrix}\right\rangle.
\end{aligned}
$$

Step 2c. We calculate the following partial sum.

$$
\begin{aligned}
& && \frac{1}{\#\xi!\#\eta!}\left(\sum_{i,j\in\eta,\, i<j}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) \\
&\overset{(4.3.8),\,(y\ i)(y'\ j)}{=} && \frac{1}{\#\xi!\#\eta!}\binom{\#\eta}{2}\varepsilon_{yy'}\left\langle\begin{matrix} a & b^{yy',} \\ y & \\ y' & \end{matrix}\right\rangle\circ(\xi\cup\eta) \\
&\overset{(4.4.2)}{=} && -\binom{\#\xi+2}{2}\sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & (b^{yy',})^{\eta',\bar{\xi}\backslash x} \\ y & \\ y' & \end{matrix}\right\rangle
\end{aligned}
$$

Writing

$$
\begin{aligned}
A &:= \sum_{x\in\bar{\xi}}\left\langle\begin{matrix} a^{\bar{\xi}\backslash x,\eta'} & (b^{yy',})^{\eta',\bar{\xi}\backslash x} \\ y & \\ y' & \end{matrix}\right\rangle \\
B &:= \sum_{i\in b\backslash\eta}\varepsilon_{iy'}\left\langle\begin{matrix} a^{\bar{\xi},\tilde{\eta}} & (b^{iy',})^{\tilde{\eta},\bar{\xi}} \\ i & \\ y' & \end{matrix}\right\rangle
\end{aligned}
$$

we obtain, remarking that $\#\xi = n-k-1-\#\eta$ and $\#b = k+2$,

$$
\begin{aligned}
\frac{1}{\#\xi!\#\eta!}\left(\sum_{i,j\in b,\, i<j}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) &= A\left(-\binom{\#b\backslash\eta}{2}+(\#\xi+1)\#b\backslash\eta-\binom{\#\xi+2}{2}\right) \\
&\quad +B\,(\#b\backslash\eta-1-(\#\xi+1)) \\
&= -\binom{n-2k-1}{2}A-(n-2k-1)B.
\end{aligned}
$$

Case $\eta=(y')$ consists of a single element. In particular, $\xi = a$ and $\bar{\xi}=\emptyset$. This case is only formally distinct from the former for the lack of y.

Step 2a′. We calculate the following partial sum.

$$
\begin{aligned}
& && \frac{1}{\#\xi!}\left(\sum_{i,j\in b\backslash\eta,\, i<j}\varepsilon_{ij}\left\langle\begin{matrix} a & b^{ij,} \\ i & \\ j & \end{matrix}\right\rangle\right)\circ(\xi\cup\eta) \\
&\overset{(4.4.2)}{=} && \sum_{i,j\in b\backslash\eta,\, i<j}\varepsilon_{ij}\left(\left\langle\begin{matrix} a & (b^{ij,})^{y',j} \\ i & \\ y' & \end{matrix}\right\rangle+\left\langle\begin{matrix} a & (b^{ij,})^{y',i} \\ y' & \\ j & \end{matrix}\right\rangle\right) \\
&= && \sum_{i,j\in b\backslash\eta,\, i<j}\left(\varepsilon_{iy'}\left\langle\begin{matrix} a & b^{iy',} \\ i & \\ y' & \end{matrix}\right\rangle-\varepsilon_{y'j}\left\langle\begin{matrix} a & b^{y'j,} \\ y' & \\ j & \end{matrix}\right\rangle\right) \\
&= && (\#b-2)\sum_{i\in b\backslash\eta}\varepsilon_{iy'}\left\langle\begin{matrix} a & b^{iy',} \\ i & \\ y' & \end{matrix}\right\rangle.
\end{aligned}
$$

Step 2b′. We calculate the following partial sum.

$$\frac{1}{\#\xi!}\left(\sum_{j\in b\setminus\eta}\varepsilon_{y'j}\left\langle\begin{array}{ll} a & b^{y'j,} \\ y' & \\ j & \end{array}\right\rangle\right)\circ(\xi\cup\eta)=(\#\xi+1)\sum_{j\in b\setminus\eta}\varepsilon_{y'j}\left\langle\begin{array}{ll} a & b^{y'j,} \\ y' & \\ j & \end{array}\right\rangle$$

Note that

$$(\#b-2)-(\#\xi+1)=-(n-2k-1).$$

Remark 4.4.4 The specializations of the generic morphisms obtained in (4.4.1) and (4.4.3) are nonzero, as can be seen by regarding standard tableaux.

Remark 4.4.5 There exist the following two-box-shift morphisms, which are predicted by the result of Carter and Payne [CP 80] and which have been calculated directly by computer ([3]). We drop the brackets.

$$S^{(3,3,2)}/5 \longrightarrow S^{(2,2,2,1,1)}/5$$

$$\begin{array}{lll} 1&4&7\\ 2&5&8\\ 3&6& \end{array} \longrightarrow -\begin{array}{ll}1&6\\2&7\\3&8\\4&\\5&\end{array}+\begin{array}{ll}1&5\\2&7\\3&8\\4&\\6&\end{array}-\begin{array}{ll}1&4\\2&7\\3&8\\5&\\6&\end{array}+\begin{array}{ll}1&5\\2&6\\3&8\\4&\\7&\end{array}-\begin{array}{ll}1&4\\2&6\\3&8\\5&\\7&\end{array}$$

$$+\begin{array}{ll}1&4\\2&5\\3&8\\6&\\7&\end{array}-\begin{array}{ll}1&5\\2&6\\3&7\\4&\\8&\end{array}+\begin{array}{ll}1&4\\2&6\\3&7\\5&\\8&\end{array}-\begin{array}{ll}1&4\\2&5\\3&7\\6&\\8&\end{array}+2\cdot\begin{array}{ll}1&4\\2&5\\3&6\\7&\\8&\end{array}$$

$$S^{(3,3,1,1)}/3 \longrightarrow S^{(2,2,1,1,1,1)}/3$$

$$\begin{array}{lll} 1&5&7\\ 2&6&8\\ 3&&\\ 4&& \end{array} \longrightarrow -\begin{array}{ll}1&7\\2&8\\3&\\4&\\5&\\6&\end{array}+\begin{array}{ll}1&6\\2&8\\3&\\4&\\5&\\7&\end{array}-\begin{array}{ll}1&5\\2&8\\3&\\4&\\6&\\7&\end{array}-\begin{array}{ll}1&6\\2&7\\3&\\4&\\5&\\8&\end{array}+\begin{array}{ll}1&5\\2&7\\3&\\4&\\6&\\8&\end{array}-\begin{array}{ll}1&5\\2&6\\3&\\4&\\7&\\8&\end{array}$$

$$S^{(4,4)}/5 \longrightarrow S^{(3,3,1,1)}/5$$

$$\begin{array}{llll} 1&3&5&7\\ 2&4&6&8 \end{array} \longrightarrow -2\cdot\begin{array}{lll}1&5&7\\2&6&8\\3&&\\4&&\end{array}-\begin{array}{lll}1&4&7\\2&6&8\\3&&\\5&&\end{array}+\begin{array}{lll}1&3&7\\2&6&8\\4&&\\5&&\end{array}+\begin{array}{lll}1&4&7\\2&5&8\\3&&\\6&&\end{array}-\begin{array}{lll}1&3&7\\2&5&8\\4&&\\6&&\end{array}$$

$$-2\cdot\begin{array}{lll}1&3&7\\2&4&8\\5&&\\6&&\end{array}+\begin{array}{lll}1&4&6\\2&5&8\\3&&\\7&&\end{array}-\begin{array}{lll}1&3&6\\2&5&8\\4&&\\7&&\end{array}-\begin{array}{lll}1&3&6\\2&4&8\\5&&\\7&&\end{array}-\begin{array}{lll}1&4&5\\2&6&8\\3&&\\7&&\end{array}$$

$$+\begin{array}{lll}1&3&5\\2&6&8\\4&&\\7&&\end{array}+\begin{array}{lll}1&3&5\\2&4&8\\6&&\\7&&\end{array}-\begin{array}{lll}1&4&6\\2&5&7\\3&&\\8&&\end{array}+\begin{array}{lll}1&3&6\\2&5&7\\4&&\\8&&\end{array}+\begin{array}{lll}1&3&6\\2&4&7\\5&&\\8&&\end{array}$$

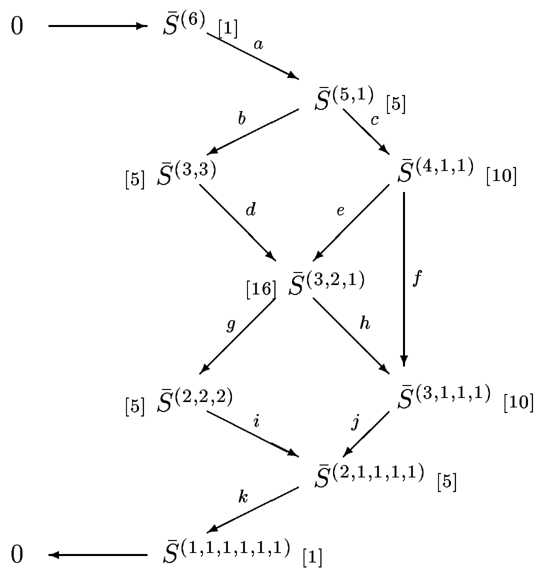
$$+\begin{array}{lll}1&4&5\\2&6&7\\3&&\\8&&\end{array}-\begin{array}{lll}1&3&5\\2&6&7\\4&&\\8&&\end{array}-\begin{array}{lll}1&3&5\\2&4&7\\6&&\\8&&\end{array}-2\cdot\begin{array}{lll}1&3&5\\2&4&6\\7&&\\8&&\end{array}$$

[3] Cf. http://xxx.lanl.gov/abs/math.RT/0003083.

4.4.2 Approximating $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ via specializations

We shall investigate to what extent the specializations of the generic morphisms already exhibited give the ties describing $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ (cf. S 2.3.3). This may be considered as the failure of an attempt to describe the quasiblock 11 via specializations of generic morphisms, say, up to the 9-tie, in a similar manner to the examples given in (S 4.2.2). We replace direct matrix calculations by usage of elementary properties of the morphisms, viz. their ranks and various commutativities.

Consider the following diagram of morphisms modulo 3, in which we abbreviate $\bar{S}^\lambda := S^\lambda/3$. The number in brackets indicates the dimension of the respective Specht module over $\mathbf{F}_3$.



a, c, f, j and k form the long exact hook sequence (4.2.4), taken modulo 3. e, h, d and g are further specializations of the generic morphism in (4.3.31). b is the specialization of the generic morphism in (4.4.1), i is the negative of the specialization of the generic morphism in (4.4.3). We have

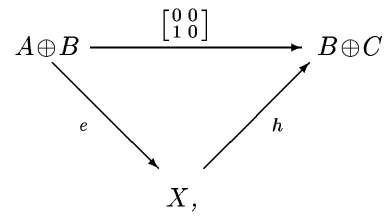
$$\begin{aligned} ab &= 0 \\ bd &= ce \\ eh &= f \\ dg &= 0 \\ dh &= 0 \\ eg &= 0 \\ gi &= hj \\ ik &= 0 \end{aligned}$$

as can be checked on a single polytabloid generating the Specht module (cf. S 4.3.5). It would be desirable to have general statements of this kind.

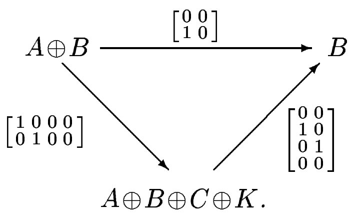
The ranks of the linear maps are calculated resp. known (proof of 4.2.4) to be

$$\begin{aligned} \operatorname{rk} a &= 1 \\ \operatorname{rk} b &= 4 \\ \operatorname{rk} c &= 4 \\ \operatorname{rk} d &= 5 \\ \operatorname{rk} e &= 10 \\ \operatorname{rk} f &= 6 \\ \operatorname{rk} g &= 5 \\ \operatorname{rk} h &= 10 \\ \operatorname{rk} i &= 4 \\ \operatorname{rk} j &= 4 \\ \operatorname{rk} k &= 1 \end{aligned}$$

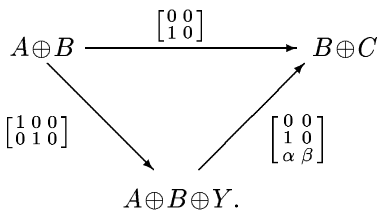
Lemma 4.4.6 (of linear algebra type) *Suppose given a commutative triangle in an abelian category of the form*



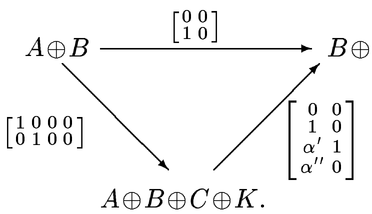
where e is a split monomorphism and h is a split epimorphism. Then X can be replaced isomorphically by $A \oplus B \oplus C \oplus K$ for some object K such that the morphisms become



Using that e is a split monomorphism and that the triangle commutes we may substitute X isomorphically to obtain



Writing down a coretraction retracted by the substitute of h yields β to be a split epimorphism. Substituting Y isomorphically then gives

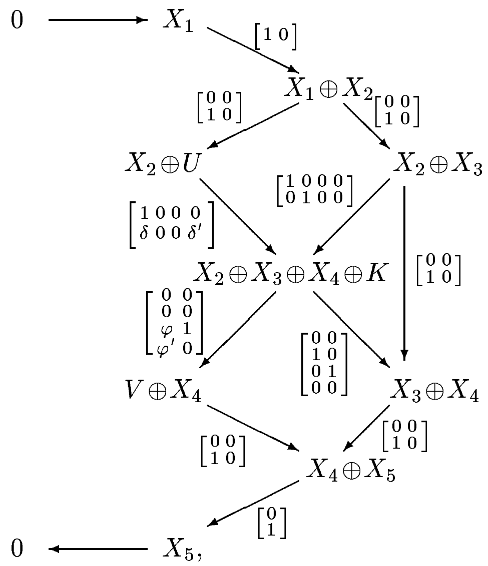


Isomorphic substitution by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \alpha' & 1 & 0 \\ 0 & \alpha'' & 0 & 1 \end{bmatrix}$$

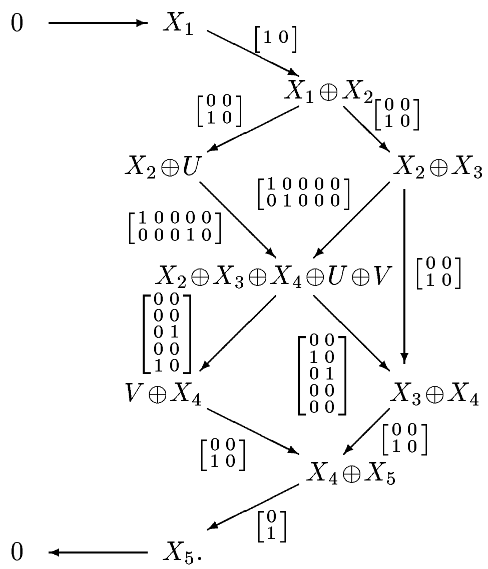
yields the result.

Using the commutativities stated above and applying (4.4.6) to $eh = f$ we may substitute our initial diagram isomorphically by the following diagram of **vector spaces over $\mathbf{F}_3$.**

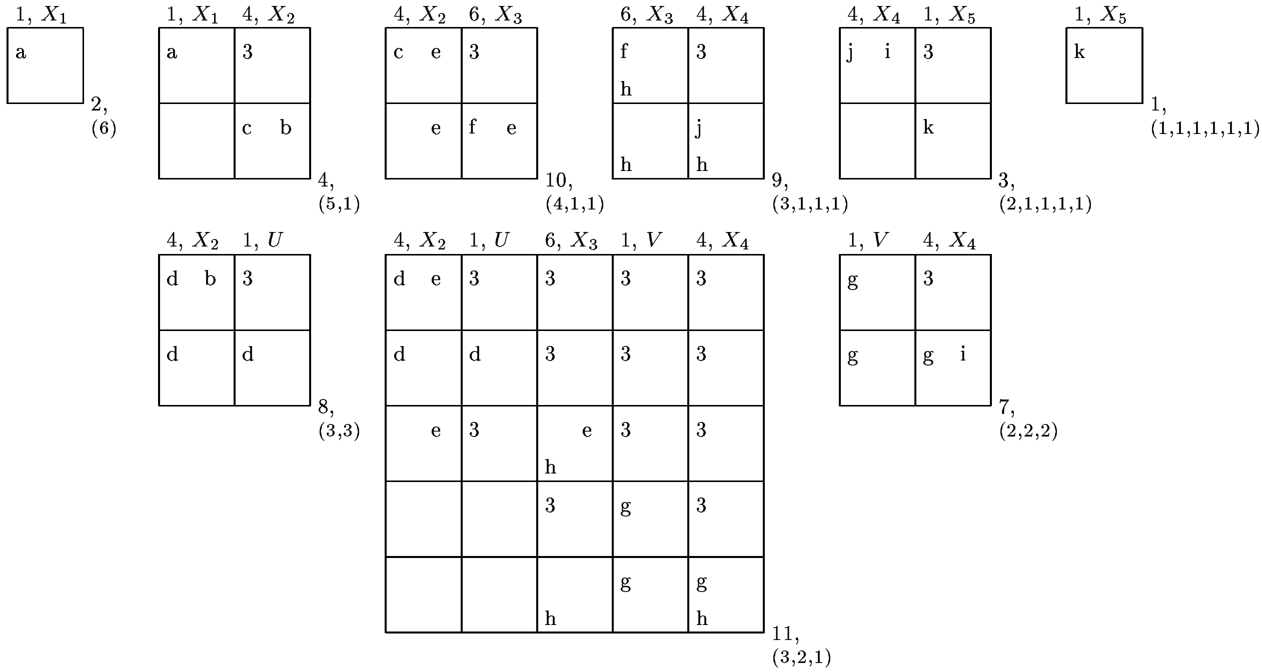


in which however the morphisms are $\mathcal{S}_6$-linear - the respective module structure given by 'transport de structure' -, only the direct sum decompositions are not.

Now ranks and $dg = 0$ yield δ' and φ' to constitute a short exact sequence. Isomorphic substitution according to this short exact sequence as well as via $\left[\begin{smallmatrix}1&0\\ \delta&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\ \varphi&1\end{smallmatrix}\right]$ yields the diagram



Now choosing integral inverse image decompositions (A.2.1) we obtain the following ties caused by these morphisms - i.e. resulting from the diagram expressing the linear map to be $\mathcal{S}_6$-linear modulo 3 - and denoted by the same letter, except for the single ties. f is redundant since $f = gh$. The numbering of the quasiblocks is that of (S 2.3.1), but we have also recorded the respective partition, and similarly, we have recorded not only the Morita multiplicities but also the names of the corresponding linear summands.



a	x^2	$\equiv_3$	x^4
b	x^4	$\equiv_3$	x^8
c	x^4	$\equiv_3$	x^{10}
d	x^8	$\equiv_3$	x^{11}
e	x^{10}	$\equiv_3$	x^{11}
f	x^{10}	$\equiv_3$	x^9
g	x^{11}	$\equiv_3$	x^7
h	x^{11}	$\equiv_3$	x^9
i	x^7	$\equiv_3$	x^3
j	x^9	$\equiv_3$	x^3
k	x^3	$\equiv_3$	x^1

As usual, for accordance with (2.3.3) one should check that the morphisms appearing there implicitely conincide with the morphisms used here. The precise statement of what we have just obtained is that there exist integral bases of the Specht lattices such that the image of the corresponding embedding of $(\mathbf{Z}\mathcal{S}_6)_{[3]}$ into a product of integral matrix rings Γ is contained in the ring just described. Which has index 3^{397} in Γ, whereas the index of that embedding is 3^{558} (S 2.3.1).

4.5 Table of morphisms

For ease of reading, we list and describe in an informal manner the generic modular morphisms between Specht lattices exhibited so far.

combinatorial situation	type of formula	modulus	reference
the target partition arises from the start by the shift of an arbitrary box on the edge to an arbitrary edge position further down to the left (provided the resulting figure represents a partition)	push through the entry on the withdrawn box position to the new position stepwise and form a linear combination in the possibilities of doing so the coefficients of which are polynomial in the combinatorial data	the path length covered by the moved box	(4.3.31), in case of hooks, alternatively (4.2.3)
from the single row to the partition with two boxes shifted into the second row, and mutatis mutandis with columns arbitrarily attached to the left (provided the resulting figure represents a partition)	a sum over two entry shifts minus a sum over one entry shifts involving this entry as coefficient	the row length minus one, divided by two if possible, mutatis mutandis in case of attached columns	(4.4.1)
from a partition consisting of two columns to the partition arising from it by shifting two boxes from the second to the first column (provided the resulting figure represents a partition)	a signed sum over two ordered entries from the second column appended to the first column	$n - 2k - 1$, divided by two if possible, where k is the length of the second column in the target partition	(4.4.3)
between hooks with certain parameters required to be even	the sum of the standard polytabloids in the target	2	(4.2.11)
from a hook with even column length to its transpose	a sum over one-entry-replacements in the transposed polytabloid	2	(4.2.13)
two-box-shift downwards between hooks, n odd	the sum over two ordered entries in the row appended to the column	2	(4.2.14)
two-box-shift upwards between hooks	the sum over two ordered entries in the column appended to the row	2	(4.2.15)

Chapter 5

The truss

We shall construct the **truss** (German: Gebälk), which is a certain combinatorially given lattice over an integral path algebra the quiver of which can be depicted as a binary double tree with some vertices identified, whence its name. The truss gives a complete set of ties for the inclusion $\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda}$ via (5.3.15). Thus the (non precisely posed) problem of finding a normal form for the truss turns out to be equivalent to our initial (and likewise non precisely posed) problem of finding a satisfactory embedding in the sense of (S 0.1.2).

Let it be remarked that the top part of the truss of $\mathbf{Z}\mathcal{S}_n$ arises from the truss of $\mathbf{Z}\mathcal{S}_{n-1}$ via induction (cf. 5.3.3, 5.3.5, 5.3.6, 5.3.7, 5.3.8).

James has discovered short exact sequences of $\mathbf{Z}\mathcal{S}_n$-lattices, the **James extensions** [J 78, 17.13], which in particular may be used to filter $\mathbf{Z}\mathcal{S}_n$ by Specht lattices. Such a filtration a priori suffices to give a complete set of ties, provided the extensions involved in this filtration are sufficiently well known (cf. C 3).

More precisely, it is possible to write $M^{(1^n)}$ as iterated extension, starting with Specht lattices, and using only James extensions, which then yields such a filtration via pullbacks. That procedure of unscrewing gives rise to the binary tree mentioned above. The information needed of an occurring James extension in order to be able to read off the ties is a retraction of its inclusion up to a nonzero integral scalar factor, which is divided by the order of the element in Ext^1 represented by such an extension. This can be done in a combinatorial manner along the lines of the construction of the box shift morphism (S 4.3). Thus this chapter may be viewed as a corollary to James' discovery and, as we present it, to the proof of (4.3.31). However, the latter dependence is merely due to the order in which we proceed, in view of the overlap of arguments. This overlap consists of parts of (S 4.3.3) and of the overall idea of (S 4.3). We shall need to recall slightly modified methods and assertions from there.

Let n be a natural number.

5.1 The James extension

Since we need the integral version (5.1.18) of the James extension [J 78, 17.13], we review the according part of [J 78]. We derive it from the combinatorial result [J 78, 15.14] by an application of James' arguments to our slightly modified assumption.

Definition 5.1.1 *A* **prepartition** ν *of* n *is a map*

$$\begin{array}{ccc} \mathbf{N} & \xrightarrow{\nu} & \mathbf{N} \\ i & \longrightarrow & \nu_i \end{array}$$

such that $\sum_{i\in\mathbf{N}} \nu_i = n$. *Replacing* λ *by* ν, *(4.1.1) carries over verbatim until the definition of the* $\mathbf{Z}\mathcal{S}_n$*-lattice* M^ν, *which as a* $\mathbf{Z}$*-module is free on the set of* ν*-tabloids.*

Let ν be a prepartition of n, let λ be a partition of some natural number $\leq n$ such that

$$\lambda_i \leq \nu_i$$

for all $i \in \mathbf{N}$. For short, we write $\lambda \subseteq \nu$ for such a situation.

Definition 5.1.2 *Let* $[a]$ *be a* ν*-tableau. Let* $C_{a,\lambda}$ *be the column stabilizer of* $[a]$ *which moves only entries inside* λ, *i.e.*

$$C_{a,\lambda} := \{\sigma \in \mathcal{S}_n \mid a_{ij}\sigma \in a_{*j} \text{ for all } i,j,\ a_{ij}\sigma = a_{ij} \text{ for } j > \lambda_i\}.$$

We define a $\lambda \subseteq \nu$**-semitabloid** *to be an element of* M^ν *of the form*

$$\langle a\rangle_\lambda := \sum_{\sigma\in C_{a,\lambda}} \{a\}\sigma\varepsilon_\sigma$$

Let the **James lattice** $S^{\lambda\subseteq\nu}$ *be the sublattice of* M^λ *generated over* $\mathbf{Z}$ *by the* $\lambda \subseteq \nu$*-semitabloids.*

Note that for $\rho \in \mathcal{S}_n$ we have $C_{a\rho,\lambda} = (C_{a,\lambda})^\rho$ and thus $(\langle a\rangle_\lambda)\rho = \langle a\rho\rangle_\lambda$.

The notion of a James lattice is a common generalization both of the $\mathbf{Z}\mathcal{S}_n$-lattice M^ν ($\lambda = \emptyset$) and of the Specht lattice S^ν ($\lambda = \nu$).

Remark 5.1.3 *Let* $\tilde{\lambda}_1 := \nu_1$, $\tilde{\lambda}_i := \lambda_i$ *for* $i \geq 2$. *Then*

$$S^{\lambda\subseteq\nu} = S^{\tilde{\lambda}\subseteq\nu}$$

as $\mathbf{Z}\mathcal{S}_n$*-sublattices of* M^ν.

As a corollary to the Garnir relations for the λ-polytabloids in the Specht lattice S^λ (4.1.4) we obtain the

Corollary 5.1.4 (Garnir relations for $\lambda \subseteq \nu$-semitabloids) *Let* $[a]$ *be a* ν*-tableau. Fix* $j < k$. *Let* $\xi \subseteq \{a_{ij} \mid i \leq \lambda'_j\}$ *resp.* $\eta \subseteq \{a_{ik} \mid i \leq \lambda'_k\}$ *be a subset of the column* j *resp.* k *inside* λ *such that*

$$\#\xi + \#\eta > \lambda'_j.$$

For a subset $\zeta \subseteq [1,n]$ *we denote by* $\mathcal{S}_\zeta$ *the subgroup of* $\mathcal{S}_n$ *fixing the elements outside* ζ, *i.e.* $S_\zeta := C_{\mathcal{S}_n}([1,n]\backslash\zeta)$. *We obtain*

$$\sum_{\sigma\in\mathcal{S}_\xi\times\mathcal{S}_\eta\backslash\mathcal{S}_{\xi\cup\eta}} \langle a\rangle_\lambda\sigma\varepsilon_\sigma = 0.$$

As a variant, we dispose of the following Garnir relation for a λ-column and a single ν-element, not necessarily inside λ.

Corollary 5.1.5 *Let $[a]$ be a ν-tableau. Fix $j < k$. Let $\xi := \{a_{ij} \mid i \leq \lambda'_j\}$, let $y := a_{lk}$ for some $l \in [1, \lambda'_j]$. Then*

$$\sum_{x\in\xi} \langle a\rangle_\lambda (x\ y) = \langle a\rangle_\lambda .$$

Instead of modifying the argument for the ordinary Garnir relation (4.1.4), we prefer to argue directly, so, in particular, we reprove the case $l \leq \lambda'_k$. Let $z := a_{lj}$.

$$\begin{array}{rcl}
\sum_{x\in\xi} \langle a\rangle_\lambda (x\ y) & = & \sum_{x\in\xi} \sum_{\sigma\in C_{a,\lambda}} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& = & \sum_{x\in\xi} \sum_{\sigma\in C_{a,\lambda},\ z\sigma = x} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& & + \sum_{x\in\xi} \sum_{\sigma\in C_{a,\lambda},\ z\sigma > x} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& & + \sum_{x\in\xi} \sum_{\sigma\in C_{a,\lambda},\ z\sigma < x} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& = & \sum_{x\in\xi} \sum_{\sigma\in C_{a,\lambda},\ z\sigma = x} \{a\}\sigma\varepsilon_\sigma \\
& & + \sum_{x,u\in\xi,\ u>x} \sum_{\sigma\in C_{a,\lambda},\ z\sigma = u} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& & + \sum_{x,u\in\xi,\ u<x} \sum_{\sigma\in C_{a,\lambda},\ z\sigma = u} \{a\}\sigma\varepsilon_\sigma (u\ y)(x\ y) \\
& \overset{\sigma' = \sigma(x\ u)}{=} & \langle a\rangle_\lambda \\
& & + \sum_{x,u\in\xi,\ u>x} \sum_{\sigma\in C_{a,\lambda},\ z\sigma = u} \{a\}\sigma\varepsilon_\sigma (x\ y) \\
& & - \sum_{x,u\in\xi,\ x>u} \sum_{\sigma'\in C_{a,\lambda},\ z\sigma' = x} \{a\}\sigma'\varepsilon_{\sigma'} (u\ y) \\
& = & \langle a\rangle_\lambda .
\end{array}$$

Assume given $z \geq 2$ such that $\lambda_z < \lambda_{z-1} = \nu_{z-1}$ and such that $\lambda_z < \nu_z$.

Notation 5.1.6 Let

$$(\lambda A_z)_i := \begin{cases} \lambda_i + 1 & \text{for } i = z \\ \lambda_i & \text{for } i \neq z \end{cases}$$

define a partition λA_z of some number $\leq n$, A for 'add'. Let

$$(\nu R_z)_i := \begin{cases} \nu_{z-1} + (\nu_z - \lambda_z) & \text{for } i = z-1 \\ \lambda_z & \text{for } i = z \\ \nu_i & \text{for } i \neq z-1, z \end{cases}$$

define a prepartition νR_z of n, R for 'raise'. For a ν-tableau $[a]$ we let the νR_z-tableau $[aR_z]$ be defined by

$$\begin{array}{rcll}
(aR_z)_{ij} & := & a_{ij} & \text{for}\quad j \leq \min(\nu_i, (\nu R_z)_i) \\
(aR_z)_{z-1,\,\nu_{z-1}+j} & := & a_{z,\lambda_z+j} & \text{for}\quad j \in [1, \nu_z - \lambda_z],
\end{array}$$

i.e. by 'shifting the $\nu\backslash\nu R_z$-part of $[a]$ one row up while retaining the order'.

The possibility of νR_z not being a partition even in case ν is a partition forces us to work with prepartitions.

Remark 5.1.7 *We have an embedding of $\mathbf{Z}S_n$-sublattices of M^ν*

$$S^{\lambda A_z \subseteq \nu} \subseteq S^{\lambda \subseteq \nu}$$

For a ν-tableau $[a]$ we may write

$$\begin{aligned} \langle a\rangle_{\lambda A_z} &= \sum_{\sigma \in C_{a,\lambda A_z}} \{a\}\sigma\varepsilon_\sigma \\ &= \sum_{\sigma \in C_{a,\lambda}\backslash C_{a,\lambda A_z}} (\sum_{\tau \in C_{a,\lambda}} \{a\}\tau\varepsilon_\tau)\sigma\varepsilon_\sigma \\ &= \sum_{\sigma \in C_{a,\lambda}\backslash C_{a,\lambda A_z}} \langle a\rangle_\lambda \sigma\varepsilon_\sigma . \end{aligned}$$

Example 5.1.8 Let $n = 9$, $z = 3$, $\nu = (3,3,3)$, $\lambda = (3,3,1)$,

$$a = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

Then $\lambda A_3 = (3,3,2)$, $\nu R_3 = (3,5,1)$,

$$aR_3 = \begin{matrix} 1 & 2 & 3 & & \\ 4 & 5 & 6 & 8 & 9 \\ 7 & & & & \end{matrix}$$

We need some combinatorial notation in order to generalize the notion of a standard polytabloid. It is not as straightforward as one may hope (cf. 5.1.15).

Definition 5.1.9 ([J 78, 15.2]) *A* **sequence of type** ν *is a map*

$$\begin{aligned} [1,n] &\xrightarrow{s} \mathbf{N} \\ i &\longrightarrow s_i \end{aligned}$$

such that $\#s^{-1}(j) = \nu_j$. *The subset* $G_s \subseteq [1,n]$ *of* **good terms** *of the sequence* s *is constructively determined by the conditions*

(i) $s^{-1}(1) \subseteq G_s$,

(ii) in case $s_i \geq 2$, i *is in* G_s *iff*

$$\#\left(s^{-1}(s_i - 1) \cap G_s \cap [1, i-1]\right) > \#\left(s^{-1}(s_i) \cap G_s \cap [1, i-1]\right).$$

Let $\mathrm{seq}(\lambda \subseteq \nu)$ *be the set of sequences* s *of type* ν *such that, for all* $j \geq 0$,

$$s^{-1}(j) \cap G_s \geq \lambda_j .$$

More specifically, the prepartition μ *of some number* $\leq n$ *given by* $\mu_i := \#(G_s \cap s^{-1}(i))$ *is called the* **subtype** *of* s. *(In fact,* μ *is a partition).*

Example 5.1.10 Let $n = 5$, $z = 3$, $\nu = (2,2,1)$, $\lambda = (2,2)$, $\lambda A_3 = (2,2,1)$, $\nu R_3 = (2,3)$. We list the sequences of type ν together with their subtypes. In the sequences in

$$\text{seq}(\lambda \subseteq \nu) \backslash \text{seq}(\lambda A_z \subseteq \nu)$$

we replace the values $s_i = z$ for $i \notin G_s$ by $s'_i = z - 1$ and list the resulting sequence s' in the third column. I.e. we replace the value 3 by 2 in case the sequence has subtype $(2,2)$.

sequence s	subtype	replaced sequence s'
11223	$(2,2,1)$	
11232	$(2,2,1)$	
11322	$(2,2)$	11222
12123	$(2,2,1)$	
12132	$(2,2,1)$	
12213	$(2,1,1)$	
12231	$(2,1,1)$	
12312	$(2,2,1)$	
12321	$(2,1,1)$	
13122	$(2,2)$	12122
13212	$(2,2)$	12212
13221	$(2,1)$	
21123	$(2,1,1)$	
21132	$(2,1)$	
21213	$(2,1,1)$	
21231	$(2,1,1)$	
21312	$(2,1)$	
21321	$(2,1)$	
22113	(2)	
22131	(2)	
22311	(2)	
23112	$(2,1)$	
23121	$(2,1)$	
23211	(2)	
31122	$(2,2)$	21122
31212	$(2,2)$	21212
31221	$(2,1)$	
32112	$(2,1)$	
32121	$(2,1)$	
32211	(2)	

We list the sequences of type νR_z together with their subtypes

sequence	subtype
11222	$(2,2)$
12122	$(2,2)$
12212	$(2,2)$
12221	$(2,1)$
21122	$(2,2)$
21212	$(2,2)$
21221	$(2,1)$
22112	$(2,1)$
22121	$(2,1)$
22211	(2)

and recognize that in this example there is a bijection from $\text{seq}(\lambda \subseteq \nu)\backslash\text{seq}(\lambda A_z \subseteq \nu)$ to $\text{seq}(\lambda \subseteq \nu R_z)$ given by the replacement described above. This is in fact true in general, as has been discovered by JAMES [J 78, 15.14] (cf. 5.1.11).

The following combinatorial result of JAMES is the key to the James extension [J 78, 17.13] as well as, independently, to the Littlewood-Richardson rule [J 78, 16.4]. We cite it without proof.

Theorem 5.1.11 (James, **[J 78, 15.14]**)

$$\#\mathrm{seq}(\lambda \subseteq \nu) = \#\mathrm{seq}(\lambda A_z \subseteq \nu) + \#\mathrm{seq}(\lambda \subseteq \nu R_z).$$

Lemma 5.1.12 ([J 78, 17.6, 17.9]) *Let $s \in \mathrm{seq}(\lambda \subseteq \nu)$. Let $[a_s]$ be a ν-tableau constructed in the following manner. The j-th row of $[a_s]$ is filled with $s^{-1}(j)$ such that $s^{-1}(j) \cap G_s$ appears increasingly from the left and such that $s^{-1}(j) \backslash G_s$ appears increasingly from the right. Then*

$$(\langle a_s \rangle_\lambda \mid s \in \mathrm{seq}(\lambda \subseteq \nu)) \subseteq S^{\lambda \subseteq \nu}$$

is a **Z***-linear independent tuple, in particular, $s \longrightarrow \langle a_s \rangle_\lambda$ is injective. Moreover, its* **Z***-linear span is a pure* **Z***-sublattice of M^ν.*

In fact, it will turn out to be a **Z**-linear basis of $S^{\lambda \subseteq \nu}$ (cf. 5.1.18).

We **claim** that for $s \in \mathrm{seq}(\lambda \subseteq \nu)$, a_s is standard inside λ, i.e. that (i) $[a_s]_{i-1,j} < [a_s]_{ij}$ for $i \in [2, \lambda'_j]$ and that (ii) $[a_s]_{i,j-1} < [a_s]_{ij}$ for $j \in [2, \lambda_i]$. Assume $x := [a_s]_{i-1,j} > [a_s]_{ij} =: y$ for some $i \in [2, \lambda'_j]$, so that $s_x = i-1$, $s_y = i$. Since x and y are in G_s we would obtain

$$\begin{array}{rcl} j-1 & = & \#\,(s^{-1}(i-1) \cap G_s \cap [1, x-1]) \\ & \geq & \#\,(s^{-1}(i-1) \cap G_s \cap [1, y-1]) \\ & > & \#\,(s^{-1}(i) \cap G_s \cap [1, y-1]) \\ & = & j-1. \end{array}$$

Consider the total order on the ν-tabloids in which the largest entry x in different rows decides the order of two tabloids $\{a\}$ and $\{b\}$ as follows. If x is higher in $\{a\}$ than in $\{b\}$, then $\{a\}$ is smaller than $\{b\}$. Let a be a tableau satisfying $a_{i-1,j} < a_{ij}$ for $i \in [2, \lambda'_j]$. In the defining sum

$$\langle a \rangle_\lambda = \sum_{\sigma \in C_{a,\lambda}} \{a\} \sigma \varepsilon_\sigma,$$

$\{a\}$ is the largest occurring summand. Since the map $s \longrightarrow \{a_s\}$ is injective, for different sequences yield different distributions over the rows, the matrix representing the elements $\langle a \rangle_\lambda$, in terms of the tabloid basis ordered as just described, can be written in a lower triangular manner with entries $\in \{-1, 0, 1\}$.

Remark 5.1.13 *The equations*

$$\begin{array}{rcl} \#\mathrm{seq}((0) \subseteq \nu) & = & \mathrm{rk}\, M^\nu \\ \#\mathrm{seq}(\lambda \subseteq \lambda) & = & \mathrm{rk}\, S^\lambda \end{array}$$

hold.

The first equation results from the bijection from $\mathrm{seq}((0) \subseteq \nu)$ to the set of ν-tabloids as given in (5.1.12).

The second equation, which we won't use but reprove further down, results from the bijection from $\mathrm{seq}(\lambda \subseteq \lambda)$ to the set of standard λ-polytabloids as given in (5.1.12). In fact, surjectivity follows, in the notation used there, by assuming $[a]$ to be standard and, using induction on y, by considering

$$\begin{array}{rcl} \#\,(s^{-1}(i) \cap [1, y-1]) & = & \#\,(s^{-1}(i-1) \cap [1, x-1]) \\ & < & \#\,(s^{-1}(i-1) \cap [1, x]) \\ & \leq & \#\,(s^{-1}(i-1) \cap [1, y-1]) \end{array}$$

in order to prove $y \in G_s$, so that eventually $G_s = [1, n]$ results.

Example 5.1.14 Let $\nu = (3,3)$, $\lambda = (3,2)$. We list $\operatorname{seq}(\lambda \subseteq \nu)$ together with the according ν-tableaux.

111222 $\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}$
112122 $\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}$
112212 $\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}$
112221 $\begin{smallmatrix}1&2&6\\3&4&5\end{smallmatrix}$
121122 $\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}$
121212 $\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}$
121221 $\begin{smallmatrix}1&3&6\\2&4&5\end{smallmatrix}$
122112 $\begin{smallmatrix}1&4&5\\2&6&3\end{smallmatrix}$
122121 $\begin{smallmatrix}1&4&6\\2&5&3\end{smallmatrix}$
211122 $\begin{smallmatrix}2&3&4\\5&6&1\end{smallmatrix}$
211212 $\begin{smallmatrix}2&3&5\\4&6&1\end{smallmatrix}$
211221 $\begin{smallmatrix}2&3&6\\4&5&1\end{smallmatrix}$
212112 $\begin{smallmatrix}2&4&5\\3&6&1\end{smallmatrix}$
212121 $\begin{smallmatrix}2&4&6\\3&5&1\end{smallmatrix}$.

Example 5.1.15 (dangerous bend) It is possible that a tableau which is standard inside λ does **not** occur as a_s for some $s \in \operatorname{seq}(\lambda \subseteq \nu)$. Let $\nu = (2,2)$, $\lambda = (2,1)$. We list $\operatorname{seq}(\lambda \subseteq \nu)$ together with the according ν-tableaux,

1122 $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$
1212 $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$
1221 $\begin{smallmatrix}1&4\\2&3\end{smallmatrix}$
2112 $\begin{smallmatrix}2&3\\4&1\end{smallmatrix}$
2121 $\begin{smallmatrix}2&4\\3&1\end{smallmatrix}$,

and notice that $\begin{smallmatrix}1&2\\4&3\end{smallmatrix}$ does not occur. Note that $s \longrightarrow \{a_s\}$ couldn't be injective if it did.

Proposition 5.1.16 ([J 78, 17.10, 17.12]) *There is a* $\mathbf{Z}\mathcal{S}_n$*-linear epimorphism*

$$\begin{array}{rcl} S^{\lambda\subseteq\nu} & \longrightarrow & S^{\lambda\subseteq\nu R_z} \\ \langle a\rangle_\lambda & \longrightarrow & \langle aR_z\rangle_\lambda \end{array}$$

which annihilates $S^{\lambda A_z\subseteq\nu}$ *(cf. 5.1.7).*

Given a ν-tableau $[a]$, let

$$\begin{array}{rcll} \xi & := & \{a_{zj} \mid j \in [1,\lambda_z]\} & = \{(aR_z)_{zj} \mid j \in [1,(\nu R_z)_z]\} \\ \eta & := & \{a_{zj} \mid j \in [\lambda_z+1,\nu_z]\} & = \{(aR_z)_{z-1,j} \mid j \in [\nu_{z-1}+1,(\nu R_z)_{z-1}]\}. \end{array}$$

Consider the $\mathbf{Z}\mathcal{S}_n$-morphism

$$\begin{array}{rcl} F^\nu & \longrightarrow & M^{\nu R_z} \\ [a] & \longrightarrow & \sum_{\sigma\in\mathcal{S}_\xi\times\mathcal{S}_\eta\backslash\mathcal{S}_{\xi\cup\eta}}\{aR_z\}\sigma \end{array}$$

(cf. 4.1.1, 5.1.6) which factors over

$$\begin{array}{rcl} M^\nu & \xrightarrow{\psi} & M^{\nu R_z} \\ \{a\} & \longrightarrow & \sum_{\sigma\in\mathcal{S}_\xi\times\mathcal{S}_\eta\backslash\mathcal{S}_{\xi\cup\eta}}\{aR_z\}\sigma. \end{array}$$

since $a_{z*} = \xi \cup \eta$.

We evaluate

$$\begin{array}{rcl} \langle a\rangle_\lambda\psi & = & \sum_{\rho\in C_{a,\lambda}} (\{a\}\psi)\rho\varepsilon_\rho \\ & = & \sum_{\rho\in C_{a,\lambda}} (\sum_{\sigma\in\mathcal{S}_\xi\times\mathcal{S}_\eta\backslash\mathcal{S}_{\xi\cup\eta}}\{aR_z\}\sigma)\rho\varepsilon_\rho \end{array}$$

and **claim** that for $(\mathcal{S}_\xi \times \mathcal{S}_\eta)\sigma \neq (\mathcal{S}_\xi \times \mathcal{S}_\eta)$ the summand

$$\sum_{\rho\in C_{a,\lambda}} \{aR_z\}\sigma\rho\varepsilon_\rho$$

vanishes. In fact, let $x \in \xi$ such that $x \in \eta\sigma$, and, writing $x =: a_{zj}$, let $y := a_{z-1,j}$. We calculate

$$\begin{array}{rcl} \sum_{\rho\in C_{a,\lambda}} \{aR_z\}\sigma\rho\varepsilon_\rho & = & \sum_{\rho\in C_{a,\lambda},\, x\rho>y\rho} (\{aR_z\}\sigma\rho\varepsilon_\rho - \{aR_z\}\sigma(x\ y)\rho\varepsilon_\rho) \\ & = & \sum_{\rho\in C_{a,\lambda},\, x\rho>y\rho} (\{aR_z\}\sigma\rho\varepsilon_\rho - \{aR_z\}(x\sigma^{-1}\ y)\sigma\rho\varepsilon_\rho) \\ & = & \sum_{\rho\in C_{a,\lambda},\, x\rho>y\rho} 0. \end{array}$$

Thus

$$\begin{array}{rcl} \langle a\rangle_\lambda\psi & = & \sum_{\rho\in C_{a,\lambda}} \{aR_z\}\rho\varepsilon_\rho \\ & = & \sum_{\rho\in C_{aR_z,\lambda}} \{aR_z\}\rho\varepsilon_\rho \\ & = & \langle aR_z\rangle_\lambda. \end{array}$$

It remains to be seen that $\langle a\rangle_{\lambda A_z}\psi = 0$. We modify the argument just given by remarking that now for **any** σ there is an $x := a_{zj}$ with $j \in [1, \lambda_z + 1]$ such that $x \in \eta\sigma$.

Lemma 5.1.17 *Let* $X \xrightarrow{f} Y$ *be a morphism of* $\mathbf{Z}$*-lattices. If* $\dim_{\mathbf{F}_p} \mathrm{Im}\,(X/p \xrightarrow{f} Y/p)$ *is independent of the prime* p*, then* $\mathrm{Im}\, f$ *is a pure sublattice of* Y*.*

Write f in elementary divisor form (A.1.1).

Theorem 5.1.18 (James, [J 78, 17.13]) *The sequence of* $\mathbf{Z}\mathcal{S}_n$*-lattices*

$$\begin{array}{ccccccc} 0 \longrightarrow & S^{\lambda A_z \subseteq \nu} & \longrightarrow & S^{\lambda\subseteq\nu} & \longrightarrow & S^{\lambda\subseteq\nu R_z} & \longrightarrow 0, \\ & \langle a\rangle_{\lambda A_z} & \longrightarrow & \langle a\rangle_{\lambda A_z} & & & \\ & & & \langle a\rangle_\lambda & \longrightarrow & \langle aR_z\rangle_\lambda & \end{array}$$

called the **James extension,** *is short exact. Moreover,* $\mathrm{rk}\, S^{\lambda\subseteq\nu} = \#\mathrm{seq}(\lambda \subseteq \nu)$*, and the tuple*

$$(\langle a_s\rangle_\lambda | s \in \mathrm{seq}(\lambda \subseteq \nu))$$

forms a basis of $S^{\lambda\subseteq\nu}$ *(cf. 5.1.12).*

In order to apply a rank argument, we need to know the left hand side inclusion to be pure. So we first reduce modulo a prime p, using the analogous definition of the James module $S^{\lambda\subseteq\nu}_{\mathbf{F}_p}$ over $\mathbf{F}_p\mathcal{S}_n$ as being generated by the $\lambda \subseteq \nu$-semitabloids inside M^ν/p. NB we do **not** know yet that $S^{\lambda \subseteq \nu}/p = S^{\lambda \subseteq \nu}_{\mathbf{F}_p}$ but only a surjection of the former onto the latter.

The above sequence also exists for the James modules over $\mathbf{F}_p$, as can be seen by the construction used in (5.1.16), moreover, composition is zero, the left hand side morphism is an inclusion and the right hand side morphism is surjective.

The elements listed in (5.1.12) are linearly independent over $\mathbf{Z}$ and span a pure sublattice of M^ν. Hence their images in $S^{\lambda \subseteq \nu}_{\mathbf{F}_p} \subseteq M^\nu/p$ are linearly independent over $\mathbf{F}_p$.

Since every pair $\lambda \subseteq \nu$, $\lambda_1 = \nu_1$, can be turned into a pair of type $(\mu_1) \subseteq \mu$, μ being a prepartition, by a sequence of inverse R_ζ-operations followed by a sequence of inverse A_ζ-operations for various $\zeta \geq 1$ while taking care of the $(\lambda_{\zeta-1} = \nu_{\zeta-1})$-condition, we may assume by induction and by (5.1.13) the equality $\dim S^{\lambda \subseteq \nu}_{\mathbf{F}_p} = \#\mathrm{seq}(\lambda \subseteq \nu)$ to hold in order to prove that the inequalities $\dim S^{\lambda A_z \subseteq \nu}_{\mathbf{F}_p} \geq \#\mathrm{seq}(\lambda A_z \subseteq \nu)$ as well as $\dim S^{\lambda \subseteq \nu R_z}_{\mathbf{F}_p} \geq \#\mathrm{seq}(\lambda \subseteq \nu R_z)$ are equalities.

$$
\begin{array}{rcl}
\#\mathrm{seq}(\lambda \subseteq \nu) & = & \dim S^{\lambda \subseteq \nu}_{\mathbf{F}_p} \\
& \geq & \dim S^{\lambda A_z \subseteq \nu}_{\mathbf{F}_p} + \dim S^{\lambda \subseteq \nu R_z}_{\mathbf{F}_p} \\
& \geq & \#\mathrm{seq}(\lambda A_z \subseteq \nu) + \#\mathrm{seq}(\lambda \subseteq \nu R_z) \\
& \overset{(5.1.11)}{=} & \#\mathrm{seq}(\lambda \subseteq \nu).
\end{array}
$$

Actually, we use only $\#\mathrm{seq}(\lambda A_z \subseteq \nu) + \#\mathrm{seq}(\lambda \subseteq \nu R_z) \geq \#\mathrm{seq}(\lambda \subseteq \nu)$, i.e. welldefinedness and injectivity of a map $\mathrm{seq}(\lambda \subseteq \nu) \backslash \mathrm{seq}(\lambda A_z \subseteq \nu) \longrightarrow \mathrm{seq}(\lambda \subseteq \nu R_z)$.

Applying (5.1.17) to the map

$$
\begin{array}{rcl}
F^\nu & \longrightarrow & M^\nu \\
[a] & \longrightarrow & \langle a \rangle_{\lambda A_z}
\end{array}
$$

etc. we see that the inclusions $S^{\lambda A_z \subseteq \nu} \subseteq S^{\lambda \subseteq \nu} \subseteq M^\nu$ are pure. Therefore, $S^{\lambda \subseteq \nu}/p = S^{\lambda \subseteq \nu}_{\mathbf{F}_p}$, so that $\mathrm{rk}\, S^{\lambda \subseteq \nu} = \#\mathrm{seq}(\lambda \subseteq \nu)$. A comparison of ranks shows the James extension to be short exact.

Remark 5.1.19 There are further sublattices of M^ν, given by summing up alternatingly over place permutation actions for an arbitrary subdivision of ν into parts of columns (to make it nonzero), and inclusions between them by fusing vertically such parts. I do not know in which cases the cokernel of such an inclusion is again of such a form.

5.2 Retracting the James extension up to an integer, simple case

We keep the notation of (S 5.1), but specialize to the case that ν is a partition and that there is a $z \geq 2$ such that

$$\lambda_i = \begin{cases} \nu_z - 1 & \textbf{for } i = z \\ \nu_i & \textbf{for } i \neq z. \end{cases}$$

Let $k := \nu_1$.

The James extension (5.1.18)

$$(0 \longrightarrow S^{\nu} \longrightarrow S^{\lambda \subseteq \nu} \longrightarrow S^{\lambda \subseteq \nu R_z} \longrightarrow 0) \in \mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^{\nu})$$

represents an element of finite order in Ext^1 (cf. A.3.3), whence it allows a retraction up to

$$m = (S^{\nu} \longrightarrow S^{\lambda \subseteq \nu} \longrightarrow S^{\nu}),$$

where m is a nonzero integer divisible by this order. In our particularly simple situation we shall exhibit such a retraction in a combinatorial manner. Curiously, its formula is similar to that of the morphism exhibited in (4.3.31). I tend to consider (5.2.9) to be the reason for the sums of type f_e to occur (cf. 5.2.1), being 'potential retractions up to an integer'.

The reader might wish to have seen some illustration in advance (S 5.2.5).

5.2.1 Preparation

Notation 5.2.1 Let

$$[a] = [a_1 \ldots a_k]$$

be a ν-tableau, where a_i denotes its i-th column and $a_{i,j}$ the entry in the i-th column and the j-th row. Note that this means a **change of notation** compared to (S 5.1), which is convenient to handle columns. Let $g := \nu_z$,

$$y := a_{\nu_z, z},$$

so that y is the 'element in ν but not in λ', situated in column g and in row z.

Let e be a function

$$\begin{array}{rcl} [g+1, k] & \xrightarrow{e} & \{0, 1\} \\ j & \longrightarrow & e_j, \end{array}$$

let

$$\begin{array}{rcl} [1, l] & \xrightarrow{i} & [g+1, k] \\ j & \longrightarrow & i_j \end{array}$$

be the strictly monotone function of which e is the characteristic function, i.e. $l := \#e^{-1}(1)$, $j \in i_{[1,l]} :\Longleftrightarrow e_j = 1$. Extend e to $e_g := 1$ and, accordingly, i to $i_0 := g$. Finally, extend e to $[1, k]$ by zero.

Let

$$\boxed{\begin{array}{rcl} F^{\nu} & \xrightarrow{f_e} & S^{\nu} \\ [a] & \longrightarrow & \sum\limits_{x_j \in a_{i_j}} \left\langle \ldots a_{i_0}^{y, x_1} \ldots a_{i_1}^{x_1, x_2} \ldots a_{i_l}^{x_l, y} \ldots \right\rangle. \end{array}}$$

Without further specification, '$x_j \in a_{i_j}$' means '$x_j \in a_{i_j},\ j \in [1, l]$'.

Lemma 5.2.2 *The kernel of*

$$\begin{array}{rcl} F^\nu & \longrightarrow & S^{\lambda\subseteq\nu} \\ [a] & \longrightarrow & \langle a\rangle_\lambda \end{array}$$

is generated over $\mathbf{Z}\mathcal{S}_n$ *by the* **signed column transpositions inside** λ, *by the* **Garnir relations for** y *of the form* $G'_{a,\lambda,1}$ *and by the* **one-step Garnir relations inside** λ, *denoted* $G_{a,\lambda,\xi,\eta}$.

Signed column transpositions inside λ *are elements of the form*

$$[a_1 \ldots a_j \ldots a_k] + [a_1 \ldots a_j \ldots a_k](s\ t)$$

where $s = a_{j,p}$, $t = a_{j,q}$ *for some* $p, q \in [1, \lambda'_j]$, $p \neq q$.

Garnir relations for y *are elements of the form*

$$G'_{a,\lambda,j} := [\ldots a_j \ldots a_g \ldots] - \sum_{x \in a_j} [\ldots a_j^{x,y} \ldots a_g^{y,x} \ldots]$$

where $j < g$. *In case* $g = 1$ *we set* $G'_{a,\lambda,1} := 0$.

Let $j \in [1, k-1]$. *Let* $\xi \subseteq \{a_{j,i} \mid i \leq \lambda'_j\}$, $\eta \subseteq \{a_{j+1,i} \mid i \leq \lambda'_{j+1}\}$ *be given such that*

$$\#\xi + \#\eta > \lambda'_j.$$

A **one-step Garnir relation inside** λ *is an element of the form*

$$G_{a,\lambda,\xi,\eta} := \sum_{\sigma \in \mathcal{S}_\xi \times \mathcal{S}_\eta \backslash \mathcal{S}_{\xi\cup\eta}} [a]\sigma\varepsilon_\sigma.$$

The elements of these three kinds in fact lie in that kernel (5.1.4, 5.1.5), so that (F^ν modulo the submodule generated by them)$=: \bar{F}^\nu$ surjects onto $S^{\lambda\subseteq\nu}$. Therefore, by (5.1.18), it would suffice to show that $([a_s] | s \in \mathrm{seq}(\lambda \subseteq \nu))$ generates $\bar{F}^\nu$. We proceed in different way.

It suffices to exhibit a tuple in F^ν which remains linearly independent in $S^{\lambda\subseteq\nu}$ and which generates $\bar{F}^\nu$ $\mathbf{Z}$-linearly. For then the same argument applies, i.e. the induced surjection from $\bar{F}^\nu$ to $S^{\lambda\subseteq\nu}$ maps a $\mathbf{Z}$-generating tuple to a $\mathbf{Z}$-linearly independent tuple, hence a $\mathbf{Z}$-basis to a $\mathbf{Z}$-basis.

$[a]$ is (provisonally) called z-λ-**standard** if

$$\begin{array}{rl} a_{i,j} < a_{i',j} & \text{for } j \geq 1 \text{ and } i, i' \in [1, \lambda_j],\ i < i' \\ a_{i,j} < a_{i,j'} & \text{for } i \geq 1 \text{ and } j, j' \in [1, \lambda'_i] \backslash \{z\},\ j < j' \\ a_{i,j} < a_{i,z} & \text{for } i \in [1, \lambda_z] \text{ and } j \in [1, \lambda'_i] \backslash \{z\}. \end{array}$$

This is, we 'think of the zth row as of the last one'. Accordingly, we introduce a total order on the ν-tabloids by declaring $\{a\}$ to be smaller than $\{b\}$ if the largest entry x which is in different rows decides their ordering as follows. If its row position in $\{a\}$ (resp. $\{b\}$) is z, then $\{b\}$ is smaller than $\{a\}$ (resp. $\{a\}$ is smaller than $\{b\}$). If the row position of x neither in $\{a\}$ nor in $\{b\}$ is z, then $\{a\}$ is smaller than $\{b\}$ iff x is higher in $\{a\}$ than in $\{b\}$.

Consider the row equivalence classes of the set of z-λ-standard ν-tableaux, i.e. the orbits under the respective row stabilizer, and let $[a]$ represent such a class. Assume y' to be the minimal element of the z-th row. Replacing y by y' and ordering the first $g-1$ entries of the z-th row increasingly yields a z-λ-standard ν-tableau again. Choose from the row equivalence class this ν-tableau and consider the tuple T formed by them.

Letting $[a]$ be such a chosen tableau, we observe that $\{a\}$ is the maximal occurring element in the defining sum of $\langle a\rangle_\lambda$. Moreover, the tabloids corresponding to the chosen tableaux differ pairwise by construction. Therefore, the tuple formed by the chosen tableaux is linearly independent when regarded in $S^{\lambda\subseteq\nu}$.

We **claim** that the tuple T (of y-minimal z-λ-standard tableaux) generates $\bar{F}^\nu$ $\mathbf{Z}$-linearly. Using signed column transpositions inside λ and one-step Garnir relations to standardize the λ-area by the method described in (4.3.2), we see that the z-λ-standard tableaux generate $\bar{F}^\nu$. Moreover, we see that it is possible in $\bar{F}^\nu$ to write an arbitrary tableau as a linear combination of z-λ-standard tableaux of the same y-value.

We perform an induction over y.

Start of the induction, $y = 1$.

Step of the induction. Given a z-λ-standard tableau a. Assume y not to be minimal in the z-th row and let $y' = a_{1,z} < y$ the minimal element of the z-th row, whence $g \geq 2$. The Garnir relation for y given by $G'_{a,\lambda,1}$ has as its negative summands tableaux with smaller y-value than $[a]$.

Example 5.2.3 Let $\nu = (2,2)$, let $\lambda = (2,1)$. The tuple T appearing in the proof of (5.2.2) consists of the elements

$$\begin{smallmatrix}1&2\\4&3\end{smallmatrix},\ \begin{smallmatrix}1&3\\4&2\end{smallmatrix},\ \begin{smallmatrix}1&4\\3&2\end{smallmatrix},\ \begin{smallmatrix}2&3\\4&1\end{smallmatrix},\ \begin{smallmatrix}2&4\\3&1\end{smallmatrix},$$

which differ from those chosen via sequences in (5.1.15).

Remark 5.2.4 I do not know whether the relations exibited above mutatis mutandis suffice to generate the kernel of $F^\nu \longrightarrow S^{\lambda\subseteq\nu}$ for a general pair $\lambda \subseteq \nu$.

Lemma 5.2.5 *The signed column transpositions inside λ vanish under f_e.*

This is the same calculation as in (4.3.3, 'Case $p \in [g+1,k]$').

Lemma 5.2.6 *The Garnir relation for y given by $G_{a,\lambda,j}$, $j < g$, vanishes under f_e.*

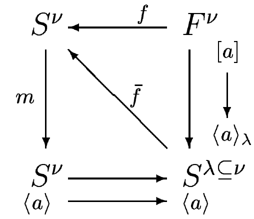
This follows by (5.1.5).

5.2.2 Strategy

Orientation 5.2.7 We shall exhibit a $\mathbf{Z}$-linear combination

$$F^\nu \xrightarrow{f:=\sum_e u_e f_e} S^\nu$$

of the maps f_e (5.2.1), where e runs over the maps $[g+1,k] \xrightarrow{e} \{0,1\}$, which allows a commutative diagram



for some integer m that depends on the combinatorial data and which will result from the calculation. Using (5.2.2, 5.2.5, 5.2.6), it remains to evaluate the expressions

$$G_{a,\lambda,\xi,\eta} f_e = \frac{1}{\#\xi!\#\eta!} \sum_{x_j \in a_{i_j}} \langle \ldots a_g^{y,x_1} \ldots a_{i_1}^{x_1,x_2} \ldots a_{i_l}^{x_l,y} \rangle \circ (\xi \cup \eta)$$

and combine them $\mathbf{Z}$-linearly over e to yield zero, where the coefficients may not depend on ξ and η.

Suppose given a ν-tableau a. Let $p \in [1, k-1]$, let $\xi \subseteq a_p$, $\eta \subseteq a_{p+1}$ such that $y \notin \xi$, $y \notin \eta$ and such that

$$\#\xi + \#\eta = \#\lambda'_p + 1.$$

We have to distinguish seven cases.

(I) $e_p = 1$, $e_{p+1} = 1$.

- (i) $p = g = i_0$, $p+1 = g+1 = i_1$.
- (ii) $p = i_s$, $p+1 = i_{s+1}$, $s \in [1, l-1]$.

(II) $e_p = 1$, $e_{p+1} = 0$.

- (i) $p = g = i_0$, $e_{g+1} = 0$.
- (ii) $p = i_s$, $s \in [1, l]$, $e_{p+1} = 0$.

(III) $e_p = 0$, $e_{p+1} = 1$.

- (i) $g \geq 2$, $p = g - 1$, $p + 1 = i_0 = g$.
- (ii) $e_p = 0$, $p + 1 = i_s$, $s \in [1, l]$.

(IV) $e_p = 0$, $e_{p+1} = 0$.

Let $a'_g := a_g \backslash y$. For a subset $\xi \subseteq a_p$, let $\bar{\xi} := a_p \backslash \xi$ in case $p \neq g$, let $\bar{\xi} := a'_g \backslash \xi$ in case $p = g$. Note that in the latter case we stipulate $\xi \subseteq a'_g$.

We start by recalling a particular Garnir relation (4.1.4).

Lemma 5.2.8 *Given a ν-tableau $[a]$, $p, q \in [1, k]$, $p < q$, $c \in a_q$, $d \in a_p$, we have*

$$\begin{array}{rcl} \sum_{b \in a_p} \langle \ldots a_p^{b,c} \ldots a_q^{c,b} \ldots \rangle & = & \langle \ldots a_p \ldots a_q \ldots \rangle . \\ \sum_{b \in a_p \backslash d} \langle \ldots a_p^{b,c} \ldots a_q^{c,b} \ldots \rangle & = & \langle \ldots a_p \ldots a_q \ldots \rangle - \langle \ldots a_p^{d,c} \ldots a_q^{c,d} \ldots \rangle . \end{array}$$

Lemma 5.2.9 (potential retractions) *Suppose given an integral linear combination*

$$F^\nu \xrightarrow{f := \sum_e u_e f_e} S^\nu$$

factorizing as in (5.2.7). Then

$$(S^\nu \xrightarrow{\iota} S^{\lambda \subseteq \nu} \xrightarrow{\bar{f}} S^\nu) = \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} u_e \nu'_{i_l},$$

where, as in (5.2.1), $i_l = \max\{e^{-1}(1)\}$*, including* $e_g = 1$*, and where* ι *is the inclusion of the James extension (5.1.18).*

We calculate using the language of (4.3.1).

$$
\begin{array}{rl}
\langle a\rangle \xrightarrow{\;\iota\;} & \langle a\rangle \\
= & \displaystyle\sum_{\sigma\in C_a} \{a\}\sigma\varepsilon_\sigma \\
= & \displaystyle\sum_{\sigma\in C_a,\; y\sigma=y} \{a\}\sigma\varepsilon_\sigma \\
& \displaystyle + \sum_{x_0\in a'_g}\;\sum_{\sigma\in C_a,\; y\sigma=x_0} \{a\}\sigma\varepsilon_\sigma \\
\overset{\sigma'=(x_0\;y)\sigma}{=} & \displaystyle\sum_{\sigma\in C_{a,\lambda}} \{a\}\sigma\varepsilon_\sigma \\
& \displaystyle - \sum_{x_0\in a'_g}\;\sum_{\sigma'\in C_{a(x_0\;y),\lambda}} \{a\}(x_0\;y)\sigma'\varepsilon_{\sigma'} \\
= & \displaystyle\langle a\rangle_\lambda - \sum_{x_0\in a'_g} \left\langle \begin{matrix} \cdots & (a'_g)^{x_0,y} & \cdots \\ & x_0 & \end{matrix} \right\rangle_\lambda \\
\xrightarrow{\sum_e u_e f_e} & \displaystyle\sum_e u_e \Bigg(\sum_{x_j\in a_{i_j},\; j\in[1,l]} \langle \ldots\, a_{i_0}^{y,x_1} \ldots\, a_{i_1}^{x_1,x_2} \ldots\, a_{i_l}^{x_l,y} \ldots\rangle \\
& \displaystyle - \sum_{x_0\in a'_{i_0},\; x_j\in a_{i_j},\; j\in[1,l]} \left\langle \begin{matrix} \cdots & (a'_{i_0})^{x_0,y} & \cdots & a_{i_1}^{x_1,x_2} & \cdots & a_{i_l}^{x_l,x_0} \\ & x_1 & & & & \end{matrix} \right\rangle \Bigg) \\
= & \displaystyle\sum_e u_e \Bigg(\sum_{x_j\in a_{i_j},\; j\in[1,l]} \langle \ldots\, a_{i_0}^{y,x_1} \ldots\, a_{i_1}^{x_1,x_2} \ldots\, a_{i_l}^{x_l,y} \ldots\rangle \\
& \displaystyle - \sum_{x_0\in a'_{i_0},\; x_j\in a_{i_j},\; j\in[1,l]} \left\langle \begin{matrix} \cdots & (a'_{i_0})^{x_0,y} & \cdots & a_{i_1}^{x_1,x_2} & \cdots & (a_{i_l}^{x_l,y})^{y,x_0} \\ & x_1 & & & & \end{matrix} \right\rangle \Bigg) \\
\overset{(5.2.8)}{=} & \displaystyle\sum_e u_e \Bigg(\sum_{x_j\in a_{i_j},\; j\in[1,l]} \langle \ldots\, a_{i_0}^{y,x_1} \ldots\, a_{i_1}^{x_1,x_2} \ldots\, a_{i_l}^{x_l,y} \ldots\rangle \\
& \displaystyle - \sum_{x_j\in a_{i_j},\; j\in[1,l]} \left\langle \begin{matrix} \cdots & a'_{i_0} & \cdots & a_{i_1}^{x_1,x_2} & \cdots & a_{i_l}^{x_l,y} \\ & x_1 & & & & \end{matrix} \right\rangle \\
& \displaystyle + \sum_{x_j\in a_{i_j},\; j\in[1,l]} \left\langle \begin{matrix} \cdots & a'_{i_0} & \cdots & a_{i_1}^{x_1,x_2} & \cdots & a_{i_l}^{x_l,x_1} \\ & y & & & & \end{matrix} \right\rangle \Bigg) \\
= & \displaystyle\sum_e u_e \Bigg(\sum_{x_j\in a_{i_j},\; j\in[1,l]} \langle \ldots\; a_{i_0} \;\ldots\; a_{i_1}^{x_1,x_2} \;\ldots\; a_{i_l}^{x_l,x_1} \rangle \Bigg) \\
\overset{(5.2.8)}{=} & \displaystyle\sum_e u_e \Bigg(\sum_{x_j\in a_{i_j},\; j\in[2,l]} \langle \ldots\; a_{i_0} \;\ldots\; a_{i_2}^{x_2,x_3} \;\ldots\; a_{i_l}^{x_l,x_2} \rangle \Bigg) \\
\overset{(5.2.8)\text{'s}}{=} & \displaystyle\sum_e u_e \Bigg(\sum_{x_l\in a_{i_l}} \langle \ldots\; a_{i_0} \;\ldots\; a_{i_l}^{x_l,x_l} \rangle \Bigg) \\
= & \displaystyle\Bigg(\sum_e u_e \nu'_{i_l} \Bigg) \langle a\rangle .
\end{array}
$$

Notation 5.2.10 Suppose given $\xi \in a'_g$, $\eta \in a_{g+1}$ such that $\#\xi + \#\eta = \nu'_g$, $w \in \eta$, and a strictly increasingly ordered tuple $(u_1, \ldots, u_t) \subseteq [g+2, k]$, possibly empty, with characteristic function

$$[g+1, k] \xrightarrow{\;e\;} \{0, 1\}.$$

Denote

$$A^w_{e,\xi,\eta} := A(u_1,\ldots,u_t)^w_{\xi,\eta} := \sum_{x_j\in a_{u_j},\; j\in[1,t]} \left\langle \begin{array}{ccccccc} \cdots & (a'_g)^{\bar{\xi},\eta\setminus w} & (a_{g+1}^{\eta\setminus w,\bar{\xi}})^{w,x_1} & \cdots & a_{u_1}^{x_1,x_2} & \cdots & a_{u_t}^{x_t,y} \;\cdots \\ & & w & & & & \end{array} \right\rangle .$$

So in particular, in case $t = 0$ we obtain

$$A()^w_{\xi,\eta} = \left\langle \begin{array}{cccc} \cdots & (a'_g)^{\bar{\xi},\eta\setminus w} & (a_{g+1}^{\eta\setminus w,\bar{\xi}})^{w,y} & \cdots \\ & & w & \end{array} \right\rangle .$$

Notation 5.2.11 Suppose given $\xi \subseteq a_p$, $\eta \subseteq a_{p+1}$, $p \in [g+1, k-1]$, such that $\#\xi + \#\eta = \nu'_p + 1$, $w \in \eta$, and a strictly increasingly ordered tuple $(u_1,\ldots,u_t) \subseteq [g+1,k]\backslash\{p,p+1\}$, possibly empty, with characteristic function $[g+1,k] \xrightarrow{e} \{0,1\}$ (so $e_p = e_{p+1} = 0$). Denote

$$B^w_{e,\xi,\eta} := B(u_1,\ldots,u_t)^w_{\xi,\eta} :=$$
$$\frac{1}{\nu'_p!} \sum_{x_j\in a_{u_j},\; j\in[1,t]} \left\langle \cdots a_g^{y,x_1} \cdots a_{u_{q-1}}^{x_{q-1},w} \cdots a_p^{\bar{\xi},\eta\setminus w} (a_{p+1}^{\eta\setminus w,\bar{\xi}})^{w,x_q} \cdots a_{u_q}^{x_q,x_{q+1}} \cdots a_{u_t}^{x_t,y} \cdots \right\rangle \circ (\xi \cup \eta).$$

So in particular, in case $t = 0$ we obtain

$$B()^w_{\xi,\eta} = \frac{1}{\nu'_p!} \left\langle \cdots a_g^{y,w} \cdots a_p^{\bar{\xi},\eta\setminus w} (a_{p+1}^{\eta\setminus w,\bar{\xi}})^{w,y} \cdots \right\rangle .$$

5.2.3 Calculations

We shall refer to the calculations in (S 4.3.3) in case this is possible after an obvious modification.

Calculation 5.2.12 *We treat the case* (I.i), *i.e.* $\xi \subseteq a'_g = a'_{i_0}$, $\eta \subseteq a_{g+1} = a_{i_1}$, $\#\xi + \#\eta = \nu'_g$. *Choose* $w \in \eta$. *We obtain*

$$\begin{array}{rcl} G_{a,\lambda,\xi,\eta} f_e & \overset{(4.3.13)}{=} & (1 + \nu'_g - \nu'_{g+1}) \displaystyle\sum_{x_j\in a_{i_j},\; j\in[2,l]} \left\langle \begin{array}{ccccccc} \cdots & (a'_g)^{\bar{\xi},\eta\setminus w} & (a_{i_1}^{\eta\setminus w,\bar{\xi}})^{w,x_2} & \cdots & a_{i_2}^{x_2,x_3} & \cdots & a_{i_l}^{x_l,y} \;\cdots \\ & & w & & & & \end{array} \right\rangle \\ & \overset{(5.2.10)}{=} & (1 + \nu'_g - \nu'_{g+1}) A(i_2,\ldots,i_l)^w_{\xi,\eta}. \end{array}$$

Calculation 5.2.13 *We treat the case* (I.ii), *i.e.* $\xi \subseteq a_p = a_{i_s}$, $\eta \subseteq a_{p+1} = a_{i_{s+1}}$, $s \in [1, l-1]$, $\#\xi + \#\eta = \nu'_{i_s} + 1$. *Choose* $w \in \eta$. *We obtain*

$$\begin{array}{rl} & G_{a,\lambda,\xi,\eta} f_e \\ \overset{(4.3.15)}{=} & -\frac{1+\nu'_p-\nu'_{p+1}}{\nu'_p!} \displaystyle\sum_{x_j\in a_{i_j},\; j\neq s,s+1} \left\langle a_g^{y,x_1} \cdots a_{i_{s-1}}^{x_{s-1},w} \cdots a_{i_s}^{\bar{\xi},\eta\setminus w} (a_{i_{s+1}}^{\eta\setminus w,\bar{\xi}})^{w,x_{s+2}} \cdots a_{i_{s+2}}^{x_{s+2},x_{s+3}} \cdots a_{i_l}^{x_l,y} \right\rangle \circ (\xi \cup \eta) \\ \overset{(5.2.11)}{=} & -(1 + \nu'_p - \nu'_{p+1}) B(i_1,\ldots,\hat{i}_s,\hat{i}_{s+1},\ldots,i_l)^w_{\xi,\eta}. \end{array}$$

Calculation 5.2.14 *We treat the case* (II.i), *i.e.* $\xi \subseteq a'_g = a'_{i_0}$, $\eta \subseteq a_{g+1}$, $e_{g+1} = 0$, $\#\xi + \#\eta = \nu'_g$. *Choose* $w \in \eta$. *We obtain*

$$\begin{array}{rcl} G_{a,\lambda,\xi,\eta} f_e & \overset{(4.3.17)}{=} & \displaystyle\sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccccc} \cdots & (a'_g)^{\bar{\xi},\eta\setminus w} & (a_{g+1}^{\eta\setminus w,\bar{\xi}})^{w,x_1} & \cdots & a_{i_1}^{x_1,x_2} & \cdots & a_{i_l}^{x_l,y} \;\cdots \\ & & w & & & & \end{array} \right\rangle \\ & \overset{(5.2.10)}{=} & A(i_1,\ldots,i_l)^w_{\xi,\eta}. \end{array}$$

Calculation 5.2.15 *We treat the case* (II.ii), *i.e.* $\xi \subseteq a_p = a_{i_s}$, $s \in [1,l]$, $\eta \subseteq a_{p+1}$, $e_{p+1} = 0$, $\#\xi + \#\eta = \nu_{i_s} + 1$. *Choose* $w \in \eta$. *We obtain*

$$\begin{array}{rl} & G_{a,\lambda,\xi,\eta} f_e \\ \overset{(4.3.18)}{=} & -\frac{1}{\nu'_p!} \displaystyle\sum_{x_j\in a_{i_j},\; j\neq s} \left\langle \cdots a_g^{y,x_1} \cdots a_{i_{s-1}}^{x_{s-1},w} \cdots a_{i_s}^{\bar{\xi},\eta\setminus w} (a_{p+1}^{\eta\setminus w,\bar{\xi}})^{w,x_{s+1}} \cdots a_{i_{s+1}}^{x_{s+1},x_{s+2}} \cdots a_{i_l}^{x_l,y} \cdots \right\rangle \circ (\xi \cup \eta) \\ \overset{(5.2.11)}{=} & -B(i_1,\ldots,\hat{i}_s,\ldots,i_l)^w_{\xi,\eta}. \end{array}$$

Calculation 5.2.16 *We treat the case* (III.i)*, i.e.* $\xi \subseteq a_{g-1}$*,* $\eta \subseteq a'_g$*,* $\#\xi + \#\eta = \nu'_{g-1} + 1$*, to obtain*

$$G_{a,\lambda,\xi,\eta} f_e \overset{\text{Garnir, (4.1.4)}}{=} 0.$$

Calculation 5.2.17 *We treat the case* (III.ii)*, i.e.* $\xi \subseteq a_p$*,* $e_p = 0$*,* $\eta \subseteq a_{p+1} = a_{i_s}$*,* $s \in [1, l]$*,* $\#\xi + \#\eta = \nu'_p + 1$*. Choose* $w \in \eta$*. We obtain*

$$\begin{aligned} & G_{a,\lambda,\xi,\eta} f_e \\ \overset{(4.3.21)}{=} & \frac{1}{\nu'_p!} \sum_{x_j \in a_{i_j},\, j \neq s} \left\langle \ldots a_g^{y,x_1} \ldots a_{i_{s-1}}^{x_{s-1},w} \ldots a_p^{\bar{\xi},\eta\backslash w} (a_{i_s}^{\eta\backslash w,\bar{\xi}})^{w,x_{s+1}} \ldots a_{i_{s+1}}^{x_{s+1},x_{s+2}} \ldots a_{i_l}^{x_l,y} \right\rangle \circ (\xi \cup \eta) \\ \overset{(5.2.11)}{=} & B(i_1, \ldots, \hat{i}_s, \ldots, i_l)^w_{\xi,\eta}. \end{aligned}$$

Calculation 5.2.18 *We treat the case* (IV)*, i.e.* $\xi \subseteq a_p$*,* $\eta \subseteq a_{p+1}$*,* $e_p = 0$*,* $e_{p+1} = 0$*,* $\#\xi + \#\eta = \nu'_p + 1$*, to obtain*

$$G_{a,\lambda,\xi,\eta} f_e \overset{\text{Garnir, (4.1.4)}}{=} 0.$$

5.2.4 Polynomial Coefficients

We use the notation introduced in (4.3.25).

Proposition 5.2.19 *Let*

$$\boxed{Y_j := -(\nu'_g - g) + (\nu'_j - j)}$$

for $j \in [g, k]$*, so* $Y_g = 0$*. Let*

$$f^0 := \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} Y^{1-e} f_e.$$

For a ν*-tableau* a*, for* $p \in [1, k-1]$ *and for* $\xi \subseteq a_p$*,* $\eta \subseteq a_{p+1}$ *such that* $y \notin \xi$*,* $y \notin \eta$ *and such that* $\#\xi + \#\eta = \lambda'_p + 1$ *we have*

$$G_{a,\lambda,\xi,\eta} f^0 = 0.$$

Hence f^0 *induces a morphism of* $\mathbf{Z}\mathcal{S}_n$*-lattices*

$$S^{\lambda \subseteq \nu} \xrightarrow{\bar{f}^0} S^\nu$$

such that the composition with the inclusion of the James extension (5.1.18) turns out to be

$$(S^\nu \longrightarrow S^{\lambda \subseteq \nu} \xrightarrow{\bar{f}^0} S^\nu) = \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} Y^{1-e} \nu'_{i_l}$$

where, as in (5.2.1), $i_l = \max\{e^{-1}(1)\}$*, including* $e_g = 1$ *(5.2.7, 5.2.9).*

Choose $w \in \eta$.

Case $p = g$**,** i.e. (I.i) or (II.i). Let $[g+2, k] \xrightarrow{e'} \{0, 1\}$ be given. From (5.2.12, 5.2.14) we take

$$\begin{aligned} & G_{a,\lambda,\xi,\eta}(Y^{1-[1e']} f_{[1e']} + Y^{1-[0e']} f_{[0e']}) \\ = \; & (1 \cdot (1 + \nu'_g - \nu'_{g+1}) + Y_{g+1} \cdot 1) Y^{1-e'} A^w_{[0e'],\xi,\eta} \\ = \; & ((1 + \nu'_g - \nu'_{g+1}) - (\nu'_g - g) + (\nu'_{g+1} - (g+1))) Y^{1-e'} A^w_{[0e'],\xi,\eta} \\ = \; & 0. \end{aligned}$$

Case $p \in [g+1, k-1]$**,** i.e. (I.ii), (II.ii), (III.ii) or (IV). Let $[g+1,k]\backslash\{p,p+1\} \xrightarrow{e'} \{0,1\}$ be given. From (5.2.13, 5.2.15, 5.2.17, 5.2.18) we take

$$\begin{aligned} & G_{a,\lambda,\xi,\eta}(Y^{1-[11e']}f_{[11e']} + Y^{1-[10e']}f_{[10e']} + Y^{1-[01e']}f_{[01e']} + Y^{1-[00e']}f_{[00e']}) \\ = \ & (1 \cdot (-(1+\nu'_p - \nu'_{p+1})) + Y_{p+1} \cdot (-1) + Y_p \cdot 1 + Y_p Y_{p+1} \cdot 0) Y^{1-e'} B^w_{[00e'],\xi,\eta} \\ = \ & (-(1+\nu'_p - \nu'_{p+1}) + (\nu'_g - g) - (\nu'_{p+1} - (p+1)) - (\nu'_g - g) + (\nu'_p - p)) Y^{1-e'} B^w_{[00e'],\xi,\eta} \\ = \ & 0. \end{aligned}$$

Case $p \in [1, g-1]$**,** i.e. (III.i) or (IV). Let $[g+1,k] \xrightarrow{e} \{0,1\}$ be given. From (5.2.16, 5.2.18) we take

$$G_{a,\lambda,\xi,\eta} f_e = 0.$$

We shall simplify the formula for the composition of the inclusion of the James extension with f^0 (5.2.19) which we obtain by (5.2.9).

Remark 5.2.20 *For $j \in [g+1,k]$ we have*

$$\nu'_j + (\nu'_g - g + j - 1)Y_j = (Y_j + 1)(\nu'_g - g + (j+1) - 1).$$

Lemma 5.2.21 *Let $\kappa \in [g+1, k+1]$. We obtain*

$$\left(\sum_{[\kappa,k] \xrightarrow{e} \{0,1\}} Y^{1-e} \nu'_{i_l} \right) - \left(\prod_{j \in [\kappa-1,k]} Y_j \right) = (\nu'_g + k - g) \prod_{j \in [\kappa,k]} (Y_j + 1).$$

where $i_l := \kappa - 1$ for $e = 0$ or $e = \emptyset$ (i.e. $\kappa = k+1$).

We sort the left hand side sum according to the occurring term ν'_j, $j \in [\kappa-1, k]$, which has as its coefficient

$$\left(\sum_{[\kappa,j-1] \xrightarrow{e} \{0,1\}} Y^{1-e} \right) \left(\prod_{i \in [j+1,k]} Y_j \right) = \left(\prod_{i \in [\kappa,j-1]} (Y_i + 1) \right) \left(\prod_{i \in [j+1,k]} Y_i \right)$$

and rewrite it as

$$\begin{aligned} & -\textstyle\prod_{j\in[\kappa-1,k]} Y_j + \sum_{j \in [\kappa-1,k]} \left(\prod_{i \in [\kappa,j-1]} (Y_i+1) \right) \cdot \nu'_j \cdot \left(\prod_{i \in [j+1,k]} Y_i \right) \\ = \ & (\nu'_g - g + \kappa - 1) Y_\kappa \textstyle\prod_{j \in [\kappa+1,k]} Y_j \\ & + \nu'_\kappa \textstyle\prod_{j \in [\kappa+1,k]} Y_j \\ & + \textstyle\sum_{j \in [\kappa+1,k]} \left(\prod_{i \in [\kappa,j-1]} (Y_i+1) \right) \cdot \nu'_j \cdot \left(\prod_{i \in [j+1,k]} Y_i \right) \\ \overset{(5.2.20)}{=} \ & (Y_\kappa + 1)(\nu'_g - g + (\kappa+1) - 1) Y_{\kappa+1} \textstyle\prod_{j \in [\kappa+2,k]} Y_j \\ & + (Y_\kappa + 1)\nu'_{\kappa+1} \textstyle\prod_{i \in [\kappa+2,k]} Y_i \\ & + \textstyle\sum_{j \in [\kappa+2,k]} \left(\prod_{i \in [\kappa,j-1]} (Y_i+1) \right) \cdot \nu'_j \cdot \left(\prod_{i \in [j+1,k]} Y_i \right) \\ \overset{(5.2.20)}{=} \ & (Y_\kappa + 1)(Y_{\kappa+1}+1)(\nu'_g - g + (\kappa+2) - 1) Y_{\kappa+2} \textstyle\prod_{j \in [\kappa+3,k]} Y_j \\ & + (Y_\kappa + 1)(Y_{\kappa+1}+1)\nu'_{\kappa+2} \textstyle\prod_{i \in [\kappa+3,k]} Y_i \\ & + \textstyle\sum_{j \in [\kappa+3,k]} \left(\prod_{i \in [\kappa,j-1]} (Y_i+1) \right) \cdot \nu'_j \cdot \left(\prod_{i \in [j+1,k]} Y_i \right) \\ \overset{(5.2.20)}{=} \ & \\ \dots & \\ \overset{(5.2.20)}{=} \ & (Y_\kappa + 1)(Y_{\kappa+1}+1) \cdots (Y_{k-1}+1)(\nu'_g - g + k - 1) Y_k \\ & + (Y_\kappa + 1)(Y_{\kappa+1}+1) \cdots (Y_{k-1}+1)\nu'_k \\ \overset{(5.2.20)}{=} \ & \left(\textstyle\prod_{j \in [\kappa,k]} (Y_j + 1) \right) (\nu'_g - g + k). \end{aligned}$$

We shall exhibit a redundant factor.

Lemma 5.2.22 *Given* $p \in [g+1,k]$, $[g+1,k]\backslash\{p,p+1\} \xrightarrow{e'} \{0,1\}$ *such that* $\nu'_p = \nu'_{p+1}$, *we have*

$$f_{[10e']} = f_{[11e']}.$$

We apply a Garnir relation (4.1.4), cf. (4.3.28).

Lemma 5.2.23 *Let*

$$\boxed{r := \textstyle\prod_{j\in[g+1,k-1],\,\nu'_j=\nu'_{j+1}} Y_j}\;.$$

Then $S^{\lambda\subseteq\nu} \xrightarrow{f^0} S^\nu$ *is divisible by* r. *Let* $f := f^0/r$. *Note that we may also write*

$$r = \prod_{j\in[g+2,k],\,\nu'_{j-1}=\nu'_j} (Y_j+1).$$

In the part of the proof of (4.3.29) concerning 'f', we replace 'X' by 'Y' and '(4.3.28)' by '(5.2.22)'.

Lemma 5.2.24 f *is indivisible.*

Let $[\breve{a}]$ be the ν-tableau which has $y = n$ as its maximal entry, which is ordered increasingly down columns and for which $i < i'$, $\breve{a}_{i,j} \neq y$ and $\breve{a}_{i',j'} \neq y$ implies $\breve{a}_{i,j} < \breve{a}_{i',j'}$. In particular, $[\breve{a}]$ corresponds to a sequence in seq$(\lambda \subseteq \nu)$ (5.1.12). Suppose given $[g+1,k] \xrightarrow{e} \{0,1\}$ such that for $p, q \in [g+1,k]$, $p < q$ and $\nu'_p = \nu'_q$ we have $e_p \leq e_q$. The summands of

$$[\breve{a}]f_e = \sum_{x_j\in\breve{a}_{i_j}} \left\langle \begin{matrix} \ldots\breve{a}'_g\ldots\breve{a}_{i_1}^{x_1,x_2}\ldots\breve{a}_{i_2}^{x_2,x_3}\ldots\breve{a}_{i_l}^{x_l,y}\ldots \\ x_1 \end{matrix} \right\rangle$$

are standard ν-polytabloids up to sign. Note that, after ordering columns, 'the new place of y is a proper corner'. Since we may write the image of $[\breve{a}]$ under f as an integral linear combination of such elements (5.2.22), and since the occurring standard polytabloids are pairwise different because of different fillings of the columns, we are reduced to consider a chosen such e and to prove indivisibility for the corresponding summand

$$\frac{1}{r}\left(\sum_{e'\in E(e)} Y^{1-e'}\right)[\breve{a}]f_e,$$

where

$$E(e) := \{[g{+}1,k] \xrightarrow{e'} \{0,1\} \mid \forall i \in [g{+}1,k]\ (e'_i = e_i \vee \exists j \in [g{+}1,i{-}1]\ (\nu'_j = \nu'_i \wedge e'_j = e_j = 1))\}.$$

However, for $e = 11\ldots1$ we obtain

$$\sum_{e'\in E(e)} Y^{1-e'} \overset{(4.3.27)}{=} \prod_{i\in[g+2,k],\,\nu'_{i-1}=\nu'_i} (Y_i+1) \overset{(5.2.23)}{=} r.$$

We summarize to the

Theorem 5.2.25 *Keep the notation from (5.2.1, 5.2.19, 5.2.23). The* $\mathbf{Z}\mathcal{S}_n$*-linear map*

$$\boxed{f = \frac{1}{r} \sum_{[g+1,k] \xrightarrow{e} \{0,1\}} Y^{1-e} f_e : F^\nu \longrightarrow S^\nu}$$

factors over

$$S^{\lambda \subseteq \nu} \xrightarrow{\bar{f}} S^\nu,$$

which is indivisible, i.e. $\bar{f} \not\equiv_s 0$ *for* $s \geq 2$*, and which composes to*

$$(S^\nu \longrightarrow S^{\lambda \subseteq \nu} \xrightarrow{\bar{f}} S^\nu) = (\nu'_g + k - g) \prod_{j \in [g+1,k],\, \nu'_{j-1} > \nu'_j} (Y_j + 1) \;\; =: \;\; m^{\lambda \subseteq \nu}$$

with the inclusion of the James extension (5.1.18).

This follows from (5.2.19, 5.2.23, 5.2.24, 5.2.21 for $\kappa = g + 1$).

Corollary 5.2.26 *The James extension (5.1.18)*

$$J := (0 \longrightarrow S^\nu \longrightarrow S^{\lambda \subseteq \nu} \longrightarrow S^{\lambda \subseteq \nu R_z} \longrightarrow 0) \in \mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^\nu)$$

has order $m := |m^{\lambda \subseteq \nu}|$.

Let m' be the order of J, so that m' divides m (5.2.25). Note that $\mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^\nu) = 0$ since $S^{\lambda \subseteq \nu R_z}$ has a filtration of Specht lattices to partitions strictly dominating ν (5.1.18, [J 78, 3.2]). The exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^\nu/m') \longrightarrow \mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^\nu) \xrightarrow{m'} \mathrm{Ext}^1_{\mathbf{Z}\mathcal{S}_n}(S^{\lambda \subseteq \nu R_z}, S^\nu)$$

yields a (unique) morphism $S^{\lambda \subseteq \nu R_z} \longrightarrow S^\nu/m'$ which pulls back

$$0 \longrightarrow S^\nu \xrightarrow{m'} S^\nu \longrightarrow S^\nu/m' \longrightarrow 0$$

to J, so in particular it yields a retraction $S^{\lambda \subseteq \nu} \xrightarrow{f'} S^{\lambda \subseteq \nu R_z}$ of J up to m'. Since the morphism $S^{\lambda \subseteq \nu R_z} \longrightarrow S^\nu/m$ which pulls back

$$0 \longrightarrow S^\nu \xrightarrow{m} S^\nu \longrightarrow S^\nu/m \longrightarrow 0$$

to J is likewise unique, we obtain

$$f \equiv_m \frac{m}{m'} f',$$

thus

$$f \equiv_{m/m'} 0,$$

whence $m = m'$ by indivisibility of f (5.2.25).

Remark 5.2.27 *Assume* νR_z *to be a partition. The composition*

$$S^{\nu R_z} \longrightarrow S^{\lambda \subseteq \nu R_z} \xrightarrow{\bar{f}} S^\nu/m^{\lambda \subseteq \nu}$$

of the inclusion of the James extension (5.1.18) with the morphism induced on the cokernels by the morphism constructed in (5.2.25), again denoted by $\bar{f}$ *by abuse of notation, coincides with the morphism constructed in (4.3.31) up to sign and in the direct limit (this is, we adjust the modulus).*

Let $\kappa := \nu_{z-1}$. We exclude the case $\kappa = k$, $z = 2$, in which the assertion holds true, and assume $\kappa < k$ in the sequel. We shall pretend $S^{\lambda\subseteq\nu R_z} \xrightarrow{f_e} S^\nu/m^{\lambda\subseteq\nu}$ to be well defined, $[g+1,k] \xrightarrow{e} \{0,1\}$, which becomes correct as soon as we sum up as in (5.2.25). We do so in order not to carry this sum through the calculation. Moreover, we denote the map called f (resp. f_e) in (4.3.31) (resp. (4.3.1)) by φ (resp. φ_e).

We send a νR_z-polytabloid $\langle aR_z\rangle$, which we may assume to take this shape, first to

$$
\begin{array}{rcl}
S^{\nu R_z} & \longrightarrow & S^{\lambda\subseteq\nu R_z} \\
\langle aR_z\rangle & \longrightarrow & \langle aR_z\rangle \\
& = & \left\langle\begin{array}{ccccc} \cdots & a'_g & \cdots & a_{\kappa+1} & \cdots \\ & & & y & \end{array}\right\rangle \\
& & - \sum\limits_{x\in a_{\kappa+1}} \left\langle\begin{array}{ccccc} \cdots & a'_g & \cdots & a^{x,y}_{\kappa+1} & \cdots \\ & & & x & \end{array}\right\rangle .
\end{array}
$$

Case 1. $\kappa+1 \leq i_l$.

Subcase 1.1. $e_{\kappa+1} = 0$, in particular, $\kappa+1 < i_l$. We send the element $\langle aR_z\rangle$ via f_e further to the following element of $S^\nu/m^{\lambda\subseteq\nu}$.

$$
\begin{aligned}
& \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccccc} \cdots & a'_g & \cdots & a^{x_1,x_2}_{i_1} & \cdots & a_{\kappa+1} & \cdots & a^{x_l,y}_{i_l} & \cdots \\ & x_1 & & & & & & & \end{array}\right\rangle \\
& - \sum_{x\in a_{\kappa+1}} \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccccc} \cdots & a'_g & \cdots & a^{x_1,x_2}_{i_1} & \cdots & a^{x,y}_{\kappa+1} & \cdots & a^{x_l,x}_{i_l} & \cdots \\ & x_1 & & & & & & & \end{array}\right\rangle \\
\overset{\text{Garnir, (4.1.4)}}{=} \; & 0.
\end{aligned}
$$

Subcase 1.2. $e_{\kappa+1} = 1$, $\kappa+1 < i_l$. Let $i_s := \kappa+1$. We send the element $\langle aR_z\rangle$ via f_e further to the following element of $S^\nu/m^{\lambda\subseteq\nu}$.

$$
\begin{aligned}
& \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccc} \cdots & a'_g & \cdots & a^{x_s,x_{s+1}}_{i_s} & \cdots & a^{x_l,y}_{i_l} & \cdots \\ & x_1 & & & & & \end{array}\right\rangle \\
(x_s \in a^{x,y}_{i_s}\backslash y) \quad & - \sum_{x_s,x\in a_{i_s},\ x\neq x_s} \; \sum_{x_j\in a_{i_j},\ j\neq s} \left\langle\begin{array}{ccccccc} \cdots & a'_g & \cdots & (a^{x,y}_{i_s})^{x_s,x_{s+1}} & \cdots & a^{x_l,x}_{i_l} & \cdots \\ & x_1 & & & & & \end{array}\right\rangle \\
(x_s = y) \quad & - \sum_{x\in a_{i_s}} \; \sum_{x_j\in a_{i_j},\ j\neq s} \left\langle\begin{array}{ccccccccc} a'_g & \cdots & a^{x_{s-1},y}_{i_{s-1}} & \cdots & a^{x,x_{s+1}}_{i_s} & \cdots & a^{x_{s+1},x_{s+2}}_{i_{s+1}} & \cdots & a^{x_l,x}_{i_l} \\ x_1 & & & & & & & & \end{array}\right\rangle \\
\overset{2\times\ \text{Garnir, (4.1.4)}}{=} \quad & \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccc} \cdots & a'_g & \cdots & a^{x_s,x_{s+1}}_{i_s} & \cdots & a^{x_l,y}_{i_l} & \cdots \\ & x_1 & & & & & \end{array}\right\rangle \\
& - \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccc} \cdots & a'_g & \cdots & a^{x_s,x_{s+1}}_{i_s} & \cdots & a^{x_l,y}_{i_l} & \cdots \\ & x_1 & & & & & \end{array}\right\rangle \\
(\text{`}x = x_{s+1}\text{'}) \quad & + \sum_{x_j\in a_{i_j}} \left\langle\begin{array}{ccccccccc} \cdots & a'_g & \cdots & a^{x_s,y}_{i_s} & \cdots & a^{x_{s+1},x_{s+2}}_{i_{s+1}} & \cdots & a^{x_l,x_{s+1}}_{i_l} & \cdots \\ & x_1 & & & & & & & \end{array}\right\rangle \\
& - \sum_{x_j\in a_{i_j},\ j\neq s} \left\langle\begin{array}{ccccccccccc} \cdots & a'_g & \cdots & a^{x_{s-1},y}_{i_{s-1}} & \cdots & a_{i_s} & \cdots & a^{x_{s+1},x_{s+2}}_{i_{s+1}} & \cdots & a^{x_l,x_{s+1}}_{i_l} & \cdots \\ & x_1 & & & & & & & & & \end{array}\right\rangle \\
\overset{(5.2.8)}{=} \quad & \sum_{x_j\in a_{i_j},\ j\in[1,s]} \nu'_{i_l} \cdot \left\langle\begin{array}{ccccc} \cdots & a'_g & \cdots & a^{x_s,y}_{i_s} & \cdots \\ & x_1 & & & \end{array}\right\rangle \\
& - \sum_{x_j\in a_{i_j},\ j\in[1,s-1]} \nu'_{i_l} \cdot \left\langle\begin{array}{ccccccc} \cdots & a'_g & \cdots & a^{x_{s-1},y}_{i_{s-1}} & \cdots & a_{i_s} & \cdots \\ & x_1 & & & & & \end{array}\right\rangle \\
= \quad & (-1)^{\nu'_{\kappa+1}+1} \nu'_{i_l} \cdot \langle aR_z\rangle \varphi_{e|_{[g+1,\kappa]}} ,
\end{aligned}
$$

where the sum involving '$j \in [1, s-1]$' is to be read as a single summand in case $s = 1$.

Subcase 1.3. $e_{\kappa+1} = 1$, $\kappa+1 = i_l$. We send the element $\langle aR_z\rangle$ via f_e further to the following element of $S^\nu/m^{\lambda\subseteq\nu}$.

$$\begin{array}{lll}
 & & \sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccc} \cdots & a'_g & \cdots & a_{i_l}^{x_l,y} & \cdots \\ & x_1 & & & \end{array} \right\rangle \\
(x_l \in a_{i_l}^{x,y}\backslash y) & - & \sum_{x_l,x\in a_{i_l},\, x\neq x_l} \sum_{x_j\in a_{i_j},\, j\neq l} \left\langle \begin{array}{ccccc} \cdots & a'_g & \cdots & (a_{i_l}^{x,y})^{x_l,x} & \cdots \\ & x_1 & & & \end{array} \right\rangle \\
(x_l = y) & - & \sum_{x\in a_{i_l}} \sum_{x_j\in a_{i_j},\, j\neq l} \left\langle \begin{array}{ccccccc} \cdots & a'_g & \cdots & a_{i_{l-1}}^{x_{l-1},y} & \cdots & (a_{i_l}^{x,y})^{y,x} & \cdots \\ & x_1 & & & & & \end{array} \right\rangle \\
 & = & \sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccc} \cdots & a'_g & \cdots & a_{i_l}^{x_l,y} & \cdots \\ & x_1 & & & \end{array} \right\rangle \\
 & & +(\nu'_{i_l}-1)\cdot \sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccc} \cdots & a'_g & \cdots & a_{i_l}^{x_l,y} & \cdots \\ & x_1 & & & \end{array} \right\rangle \\
 & & -\nu'_{i_l}\cdot \sum_{x_j\in a_{i_j},\, j\neq l} \left\langle \begin{array}{ccccccc} \cdots & a'_g & \cdots & a_{i_{l-1}}^{x_{l-1},y} & \cdots & a_{i_l} & \cdots \\ & x_1 & & & & & \end{array} \right\rangle \\
 & = & (-1)^{\nu'_{\kappa+1}+1}\nu'_{i_l}\cdot\langle aR_z\rangle\varphi_{e|_{[g+1,\kappa]}}.
\end{array}$$

Case 2. $\kappa+1 > i_l$. We send the element $\langle aR_z\rangle$ via f_e further to the following element of $S^\nu/m^{\lambda\subseteq\nu}$.

$$\begin{array}{ll}
 & \sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccccc} \cdots & a'_g & \cdots & a_{i_l}^{x_l,y} & \cdots & a_{\kappa+1} & \cdots \\ & x_1 & & & & & \end{array} \right\rangle \\
 & -\sum_{x\in a_{\kappa+1}} \sum_{x_j\in a_{i_j}} \left\langle \begin{array}{ccccccc} \cdots & a'_g & \cdots & a_{i_l}^{x_l,x} & \cdots & a_{\kappa+1}^{x,y} & \cdots \\ & x_1 & & & & & \end{array} \right\rangle \\
= & -(-1)^{\nu'_{\kappa+1}+1}\langle aR_z\rangle\varphi_{e|_{[g+1,\kappa]}}.
\end{array}$$

Therefore $\langle aR_z\rangle$ is sent via $S^{\lambda\subseteq\nu R_z}\xrightarrow{\bar f} S^\nu/m^{\lambda\subseteq\nu}$ to the following sum, up to sign. For $e''=0$ we let $i_l := \kappa+1$.

$$\begin{array}{ll}
 & \frac{1}{r}\left[\sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} \sum_{[\kappa+2,k]\xrightarrow{e''}\{0,1\}} Y^{1-e'}Y^{1-e''}\nu'_{i_l}\langle aR_z\rangle\varphi_{e'} \right. \\
 & \left. - \sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} Y^{1-e'}\left(\prod_{j\in[\kappa+1,k]} Y_j\right)\langle aR_z\rangle\varphi_{e'} \right] \\
= & \frac{1}{r}\sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} Y^{1-e'}\left[\sum_{[\kappa+2,k]\xrightarrow{e''}\{0,1\}} Y^{1-e''}\nu'_{i_l} - \prod_{j\in[\kappa+1,k]} Y_j\right]\langle aR_z\rangle\varphi_{e'} \\
\overset{(5.2.21,\kappa<k)}{=} & \frac{1}{r}\sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} Y^{1-e'}\left[(\nu'_g+k-g)\prod_{j\in[\kappa+2,k]}(Y_j+1)\right]\langle aR_z\rangle\varphi_{e'} \\
= & (\nu'_g+k-g)\,\frac{\prod_{j\in[\kappa+2,k],\,\nu'_{j-1}\neq\nu'_j}(Y_j+1)}{\prod_{j\in[g+2,\kappa+1],\,\nu'_{j-1}=\nu'_j}(Y_j+1)} \sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} Y^{1-e'}\langle aR_z\rangle\varphi_{e'}\,.
\end{array}$$

Adjusting the modulus, i.e. replacing $m^{\lambda\subseteq\nu}$ by $m' := \prod_{j\in[g+1,\kappa+1],\, \nu'_{j-1}>\nu'_j}(Y_j + 1)$, we obtain the according image of $\langle aR_z\rangle$ to be

$$\frac{1}{\prod_{j\in[g+1,\kappa],\, \nu'_j=\nu'_{j+1}} Y_j} \sum_{[g+1,\kappa]\xrightarrow{e'}\{0,1\}} Y^{1-e'}\langle aR_z\rangle\varphi_{e'},$$

welldefinedness of the adjustment to be seen below. Note that for $j \in [g+1,\kappa]$ we have $Y_j = -(\nu'_g-g)+(\nu'_g-1-j) = g-j-1$, and that $Y_{\kappa+1} = -(\nu'_g-g)+(\nu'_g-2-(\kappa+1)) = g-\kappa-3$. Hence $m' = (g-(g+1))(g-\kappa-2) = \kappa-g+2$ and the image of $\langle aR_z\rangle$ becomes

$$\frac{1}{\prod_{j\in[g+1,\kappa-1]} Y_j} \sum_{[g+1,\kappa]\xrightarrow{e}\{0,1\}} Y^{1-e}\langle aR_z\rangle\varphi_e = Y_\kappa\langle aR_z\rangle\varphi_{e^{\kappa+1}} + \sum_{\tau\in[g+1,\kappa]}\langle aR_z\rangle\varphi_{e^\tau}$$

in the notation and using the argument of the proof of (4.3.29), which is to be compared to the image

$$\frac{1}{\prod_{j\in[g+1,\kappa-1]} X_j} \sum_{[g+1,\kappa]\xrightarrow{e}\{0,1\}} X^{1-e}\langle aR_z\rangle\varphi_e = X_\kappa\langle aR_z\rangle\varphi_{e^{\kappa+1}} + \sum_{\tau\in[g+1,\kappa]}\langle aR_z\rangle\varphi_{e^\tau}$$

under the morphism from (4.3.31). However, $X_\kappa - Y_\kappa = 1-(g-\kappa-1) = m'$.

5.2.5 Illustration

We give an example in order to illustrate what the summands of the image of a semitabloid under $\bar{f}$ look like. Let $\lambda := (3,2,1)$, $\nu = (3,2,1,1)$, i.e.

```
g   k
x x x
x x
x
y
```

We find $g = 1$, $k = 3$, $\nu'_1 = 4$, $\nu'_2 = 2$, $\nu'_3 = 1$. Thus $Y_2 = -(4-1)+(2-2) = -3$, $Y_3 = -(4-1)+(1-3) = -5$, and therefore

$$f = 15\cdot f_{00} - 3\cdot f_{01} - 5\cdot f_{10} + 1\cdot f_{11}.$$

This is, we map, dropping the brackets,

$$S^{(3,2,1)\subseteq(3,2,1,1)} \xrightarrow{\bar{f}} S^{(3,2,1,1)}$$

$$\begin{matrix}1&5&7\\2&6\\3\\4\end{matrix} \longrightarrow 15\cdot\begin{matrix}1&5&7\\2&6\\3\\4\end{matrix} -3\cdot\begin{matrix}1&5&4\\2&6\\3\\7\end{matrix} -5\cdot\left(\begin{matrix}1&4&7\\2&6\\3\\5\end{matrix} + \begin{matrix}1&5&7\\2&4\\3\\6\end{matrix}\right) + \left(\begin{matrix}1&7&4\\2&6\\3\\5\end{matrix} + \begin{matrix}1&5&4\\2&7\\3\\6\end{matrix}\right),$$

giving a retraction to the inclusion $S^{(3,2,1,1)} \longrightarrow S^{(3,2,1)\subseteq(3,2,1,1)}$ of the James extension up to

$$m^{(3,2,1)\subseteq(3,2,1,1)} = (\nu'_g + k - g)(Y_2+1)(Y_3+1) = 48,$$

which is best possible, since being its order in Ext^1.

5.3 Construction of the truss

We consider a filtration of the regular lattice $M^{(1^n)}$ built up of James extensions in the following sense. First, consider a James extension with middle term $M^{(1^n)}$. Second, consider two James extensions with middle terms the kernel and the cokernel of the James extension of the first step. Third, consider four James extensions with middle terms the kernels and the cokernels of the James extensions of the second step. And so on. In case an end term of an occurring James extension is a Specht lattice, we let the resulting tree end at this point. This yields a finite tree, in which each Specht lattice S^λ occurs with multiplicity $\mathrm{rk}\, S^\lambda$ as an end point, as to be seen rationally. We obtain a filtration of the regular lattice by Specht lattice quotients by pulling back an inclusion of an occurring James extension to a filtration step of the regular lattice. Whereas we shall not make use of that filtration explicitely, but only of these James extensions, the possibility of having such a filtration at our disposal in principle has been our starting point.

Having constructed the system of occurring James extensions together with retractions up to the Ext^1-order, we need to find a tuple of bases for the occurring James lattices such that the system of morphisms is of 'simplest possible shape'. I.e. we have to identify the points of our tree belonging to the same James lattice and to solve the 'normal form problem' for the module over this tree quotient which is given by that system of morphisms. This module is called the **truss** of $\mathbf{Z}\mathcal{S}_n$, according to the geometric shape of its underlying quiver as a quotient of a binary tree (with double edges) (cf. S 5.4). Strictly speaking, we do not deal with a normal form problem, i.e. with a classification of indecomposables, which would hardly be possible, but with a (non well defined) normalization problem for a **single** module. Given a tuple of bases for the occurring James lattices, we can read off a complete set of ties describing the embedding

$$\mathbf{Z}\mathcal{S}_n \hookrightarrow \prod_\lambda (\mathbf{Z})_{n_\lambda}$$

from the truss, using its system of morphisms (5.3.15). Therefore, our original problem of finding a satisfactory embedding (S 0.1.2) is converted into a normal form problem for a single module over a path algebra, i.e. to a problem of 'simultaneous linear algebra', similar to, but more intricate than (S 4.4.2).

Suppose given $\lambda \subseteq \nu$.

Lemma 5.3.1 (row switch) *Let s be such that $\lambda_s = \lambda_{s+1}$. Define $\bar\nu$ by setting $\bar\nu_i := \nu_{i(s\ s+1)}$ for $i \geq 1$, analogously $\bar\lambda$, analogously the $\bar\nu$-tableau $\bar a$, given a ν-tableau a. There is an isomorphism*

$$\begin{array}{ccc} S^{\lambda\subseteq\nu} & \xrightarrow{\sim} & S^{\bar\lambda\subseteq\bar\nu} \\ \langle a\rangle_\lambda & \longrightarrow & \langle \bar a\rangle_{\bar\lambda} \end{array}$$

This isomorphism is defined already on the level $M^\nu \xrightarrow{\sim} M^{\bar\nu}$ and restricts to the James lattices in both directions.

Remark 5.3.2 (vertical box shift) *Given $\lambda \subseteq \nu$, z as in (S 5.2), we may use (5.3.1) to substitute $S^{\lambda\subseteq\nu R_z}$ isomorphically by a James lattice to $\lambda \subseteq \tilde\nu$, $\tilde\nu$ being a partition, not merely a prepartition, by shifting the rightmost box in the $(z-1)$-st row upwards as far as possible.*

The following lemma is the reason for the top part of the truss of $\mathbf{Z}\mathcal{S}_n$ to arise from the truss of $\mathbf{Z}\mathcal{S}_{n-1}$ (cf. S 5.4).

Lemma 5.3.3 (lengthening the first column) *Suppose given $\lambda \subseteq \nu$, z as in (S 5.2). Let $n \leq \hat{n}$, let*

$$\hat{\nu}_i := \begin{cases} \nu_i & \text{for } i \in [1, \nu'_1] \\ 1 & \text{for } i \in [\nu'_1 + 1, \nu'_1 + \hat{n} - n] \end{cases}$$

for some $\hat{\nu}'_1 \geq \nu'_1$. The James extension

$$0 \longrightarrow S^{\lambda A_z \underline{\subseteq} \hat{\nu}} \longrightarrow S^{\lambda \underline{\subseteq} \hat{\nu}} \longrightarrow S^{\lambda \underline{\subseteq} \hat{\nu} R_z} \longrightarrow 0.$$

arises up to an isomorphism of exact sequences from the James extension

$$0 \longrightarrow S^{\nu} \longrightarrow S^{\lambda \underline{\subseteq} \nu} \longrightarrow S^{\lambda \underline{\subseteq} \nu R_z} \longrightarrow 0,$$

by induction from $\mathcal{S}_n$, considered as the centralizer of $[n+1, \hat{n}]$ inside $\mathcal{S}_{\hat{n}}$, to $\mathcal{S}_{\hat{n}}$. Consequently, the retraction up to $m^{\lambda \underline{\subseteq} \nu}$ of the inclusion of the sequence to $\lambda \subseteq \nu$, z, induced up from $\mathcal{S}_n$ to $\mathcal{S}_{\hat{n}}$, yields a retraction up to $m^{\lambda \underline{\subseteq} \nu}$ of the inclusion of the sequence to $\lambda \subseteq \hat{\nu}$, z.

For a ν-tableau a, let the $\hat{\nu}$-tableau $\hat{a}$ be defined by filling up the first column, i.e. by

$$\hat{a}_{ij} := \begin{cases} a_{ij} & \text{for } i \in [1, \nu'_1] \\ \hat{n} - (\hat{\nu}'_1 - i) & \text{for } i \in [\nu'_1 + 1, \hat{\nu}'_1],\ j = 1. \end{cases}$$

There is an epimorphism

$$\begin{array}{ccccc} S^{\lambda \underline{\subseteq} \nu} & \otimes_{\mathbf{Z}\mathcal{S}_n} & \mathbf{Z}\mathcal{S}_{\hat{n}} & \longrightarrow & S^{\lambda \underline{\subseteq} \hat{\nu}} \\ \langle a \rangle_\lambda & \otimes & \sigma & \longrightarrow & \langle \hat{a} \rangle_\lambda\, \sigma, \end{array}$$

given by the restriction of $M^{\nu} \longrightarrow M^{\hat{\nu}}|_{\mathcal{S}_n}$ to the James lattices. For to prove injectivity it suffices to show equality of ranks, so that we may assume by induction that $\hat{n} = n + 1$. In a sequence in $s(\lambda \subseteq \hat{\nu})$ (cf. 5.1.9) we may move the entry $\hat{\nu}'_1$ arbitrarily without leaving $s(\lambda \subseteq \hat{\nu})$. Therefore $\#s(\lambda \subseteq \hat{\nu}) = (n+1) \cdot \#s(\lambda \subseteq \nu)$, so that we may apply (5.1.18).

Now this isomorphism is applicable to the terms and compatible with the maps of the James extension attached to $\lambda \subseteq \nu$, z (5.1.18).

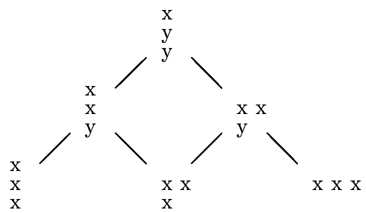
Proposition 5.3.4 *A pair $\lambda \subseteq \nu$ for which λ is a partition and ν is a partition of n, for which $\lambda_1 = \nu_1$, for which $\nu_i \leq 1$ for $i \geq \lambda'_1$ and for which*

$$\#\{i \in [1, \lambda'_1] \mid \lambda_i < \nu_i\} \leq 1$$

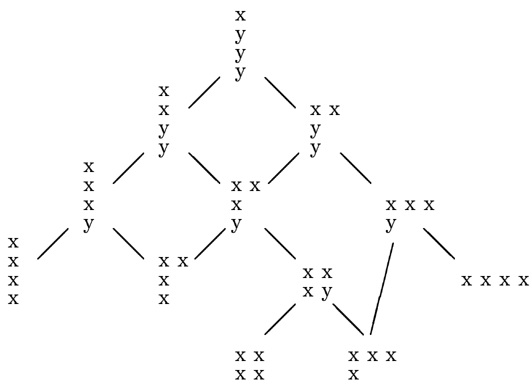
holds, is called an **occurring pair.** *For any occurring pair either we have $\lambda = \nu$ or the corresponding James lattice can be written as the middle term of a James extension with outer terms being attached to occurring pairs, up to (5.3.2, 5.1.3). (5.2.25, 5.3.3) give a retraction up to a nonzero integer to this James extension. Moreover, starting with $M^{(1^n)} = S^{(1) \underline{\subseteq} (1^n)}$, the binary tree arising from this process is finite. The set of occurring pairs in this tree coincides with the set of occurring pairs.*

Let z be minimal with $\lambda_i < \nu_i$. The kernel of the corresponding James extension is attached to an occurring pair without changes. The pair to which the cokernel is attached possibly needs vertical shifting of the rightmost box in the $(z-1)$-st row via (5.3.2) and rewriting via (5.1.3).

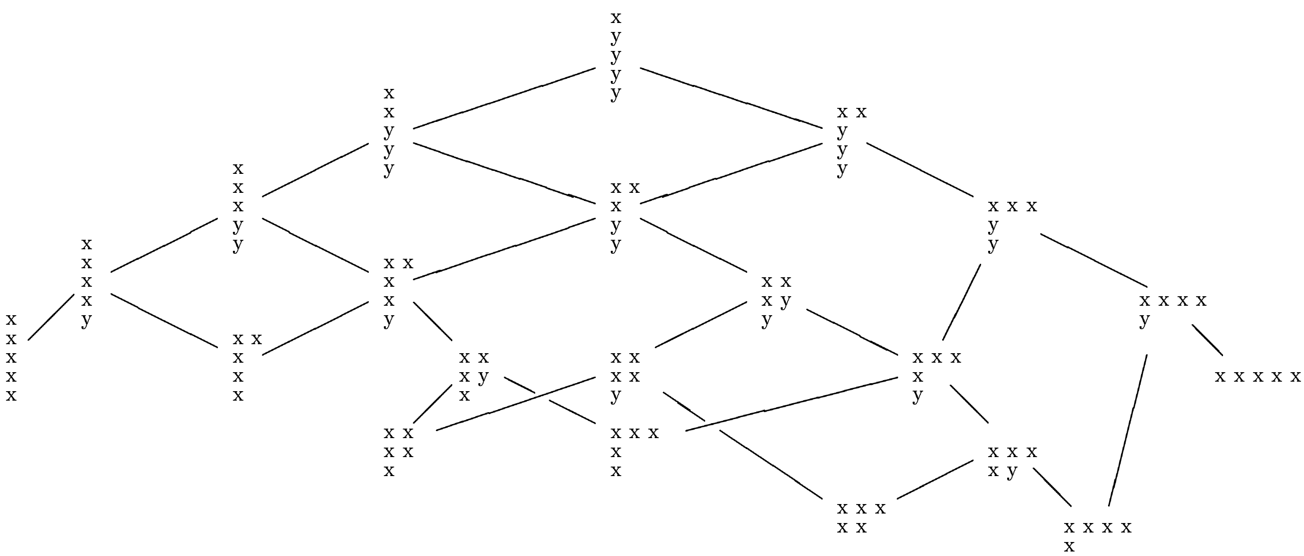
Example 5.3.5 ($n = 3$) The following diagram depicts a way how to unscrew $M^{(1^3)}$ via James extensions. The respective λ-region is indicated via x's, the ν-region via x's and y's. Starting from such a pair, the James lattice attached to it can be written as a middle term of a James extension with kernel attached to the pair to the left and down, and with cokernel attached to the pair to the right and down.



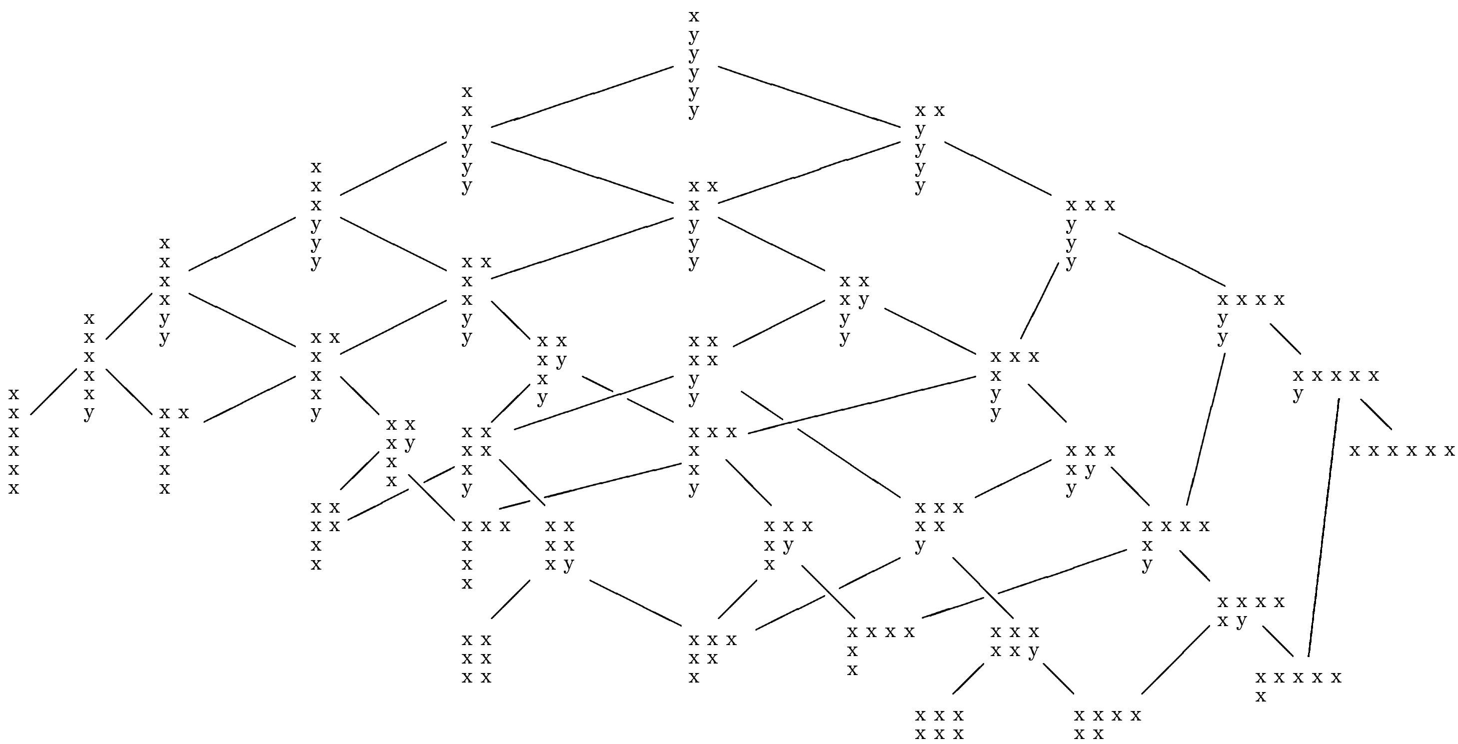
Example 5.3.6 ($n = 4$) The following diagram depicts a way how to unscrew $M^{(1^4)}$ via James extensions.



Example 5.3.7 ($n = 5$) The following diagram depicts a way how to unscrew $M^{(1^5)}$ via James extensions.



Example 5.3.8 ($n = 6$) The following diagram depicts a way how to unscrew $M^{(1^6)}$ via James extensions.



Notation 5.3.9 (the binary tree) Let E be the set of pairs

$$(h, [1, h] \xrightarrow{e} \{0, 1\}),$$

where $h \geq 0$. We drop the h in the notation and refer to such a pair as e, and to its datum h as h_e. To the value of e at $i \in [1, h_e]$ we refer as e_i. For the unique map e in case $h = 0$ we write $[1, 0] \xrightarrow{\emptyset} \{0, 1\}$. For $h \geq 1$ we write e.g. 010 for the map e with $h_e = 3$, $e_1 = 0$, $e_2 = 1$, $e_3 = 0$.

Given $[1, h_e] \xrightarrow{e} \{0, 1\}$, we let

$$\begin{array}{ccc} [1, h_e + 1] & \xrightarrow{e0} & \{0, 1\} \\ [1, h_e + 1] & \xrightarrow{e1} & \{0, 1\} \end{array}$$

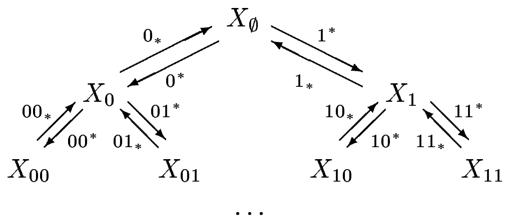
be defined by $e0|_{[1,h_e]} := e1|_{[1,h_e]} := e$, $e0_{h_e+1} := 0$, $e1_{h_e+1} := 1$. In particular, we have $h_{e0} = h_{e1} = h_e + 1$.

A **binary double tree** X with support in a subset $E' \subseteq E$ and with values in a category $\mathcal{C}$ is defined to be a diagram whose objects are indexed as X_e, $e \in E'$, and whose morphisms run as follows.

$$\begin{array}{ccc} X_{e0} & \xrightarrow{e0_*} & X_e \\ X_{e1} & \xrightarrow{e1_*} & X_e \\ X_e & \xrightarrow{e0^*} & X_{e0} \\ X_e & \xrightarrow{e1^*} & X_{e1}, \end{array}$$

provided the respective indices are both in E'.

Example 5.3.10 For example, the 'two upper layers' of a binary double tree as in (5.3.9), for $E' = E$ say, yield a diagram that can be depicted as follows.



In the sequel, we shall define our $e1_*$ out of $e0_*$, $e0^*$, $e1^*$ by the following

Lemma 5.3.11 *Let $m \geq 1$, let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a_*} & Y & \xrightarrow{b^*} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle a^*} & & \downarrow & & \\ 0 & \longrightarrow & X & \xrightarrow{m} & X & \longrightarrow & X/m & \longrightarrow & 0 \end{array}$$

be a morphism of short exact sequences of $\mathbf{Z}\mathcal{S}_n$-lattices. Then there exists a unique $\mathbf{Z}\mathcal{S}_n$-morphism $Z \xrightarrow{b_*} Y$ such that

$$\begin{array}{rcl} \begin{pmatrix} a_* \\ b_* \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix} & = & \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \\ \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} a_* \\ b_* \end{pmatrix} & = & m. \end{array}$$

Uniqueness of the rational inverse together with a comparison of ranks shows that it suffices to find a $\mathbf{Z}\mathcal{S}_n$-linear map b_* which satisfies the first equation. Let Z' be the kernel of a^*. The restriction of b^* maps Z' injectively into Z, this inclusion being the kernel of $Z \longrightarrow X/m$, so that we obtain $mZ \subseteq Z'b^*$. Let $Z \xrightarrow{b_*} Y$ be defined by sending z to

$$mz \in mZ \subseteq Z'b^* \overset{b^*}{\underset{\sim}{\longleftarrow}} Z' \subseteq Y.$$

Then $b_*a^* = 0$ as well as $b_*b^* = m$.

Definition 5.3.12 (the truss, yet unglued) Let $n \geq 1$. Define inductively a subset $E_n \subseteq E$, a subset $\partial E_n \subseteq E_n$, a surjective map

$$\begin{array}{ccc} E_n & \xrightarrow{\pi} & \{\text{ occurring pairs }\} \\ e & \longrightarrow & \lambda^e \subseteq \nu^e \end{array}$$

(cf. 5.3.4) as well as a binary double tree T with support in E_n as follows.

Let $\emptyset \subseteq E_n$, let $\lambda^\emptyset := (1)$, let $\nu^\emptyset := (1^n)$, let $T_\emptyset := S^{\lambda^\emptyset \subseteq \nu^\emptyset}$.

Suppose given $e \in E_n$ such that $\lambda^e \subseteq \nu^e$ has been defined.

Case $\lambda^e = \nu^e$. Let any map e' restricting to $e0$ or $e1$ be excluded from E_n, let e belong to the **boundary** ∂E_n.

In order to prolong T (artificially) to the whole of E, suppose given e' with $h_e < h_{e'}$ such that $e'|_{[1,h_e]} = e$.

In case $1 \in e_{[h_e+1,h_{e'}]}$, i.e. in case 'there is a 1 in $(e'\backslash e)$', let $T_{e'} := 0$. Otherwise, let $T_{e'} := T_e$, and, moreover, let

$$\begin{array}{rcl}(T_{e'0} \stackrel{e'0_*}{\longrightarrow} T_{e'}) & := & 1 \\ (T_{e'} \stackrel{e'0^*}{\longrightarrow} T_{e'0}) & := & 1.\end{array}$$

Case $\lambda^e \neq \nu^e$. Let $e0$ and $e1$ be contained in E_n. Let z be minimal such that $\lambda^e_z < \nu^e_z$, so in particular $z \geq 2$. Provisionally, let

$$\begin{array}{rcl}\lambda^{e0} & := & \lambda^e A_z \\ \nu^{e0} & := & \nu^e \\ \tilde{\lambda}^{e1} & := & \lambda^e \\ \tilde{\nu}^{e1} & := & \nu^e R_z \\ T_{e0} & := & S^{\lambda^{e0}\subseteq\nu^{e0}} \\ \tilde{T}_{e1} & := & S^{\tilde{\lambda}^{e1}\subseteq\tilde{\nu}^{e1}},\end{array}$$

let

$$0 \longrightarrow T_{e0} \stackrel{e0_*}{\longrightarrow} T_e \stackrel{(e1^*)^\sim}{\longrightarrow} \tilde{T}_{e1} \longrightarrow 0$$

be the James extension for $\lambda^e \subseteq \nu^e$, z (5.1.18). Let $e0^*$ be the retraction to $e0_*$ up to $m^{\lambda^e\subseteq\nu^e}$ given in (5.2.25), let $(e1_*)^\sim$ be as constructed in (5.3.11).

In case $\tilde{\nu}^{e1}$ is not a partition, we replace it via (5.3.2) by a partition and accordingly substitute the James extension isomorphically, keeping the notation by abuse.

In case we obtain $\tilde{\lambda}^{e1}_1 < \tilde{\nu}^{e1}_1$, we replace $\tilde{\lambda}^{e1}_1$ by $\tilde{\nu}^{e1}_1$ by (5.1.3), keeping the notation by abuse.

Now we remove all twiddles.

Remark 5.3.13 We note that E_n is finite and that $E_{n-1} \subseteq E_n$ for $n \geq 2$ (cf. 5.3.5, 5.3.6, 5.3.7, 5.3.8). Furthermore, we note that by construction we have

$$\begin{array}{rcl}\begin{pmatrix} e0_* \\ e1_* \end{pmatrix}\begin{pmatrix} e0^* & e1^* \end{pmatrix} & = & \begin{pmatrix} m^{\lambda^e\subseteq\nu^e} & 0 \\ 0 & m^{\lambda^e\subseteq\nu^e} \end{pmatrix} \\ \begin{pmatrix} e0^* & e1^* \end{pmatrix}\begin{pmatrix} e0_* \\ e1_* \end{pmatrix} & = & m^{\lambda^e\subseteq\nu^e}.\end{array}$$

whenever $e, e0, e1 \in E_n$, and that

$$0 \longrightarrow T_{e0} \stackrel{e0_*}{\longrightarrow} T_e \stackrel{e1^*}{\longrightarrow} T_{e1} \longrightarrow 0$$

is short exact. We have a filtration

$$M^{(1^n)} = T_\emptyset \hookrightarrow T_0 \oplus T_1 \hookrightarrow T_{00} \oplus T_{01} \oplus T_{10} \oplus T_{11} \hookrightarrow \cdots \hookrightarrow \bigoplus_{e\in E,\, h_e=h} T_e \hookrightarrow \bigoplus_{e\in E,\, h_e=h+1} T_e \hookrightarrow \cdots$$

given by

$$T_e \xrightarrow{(e0^*\ e1^*)} T_{e0} \oplus T_{e1}.$$

This filtration stabilizes at a finite stage $h = H$ at a direct sum of Specht lattices, isomorphic to $\prod_\lambda \mathrm{End}_{\mathbf{Z}} S^\lambda$.

We do some general gymnastics on filtrations of lattices.

Remark 5.3.14 Suppose given a full embedding of **Z**-orders $\Lambda \subseteq \Gamma$. Note that there is at most one way to extend the operation of Λ on a Λ-lattice X to an operation of Γ, since $\Lambda \hookrightarrow \Gamma$ is an epimorphism of **Z**-orders. For Γ-lattices X and Y, the restriction

$${}_{\Gamma}(X,Y) \longrightarrow {}_{\Lambda}(X|_{\Lambda}, Y|_{\Lambda})$$

is bijective. The morphisms in the following are meant to be Λ-linear.

Let $g \in \Gamma$, let $\Lambda[g]$ be the smallest suborder of Γ containing Λ and g. Let X be a Λ-lattice, let Y be a $\Lambda[g]$-lattice, let

$$X \hookrightarrow Y$$

be a full inclusion. g is said to **respect** this inclusion if $gX \subseteq X$, or, in other words, if the rational conjugation of g by $X \hookrightarrow Y$ remains integral. In this case, X is a $\Lambda[g]$-lattice.

In case $X \simeq \Lambda$ and $Y \simeq \Gamma$ we obtain a factorization

$$(X \hookrightarrow Y) = (X \hookrightarrow \Gamma X \hookrightarrow Y),$$

the first step of which is isomorphic to

$$(X \hookrightarrow \Gamma X) \simeq (\Lambda \tilde{\otimes}_{\Lambda} X \hookrightarrow \Gamma \tilde{\otimes}_{\Lambda} X) \simeq (\Lambda \hookrightarrow \Gamma)$$

(cf. B.1.11). $\Gamma X \hookrightarrow Y$ is in fact Γ-linear. Hence $g \in \Gamma$ respects $X \hookrightarrow Y$ iff g respects $\Lambda \hookrightarrow \Gamma$. But this is the case if and only if g is already contained in Λ, since, in particular, $g \cdot 1_{\Lambda} \in \Lambda$ ensues.

Let

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_N$$

be a finite filtration of Λ-lattices, consisting of full inclusions, let $X_0 \simeq \Lambda$, $X_N \simeq \Gamma$. We **claim** that g is in Λ iff it respects each stage of the filtration, where the latter assertion is well defined when starting from the right.

In case g is in Λ, it respects each stage of the filtration. In case g respects each stage $X_i \hookrightarrow X_{i+1}$ of the filtration, starting from the right, it eventually respects $X_0 \hookrightarrow X_N$, thus g is contained in Λ by what we have remarked above.

Theorem 5.3.15 *Let Λ be the image of the embedding of $\mathbf{Z}\mathcal{S}_n$ into $\prod_\lambda \mathrm{End}_{\mathbf{Z}} S^\lambda =: \Gamma$, sending a group element to the tuple of its operations. Let $g = (g^\lambda)_\lambda \in \Gamma$. Define inductively operations of g on $\mathbf{Q}T_e$ for $e \in E_n$ as follows.*

For $e \in \partial E_n$, we define

$$g_e := g^{\nu_e}.$$

For $e \in E_n \backslash \partial E_n$, we may assume the operations g_{e0} resp. g_{e1} on $\mathbf{Q}T_{e0}$ resp. $\mathbf{Q}T_{e1}$ to be defined. Let

$$g_e := \frac{1}{m^{\lambda^e \subseteq \nu^e}} \begin{pmatrix} e0^* & e1^* \end{pmatrix} \begin{pmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{pmatrix} \begin{pmatrix} e0_* \\ e1_* \end{pmatrix}.$$

$g \in \Gamma$ is contained in Λ iff g_e is integral for all $e \in E_n$.

Consider the filtration appearing in (5.3.13). By (5.3.14), g is contained in Λ iff it respects each step of that filtration, i.e. each summand

$$T_e \xrightarrow{\begin{pmatrix} e0^* & e1^* \end{pmatrix}} T_{e0} \oplus T_{e1}$$

of each step. In case $e \notin E_n \backslash \partial E_n$, this inclusion is just the identity, by construction. In case $e \in E_n \backslash \partial E_n$, the operation of g induced on $\mathbf{Q}T_e$ by this inclusion is given by

$$\begin{pmatrix} e0^* & e1^* \end{pmatrix} \begin{pmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{pmatrix} \begin{pmatrix} e0^* & e1^* \end{pmatrix}^{-1} = \frac{1}{m^{\lambda^e \subseteq \nu^e}} \begin{pmatrix} e0^* & e1^* \end{pmatrix} \begin{pmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{pmatrix} \begin{pmatrix} e0_* \\ e1_* \end{pmatrix} = g_e.$$

Remark 5.3.16 Consider, for $e \in E_n \backslash \partial E_n$, the corresponding summand of a filtration step of the filtration appearing in (5.3.13)

$$T_e \xrightarrow{(e0^* \; e1^*)} T_{e0} \oplus T_{e1}.$$

We write T_e as $T_{e0} \lhd T_{e1}$, $e0_*$ as $(1\,0)$, $e1^*$ as $\binom{0}{1}$ (cf. S 3.1). By $e0_* e0^* = m^{\lambda^e \subseteq \nu^e} =: m$ we obtain $e0^* = \binom{m}{f}$ for some $\mathbf{Z}$-linear map

$$T_{e1} \xrightarrow{f} T_{e0}$$

which gives the $\mathbf{Z}\mathcal{S}_n$-linear map induced on the cokernels

$$T_{e1} \xrightarrow{f} T_{e0}/m.$$

The condition on $\left(\begin{smallmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{smallmatrix}\right)$, supposed integral, to respect the inclusion above is equivalent to the condition that

$$\left(\begin{smallmatrix} m & 0 \\ f & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} g_{e0} & 0 \\ 0 & g_{e1} \end{smallmatrix}\right) \left(\begin{smallmatrix} m & 0 \\ f & 1 \end{smallmatrix}\right)^{-1}$$

be integral, i.e. to

$$f g_{e0} \equiv_m g_{e1} f.$$

This is, we may as well express the integrality condition in (5.3.15) as a congruence given by some modular morphism.

Remark 5.3.17 *The product*

$$\prod_{e \in E \backslash \partial E} (m^{\lambda^e \subseteq \nu^e})^{\operatorname{rk} S^{\lambda^{e0} \subseteq \nu^{e0}}}$$

is divisible by the total index

$$\sqrt{\frac{n!^{n!}}{\prod_\lambda n_\lambda^{(n_\lambda^2)}}}.$$

The quotient contains the 'amount of redundant ties' given by our system of modular morphisms.

Note that we may factor

$$T_\emptyset \subseteq \Gamma T_\emptyset \subseteq \bigoplus_{e \in \partial E_n} T_e.$$

The index of the left hand side inclusion is just the total index (1.1.4). The index of the composition is given by the product of the indices of the inclusions

$$T_e \xrightarrow{(e0^* \; e1^*)} T_{e0} \oplus T_{e1},$$

e running over $E_n \backslash \partial E_n$. But the cokernel of such an inclusion is given by $T_{e0}/m^{\lambda^e \subseteq \nu^e}$ (cf. 5.3.11).

For example, consider the case $n = 4$, of total index $2^{34}3^3$ (S 2.1.1). We have

$$E_4 \backslash \partial E_4 = \{\emptyset; 0; 1; 00; 01, 10; 11; 011, 101\}$$

(cf. 5.3.6). Accordingly, the factors of the product of the Ext^1-orders are given by

$$2^{12}; 3^4; 3^8; 4^1; 8^{2\cdot 3}; 4^3; 2^{2\cdot 2},$$

whence the quotient of this product by the total index is

$$(2^{42}3^{12})/(2^{34}3^3) = 2^8 3^9.$$

Since it is desirable that the maps $e0_*$, $e0^*$, $e1_*$, $e1^*$, $e \in E_n \backslash \partial E_n$, take a simple form, we collect them to a module over a path algebra in such a way that an isomorphism of this module to another one corresponds to a tuple of integral base changes in the lattices $S^{\lambda \subseteq \nu}$ for occurring pairs $\lambda \subseteq \nu$.

Definition 5.3.18 (The truss) *Let Q_n be the quiver arising from E_n, equipped with arrows*

$$\begin{array}{rcl} e0 & \xrightarrow{e0_{(*)}} & e \\ e1 & \xrightarrow{e1_{(*)}} & e \\ e & \xrightarrow{e0^{(*)}} & e0 \\ e & \xrightarrow{e1^{(*)}} & e1 \end{array}$$

for $e \in E_n \backslash \partial E_n$, by identification of vertices e and e' iff $\lambda_e = \lambda_{e'}$ and $\nu_e = \nu_{e'}$. Then the lattices T_e and the morphisms $e0_*$, $e0^*$, $e1_*$, $e1^*$, $e \in E_n \backslash \partial E_n$, given as in (5.3.12), form a module over the integral path algebra $\mathbf{Z}Q_n$ by attaching e to T_e, $e0_{(*)}$ to $e0_*$, $e1_{(*)}$ to $e1_*$, $e0^{(*)}$ to $e0^*$ and $e1^{(*)}$ to $e1^*$. This module is called the **truss** *of $\mathbf{Z}\mathcal{S}_n$. It is again denoted by T.*

For the shape of Q_n for $n = 3,\ 4,\ 5,\ 6$, we may replace in (5.3.5, 5.3.6, 5.3.7, 5.3.8) each edge by one arrow upwards and one arrow downwards.

5.4 Small cases

We give normal forms for the trusses of $\mathbf{Z}\mathcal{S}_3$ and of $\mathbf{Z}\mathcal{S}_4$ (almost). We 'cheat' in that we use the satisfactory embedding of the $\mathbf{Z}\mathcal{S}_4$ obtained in (S 2.1.1), in order to get acquainted to the problem and to adjust our expectations concerning a possible solution.

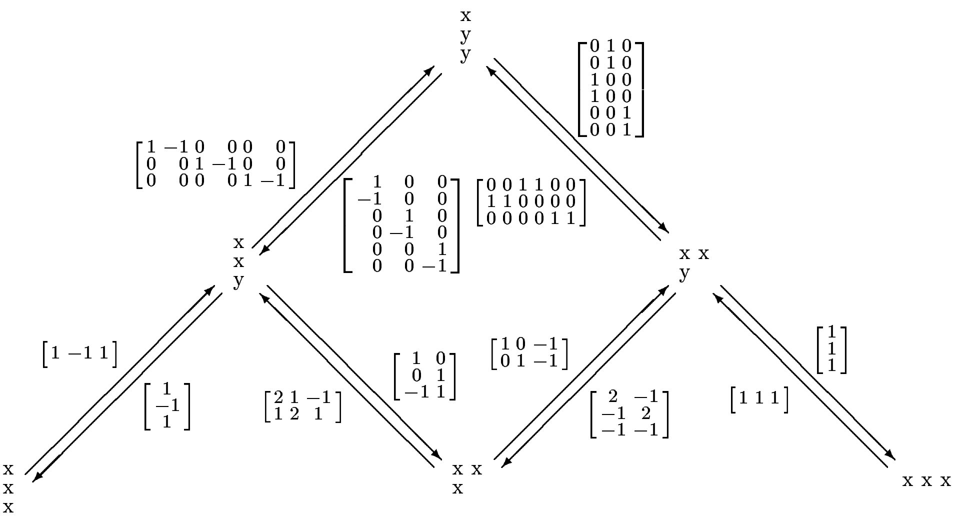
It might be possible to use a normal form for the truss over Q_{n-1} to facilitate the search for a normal form for the truss over Q_n, given that the top part of the latter is obtained by induction of the former along $\mathcal{S}_{n-1} \leq \mathcal{S}_n$. But I do not know how to follow the according suggestion to 'simplify from top to bottom'. Moreover, I lack an appropriate naive localization technique.

5.4.1 The truss of $\mathbf{Z}\mathcal{S}_3$

With respect to the semitabloid bases, brackets dropped,

$$\begin{array}{ll}
S^{(1,0,0)\subseteq(1,1,1)} & \begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}2\\1\\3\end{smallmatrix}\ \begin{smallmatrix}1\\3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\3\\1\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix} \\[2ex]
S^{(1,1,0)\subseteq(1,1,1)} & \begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\3\\2\end{smallmatrix}\ \begin{smallmatrix}2\\3\\1\end{smallmatrix} \\[2ex]
S^{(2,0)\subseteq(2,1)} & \begin{smallmatrix}1&3\\2&\end{smallmatrix}\ \begin{smallmatrix}1&2\\3&\end{smallmatrix}\ \begin{smallmatrix}2&3\\1&\end{smallmatrix} \\[2ex]
S^{(1,1,1)\subseteq(1,1,1)} & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \\[2ex]
S^{(2,1)\subseteq(2,1)} & \begin{smallmatrix}1&3\\2&\end{smallmatrix}\ \begin{smallmatrix}1&2\\3&\end{smallmatrix} \\[2ex]
S^{(3)\subseteq(3)} & 1\ 2\ 3\,,
\end{array}$$

the truss of $\mathbf{Z}\mathcal{S}_3$ takes the following shape.



We use the following base change matrices, in which the new basis elements are recorded as row vectors in terms of the bases given above,

$$S^{(1,0,0)\subseteq(1,1,1)} \quad \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$S^{(1,1,0)\subseteq(1,1,1)} \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

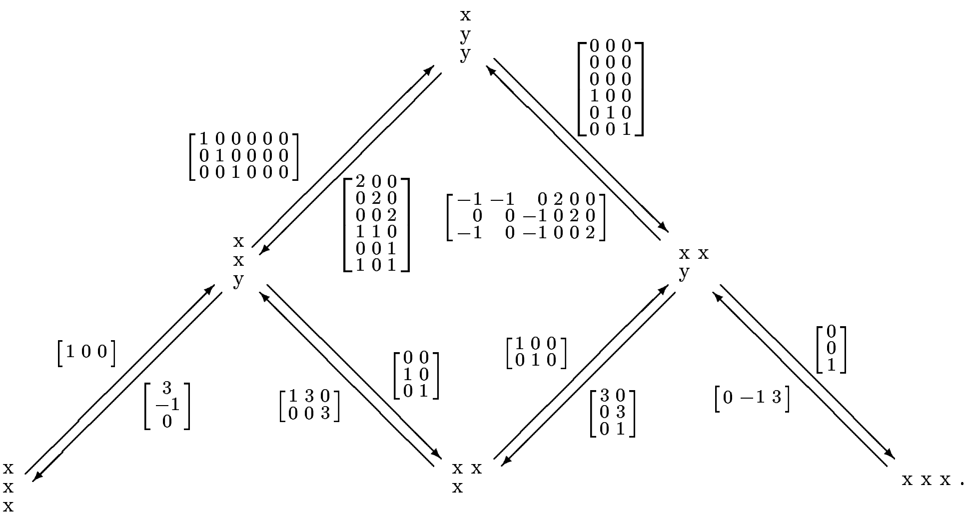
$$S^{(2,0)\subseteq(2,1)} \quad \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$S^{(1,1,1)\subseteq(1,1,1)} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$S^{(2,1)\subseteq(2,1)} \quad \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$S^{(3)\subseteq(3)} \quad \begin{bmatrix} 1 \end{bmatrix},$$

to obtain the normal form



As $m^{\lambda\subseteq\nu}$'s we have

$$\begin{aligned} m^{(1,0,0)\subseteq(1,1,1)} &= 2 \\ m^{(1,1,0)\subseteq(1,1,1)} &= 3 \\ m^{(2,0)\subseteq(2,1)} &= 3. \end{aligned}$$

Suppose given

$$x^1_{11} \times \begin{pmatrix} x^2_{11} & x^2_{12} \\ x^2_{21} & x^2_{22} \end{pmatrix} \times x^3_{11} \in \mathbf{Z} \times (\mathbf{Z})_2 \times \mathbf{Z},$$

the factors ordered $(1,1,1)$, $(2,1)$, (3). According to (5.3.15), the operation on $S^{(1,1,0)\subseteq(1,1,1)}$ is given by

$$\frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x^1_{11} & 0 & 0 \\ 0 & x^2_{11} & x^2_{12} \\ 0 & x^2_{21} & x^2_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} x^1_{11} & 0 & 0 \\ \frac{1}{3}(x^2_{11}-x^1_{11}) & x^2_{11} & x^2_{12} \\ \frac{1}{3}x^2_{21} & x^2_{21} & x^2_{22} \end{bmatrix}.$$

The operation on $S^{(2,0)\subseteq(2,1)}$ is given by

$$\frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x^2_{11} & x^2_{12} & 0 \\ x^2_{21} & x^2_{22} & 0 \\ 0 & 0 & x^3_{11} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} x^2_{11} & x^2_{12} & 0 \\ x^2_{21} & x^2_{22} & 0 \\ \frac{1}{3}x^2_{21} & \frac{1}{3}(x^2_{22}-x^3_{11}) & x^3_{11} \end{bmatrix}.$$

So, finally, the operation on $S^{(1,0,0)\subseteq(1,1,1)}$ is given by

$$\frac{1}{2}\begin{bmatrix} 2&0&0&0&0&0 \\ 0&2&0&0&0&0 \\ 0&0&2&0&0&0 \\ 1&1&0&1&0&0 \\ 0&0&1&0&1&0 \\ 1&0&1&0&0&1 \end{bmatrix}\begin{bmatrix} x^1_{11} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}(x^2_{11}-x^1_{11}) & x^2_{11} & x^2_{12} & 0 & 0 & 0 \\ \frac{1}{3}x^2_{21} & x^2_{21} & x^2_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & x^2_{11} & x^2_{12} & 0 \\ 0 & 0 & 0 & x^2_{21} & x^2_{22} & 0 \\ 0 & 0 & 0 & \frac{1}{3}x^2_{21} & \frac{1}{3}(x^2_{22}-x^3_{11}) & x^3_{11} \end{bmatrix}\begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ -1&-1&0&2&0&0 \\ 0&0&-1&0&2&0 \\ -1&0&-1&0&0&2 \end{bmatrix}$$

$$= \begin{bmatrix} x^1_{11} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}(x^2_{11}-x^1_{11}) & x^2_{11} & x^2_{12} & 0 & 0 & 0 \\ \frac{1}{3}x^2_{21} & x^2_{21} & x^2_{22} & 0 & 0 & 0 \\ -\frac{1}{3}(x^2_{11}-x^1_{11}) & 0 & 0 & x^2_{11} & x^2_{12} & 0 \\ -\frac{1}{3}x^2_{21} & 0 & 0 & x^2_{21} & x^2_{22} & 0 \\ \frac{1}{2}(x^1_{11}-x^3_{11}) & \frac{1}{3}x^2_{21} & \frac{1}{3}(x^2_{22}-x^3_{11}) & \frac{1}{3}x^2_{21} & \frac{1}{3}(x^2_{22}-x^3_{11}) & x^3_{11} \end{bmatrix}.$$

The ties known from (S 0.2) result.

5.4.2 The truss of $\mathbf{Z}\mathcal{S}_4$

With respect to the semitabloid bases

$$S^{(1,0,0,0)\subseteq(1,1,1,1)} \quad \begin{matrix} 1&2&1&3&3&2&1&2&1&4&4&2&1&4&1&3&3&4&4&2&4&3&3&2 \\ 2&1&3&1&2&3&2&1&4&1&2&4&4&1&3&1&4&3&2&4&3&4&2&3 \\ 3&3&2&2&1&1&4&4&2&2&1&1&3&3&4&4&1&1&3&3&2&2&4&4 \\ 4&4&4&4&4&4&3&3&3&3&3&3&2&2&2&2&2&2&1&1&1&1&1&1 \end{matrix}$$

$$S^{(1,1,0,0)\subseteq(1,1,1,1)} \quad \begin{matrix} 1&1&3&1&1&4&1&1&3&4&4&3 \\ 2&3&2&2&4&2&4&3&4&2&3&2 \\ 3&2&1&4&2&1&3&4&1&3&2&4 \\ 4&4&4&3&3&3&2&2&2&1&1&1 \end{matrix}$$

$$S^{(2,0,0)\subseteq(2,1,1)} \quad \begin{smallmatrix}1&2\\3\\4\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\\4\end{smallmatrix}\ \begin{smallmatrix}3&2\\1\\4\end{smallmatrix}\ \begin{smallmatrix}1&2\\4\\3\end{smallmatrix}\ \begin{smallmatrix}1&4\\2\\3\end{smallmatrix}\ \begin{smallmatrix}4&2\\1\\3\end{smallmatrix}\ \begin{smallmatrix}1&4\\3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\4\\2\end{smallmatrix}\ \begin{smallmatrix}3&4\\1\\2\end{smallmatrix}\ \begin{smallmatrix}4&2\\3\\1\end{smallmatrix}\ \begin{smallmatrix}4&3\\2\\1\end{smallmatrix}\ \begin{smallmatrix}3&2\\4\\1\end{smallmatrix}$$

$$S^{(1,1,1,0)\subseteq(1,1,1,1)} \quad \begin{matrix} 1&1&1&4 \\ 2&2&4&2 \\ 3&4&3&3 \\ 4&3&2&1 \end{matrix}$$

$$S^{(2,1,0)\subseteq(2,1,1)} \quad \begin{smallmatrix}1&2\\3\\4\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\\4\end{smallmatrix}\ \begin{smallmatrix}1&2\\4\\3\end{smallmatrix}\ \begin{smallmatrix}1&4\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1&4\\3\\2\end{smallmatrix}\ \begin{smallmatrix}1&3\\4\\2\end{smallmatrix}\ \begin{smallmatrix}4&2\\3\\1\end{smallmatrix}\ \begin{smallmatrix}4&3\\2\\1\end{smallmatrix}$$

$$S^{(3,0)\subseteq(3,1)} \quad \begin{smallmatrix}1&2&3\\4\end{smallmatrix}\ \begin{smallmatrix}1&2&4\\3\end{smallmatrix}\ \begin{smallmatrix}1&4&3\\2\end{smallmatrix}\ \begin{smallmatrix}4&2&3\\1\end{smallmatrix}$$

$$S^{(1,1,1,1)\subseteq(1,1,1,1)} \quad \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$S^{(2,1,1)\subseteq(2,1,1)} \quad \begin{smallmatrix}1&2\\3\\4\end{smallmatrix}\ \begin{smallmatrix}1&3\\2\\4\end{smallmatrix}\ \begin{smallmatrix}1&4\\2\\3\end{smallmatrix}$$

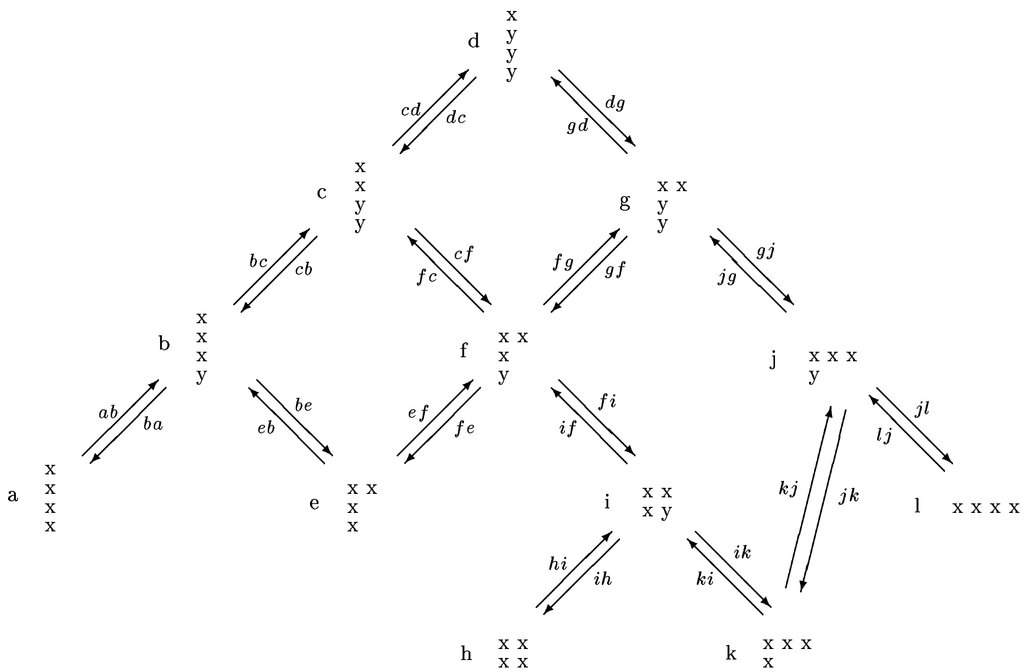
$$S^{(2,1)\subseteq(2,2)} \quad \begin{smallmatrix}1&2\\3&4\end{smallmatrix}\ \begin{smallmatrix}1&3\\2&4\end{smallmatrix}\ \begin{smallmatrix}1&4\\2&3\end{smallmatrix}\ \begin{smallmatrix}2&3\\4&1\end{smallmatrix}\ \begin{smallmatrix}2&4\\3&1\end{smallmatrix}$$

$$S^{(4)\subseteq(4)} \quad \begin{smallmatrix}1&2&3&4\end{smallmatrix}$$

$$S^{(2,2)\subseteq(2,2)} \quad \begin{smallmatrix}1&2\\3&4\end{smallmatrix} \; \begin{smallmatrix}1&3\\2&4\end{smallmatrix}$$

$$S^{(3,1)\subseteq(3,1)} \quad \begin{smallmatrix}1&2&3\\4\end{smallmatrix} \; \begin{smallmatrix}1&2&4\\3\end{smallmatrix} \; \begin{smallmatrix}1&4&3\\2\end{smallmatrix}$$

the maps of the truss



have the following shape. A block diagonal matrix carrying m times the block A will be denoted by A^m.

$$\begin{aligned}
cd &= \begin{bmatrix} 1 & -1 \end{bmatrix}^{12} \\
dc &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{12} \\
dg &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{12} \\
gd &= \begin{bmatrix} 1 & 1 \end{bmatrix}^{12} \\
bc &= \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^{4} \\
cb &= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}^{4} \\
cf &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}^{4} \\
fc &= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix}^{4} \\
fg &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}^{4} \\
gf &= \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ -1 & -1 \end{bmatrix}^{4} \\
gj &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^{4} \\
jg &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{4} \\
ab &= \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \\
ba &= \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \\
be &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \\
eb &= \begin{bmatrix} -1 & 1 & -3 & 1 \\ 1 & 3 & -1 & -1 \\ 3 & 1 & 1 & 1 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
ef &= \begin{bmatrix} 1&0&-1&0&0&0&-1&0\\ 0&1&0&0&0&-1&0&-1\\ 0&0&0&1&-1&0&1&-1 \end{bmatrix}\\
fe &= \begin{bmatrix} 3&0&1\\ 0&3&-1\\ -3&1&0\\ 0&-1&3\\ -1&0&-3\\ 1&-3&0\\ -2&-1&1\\ -1&-2&-1 \end{bmatrix}\\
fi &= \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&0&0&1&-1\\ 0&0&1&0&0\\ 0&0&1&0&1\\ 0&1&0&1&0\\ 0&0&0&-1&1\\ 0&0&0&-1&0 \end{bmatrix}\\
if &= \begin{bmatrix} 5&0&3&-1&1&0&2&1\\ 0&5&0&1&-1&3&1&2\\ 0&1&0&5&3&-1&-3&2\\ -1&-2&1&-1&1&2&0&-5\\ 1&-1&-1&-2&2&1&5&-5 \end{bmatrix}\\
kj &= \begin{bmatrix} 1&0&0&-1\\ 0&1&0&-1\\ 0&0&1&-1 \end{bmatrix}\\
jk &= \begin{bmatrix} 3&-1&-1\\ -1&3&-1\\ -1&-1&3\\ -1&-1&-1 \end{bmatrix}\\
jl &= \begin{bmatrix} 1\\ 1\\ 1\\ 1 \end{bmatrix}\\
lj &= \begin{bmatrix} 1&1&1&1 \end{bmatrix}\\
hi &= \begin{bmatrix} 1&0&-1&0&-1\\ 0&1&-1&0&0 \end{bmatrix}\\
ih &= \begin{bmatrix} 1&0\\ 0&1\\ 0&-1\\ -1&0\\ -1&1 \end{bmatrix}\\
ik &= \begin{bmatrix} 0&1&0\\ 0&0&1\\ 0&0&1\\ 1&0&-1\\ 0&1&-1 \end{bmatrix}\\
ki &= \begin{bmatrix} 1&1&0&2&-1\\ 1&0&1&0&1\\ 0&1&1&0&0 \end{bmatrix}.
\end{aligned}
$$

The following calculations use MAPLE, in particular, ISMITH. We perform base changes by the following matrices, whose rows contain the new basis elements in terms of the ones listed above.

$$
S^{(1,0,0,0)\subseteq(1,1,1,1)} \quad
\begin{bmatrix}
1&-1&-1&1&-1&1&-1&1&1&-1&1&-1&-1&1&1&-1&1&-1&-1&1&1&-1&1&-1\\
1&-1&-1&1&-1&1&0&0&0&0&0&0&1&-1&-1&1&-1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-1&-1&1&-1&1&-1&1&1&-1&1&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&1&1&-1&1&-1&0&0&0&0&0&0\\
0&0&-1&1&0&0&1&-1&1&-1&0&0&0&0&-1&1&0&0&0&0&1&-1&-1&1\\
1&-1&1&-1&0&0&0&0&-1&1&0&0&-1&1&0&0&0&0&-1&1&-1&1&0&0\\
0&0&1&-1&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&-1&1&0&0\\
0&0&1&-1&1&-1&0&0&0&0&-1&1&0&0&-1&1&1&-1&1&-1&1&-1&-1&1\\
1&-1&1&-1&0&0&-1&1&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&1&1&-1&-1&1&0&0&0&0&2&-2&-1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&1&-1&0&0&0&0&0&0\\
1&-1&0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0&1&-1\\
0&-1&-1&1&0&1&1&0&1&0&-1&-1&0&-1&-1&1&0&1&1&1&0&-1&-1&0\\
1&0&1&0&-1&-1&0&-1&-1&1&0&1&-1&1&0&-1&1&0&-1&0&-1&0&1&1\\
1&0&0&0&-1&0&-1&0&0&0&1&0&-1&1&1&-1&1&-1&-1&0&0&0&1&0\\
1&-1&0&1&-1&0&0&1&0&-1&0&0&-1&-1&0&1&1&0&1&1&1&-1&-1&-1\\
1&0&1&0&-1&-1&0&0&-1&0&0&1&-1&0&0&0&1&0&0&0&0&0&0&0\\
1&-1&-1&1&-1&1&0&2&0&-2&0&0&-1&-1&1&1&1&-1&-1&1&0&-1&1&0\\
0&0&0&0&0&0&1&0&-1&0&-1&1&-1&0&1&0&1&-1&0&0&0&0&0&0\\
1&-1&0&1&-1&0&-1&0&1&0&1&-1&-1&1&1&-1&1&-1&-1&0&1&0&1&-1\\
1&1&1&1&1&0&0&0&0&-2&-1&1&-1&-1&-1&1&1&-2&0&0&1&-1&-1&1\\
1&0&1&0&0&0&0&0&-1&0&-1&1&-1&0&0&0&1&-1&0&0&0&0&0&0\\
1&-1&0&0&-1&0&-1&0&0&0&1&-1&0&1&1&0&1&0&-1&0&0&0&1&-1\\
2&-1&2&-1&-1&-1&-1&0&-1&1&1&0&-1&1&0&0&1&0&-1&0&-1&1&1&0
\end{bmatrix}
$$

$$
S^{(1,1,0,0)\subseteq(1,1,1,1)} \quad
\begin{bmatrix}
1&-1&-1&-1&1&1&-1&1&1&-1&1&1\\
1&-1&-1&0&0&0&1&-1&-1&0&0&0\\
0&0&0&1&-1&-1&-1&1&1&0&0&0\\
0&0&0&0&0&0&-1&1&1&0&0&0\\
0&-1&0&1&1&0&0&-1&0&0&1&-1\\
1&1&0&0&-1&0&-1&0&0&-1&-1&0\\
0&1&0&0&-1&0&0&0&0&0&-1&0\\
0&1&1&0&0&-1&0&-1&1&1&1&-1\\
1&1&0&-1&0&0&0&-1&0&0&0&0\\
0&0&0&-1&1&-1&0&0&2&-1&0&0\\
0&0&0&0&0&-1&0&0&1&0&0&0\\
1&0&0&0&0&1&0&0&0&0&0&1
\end{bmatrix}
$$

$$
S^{(2,0,0)\subseteq(2,1,1)} \quad
\begin{bmatrix}
-1&0&1&1&1&-2&-1&0&1&2&-1&-1\\
1&1&-2&-1&0&1&0&-1&1&-1&-1&2\\
1&0&-1&-1&0&1&0&0&0&-1&0&1\\
0&1&-1&1&-1&0&-2&1&1&2&0&-2\\
1&1&-2&0&-1&1&-1&0&1&0&0&0\\
0&0&0&2&-2&0&-2&2&0&0&-1&1\\
0&0&0&1&-1&0&-1&1&0&0&0&0\\
0&1&-1&-1&1&0&0&0&0&-1&1&0\\
2&2&1&0&-2&0&-2&0&-1&0&0&0\\
1&1&0&0&-1&0&-1&0&0&0&0&0\\
0&0&-1&-1&0&0&1&1&1&-1&0&0\\
1&1&-2&-1&0&1&0&0&1&-1&0&1
\end{bmatrix}
$$

$$S^{(1,1,1,0)\subseteq(1,1,1,1)} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$S^{(2,1,0)\subseteq(2,1,1)} \quad \begin{bmatrix} -1 & 0 & 1 & 1 & -1 & 0 & 2 & -1 \\ 1 & 1 & -1 & 0 & 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & -2 & 1 & 2 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & -2 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$S^{(3,0)\subseteq(3,1)} \quad \begin{bmatrix} 5 & -2 & -3 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S^{(2,1)\subseteq(2,2)} \quad \begin{bmatrix} 1 & 2 & -3 & 0 & -1 \\ 1 & 1 & -2 & 0 & -1 \\ 2 & 2 & -4 & 5 & -4 \\ 1 & 1 & -2 & 2 & -2 \\ -1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

$$S^{(1,1,1,1)\subseteq(1,1,1,1)} \quad \begin{bmatrix} 1 \end{bmatrix}$$

$$S^{(2,1,1)\subseteq(2,1,1)} \quad \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$S^{(2,2)\subseteq(2,2)} \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$S^{(3,1)\subseteq(3,1)} \quad \begin{bmatrix} 5 & -2 & -3 \\ 2 & -1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$S^{(4)\subseteq(4)} \quad \begin{bmatrix} 1 \end{bmatrix}$$

This yields the following matrices for the maps of the truss, which we sort blockwise according to the Specht lattices the James lattices are glued from, if necessary.

$$cd = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$dc = \left[\begin{array}{c|ccc|ccc|cc|ccc}
2&0&0&0&0&0&0&0&0&0&0&0 \\
0&2&0&0&0&0&0&0&0&0&0&0 \\
0&0&2&0&0&0&0&0&0&0&0&0 \\
0&0&0&2&0&0&0&0&0&0&0&0 \\
0&0&0&0&2&0&0&0&0&0&0&0 \\
0&0&0&0&0&2&0&0&0&0&0&0 \\
0&0&0&0&0&0&2&0&0&0&0&0 \\
0&0&0&0&0&0&0&2&0&0&0&0 \\
0&0&0&0&0&0&0&0&2&0&0&0 \\
0&0&0&0&0&0&0&0&0&2&0&0 \\
0&0&0&0&0&0&0&0&0&0&2&0 \\
0&0&0&0&0&0&0&0&0&0&0&2 \\
\hline
0&1&0&0&1&0&0&0&0&0&0&0 \\
0&0&1&0&0&1&0&0&0&0&0&0 \\
1&0&0&1&0&0&1&0&0&0&0&0 \\
\hline
1&1&0&0&0&0&0&1&0&0&0&0 \\
0&0&1&0&0&0&0&0&1&0&0&0 \\
\hline
1&1&0&0&0&0&0&0&0&1&0&0 \\
0&0&1&0&0&0&0&0&0&0&1&0 \\
1&0&0&1&0&0&0&0&0&0&0&1 \\
\hline
0&0&0&0&1&0&0&1&0&1&0&0 \\
0&0&1&0&0&0&0&0&1&0&1&0 \\
1&0&0&0&0&0&1&0&0&0&0&1 \\
\hline
0&0&0&1&0&1&1&0&1&0&0&1
\end{array}\right]$$

$$dg = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$gd = \left[\begin{array}{r|rrr|rrr|rr|rrr|cccccccccccc} 0&-1&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\ -1&0&0&-1&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\ \hline -1&-1&0&0&0&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0\\ \hline -1&-1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0&0\\ -1&0&0&-1&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&2&0&0&0&0\\ \hline 0&0&0&0&-1&0&0&-1&0&-1&0&0&0&0&0&0&0&0&0&0&2&0&0&0\\ 0&0&-1&0&0&0&0&0&-1&0&-1&0&0&0&0&0&0&0&0&0&0&2&0&0\\ -1&0&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&2&0\\ \hline 0&0&0&-1&0&-1&-1&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&2 \end{array}\right]$$

$$bc = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$cb = \left[\begin{array}{r|rrr} 3&0&0&0\\ 0&3&0&0\\ 0&0&3&0\\ 0&0&0&3\\ \hline 0&1&0&0\\ 0&0&1&0\\ -1&0&0&1\\ \hline -1&-1&0&0\\ 0&0&-1&0\\ \hline 1&-1&0&0\\ 0&0&1&0\\ 1&0&0&-1 \end{array}\right]$$

$$cf = \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$fc = \left[\begin{array}{r|rrr|cccccccc} 0&-1&0&0&3&0&0&0&0&0&0&0\\ 0&0&-1&0&0&3&0&0&0&0&0&0\\ 1&0&0&-1&0&0&3&0&0&0&0&0\\ \hline 1&1&0&0&0&0&0&3&0&0&0&0\\ 0&0&1&0&0&0&0&0&3&0&0&0\\ \hline -1&1&0&0&0&0&0&0&0&3&0&0\\ 0&0&-1&0&0&0&0&0&0&0&3&0\\ -1&0&0&1&0&0&0&0&0&0&0&3 \end{array}\right]$$

$$fg = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

$$gf = \left[\begin{array}{rrr|rr|rrr} 3&0&0&0&0&0&0&0\\ 0&3&0&0&0&0&0&0\\ 0&0&3&0&0&0&0&0\\ 0&0&0&3&0&0&0&0\\ 0&0&0&0&3&0&0&0\\ 0&0&0&0&0&3&0&0\\ 0&0&0&0&0&0&3&0\\ 0&0&0&0&0&0&0&3\\ \hline -1&0&0&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&1&0&0&0&0&1\\ \hline 0&1&1&0&1&0&0&1 \end{array}\right]$$

$$gj = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}$$

$$jg = \left[\begin{array}{ccc|cc|ccc|cccc} 1&0&0&-1&0&-1&0&0&3&0&0&0 \\ 0&0&0&0&-1&0&-1&0&0&3&0&0 \\ 0&0&-1&0&0&0&0&-1&0&0&3&0 \\ \hline 0&-1&-1&0&-1&0&0&-1&0&0&0&3 \end{array}\right]$$

$$ab = \begin{bmatrix} 1&0&0&0 \end{bmatrix}$$

$$ba = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$be = \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

$$eb = \begin{bmatrix} 0&4&0&0 \\ 0&0&4&0 \\ -1&0&0&4 \end{bmatrix}$$

$$ef = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{bmatrix}$$

$$fe = \left[\begin{array}{ccc} 8&0&0 \\ 0&8&0 \\ 0&0&8 \\ \hline 4&0&0 \\ 0&4&0 \\ \hline 1&0&0 \\ 0&-1&0 \\ 0&0&4 \end{array}\right]$$

$$fi = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}$$

$$if = \left[\begin{array}{ccc|ccccc} -4&0&0&8&0&0&0&0 \\ 0&-4&0&0&8&0&0&0 \\ \hline -1&0&0&0&0&8&0&0 \\ 0&1&0&0&0&0&8&0 \\ 0&0&-4&0&0&0&0&8 \end{array}\right]$$

$$kj = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}$$

$$jk = \begin{bmatrix} 4&0&0 \\ 0&4&0 \\ 0&0&4 \\ 0&0&1 \end{bmatrix}$$

$$jl = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$lj = \begin{bmatrix} 0&0&-1&4 \end{bmatrix}$$

$$hi = \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix}$$

$$ih = \begin{bmatrix} 2&0 \\ 0&2 \\ 1&0 \\ 0&1 \\ 0&0 \end{bmatrix}$$

$$ik = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

$$ki = \begin{bmatrix} -1&0&2&0&0 \\ 0&-1&0&2&0 \\ 0&0&0&0&2 \end{bmatrix}.$$

In the language of (S 2.1.1), the occurring James extensions cause the follwing ties, with some redundancies cut down already.

$$
\begin{array}{ll}
S^{(2,1)\subseteq(2,2)} & \left\{\begin{array}{rcl}
x_{11}^4 & \equiv_2 & x_{11}^5 \\
x_{12}^4 & \equiv_2 & x_{12}^5 \\
x_{21}^4 & \equiv_2 & x_{21}^5 \\
x_{22}^4 & \equiv_2 & x_{22}^5 \\
x_{31}^4 & \equiv_2 & 0 \\
x_{32}^4 & \equiv_2 & 0
\end{array}\right. \\
S^{(1,1,1,0)\subseteq(1,1,1,1)} & \left\{\begin{array}{rcl}
x_{13}^3 & \equiv_4 & 0 \\
x_{23}^3 & \equiv_4 & 0 \\
x_{33}^3 & \equiv_4 & x_{11}^1
\end{array}\right. \\
S^{(3,0)\subseteq(3,1)} & \left\{\begin{array}{rcl}
x_{31}^4 & \equiv_4 & 0 \\
x_{32}^4 & \equiv_4 & 0 \\
x_{33}^4 & \equiv_4 & x_{11}^2
\end{array}\right. \\
S^{(2,1)\subseteq(2,2)} & \left\{\begin{array}{rcl}
x_{11}^3 & \equiv_2 & x_{11}^5 \\
x_{12}^3 & \equiv_2 & x_{12}^5 \\
x_{13}^3 & \equiv_2 & 0 \\
x_{21}^3 & \equiv_2 & x_{21}^5 \\
x_{22}^3 & \equiv_2 & x_{22}^5 \\
x_{23}^3 & \equiv_2 & 0 \\
2x_{11}^5 & \equiv_8 & x_{11}^4 + x_{11}^3 \\
2x_{12}^5 & \equiv_8 & 3x_{12}^4 + x_{12}^3 \\
0 & \equiv_8 & -4x_{13}^4 + x_{13}^3 \\
2x_{21}^5 & \equiv_8 & x_{21}^4 - x_{21}^3 \\
2x_{22}^5 & \equiv_8 & 3x_{22}^4 - x_{22}^3 \\
0 & \equiv_8 & -4x_{23}^4 - x_{23}^3 \\
0 & \equiv_8 & x_{31}^4 + 4x_{31}^3 \\
0 & \equiv_8 & 3x_{32}^4 + 4x_{32}^3 \\
0 & \equiv_8 & -4x_{33}^4 + 4x_{33}^3
\end{array}\right. \\
S^{(1,1,0,0)\subseteq(1,1,1,1)} & \left\{\begin{array}{rcl}
x_{11}^1 & \equiv_3 & x_{11}^5 \\
0 & \equiv_3 & x_{21}^5
\end{array}\right. \\
S^{(2,0,0)\subseteq(2,1,1)} & \left\{\begin{array}{rcl}
x_{11}^2 & \equiv_3 & x_{22}^5 \\
0 & \equiv_3 & x_{21}^5
\end{array}\right. \\
S^{(1,0,0,0)\subseteq(1,1,1,1)} & \left\{\; x_{11}^1 + 3x_{11}^2 \;\equiv_8\; x_{33}^3 + 3x_{33}^4\right.
\end{array}
$$

We may sort these ties to obtain the system

$$
\begin{array}{rclclcl}
x_{11}^4 & \equiv_2 & x_{11}^5 & \equiv_2 & x_{11}^3 & & \\
x_{12}^4 & \equiv_2 & x_{12}^5 & \equiv_2 & x_{12}^3 & & \\
x_{21}^4 & \equiv_2 & x_{21}^5 & \equiv_2 & x_{21}^3 & & \\
x_{22}^4 & \equiv_2 & x_{22}^5 & \equiv_2 & x_{22}^3 & & \\
x_{11}^1 & \equiv_4 & x_{33}^3 & \equiv_2 & x_{33}^4 & \equiv_4 & x_{11}^2 \\
x_{11}^4 + x_{11}^3 & \equiv_8 & 2x_{11}^5 & & & & \\
x_{12}^4 - x_{12}^3 & \equiv_8 & 2x_{12}^5 & & & & \\
x_{21}^4 - x_{21}^3 & \equiv_8 & 2x_{21}^5 & & & & \\
x_{22}^4 + x_{22}^3 & \equiv_8 & 2x_{22}^5 & & & & \\
4x_{13}^4 & \equiv_8 & x_{13}^3 & & & & \\
4x_{23}^4 & \equiv_8 & x_{23}^3 & & & & \\
x_{31}^4 & \equiv_8 & 4x_{31}^3 & & & & \\
x_{32}^4 & \equiv_8 & 4x_{32}^3 & & & &
\end{array}
$$

$$
x_{11}^1 - x_{11}^2 \equiv_8 x_{33}^3 - x_{33}^4
$$

$$
\begin{array}{rcl}
x_{11}^1 & \equiv_3 & x_{11}^5 \\
x_{11}^2 & \equiv_3 & x_{22}^5 \\
0 & \equiv_3 & x_{21}^5 .
\end{array}
$$

Note that we have employed the actual Specht lattice $S^{(3,1)}$, whereas in (S 2.1.1) we used its dual $S^{(2,1,1),-}$. Presumably, a base change on $S^{(3,1)}$ via $\begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix}$ would yield a slightly better presentation still.

Chapter 6

Gram matrices

Inclusions of simple lattices cause ties by the requirement that the conjugation with this inclusion be integral. A prominent role amongst them is taken by the inclusion of the Specht lattice into its dual, described, in terms of linear algebra, by the Gram matrix of the invariant bilinear form on the Specht lattice. Suborders of matrix rings described by a single embedding of simple lattices are called Gram orders. However, note that a quasiblock cannot possibly be a Gram order in case the decomposition numbers of its Specht module are not contained in $\{0,1\}$, since then it has to have a nonsimple indecomposable projective module (cf. S 0.3, E.1.24).

6.1 Gram orders

Let R be a discrete valuation ring of characteristic zero with maximal ideal (π), valuation v and field of fractions K.

Definition 6.1.1 *Let $m \geq 1$, let $G \in (K)_m$ such that* $\det G \neq 0$. *Let*

$$\Lambda_G = \{A \in (R)_m | G^{-1}AG \in (R)_m\} = (R)_m \cap {}^G(R)_m \subseteq (K)_m.$$

A **Gram order** *is an R-order isomorphic to an R-order of the form Λ_G.*

Remark 6.1.2 Suppose given a Gram order Λ_G. We may assume G to be integral. Write

$$G = SDT,$$

where S, T are in $\mathrm{SL}(R)$ and D is a main diagonal matrix in elementary divisor form (A.1). Then

$$\begin{array}{ccc} \Lambda_G & \xrightarrow{\sim} & \Lambda_D \\ A & \longrightarrow & A^S. \end{array}$$

Hence we may restrict ourselves to the consideration of main diagonal matrices G in elementary divisor form.

Definition 6.1.3 *Let $g := (g_1, \ldots, g_m)$ be a tuple of integers, ordered increasingly, $g_i \leq g_j$ for $i \leq j$. Let $G \in (R)_m$ be the main diagonal matrix with diagonal $(\pi^{g_1}, \ldots, \pi^{g_m})$. Let*

$$\Lambda_g := \Lambda_G = \{(\alpha_{ij}) \in (R)_m \mid \alpha_{ij} \in (\pi^{g_i - g_j}) \text{ for } i \geq j\}.$$

An integer, when viewed as a tuple, stands for the constant tuple. Note that for $c \in \mathbf{Z}$ the tuples g and $g+c$ yield the same R-order, $\Lambda_g = \Lambda_{g+c}$.

Definition 6.1.4 There is a simple Λ_g-lattice N_γ for each increasingly ordered tuple $\gamma := (\gamma_1, \ldots, \gamma_m)$ of integers such that

$$\gamma_j - \gamma_i \leq g_j - g_i$$

for each pair $i \leq j \in [1, m]$, defined as R-linear span of $\{\pi^{\gamma_1} e_1, \ldots, \pi^{\gamma_m} e_m\}$ inside a column of $(R)_m$, e_i being the column having entries $e_{i,j} = \partial_{ij}$.

In particular we dispose of the lattices N_0 and N_g.

We denote by $[\gamma]$ the residue class of an increasing tuple γ modulo addition of constants, and by Γ the set formed by these. There is a partial order on Γ, given by

$$[\gamma] \leq [\gamma'] :\Longleftrightarrow \gamma_i - \gamma_j \leq \gamma'_i - \gamma'_j \text{ for all } i \leq j.$$

In this section, g resp. γ will be used to denote a tuple of integers as in (6.1.3) resp. as in (6.1.4).

Example 6.1.5 *Let $m = 3$, let $g = (0, 1, 3)$. We obtain*

$$\Lambda_g = \begin{bmatrix} R & R & R \\ \pi & R & R \\ \pi^3 & \pi^2 & R \end{bmatrix}.$$

We dispose e.g. of

$$N_0 = N_{(0,0,0)} = \begin{bmatrix} R \\ R \\ R \end{bmatrix}, N_{(0,1,1)} = \begin{bmatrix} R \\ \pi \\ \pi \end{bmatrix}, N_g = N_{(0,1,3)} = \begin{bmatrix} R \\ \pi \\ \pi^3 \end{bmatrix}.$$

Remark 6.1.6 (intrinsic characterization) *The full suborder $\Lambda \subseteq (R)_m$ is a Gram order iff the indecomposable projective Λ-lattices are simple and there exist projective lattices X and Y such that the inclusion $Y \hookrightarrow X$, which is existent and unique up to scalar, is such that an element $\xi \in \mathrm{End}_R X \supseteq \Lambda$ restricts to Y iff it is contained in Λ.*

Note that Krull-Schmidt holds for projective Λ-lattices (D.1.7, or, a bit oversized, C.2.15), so that one merely has to consider the inclusions of the lattices given by the columns of Λ. Cf. e.g. (6.1.29).

If Λ is a Gram order, say $\Lambda \simeq \Lambda_g$, then take $X = N_0$ and $Y = N_g$ or vice versa.

In case the conditions are satisfied, we conjugate rationally with $Y \longrightarrow X$.

Observation 6.1.7 *Gram orders typically arise from invariant bilinear forms.*

Let p be a prime. Consider the Specht lattice $S^\lambda_{(p)}$ over $R := \mathbf{Z}_{(p)}$ to the partition λ of n. It comes equipped with a nondegenerate $\mathcal{S}_n$-invariant bilinear form

$$S^\lambda_{(p)} \otimes_R S^\lambda_{(p)} \xrightarrow{(-,=)} R$$

which gives an embedding of $R\mathcal{S}_n$-lattices

$$S^\lambda_{(p)} \xrightarrow{(-,=)} (S^\lambda_{(p)})^*$$

having as matrix the Gram matrix G^λ of the bilinear form [J 78, < 4.8], cf. [Se 77, 13.2]. Since multiplication by group elements on $S^\lambda_{(p)}$ as well as on $(S^\lambda_{(p)})^*$ is an integral operation, we obtain an inclusion of the local quasiblock

$$Q^\lambda_{(p)} \stackrel{i}{\hookrightarrow} \Lambda_{(G^\lambda)^{-1}}.$$

Remark 6.1.8 Conversely, consider the Gram order Λ_g, let G be the according main diagonal matrix with diagonal $(\pi^{g_1},\ldots,\pi^{g_m})$. There is an antiinvolution

$$\begin{array}{rcl} \Lambda_G & \stackrel{\alpha}{\longrightarrow} & \Lambda_G \\ A & \longrightarrow & A^- := (A^G)^t \end{array}$$

which may be used to define a structure as a left lattice over Λ_G on the R-linear dual N_γ^* via

$$(y)(Af) := (A^- y)f$$

for $y \in N_\gamma$, $f \in N_\gamma^*$, $A \in \Lambda_g$, giving an antiinvolution on Λ_G-lat, since multiplication on the dual of the dual is given by

$$\begin{array}{rcl} (f)(A\cdot \mathrm{eva}_y) & = & (A^- f)\mathrm{eva}_y \\ & = & (y)(A^- f) \\ & = & ((A^-)^- y)f \\ & = & (f)\mathrm{eva}_{Ay}, \end{array}$$

where $f \in N_\gamma^*$, $A \in \Lambda_g$, and where eva_y denotes the evaluation at $y \in N_\gamma$.

There is some kind of an 'invariant bilinear form' on N_0, viz. the Λ_g-morphism

$$\begin{array}{rcl} N_0 & \longrightarrow & N_0^* \\ x & \longrightarrow & x^t G'(-), \end{array}$$

where G' stands for the main diagonal matrix with diagonal $(\pi^{g_m - g_1},\ldots,\pi^{g_m - g_m})$, i.e. $G' = \pi^{g_m} G^{-1}$. Λ_g-linearity follows for $A \in \Lambda_g$ and $x, y \in N_0$ from

$$(Ax)^t G' y = x^t G' (A^G)^t y.$$

More generally, there is an isomorphism

$$\begin{array}{rcl} N_\gamma & \stackrel{\sim}{\longrightarrow} & N^*_{g-\gamma-g_m} \\ x & \longrightarrow & x^t G'(-), \end{array}$$

whence $N_\gamma^* \simeq N_{g-\gamma}$.

Remark 6.1.9 (coinciding antiinvolutions) We roughly follow [Se 77, 13.2]. Assume $R \subseteq \mathbf{C}$.

Let H be a finite group. Let χ be a **real valued** ordinary irreducible character of H such that the corresponding representation ρ can be realized over R. Let $Q \subseteq (R)_m$ the quasiblock of RG belonging to this representation. The antiinvolution of the group ring, induced by the inversion of the group elements, induces an antiinvolution α on Q since the corresponding rational central primitive idempotent

$$\varepsilon = \frac{m}{|H|}\sum_h \chi(h^{-1})h$$

is invariant under this antiinvolution.

Note that an invariant bilinear form on the column X of $(R)_m$ is unique up to scalar, since it gives a morphism $X \longrightarrow X^*$. There exists such a form given by, say, $\sum_h \rho(h)^t\rho(h)$, which is nondegenerate since positive definite. The condition on $G \in (R)_m$ to be the Gram matrix of an invariant bilinear form is, $x, y \in X$, $h \in H$,

$$(\rho(h)x)^t G y = x^t G(\rho(h^{-1})y),$$

i.e.

$$\rho(h)^t G = G\rho(h^{-1}).$$

By knowledge of Aut $(K)_m$ (D.2.2), the antiinvolution α is given by transposition followed by conjugation by a matrix $\tilde{G} \in (K)_m$. The condition on $\tilde{G}$ to furnish α in this manner is thus

$$(\rho(h)^t)^{\tilde{G}} = \rho(h^{-1}),$$

whence G and $\tilde{G}$ differ by a scalar. Therefore, the antiinvolutions α and $(A \longrightarrow (A^t)^G)$ coincide. Note that α being an involution implies $G^{-1}G^t$ to be a scalar, of value ± 1.

In particular, note that besides the implication $A \in Q \Longrightarrow {}^G A \in (R)_m$ we now also dispose of the implication $A \in Q \Longrightarrow (A^t)^G \in Q$. Note that $A \in Q \Longrightarrow (A^t)^G \in (R)_m \cap (R)_m^G$ already holds by the former implication. It seems to be hard to use this newly found implication.

In case of the quasiblock $Q_{(2)}^{(3,1,1)}$ of $\mathbf{Z}_{(2)}\mathcal{S}_5$, the Gram matrix, written in a basis such that the quasiblock takes a form as in (2.2.4), cannot be in elementary divisor form (cf. 6.3). This example also shows that it is possible that $A \in Q$ and $({}^G A)^t \in Q$, whereas ${}^G A \notin Q$, in a case in which the Gram matrix is a scalar multiple of a $SL_m(R)$-element.

Remark 6.1.10 We shall calculate the outer automorphism group of $\Lambda_{(0,k)} = \begin{pmatrix} R & R \\ \pi^k & R \end{pmatrix}$ as an order over R. Assume $k \geq 1$. Right conjugation by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$ yields the following results (cf. D.2.2).

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} &= \frac{1}{ad-bc}\begin{pmatrix} ad & bd \\ -ac & -bc \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} &= \frac{1}{ad-bc}\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\ \pi^k & 0 \end{pmatrix}^{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} &= \frac{\pi^k}{ad-bc}\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}
\end{aligned}$$

The automorphism group of $\Lambda_{(0,k)}$ is given, modulo scalars, by those matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which these results lie in $\Lambda_{(0,k)}$. By multiplication with a scalar we may assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to lie in $(R)_2$ and one of its entries to be equal to 1.

Case $a = 1$**.** Our results specialize to

$$\frac{1}{d-bc}\begin{pmatrix} d & bd \\ -c & -bc \end{pmatrix}, \frac{1}{d-bc}\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}, \frac{\pi^k}{d-bc}\begin{pmatrix} -b & -b^2 \\ 1 & b \end{pmatrix},$$

whence $v(d-bc) = 0$, so that $v(c) \geq k$ ensues, i.e. the automorphism turns out to be inner.

Case $d = 1$**.** Our results specialize to

$$\frac{1}{a-bc}\begin{pmatrix} a & b \\ -ac & -bc \end{pmatrix}, \frac{1}{a-bc}\begin{pmatrix} c & 1 \\ -c^2 & -c \end{pmatrix}, \frac{\pi^k}{a-bc}\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix}$$

whence $v(a-bc) = 0$, so that $v(c) \geq k$ ensues, i.e. the automorphism turns out to be inner.

Case $c = 1$**.** Our results specialize to

$$\frac{1}{ad-b}\begin{pmatrix} ad & bd \\ -a & -b \end{pmatrix}, \frac{1}{ad-b}\begin{pmatrix} d & d^2 \\ -1 & -d \end{pmatrix}, \frac{\pi^k}{ad-b}\begin{pmatrix} -ab & -b^2 \\ a^2 & ab \end{pmatrix},$$

which would require $v(ad-b) \leq -k$, which is impossible.

Case $b = 1$**.** Our results specialize to

$$\frac{1}{ad-c}\begin{pmatrix} ad & d \\ -ac & -c \end{pmatrix}, \frac{1}{ad-c}\begin{pmatrix} cd & d^2 \\ -c^2 & -cd \end{pmatrix}, \frac{\pi^k}{ad-c}\begin{pmatrix} -a & -1 \\ a^2 & a \end{pmatrix},$$

whence $v(ad-c) \in [0, k]$.

Subcase $v(ad-c)=0$. $2v(c)\geq k$ yields $v(a)=0$ because of $v(ad-c)=0$. Now $v(a)+v(c)\geq k$ yields $v(c)\geq k$, whence the automorphism is inner.

Subcase $1\leq v(ad-c)=:\Delta\leq k-1$. From $v(c)\geq\Delta$, $v(d)\geq\Delta$ and $2v(a)\geq\Delta$ we deduce $v(c)=\Delta$, for otherwise $v(ad-c)>\Delta$. But then $2\Delta=2v(c)\geq\Delta+k$ shows that this subcase cannot occur.

Subcase $v(ad-c)=k$. As before, from $v(c)\geq k$, $v(d)\geq k$ and $2v(a)\geq k$ we deduce $v(c)=k$, for otherwise $v(ad-c)>k$. Hence $v(a)+v(c)\geq 2k$ gives $v(a)\geq k$. We dispose of the automorphism given by right conjugation by $\left(\begin{smallmatrix}0&1\\ \pi^k&0\end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix}\pi^{-k}c&\pi^{-k}d\\ a&1\end{smallmatrix}\right)$ is a unit in $\Lambda_{(0,k)}$, the product

$$\left(\begin{smallmatrix}0&1\\ \pi^k&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\pi^{-k}c&\pi^{-k}d\\ a&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}a&1\\ c&d\end{smallmatrix}\right)$$

shows that these automorphisms all belong to the coset modulo inner automorphisms represented by $\left(\begin{smallmatrix}0&1\\ \pi^k&0\end{smallmatrix}\right)$.

The result therefore is $\operatorname{Out}\Lambda_{(0,k)}\simeq C_2$ for $k\geq 1$ (and trivial in case $k=0$).

Probably, a more conceptual way to argue is needed in order to deal with the outer automorphism groups of general Gram orders, which are of interest because they induce autoequivalences on the respective category of lattices. For instance, conjugation by $\left(\begin{smallmatrix}0&1\\ \pi^k&0\end{smallmatrix}\right)$ yields an autoequivalence on $\Lambda_{(0,k)}$ which sends $N_{(0,l)}$ to $N_{(0,k-l)}$, which however can't coincide with duality since it is covariant (cf. 6.1.8). I do not quite understand yet the composition of the duality followed by the autoequivalence just derived. E.g. for $0\leq l\leq m\leq k$ it sends the inclusion $N_{(0,m}\subseteq N_{(0,l)}$, up to isomorphic substitution, to the embedding $N_{(0,l)}\overset{\pi^{m-l}}{\hookrightarrow}N_{(0,m)}$.

Remark 6.1.11 Using the 'thickened main diagonal' as generators we obtain

$$\Lambda_g \xleftarrow{\sim} \left(E_1 \underset{B_1}{\overset{A_1}{\rightleftarrows}} \cdots \quad E_m\right)/(A_iB_i=\pi^{g_{i+1}-g_i}E_i,\, B_iA_i=\pi^{g_{i+1}-g_i}E_{i+1}\mid i\in[1,m-1]),$$

whence the according reductions modulo powers of π.

Proposition 6.1.12 (the sublattice lattice) *The sublattices of the Λ_g-lattice N_0 are of the form N_γ such that $[0]\leq[\gamma]\leq[g]$.*

NB not only up to isomorphism, but 'physically', i.e. as inclusions up to isomorphism over N_0.

This proposition should also follow from [P 80/1, (I.8)].

Given a sublattice X of N_0, we can, by choice of a basis of X, write its inclusion $X\hookrightarrow N_0$ as a lower triangular matrix $I=(\alpha_{ij})\in(R)_m$, $\alpha_{ij}=0$ for $i<j$, $\alpha_{ii}\neq 0$, $v(\alpha_{ij})<v(\alpha_{ii})$ or $\alpha_{ij}=0$ for $i>j$.

For $i\geq j\in[1,m]$, let $M^{ij}\in(R)_m$ have entry $M^{ij}_{ij}=\pi^{g_i-g_j}$, zero elsewhere. Let, for $i\leq j\in[1,m]$, $M^{ij}\in(R)_m$ have entry $M^{ij}_{ij}=1$, zero elsewhere. Note that since $M^{ij}\in\Lambda_g$, $I^{-1}M^{ij}I$ is integral, giving the operation of Λ_g on X.

We prove by diagonalwise induction that $\alpha_{ij}=0$ if $i\neq j$.

Start of the induction. We **claim** that $\alpha_{ij}=0$ for $i-j=1$. The entry

$$I^{-1}_{j+1,j}=-\alpha_{jj}^{-1}\alpha_{j+1,j+1}^{-1}\alpha_{j+1,j}$$

shows that $(I^{-1}M^{jj}I)_{j+1,j}=-\alpha_{j+1,j+1}^{-1}\alpha_{j+1,j}$, forcing $\alpha_{j+1,j}$ to be zero.

Step of the induction. We **claim** that $\alpha_{ij} = 0$ for $i - j = d$, $d \geq 2$ and assume the assertion known for $i - j \in [1, d-1]$. Now the entry

$$I^{-1}_{j+d,j} = -\alpha_{jj}^{-1}\alpha_{j+d,j+d}^{-1}\alpha_{j+d,j}$$

shows that $(I^{-1}M^{jj}I)_{j+d,j} = -\alpha_{j+d,j+d}^{-1}\alpha_{j+d,j}$, forcing $\alpha_{j+d,j}$ to be zero.

For $i \leq j$ we deduce from

$$(I^{-1}M^{ij}I)_{ij} = \alpha_{ii}^{-1}\alpha_{jj}$$

that $v(\alpha_{ii}) \leq v(\alpha_{jj})$.

For $i \leq j$ we deduce from

$$(I^{-1}M^{ji}I)_{ji} = \alpha_{jj}^{-1}\pi^{g_j - g_i}\alpha_{ii}$$

that $v(\alpha_{jj}) - v(\alpha_{ii}) \leq g_j - g_i$.

We shall investigate radical series of Gram orders. We restrict our attention to those with Morita multiplicities 1. **Assume $g = (g_1, \dots, g_m)$ to be a strictly increasingly ordered tuple of integers with $g_1 = 0$.**

Remark 6.1.13 The columns of Λ_g yield a decomposition into nonisomorphic indecomposable projectives since any morphism between them is a scalar.

Lemma 6.1.14 *The radical of Λ_g is given by*

$$\mathfrak{r}\Lambda_g = \{(\alpha_{kl}) \in \Lambda_g \subseteq (R)_m \mid v(\alpha_{kk}) \geq 1 \text{ for } k \in [1, m]\}.$$

(6.1.13) taken under consideration, this follows from (E.1.20 iii).

Lemma 6.1.15 (periodicity) *Assume R/π to be finite. Let $\Lambda \subseteq (R)_m$ be a full suborder. There are positive integers i, k and l such that*

$$\mathfrak{r}^{i+k}\Lambda = \pi^l\mathfrak{r}^i\Lambda.$$

Λ is a simple lattice over $\Lambda \otimes_R \Lambda^o$, of which the radical powers $\mathfrak{r}^i\Lambda$ are $\Lambda \otimes_R \Lambda^o$-sublattices, whose isomorphism classes in turn form a finite set by (D.2.6). So at least two of them must be isomorphic, whence their inclusion corresponds to a scalar multiplication (cf. E.2.1).

Lemma 6.1.16 *Assume $g_{j+1} - g_j \geq 2$ for $j \in [1, m-1]$. Let $i \geq 0$ and let*

$$\varphi^i_{kl} := \begin{cases} \max(0, i + k - l) & \text{for } k \leq l \\ \max(0, i + l - k) + g_k - g_l & \text{for } k > l. \end{cases}$$

Then

$$\mathfrak{r}^i\Lambda_g = \{(\alpha_{kl}) \in (R)_m \mid v(\alpha_{kl}) \geq \varphi^i_{kl} \text{ for } k, l \in [1, m]\}.$$

We proceed by induction, starting from the cases $i = 0$ and $i = 1$ (6.1.14). Assuming the radical power to be calculated by our formula for the exponent $i \geq 1$, we **claim** this formula to hold in case $i + 1$, i.e.

$$\min(\varphi^1_{ks} + \varphi^i_{sl} \mid s \in [1, m]) \overset{!}{=} \varphi^{i+1}_{kl}.$$

We fix k and l and evaluate the left hand side.

Case $k \le l$**.**

Assume $s < k \le l$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= g_k - g_s + \max(0, i+s-l)\\ &\ge 2 + \max(0, i+k-1-l)\\ &\ge \max(0, (i+1)+k-l).\end{aligned}$$

Assume $s = k \le l$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= 1 + \max(0, i+k-l)\\ &\ge \max(0, (i+1)+k-l).\end{aligned}$$

Assume $k < s \le l$.

$$\varphi^1_{ks} + \varphi^i_{sl} = 0 + \max(0, i+s-l),$$

which is minimal at $s = k+1$, taking value $\max(0, (i+1)+k-l)$.

Assume $k \le l < s$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= 0 + \max(0, i+l-s) + g_s - g_l\\ &\ge \max(0, i+l-(l+1)) + 2\\ &= i+1\\ &\ge \max(0, (i+1)+k-l).\end{aligned}$$

Case $k > l$**.**

Assume $s \le l < k$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= g_k - g_s + \max(0, i+s-l)\\ &\ge g_k - g_l + i\\ &\ge \max(0, (i+1)+l-k) + g_k - g_l.\end{aligned}$$

Assume $l < s < k$.

$$\varphi^1_{ks} + \varphi^i_{sl} = g_k - g_s + \max(0, i+l-s) + g_s - g_l,$$

which is minimal at $s = k-1$, taking value $\max(0, (i+1)+l-k) + g_k - g_l$.

Assume $l < k = s$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= 1 + \max(0, i+l-k) + g_k - g_l\\ &\ge \max(0, (i+1)+l-k) + g_k - g_l.\end{aligned}$$

Assume $l < k < s$.

$$\begin{aligned}\varphi^1_{ks} + \varphi^i_{sl} &= 0 + \max(0, i+l-s) + g_s - g_l\\ &\ge \max(0, i+l-(k+1)) + 2 + g_k - g_l\\ &\ge \max(0, (i+1)+l-k) + g_k - g_l.\end{aligned}$$

Corollary 6.1.17 *We keep the assumptions of (6.1.16) and obtain*

$$\mathfrak{r}^m \Lambda_g = \pi \mathfrak{r}^{m-1} \Lambda_g.$$

For a matrix size of $m \le 4$, we treat the remaining cases. **We abbreviate** $[s, t] := \min(s, t)$**.**

Lemma 6.1.18 *Let* $m = 2$*, let* $g = (0,1)$*, i.e.* $\Lambda_g = \left[\begin{smallmatrix} R & R \\ \pi & R \end{smallmatrix}\right]$*. Then, by (6.1.14),* $\mathfrak{r}\Lambda_g = \left[\begin{smallmatrix} \pi & R \\ \pi & \pi \end{smallmatrix}\right]$ *(6.1.14), whence* $\mathfrak{r}^2\Lambda_g = \pi\Lambda_g$*.*

Lemma 6.1.19 *Let* $m = 3$*, let* $g = (0,1,2)$*, i.e.* $\Lambda_g = \begin{bmatrix} R & R & R \\ \pi & R & R \\ \pi^2 & \pi & R \end{bmatrix}$*. Then, by (6.1.14),*

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R \\ \pi & \pi & R \\ \pi^2 & \pi & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R \\ \pi^2 & \pi & \pi \\ \pi^2 & \pi^2 & \pi \end{bmatrix}, \ \mathfrak{r}^3\Lambda_g = \begin{bmatrix} \pi^2 & \pi & \pi \\ \pi^2 & \pi^2 & \pi \\ \pi^3 & \pi^2 & \pi^2 \end{bmatrix} = \pi\mathfrak{r}\Lambda_g.$$

Lemma 6.1.20 *Let* $m = 3$*,* $a \geq 2$*,* $g = (0,1,1+a)$*, i.e.* $\Lambda_g = \begin{bmatrix} R & R & R \\ \pi & R & R \\ \pi^{a+1} & \pi^a & R \end{bmatrix}$*, whence, by (6.1.14),*

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R \\ \pi & \pi & R \\ \pi^{a+1} & \pi^a & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R \\ \pi^2 & \pi & \pi \\ \pi^{a+1} & \pi^{a+1} & \pi^2 \end{bmatrix}.$$

By induction we see that for $i \geq 1$

$$\mathfrak{r}^{2i}\Lambda_g = \begin{bmatrix} \pi^i & \pi^i & \pi^{i-1} \\ \pi^{i+1} & \pi^i & \pi^i \\ \pi^{a+i} & \pi^{a+i} & \pi^{[2i,i-1+a]} \end{bmatrix},$$

whence $\mathfrak{r}^{2a}\Lambda_g = \pi\mathfrak{r}^{2a-2}\Lambda_g$*.*

The case $g = (0,a,a+1)$ follows by transposition and conjugation.

Lemma 6.1.21 *Let* $m = 4$*, let* $g = (0,1,2,3)$*, i.e.*

$$\Lambda_g = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi^2 & \pi & R & R \\ \pi^3 & \pi^2 & \pi & R \end{bmatrix}.$$

Then, by (6.1.14),

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R & R \\ \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^3 & \pi^2 & \pi & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^2 & \pi^2 & \pi & \pi \\ \pi^3 & \pi^2 & \pi^2 & \pi \end{bmatrix}, \ \mathfrak{r}^3\Lambda_g = \begin{bmatrix} \pi^2 & \pi & \pi & R \\ \pi^2 & \pi^2 & \pi & \pi \\ \pi^3 & \pi^2 & \pi^2 & \pi \\ \pi^3 & \pi^3 & \pi^2 & \pi^2 \end{bmatrix}, \ \mathfrak{r}^4\Lambda_g = \begin{bmatrix} \pi^2 & \pi^2 & \pi & \pi \\ \pi^3 & \pi^2 & \pi^2 & \pi \\ \pi^3 & \pi^3 & \pi^2 & \pi^2 \\ \pi^4 & \pi^3 & \pi^3 & \pi^2 \end{bmatrix} = \pi\mathfrak{r}^2\Lambda_g.$$

Lemma 6.1.22 *Let* $m = 4$*,* $a \geq 2$*,* $g = (0,1,2,a+2)$*, i.e.*

$$\Lambda_g = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi^2 & \pi & R & R \\ \pi^{a+2} & \pi^{a+1} & \pi^a & R \end{bmatrix}.$$

Then, by (6.1.14),

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R & R \\ \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^{a+2} & \pi^{a+1} & \pi^a & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^2 & \pi^2 & \pi & \pi \\ \pi^{a+2} & \pi^{a+1} & \pi^{a+1} & \pi^2 \end{bmatrix}.$$

By induction we see that for $i \geq 1$

$$\mathfrak{r}^{2i}\Lambda_g = \begin{bmatrix} \pi^i & \pi^i & \pi^{i-1} & \pi^{i-1} \\ \pi^{i+1} & \pi^i & \pi^i & \pi^{i-1} \\ \pi^{i+1} & \pi^{i+1} & \pi^i & \pi^i \\ \pi^{a+i+1} & \pi^{a+i} & \pi^{a+i} & \pi^{[2i,a+i-1]} \end{bmatrix},$$

whence $\mathfrak{r}^{2a}\Lambda_g = \pi\mathfrak{r}^{2a-2}\Lambda_g$*.*

The case $g = (0,a,a+1,a+2)$ follows by transposition and conjugation.

Lemma 6.1.23 *Let* $m = 4$*,* $a \geq 2$*,* $g = (0,1,a+1,a+2)$*, i.e.*

$$\Lambda_g = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi^{a+1} & \pi^a & R & R \\ \pi^{a+2} & \pi^{a+1} & \pi & R \end{bmatrix}.$$

Then, by (6.1.14),

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R & R \\ \pi & \pi & R & R \\ \pi^{a+1} & \pi^a & \pi & R \\ \pi^{a+2} & \pi^{a+1} & \pi & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^{a+1} & \pi^{a+1} & \pi & \pi \\ \pi^{a+2} & \pi^{a+1} & \pi^2 & \pi \end{bmatrix}, \ \mathfrak{r}^4\Lambda_g = \begin{bmatrix} \pi^2 & \pi^2 & \pi & \pi \\ \pi^3 & \pi^2 & \pi^2 & \pi \\ \pi^{a+2} & \pi^{a+2} & \pi^2 & \pi^2 \\ \pi^{a+3} & \pi^{a+2} & \pi^3 & \pi^2 \end{bmatrix} = \pi\mathfrak{r}^2\Lambda_g.$$

Lemma 6.1.24 *Let $m = 4$, $a, b \geq 2$, $g = (0, 1, a+1, a+b+1)$, i.e.*

$$\Lambda_g = \begin{bmatrix} R & R & R & R \\ \pi & R & R & R \\ \pi^{a+1} & \pi^a & R & R \\ \pi^{a+b+1} & \pi^{a+b} & \pi^b & R \end{bmatrix}.$$

Then, by (6.1.14),

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R & R \\ \pi & \pi & R & R \\ \pi^{a+1} & \pi^a & \pi & R \\ \pi^{a+b+1} & \pi^{a+b} & \pi^b & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi & \pi & R & R \\ \pi^2 & \pi & \pi & R \\ \pi^{a+1} & \pi^{a+1} & \pi^2 & \pi \\ \pi^{a+b+1} & \pi^{a+b} & \pi^{b+1} & \pi^2 \end{bmatrix}.$$

By induction we see that for $i \geq 1$

$$\mathfrak{r}^{2i}\Lambda_g = \begin{bmatrix} \pi^i & \pi^i & \pi^{i-1} & \pi^{i-1} \\ \pi^{i+1} & \pi^i & \pi^i & \pi^{i-1} \\ \pi^{a+i} & \pi^{a+i} & \pi^{[2i, a+i-1]} & \pi^{[2i-1, a+i-1]} \\ \pi^{a+b+i} & \pi^{a+b+i-1} & \pi^{b+[2i-1, a+i-1]} & \pi^{[2i, a+b+i-2]} \end{bmatrix},$$

whence $\mathfrak{r}^{2a+2b-2}\Lambda_g = \pi\mathfrak{r}^{2a+2b-4}\Lambda_g$.

The case $g = (0, b, a+b, a+b+1)$ follows by transposition and conjugation.

Lemma 6.1.25 *Let $m = 4$, $a, b \geq 2$, $g = (0, a, a+1, a+b+1)$, i.e.*

$$\Lambda_g = \begin{bmatrix} R & R & R & R \\ \pi^a & R & R & R \\ \pi^{a+1} & \pi & R & R \\ \pi^{a+b+1} & \pi^{b+1} & \pi^b & R \end{bmatrix}.$$

Then, by (6.1.14),

$$\mathfrak{r}\Lambda_g = \begin{bmatrix} \pi & R & R & R \\ \pi^a & \pi & R & R \\ \pi^{a+1} & \pi & \pi & R \\ \pi^{a+b+1} & \pi^{b+1} & \pi^b & \pi \end{bmatrix}, \ \mathfrak{r}^2\Lambda_g = \begin{bmatrix} \pi^2 & \pi & R & R \\ \pi^{a+1} & \pi & \pi & R \\ \pi^{a+1} & \pi^2 & \pi & \pi \\ \pi^{a+b+1} & \pi^{b+1} & \pi^{b+1} & \pi^2 \end{bmatrix}.$$

By induction we see that for $i \geq 1$

$$\mathfrak{r}^{2i}\Lambda_g = \begin{bmatrix} \pi^{[2i, a+i-1]} & \pi^i & \pi^{i-1} & \pi^{i-1} \\ \pi^{a+i} & \pi^i & \pi^i & \pi^{i-1} \\ \pi^{a+i} & \pi^{i+1} & \pi^i & \pi^i \\ \pi^{a+b+i} & \pi^{b+i} & \pi^{b+i} & \pi^{[2i, b+i-1]} \end{bmatrix},$$

whence, letting $c := \max(a, b)$, $\mathfrak{r}^{2c}\Lambda_g = \pi\mathfrak{r}^{2c-2}\Lambda_g$.

Question 6.1.26 *Let Λ be a Gram order. Does there exist an $s \geq 0$ and a $k \in \{1, 2\}$ such that*

$$\mathfrak{r}^{s+k}\Lambda = \pi\mathfrak{r}^s\Lambda?$$

Let $\Lambda \subseteq (R)_m$ be a full suborder. Does there exist an $s \geq 0$ and a $k \geq 1$ such that this equation holds? Cf. (6.1.15, 6.1.27).

Let $Q^{\lambda}_{(p)}$ be the quasiblock of $\mathbf{Z}S_n$ to the partition λ, localized at the prime p. Assume its indecomposable projectives to be simple lattices. Recall that $Q^{\lambda}_{(p)} \subseteq \Lambda_{(G^\lambda)^{-1}}$ (6.1.7). Does there exist an integer $m \geq 0$ such that

$$\mathfrak{r}^m Q^{\lambda}_{(p)} \ = \ \mathfrak{r}^m \Lambda_{(G^\lambda)^{-1}} \ ?$$

I.e. does a Gram order determine all but a finite part of the radical series? [1].

The assumption on the indecomposable projectives is not superfluous, as $Q^{(3,1,1)}_{(2)}$ shows (6.1.27).

[1]A counterexample to the question for $m = 1$ fixed can be found on p. 116 f. of G. NEBE, *Orthogonale Darstellungen endlicher Gruppen und Gruppenringe*, Verlag Mainz, Aachen, see also `http://www.mathematik.uni-ulm.de/ReineM/nebe/pl.html`. (6.1.26) is an attempt to formalize the question to which extent the Gram matrix determines the quasiblock.

We shall discuss the sublattice lattices of the quasiblocks which are not Gram orders discovered so far.

Example 6.1.27
From (S 2.2.4) we take that $Q_{(2)}^{(3,1,1)}$ is Morita equivalent to, $R = \mathbf{Z}_{(2)}$,

$$\Lambda := \begin{matrix} R & 2 & 2 \\ R & R & 2 \\ R & R & R \end{matrix} \quad (2)$$

with Morita multiplicities $(1,4,1)$. Using (E.1.20 iii, E.1.30), the radical series of Λ is given by

$$\mathfrak{r}\Lambda = \begin{bmatrix} 2 & 2 & 2 \\ R & 2 & 2 \\ R & R & 2 \end{bmatrix}, \; \mathfrak{r}^2\Lambda = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 2 & 2 \\ R & 2 & 2 \end{bmatrix}, \; \mathfrak{r}^3\Lambda = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}, \; \mathfrak{r}^4\Lambda = \begin{bmatrix} 4 & 4 & 4 \\ 2 & 4 & 4 \\ 2 & 2 & 4 \end{bmatrix} = 2\mathfrak{r}\Lambda.$$

The submodule lattices of $S^{(3,1,1)}$ and of its Morita correspondent

$$Y := \begin{bmatrix} R \\ R \\ R \end{bmatrix}$$

are isomorphic via Morita equivalence.

Let I be the inclusion matrix of a sublattice $X \subseteq Y$, which we may assume to be of the form

$$I = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

where $v(\alpha_{ij}) < v(\alpha_{ii})$ or $\alpha_{ij} = 0$ for $i > j$. Writing $(-)^-$ for $(-)^{-1}$, we obtain

$$I^- = \begin{bmatrix} \alpha_{11}^- & 0 & 0 \\ -\alpha_{11}^-\alpha_{22}^-\alpha_{21} & \alpha_{22}^- & 0 \\ \alpha_{11}^-\alpha_{22}^-\alpha_{33}^-\alpha_{32}\alpha_{21} - \alpha_{11}^-\alpha_{33}^-\alpha_{31} & -\alpha_{22}^-\alpha_{33}^-\alpha_{32} & \alpha_{33}^- \end{bmatrix}.$$

The necessary calculation, as in (6.1.12), amounts to check the condition of integrality for I^-MI for M running over a basis of Λ.

First of all, $M = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ yields the integrality of

$$I^{-1}MI = \begin{bmatrix} 1 & 0 & 0 \\ \alpha_{22}^-\alpha_{21} & 0 & 0 \\ \alpha_{22}^-\alpha_{33}^-\alpha_{32}\alpha_{21} & \alpha_{33}^-\alpha_{32} & 1 \end{bmatrix}$$

from which we take $\alpha_{21} = 0$ and $\alpha_{32} = 0$. Inserting these values we obtain the following list of implications.

$$\begin{array}{rcl} M = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right) & \Longrightarrow & \text{(i)} \;\; \alpha_{33} \mid 2\alpha_{31} \\ M = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) & \Longrightarrow & \text{(ii)} \;\; \alpha_{22} \mid \alpha_{11} \\ M = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right) & \Longrightarrow & \text{(iii)} \;\; \alpha_{33} \mid \alpha_{22} \\ M = \left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) & \Longrightarrow & \text{(iv)} \;\; \alpha_{11} \mid 2\alpha_{31} \\ & & \text{(v)} \;\; \alpha_{11}\alpha_{33} \mid 2\alpha_{31}^2 \\ & & \text{(vi)} \;\; \alpha_{11} \mid 2\alpha_{33} \end{array}$$

Assume that $\alpha_{31} \neq 0$. Then (i) forces $v(\alpha_{33}) = v(\alpha_{31}) + 1$, for $v(\alpha_{31}) < v(\alpha_{33})$. But $v(\alpha_{33}) \leq v(\alpha_{11})$ by (ii) and (iii), and $v(\alpha_{11}) \leq v(\alpha_{31}) + 1 = v(\alpha_{33})$ by (iv), so, by (iii) and (ii), $v(\alpha_{11}) = v(\alpha_{22}) = v(\alpha_{33}) = v(\alpha_{31}) + 1$. Hence (v) reads $2v(\alpha_{31}) + 2 \leq 2v(\alpha_{31}) + 1$, which is a **contradiction.**

Therefore $\alpha_{31} = 0$. Thus, by (ii), (iii), (vi), the sublattices not contained in $2Y$ are given by the columns of Λ, viz. by

$$\begin{bmatrix} R \\ R \\ R \end{bmatrix}, \begin{bmatrix} 2 \\ R \\ R \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ R \end{bmatrix}.$$

Example 6.1.28
From (S 2.3.5) we take that $Q_{(2)}^{(3,1,1,1)}$ is Morita equivalent to, $R = \mathbf{Z}_{(2)}$,

$$\Lambda := \begin{matrix} R & 2 & 2 & 2 \\ R & R & 2 & 2 \\ R & R & R & 2 \\ R & R & R & R \end{matrix} \quad \text{(entries (1,3), (1,4), (2,4), (3,4) joined to a circled } 2\text{)}$$

with Morita multiplicities $(4, 1, 4, 1)$, thus quite similar to (6.1.27). Let the inclusion of a sublattice given by

$$I = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix},$$

where $v(\alpha_{ij}) < v(\alpha_{ii})$ or $\alpha_{ij} = 0$ for $i > j$. The integrality condition on $I^{-}MI$ yields for $M = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}$ that $\alpha_{21} = 0$, $\alpha_{32} = 0$ and $\alpha_{43} = 0$, then, inserting these values, $\alpha_{41} = 0$. Conjugating $M = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}$ we obtain $\alpha_{31} = 0$. Using the arguments of (6.1.27) now, we see that $\alpha_{42} = 0$ and moreover, that $\alpha_{44} \mid \alpha_{33} \mid \alpha_{22} \mid \alpha_{11} \mid 2\alpha_{44}$. Therefore the sublattices of the column Y of $(R)_4$ not contained in $2Y$ are given by the columns of Λ.

Example 6.1.29
From (S 2.3.3) we take that $Q_{(2)}^{(3,2,1)}$ is Morita equivalent to, $R = \mathbf{Z}_{(3)}$,

$$\begin{matrix} R & 3 & 3 & 3 & 9 \\ R & R & 3 & 3 & 3 \\ R & 3 & R & 3 & 3 \\ R & 3 & 3 & R & 3 \\ R & R & R & R & R \end{matrix}$$

with Morita multiplicities $(4, 1, 6, 1, 4)$, quite similar to $Q_{(3)}^{(3,2,1,1}$ (S F.5). Since all main diagonal idempotents of $(R)_n$ are contained in L, we obtain, as in (6.1.12), that we may assume the embedding of a sublattice to be given by a main diagonal matrix with diagonal $(\alpha_{11}, \alpha_{22}, \alpha_{33}, \alpha_{44}, \alpha_{55})$. As usual, we obtain $\alpha_{55} \mid \alpha_{22}, \alpha_{33}, \alpha_{44} \mid \alpha_{11}$ and $\alpha_{11} \mid 3\alpha_{22}, 3\alpha_{33}, 3\alpha_{44} \mid 9\alpha_{55}$, whence the following list of sublattices of the column Y of $(R)_5$ which are not contained in $3Y$.

$$\begin{bmatrix} R \\ R \\ R \\ R \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ R \\ R \\ R \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ R \\ R \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ R \\ 3 \\ R \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ R \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ R \\ R \\ 3 \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ R \\ 3 \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ R \\ 3 \\ 3 \\ R \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ R \end{bmatrix}, \begin{bmatrix} 9 \\ 3 \\ 3 \\ 3 \\ R \end{bmatrix}.$$

In particular, $Q_{(2)}^{(3,2,1)}$ is not a Gram order, using that all embeddings of simple projective lattices are given by main diagonal matrices (cf. 6.1.6).

6.2 Duality

To begin with, we need to rewrite [J 78, 6.7] in our integral context, without claiming originality. Then we use the duality/transposition/alternation game to obtain information about the Gram matrix of the invariant bilinear form on a Specht lattice.

Let n be a natural number. Let λ be a partition of n, let t be a λ-tableau. Retain the notation from (4.1.1, S 6.1). Denote by $S^{\lambda,-} := S^{(1^n)} \otimes_{\mathbf{Z}} S^\lambda$ the tensor product of S^λ with the alternating lattice $S^{(1^n)}$ on which $s \in \mathcal{S}_n$ acts as ε_s. There is a $\mathcal{S}_n$-invariant symmetric positive definite bilinear form $(-,=)$ on M^λ, given by $(\{a\},\{b\}) := \partial_{\{a\},\{b\}}$, which restricts to the positive definite, thus nondegenerate $\mathcal{S}_n$-invariant bilinear form attached to S^λ. We denote

$$\begin{aligned} S^{\lambda,\perp} &:= \{x \in M^\lambda \mid (x, S^\lambda) = 0\}, \\ \kappa_t &:= \sum_{s \in C_t} \varepsilon_s s, \\ \rho_t &:= \sum_{s \in R_t} s. \end{aligned}$$

Lemma 6.2.1 (James, **[J 78, 4.8]**) *If U is a pure $\mathbf{Z}\mathcal{S}_n$-sublattice of M^λ, then either $S^\lambda \subseteq U$ or $U \subseteq S^{\lambda,\perp}$.*

For $u \in U$ we have $u\kappa_t = \alpha_u \langle t \rangle$ for some $\alpha \in \mathbf{Z}$ [J 78, 4.6, 4.7].

In **case** $\alpha_u = 0$ for all $u \in U$, we obtain

$$\begin{aligned} 0 &= (u\kappa_t, \{t\}) \\ &= (u, \{t\}\kappa_t) \\ &= (u, \langle t \rangle), \end{aligned}$$

so $U \subseteq S^{\lambda,\perp}$ since $\langle t \rangle$ generates S^λ over $\mathbf{Z}\mathcal{S}_n$ and the form is $\mathcal{S}_n$-invariant.

In **case** there is a $u \in U$ with $u\kappa_t = \alpha_u \langle t \rangle \neq 0$, we conclude from $\alpha_u \langle t \rangle \in U$ and the inclusion $U \subseteq M^\lambda$ being pure that $\langle t \rangle \in U$, hence $S^\lambda \subseteq U$ since $\langle t \rangle$ generates S^λ over $\mathbf{Z}\mathcal{S}_n$.

Lemma 6.2.2 *The epimorphism*

$$\begin{aligned} F^\lambda &\longrightarrow S^{\lambda',-} \\ [t] &\longrightarrow 1 \otimes \langle t' \rangle \end{aligned}$$

factors over

$$M^\lambda \xrightarrow{\Theta} S^{\lambda',-}.$$

M^λ may be regarded as the quotient of F^λ modulo unsigned row transpositions. Given a row transposition $(u\ v)$, i.e. u and v assumed to lie in the same column of t', we obtain

$$\begin{aligned} (1 \otimes \langle t' \rangle)(u\ v) &= \varepsilon_{(u\ v)} \otimes \langle t'(u\ v) \rangle \\ &= 1 \otimes \langle t' \rangle. \end{aligned}$$

Lemma 6.2.3 *The sublattices* $S^{\lambda,\perp}$ *and* Kern Θ *of* M^λ *coincide.*

Since the inclusion Kern $\Theta \subseteq M^\lambda$ is pure, by (6.2.1) it suffices to show for Kern $\Theta \subseteq S^{\lambda,\perp}$ that $S^\lambda\Theta \neq 0$, i.e. that

$$\begin{aligned}(\sum_{s\in C_t}\varepsilon_s\{t\}s)\Theta &= \sum_{s\in R_{t'}}\varepsilon_s\otimes\varepsilon_s\langle t's\rangle \\ &= 1\otimes\langle t'\rangle\rho_{t'} \\ &\neq 0.\end{aligned}$$

Since both Kern Θ and $S^{\lambda,\perp}$ are pure $\mathbf{Z}\mathcal{S}_n$-sublattices in M^λ of the same rank (rk M^λ − rk S^λ) this inclusion even suffices to prove the initial assertion.

But

$$\begin{aligned}(\{t'\},\langle t'\rangle\rho_{t'}) &= (\{t'\}\rho_{t'},\langle t'\rangle) \\ &= |R_{t'}|(\{t'\},\langle t'\rangle) \\ &= |R_{t'}|.\end{aligned}$$

Lemma 6.2.4 *The canonical morphism*

$$\begin{array}{rcl} M^\lambda/S^{\lambda,\perp} & \longrightarrow & S^{\lambda,*} \\ \{t\} & \longrightarrow & (\{t\},-) \end{array}$$

is an isomorphism of $\mathbf{Z}\mathcal{S}_n$*-lattices.*

Injectivity ensues by definition of $S^{\lambda,\perp}$. For to see surjectivity we need the inclusion $S^\lambda \subseteq M^\lambda$ to be pure, thus inducing a surjection $M^{\lambda,*} \longrightarrow S^{\lambda,*}$.

But this follows from [J 78, 8.3], using the basis consisting of standard polytabloids.

Proposition 6.2.5 *The map determined by*

$$\begin{array}{rcl} S^{\lambda',-} & \longrightarrow & S^{\lambda,*} \\ 1\otimes\langle t'\rangle & \longrightarrow & (\{t\},-) \end{array}$$

is a well defined isomorphism of $\mathbf{Z}\mathcal{S}_n$*-lattices.*

This ensues from (6.2.3, 6.2.4). Note that the sign of this map depends on the choice of the λ-tableau $[t]$.

Question 6.2.6 ([2]) *Given a* $\mathbf{Z}\mathcal{S}_n$*-morphism*

$$S^\lambda/m \xrightarrow{f} S^\mu/m$$

we may dualize $\mathbf{Z}/m$*-linearly and alternate to obtain*

$$S^{\lambda,*,-}/m \xleftarrow{f^{*,-}} S^{\mu,*,-}/m,$$

which we may substitute isomorphically by virtue of (6.2.5) by

$$S^{\lambda'}/m \xleftarrow{f'} S^{\mu'}/m,$$

which we take as the definition of the transpose f'.

---

[2] C. Ringel asked for the behaviour of the morphism in (4.3.31) under dualization.

Consider the situation of a one-box-shift downwards from λ to μ as in (4.3.31) and note that λ' arises from μ' by a downwards shift of one box, too. Do the according specializations of the generic morphism in (4.3.31) correspond, up to sign, under transposition? Moreover, is the transpose of a specialization of the generic morphism given in (4.4.3) a specialization of the generic morphism given in (4.4.1)? Cf. (4.3.33).

If so, the elementary divisors over $\mathbf{Z}/m$ would coincide and the composition properties would dualize.

Lemma 6.2.7 ([J 78, 23.2 ii]) *Let $n_\lambda := \operatorname{rk} S^\lambda$. Then*

$$\{t\}\kappa_t\rho_t\kappa_t = \frac{n!}{n_\lambda}\{t\}\kappa_t.$$

By [J 78, 4.6, 4.7], $\{t\}\kappa_t\rho_t\kappa_t$ is a scalar multiple of $\{t\}\kappa_t$, say $\{t\}\kappa_t\rho_t\kappa_t = \alpha \cdot \{t\}\kappa_t$ where $\alpha \in \mathbf{Z}$.

Since by [J 78, > 4.13] we have

$$\begin{array}{ccc} M^\lambda & \xrightarrow{\sim} & \rho_t\mathbf{Z}\mathcal{S}_n \\ \{t\} & \longrightarrow & \rho_t \end{array}$$

the assertion is equivalent to

$$(\rho_t\kappa_t)^2 = \frac{n!}{n_\lambda}(\rho_t\kappa_t),$$

whereas we know that $(\rho_t\kappa_t)^2 = \alpha(\rho_t\kappa_t)$.

Following e.g. [Rog 74, 4.10], we calculate the trace of

$$\mathbf{Q}\mathcal{S}_n \xrightarrow{\rho_t\kappa_t(-)} \mathbf{Q}\mathcal{S}_n$$

twice. Since $C_t \cap R_t = 1$, the 1-coefficient of $\rho_t\kappa_t$ is 1, whence, using the canonical basis of $\mathbf{Q}\mathcal{S}_n$, the trace equals $n!$. The isomorphism of M^λ with $\rho_t\mathbf{Z}\mathcal{S}_n$ just cited restricts to the isomorphism $S^\lambda \xrightarrow{\sim} \rho_t\kappa_t\mathbf{Z}\mathcal{S}_n$. Writing $\mathbf{Q}\mathcal{S}_n = \rho_t\kappa_t\mathbf{Q}\mathcal{S}_n \oplus V$ as vectorspaces, $\rho_t\kappa_t(-)$ reads $(-)\left(\begin{smallmatrix}\alpha & 0 \\ * & 0\end{smallmatrix}\right)$. Consequently, we obtain

$$\operatorname{tr} \rho_t\kappa_t(-) \overset{1.}{=} n! \overset{2.}{=} \alpha \cdot n_\lambda.$$

Lemma 6.2.8 *Let G^λ be the Gram matrix of the $\mathcal{S}_n$-invariant bilinear form attached to S^λ written in the standard basis given by the standard polytabloids. Then there are $\mathrm{GL}_{n_\lambda}(\mathbf{Z})$-elements A^λ such that*

$$A^{\lambda'}G^{\lambda'}A^\lambda G^\lambda = \frac{n!}{n_\lambda}.$$

G^λ is the matrix of the morphism

$$\begin{array}{ccc} S^\lambda & \longrightarrow & S^{\lambda,*} \\ \langle t\rangle & \longrightarrow & (\langle t\rangle, -) \end{array}$$

when using the standard basis and its dual, respectively.

Consider the composition with the isomorphism from (6.2.5)

$$\begin{array}{lll} S^\lambda \longrightarrow S^{\lambda,*} \xleftarrow{\sim} & S^{\lambda',-} \\ \langle t\rangle \quad \longrightarrow & (1\otimes\langle t'\rangle)\kappa_t = 1\otimes\langle t'\rangle\rho_{t'}. \end{array}$$

The matrix of the inverse of $S^{\lambda',-} \xrightarrow{\sim} S^{\lambda,*}$ with respect to the standard basis and its dual denoted by $A^{\lambda'}$, we obtain the matrix of this morphism to be $A^\lambda G^\lambda$.

Now composition yields

$$\begin{array}{lllll} S^\lambda & \longrightarrow & S^{\lambda',-} & \longrightarrow & S^\lambda \\ \langle t\rangle & \longrightarrow & (1\otimes\langle t'\rangle)\kappa_t & & \\ & & 1\otimes\langle t'\rangle & \longrightarrow & \langle t\rangle\rho_t \\ & & (1\otimes\langle t'\rangle)\kappa_t & \longrightarrow & \langle t\rangle\rho_t\kappa_t = \{t\}\kappa_t\rho_t\kappa_t \end{array}$$

whence the result by (6.2.7).

The nontriviality of A^λ corresponds to the 'annoying' phenomenon mentioned in [J 78, 8.1] that $(\{a\},\langle b\rangle)$ may be nonzero also for $[a]$ and $[b]$ being different standard tableaux.

Definition 6.2.9 *Let* $M \in (R)_m$, $\det M \neq 0$. *Let*

$$d_i^M$$

denote the elementary divisors of M, *ordered increasingly,* $d_i^M \mid d_j^M$ *for* $i \le j$.

Proposition 6.2.10 *Retain the notation from (6.2.8). We obtain*

$$d_i^{G^\lambda} d_{n_\lambda+1-i}^{G^{\lambda'}} = \pm\frac{n!}{n_\lambda}$$

for $i \in [1, n_\lambda]$.

This ensues from (6.2.8), using elementary divisors of the inverse.

6.3 List of elementary divisors

We list the local elementary divisors of the Gram matrices of the $\mathcal{S}_n$-invariant bilinear form on the Specht module in the following form. E.g. at the prime p the tuple $(1,0,1,3,2)$ translates into the local elementary divisor tuple $(p^0,p^2,p^3,p^3,p^3,p^4,p^4)$ [3].

$n = 4$.

partition	divisors at 2	divisors at 3
(2,1,1)	(0,1,0,2)	(3)
(2,2)	(0,2)	(1,1)

$n = 5$.

partition	divisors at 2	divisors at 3	divisors at 5
(2,1,1,1)	(0,4)	(0,4)	(1,3)
(2,2,1)	(0,0,1,4)	(4,1)	(5)
(3,1,1)	(0,6)	(6)	(3,3)

[3]A. MATHAS supplied me with an alternative routine for the elementary divisors, based on SPECHT under GAP. Moreover, I used his routine SCHAPER to compare [4].

[4]Cf. A. MATHAS, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, Appendix C.

$n = 6.$

partition	divisors at 2	divisors at 3	divisors at 5
(2,1,1,1,1)	(0,0,0,1,4)	(0,1,4)	(5)
(2,2,1,1)	(0,0,4,1,4)	(9)	(1,8)
(2,2,2)	(0,0,1,4)	(0,4,1)	(5)
(3,1,1,1)	(0,4,6)	(0,4,6)	(10)
(3,2,1)	(16)	(4,8,4)	(8,8)

$n = 7.$

partition	divisors at 2	divisors at 3	divisors at 5	divisors at 7
(2,1,1,1,1,1)	(0,0,0,6)	(0,6)	(0,6)	(1,5)
(2,2,1,1,1)	(0,0,0,14)	(0,1,13)	(6,8)	(14)
(2,2,2,1)	(0,0,6,8)	(0,13,1)	(1,13)	(14)
(3,1,1,1,1)	(0,0,0,15)	(0,15)	(15)	(5,10)
(3,2,1,1)	(0,14,0,1,20)	(13,2,20)	(35)	(35)
(3,2,2)	(0,0,1,20)	(15,6)	(13,8)	(21)
(4,1,1,1)	(0,20)	(0,20)	(20)	(10,10)

$n = 8.$

partition	divisors at 2	divisors at 3	divisors at 5	divisors at 7
(2,1,1,1,1,1,1)	(0,0,0,0,1,0,0,6)	(0,0,7)	(0,7)	(7)
(2,2,1,1,1,1)	(0,0,0,0,6,14)	(0,7,13)	(20)	(1,19)
(2,2,2,1,1)	(0,0,0,14,6,8)	(0,0,28)	(7,21)	(28)
(2,2,2,2)	(0,0,0,0,6,8)	(0,13,1)	(1,13)	(14)
(3,1,1,1,1,1)	(0,0,0,6,0,0,15)	(0,21)	(0,21)	(21)
(3,2,1,1,1)	(0,64)	(0,29,35)	(21,43)	(19,45)
(3,2,2,1)	(0,0,6,8,14,2,40)	(28,35,7)	(70)	(70)
(3,3,1,1)	(0,0,14,42)	(13,8,35)	(43,13)	(56)
(3,3,2)	(0,0,0,42)	(21,21)	(21,21)	(42)
(4,1,1,1,1)	(0,0,0,15,0,0,20)	(0,35)	(35)	(35)
(4,2,1,1)	(0,14,0,62,0,14)	(90)	(90)	(45,45)

$n = 9.$

partition	divisors at 2	divisors at 3	divisors at 5	divisors at 7
(2,1,1,1,1,1,1,1)	(0,0,0,0,8)	(0,0,1,0,7)	(0,8)	(0,8)
(2,2,1,1,1,1,1)	(0,0,0,0,0,1,0,26)	(0,27)	(0,27)	(8,19)
(2,2,2,1,1,1)	(0,0,0,48)	(0,0,7,41)	(27,21)	(1,47)
(2,2,2,2,1)	(0,0,0,0,0,26,16)	(0,0,41,1)	(8,34)	(42)
(3,1,1,1,1,1,1)	(0,0,0,0,28)	(0,0,7,0,21)	(0,28)	(28)
(3,2,1,1,1,1)	(0,0,0,26,0,1,0,78)	(0,7,63,35)	(105)	(105)
(3,2,2,1,1)	(0,0,0,26,94,2,40)	(162)	(28,134)	(47,115)
(3,2,2,2)	(0,0,26,18,40)	(0,41,36,7)	(1,83)	(84)
(3,3,1,1,1)	(0,0,78,42)	(0,42,43,35)	(120)	(19,101)
(3,3,2,1)	(0,0,0,8,160)	(41,43,63,21)	(134,34)	(168)
(3,3,3)	(0,0,0,42)	(0,21,21)	(21,21)	(42)
(4,1,1,1,1,1)	(0,0,0,56)	(0,21,0,35)	(0,56)	(56)
(4,2,1,1,1)	(0,78,0,1,40,2,68)	(0,189)	(56,133)	(189)
(4,2,2,1)	(0,0,56,160)	(189,27)	(83,133)	(115,101)
(5,1,1,1,1)	(0,0,0,70)	(0,35,0,35)	(70)	(70)

Cf. [J 78, 23.8, 23.9].

Appendix A

Elementary divisors

A.1 Elementary Divisor Theorem for principal ideal domains

We recall a constructive proof of the Elementary Divisor Theorem for a principal ideal domain - constructive under the assumption that the greatest common divisor is given constructively - , since it it used for a considerable number of arguments in the text as well as for almost all computer calculations needed in the progress of this work.

For the history of this theorem, due to S. Smith, see [St 12, sec. 68].

It is not quite consequent not to presuppose any prerequisites on principal ideal domains, but basic knowledge on Dedekind domains further down (S A.4).

Let R be a principal ideal domain.

A nonzero element a is called **irreducible** if $a = bc$ implies $(b) = (1)$ or $(c) = (1)$. Hence a is irreducible iff (a) is maximal, for $(a) \subseteq (b) \subseteq (1)$ implies $(b) = 1$ or $(a) = (b)$. In particular, irreducible elements are **prime**, i.e. they generate a nonzero prime ideal. And conversely, prime elements are irreducible.

In order to see that each element of R allows a product decomposition into irreducible elements, we let the ideal $(a) \neq (0)$ be maximal with respect to the property that a does not allow such a decomposition, under the **assumption** of the existence of a nonzero ideal of this kind. In particular, since a itself is not irreducible there exists a factorization $a = bc$ with $(b), (c) \supset (a)$, which is a **contradiction,** for b and c do have factorizations into irreducibles.

A decomposition into prime ideals is unique, since we may cancel and then use that the elements generate prime ideals.

In order to avoid confusion, we also write $(x_1, \ldots, x_m) =: R(x_1, \ldots, x_m)$ for the ideal of R generated by the x_i's.

Proposition A.1.1 *Let $A = (a_{ij})_{i\in[1,\mu],j\in[1,\nu]}$ be a matrix in $(R)_{\mu\times\nu}$, $\mu, \nu \geq 2$. There exist matrices $S \in \mathrm{SL}_\mu(R)$, $T \in \mathrm{SL}_\nu(R)$ such that SAT is a main diagonal matrix such that the entry at ii divides the entry at jj for $i \leq j$. These entries are determined up to units in R and are called the* **elementary divisors** *of A.*

(i). **Cleaning of the first column.** We **claim** that we can multiply (a_{ij}) from the left by a matrix in $\mathrm{SL}_\mu(R)$, such that (1) the result (a'_{ij}) has $R(a'_{11}) = R(a_{11}, \ldots, a_{\mu 1})$ and (2) $a'_{j1} = 0$ for $j \in [2, \mu]$. If (1) is satisfied, (2) can be achieved by a further multiplication with an $\mathrm{SL}_\mu(R)$-element. Moreover, by construction we will be free to choose the ideal generator a'_{11}.

Let x be a generator of $R(a_{11}, \ldots, a_{\mu 1})$. Write $x = \sum_{j\in[1,\mu]} s_j a_{j1}$ and note that $R(s_1, \ldots, s_\mu) = R(1)$ since we may divide each summand by x. In order to prove our claim it suffices to show that we can realize a tuple $(s_1, \ldots, s_\mu)$ which generates $R(1)$ as a row of an element of $\mathrm{SL}_\mu(R)$. This is true for $\mu = 2$ by

$$1 = s_1 b_1 + s_2 b_2 = \det\begin{pmatrix} s_1 & s_2 \\ -b_2 & b_1 \end{pmatrix}.$$

By associativity of the greatest common divisor - obtained using unique factorization into prime elements - it suffices to apply the transpose of the assertion just shown, viz. the possible multiplication of a two element row with an $SL_2(R)$-element from the right such that the gcd and a zero arise, to the last two entries of our row $(s_1, \ldots, s_\mu)$, and to apply induction.

(i') **Cleaning of the first row,** transposed to (i).

(ii). For $(x) = \prod_{(p)}(p)^{\xi_p}$, let $V(x) := \sum_{(p)} \xi_p$ be the **total value** of (x), where (p) runs over the nonzero prime ideals. Cleaning of the first column either strictly decreases the value of the top left entry or it is possible with the top left entry as pivot element, so that in particular the first row is not affected. Hence the process of **alternating** the cleaning of the first column and the cleaning of the first row ends up with a cleaned first row **and** a cleaned first column.

(iii). In case that now the top left entry does not divide every entry, we clean the column and the row the entry which is not divided sits in, without affecting the first column or the first row, which are already cleaned. Now we add that column to the first column, which now contains two nonzero elements whose gcd has a total value strictly smaller than the top left entry. Entering the (i, i')-loop again at this stage therefore yields a finite algorithm, resulting in a matrix with cleaned first row and column and top left entry dividing all the others.

(iv). Iterating, i.e. applying (i-iii) to the respective remaining matrix to be diagonalized, we obtain a diagonal matrix $D = (d_{ij}) = SAT$, $S \in \mathrm{SL}_\mu(R)$, $T \in \mathrm{SL}_\nu(R)$ with diagonal entries dividing each other consecutively, $d_{ii} \mid d_{i+1,i+1}$.

Uniqueness of the elementary divisors follows by regarding A as a morphism from R^μ to R^ν. We write the nonzero (d_{ii})'s as $(d_{ii}) = \prod_{(p)}(p)^{\alpha_{p,i}}$. Localizing at (p), the cokernel of $A_{(p)}$ is isomorphic to $(\bigoplus_i R/(p^{\alpha_{p,i}})) \oplus R^k$, $k \geq 0$. Both summands, and so in particular the rank k of the torsionfree summand, are independent of the chosen bases of R^μ and R^ν. Moreover, denoting the torsionfree summand by $T_{(p)}$, we have for $j \geq 1$

$$\dim_{R/(p)} p^{j-1}T_{(p)}/p^jT_{(p)} = \#\{i \mid \alpha_{p,i} \geq j\},$$

so that the $\alpha_{p,i}$'s are determined, which in turn determine the elementary divisors (d_{ii}).

Consider the proof in the particular case of $A \in \mathrm{SL}_\mu(R)$. Note that our construction uses only $\mathrm{SL}_2(R)$-operations, embedded in various canonical ways into $\mathrm{SL}_\mu(R)$. Since

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and since we have, for $u \in R^*$,

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that we are free to choose ideal generators for (d_{ii}) for $i \in [1, \mu-1]$, the implication ($\mathrm{SL}_2(R)$ is generated by elementary matrices $\Longrightarrow$ $\mathrm{SL}_\mu(R)$ is generated by elementary matrices) follows. Its condition is met for an euclidean domain, for we can use division with remainder to reduce the first row to $(1\ 0)$ via column operations. In particular, $\mathrm{SL}_\mu(\mathbf{Z})$ is generated by elementary matrices.

A.2 Chinese Remainder Theorem, version SL

Let R be a Dedekind domain. By $\mathfrak{p}, \mathfrak{q}$ we denote nonzero prime ideals of R.

Lemma A.2.1 ([1]) *Let $\mathfrak{b}$ be a nonzero ideal in R, let $m \geq 1$. Then the residue class morphism*

$$\mathrm{SL}_m(R) \longrightarrow \mathrm{SL}_m(R/\mathfrak{b})$$

is surjective.

The problem is that we do not know whether $\mathrm{SL}_m(R/\mathfrak{b})$ is generated by elementary matrices, i.e. by matrices with main diagonal constant 1 and a single nonzero non main diagonal entry (cf. S A.1).

[1] S. König helped to simplify the proof.

Note that for a direct product of commutative rings A and B we have $\mathrm{SL}_m(A\times B) \xrightarrow{\sim} \mathrm{SL}_m(A)\times\mathrm{SL}_m(B)$, so that the target of our morphism may be replaced by $\prod_{\mathfrak{p}} \mathrm{SL}_m(R/\mathfrak{p}^{\beta_\mathfrak{p}})$ by the Chinese Remainder Theorem, where $\mathfrak{b} = \prod_{\mathfrak{p}} \mathfrak{p}^{\beta_\mathfrak{p}}$, $\mathfrak{p}$ running over a finite set.

Suppose given $M \in \mathrm{SL}_m(R/\mathfrak{b})$. Choose entrywise an inverse image of $M \in \mathrm{SL}(R/\mathfrak{p}^{\beta_\mathfrak{p}})$ in $\mathrm{GL}_m(R_\mathfrak{p})$ and write it by the Elementary Divisor Theorem (A.1.1) as a product of elementary matrices, and one extra factor in between being a main diagonal matrix D having as entries 1's except for a unit of $R_\mathfrak{p}$ in the lower right corner. Changing the inverse image without changing its image in $\mathrm{SL}_m(R/\mathfrak{p}^{\beta_\mathfrak{p}})$ we may assume that these elementary matrices lie in $\mathrm{SL}_m(R)$. Moreover, since $\det D \equiv_{\mathfrak{p}^{\beta_\mathfrak{p}}} 1$ we may assume D to vanish. Furthermore, by the Chinese Remainder Theorem we may assume the non main diagonal entries of the elementary matrices occurring in our product to vanish modulo $\mathfrak{q}^{\beta_\mathfrak{q}}$ for $\mathfrak{q} \neq \mathfrak{p}$. Hence the product over $\mathfrak{p}$ of our products of elementary matrices is in $\mathrm{SL}_m(R)$ and maps to $M \in \mathrm{SL}_m(R/\mathfrak{p}^{\beta_\mathfrak{p}})$ for all $\mathfrak{p}$, and therefore to $M \in \mathrm{SL}_m(R/\mathfrak{b})$.

A.3 Some Ext-preliminaries

Let $R \longrightarrow S$ be a flat morphism of commutative noetherian rings. Denote $S(-) := S \otimes_R -$. Let A be a R-algebra, finitely generated as an R-module. Let X and Y be finitely generated A-modules.

Lemma A.3.1 *Let $i \geq 0$. $\mathrm{Ext}^i_A(X,Y)$ is a finitely generated R-module.*

Since R is noetherian, we may resolve X with projective modules which are finitely generated as R-modules.

Lemma A.3.2 *Let $i \geq 0$. The natural transformation*

$$\mathrm{Ext}^i_A(X,Y) \xrightarrow{S(-)} \mathrm{Ext}^i_{SA}(SX,SY).$$

induces a natural isomorphism

$$S\mathrm{Ext}^i_A(X,Y) \xrightarrow{\sim} \mathrm{Ext}^i_{SA}(SX,SY).$$

Using flatness of $R \longrightarrow S$ we pass to the level of projective resolutions. Now

$$S\mathrm{Hom}_A(P,Y) \longrightarrow \mathrm{Hom}_{SA}(SP,SY)$$

is an isomorphism for P finitely generated projective, since the assertion is true for $P = A$ and extends by naturality and additivity to the required generality.

Corollary A.3.3 *In case R is an integral domain with field of fractions K and KA is semisimple, $\mathrm{Ext}^i_A(X,Y)$ is a finitely generated torsion module over R for $i \geq 1$.*

A.4 The Steinitz-Chevalley Elementary Divisor Theorem for Dedekind domains

For us, this generalization of (A.1.1) is mainly of importance because the structure of finitely generated torsion modules over Dedekind domains ensues. We follow CHEVALLEY's artistic proof [C 36, App. II, Th. 1], for whose understanding [CR 62] was helpful. CHEVALLEY's assertion is more general than STEINITZ', the latter treating only the case of M being finitely generated free over R [St 12, §41]. For further historical comments cf. [C 36].

Let R be a Dedekind domain with field of fractions K. By $\mathfrak{p},\mathfrak{q}$ we denote nonzero prime ideals of R. Writing the expression $x/y \in K$ it is understood that $x, y \in R$ and $y \neq 0$. Let M be a finitely generated torsion free R-module, let $N \subseteq M$ be a submodule.

The following basic facts on Dedekind domains will be used, cf. [S 68].

(A) A fractional ideal in K has a unique decomposition as a product of integral powers of prime ideals.

(B) $R_{\mathfrak p}$ is a discrete valuation ring.

Lemma A.4.1 *Let X, Y be finitely generated R-modules.*

(i) We have a natural isomorphism

$$({}_{R}(X,Y))_{\mathfrak p} \xrightarrow{\sim} {}_{R_{\mathfrak p}}(X_{\mathfrak p}, Y_{\mathfrak p}).$$

(ii) $X = 0$ iff $X_{\mathfrak p} = 0$ for all $\mathfrak p$.

(iii) A sequence is exact iff it is exact at all $\mathfrak p$.

(iv) X is projective iff $X_{\mathfrak p}$ is projective for all $\mathfrak p$.

(v) An ideal $\mathfrak a \subseteq R$ is a projective R-module.

(i). The reader might amuse himself in giving a direct proof, alternatively to (A.3.2).

(ii). $X_{\mathfrak p} = 0$ is equivalent to the annihilator of X being not contained in $\mathfrak p$.

(iii). Regard the homology.

(iv). Use (A.3.2) or use (i) and (iii).

(v). $\mathfrak a_{\mathfrak p}$ is a principal ideal.

Definition A.4.2 *The* **pure closure of** N **in** M *is defined to be*

$$\bar N := \{m \in M \mid \text{there is a } y \in R\backslash\{0\} \text{ such that } ym \in N\}.$$

$M/\bar N$ is torsionfree, whence the name.

Lemma A.4.3 *M is isomorphic to a direct sum of ideals of R. In particular, M is projective.*

Let $m \in M$ be a nonzero element. Let

$$U := \{u \in M \mid \text{there is a } x/y \in K \text{ such that } yu = xm\}$$

be the pure closure of the submodule generated by m.

By uniqueness of x/y for a given $u \in U$ and conversely, U is isomorphic to the fractional ideal

$$\mathfrak u := \{x/y \in K \mid \text{there is a } u \in U \text{ such that } yu = xm\},$$

which **is** a fractional ideal via this isomorphism, M being noetherian. A fractional ideal, however, is isomorphic to, say, the ideal obtained by multiplying with a common denominator of its generators.

By induction on the rank of M and in view of (A.4.1 v) it remains to be remarked that M/U is torsionfree.

Corollary A.4.4 *The pure closure of a submodule of M is a direct summand of M.*

Lemma A.4.5 (principal ideal approximation) *Suppose given a nonzero ideal $\mathfrak a \subseteq R$. Fix a finite set of nonzero prime ideals $\{\mathfrak p_1, \dots, \mathfrak p_k\}$. There is an element $x \in \mathfrak a$ with*

$$v_{\mathfrak p_i}(x) = v_{\mathfrak p_i}(\mathfrak a)$$

for $i \in [1, k]$.

In particular, choosing two elements in this manner, the set of primes for the first being the set of prime ideal divisors of $\mathfrak a$, the set of primes for the second being the set prime ideal divisors of $\mathfrak a$ united with the set of prime ideal divisors of the first element, we see that each ideal of R is generated by two elements.

We may assume the prime divisors of $\mathfrak a$ to be contained in $\{\mathfrak p_1, \dots, \mathfrak p_k\}$. Regard the diagram

$$\begin{array}{ccc} R/\mathfrak{a} & \xrightarrow{\sim} & R/\mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{a})} \times \cdots \times R/\mathfrak{p}_k^{v_{\mathfrak{p}_k}(\mathfrak{a})} \\ \uparrow & & \uparrow \\ R/\mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_k & \xrightarrow{\sim} & R/\mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{a})+1} \times \cdots \times R/\mathfrak{p}_k^{v_{\mathfrak{p}_k}(\mathfrak{a})+k} \end{array}$$

and choose elements $x_i \in \mathfrak{p}_i^{v_{\mathfrak{p}_i}(\mathfrak{a})} \backslash \mathfrak{p}_i^{v_{\mathfrak{p}_i}(\mathfrak{a})+1}$ for $i \in [1, k]$.
NB in case $\mathfrak{a}$ is not a principal ideal, x has to have valuations > 0 away from this set, as long as it includes the prime divisors of $\mathfrak{a}$.

Lemma A.4.6 *Suppose given nonzero ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$. Let $x \in \mathfrak{a}$ be an element having the same valuations as $\mathfrak{a}$ at all prime ideal divisors of $\mathfrak{b}$ (A.4.5). We obtain an isomorphism*

$$\begin{array}{ccc} R/\mathfrak{b} & \xrightarrow{\sim} & \mathfrak{a}/\mathfrak{a}\mathfrak{b} \\ 1 & \longrightarrow & x. \end{array}$$

Localizing at a prime ideal not occurring in $\mathfrak{b}$, we obtain $0 \xrightarrow{\sim} 0$, localizing at a prime ideal $\mathfrak{p}$ occurring in $\mathfrak{b}$, we obtain the isomorphism, $\pi \in R$ denoting an element with $v_\mathfrak{p}(\pi) = 1$,

$$\begin{array}{ccc} R_\mathfrak{p}/(\pi^{v_\mathfrak{p}(\mathfrak{b})}) & \xrightarrow{\sim} & (\pi^{v_\mathfrak{p}(\mathfrak{a})})/(\pi^{v_\mathfrak{p}(\mathfrak{a})+v_\mathfrak{p}(\mathfrak{b})}) \\ 1 & \longrightarrow & \pi^{v_\mathfrak{p}(\mathfrak{a})} u_\mathfrak{p}, \end{array}$$

$u_\mathfrak{p}$ being a unit in $R_\mathfrak{p}$ so that the result follows from (A.4.1 iii).

Corollary A.4.7 *Keep the notation from (A.4.6). We obtain*

$$\mathfrak{a} \oplus \mathfrak{b} \simeq R \oplus \mathfrak{a}\mathfrak{b}.$$

In particular, letting $\mathfrak{b} = \mathfrak{a}^{-1}(y)$ for some $0 \neq y \in \mathfrak{a}$, we see that each ideal of R is generated by two elements.
The pushout (A.4.6)

$$\begin{array}{ccccc} \mathfrak{a}\mathfrak{b} & \longrightarrow & \mathfrak{a} & & x \\ \uparrow & \lrcorner & \uparrow & & \uparrow \\ \mathfrak{b} & \longrightarrow & R & & 1 \end{array}$$

is also a pullback. Its short exact diagonal sequence splits by projectivity of $\mathfrak{a}$ (A.4.1 v).

Theorem A.4.8 (Steinitz-Chevalley Elementary Divisor Theorem)

(i) There is an isomorphism

$$M \xrightarrow{\sim} \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_\alpha$$

which restricts to an isomorphism

$$N \xrightarrow{\sim} \mathfrak{b}_1\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{b}_\alpha\mathfrak{a}_\alpha$$

for suitable ideals $\mathfrak{a}_i, \mathfrak{b}_i \subseteq R$, such that $\mathfrak{b}_i \supseteq \mathfrak{b}_{i+1}$ for $i \in [1, \alpha - 1]$.

(ii) The ideals $\mathfrak{b}_i$, called the **elementary divisors** *of the inclusion $N \subseteq M$, are independent of the choice of the isomorphism.*

We may restrict ourselves to the case $\bar{N} = M$ by (A.4.4, A.4.3).
(i). Choose $0 \neq r \in R$ such that $rM \subseteq N \subseteq M$. Let $\mathfrak{f}_r := \mathrm{Ann}_R(N/rM)$, let $\mathfrak{e} := r\mathfrak{f}_r^{-1}$. Since $r \in \mathfrak{f}_r$, $\mathfrak{e}$ is an ideal in R.

(In case **given** M and N as in the result, we obtain $\mathfrak{e} = \mathfrak{b}_1$.)
Note that

$$N \subseteq \mathfrak{e}M,$$

and that $\mathfrak{e}$ is contained in any ideal with this property (showing $\mathfrak{e}$ to be independent of r). In particular, we have $N \not\subseteq \mathfrak{q}\mathfrak{e}M$ for any $\mathfrak{q}$.
Decompose

$$(r) = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(r)}$$

into prime ideals. For a prime ideal $\mathfrak{p}$ occurring in r, we choose

$$n_{\mathfrak{p}} \in N \backslash \mathfrak{p}\mathfrak{e}M.$$

By the Chinese Remainder Theorem, for $\mathfrak{p}$ occurring in r we choose a $\nu_{\mathfrak{p}} \in R$ such that for $\mathfrak{q}$ occurring in r we have valuations

$$v_{\mathfrak{q}}(\nu_{\mathfrak{p}}) = \begin{cases} 0 & \text{for} \quad \mathfrak{q} = \mathfrak{p} \\ 1 & \text{for} \quad \mathfrak{q} \neq \mathfrak{p} \end{cases}$$

Since $\nu_{\mathfrak{p}} n_{\mathfrak{p}} \notin \mathfrak{p}\mathfrak{e}M$ (invert $\nu_{\mathfrak{p}}$ in $R/\mathfrak{p}$) whereas $\nu_{\mathfrak{p}} n_{\mathfrak{p}} \in \mathfrak{q}N \subseteq \mathfrak{q}\mathfrak{e}M$ for $\mathfrak{q} \neq \mathfrak{p}$, we obtain

$$n := \sum_{\mathfrak{p} \text{ occ. in } r} \nu_{\mathfrak{p}} n_{\mathfrak{p}} \in N \backslash \bigcup_{\mathfrak{p} \text{ occ. in } r} \mathfrak{p}\mathfrak{e}M.$$

(In case **given** M and N as in the result, we find such an element n e.g. as an element of $\mathfrak{b}_1\mathfrak{a}_1$ whose valuations coincide with those of $\mathfrak{b}_1\mathfrak{a}_1$ at all prime ideals occurring in r.)
The R-submodules of K

$$\begin{array}{rcl} \mathfrak{n} & := & \{x/y \in K \mid xn \in yN\} \\ \mathfrak{m} & := & \{x/y \in K \mid xn \in yM\} \end{array}$$

are fractional ideals, since there are injections $\mathfrak{n} \xrightarrow{\sim} N_1 \subseteq N$ and $\mathfrak{m} \xrightarrow{\sim} M_1 \subseteq M$ by uniqueness of, say, m in $xn = ym$ for a given $x/y \in K$. We **claim** that

$$\mathfrak{m}\mathfrak{e} = \mathfrak{n},$$

which then implies that

$$M_1\mathfrak{e} = N_1.$$

$\mathfrak{m} \supseteq \mathfrak{n}\mathfrak{e}^{-1}$ follows by $N \subseteq \mathfrak{e}M$, for given $xn = yem$, $x/y \in K$, $n \in N$, $m \in M$, $e \in \mathfrak{e}$, and given $u/v \in K$ with $ue/v \in R$, we get $(xu)n = yuem = (yv)(ue/v)m$, hence $(x/y)(u/v) \in \mathfrak{m}$.
Let $\mathfrak{c} := (\mathfrak{m}\mathfrak{e})^{-1}\mathfrak{n} \subseteq R$. Note that $rM \subseteq N$ implies $r\mathfrak{m} \subseteq \mathfrak{n}$, so that

$$(r) \subseteq \mathfrak{n}\mathfrak{m}^{-1} = \mathfrak{c}\mathfrak{e},$$

therefore a prime ideal factor $\mathfrak{p}$ of $\mathfrak{c}$ occurs in r, if existent. But in this case we had, since

$$1 \in \mathfrak{n} = \mathfrak{m}\mathfrak{e}\mathfrak{c} \subseteq \mathfrak{m}\mathfrak{e}\mathfrak{p},$$

i.e.

$$1 = \sum (x_i/y)e_i p_i,$$

where $x_i/y \in K$ with $x_i n = y m_i$ for some $m_i \in M$, $e_i \in \mathfrak{e}$, $p_i \in \mathfrak{p}$, that

$$yn = \sum x_i e_i p_i n = y(\sum e_i p_i m_i),$$

thus

$$n = \sum e_i p_i m_i \in \mathfrak{e}\mathfrak{p}M,$$

which we have excluded however.
Since M_1 is the pure closure in M of the submodule generated by n, it has, by (A.4.4), a complement

$$M = M_1 \oplus M_2.$$

We **claim** that

$$N = N_1 \oplus (N \cap M_2).$$

For to show that N_1 and $N \cap M_2$ generate N, we regard an element $m_1 + m_2 \in N$, where $m_1 \in M_1$ and $m_2 \in M_2$. Now $m_1 + m_2 \in \mathfrak{e}M = \mathfrak{e}M_1 \oplus \mathfrak{e}M_2$ implies $m_1 \in \mathfrak{e}M_1 = N_1$ (this equality being crucial), thus also $m_2 \in N$.

By induction on the rank of M we may assume given the decomposition

$$\begin{array}{rcl} M_2 & \xrightarrow{\sim} & \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_\alpha \\ N \cap M_2 & \xrightarrow{\sim} & \mathfrak{b}_2\mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{b}_\alpha\mathfrak{a}_\alpha, \end{array}$$

the second isomorphism being the restriction of the first, with $\mathfrak{b}_i \supseteq \mathfrak{b}_{i+1}$ for $i \in [2, \alpha - 1]$. Letting $\mathfrak{b}_1 := \mathfrak{e}$ it remains to be shown in case $\alpha \geq 2$ that $\mathfrak{e} \supseteq \mathfrak{b}_2$. But $N \subseteq \mathfrak{e}M$ implies $\mathfrak{b}_2\mathfrak{a}_2 \subseteq \mathfrak{e}\mathfrak{a}_2$.

(ii). In view of (A.4.6) we may write the quotient M/N as

$$M/N \xrightarrow{\sim} R/\mathfrak{b}_1 \oplus \cdots \oplus R/\mathfrak{b}_\alpha$$

and compare to

$$M/N \xrightarrow{\sim} R/\mathfrak{b}'_1 \oplus \cdots \oplus R/\mathfrak{b}'_{\alpha'},$$

where $\mathfrak{b}_i \supseteq \mathfrak{b}_{i+1}$ for $i \in [1, \alpha - 1]$ and $\mathfrak{b}'_i \supseteq \mathfrak{b}'_{i+1}$ for $i \in [1, \alpha' - 1]$. Appending trivial quotients, we may assume $\alpha = \alpha'$. We have to prove $v_\mathfrak{p}(\mathfrak{b}_i) = v_\mathfrak{p}(\mathfrak{b}'_i)$ for all prime ideals $\mathfrak{p}$. However, these valuations are determined by, $\pi \in R$ denoting an element with $v_\mathfrak{p}(\pi) = 1$,

$$\dim_{R_\mathfrak{p}/(\pi)} \left[\pi^{i-1}(M/N)_\mathfrak{p}/\pi^i(M/N)_\mathfrak{p}\right] = \#\{j \in [1, \alpha] \mid v_\mathfrak{p}(\mathfrak{b}_j) \geq i\}.$$

Corollary A.4.9 *A finitely generated R-module X can be written as*

$$X \xrightarrow{\sim} R/\mathfrak{b}_1 \oplus \cdots \oplus R/\mathfrak{b}_s \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_t,$$

where $\mathfrak{b}_i$ and $\mathfrak{a}_j$ are ideals of R.

Apply (A.4.8) to the inclusion of a kernel of a surjection from a direct sum of copies of R to X and consider (A.4.6).

Corollary A.4.10 *A finitely generated R-module has a decomposition*

$$X = X_0 \oplus \bigoplus_\mathfrak{p} X_\mathfrak{p}$$

with X_0 torsionfree, $\mathrm{Ann}_R X_\mathfrak{p} = \mathfrak{p}^{i_\mathfrak{p}}$, $X_\mathfrak{p} \neq 0$ only for a finite number of $\mathfrak{p}$'s.

We call $X_\mathfrak{p}$ the **$\mathfrak{p}$-part** *and $\bigoplus_{\mathfrak{q} \neq \mathfrak{p}} X_\mathfrak{q}$ the* **$\mathfrak{p}'$-part** *of X.*

Moreover, the $\mathfrak{p}$-parts $X_\mathfrak{p}$ of X are uniquely determined as submodules. A morphism $X \xrightarrow{f} Y$ maps $X_\mathfrak{p}$ into $Y_\mathfrak{p}$. The torsionfree part X_0 is determined up to isomorphism as the quotient of X by its torsion part.

Existence of such a decomposition follows from (A.4.9) and the Chinese Remainder Theorem. Uniqueness and preservation of the $\mathfrak{p}$-part by morphisms follow from the characterization

$$X_\mathfrak{p} = \{x \in X \mid \mathrm{Ann}_R x \text{ is a power of } \mathfrak{p}\}.$$

Corollary A.4.11 *In case R is the ring of algebraic integers in a number field K, a finitely generated R-module X with $KX = 0$ is finite.*

By (A.4.9) it suffices to show that $R/\mathfrak{b}$ is finite for an ideal $\mathfrak{b} \subseteq R$, which follows by $\mathfrak{b} \cap \mathbf{Z} \neq 0$ (consider e.g. the constant term of a minimal polynomial over $\mathbf{Z}$ of an element in $\mathfrak{b}$).

Remark A.4.12

Let R be the ring of algebraic integers in a number field K, let $S \subseteq R$ be a multiplicative subset. Since $S^{-1}R/R$ is torsion, $S^{-1}R$ is not finitely generated over R as long as $S^{-1}R/R$ is infinite (A.4.11). Hence in this case, $S^{-1}R$ is a fortiori not finitely generated. In particular, let $x \in R\backslash\mathfrak{p}$ be a nonunit with, say, $v_\mathfrak{q}(x) \geq 1$. The difference of two elements of $\{x^{-1}, x^{-2}, \ldots\} \subseteq R_x$ has a negative valuation at $\mathfrak{q}$, thus R_x/R is infinite and therefore R_x is not finitely generated over R. A fortiori, K is not finitely generated over R. Since given $\mathfrak{p}$ there exists a nonunit in $R\backslash\mathfrak{p}$, also $R_\mathfrak{p}$ is not finitely generated over R.

Remark A.4.13 We justify the name Elementary Divisor Theorem for (A.4.8) by deducing the Elementary Divisor Theorem for principal ideal domains from (A.4.8).

Let R be a principal ideal domain. Let $A \in (R)_{\mu\times\nu}$ be a matrix over R. Let $M = R^\nu$, regarded as a row, let $N \subseteq M$ be its submodule generated by the rows of A. By (A.4.8) we obtain isomorphisms, the latter being a restriction of the former,

$$\begin{array}{ccc} M & \xleftarrow[\sim]{f} & R^\nu \\ N & \xleftarrow{\sim} & \bigoplus_{i=1}^{\nu}(b_i). \end{array}$$

Denote by ε_i the standard basis of R^ν. Let

$$(\varepsilon_i)f =: \sum_{j=1}^{\nu} \xi_{ij}\varepsilon_j,$$

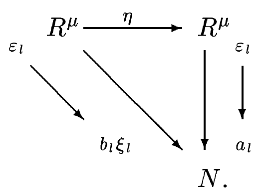
let $\sum_m \xi_{im}\bar{\xi}_{mj} = \partial_{ij}$. There is a matrix $\eta = (\eta_{ij}) \in (R)_\mu$ such that for $m \in [1,\nu]$

$$\sum_{l=1}^{\mu} \eta_{il}a_{lm} = b_i\xi_{im},$$

whence

$$\sum_{l,m} \eta_{il}a_{lm}\bar{\xi}_{mj} = \sum_m b_i\xi_{im}\bar{\xi}_{mj} = b_i\partial_{ij}.$$

We **claim** that η may be chosen to be invertible. η has to make the following diagram commute, in which the downwards morphisms are split epimorphisms



Replacing the morphisms $R^\mu \longrightarrow N$ isomorphically by canonical projections $N\oplus N' \xrightarrow{\binom{1}{0}} N$ (i.g. by different substitution isomorphisms) we obtain as substituted condition on the horizontal morphism that it be of the form

$$\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right),$$

whence the claim.

Example A.4.14 Given an ideal $\mathfrak{a} \subseteq R$. We shall construct a matrix whose rows generate a submodule of the free module on $(1\;0)$ and $(0\;1)$ having elementary divisors $\mathfrak{a}$ and $(y)\mathfrak{a}^{-1}$, where $y \in \mathfrak{a}^2$. Note that $\mathfrak{a}^{-1}(y) \subseteq \mathfrak{a}$. Let $x \in \mathfrak{a}^{-1}(y)$ such that $v_\mathfrak{p}(x) = v_\mathfrak{p}(\mathfrak{a}^{-1}y)$ for all prime ideal divisors $\mathfrak{p}$ of $\mathfrak{a}$ (A.4.5). Choose $s \in \mathfrak{a}(y)^{-1}$, $t \in \mathfrak{a}$ such that

$$sx + t = 1,$$

existent by surjectivity of $\mathfrak{a}\oplus\mathfrak{a}(y)^{-1} \xrightarrow{\binom{1}{x}} R$, which can be seen locally. Letting matrices act on the right,

$$R\oplus(y) \xrightarrow{\left(\begin{smallmatrix} t & x \\ -s & 1 \end{smallmatrix}\right)} \mathfrak{a}\oplus\mathfrak{a}^{-1}(y)$$

is inverted by $\left(\begin{smallmatrix} 1 & -x \\ s & t \end{smallmatrix}\right)$. Therefore the matrix

$$R\oplus R \xrightarrow{\left(\begin{smallmatrix} t & x \\ -ys & y \end{smallmatrix}\right)} R\oplus R$$

gives the required example.

Appendix B

Two tools

The lattice tensor product over an order is defined as the torsion free part of the tensor product. Moreover, we introduce the Higman ideal, i.e. of the annihilator ideal of Ext^1.

B.1 The lattice tensor product

Let R be a Dedekind domain with field of fractions K (to which we refer by 'rational'). Let Λ and Δ be orders over R, i.e. R-algebras which are finitely generated projective as modules over R. By $\mathfrak{p}, \mathfrak{q}$ we denote nonzero prime ideals of R. p denotes a prime element of R except stated otherwise.
A Λ-lattice is a Λ-module which is finite projective over R. A Λ-Δ-bilattice over R is a $\Lambda \otimes_R \Delta^o$-lattice. We abbreviate $K \otimes_R -$ by $K(-)$. A simple lattice is a lattice X such that KX is a simple $K\Lambda$-module. A pure monomorphism of Λ-lattices has a torsionfree quotient, a full monomorphism has a torsion quotient.

Example B.1.1 Let $R = \mathbf{Z}$, let $\Lambda = \{x \times y \mid x \equiv_2 y\} \subseteq \mathbf{Z} \times \mathbf{Z}$. Then

$$0 \neq t := (0 \times 1) \otimes (1 \times 0) \in (\mathbf{Z} \times \mathbf{Z}) \otimes_\Lambda (\mathbf{Z} \times 0),$$

whereas $2t = 0$. For to see this, we regard the Λ-bilinear map

$$\begin{array}{ccccc} (\mathbf{Z} \times \mathbf{Z}) & \times & (\mathbf{Z} \times 0) & \longrightarrow & \mathbf{Z}/2 \\ (a \times b) & \times & (c \times 0) & \longrightarrow & bc \end{array}$$

which sends t to 1.

Lemma B.1.2 *The inclusion*

$$\Lambda\text{-lat} \longrightarrow \Lambda\text{-mod}$$

has a left adjoint, denoted by $(\tilde{-})$ *and called* **lattification**. *Which is compatible with the forgetful functors from Λ to R.*
Note that for X in Λ-mod there exists a short exact sequence

$$0 \longrightarrow X' \longrightarrow X \xrightarrow{\varepsilon} \tilde{X} \longrightarrow 0,$$

split over R (A.4.10), with X' torsion over R and $\tilde{X}$ torsionfree over R. It follows that this sequence is functorial, that $X \xrightarrow{\varepsilon} \tilde{X}$ is the unit and $Y =\!= Y$, $Y \in \Lambda$-lat, is the counit of the required adjunction.

Definition B.1.3 *Let, for X a left Λ-lattice and Y a right Λ-lattice,*

$$X \tilde{\otimes}_\Lambda Y := (X \otimes_\Lambda Y)\tilde{}$$

be their **lattice tensor product**, *or just* **tensor product** *as long as no confusion may arise. Accordingly, its elements are generated by elements of the form*

$$x\tilde{\otimes}y := (x \otimes y)\varepsilon.$$

It ensues that the lattice tensor product is an addititve functor in both variables.

Lemma B.1.4 *Let X a left Λ-lattice and Y a right Λ-lattice. Let U be an R-lattice. The Λ-bilinear maps*

$$X \times Y \xrightarrow{\Phi} U,$$

i.e. Φ being R-linear in both variables and $\Phi(x\lambda, y) = \Phi(x, \lambda y)$, are in bijection to the R-linear maps

$$X\tilde{\otimes}_\Lambda Y \xrightarrow{\Phi'} U,$$

where Φ' arises from Φ via the restriction of the induced map on the ordinary tensor product.
Note that

$${}_R(X \otimes_\Lambda Y, U) \xrightarrow{\sim} {}_R(X\tilde{\otimes}_\Lambda Y, U)$$

via restriction (B.1.2).

Lemma B.1.5 *Let X be a Λ-Δ-bilattice over R, let Y be a left Δ-lattice. $X\tilde{\otimes}_\Delta Y$ is a left Λ-lattice via*

$$\begin{array}{ccccccc} X & \tilde{\otimes}_\Lambda & Y & \xrightarrow{(\lambda(-))^{\sim}} & X & \tilde{\otimes}_\Lambda & Y \\ x & \tilde{\otimes} & y & \longrightarrow & (\lambda x) & \tilde{\otimes} & y. \end{array}$$

Lemma B.1.6 *Let X be a right Λ-lattice, let Y be a Λ-Δ-bilattice over R, let Z be a left Δ-lattice. Then there is an isomorphism*

$$\begin{array}{ccccccccccc} (X & \tilde{\otimes}_\Lambda & Y) & \tilde{\otimes}_\Delta & Z & \xrightarrow{\sim} & X & \tilde{\otimes}_\Lambda & (Y & \tilde{\otimes}_\Delta & Z) \\ (x & \tilde{\otimes} & y) & \tilde{\otimes} & z & \longrightarrow & x & \tilde{\otimes} & (y & \tilde{\otimes} & z) \\ (x & \tilde{\otimes} & y) & \tilde{\otimes} & z & \longleftarrow & x & \tilde{\otimes} & (y & \tilde{\otimes} & z) \end{array}$$

over R.
By (B.1.4, B.1.5), the first map is well defined. Dito the second.

Lemma B.1.7 *Let X be a Λ-Δ-bilattice over R. Then*

$$\Lambda\text{-lat} \xrightarrow{{}_\Lambda(X,-)} \Delta\text{-lat}$$

has a left adjoint

$$\Delta\text{-lat} \xrightarrow{X\tilde{\otimes}_\Delta -} \Lambda\text{-lat}.$$

In particular, we see that $X\tilde{\otimes}_\Delta -$ is right exact.
For Y a left Δ-lattice and Z a left Λ-lattice we have the isomorphisms (B.1.2)

$${}_\Delta(Y, {}_\Lambda(X, Z)) \xrightarrow{\sim} {}_\Lambda(X \otimes_\Delta Y, Z) \xrightarrow{\sim} {}_\Lambda(X\tilde{\otimes}_\Delta Y, Z).$$

Lemma B.1.8 *Let X be a right Λ-lattice, let Y be a left Λ-lattice, let $S \subseteq R$ be a multiplicative subset. Then*

$$\begin{array}{ccccccc} S^{-1}(X & \tilde{\otimes}_\Lambda & Y) & \xrightarrow{\sim} & S^{-1}X & \tilde{\otimes}_{S^{-1}\Lambda} & S^{-1}Y \\ (1/s)(x & \tilde{\otimes} & y) & \longrightarrow & (x/s) & \tilde{\otimes} & y \\ (1/(st))(x & \tilde{\otimes} & y) & \longleftarrow & (x/s) & \tilde{\otimes} & (y/t). \end{array}$$

Lemma B.1.9 *Let X be a right Λ-lattice, let $Y \longrightarrow Z$ be a injective morphism of left Λ-lattices. Then*

$$X\tilde{\otimes}_\Lambda Y \longrightarrow X\tilde{\otimes}_\Lambda Z$$

is injective. It preserves monomorphisms, pure epimorphisms and cokernels (taken in Λ-lat*).*
We regard the commutative diagram with vertical and upper injections (B.1.8)

$$\begin{array}{ccc}
KX \otimes_{K\Lambda} KY & \longrightarrow & KX \otimes_{K\Lambda} KZ \\
\wr\uparrow & & \uparrow\wr \\
K(X \tilde{\otimes}_\Lambda Y) & \longrightarrow & K(X \tilde{\otimes}_\Lambda Z) \\
\uparrow & & \uparrow \\
X \tilde{\otimes}_\Lambda Y & \longrightarrow & X \tilde{\otimes}_\Lambda Z
\end{array}$$

Example B.1.10 (dangerous bend) *In general, $X\tilde{\otimes}_\Lambda -$ does not preserve short exact sequences. In particular, it does not necessarily preserve pure monomorphisms.*

Keep the notation of (B.1.1). The short exact sequence

$$0 \longrightarrow 2\mathbf{Z} \times 0 \longrightarrow \Lambda \longrightarrow 0 \times \mathbf{Z} \longrightarrow 0$$

is mapped under $\Gamma\tilde{\otimes}_\Lambda -$ to

$$2\mathbf{Z} \times 0 \longrightarrow \Gamma \longrightarrow 0 \times \mathbf{Z} \longrightarrow 0.$$

Lemma B.1.11 *Let X be a right Λ-sublattice of $K\Lambda$. Let Y be a left Λ-lattice. Let $XY \subseteq KY$ be additively generated by products of the form xy, $x \in X, y \in Y$. Then*

$$\begin{array}{ccccc}
X & \tilde{\otimes}_\Lambda & Y & \xrightarrow{\sim} & XY \\
x & \tilde{\otimes} & y & \longrightarrow & xy.
\end{array}$$

We have to show that the map is injective. Regard the diagram with vertical and upper injections

$$\begin{array}{ccc}
KX \otimes_{K\Lambda} KY & \longrightarrow & K\Lambda \otimes_{K\Lambda} KY \\
\uparrow & & \uparrow \\
X \tilde{\otimes}_\Lambda Y & \longrightarrow & XY
\end{array}$$

Lemma B.1.12 *Let $\Lambda \subseteq \Delta$ be a full inclusion of R-orders, let X be a left Λ-lattice. Then*

$$\begin{array}{ccc}
X & \longrightarrow & \Delta\tilde{\otimes}_\Lambda X \\
x & \longrightarrow & 1\tilde{\otimes}x
\end{array}$$

is a full inclusion of R-lattices.

Tensoring with K over R we may factor this map rationally as

$$K \otimes_R X \xrightarrow{\sim} K \otimes_R \Lambda \otimes_\Lambda X \xrightarrow{\sim} K \otimes_R \Delta \otimes_\Lambda X \xrightarrow{\sim} K \otimes_R (\Delta\tilde{\otimes}_\Lambda X).$$

Remark B.1.13 Let $\Lambda \subseteq \Gamma$ be a full inclusion of lattices, let e be an idempotent of Γ, let X be a left lattice over Λ. Consider the R-linear submodule $eX \subseteq \Gamma X$. We obtain an isomorphism

$$\begin{array}{ccccc}
e\Lambda & \tilde{\otimes}_\Lambda & X & \xrightarrow{\sim} & eX \\
ea & \tilde{\otimes} & x & \longrightarrow & eax \\
e & \tilde{\otimes} & x & \longleftarrow & ex,
\end{array}$$

which is in well defined in the direction $\longleftarrow$ since $ex = 0$ implies $\pi^m(e\tilde{\otimes}x) = 1\tilde{\otimes}\pi^m ex = 0$, m chosen large enough for $\pi^m e \in \Lambda$.

B.2 The Higman ideal

Keep the assumptions from (S B.1).

Definition B.2.1 *The* **Higman ideal** *of Λ, $\mathrm{Higman}(\Lambda) \subseteq R$, is defined to be the annihilator in R of the functor $\mathrm{Ext}^1_\Lambda(-,=)$ from lattices over Λ to R-modules. In other words,*

$$\mathrm{Higman}(\Lambda) := \{a \in R \mid a\mathrm{Ext}^1_\Lambda(X,Y) = 0 \textit{ for all } \Lambda\textit{-lattices } X \textit{ and } Y\}.$$

Remark B.2.2 *A Λ-lattice X is projective iff $X_\mathfrak{p}$ is projective over $\Lambda_\mathfrak{p}$ for all prime divisors $\mathfrak{p}$ of* Higman(Λ). *Any $\Lambda_\mathfrak{p}$-lattice is the localization at $\mathfrak{p}$ of some Λ-lattice. In fact, choose a set of $R_\mathfrak{p}$-linear generators of the $\Lambda_\mathfrak{p}$-lattice and consider its Λ-linear span inside, which is a lattice by (A.4.9, A.4.1 v). Therefore, we obtain*

$$\mathrm{Higman}(\Lambda)_\mathfrak{p} = \mathrm{Higman}(\Lambda_\mathfrak{p})$$

by (A.3.2), since, in general, for a finitely generated R-module M

$$(\mathrm{Ann}_R M)_\mathfrak{p} = \mathrm{Ann}_{R_\mathfrak{p}} M_\mathfrak{p}$$

(A.4.9).

Lemma B.2.3 Higman(Λ) *contains $\mathfrak{a}$. In particular, the Higman ideal does not vanish.*
We may assume R to be a discrete valuation ring with maximal ideal (π), $\mathfrak{a} = (\pi^\alpha)$ (B.2.2). Suppose given an extension

$$0 \longrightarrow X \longrightarrow E \xrightarrow{f} Y \longrightarrow 0.$$

of Λ-lattices.
We **claim** that it is annihilated by π^α as an element of Ext^1, which can be expressed by saying that there is a Λ-morphism $Y \xrightarrow{s} E$ with $sf = \pi^\alpha$, as can be seen by taking the pullback of this sequence along π^α. Writing shorthand $\Lambda(-)$ for the tautological $\Lambda\tilde{\otimes}_\Lambda-$ and $\Gamma(-)$ for $\Gamma\tilde{\otimes}_\Lambda-$ (B.1.3, cf. B.1.11), we obtain a commutative diagram of Λ-lattices

$$\begin{array}{ccc} \Lambda E & \xrightarrow{\Lambda f} & \Lambda Y \\ \downarrow & & \downarrow \\ \Gamma E & \xrightarrow{\Gamma f} & \Gamma Y \end{array}$$

Note that Γf is a pure and thus split epimorphism (B.1.9). Choose a splitting $t(\Gamma f) = 1$. Regard ΓY as a subset of $K\Lambda Y = K\Gamma Y$ and consider the inclusions

$$\Lambda Y \subseteq \Gamma Y \subseteq \pi^{-\alpha}\Lambda Y \subseteq K\Lambda Y.$$

Let s be the restriction of t to $Y = \Lambda Y$. Since the inclusion $\Lambda Y \subseteq \pi^{-\alpha}\Lambda Y$ is isomorphic to the homothety $Y \xrightarrow{\pi^\alpha} Y$, the result follows.

Example B.2.4
(a) Let G be a finite group, let $R = \mathbf{Z}$. Higman$(\mathbf{Z}G) = (|G|)$, where $\supseteq$ follows e.g. by (1.1.1, B.2.3) and $\subseteq$ follows by considering the augmentation sequence, noting that for the trivial lattice $\mathbf{Z}$ we have $\mathrm{Hom}_{\mathbf{Z}G}(\mathbf{Z}, \mathbf{Z}G) \simeq \mathbf{Z}$.
(b) Let R be a discrete valuation ring with maximal ideal (π).
(i) The Higman ideal of $\Lambda := \left(\begin{smallmatrix} R & R \\ \pi & R \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right) =: \Gamma$ is zero since the simple lattices are projective (6.1.12), although $\mathfrak{a} = (\pi)$.
(ii) Let $\alpha \geq 1$. We **claim** that the Higman ideal of $\Lambda := \left(\begin{smallmatrix} R & R \\ \pi^{2\alpha} & R \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right) =: \Gamma$ equals (π^α), whereas $\mathfrak{a} = (\pi^{2\alpha})$. Since Λ is isomorphic to $\left(\begin{smallmatrix} R & \pi^\alpha \\ \pi^\alpha & R \end{smallmatrix}\right)$, (B.2.3) yields (π^α) to be contained in the Higman ideal. On the other hand, we have

$$\mathrm{Ext}^1_\Lambda\left(\left(\begin{smallmatrix} R \\ \pi^\alpha \end{smallmatrix}\right), \left(\begin{smallmatrix} R \\ \pi^\alpha \end{smallmatrix}\right)\right) = R/\pi^\alpha,$$

as can be taken from the first step of the projetive resolution

$$0 \longrightarrow \left(\begin{smallmatrix} R \\ \pi^\alpha \end{smallmatrix}\right) \xrightarrow{(\pi^\alpha\ -1)} \left(\begin{smallmatrix} R \\ \pi^{2\alpha} \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} R \\ R \end{smallmatrix}\right) \xrightarrow{\left(\begin{smallmatrix} 1 \\ \pi^\alpha \end{smallmatrix}\right)} \left(\begin{smallmatrix} R \\ \pi^\alpha \end{smallmatrix}\right) \longrightarrow 0.$$

Appendix C

Krull-Schmidt

C.1 Historical remark

Remark C.1.1
The historical development of the Krull-Schmidt Theorem is roughly as follows.

G. Frobenius and L. Stickelberger published a result known today as the Main Theorem on Finite Abelian Groups [J. Crelle 86, p. 217-262, espec. p. 236 II., p. 242 II., 1879].

R. Remak extended this result on unique decompositions to direct product decompositions of finite, but not necessarily abelian groups. Uniqueness here means that given $G = \prod_i G_i = \prod_j G'_j$, there exists an $\alpha \in \operatorname{Aut} G$ such that $\alpha(G_i) = G'_{i\sigma}$, for a suitable bijection σ, and such that each $g^{-1}\alpha(g)$ is central. [J. Crelle 139, p. 293-308, 1911].

W. Krull proved the result known today as the Krull-Schmidt Theorem for modules which are both noetherian and artinian over an arbitrary ring, i.e. he showed the decomposition of such a module into indecomposables to be unique up to permutation and isomorphic substitution [Math. Z. 23, p. 161-196, 1925].

O. Schmidt found the smallest common generalization of the theorems of Remak and Krull [Math. Z. 29, p. 34-41, 1929].

H. Fitting simplified in Schmidt's treatment a lemma via the introduction of what is known today as Fitting's Lemma [Math. Z. 39, p. 16-30, espec. p. 19, Hilfssatz 3, 1935].

G. Azumaya established the Krull-Schmidt Theorem in the following form: in case the endomorphism rings of the indecomposable modules over a ring are local, (under some finiteness conditions) the decomposition of a module into indecomposable direct summands exists uniquely up to permutation and isomorphic substitution [J. Jap. Math. 29, p. 525-547, 1947].

Now we restrict our attention to the further development concerning general results on Krull-Schmidt for lattices over orders.

Z. Borevich - D. Faddeev, R. Swan, I. Reiner and G. Azumaya obtained independently the validity of the Krull-Schmidt theorem for lattices over an order over a complete discrete valuation ring [for references cf. CR 62, Th. (76.26)]. This assertion results from the endomophism rings of the indecomposables being local, which is the modular assertion lifted to the order via idempotents. J. M. Maranda contributed the necessary preparational assertions [Can. J. Math. 5, p. 344-355, 1953; Can. J. Math. 7, p. 516-526, 1955].

A. Heller proved that one can pull down Krull-Schmidt from the complete to the noncomplete case provided the R-order Λ becomes a direct product of matrix rings over K when tensored with the field of fractions K of R (i.e. $K\Lambda$ split semisimple) [Proc. Nat. Acad. Sci. 47, p. 1194-1197, 1961].

We give an account of Heller's variant (C.2.15) of Krull's achievement (C.2.14), following [CR 62, §76] rather closely.

C.2 Krull-Schmidt, sub split semisimple, local, non-complete

Let R be a discrete valuation ring with maximal ideal (π), field of fractions K and valuation v. Let Λ be a sub split semisimple R-order, i.e. assume a full inclusion of R-orders of the form $\Lambda \subseteq \prod_i (R)_{m_i}$ to exist. Let $\mathrm{Higman}(\Lambda) =: (\pi^h)$ (B.2.1). By X, Y we denote left Λ-lattices.

Let $\hat{R} = \varprojlim R/\pi^i$ denote the completion of R at π, the elements of which we denote as matching tuples of representatives (r_i), $i \in \mathbf{N}$, subject to $r_{i+j} \equiv_{\pi^i} r_i$ for $j \geq 0$. Let $\hat{K}$ be the quotient field of $\hat{R}$. Let $\hat{X} := \hat{R} \otimes_R X$.

We deal with left modules, left noetherianity etc. without mentioning 'left'.

Remark C.2.1 *Let U be a finitely generated free R-module. We write an element of $\varprojlim U/\pi^i$ as a matching tuple of representatives (u_i). The induced morphism*

$$\begin{array}{ccc} \hat{U} & \longrightarrow & \varprojlim U/\pi^i \\ (r_i) \otimes u & \longrightarrow & (r_i u) \end{array}$$

is an isomorphism.

By naturality and additivity, it suffices to see this for $U = R$.

Remark C.2.2 *Suppose given a sequence of R-linear morphisms $(X \xrightarrow{f_i} Y)_{i \in \mathbf{N}}$ such that for $i \geq 0$*

$$(\lambda x) f_i \equiv_{\pi^i} \lambda(x f_i)$$

for $\lambda \in \Lambda$ and such that

$$f_{i+j} \equiv_{\pi^i} f_i$$

for $j \geq 0$. We obtain a $\hat{\Lambda}$-morphism

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\ (x_i) & \longrightarrow & (x_i f_i) \end{array}$$

where the elements of $\hat{X}$ and $\hat{Y}$ are denoted as matching tuples of representatives (cf. C.2.1). This is a well defined $\hat{R}$-linear map as the inverse limit of a family of maps.

$\hat{\Lambda}$ operates, say, on $\hat{X}$ via

$$(\lambda_i)(x_i) = (\lambda_i x_i),$$

so that $\hat{f}$ becomes $\hat{\Lambda}$-linear by the assumption made above.

Lemma C.2.3 *Let $k \geq h + 1$. Let*

$$X \xrightarrow{f} Y$$

be a R-linear map with π^k dividing $\lambda(xf) - (\lambda x)f$ for each $\lambda \in \Lambda$ and each $x \in X$. There exists a Λ-morphism

$$X \xrightarrow{f'} Y$$

such that $f' \equiv_{\pi^{k-h}} f$.

We write $Y \lhd X$ for the direct sum of X and Y as R-lattices, carrying the structure of a Λ-lattice given by a certain extension of X by Y (to be constructed). Pulling back twice, we obtain, using $\pi^h \mathrm{Ext}^1_\Lambda = 0$ on lattices,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{(1\;0)} & Y\oplus X & \xrightarrow{\binom{0}{1}} & X & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ \partial & \pi^h \end{pmatrix}} & & \Big\downarrow{\scriptstyle \pi^h} & & \\
0 & \longrightarrow & Y & \xrightarrow{(1\;0)} & Y\lhd X & \xrightarrow{\binom{0}{1}} & X & \longrightarrow & 0 \\
 & & \Big\| & & \Big\downarrow{\scriptstyle \begin{pmatrix} \pi^k \\ \varphi \end{pmatrix}} & & \Big\downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Y & \xrightarrow{\pi^k} & Y & \longrightarrow & Y/\pi^k & \longrightarrow & 0
\end{array}
$$

We write **left** multiplication with λ on X as $\lambda_X := \lambda(-)$ on the **right.** Etc. The operation of λ on $X\lhd Y$ is described by a matrix of the form

$$\begin{pmatrix} \lambda_Y & 0 \\ \delta & \lambda_X \end{pmatrix}$$

since the horizontal maps are Λ-morphisms. The lower middle vertical map being a Λ-morphism means that

$$\begin{pmatrix} \lambda_Y & 0 \\ \delta & \lambda_X \end{pmatrix}\begin{pmatrix} \pi^k \\ \varphi \end{pmatrix} = \begin{pmatrix} \pi^k \\ \varphi \end{pmatrix}\lambda_Y,$$

i.e. $\delta\pi^k + \lambda_X\varphi = \varphi\lambda_Y$. The upper middle vertical map being a Λ-morphism means that

$$\begin{pmatrix} 1 & 0 \\ \partial & \pi^h \end{pmatrix}\begin{pmatrix} \lambda_Y & 0 \\ \delta & \lambda_X \end{pmatrix} = \begin{pmatrix} \lambda_Y & 0 \\ 0 & \lambda_X \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \partial & \pi^h \end{pmatrix},$$

i.e. $\partial\lambda_Y + \pi^h\delta = \lambda_X\partial$, so that we obtain, denoting $f' := \varphi + \pi^{k-h}\partial$,

$$
\begin{array}{rcl}
\lambda_X f' - f'\lambda_Y & = & \lambda_X(\varphi + \pi^{k-h}\partial) - (\varphi + \pi^{k-h}\partial)\lambda_Y \\
 & = & -\delta\pi^k + \pi^{k-h}\pi^h\delta \\
 & = & 0.
\end{array}
$$

Moreover,

$$f' = \varphi + \pi^{k-h}\partial \equiv_{\pi^{k-h}} \varphi \equiv_{\pi^k} f.$$

Lemma C.2.4 (Maranda) *Assume X/π^{h+1} and Y/π^{h+1} to be isomorphic as Λ-modules. Then X and Y are isomorphic as Λ-lattices.*

Let $X \xrightarrow{f} Y$ be an R-linear map giving the isomorphism modulo π^{h+1}. f satisfies the requirement of (C.2.3) with $k = h+1$ whence we can find a Λ-morphism $X \xrightarrow{f'} Y$ such that

$$
\begin{array}{ccc}
X & \longrightarrow & X/\pi \\
\Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f}\,\wr \\
Y & \longrightarrow & Y/\pi
\end{array}
$$

commutes. A first application of Nakayama's Lemma yields surjectivity of f'. Let C be a R-linear complement to the kernel $K_{f'}$ of f'. A second application of Nakayama's Lemma forces $C = X$.

Lemma C.2.5 *Let S be a commutative ring. The canonical morphism*

$$S[X]/(X^2 - X)^2 \longrightarrow S[X]/(X^2 - X)$$

is a retraction in the category of S-algebras.

In other words, we claim that there exists an S-algebra endomorphism of $S[X]$ which sends the ideal $(X^2 - X)$ to $(X^2 - X)^2$ and which induces the identity on $S[X]/(X^2 - X)$.

The following arguments use characteristic zero. However, the image polynomial of X exhibited this way gives a coretraction in all characteristics - e.g. choose a surjective ring morphism $S' \longrightarrow S$ with S' having characteristic zero (use a large enough polynomial ring over $\mathbf{Z}$) to pull the result down to S via $S \otimes_{S'} -$.

Note that $(f(X)^2 - f(X))' = (2f(X) - 1)f'(X)$. We have to find a polynomial $f(X) \in S[X]$ such that

$$\begin{array}{rcl} f(0)^2 - f(0) &=& 0 \\ f(1)^2 - f(1) &=& 0 \\ (2f(0) - 1)f'(0) &=& 0 \\ (2f(1) - 1)f'(1) &=& 0 \\ f(0) - 0 &=& 0 \\ f(1) - 1 &=& 0, \end{array}$$

i.e. we have to find a $f(X) = X^2 g(X)$ such that

$$\begin{array}{rcl} g(1) &=& 1 \\ g'(1) + 2 &=& 0 \end{array}$$

For instance, take $g(X) = -2X + 3$, i.e. $f(X) = 3X^2 - 2X^3$.

Lemma C.2.6 *Let Δ be an R-order. Let $e \in \Delta$ be such that $e^2 - e \in \pi^k\Delta$. Then there exists an $e' \in \Delta$ such that $e'^2 - e' \in \pi^{2k}\Delta$ and such that $e' \equiv_{\pi^k} e$.*

Using (C.2.5), we let f be a coretraction to

$$R[X]/(X^2 - X)^2 \longrightarrow R[X]/(X^2 - X),$$

and we set $e' := f(e)$. Then

$$f(e)^2 - f(e) = u(e)(e^2 - e)^2 \in \pi^{2k}\Delta$$

for some $u(X) \in R[X]$, and furthermore

$$f(e) - e = v(e)(e^2 - e) \in \pi^k\Delta$$

for some $v(X) \in R[X]$.

Corollary C.2.7 *Let Δ be an R-order. $\hat{\Delta}$ does not contain nontrivial idempotents iff Δ/π does not contain nontrivial idempotents.*

Lemma C.2.8 *$\hat{X}$ is indecomposable iff X/π^{h+1} is indecomposable.*

Assume X/π^{h+1} to be decomposable, i.e. assume given a nontrivial idempotent $X/\pi^{h+1} \xrightarrow{e} X/\pi^{h+1}$. (C.2.3) endows us with an Λ-endomorphism $X \xrightarrow{e'} X$ such that $e' \equiv_\pi e$. e modulo π remains nontrivial, since summands do not vanish.

Hence we may apply (C.2.6) iteratedly to $\Delta = \mathrm{End}_\Lambda X$, starting with e', to obtain a sequence of Λ-linear endomorphisms $X \xrightarrow{e'_i} X$ with $e'_1 = e'$, $e_{i+j} \equiv_{\pi^i} e_i$ for $j \geq 0$ and such that $e_i'^2 \equiv_{\pi^i} e'_i$.

This yields a nontrivial idempotent endomorphism of $\hat{X}$ (C.2.2).

Lemma C.2.9 *Let K'/K be a field extension. Each $K'\Lambda$-module M' arises from a $K\Lambda$-module M via scalar extension*

$$K' \otimes_K M \xrightarrow{\sim} M'.$$

In other words, we may find a K'-basis of M' for which the matrices of the Λ-operation have entries in K.

Since Λ is **sub split semisimple,** the assertion is true for an indecomposable $K'\Lambda$-module. In fact, this module is isomorphic to a column in a product of matrix rings over K', arising from a product of matrix rings over K by entrywise scalar extension.

Proposition C.2.10 (Heller's Lemma, [H 61, 2.5]) *Each $\hat{\Lambda}$-lattice U allows an isomorphism $\hat{X} \xrightarrow{\sim} U$ for some Λ-lattice X.*

By (C.2.9), we find a $K\Lambda$-module V such that $\hat{K} \otimes_{\hat{R}} U \simeq \hat{K} \otimes_K V$ as $\hat{K}\Lambda$-modules. Let $\{u_1, \ldots, u_n\}$ be a $\hat{R}$-basis of U. Regard V as a $K\Lambda$-submodule of $\hat{K} \otimes_{\hat{R}} U$ we choose a K-basis

$$\sum_j a_{ij} u_j,$$

where $A := (a_{ij}) \in (\hat{K})_n$ is an invertible matrix. We **claim** that the Λ-submodule

$$X := V \cap U.$$

contains a R-basis which is a $\hat{R}$-basis of U, thus proving the proposition. Choose $B = (b_{ij}) \in (K)_n$ with $B \equiv_{\pi^N} A^{-1}$, where N is to be taken strictly larger than the negative of the minimal valuation of the entries of A, so that $BA \equiv_{\pi} A^{-1}A = E$. In particular, BA is contained in $(\hat{R})_n$ and is invertible there, for its determinant is contained in $1 + \pi\hat{R}$. The elements

$$x_i := \sum_{j,k} b_{ij} a_{jk} u_k$$

thus form a K-basis of V and an $\hat{R}$-basis of U. Since we can write each element of $\hat{K}$ uniquely as a product of a unit in $\hat{R}$ and a power of π we have $\hat{R} \cap K = R$. Therefore the coefficients of an element of X with respect to the basis $\{x_i\}$ lie in R.

Lemma C.2.11 *X is indecomposable iff X/π^{h+1} is indecomposable.*

If X/π^{h+1} is decomposable, so is $\hat{X}$ by (C.2.8). Writing

$$\hat{X} = \hat{X}_1 \oplus \hat{X}_2 = (X_1 \oplus X_2)\hat{},$$

with X_1, X_2 nonzero, which is possible by Heller's Lemma (C.2.10), we conclude by (C.2.4) from

$$X/\pi^{h+1} = (X_1 \oplus X_2)/\pi^{h+1}$$

that there is a decomposition

$$X \simeq X_1 \oplus X_2$$

into nonzero Λ-lattices. NB we may not assert equality here, for e.g. there is no reason why the Λ-submodules X_1 and X_2 of $\hat{X}$ should be contained in X.

Remark C.2.12 *Substituting (C.2.7) for (C.2.8) in (C.2.11), we obtain that a sub split semisimple R-order contains nontrivial idempotents iff this is the case modulo π. In other words, primitive idempotents remain primitive modulo π.*

Remark C.2.13 We'd like to stress that the assertion of (C.2.11) merely involves Λ, whereas its proof needs the actual completion $\hat{\Lambda}$, which apparently cannot be substituted by 'Λ/π^N, N large', since Heller's Lemma hinges on the fact that $\hat{K}$ is a field extension of K.

(C.2.10) is the only place in which we needed the assumption on Λ to be sub split semisimple.

Problem. Assume a nontrivial idempotent endomorphism of X/π^h to be known explicitely, as an R-linear matrix, say. **Construct** a nontrivial idempotent endomorphism of X.

Lemma C.2.14 (Krull-Schmidt, artinian and noetherian) *Let A be a noetherian and artinian ring. Then the decomposition of a finitely generated A-module M into indecomposables is possible and unique up to a permutation and isomorphic substitution of the summands.*

Finitely generated A-modules are noetherian and artinian. Using this, we may apply the Circonference Lemma to the composition $f^k f^k = f^{2k}$, m large, to prove nilpotence of an endomorphism of an indecomposable module which is not an automorphism. Writing down a geometric series, we thus see that either

f or $1-f$ is an automorphism. Via composition, this also holds with an automorphism instead of 1. We conclude that the nonautomorphisms are closed under addition.

The compositions $M_1 \longrightarrow M'_j \longrightarrow M_1$ arising from decompositions $\bigoplus M_i = M = \bigoplus M'_j$ into indecomposables sum up over j to 1_{M_1}. Hence there is an automorphism amongst them, yielding, say, $M_1 \xrightarrow{\sim} M'_1$. It remains to exhibit an automorphism of M which restricts to $M_1 \xrightarrow{\sim} M'_1$ [1]. Consider

$$\vartheta := (M \longrightarrow M'_1 \xleftarrow{\sim} M_1 \longrightarrow M \xrightarrow{1-p'_1} M),$$

where p'_1 denotes the projection to M'_1. $M \xrightarrow{1-\vartheta} M$ restricts to $M_1 \xrightarrow{\sim} M'_1$. Moreover, $1 = (1-\vartheta)(1+\vartheta)$.

Theorem C.2.15 (Krull-Schmidt, noncomplete) *The decomposition of a Λ-lattice into indecomposables is possible and unique, up to a permutation and isomorphic substitution of the summands.*

Two decompositions into indecomposables remain decompositions into indecomposables modulo π^{h+1} (C.2.11), hence the summands modulo π^{h+1} are pairwise isomorphic after a permutation (C.2.14), hence the summands themselves are pairwise isomorphic after a permutation (C.2.4).

Proposition C.2.16 (Krull-Schmidt, complete) *Let Δ be an $\hat{R}$-order (not necessarily sub split semisimple). The decomposition of a Δ-lattice into indecomposables is possible and unique, up to a permutation and isomorphic substitution of the summands.*

Two decompositions into indecomposables remain decompositions into indecomposables modulo π^{h+1} (C.2.8), hence the summands modulo π^{h+1} are pairwise isomorphic after a permutation (C.2.14), hence the summands are pairwise isomorphic after a permutation (C.2.4).

C.3 Counterexamples

Out of interest, we also give an account of two well known examples which show the limitations of Krull-Schmidt. We specialize [CR 81, 36.3] to a single counterexample to Krull-Schmidt in case K is not a splitting field for the finite group G. Such a counterexample has been found, but has not been written down in detail, by Berman and Gudikov [Integral Representations of Finite Groups, Sov. Math. Dokl. 3, p. 1172-1174, 1962]. Moreover, we recall Roggenkamp's counterexample to Krull-Schmidt for projectives over orders over a local ring [Rog 70, VI].

Example C.3.1 Let $G := C_7 \times C_2 = \langle a \mid a^7 = 1\rangle \times \langle b \mid b^2 = 1\rangle$, let $R := \mathbf{Z}_{(2)}$. Krull-Schmidt fails for RG-lattices.

Denote by ζ a seventh primitive root of unity in $\mathbf{C}$.

$(2) \in R[\zeta]$ decomposes according to the seventh cyclotomic polynomial in $\mathbf{F}_2[X]$, viz.

$$\Phi_7(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \equiv_2 (X^3 + X + 1)(X^3 + X^2 + 1),$$

where the factors are coprime by

$$X(X^3 + X + 1) + (X+1)(X^3 + X^2 + 1) \equiv_2 1,$$

which gives a decomposition of the zero ideal in

$$\mathbf{F}_2[X]/\Phi_7(X) = \mathbf{F}_2[X]/(X^3 + X + 1) \times \mathbf{F}_2[X]/(X^3 + X^2 + 1) \simeq \mathbf{F}_8 \times \mathbf{F}_8$$

into the prime ideals

$$\begin{array}{rcl} 0 &=& (\,(X+1)(X^3+X^2+1)\,)\cdot(\,X(X^3+X+1)\,) \\ &=& (\,X^4+X^2+X+1\,)\cdot(\,X^4+X^2+X\,). \end{array}$$

[1] The following device is taken from [Be 91, 1.4.3].

Taking inverse images, (2) decomposes in $R[\zeta]$ into the prime ideals

$$(2) = (2,\ \underbrace{\zeta^4+\zeta^2+\zeta+1}_{=:s}) \cdot (2,\ \underbrace{\zeta^4+\zeta^2+\zeta}_{=:t}).$$

Note that $st = -2$, hence even

$$(2) = (s)(t).$$

We obtain a ring morphism

$$\begin{array}{rcccl} R(C_7 \times C_2) & \longrightarrow & R[\zeta][X]/(X^2-1) & \xrightarrow{\sim} & R[\zeta] \times R[\zeta] \\ a & \longrightarrow & \zeta & \longrightarrow & (\zeta, \zeta) \\ b & \longrightarrow & X & \longrightarrow & (1, -1), \end{array}$$

the image of which is described by

$$A := \{(x, y) \subseteq R[\zeta] \times R[\zeta] \mid x \equiv_{(2)} y\}.$$

We are reduced to find a counterexample to Krull-Schmidt for A-lattices. Consider the A-lattices

$$\begin{array}{rcl} M & := & \{(x,y) \in R[\zeta] \times R[\zeta] \mid x \equiv_{(s)} y\} \\ N & := & \{(x,y) \in R[\zeta] \times R[\zeta] \mid x \equiv_{(t)} y\} \\ X & := & \{(x,0) \in R[\zeta] \times R[\zeta]\} \end{array}$$

Since $R[\zeta]$ (but **not** $\hat{R}[\zeta]$, cf. C.2.7) is an integral domain, we see via idempotents that M, N and X are indecomposable. Let

$$\begin{array}{rcl} X & \xrightarrow{\sigma} & M \\ (1,0) & \longrightarrow & (s,0) \\ X & \xrightarrow{\tau} & N \\ (1,0) & \longrightarrow & (t,0) \\ M & \xrightarrow{\varphi} & X \\ (x,y) & \longrightarrow & (x,0) \\ N & \xrightarrow{\psi} & X \\ (x,y) & \longrightarrow & (x,0) \end{array}$$

denote some A-morphisms and observe that $\left(\begin{smallmatrix}\sigma & \tau\end{smallmatrix}\right)\left(\begin{smallmatrix}\varphi \\ -\psi\end{smallmatrix}\right) = 1$, so that X is a direct summand of $M \oplus N$.

Example C.3.2 Maintain the notation of (C.3.1). Let

$$\Lambda := \begin{bmatrix} R[\zeta] & R[\zeta] \\ (2) & R[\zeta] \end{bmatrix}$$

map to a sum of Λ-lattices as follows,

$$\begin{array}{rcl} \Lambda & \longrightarrow & \begin{bmatrix} R[\zeta] \\ (s) \end{bmatrix} \oplus \begin{bmatrix} R[\zeta] \\ (t) \end{bmatrix} \\ 1 & \longrightarrow & \begin{bmatrix} 1 \\ s \end{bmatrix} \oplus \begin{bmatrix} 1 \\ t \end{bmatrix}, \end{array}$$

surjectively (direct calculation, using $s - t = 1$, $st = -2$), hence injectively. An arbitrary Λ-linear map

$$\begin{bmatrix} R[\zeta] \\ R[\zeta] \end{bmatrix} \longrightarrow \begin{bmatrix} R[\zeta] \\ (s) \end{bmatrix},$$

is a scalar multiplication, since it is a scalar multiplication when tensored with $\mathbf{Q}$. In particular, it cannot be surjective. Dito for t instead of s. So Λ has two essentially different decompositions into indecomposable **projectives.**

Appendix D

$\mathfrak{p}$-orders

We collect a few basic facts on the genus question for orders as far as necessary in order to make precise the meaning of the 'absence of genus phenomena' in the naive localizations (D.2.10) of $\mathbf{Z}\mathcal{S}_n$ for $n \leq 6$.

The attentive reader will surely recognize that we have chosen our assumptions in this appendix in such a manner that no serious difficulties can arise. Moreover, we adhocisize several statements, not because we do not appreciate the more general framework of maximal orders, Whitehead groups etc., but because already as it is, this appendix is longer than expected to be. Basically, we follow [CR 62] and [Rog 70]. In (D.5.11) we give a criterion for certain R-orders to be homogenus. Besides this, we do not claim originality.

All conventions we make in this appendix (A D) remain valid from the place we state them on to the end of (A D), in particular, they are valid in the following sections. Exceptions are explicitely stated. A list of conventions can be found at the end of (A D).

D.1 Homogenus rings

By a module over a ring we understand a left module, except stated otherwise. Finite projective stands for finitely generated projective module. A-proj **denotes the category of finite projectives over** A**. Indecomposable projective stands for finitely generated indecomposable projective module.** $\mathrm{ip}(A)$ **denotes the set of isomorphism classes of indecomposable projectives over** A**. We say that Krull-Schmidt holds in** A-proj **if the decomposition of** $P \in A$-proj **into indecomposable projectives is unique up to permutation of the summands and up to isomorphism. The unit group of a ring** A **is denoted by** A^***.**

Definition D.1.1 *Let A be a ring.*

The indecomposable projectives P and Q over A are said to lie **in the same genus$'$** *if $P^k \simeq Q^k$ for some $k \geq 1$.*

The ring A is called **homogenus** *('of homogeneous genus') if there exists an orthogonal decomposition into primitive idempotents*

$$1 = \sum_{i=1}^{s} e_i$$

such that Ae_i and Ae_j are in the same genus$'$ iff they are isomorphic.

A **Morita reduction** *of a ring A is the endomorphism ring B of the direct sum of a set of representatives for the genus$'$-classes of the set of indecomposable projectives occurring in a decomposition of A into indecomposable projectives.*

In case A is homogenus, we can reconstruct it from B, a decomposition of B into indecomposable projectives and the **Morita multiplicities,** *i.e. the cardinalities of the genus′ classes of the given decomposition of A.*

NB it may well happen that A is homogenus although there exists a decomposition that does not fulfill the requirements (cf. D.1.4, D.5.6).

Remark D.1.2 We shall see that for an R-order Λ fully embedded into a product of matrix rings over some Dedekind domain R, the notions of genus′ and of genus coincide (D.2.21), the latter being defined in (D.2.13).

Example D.1.3 (the main example) We verify in (C 2) by direct calculation that there exists an embedding into a direct product of integral matrix rings with respect to which the naive localization $(\mathbf{Z}\mathcal{S}_n)_{[p]}$ is homogenus for $n \leq 6$ and for p a prime divisor of $n!$. (The naive localization $(-)_{[p]}$ with respect to such an embedding will be defined in (D.2.10)). Moreover, for $n = p$ prime, $\mathbf{Z}\mathcal{S}_p$ allows such an embedding with respect to which $(\mathbf{Z}\mathcal{S}_p)_{[p]}$ is homogenus (4.2.8). I do not know whether this is true in general.

Example D.1.4 (the typical one) We refer to a result further down to verify the assertion made in this example.

Let

$$\begin{aligned} A &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \;\middle|\; \begin{pmatrix} a & 2b \\ c & 2d \end{pmatrix} \equiv_5 \begin{pmatrix} a' & b' \\ 2c' & 2d' \end{pmatrix} \right\} \\ &\subseteq (\mathbf{Z})_2 \times (\mathbf{Z})_2 \\ B &:= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \times \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b & 2c \\ d & e & 2f \\ g & h & 2i \end{pmatrix} \equiv_5 \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ 2g' & 2h' & 2i' \end{pmatrix} \right\} \\ &\subseteq (\mathbf{Z})_3 \times (\mathbf{Z})_3. \end{aligned}$$

A is not homogenus, but B is, as we shall see in (D.5.11). But localized at 5, A becomes homogenus, too, see (D.1.6, C.2.15).

We will come back to the ring A in (D.5.14). Though small, this is a quite typical example and might be kept in mind throughout (cf. D.2.11). A and B are 5-orders in the sense of (D.2.8).

Remark D.1.5 *The annihilator ideals of indecomposable projectives in the same genus′ coincide.*

Remark D.1.6 *If Krull-Schmidt holds in A-*proj*, A is homogenus.*

Lemma D.1.7 *Let A be a ring. If $1 \in A$ has an orthogonal decomposition $1 = \sum_i e_i$ into primitive idempotents such that e_iAe_i is local - i.e. the nonunits form an ideal -, then Krull-Schmidt holds in A-*proj.

By BENSON's device (proof of C.2.14), the decomposition of a sum of projectives of the form Ae_i into indecomposable projectives is unique up to permutation and isomorphism. A finite projective over A is a summand of A^n, thus a sum of certain Ae_i's. Hence this uniqueness also applies to a decomposition of this finite projective into indecomposable projectives.

D.2 Naive Localization

Let R be a Dedekind domain with field of fractions K (to which we refer by 'rational') such that $R/\mathfrak{p}$ is finite as a set for each nonzero prime ideal $\mathfrak{p} \subseteq R$. By $\mathfrak{p}, \mathfrak{q}$ we denote nonzero prime ideals of R. Assume K to have finite class number, i.e. assume the set of isomorphism classes of ideals in R to be finite.

An R-order is an R-algebra which is finite projective as an R-module. Let Λ be a full (i.e. rationally equal) R-suborder of a direct product of matrix rings over R, $\Lambda \subset \Gamma := \prod_i (R)_{m_i}$ being strictly included. We fix this embedding throughout. Such an order Λ we call sub split semisimple over R. Γ/Λ is a torsion R-module with annihilator $\mathfrak{a}$ in R.

We abbreviate $K \otimes_R -$ by $K(-)$. A lattice over Λ is a Λ-module that is finite projective over R. A simple Λ-lattice is a Λ-lattice X with KX being a simple $K\Lambda$-module. A pure monomorphism of Λ-lattices has a torsionfree quotient, a full monomorphism has a torsion quotient, a pure epimorphism is surjective.

Remark D.2.1 Since $K\Lambda$ is a product of matrix rings, Krull-Schmidt holds for $\Lambda_\mathfrak{p}$-lattices (C.2.15). So in particular Krull-Schmidt holds for $\Lambda_\mathfrak{p}$-proj, whence $\Lambda_\mathfrak{p}$ is homogenus (D.1.6).

Lemma D.2.2 *Any K-algebra automorphism of $(K)_m$, $m \geq 1$, is inner.*

By Morita equivalence $(K)_m$ has only one simple module so that, given such an automorphism α, K^m and the twisted module ${}_\alpha K^m$ are isomorphic via an invertible matrix A, giving back α via conjugation.

Lemma D.2.3 *Any full embedding of R-orders $\Lambda \overset{i}{\hookrightarrow} \Gamma$ that sends the rational central primitive idempotents of Λ to the same central primitive idempotents of Γ as our chosen inclusion $\Lambda \hookrightarrow \Gamma$ can be substituted isomorphically by an inclusion $\Lambda \hookrightarrow {}^A\Gamma \subseteq K\Gamma$, where $A \in \Gamma$ is an invertible element of $K\Gamma$, and where $(\Lambda \hookrightarrow {}^A\Gamma \hookrightarrow K\Gamma) = (\Lambda \hookrightarrow K\Lambda = K\Gamma)$ is canonically embedded.*

More precisely, there is a commutative diagram

$$\begin{array}{ccccc} & & \Lambda & \longrightarrow & {}^A\Gamma \\ & & {\scriptstyle {}^A(-)}\uparrow\wr & & \wr\uparrow{\scriptstyle {}^A(-)} \\ \Lambda & \xrightarrow[\sim]{i} & i\Lambda & \longrightarrow & \Gamma. \end{array}$$

The condition on i to respect the rational matrix ring factors is merely a question of numbering the factors in the target of i appropriately.

$K(i\Lambda \xrightarrow[\sim]{i^{-1}} \Lambda)$ is a K-algebra automorphism of $K\Gamma$, which is inner by (D.2.2).

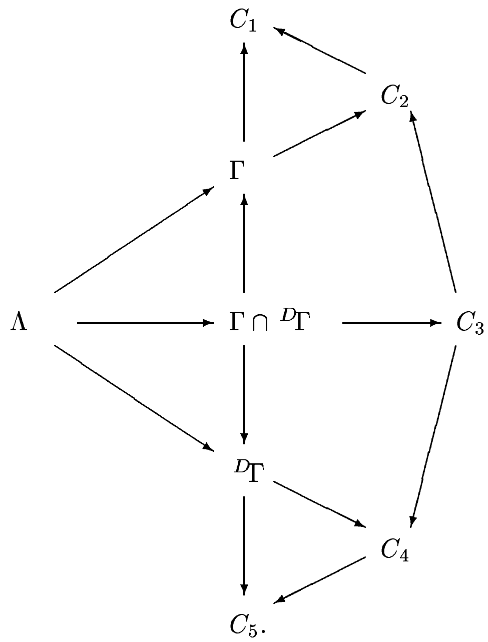
Lemma D.2.4 *Let $Y \subseteq X$ be a full inclusion of simple lattices such that Y is not contained in $\mathfrak{b}X$ for any nontrivial ideal $\mathfrak{b} \subset R$. Then $\mathfrak{a}^2 X \subseteq Y$, i.e. $\mathfrak{a}^2$ annihilates X/Y.*

We may assume R to be a discrete valuation ring with maximal ideal (π), $\mathfrak{a} = (\pi^i)$, since an R-module M vanishes iff $M_\mathfrak{p} = 0$ for all $\mathfrak{p}$. By simplicity of X we may assume $\Gamma = (R)_m$. We **claim** that $\pi^{2i}(X/Y) = 0$.

We embed X into a column L of Γ in such a way that X is not contained in πL, which is possible since $KX \simeq KL$. By the Elementary Divisor Theorem (A.1.1) we may assume after a choice of bases that there exists a main diagonal matrix $D \in \Gamma$ with main diagonal $(\pi^{s_1}, \ldots, \pi^{s_n})$, $s_1 = 0$, such that $X = DL$. Since X is a lattice over Λ, we obtain

$$\begin{aligned} \Lambda &\subseteq \{u \in (K)_m \mid uX \subseteq X\} \\ &= \{u \in (K)_m \mid uDL \subseteq DL\} \\ &= \{u \in (K)_m \mid u^D \in \Gamma\} \\ &= {}^D\Gamma, \end{aligned}$$

whence the diagram in which the C_i's are the respective cokernels (cf. 1.1.6)



C_1 and C_5 are both isomorphic to $\bigoplus_{i<j} R/\pi^{s_j - s_i}$ as modules over R. $\pi^i C_2 = 0$ implies $\pi^i C_1 = 0$ and $\pi^i C_3 = 0$, thus $\pi^{2i} C_4 = 0$.

Note that X is a column of ${}^D\Gamma$. Thus, replacing Γ by ${}^D\Gamma$, we may assume $X = L$ to be a simple Γ-lattice at the cost of merely disposing of $\pi^{2i}(\Gamma/\Lambda) = 0$.

We have $\Gamma Y = X$, since the Γ-sublattices of X are given by $\pi^j X$'s and since by assumption Y is not contained in πX. Therefore,

$$\pi^{2i} X = \pi^{2i} \Gamma Y \subseteq \Lambda Y = Y.$$

Example D.2.5 Let R be a discrete valuation ring with maximal ideal (π). For $\Lambda = \left(\begin{smallmatrix} R & \pi \\ \pi & R \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} R & R \\ R & R \end{smallmatrix}\right) = \Gamma$ and $Y = \left(\begin{smallmatrix} \pi \\ \pi^2 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} \pi \\ R \end{smallmatrix}\right) = X$ the annihilator of X/Y equals the square of $\mathfrak{a}$.

Corollary D.2.6 *The number of isomorphism classes of simple lattices over Λ is finite.*

We **claim** that the set of isomorphism classes of simple Λ-lattices rationally isomorphic to the simple Λ-lattice X is finite.

The number of nonisomorphic lattices of type $\mathfrak{b}X$, $\mathfrak{b} \subseteq K$ a fractional ideal, is finite since the class number of K is assumed to be finite and since $\mathfrak{b} \xrightarrow{\sim} \mathfrak{b}'$ over R is given by multiplication with an element of K, and thus yields $\mathfrak{b}X \xrightarrow{\sim} \mathfrak{b}'X$ over Λ.

Let Y be a Λ-lattice rationally isomorphic to X. We include $Y \subseteq X$ and choose $v \in K$ such that $Y \subseteq X \subseteq vY \subseteq KY = KX$ (A.4.5). Let $\mathfrak{c} := \{u \in K \mid uY \subseteq X\}$, being a fractional ideal since $\mathfrak{c} \subseteq Rv$. Now, if $Y \subseteq \mathfrak{b}\mathfrak{c}^{-1}X$ for some ideal $\mathfrak{b} \subseteq R$, then $\mathfrak{b}^{-1} \subseteq R$, whence $\mathfrak{b} = R$. By (D.2.4), we obtain that $Y/\mathfrak{a}^2\mathfrak{c}^{-1}X$ is a submodule of $\mathfrak{c}^{-1}X/\mathfrak{a}^2\mathfrak{c}^{-1}X$, which is finite as a set (A.4.9).

Corollary D.2.7 (Jordan-Zassenhaus **in our particular situation)** *The number of isomorphism classes of Λ-lattices X rationally isomorphic to a given $K\Lambda$-module U is finite. I.e. $K(-)$ has finite fibers on the isomorphism classes.*

We use induction on the number of simple indecomposable direct summands of U, starting with (D.2.6) in case U itself is simple. Otherwise, decompose $U = V \oplus W$ nontrivially. By induction, there are only finitely many lattices rationally isomorphic to V resp. to W. Intersecting the given Λ-lattice with V and projecting it to W, it remains to be shown that there are only finitely many isomorphism classes of extensions X of given full sublattices $Y \subseteq V$ and $Z \subseteq W$. However, there are even only finitely many elements in $\mathrm{Ext}^1_\Lambda(Z, Y)$ (A.3.3, A.4.9).

Definition D.2.8 *Λ is called a* **$\mathfrak{p}$-order** *if $\mathfrak{a} = \mathfrak{p}^i$ for some $i \geq 1$.*

Consequently, the Higman ideal of a $\mathfrak{p}$-order is a power of $\mathfrak{p}$ (B.2.3).

Note that by convention, Γ itself is not a $\mathfrak{p}$-order.

By comparison of $\Lambda \subseteq \Gamma$ and $\Lambda \subseteq {}^A\Gamma$ (cf. D.2.3) via $\Gamma \cap {}^A\Gamma$ just as in (D.2.4), the $\mathfrak{q}$-part of Γ/Λ has a cardinality independent of the chosen embedding $\Lambda \hookrightarrow \Gamma$, so that the property of being a $\mathfrak{p}$-order is independent from this choice.

Remark D.2.9 *If Λ is a $\mathfrak{p}$-order, a Λ-lattice X is projective iff $X_\mathfrak{p}$ is projective over $\Lambda_\mathfrak{p}$ (B.2.2).*

Definition D.2.10 *Let $\mathfrak{p} \subseteq R$ be a nonzero prime ideal. Let C be the cokernel of the inclusion of R-modules $\Lambda \subseteq \Gamma$, and let $C_\mathfrak{p}$ be its the $\mathfrak{p}$-part (A.4.10). Let the* **naive localization** *$\Lambda_{[\mathfrak{p}]}$ be defined as the kernel of the composition of the canonical map $\Gamma \longrightarrow C$ with the projection $C \longrightarrow C_\mathfrak{p}$, i.e.*

$$\begin{array}{ccccc} & \Lambda_{[\mathfrak{p}]} & & & \\ \nearrow & & \searrow & & \\ \Lambda & \longrightarrow & \Gamma & \longrightarrow & C \\ & & & \searrow & \downarrow \\ & & & & C_\mathfrak{p}. \end{array}$$

$\Lambda_{[\mathfrak{p}]}$ is an R-order. By construction, we have

$$\Lambda = \bigcap_{\mathfrak{p} \subseteq R} \Lambda_{[\mathfrak{p}]} \subseteq \Gamma.$$

NB $\Lambda_{[\mathfrak{p}]}$ **depends on the chosen embedding** *$\Lambda \hookrightarrow \Gamma$, a dependence which we shall not denote by* **abuse of notation** *(cf. D.2.11).*

In case $\mathfrak{p} = (p)$ is a principal prime ideal, we also denote

$$\Lambda_{[p]} := \Lambda_{[(p)]}.$$

We have to show that $\Lambda_{[\mathfrak{p}]}$ is closed under multiplication in Γ. Let $C = C_\mathfrak{p} \oplus C_{\mathfrak{p}'}$ be the decomposition of C into its $\mathfrak{p}$ and its $\mathfrak{p}'$-part (A.4.10), let $(\Gamma \longrightarrow C) =: (\Gamma \xrightarrow{(f\ g)} C_\mathfrak{p} \oplus C_{\mathfrak{p}'})$. Suppose given $x, y \in \Lambda_{[\mathfrak{p}]}$, i.e. suppose that $xf = yf = 0$. There exists an $s \in R \backslash \mathfrak{p}$ such that $sC_{\mathfrak{p}'} = 0$ (A.4.5). Thus $(sxf\ sxg) = 0$, i.e. $sx \in \Lambda$, as well as $(syf\ syg) = 0$, i.e. $sy \in \Lambda$. Hence s^2xy is contained in Λ, in particular $(s^2xy)f = 0$, whence $(xy)f = 0$, i.e. $xy \in \Lambda_{[\mathfrak{p}]}$.

Alternatively, we may describe $\Lambda_{[\mathfrak{p}]}$ as the pullback - as abelian groups as well as as rings - of $\Lambda_\mathfrak{p} \longrightarrow \Gamma_\mathfrak{p}$ and $\Gamma \longrightarrow \Gamma_\mathfrak{p}$, which also shows $\Lambda_{[\mathfrak{p}]}$ to be closed under multiplication in Γ.

Note that $\Lambda_\mathfrak{p} \xrightarrow{\sim} (\Lambda_{[\mathfrak{p}]})_\mathfrak{p}$.

Morally, $\Lambda_{[\mathfrak{p}]}$ arises from Λ by dropping the $\mathfrak{p}'$-ties without changing the ground ring R, since of the cokernel C we think as a list of ties, grouped in sublists $C_\mathfrak{q}$ of $\mathfrak{q}$-ties.

Example D.2.11 (dangerous bend) *It may happen that with respect to the embedding $\Lambda \hookrightarrow \Gamma$, the naive localization of Λ at $\mathfrak{p}$ is homogenus, whereas with respect to an embedding $\Lambda \hookrightarrow {}^A\Gamma$ for some $A \in (K\Gamma)^*$, it is not homogenus. In particular, these naive localizations are nonisomorphic.*

We refer to a result further down to verify the assertion made in this example.

We may as well regard the isomorphic substitution $\Lambda^A \longrightarrow \Gamma$ of $\Lambda \longrightarrow {}^A\Gamma$. Let

$$\Lambda := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv_5 \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right),\ b \equiv_2 0 \right\} \subseteq (\mathbf{Z})_2 \times (\mathbf{Z})_2 =: \Gamma,$$

let

$$A = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

We obtain

$$\begin{array}{rcl} \Lambda_{[5]} & := & \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \;\middle|\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv_5 \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \right\} \\ \Lambda^A_{[5]} & := & \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \;\middle|\; \left(\begin{smallmatrix} a & 2b \\ c & 2d \end{smallmatrix}\right) \equiv_5 \left(\begin{smallmatrix} a' & b' \\ 2c' & 2d' \end{smallmatrix}\right) \right\}. \end{array}$$

$\Lambda_{[5]}$ is homogenus, whereas by (D.5.11), $\Lambda^A_{[5]}$ is not.

Lemma D.2.12 *Retaining the notation of (D.2.10) we obtain*

$$\mathrm{Higman}(\Lambda)_{\mathfrak{p}} \cap R = \mathrm{Higman}(\Lambda_{\mathfrak{p}}) \cap R = \mathrm{Higman}(\Lambda_{[\mathfrak{p}]}).$$

Cf. (B.2.2).

For $\Lambda_{[\mathfrak{p}]}$-lattices X and Y, we use the formula

$$(\mathrm{Ext}^1_{\Lambda_{[\mathfrak{p}]}}(X,Y))_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda_{\mathfrak{p}}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$$

which ensues from $\Lambda_{\mathfrak{p}} \xrightarrow{\sim} (\Lambda_{[\mathfrak{p}]})_{\mathfrak{p}}$ (A.3.2). Since any $\Lambda_{\mathfrak{p}}$-lattice arises from a $\Lambda_{[\mathfrak{p}]}$-lattice via localization at $\mathfrak{p}$, the element $s \in R$ lies in $\mathrm{Higman}(\Lambda_{\mathfrak{p}})$ iff it annihilates the $\mathfrak{p}$-part of $\mathrm{Ext}^1_{\Lambda_{[\mathfrak{p}]}}(X,Y)$ for all $\Lambda_{[\mathfrak{p}]}$-lattices X and Y.

A power of $\mathfrak{p}$ annihilates $\mathrm{Ext}^1_{\Lambda_{[\mathfrak{p}]}}(X,Y)$ by (B.2.3).

Alternatively, let $\mathfrak{q} \neq \mathfrak{p}$ be a nonzero prime ideal in R. We use (A.3.2) to obtain

$$\begin{array}{rcl} \mathrm{Ext}^1_{\Lambda_{[\mathfrak{p}]}}(X,Y)_{\mathfrak{q}} & = & \mathrm{Ext}^1_{(\Lambda_{[\mathfrak{p}]})_{\mathfrak{q}}}(X_{\mathfrak{q}}, Y_{\mathfrak{q}}) \\ & = & \mathrm{Ext}^1_{\Gamma_{\mathfrak{q}}}(X_{\mathfrak{q}}, Y_{\mathfrak{q}}) \\ & = & 0. \end{array}$$

Definition D.2.13 *The Λ-lattices X and Y are said to* **lie in the same genus,** *written $X \vee Y$, if $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are isomorphic for all nonzero prime ideals $\mathfrak{p}$ of R.*

By Krull-Schmidt locally (C.2.15), two projective indecomposable lattices over Λ that lie in the same genus′ lie in the same genus. For the converse, see (D.2.21).

Lemma D.2.14 *The Λ-lattices X and Y lie in the same genus iff $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are isomorphic for all prime divisors $\mathfrak{p}$ of $\mathfrak{a}$.*

In particular, in case Λ is a $\mathfrak{p}$-order, X and Y lie in the same genus iff $X_{\mathfrak{p}} \simeq Y_{\mathfrak{p}}$.

A rational isomorphism ensues which gives the remaining required local isomorphisms by counting components, since for $\mathfrak{q} + \mathfrak{a} = R$ we have $\Lambda_{\mathfrak{q}} = \Gamma_{\mathfrak{q}}$, the lattices of which being direct sums of its columns.

Lemma D.2.15 (globalization of morphism families) *Let S be a* **finite** *set of nonzero prime ideals of R. Suppose given lattices X and Y over Λ together with morphisms*

$$X_{\mathfrak{p}} \xrightarrow[\sim]{f^{\mathfrak{p}}} Y_{\mathfrak{p}}$$

of $\Lambda_{\mathfrak{p}}$-lattices for $\mathfrak{p} \in S$. Then there is a Λ-morphism

$$X \xrightarrow{f} Y$$

such that $f_{\mathfrak{p}} \equiv_{\mathfrak{p}} f^{\mathfrak{p}}$ for all $\mathfrak{p} \in S$.

Using $({}_\Lambda(X,Y))_\mathfrak{p} \xrightarrow{\sim} {}_{\Lambda_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p})$ (A.3.2), we choose a Λ-morphism $X \xrightarrow{u^\mathfrak{p}} Y$ for each $\mathfrak{p} \in S$ such that $(u^\mathfrak{p})_\mathfrak{p} \equiv_\mathfrak{p} f^\mathfrak{p}$. This is possible for our Hom-module as well as for any R-module M, since given $m/s \in M_\mathfrak{p}$, the condition $n - (m/s) = (ns - m)/s \in \mathfrak{p}M_\mathfrak{p}$ reads $ns \equiv_\mathfrak{p} m$, solvable by invertibility of s in $R/\mathfrak{p}$.

We apply the Chinese Remainder Theorem to obtain elements $a^\mathfrak{q} \in R$ for $\mathfrak{q} \in S$ with $a^\mathfrak{q} \equiv_\mathfrak{p} \partial_{\mathfrak{p},\mathfrak{q}}$ for $\mathfrak{p}, \mathfrak{q} \in S$, and let

$$f := \sum_{\mathfrak{q} \in S} a^\mathfrak{q} u^\mathfrak{q}.$$

Then

$$\begin{array}{rcl} f_\mathfrak{p} & = & \sum_{\mathfrak{q}\in S} a^\mathfrak{q} (u^\mathfrak{q})_\mathfrak{p} \\ & \equiv_\mathfrak{p} & \sum_{\mathfrak{q}\in S} \partial_{\mathfrak{p},\mathfrak{q}} (u^\mathfrak{q})_\mathfrak{p} \\ & = & (u^\mathfrak{p})_\mathfrak{p} \\ & \equiv_\mathfrak{p} & f^\mathfrak{p}. \end{array}$$

Corollary D.2.16 *Retain the notation of (D.2.15). If some $f^\mathfrak{p}$ is an epimorphism, so is $f_\mathfrak{p}$ by Nakayama's Lemma (cf. E.1.5).*

Thus, if some $f^\mathfrak{p}$ is an isomorphism, so is $f_\mathfrak{p}$.

Suppose some $f^\mathfrak{p}$ is a split epimorphism with coretraction $g^\mathfrak{p} f^\mathfrak{p} = 1$. Globalization of $g^\mathfrak{p}$ to g with respect to $S = \{\mathfrak{p}\}$ yields $g_\mathfrak{p} f_\mathfrak{p} \equiv_\mathfrak{p} 1$. Hence $g_\mathfrak{p} f_\mathfrak{p}$ is an automorphism by Nakayama's Lemma, so that $f_\mathfrak{p}$ is a split epimorphism.

NB in case of S being, say, the set of prime divisors of $\mathfrak{a}$, for X and Y nonisomorphic but in the same genus, a globalization f of the corresponding local isomorphisms at $\mathfrak{p} \in S$ must not be an epimorphism, although it is an isomorphism ('semi')localized at $\bigcap_{\mathfrak{p}\in S}(R\backslash\mathfrak{p})$, since the cokernel of f has torsion away from S.

Moreover, for R semilocal we see that $X \vee Y$ implies $X \simeq Y$.

Lemma D.2.17 (Roiter) *For X and Y lattices over Λ lying in the same genus, there exists a short exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow T \longrightarrow 0,$$

where T is a torsion module with annihilator coprime to Higman(Λ)*, decomposing into a direct sum of simple Λ-modules with different annihilators in R. The finite set of primes $\mathfrak{p}$ for which $T_\mathfrak{p} \neq 0$ can be chosen away from any given finite set of primes S.*

Let $\mathcal{R}(X)$ be a set of representatives of isomorphism classes of lattices rationally isomorphic to X, so that in particular $X, Y \in \mathcal{R}(X)$. The set $\mathcal{R}(X)$ is finite by Jordan-Zassenhaus (D.2.7). For $U, V \in \mathcal{R}(X)$, let $\mathrm{Spec}(U,V) \subseteq \mathrm{Spec}R$ be the (possibly empty) set of primes $\mathfrak{p}$ for which an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow T \longrightarrow 0$$

exists such that T is a simple torsion Λ-module with annihilator $\mathfrak{p}$. Let S' be the union over $U, V \in \mathcal{R}(X)$ of those sets $\mathrm{Spec}(U,V)$ which are finite, joined moreover with the set of prime divisors of Higman(Λ) and with S. Note that S' is a finite set. Globalizing local isomorphisms at primes in S' (D.2.15), we obtain an embedding of X into Y with annihilator in R of the cokernel away from S'. We filter this embedding

$$X = X_s \subseteq X_{s-1} \subseteq \cdots \subseteq X_1 \subseteq X_0 = Y$$

with simple quotients X_i/X_{i+1}, having annihilator away from S'. By construction, $\mathrm{Spec}(X_{i+1}, X_i)$ is infinite. Replacing embeddings, we may assume the annihilators of the quotients to be coprime. However, the $\mathfrak{p}$-part decomposition of Y/X (A.4.10) is respected by the operation of Λ, whence the quotient fulfills our requirements.

Lemma D.2.18 *Suppose given a full embedding $X \subseteq Y$ of Λ-lattices such that the annihilator of the quotient $T := X/Y$ is coprime to* Higman(Λ)*. Let V be a Λ-lattice, let $V \xrightarrow{f} T$ be a morphism of Λ-modules.*

Then the pullback short exact sequence of

$$0 \longrightarrow X \longrightarrow Y \longrightarrow T \longrightarrow 0$$

along $V \xrightarrow{f} T$ *vanishes in* Ext^1.

By (A.4.5), there is an element $h \in \mathrm{Higman}(\Lambda)$ that is not contained in a prime ideal factor of the annihilator of T, thus it annihilates $\mathrm{Ext}^1(V, X)$ and its multiplication on T is invertible. The factorization

$$(V \xrightarrow{f} T) = (V \xrightarrow{h} V \xrightarrow{f} T \xrightarrow[\sim]{h^{-1}} T)$$

can be used to pull back the short exact sequence (X, Y, T) stepwise and to show that the result is zero.

Lemma D.2.19 *Suppose given Λ-lattices X and Y lying in the same genus, $X \vee Y$, and suppose given a Λ-lattice V whose set of rational simple components contains that of X. Then there exists a Λ-lattice U in the same genus as V such that*

$$Y \oplus U \simeq X \oplus V.$$

By (D.2.17), we may choose an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow T \longrightarrow 0$$

in which the annihilator of T is coprime to $\mathfrak{a}$, thus in particular coprime to $\mathrm{Higman}(\Lambda)$ (B.2.3), and where $T_\mathfrak{p}$ is zero or a simple torsion module over $\Lambda_\mathfrak{p}$. If it is simple, since for such a prime ideal $\mathfrak{p}$ we have $\Lambda/\mathfrak{p} = \Gamma/\mathfrak{p}$, $T_\mathfrak{p}$ is a isomorphic to a column in $\Gamma/\mathfrak{p}$ and $V/\mathfrak{p}$ is isomorphic to a direct sum of such columns, by assumption containing an isomorphic copy of $T_\mathfrak{p}$ as a summand. Collecting the resulting epimorphisms $V \longrightarrow T_\mathfrak{p}$ furnishes a short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow T \longrightarrow 0$$

whose right exactness we see locally. (D.2.14) yields $U \vee V$. Two applications of (D.2.18) yield the assertion.

> **Remark D.2.20** For $\Lambda = R$, $X = R$, $Y = \mathfrak{b} \subseteq R$ a nonzero ideal of R and $V = R$ we recover the fact that $\mathfrak{b}$ is generated by two elements.

Lemma D.2.21 *The Λ-lattices X and Y lie in the same genus iff there exists an integer $s \geq 1$ such that $X^s \simeq Y^s$. In particular, indecomposable projectives over Λ are in the same genus iff they are in the same genus′ (cf. D.1.1).*

Krull-Schmidt locally (C.2.15) allows to conclude that $X^s \simeq Y^s$ for an integer $s \geq 1$ implies $X \vee Y$.

Conversely, by (D.2.19) we see that for each $k \geq 1$ there is a Λ-lattice $Z_k \vee X$ such that

$$X^{k+1} \simeq Y^k \oplus Z_k.$$

Choose $i + 1 \leq j$ such that $Z_i \simeq Z_{2j} =: Z$, which is possible by Jordan-Zassenhaus (D.2.7). In fact, assuming the contrary, we let i run over a finite interval comprising all occurring isomorphism classes, and come thus to a contradiction for j sufficiently large. We conclude that

$$\begin{aligned} X^{2j-i} \oplus X^{i+1} &\simeq Y^{2j} \oplus Z \\ &\simeq Y^{2j-i} \oplus Y^i \oplus Z \\ &\simeq Y^{2j-i} \oplus X^{i+1}. \end{aligned}$$

Adding $X^{2j-2i-1}$, we obtain that there is a $k \geq 1$ such that

$$X^{2k} \simeq Y^k \oplus X^k.$$

Replacing X by Y^k and Y by X^k the argument just given yields an $l \geq 1$ such that

$$Y^{2kl} \simeq X^{kl} \oplus Y^{kl} \simeq X^{2kl}.$$

D.3 Jacobinski's Cancellation Theorem

We give an account of the proof of Jacobinski's Cancellation Theorem in our particular case of a sub split semisimple R-order Λ, following, according to [Rog 70], an unpublished proof of SWAN.

As a precursor, we mention the following result.

Proposition D.3.1 (I. SCHUR, **[Sch 12, §3, II]**) *Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero ideals in R and let $m \geq 1$. If $R^m \oplus \mathfrak{a} \simeq R^m \oplus \mathfrak{b}$, then $\mathfrak{a} \simeq \mathfrak{b}$.*

In view of (A.4.9) this assertion in fact is a precursor of (D.3.6).

Let $X \subseteq R^n$ be a full R-sublattice. We write the elements of R^n as rows. Define its **determinant ideal** to be

$$\det(X \subseteq R^m) := (\det(\xi_{ij})_{ij} \mid (\xi_{ij})_j \in X \subseteq R^n \text{ for all } i) \subseteq R.$$

Using suitable main diagonal matrices, we see that

$$\det(R^{m-1} \oplus \mathfrak{a} \subseteq R^m) = \mathfrak{a}.$$

We **claim** that the determinant ideal transforms composition into the product of ideals, i.e. that

$$\det(X \subseteq R^m \xrightarrow{(-)A} R^m) = \det(X \subseteq R^m)(\det A)$$

for $A \in (R)_m$, $\det A \neq 0$.

The inclusion $\subseteq$ follows from $(\eta_{ij})_j \in XA$ for all i implying $\det((\eta_{ij})_{ij} A^{-1}) \in \det(X \subseteq R^m)$. The inclusion $\supseteq$ follows from $(\xi_{ij})_j \in X$ for all i implying $\det((\xi_{ij})_{ij} A) \in \det(X \subseteq R^m \xrightarrow{(-)A} R^m)$.

We tensor the isomorphism $R^{m-1} \oplus \mathfrak{a} \xrightarrow[\sim]{f} R^{m-1} \oplus \mathfrak{b}$ with $K \otimes_R -$. Restricting the resulting map $K^m \xrightarrow[\sim]{Kf} K^m$ to R^m, and restricting also its image to $x^{-1}R^m$, $x \in R\backslash 0$, we obtain a map $R^m \xrightarrow{f'} x^{-1}R^m$ which restricts to f and which yields

$$\begin{aligned}
\mathfrak{a}(\det f') &= \det(R^{m-1} \oplus \mathfrak{a} \subseteq R^m \xrightarrow{f'} x^{-1}R^m) \\
&= \det(R^{m-1} \oplus \mathfrak{a} \xrightarrow[\sim]{f} R^{m-1} \oplus \mathfrak{b} \subseteq x^{-1}R^m) \\
&= \det(R^{m-1} \oplus \mathfrak{b} \subseteq R^m \subseteq x^{-1}R^m) \\
&= \det(R^{m-1} \oplus \mathfrak{b} \subseteq R^m)(x^m) \\
&= \mathfrak{b}(x^m).
\end{aligned}$$

Lemma D.3.2 (EICHLER, SWAN, **in our particular situation**) *Suppose given a simple Λ-module U with annihilator $\mathfrak{p}$ in R coprime to $\mathfrak{a}$, a nonzero ideal $\mathfrak{b}$ of R coprime to $\mathfrak{p}$, a lattice X over Λ and two epimorphisms*

$$X \xrightarrow{f} U$$
$$X \xrightarrow{g} U$$

Then there exists an automorphism $X \xrightarrow[\sim]{u} X$ such that $u \equiv_{\mathfrak{b}} 1_X$, and which restricts to an isomorphism on the kernels $K_f \xrightarrow[\sim]{u} K_g$. NB we do **not** *require $ug = f$.*

We remark that neither this lemma nor (D.3.6) hold for a general R-order, semisimple when tensored with K, but only under the extra assumption of the so called Eichler condition which we won't explain here. The proof in the general situation is much harder.

We shall not need the assertion $u \equiv_{\mathfrak{b}} 1$ except in a reduction step of the proof itself.

We reduce to the case $\Lambda = \Gamma$ (which we do not exclude for our present purpose). So suppose the assertion to hold for Γ and apply it in the situation of the simple Γ-module U - note that $\Lambda_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}$ -, the Γ-lattice $\Gamma X = \Gamma \tilde{\otimes}_{\Lambda} X$ (B.1.3), the ideal $\mathfrak{a}\mathfrak{b}$ coprime to $\mathfrak{p}$ and the epimorphisms

$$\Gamma X \xrightarrow{f'} U$$
$$\Gamma X \xrightarrow{g'} U$$

obtained by lifting f and g, using $X_\mathfrak{p} \xrightarrow{\sim} (\Gamma X)_\mathfrak{p}$ and the factorization $(X \longrightarrow U) = (X \longrightarrow X_\mathfrak{p} \longrightarrow U)$. We obtain an automorphism $\Gamma X \xrightarrow[\sim]{u'} \Gamma X$ such that the restriction of u' to the kernels of f' resp. of g' induces an isomorphism and such that $u' \equiv_{\mathfrak{ab}} 1$. This implies that given $x \in X$, we have $xu' - x \in \mathfrak{ab}\Gamma X \subseteq \mathfrak{b}X$. Taking for $X \xrightarrow{u} X$ the restriction of u' to X, this shows its well definedness as well as $u \equiv_\mathfrak{b} 1$. Moreover, $u'^{-1} \equiv_{\mathfrak{ab}} 1$ restricts to an inverse of u. The kernel of f is the intersection of X with the kernel of f' by the Circonference Lemma applied to $(X, \Gamma X, U)$. Dito for g. Thus, via pullback, u induces an isomorphism on the kernels.

Furthermore, we reduce to the case $\Lambda = R$. The validity of our assertion is invariant under Morita equivalences F which are compatible with R-module structure in the sense that for $r \in R$ we have $F(M \xrightarrow{rh} N) = FM \xrightarrow{rFh} FN$, for the annihilator of a module remains invariant under F, by regarding $r \cdot 1_M$, and for congruences of morphisms modulo $\mathfrak{c} \subseteq R$ are preserved, since we may write $\mathfrak{c}M$ as $\sum_{c\in\mathfrak{c}} \mathrm{Im}\,(M \xrightarrow{c} M)$ which gives $F\mathfrak{c}M = \mathfrak{c}FM$. Thus we are reduced to the case of Λ being a direct product of copies of R. Now since U is a simple module over one of the factors of Λ and since the category Λ-mod splits accordingly into a direct product of copies of R-mod, we are reduced to the case of $\Lambda = R$.

Let $U = R/\mathfrak{p}$ and let $X = \bigoplus_{i=1}^m \mathfrak{c}_i$, the $\mathfrak{c}_i$'s being nonzero ideals in R (A.4.9).

Case $m = 1$**.** This is the case in which we **cannot** achieve $ug = f$ in general. Note that $\mathfrak{c}_1/\mathfrak{c}_1\mathfrak{p} \simeq R/\mathfrak{p}$, whence ${}_R(\mathfrak{c}_1, R/\mathfrak{p}) \simeq R/\mathfrak{p}$ so that we may choose $u = 1$.

Case $m \geq 2$**.** We achieve $ug = f$ in the following manner. Suppose the epimorphism f resp. g to be given by nonzero matrices F resp. G with entries in $R/\mathfrak{p}$ written as $m \times 1$-columns. There is an element $U \in \mathrm{SL}_m(R/\mathfrak{p})$ such that $UG = F$ - let $(F\ *) \in \mathrm{SL}_m(R/\mathfrak{p})$ have F as first column, let $(G\ *) \in \mathrm{SL}_m(R/\mathfrak{p})$ and choose $U = (F\ *)(G\ *)^{-1} \in \mathrm{SL}_m(R/\mathfrak{p})$.

We modify the argument of (A.2.1). Note that ${}_R(\mathfrak{c}_i, \mathfrak{c}_j) = \mathfrak{c}_i^{-1}\mathfrak{c}_j \subseteq K$. Choose entrywise an inverse image of $U \in \mathrm{SL}_m(R/\mathfrak{p})$ in $\mathrm{Aut}\, X_\mathfrak{p} \simeq \mathrm{GL}_m(R_\mathfrak{p})$, which, by the Elemtary Divisor Theorem (A.1.1) may be assumed to be a product of elementary matrices after replacement of the remaining diagonal matrix factor by the identity without changing its image in $\mathrm{SL}_m(R/\mathfrak{p})$. Modifying the non main diagonal entries of these elementary matrices without changing their image in $\mathrm{SL}_m(R/\mathfrak{p})$, we may assume that our inverse image lies in $\mathrm{Aut}\, X \subseteq (\mathfrak{c}_i^{-1}\mathfrak{c}_j)_{ij} \subseteq (K)_m$ and that it maps, reducing modulo $\mathfrak{b}$, to the identity of $X/\mathfrak{b}X$. In fact, letting $\mathfrak{f} := \mathfrak{c}_i^{-1}\mathfrak{c}_j$, there is an epimorphism

$$\mathfrak{f} \longrightarrow \mathfrak{f}/\mathfrak{p}\mathfrak{f} \times \mathfrak{f}/\mathfrak{b}\mathfrak{f} \simeq R/\mathfrak{p} \times \mathfrak{f}/\mathfrak{b}\mathfrak{f}.$$

Lemma D.3.3 (BASS) *Let A be a ring such that $A/\mathrm{rad}\, A$ is finite as a set. Let $m \geq 3$. $\mathrm{GL}_m(A)$ is defined to be the automorphism group of A^m as a left A-module, which can be written in matrices, A^m viewed as a row. Suppose given a surjective group morphism*

$$\mathrm{GL}_m(A) \xrightarrow{f} M,$$

M being an abelian group. Then the restriction of f to $\mathrm{GL}_1(A)$, sitting in the top left corner of $\mathrm{GL}_m(A)$, the rest of the main diagonal being of constant value 1, is surjective.

In other words, we have to show that each element of $\mathrm{GL}_m(A)$ can be represented by an element of $\mathrm{GL}_1(A)$ modulo the commutator subgroup $[\mathrm{GL}_m(A), \mathrm{GL}_m(A)]$. Note that the assumption on A yields $A/\mathrm{rad}\, A$ to be a product of matrix rings over finite fields.

We **claim** that each elementary matrix $E + xE_{ij}$, having non main diagonal entry $x \in A$ at position ij, $i \neq j$, is in $[\mathrm{GL}_m(A), \mathrm{GL}_m(A)]$. In fact, choose $k \neq i$, $k \neq j$, which is possible since $m \geq 3$. We calculate

$$\begin{aligned}(E - E_{ik})(E - xE_{kj})(E + E_{ik})(E + xE_{kj}) &= (E - E_{ik} - xE_{kj} + xE_{ij})(E + E_{ik})(E + xE_{kj}) \\ &= (E - xE_{kj} + xE_{ij})(E + xE_{kj}) \\ &= E + xE_{ij}.\end{aligned}$$

Therefore it suffices to show that each element of $\mathrm{GL}_m(A)$ can be represented by an element of $\mathrm{GL}_1(A)$ modulo the normal subgroup generated by the elementary matrices.

Suppose given $(a_{ij})_{ij} \in GL_m(A)$. We **claim** that we may diagonalize by multiplication with elementary matrices from both sides to obtain a main diagonal matrix with units on the diagonal, reminiscent of the Elementary Divisor Theorem. Consider the column $(a_{11},\ldots,a_{m1})$. By left invertibility of $(a_{ij})_{ij}$, the sum of the left ideals Aa_{11} and $L := A\langle a_{21},\ldots,a_{m1}\rangle$ is A. We need to find elements $s_2,\ldots,s_m$ such that $a_{11}+\sum_{i\in[2,m]} s_i a_{i1}$ is a unit - i.e. it is left and right invertible, equivalently, its right multiplication is bijective -, for then left multiplication with $E+\sum_{i\in[2,m]} s_i E_{1i}$ yields this unit in the upper left corner, which then can be used to clean the upper row and the left column from nonzero entries. Whence the claim by induction.

In order to find such an element of L we may assume A to equal $A/\mathrm{rad}\,A$, since $u\in A$ is a unit iff $u\in A/\mathrm{rad}\,A$ is a unit - use Nakayama's Lemma to show that $(-)u$ is surjective, then split off its kernel and use Nakayama again. Now since A is semisimple, we may assume $A = Aa_{11}\oplus L$, if necessary by passing from L to a smaller left ideal. This decomposition can be written as an isomorphism of left A-modules

$$\begin{array}{ccc} Aa_{11}\oplus L & \xrightarrow{\sim} & A \\ x\oplus y & \longrightarrow & x+y. \end{array}$$

Moreover, multiplication with a_{11} yields a split exact sequence

$$0\longrightarrow L'\xrightarrow{\;i\;} A\xrightarrow{(-)a_{11}} Aa_{11}\longrightarrow 0$$

whose kernel L' is isomorphic to L by Krull-Schmidt, say, via $L'\xrightarrow[\sim]{w} L$. Let $A\xrightarrow{\;v\;} L'$ be a retraction of i, so that we obtain another isomorphism

$$A\xrightarrow[\sim]{((-)a_{11}\ \ vw)} Aa_{11}\oplus L.$$

The composition of these isomorphisms is the right multiplication with an element of A, therefore, it sends 1_A to a unit in A, viz. to $a_{11}+(1)vw$. By construction, $(1)vw$ is in L.

It remains to be remarked that a diagonal matrix (d_{ij}) with $d_{11}=d_{ii}^{-1}$ being a unit, $i\neq 1$, and with $d_{jj}=1$ for $j\notin\{1,i\}$, is a product of elementary matrices (cf. the calculation in A.1) in order to reduce to a matrix in $GL_1(A)$.

Definition D.3.4 *Let $G_0^{\mathrm{tors},\mathfrak{a}}(\Lambda)$ be the free abelian group on the isomorphism classes of the simple Λ-modules with annihilator in R coprime to $\mathfrak{a}$.*

Note that a torsion Λ-module T with annihilator coprime to $\mathfrak{a}$ has an image $[T]$ in $G_0^{\mathrm{tors},\mathfrak{a}}(\Lambda)$, letting $[T]$ be the (formal) sum of its composition factors, which is well defined by Jordan-Hölder.

Lemma D.3.5 *Let $S := \bigcap_{\mathfrak{p}\supseteq\mathfrak{a}}(R\backslash\mathfrak{p})$. For an R-module M, denote by M_S the localization of M at S. Let X be a lattice over Λ. There is a group morphism*

$$\begin{array}{ccc} GL_m(\mathrm{End}_{\Lambda_S}X_S) & \xrightarrow{\;c\;} & G_0^{\mathrm{tors},\mathfrak{a}}(\Lambda) \\ (X_S^m \xrightarrow[\sim]{\xi/s} X_S^m) & \longrightarrow & [C_\xi]-[C_s] \end{array}$$

where $X^m\xrightarrow{\;\xi\;}X^m$ is a monomorphism, $s\in S$ and where C denotes the cokernel, in particular, where C_s denotes the cokernel of $X^m\xrightarrow{\;s\;}X^m$.

By (D.3.3), an element in the image of this morphism can be written as the image of some element in GL_1, i.e. as difference of the G_0-images of the cokernels of ξ and $s\in S$ for some automorphism

$$X_S\xrightarrow[\sim]{\xi/s} X_S$$

in case $m\geq 3$.

c is well defined since for $t\in S$ the Circonference Lemma shows

$$[C_{\xi t}]-[C_{st}] = ([C_\xi]+[C_t])-([C_s]+[C_t]).$$

c is a group morphism since by the Circonference Lemma the image of $(\xi/s)(\eta/t)$ is

$$[C_{\xi\eta}]-[C_{st}] = ([C_\xi]+[C_\eta])-([C_s]+[C_t]).$$

$\mathrm{End}_{\Lambda_S}(X_S)/\mathrm{rad}\,\mathrm{End}_{\Lambda_S}(X_S)$ is finite as a set since $\mathrm{rad}\,\mathrm{End}_{\Lambda_S}(X_S)$ contains $\mathfrak{a}\mathrm{End}_{\Lambda_S}(X_S)$, the latter annihilating all simple modules of $\mathrm{End}_{\Lambda_S}(X_S)$. Hence (D.3.3) may be applied.

Theorem D.3.6 (JACOBINSKI**'s cancellation theorem, in our particular situation)**
Let X and Y be lattices over Λ in the genus of Λ such that $X \oplus \Lambda^i \simeq Y \oplus \Lambda^i$ for some $i \geq 1$. Then $X \simeq Y$.

NB in general this is false for Λ not sub split semisimple, see [Sw 62].

Using Krull-Schmidt (C.2.15) and globalizing morphisms (D.2.15), we choose a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\eta} X \longrightarrow T_1 \longrightarrow 0$$

with T_1 a torsion Λ-module with annihilator coprime to $\mathfrak{a}$.

By (D.2.19) we know Λ^i to be a summand of X^{i+1} so that $X^{i+2} \simeq Y \oplus X^{i+1}$. This yields a short exact sequence

$$0 \longrightarrow X^{i+2} \longrightarrow X^{i+2} \longrightarrow T_1 \longrightarrow 0.$$

So by (D.3.5) we obtain exact sequences

$$0 \longrightarrow X \xrightarrow{\xi} X \longrightarrow T \longrightarrow 0$$

and

$$0 \longrightarrow X \xrightarrow{s} X \longrightarrow T_2 \longrightarrow 0$$

with $s \in S := \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} (R \backslash \mathfrak{p})$ and $[T] - [T_2] = [T_1] \in G_0^{\text{tors},\mathfrak{a}}(\Lambda)$ so that, by the Circonference Lemma, the $G_0^{\text{tors},\mathfrak{a}}(\Lambda)$-images of the cokernels of

$$0 \longrightarrow Y \xrightarrow{\eta s} X \longrightarrow T' \longrightarrow 0$$

and of

$$0 \longrightarrow X \xrightarrow{\xi} X \longrightarrow T \longrightarrow 0$$

coincide. Since $\Lambda_{\mathfrak{p}} = \Gamma_{\mathfrak{p}}$ for $\mathfrak{p}$ coprime to $\mathfrak{a}$ and since therefore $T = \bigoplus_{\mathfrak{p}} T_{\mathfrak{p}}$ and $T' = \bigoplus_{\mathfrak{p}} T'_{\mathfrak{p}}$ decompose further into the components belonging to the matrix factors of $\Gamma_{\mathfrak{p}}$ and since these are Morita equivalent to $R_{\mathfrak{p}}$, we may choose filtrations

$$\begin{array}{ccccccccc} Y\eta s = Y_0 & \subseteq & Y_1 & \subseteq & \dots & \subseteq & Y_{k-1} & \subseteq & Y_k = X \\ X\xi = X_0 & \subseteq & X_1 & \subseteq & \dots & \subseteq & X_{k-1} & \subseteq & X_k = X \end{array}$$

with $X_i/X_{i-1} \simeq Y_i/Y_{i-1}$ for $i \in [1,k]$ by pulling back such filtrations of T resp. of T', using $[T] = [T']$.

Assuming $Y_i \simeq X_i$, we conclude that $Y_{i-1} \simeq X_{i-1}$ by Eichler-Swan (D.3.2).

D.4 Basics on $\mathfrak{p}$-orders

Suppose Λ to be a $\mathfrak{p}$-order (D.2.8).

Proposition D.4.1 (globalizing decompositions) *Let X and Y be lattices over Λ. A split epimorphism $X_{\mathfrak{p}} \xrightarrow{f^{\mathfrak{p}}} Y_{\mathfrak{p}}$ is, up to isomorphic substitution, the localization at $\mathfrak{p}$ of a split epimorphism $X \xrightarrow{f} Y'$.*

First, we note that this does not follow from a globalization of $f^{\mathfrak{p}}$ as in (D.2.15), cf. (D.2.16), and that we do **not** claim that $Y \simeq Y'$.

Inserting the image Y' of the composition $X \longrightarrow X_{\mathfrak{p}} \xrightarrow{f^{\mathfrak{p}}} Y_{\mathfrak{p}}$, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_{\mathfrak{p}} & \longrightarrow & X_{\mathfrak{p}} & \xrightarrow{f^{\mathfrak{p}}} & Y_{\mathfrak{p}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & Z' & \longrightarrow & X & \xrightarrow{f} & Y' & \longrightarrow & 0. \end{array}$$

Localizing this diagram at $\mathfrak{p}$, the vertical morphisms become isomorphisms. Since the morphism

$$\mathrm{Ext}^1_\Lambda(Y', Z') \xrightarrow{\sim} \mathrm{Ext}^1_{\Lambda_\mathfrak{p}}(Y'_\mathfrak{p}, Z'_\mathfrak{p})$$

induced by localization is an isomorphism (B.2.3, A.3.2, A.3.3, A.4.10), we conclude that f is a split epimorphism.

Corollary D.4.2 *A Λ-lattice X is indecomposable iff $X_\mathfrak{p}$ is indecomposable over $\Lambda_\mathfrak{p}$ (D.4.1).*

Remark D.4.3 (dangerous bend) *The argument of (D.4.1) fails when localization is replaced by completion* ([1])*.*

First we note that the argument has to fail, since Heller's Lemma (C.2.10), needed in order to pull Krull-Schmidt down from the complete to the noncomplete local case, is used only to see that a summand of the completion is the completion of some lattice, which also would ensue from the modified version of the argument of (D.4.1). And Krull-Schmidt is known to fail in the noncomplete local case, provided of course the requirements of Heller's Lemma are not met. But here we are interested in an analysis of the argument, not in a counterexample (cf. A C.3).

Let S be a noncomplete discrete valuation ring with maximal ideal (π). **We wish to see that on not necessarily finitely generated modules, completion and $\hat{S} \otimes_S -$ are nonisomorphic functors in general and that both are not suited as a replacement of localization in the argument of (D.4.1).**

Recall that a general element of the completion $\hat{M} := \varprojlim_i M/\pi^i$ of an S-module M is represented by a sequence $(m_i)_i = (m_1, m_2, \ldots)$ of elements $m_i \in M$ such that $m_i \equiv_{\pi^{i-1}} m_{i-1}$. $(m_i)_i$ represents zero iff $m_i \in \pi^i M$ for all $i \geq 1$. In this case we simply write $(m_i)_i = 0$.

Let us describe the submodule $\pi^j \hat{M} \subseteq \hat{M}$. A sequence $(\pi^j m_i)_i$ has its i-th entry in $\pi^i M$ for $i \leq j$ and in $\pi^j M$ for $i \geq j$. Any sequence representing the same element as $(\pi^j m_i)_i$ also enjoys this property. Conversely, if $(m_i)_i$ is a sequence satisfying these conditions, we shift it by j positions to the left without changing the element in $\hat{M}$ it represents. Thus we may assume that $(m_i)_i$ is such that $m_i \in \pi^j M$ and such that $m_{i+1} \equiv_{\pi^{i+j}} m_i$. Let $m_i = \pi^j m'_i$. Write $\pi^j(m'_{i+1} - m'_i) = \pi^{j+i} v_i$ for some v_i for each $i \geq 1$. Letting $u_i := m'_{i+1} - m'_i - \pi^i v_i$ we obtain $\pi^j u_i = 0$. Hence $\pi^j(m'_1, m'_2 - u_1, m'_3 - u_2 - u_1, \ldots) = (m_i)_i$ yields $(m_i)_i \in \pi^j \hat{M}$.

Define M to be **complete** if the natural transformation

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon X} & \hat{X} \\ x & \longrightarrow & (x)_i \end{array}$$

is an isomorphism εM at M. Let ι be the inclusion functor of the full subcategory of complete S-modules to the category of all S-modules. We **claim** that the completion functor factors over ι, i.e. that the completion of a module is complete. Again, let M be an S-module. We have to consider the map

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\varepsilon \hat{M}} & (\hat{M})\hat{} \\ (m_i)_i & \longrightarrow & ((m_i)_i)_j \end{array}$$

Suppose $((m_i)_i)_j = 0$, i.e. suppose $(m_i)_i \in \pi^j \hat{M}$ for all j. We conclude that $m_i \in \pi^i M$ for all i, hence that $(m_i)_i = 0$, and that therefore $\varepsilon \hat{M}$ is injective.

Suppose given $((m_{ji})_i)_j \in (\hat{M})\hat{}$. We **claim** that it equals its diagonal, i.e. that

$$((m_{ji} - m_{ii})_i)_j = 0 \in (\hat{M})\hat{},$$

thus proving surjectivity, since $m_{i+1,i+1} \equiv_{\pi^i} m_{i,i+1} \equiv_{\pi^i} m_{ii}$. We have to show that $(m_{ji} - m_{ii})_i$ is in $\pi^j \hat{M}$, i.e. that for $i \leq j$ the congruence $m_{ji} \equiv_{\pi^i} m_{ii}$ holds - which is true since

[1] This remark consists of slightly extended notes taken from a discussion with S. König. It is hoped that this discussion of a wrong argument is sufficiently justified by its 'correct looks' at first sight.

$(m_{jk})_k \equiv_{\pi^i} (m_{ik})_k$ -, and that for $i \geq j$ the congruence $m_{ji} \equiv_{\pi^j} m_{ii}$ holds - which is also satisfied since $(m_{ik})_k \equiv_{\pi^j} (m_{jk})_k$.

Let c denote the completion functor with target being the full subcategory of complete S-modules, i.e. $\hat{M} = \iota c M$. M is in the image of ι iff the unit $M \xrightarrow{\varepsilon M} \iota c M$ is an isomorphism. Let the counit $c\iota X \xrightarrow{\eta X} X$ be defined by

$$(\iota c \iota X \xrightarrow{\iota\eta X} \iota X) := (\iota X \xrightarrow{\varepsilon\iota X} \iota c \iota X)^{-1}.$$

Note that thus η is an isomorphism at all objects at which it is defined.

In order to obtain

$$c \dashv \iota$$

it remains to be seen that the composition

$$cM \xrightarrow{c\varepsilon M} c\iota c M \xrightarrow{\eta c M} cM$$

is the identity. An application of ι yields

$$(\iota c M \xrightarrow{\iota c\varepsilon M} \iota c \iota c M \xrightarrow{\iota\eta c M} \iota c M) = (\iota c M \xrightarrow{\iota c\varepsilon M} \iota c \iota c M \xleftarrow{\varepsilon\iota c M} \iota c M).$$

Thus we are reduced to verify that $\iota c \varepsilon M = \varepsilon \iota c M$. The map $\iota c\varepsilon M$ sends $(m_i)_i$ to $((m_j)_i)_j$, whereas $\varepsilon\iota c M$ sends $(m_i)_i$ to $((m_i)_i)_j$. But by the claim above on the diagonalization, $((m_i - m_j)_i)_j$ equals its diagonal $((m_i - m_i)_i)_j$.

In particular, c commutes with cokernels and ι commutes with kernels.

We would like to see that the subcategory of complete S-modules has kernels. Consider a morphism $M \xrightarrow{f} N$ of complete S-modules. Let $K_{\iota f} \xrightarrow{v} \iota M$ be the kernel of $\iota M \xrightarrow{\iota f} \iota N$ in the category of all S-modules. Since ι is full, faithful and left exact, in order to show that $(cv)(\eta M)$ is the kernel of f, it suffices to prove that $(\iota c v)(\iota \eta M)$ isomorphic to v over M, i.e. that $\varepsilon K_{\iota f}$ is an isomorphism, i.e. that $K_{\iota f}$ is complete. Since v is a monomorphism, an application of the universal property of $K_{\iota f} \xrightarrow{v} \iota M$ shows that $\varepsilon K_{\iota f}$ is a coretraction. Let L be its cokernel. Since the functor ιc is exact on split short exact sequences, and since $\iota c \varepsilon K_{\iota f} = (\iota \eta c K_{\iota f})^{-1}$ is an isomorphism, we conclude that $\iota c L = 0$. Choosing a coretraction $L \longrightarrow \iota c K_{\iota f}$ and noticing that we have a factorization $(L \longrightarrow \iota c K) = (L \xrightarrow{\varepsilon L} \iota c L \longrightarrow \iota c K_{\iota f})$ by adjunction, we obtain $L = 0$ ([2]).

Consider the short exact sequence of S-modules

$$0 \longrightarrow S \xrightarrow{\varepsilon S} \iota c S \longrightarrow \iota c S/S \longrightarrow 0,$$

where εS is injective since $\bigcap_i \pi^i S = 0$, which becomes

$$cS \underset{\sim}{\xrightarrow{c\varepsilon S}} c\iota c S \longrightarrow c(\iota c S/S) \longrightarrow 0$$

under c, right exact in the subcategory of complete S-modules, from which we conclude $c(\iota c S/S) = (\hat{S}/S)\hat{}$ to be zero. Note that c annihilates no finitely generated S-module, neither S nor S/π^i. Since S is not complete, we may choose a finitely generated nonzero submodule $M \hookrightarrow \hat{S}/S$. An application of c to this inclusion shows c and $\iota c = (-)\hat{}$ not to be left exact, for the result is a morphism with nonzero source and zero target.

Thus in the argument of (D.4.1), transcribed to completions, we may not conclude that the completion of $Y' \longrightarrow \hat{Y}$ is a monomorphism, so that we may not continue and conclude that it is an isomorphism.

Note that on **finitely generated** S-modules, such as S and S/π^i, c and $\hat{S} \otimes_S -$ are isomorphic and c is an exact functor [AM 69, 10.12]. Therefore $\hat{S} \otimes_S -$ is exact on all S-modules. In fact, an arbitrary module can be written as the direct limit of finitely generated

[2] This argument I've learnt from H. Reimann.

submodules so that in calculating $\operatorname{Tor}_1^S(\hat{S}, X)$, we may resolve $\hat{S}$ by S-projectives P_j, write X as direct limit of finitely generated X_α's and use that

$$\varinjlim_\alpha (P_j \otimes_S X_\alpha) \xleftarrow{\sim} P_j \otimes_S \varinjlim_\alpha X_\alpha$$

as well as the exactness of the direct limit to conclude that in fact $\operatorname{Tor}_1(\hat{S}, X)$ vanishes. In particular, completion and tensor product with $\hat{S}$ over S are nonisomorphic functors on the category of all S-modules.

But if one wishes to repair the transport of the argument of (D.4.1) by a replacement of completion by this tensor product, one is confronted with the sequence

$$0 \longrightarrow \hat{S} \otimes_S S \xrightarrow{\hat{S}\otimes_S \varepsilon S} \hat{S} \otimes_S \hat{S} \longrightarrow \hat{S} \otimes_S (\hat{S}/S) \longrightarrow 0,$$

for which flatness of $\hat{S}$ over S together with nonvanishing of $\hat{S} \otimes_S -$ on finitely generated modules this time shows that the cokernel is **non**zero so that $\hat{S} \otimes_S \varepsilon$ is not an isomorphism in general. In particular, we may not conclude that the middle vertical morphism in the diagram of the transcribed argument becomes an isomorphism under $\hat{S} \otimes_S -$.

Finally, we try to apply this reasoning which destroys the transcribed argument to our original argument. Let S be a commutative ring, let $\mathfrak{p}$ be a prime ideal of S. But now $S_\mathfrak{p}/S$ is a torsion S-module that has **no** finitely generated submodule isomorphic to some $S/\mathfrak{p}^i$, and thus may well be annihilated when localized at $\mathfrak{p}$.

Lemma D.4.4 *An indecomposable projective Λ-module P is a summand of Λ^2.*

$P_\mathfrak{p}$ is a direct summand of $\Lambda_\mathfrak{p}$ by its indecomposability (D.4.2) and by Krull-Schmidt (C.2.15). Lifting the corresponding split epimorphism by (D.4.1) we obtain P to be in the same genus as an indecomposable summand Λe of Λ, e being a primitive idempotent of Λ (D.2.14). Therefore, P is a summand of $(\Lambda e)^2$ (D.2.19), which itself is a summand of Λ^2.

Lemma D.4.5 *A finite projective P over Λ is a progenerator - i.e. Λ is a direct summand of some P^m - iff $P_\mathfrak{p}$ is a progenerator.*

If Λ is a direct summand of P^m, $\Lambda_\mathfrak{p}$ is a direct summand of $P_\mathfrak{p}^m$.

Conversely, suppose $\Lambda_\mathfrak{p}$ to be a summand of $P_\mathfrak{p}^m$. Hence there is a summand L of P^m in the genus of Λ by a lift of decompositions (D.4.1, D.2.14). Now Λ is a summand of L^2 (D.2.19), thus of P^{2m}.

Lemma D.4.6 *The localization map*

$$\begin{array}{ccc} \operatorname{ip}(\Lambda) & \xrightarrow{\text{loc}} & \operatorname{ip}(\Lambda_\mathfrak{p}) \\ P & \longrightarrow & P_\mathfrak{p} \end{array}$$

is well defined (D.4.2) and surjective (D.2.9 applied to a chosen Λ-lattice inside).

Remark D.4.7 *Krull-Schmidt holds in* Λ-proj *iff* loc *is bijective.*

Suppose loc to be bijective. Suppose given two decompositions of a finite projective over Λ into indecomposables. The bijection and the isomorphisms between the respective summands given locally are also given globally, for loc is bijective.

The converse follows from (D.2.14) and (D.2.19 or D.2.21).

Lemma D.4.8 (fibration of loc**)** *Let $1_\Lambda = \sum_i e_i$ be an orthogonal decomposition into primitive idempotents. Then we have the set theoretical pullback*

$$\begin{array}{ccc} \operatorname{ip}(e_i\Lambda e_i) & \xrightarrow{\text{loc}_i} & \operatorname{ip}(e_i\Lambda_\mathfrak{p} e_i) = * \\ {\scriptstyle \Lambda e_i \otimes_{e_i\Lambda e_i} -}\downarrow & \ulcorner & \downarrow{\scriptstyle \Lambda_\mathfrak{p} e_i \otimes_{e_i\Lambda_\mathfrak{p} e_i} -} \\ \operatorname{ip}(\Lambda) & \xrightarrow{\text{loc}} & \operatorname{ip}(\Lambda_\mathfrak{p}), \end{array}$$

i.e. ' loc *is fibered by primitive idempotents'. In other words, to investigate* loc *we can restrict ourselves to investigate* loc *for the local endomorphism rings of the indecomposable projectives over $\Lambda_\mathfrak{p}$.*

Note that the left vertical map is well defined since the image of an indecomposable projective over $e_i\Lambda e_i$ is sent by localization to the indecomposable projective $\Lambda_\mathfrak{p} e_i$, hence this image itself is indecomposable (D.4.2). The left vertical map is injective since it is inverted by $e_i(-)$.

It suffices see that each indecomposable projective P over Λ in the genus of Λe_i is in the image of the left vertical map. But e_iP is in fact an indecomposable projective over $e_i\Lambda e_i$, since it lies in the genus of $e_i\Lambda e_i$. The evaluation of the natural transformation

$$\begin{array}{rcl} \Lambda e_i \otimes_{e_i\Lambda e_i} e_iX & \longrightarrow & X \\ ae_i \otimes e_ix & \longrightarrow & ae_ix \end{array}$$

at $X = P$ is an isomorphism, since it is an isomorphism localized at any prime $\mathfrak{q}$, for we may substitute isomorphically $P_\mathfrak{q}$ by $\Lambda_\mathfrak{q} e_i$ on both sides of the transformation, and at the latter object it is in fact locally an isomorphism, yielding the result by naturality.

For $\Lambda_\mathfrak{p}$ local, the investigation of loc fits into the framework of class groups, which we shall review, cf. [CR 81, §31 B]. (Concerning (D.4.9) and (D.4.10), $\mathfrak{a}$ may be an arbitrary ideal of R.)

Definition D.4.9 *Recall that any Λ-lattice in the genus of Λ can be realized as a full sublattice of Λ.*

Suppose P to be a Λ-lattice in the genus of Λ inside $K\Lambda$ such that $KP = K\Lambda$.

P and Λ are isomorphic as lattices over Λ iff there is a unit $x \in (K\Lambda)^*$ (the image of $1 \in \Lambda$ under this isomorphism, note $K\Lambda x = K\Lambda$) such that $P = \Lambda x$. Mutatis mutandis localized at $\mathfrak{q}$.

We recover $P = \bigcap_\mathfrak{q} P_\mathfrak{q} \subseteq K\Lambda$. Hence, writing $P_\mathfrak{q} = \Lambda_\mathfrak{q} a_\mathfrak{q}$, $a_\mathfrak{q} \in \Lambda_\mathfrak{q}^*$ for all but finitely many $\mathfrak{q}$ (multiply P into Λ via $b \in R$ and regard the cokernel of the resulting inclusion), we obtain

$$P = \bigcap_\mathfrak{q} \Lambda_\mathfrak{q} a_\mathfrak{q}.$$

Conversely, such a tuple of elements $a_\mathfrak{q}$ yields a Λ-lattice in $K\Lambda$ that lies in the genus of Λ via this formula.

Therefore, we define the **idèle group** *of Λ to be*

$$I(\Lambda) := \{a = (a_\mathfrak{q})_\mathfrak{q} \in \prod_\mathfrak{q} (K\Lambda)^* \mid a_\mathfrak{q} \in \Lambda_\mathfrak{q}^* \textit{ almost everywhere } \}.$$

For an idèle $a \in I(\Lambda)$ we write

$$\Lambda a := \bigcap_\mathfrak{q} \Lambda_\mathfrak{q} a_\mathfrak{q},$$

so that $(\Lambda a)_\mathfrak{q} = \Lambda_\mathfrak{q} a_\mathfrak{q}$.

Lemma D.4.10 *Let $G(\Lambda) \subseteq I(\Lambda)$ be the image of the embedding of 'rational but global' units*

$$\begin{array}{rcl} (K\Lambda)^* & \longrightarrow & I(\Lambda) \\ x & \longrightarrow & (x)_\mathfrak{q}. \end{array}$$

Let $U(\Lambda) := \prod_\mathfrak{q} \Lambda_\mathfrak{q}^* \subseteq I(\Lambda)$.

Given idèles $a, b \in I(\Lambda)$, Λa is isomorphic to Λb iff the double cosets $U(\Lambda)aG(\Lambda)$ and $U(\Lambda)bG(\Lambda)$ coincide.

Consequently, if Λ is a $\mathfrak{p}$-order such that $\Lambda_\mathfrak{p}$ is local, the map $\mathrm{ip}(\Lambda) \xrightarrow{\mathrm{loc}} \mathrm{ip}(\Lambda_\mathfrak{p})$ *(D.4.6) is bijective iff $U(\Lambda)G(\Lambda) = I(\Lambda)$.*

We also write $I = I(\Lambda)$, $U = U(\Lambda)$, $G = G(\Lambda)$.

Suppose $\Lambda a \simeq \Lambda b$, i.e. there exists $x \in (K\Lambda)^*$ such that $\Lambda a = \Lambda bx$. Localizing at $\mathfrak{q}$ we obtain $\Lambda_\mathfrak{q} a_\mathfrak{q} = \Lambda_\mathfrak{q} b_\mathfrak{q} x_\mathfrak{q}$, whence $u_\mathfrak{q} \in \Lambda_\mathfrak{q}$ with $a_\mathfrak{q} = u_\mathfrak{q} b_\mathfrak{q} x_\mathfrak{q}$ exists, as well as $v_\mathfrak{q} \in \Lambda_\mathfrak{q}$ with $v_\mathfrak{q} a_\mathfrak{q} = b_\mathfrak{q} x_\mathfrak{q}$. From $a_\mathfrak{q} = u_\mathfrak{q} v_\mathfrak{q} a_\mathfrak{q}$ we deduce $1 = u_\mathfrak{q} v_\mathfrak{q}$. Altogether we obtain $a \in UbG$.

Conversely, let $u \in U$, $x \in G$, $a \in I$. Then, since the localizations coincide, $\Lambda uax = \Lambda ax \simeq \Lambda a$.

Writing an idèle a as a matrix $(a_{\mathfrak{q},i})_{\mathfrak{q},i}$, $a_{\mathfrak{q},i} \in (K)_{m_i}$, in general with infinitely many rows, the operation of G can be thought of as columnwise, the operation of U can be thought of as rowwise.

D.5 Endomorphism rings of projectives over commutative $\mathfrak{p}$-orders

We give a criterion for when two endomorphism rings of certain projectives over commutative sub split semisimple $\mathfrak{p}$-orders are isomorphic via an isomorphism fixing the rational central primitive idempotents in case R is a principal ideal domain (D.5.11).

Let R be a principal ideal domain, let $\mathfrak{p} = (p)$. Let Λ be commutative, i.e. let Λ be a full suborder of $\prod_{i=1}^{s} R =: \Gamma$. Let $\mathfrak{a} = \mathfrak{p}^{\xi}$.

Remark D.5.1 It turns out that for $n \leq 6$ quite often the endomorphism ring of an indecomposable projective over $\mathbf{Z}_{(p)}\mathcal{S}_n$ is commutative - the only exceptions are the projective covers of the trivial module $\mathbf{F}_2$ in the cases $n = 5, 6$, $p = 2$, whose endomorphism rings have a rational factor $(\mathbf{Q})_2$ (S 2.2.4, S 2.3.5). In particular, we are **not able** to calculate the class groups of those two examples. PLESKEN [P 80/1, (I.26)] gives an obstruction to this commutativity in terms of decomposition numbers.

The restriction on R is made in order not to have algebraic number theory involved, being as important as difficult. The technical reason is (D.5.3).

Definition D.5.2 *In the notation of (D.4.10), the* **class group** *of Λ is defined to be*

$$\mathrm{Cl}(\Lambda) := I/UG.$$

By Jordan-Zassenhaus (D.2.7) and by (D.4.10), $\mathrm{Cl}(\Lambda)$ *is finite.*

Lemma D.5.3 *Each coset aUG can be represented by a* **normalized** *idèle $a = (a_{\mathfrak{p}}, 1, 1, \ldots)$, where $a_{\mathfrak{p}} \in \Gamma_{\mathfrak{p}}^* = \prod_{i=1}^{s} R_{\mathfrak{p}}^*$. We write, for $\alpha \in \Gamma_{\mathfrak{p}}^*$, the corresponding normalized idèle as*

$$(\alpha) := (\alpha, 1, 1, \ldots).$$

The normalized idèles (α) and (β) coincide modulo UG iff there is a tuple $(\varepsilon_i)_{i \in [1,s]}$ of units in R and a unit $u \in \Lambda_{\mathfrak{p}}^* \subseteq \Gamma_{\mathfrak{p}}^*$ such that

$$\beta_i = \alpha_i u_i e_i.$$

for each $i \in [1, s]$.

In other words, the class group of Λ admits the description

$$\begin{array}{rcl} \mathrm{Cl}(\Lambda) & \xrightarrow{\sim} & \Gamma_{\mathfrak{p}}^*/\Lambda_{\mathfrak{p}}^*\Gamma^*. \\ a & \longrightarrow & a_{\mathfrak{p}} \\ (\alpha) & \longleftarrow & \alpha. \end{array}$$

In particular, $\mathrm{Cl}(\Lambda)$ *is a quotient of $(\Gamma/\mathfrak{p}^{\xi}\Gamma)^*$.*

Moreover, for a normalized idèle (α) we have

$$\Lambda(\alpha) = \Lambda_p \cap \Lambda_{\mathfrak{p}}\alpha = \Gamma_p \cap \Lambda_{\mathfrak{p}}\alpha = \Gamma \cap \Lambda_{\mathfrak{p}}\alpha$$

where $(-)_p$ denotes the localisation at $\{1, p, p^2, \ldots\}$. In other words,

$$\Lambda(\alpha) = \{(x_i)_i \in \Gamma \mid (x_i\alpha_i^{-1})_i \in \Lambda_{\mathfrak{p}}\}.$$

Suppose given $a \in I$. Via U we may assume that only a finite number of entries $a_{\mathfrak{q}}$ is not equal to 1.

Since Λ is a $\mathfrak{p}$-order, $\Lambda_\mathfrak{q} = \prod_{i=1}^s R_\mathfrak{q} \subseteq K\Lambda = \prod_{i=1}^s K$. R being a **principal ideal domain,** we may choose in each factor K of $K\Lambda$ an element x_i, independent of $\mathfrak{q}$, with $v_\mathfrak{q}(x_i) = -v_\mathfrak{q}(a_{\mathfrak{q},i})$ for all primes $\mathfrak{q}$ (including $\mathfrak{p}$). Finally, via U we may assume $a_\mathfrak{q} = 1$ for $\mathfrak{q} \neq \mathfrak{p}$ and $v_\mathfrak{p}(a_{\mathfrak{p},i}) = 0$ for $i \in [1, s]$, i.e. we may assume a to be normalized.

Suppose given $a \in I$, $u \in U$ and $x \in G$ such that both a and aux are normalized idèles. Then $0 = v_\mathfrak{q}(x_i)$ for all $\mathfrak{q}$, hence x_i is a unit in R.

Now, $\mathrm{Cl}(\Lambda)$ is a quotient of $(\Gamma/\mathfrak{p}^\xi\Gamma)^*$ since $\Gamma_\mathfrak{p}^* \longrightarrow \Gamma_\mathfrak{p}^*/\Lambda_\mathfrak{p}^*\Gamma^*$ sends $1+\mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p}$ to 1. In fact, $\mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p}$ is contained in the radical of $\Lambda_\mathfrak{p}$, for $\mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p} + \mathfrak{p}\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$ is nilpotent in $\Lambda_\mathfrak{p}/\mathfrak{p}\Lambda_\mathfrak{p}$. Whence $1 + \mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p}$ is contained in $\Lambda_\mathfrak{p}^*$.

The equation for $\Lambda(\alpha)$ follows from $\bigcap_{\mathfrak{q}\neq\mathfrak{p}} R_\mathfrak{q} = R_p$ and from $v_\mathfrak{p}(\alpha_i) = 0$ for the entries of a normalized idèle (α).

Proposition D.5.4 *Let a, b be idèles of Λ. Then*

$$\Lambda a \oplus \Lambda b \simeq \Lambda \oplus \Lambda ab.$$

We **claim** that we may assume $a_\mathfrak{q} \in \Gamma_\mathfrak{q}$ for $\mathfrak{q}$ belonging to a finite set Q of primes, $\mathfrak{p} \notin Q$, $a_\mathfrak{q} = 1$ for $\mathfrak{q} \notin Q \cup \{\mathfrak{p}\}$ and $a_\mathfrak{p} \in \Lambda_\mathfrak{p}$. Use G to achieve $a_\mathfrak{p} \in \Lambda_\mathfrak{p}$ via global multiplication with a large enough power of p, if necessary. Similarly, use G to achieve $a_\mathfrak{q} \in \Gamma_\mathfrak{q}$ by multiplication with a large enough power of the generator of $\mathfrak{q}$, if necessary (at most at a finite number of primes). Now, let Q be the set of primes $\mathfrak{q}$ different from $\mathfrak{p}$ with $a_\mathfrak{q} \notin \Gamma_\mathfrak{q}^*$. Outside $Q \cup \{\mathfrak{p}\}$, use U to achieve the claim. In particular, $\Lambda a \subseteq \Lambda$, as is to be seen locally.

We **claim** that we may assume $b_\mathfrak{q} = 1$ for $\mathfrak{q} \in Q$, $b_\mathfrak{p} \in 1 + \mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p}$ and $b_\mathfrak{q} \in \Gamma_\mathfrak{q}$ for $\mathfrak{q} \notin Q \cup \{\mathfrak{p}\}$. Use G to divide globally by $b_\mathfrak{p}$ so that we achieve $b_\mathfrak{p} = 1$. Use G and the Chinese Remainder Theorem to multiply globally with an element in R which achieves $b_\mathfrak{q} \in \Gamma_\mathfrak{q}$ for $\mathfrak{q} \neq \mathfrak{p}$, but which is congruent to 1 modulo $\mathfrak{p}^\xi$, whence $b_\mathfrak{p} \in 1 + \mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p}$ remains valid. Use G and the Chinese Remainder Theorem to divide globally by an element in R which achieves $b_\mathfrak{q} \in \Gamma_\mathfrak{q}^*$ for $\mathfrak{q} \in Q$, but which is congruent to 1 modulo $\mathfrak{p}^\xi$. Again, use G and the Chinese Remainder Theorem to multiply globally with an element in R which achieves $b_\mathfrak{q} \in \Gamma_\mathfrak{q}$ for $\mathfrak{q} \notin Q \cup \{\mathfrak{p}\}$, but which is congruent to 1 modulo $\mathfrak{p}^\xi$ as well as modulo $\mathfrak{q}$ for $\mathfrak{q} \in Q$. Use U at the primes in Q to achieve the claim. In particular, $\Lambda b \subseteq \Lambda$.

Now $\Lambda a + \Lambda b = \Lambda$ is to be seen locally, since at $\mathfrak{q}$, at least one of the summands equals $\Lambda_\mathfrak{q}$.

Therefore it suffices to show that $\Lambda a \cap \Lambda b = \Lambda ab$, for then the split diagonal short exact sequence of the exact square $(\Lambda ab, \Lambda a, \Lambda b, \Lambda)$ proves the assertion. The required equality is to be seen locally, using $1 + \mathfrak{p}_\mathfrak{p}^\xi\Gamma_\mathfrak{p} \subseteq \Lambda_\mathfrak{p}^*$, cf. the proof of (D.5.3).

Remark D.5.5 The group structure on the class group $\mathrm{Cl}(\Lambda)$ may be described in terms of operations on isomorphism classes of lattices. For $a, b \in I$ the isomorphism class of Λab is determined by $\Lambda a \oplus \Lambda b \simeq \Lambda \oplus \Lambda ab$, using Jacobinksi's Cancellation Theorem (D.3.6). In other words, if $K_0(\Lambda)$ denotes the free abelian group on the isomorphism classes of projective Λ-lattices modulo the relation that the formal sum equals the direct sum, we obtain an isomorphism

$$\begin{array}{rcl} K_0(\Lambda) & \xrightarrow{\sim} & \mathbf{Z} \oplus \mathrm{Cl}(\Lambda) \\ \Lambda a & \longrightarrow & 1 \oplus a. \end{array}$$

Corollary D.5.6 $\bigoplus_\sigma \Lambda a_\sigma \simeq \bigoplus_\sigma \Lambda b_\sigma$ *iff* $\prod_\sigma a_\sigma$ *equals* $\prod_\sigma b_\sigma$ *in* $\mathrm{Cl}(\Lambda)$.

In terms of normalized idèles (D.5.3) this means that $\bigoplus_\sigma \Lambda(\alpha_\sigma) \simeq \bigoplus_\sigma \Lambda(\beta_\sigma)$ iff there exists a unit u in $\Lambda_\mathfrak{p}$ and a tuple $(\varepsilon_i)_i$ of units in R such that

$$\prod_\sigma \alpha_{\sigma,i}\beta_{\sigma,i}^{-1} = u_i\varepsilon_i$$

for all i.

This tells us to what extent Krull-Schmidt fails in Λ-proj.

The assertion follows from (D.5.4) by Jacobinski's Cancellation Theorem (D.3.6).

Lemma D.5.7 *The Λ-lattice of morphisms over Λ between $\Lambda(\alpha)$ and $\Lambda(\beta)$ is given by*

$$\begin{array}{rcl} \Lambda(\beta/\alpha) & \xrightarrow{\sim} & {}_\Lambda(\Lambda(\alpha),\Lambda(\beta)) \\ x & \longrightarrow & (-)x \end{array}$$

where (α) and (β) are normalized idèles. In particular, the endomorphism ring of a direct sum of indecomposable projectives, acting on the right, has the following form

$$\mathrm{End}_\Lambda(\Lambda(\alpha^{(1)})\oplus\cdots\oplus\Lambda(\alpha^{(m)})) = \begin{bmatrix} \Lambda(\alpha^{(1)}/\alpha^{(1)}) & \Lambda(\alpha^{(2)}/\alpha^{(1)}) & \ldots & \Lambda(\alpha^{(m)}/\alpha^{(1)}) \\ \Lambda(\alpha^{(1)}/\alpha^{(2)}) & \Lambda(\alpha^{(2)}/\alpha^{(2)}) & \ldots & \Lambda(\alpha^{(m)}/\alpha^{(2)}) \\ \vdots & \vdots & & \vdots \\ \Lambda(\alpha^{(1)}/\alpha^{(m)}) & \Lambda(\alpha^{(2)}/\alpha^{(m)}) & \ldots & \Lambda(\alpha^{(m)}/\alpha^{(m)}) \end{bmatrix}.$$

We use the description of $\Lambda(\alpha)$ given in (D.5.3).

We'd like to see that the map is well defined. Suppose given $y \in \Lambda(\alpha)$. First,

$$\beta^{-1}yx = (\beta^{-1}\alpha x)(\alpha^{-1}y) \in \Lambda_{\mathfrak{p}}.$$

Second, $yx \in \Gamma$.

By a rank consideration, it remains to be shown that the map is surjective. An application of $K(-)$ shows that any morphism is given by multiplication with an element $x \in K\Gamma$, and we **claim** that such an x is already contained in $\Lambda(\beta/\alpha)$.

Let $\mathfrak{q} \neq \mathfrak{p}$. $\Gamma_{\mathfrak{q}} \xrightarrow{(-)x} \Gamma_{\mathfrak{q}}$ yields $x \in \Gamma_{\mathfrak{q}}$. $\Lambda_{\mathfrak{p}}\alpha_{\mathfrak{p}} \xrightarrow{(-)x} \Lambda_{\mathfrak{p}}\beta_{\mathfrak{p}}$ yields $\beta_{\mathfrak{p}}^{-1}\alpha_{\mathfrak{p}}x \in \Lambda_{\mathfrak{p}}$, whence in particular $x \in \Gamma_{\mathfrak{p}}$.

Lemma D.5.8

$$\begin{array}{rcl} \Lambda(\alpha)\otimes_\Lambda\Lambda(\beta) & \xrightarrow{\sim} & \Lambda(\alpha\beta) \\ x\otimes y & \longrightarrow & xy \end{array}$$

We use the description of $\Lambda(\alpha)$ given in (D.5.3). Note that for projective lattices $\otimes$ and $\tilde{\otimes}$ coincide.

The map is well defined, since $(\alpha\beta)^{-1}xy = (\alpha^{-1}x)(\beta^{-1}y) \in \Lambda_{\mathfrak{p}}$ (cf. D.5.3). By a rank consideration, it remains to be shown that the map is surjective. This in turn is seen locally, using (B.1.8), since localized at $\mathfrak{p}$ in fact $\alpha\beta$ lies in the image.

Lemma D.5.9 *Let $\Xi' \xrightarrow{\varphi} \Xi$ be a morphism of orders over Λ, let X be a Ξ-lattice, let Y be a Λ-lattice. For a left Ξ-lattice, we denote by a left lower index φ its restriction (or 'twist') via φ to a Ξ'-module. We have*

$$\begin{array}{rcl} {}_\varphi(X\,\tilde{\otimes}_\Lambda\,Y) & \xrightarrow{\sim} & ({}_\varphi X)\,\tilde{\otimes}_\Lambda\,Y \\ x\,\tilde{\otimes}\,y & \longrightarrow & x\,\tilde{\otimes}\,y \end{array}$$

as left Ξ'-lattices.

Note that for projective lattices $\otimes$ and $\tilde{\otimes}$ coincide.

Example D.5.10 (dangerous bend) Let $\Lambda = R\times R$, let $\Xi' = R\times R$, let $\Xi = R\times R$. Let $\Xi' \xrightarrow{\varphi} \Xi$ be the isomorphism which interchanges the factors. For $X = R\times 0$ and $Y = 0\times R$ we obtain on the one hand

$${}_\varphi(X\otimes_\Lambda Y) = 0$$

and on the other hand

$$({}_\varphi X)\otimes_\Lambda Y \simeq Y.$$

Theorem D.5.11 *Let $(\alpha^{(i)})$ and $(\beta^{(i)})$, $i \in [1,u]$, be normalized idèles.*

The endomorphism rings of $\bigoplus_{i\in[1,u]}\Lambda(\alpha^{(i)})$ *and of* $\bigoplus_{i\in[1,u]}\Lambda(\beta^{(i)})$ *over* Λ *are isomorphic as orders over* Λ *if and only if*

$$(\prod_{i\in[1,u]}\frac{\alpha^{(i)}}{\beta^{(i)}})\in \mathrm{Cl}(\Lambda)^u.$$

In this formula, only the upper index u *is to be read as an exponent.*

In particular, in case $\Lambda_\mathfrak{p}$ *is local, the endomorphism ring of* $\bigoplus_{i\in[1,u]}\Lambda(\alpha^{(i)})$ *is homogenus iff*

$$(\prod_{i\in[1,u]}\alpha^{(i)})\in \mathrm{Cl}(\Lambda)^u.$$

Assume this product of idèle quotients to be an u-th power. By (D.5.6) we rewrite this assumption as

$$\bigoplus_{i\in[1,u]}\Lambda(\beta^{(i)})\simeq\bigoplus_{i\in[1,u]}\Lambda(\alpha^{(i)}\gamma)$$

for some normalized idèle (γ) so that we are done by (D.5.7).

Now assume the endomorphism rings to be isomorphic via an isomorphism φ of orders over Λ.

By (D.5.4) we may assume that $\alpha^{(i)}=1$ and $\beta^{(i)}=1$ for $i\in[1,u-1]$, i.e. that only $\alpha^{(u)}=:\alpha$ and $\beta^{(u)}=:\beta$ are nontrivial. We **claim** that $(\alpha/\beta)\in\mathrm{Cl}(\Lambda)^u$.

By (D.5.7) we obtain

$$\Xi_\alpha:=\mathrm{End}_\Lambda(\Lambda\oplus\cdots\oplus\Lambda\oplus\Lambda(\alpha))=\begin{bmatrix}\Lambda & \Lambda & \dots & \Lambda(\alpha)\\ \Lambda & \Lambda & \dots & \Lambda(\alpha)\\ \vdots & \vdots & & \vdots\\ \Lambda(1/\alpha) & \Lambda(1/\alpha) & \dots & \Lambda\end{bmatrix},$$

accordingly Ξ_β. To avoid lots of dots, we now restrict ourselves **in notation** to the case $u=2$, i.e. to

$$\Xi_\alpha=\begin{bmatrix}\Lambda & \Lambda(\alpha)\\ \Lambda(1/\alpha) & \Lambda\end{bmatrix}.$$

Via restriction along the assumed isomorphism $\Xi_\alpha\xrightarrow[\sim]{\varphi}\Xi_\beta$ we obtain two decompositions of Ξ_α into indecomposable projectives which we wish to compare.

Let $Q_1:=\begin{bmatrix}\Lambda\\ \Lambda(1/\beta)\end{bmatrix}$ be the first and let $Q_u:=\begin{bmatrix}\Lambda(\beta)\\ \Lambda\end{bmatrix}$ be the last column of Ξ_β. We obtain

$$Q_1\otimes_\Lambda\Lambda(\beta)=\begin{bmatrix}\Lambda\\ \Lambda(1/\beta)\end{bmatrix}\otimes_\Lambda\Lambda(\beta)\simeq\begin{bmatrix}\Lambda & \otimes_\Lambda & \Lambda(\beta)\\ \Lambda(1/\beta) & \otimes_\Lambda & \Lambda(\beta)\end{bmatrix}\simeq\begin{bmatrix}\Lambda(\beta)\\ \Lambda\end{bmatrix}=Q_u$$

as left Ξ_β-lattices (D.5.8). Since the first column $P:=\begin{bmatrix}\Lambda\\ \Lambda(1/\alpha)\end{bmatrix}$ of Ξ_α is a progenerator with endomorphism ring Λ (use the Morita equivalence given by the definition of Ξ_α), the projective indecomposables of Ξ_α and of Λ correspond to each other via $P\otimes_\Lambda -$. The Ξ_β-lattice Q_1 restricts via φ to some lattice of the form ${}_\varphi Q_1\simeq P\otimes_\Lambda\Lambda(\gamma)$ along $\Xi_\alpha\xrightarrow[\sim]{\varphi}\Xi_\beta$, (γ) being a normalized idèle. By the calculation just performed, we obtain

$$\begin{array}{rcl}{}_\varphi Q_u & \simeq & {}_\varphi(Q_1\otimes_\Lambda\Lambda(\beta))\\ & \overset{(D.5.9)}{\simeq} & ({}_\varphi Q_1)\otimes_\Lambda\Lambda(\beta)\\ & \simeq & (P\otimes_\Lambda\Lambda(\gamma))\otimes_\Lambda\Lambda(\beta)\\ & \overset{(D.5.8)}{\simeq} & P\otimes_\Lambda\Lambda(\gamma\beta)\end{array}$$

as left Ξ_α-lattices. Passing to Λ-lattices via Morita equivalence backwards (i.e. cancelling $P\otimes_\Lambda -$), we obtain

$$\Xi_\alpha=\Lambda^{u-1}\oplus\Lambda(\alpha)\simeq\Lambda(\gamma)^{u-1}\oplus\Lambda(\gamma\beta),$$

whence the assertion by (D.5.6).

Remark D.5.12
(a). In the course of the the direct calculation of the ties of $(\mathbf{Z}\mathcal{S}_n)_{[p]}$, $n = 5, 6$, endomorphism rings as in (D.5.11) occurred. We had to conjugate them by hand to obtain a homogenus endomorphism ring in order to be able to employ the language of Morita multiplicities.

(b). Note that Jacobinski's Cancellation Theorem enters the proof of (D.5.11) via (D.5.6).

(c). In case $\Lambda_{\mathfrak{p}}$ is local, we observe already by (D.5.7) that a ring Morita equivalent and locally isomorphic to Λ is isomorphic to Λ. This is false in general for the larger R-orders Morita equivalent to Λ, viz. for the Ξ_α's in the language of (D.5.11). I don't know of an example of two nonisomorphic Morita equivalent $\mathfrak{p}$-orders which yield isomorphic **basic** local orders when localized at $\mathfrak{p}$ - dropping, of course, our assumptions of this section. Asking less formally, is the genus effect for orders merely due to the genus effect for indecomposable projectives?

Remark D.5.13 In case $\Gamma = R \times R$ we shall give a direct calculational proof of (D.5.11) which avoids usage of Jacobinksi's Cancellation Theorem (D.3.6). This seems to be difficult in bigger cases.

Let $\Lambda = \{x \times y \mid x \equiv_{p^t} y\} \subseteq R \times R$ for some $t \geq 1$. Any normalized idèle can be written as $(\alpha) = (1, \alpha_2)$ via $\Lambda_{\mathfrak{p}}^*$. By **abuse of notation,** we identify α with α_2, i.e. we regard α as an element of $R_{\mathfrak{p}}^*$. (α) is trivial in $\mathrm{Cl}(\Lambda)$ iff $\alpha \equiv_{p^t} \varepsilon$ for some $\varepsilon \in R^*$. In particular, we may assume $\alpha \in R$, for if the difference is in (p^t), the quotient is trivial in $\mathrm{Cl}(\Lambda)$.

Note that by (D.5.3), we have

$$\Lambda(\alpha) = \{x \times y \mid x\alpha \equiv_{p^t} y\} \subseteq R \times R.$$

Let M_α be the main diagonal matrix with entries $M_{\alpha,ii} = 1$ for $i \in [1, u-1]$ and $M_{\alpha,uu} = \alpha$, and keep the notation of the proof of (D.5.11). We obtain

$$\Xi_\alpha = \{X \times Y \in (\Gamma)_u \mid XM_\alpha \equiv_{p^t} M_\alpha Y\}.$$

Now assume given an $\Xi_\alpha \xrightarrow{\sim} \Xi_\beta$ as orders over Λ, realized by right conjugation with $U \times V \in \mathrm{GL}_u(K\Gamma)$ (D.2.2). In other words, assume that

$$\Xi_\alpha^{U \times V} = \Xi_\beta.$$

Since the projection of Ξ_α to each of the ring direct factors of $(\Gamma)_u$ is surjective, we may assume $U \times V \in \mathrm{GL}_u(\Gamma)$. In fact, by the Elementary Divisor Theorem (A.1.1) we may write $U = U'DU''$ with U', U'' in $\mathrm{SL}_u(R)$, D being a main diagonal matrix with entries in K, which we may assume to have 1 as its upper left entry. Let E_{ij} be the matrix having entry 1 at the position ij and zero elsewhere. By $E_{1i}D = DX$ for each $i \in [1, u]$ and for some $X \in (R)_u$, we conclude $D \in (R)_u$. By $E_{i1}D = DX$ for each $i \in [1, u]$ and for some $X \in (R)_u$, it follows that $D \in \mathrm{GL}_u(R)$.

Since for $X, Y \in (R)_u$, $XM_\alpha \equiv_{p^t} M_\alpha Y$ implies $XUM_\beta V^{-1} \equiv_{p^t} UM_\beta V^{-1}Y$, it follows that

$$M_\alpha \equiv UM_\beta V^{-1} \in \mathrm{PGL}_u(R/p^t).$$

Taking determinants, we obtain

$$\alpha \equiv_{p^t} \beta\gamma^u\varepsilon$$

for some $\gamma \in R\backslash(p)$ and some $\varepsilon \in R^*$.

Conversely, suppose given Ξ_α and Ξ_β such that such a γ and such an ε exist. Let $V = 1$. It is possible to find a $U \in \mathrm{SL}_u(R)$ such that $U \equiv_{p^t} \gamma^{-1}M_{\alpha/\beta}$ since $\mathrm{SL}_u(R) \longrightarrow \mathrm{SL}_u(R/p^t) = \mathrm{SL}_u(R_{(p)}/p^t)$ is surjective (A.2.1).

Example D.5.14 Keep the notation of (D.5.13). Let $R = \mathbf{Z}$, $p = 5$, $u = 2$, $t = 1$, yielding

$$\Xi_2 = \{X \times Y \mid X \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) \equiv_5 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right) Y\} \subseteq (\Gamma)_2 = (\mathbf{Z})_2 \times (\mathbf{Z})_2.$$

Note that Ξ_2 is not isomorphic to Ξ_1, since 2 is not $\pm$ a square in $\mathbf{Z}/5$ (cf. D.5.13). For short, denote $\Xi := \Xi_2$. We choose the following $\mathbf{Z}$-linear basis of Ξ.

$$\begin{array}{rcl} e &:=& \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \\ g &:=& \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}5&0\\0&0\end{smallmatrix}\right) \\ h &:=& \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right) \\ i &:=& \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&5\\0&0\end{smallmatrix}\right) \\ j &:=& \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \\ k &:=& \left(\begin{smallmatrix}0&0\\5&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right) \\ f &:=& \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \\ l &:=& \left(\begin{smallmatrix}0&0\\0&5\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right). \end{array}$$

Ξe is a progenerator, since we have a coretraction

$$\Xi e \xrightarrow{(h\ i)} \Xi f \oplus \Xi f$$

retracted by

$$\Xi e \xleftarrow{\left(\begin{smallmatrix}-2j+k\\ j\end{smallmatrix}\right)} \Xi f \oplus \Xi f,$$

cf. (D.2.19). Thus $e\Xi e = \{x \times y \mid x \equiv_5 y\} \subseteq \mathbf{Z} \times \mathbf{Z}$ is Morita equivalent to Ξ. Ξe is not isomorphic to Ξf since

$$\begin{array}{rcl} (ah+bi)(cj+dk) &=& \left(\left(\begin{smallmatrix}0&a\\0&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&2a+5b\\0&0\end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix}0&0\\2c+5d&0\end{smallmatrix}\right) \times \left(\begin{smallmatrix}0&0\\c&0\end{smallmatrix}\right)\right) \\ &=& e \end{array}$$

required $2c + 5d = \pm 1$ and $c = \pm 1$, hence $\pm 2 \equiv_5 \pm 1$, which is impossible.

Also note that $e\Xi e = \mathbf{Z}[X]/(X^2 - 5X)$ is not a local ring since the nonunit 2 is not contained in the maximal ideal $(5, X)$. However, it becomes local when tensored with $\mathbf{Z}_{(5)}$ since now the maximal ideal $(5, X)$ contains all nonunits.

NB it may happen that there exist finite projectives X, Y over Λ such that

$$\begin{array}{rcl} \operatorname{End} X &\not\simeq& \operatorname{End} Y \\ \operatorname{End}(X \oplus \Lambda) &\simeq& \operatorname{End}(Y \oplus \Lambda) \\ \operatorname{End}(X \oplus \Lambda^2) &\not\simeq& \operatorname{End}(Y \oplus \Lambda^2). \end{array}$$

For example, let $X = \Lambda \oplus \Lambda(2)$ and $Y = \Lambda \oplus \Lambda$: 2 is neither $\pm$ a square nor $\pm$ a fourth power in $\mathbf{Z}/5$, but it is a third power.

D.6 Examples

We calculate some nontrivial and also some trivial class groups of endomorphism rings of indecomposable projectives of $(\mathbf{Z}\mathcal{S}_n)_{[p]}$ for $n \leq 6$ in order to ensure that the reason for $(\mathbf{Z}\mathcal{S}_n)_{[p]}$ being homogenus for $n \leq 6$ with respect to some embedding into a direct product of integral matrix rings is not just an overall triviality of the class groups (cf. D.5.11, D.4.8).

Let $R = \mathbf{Z}$.

Example D.6.1 Let $2 \neq p \in \mathbf{Z}$ be a prime, let

$$\Lambda_t := \{x \times y \in \mathbf{Z} \times \mathbf{Z} \mid x \equiv_{p^t} y\} \subseteq \mathbf{Z} \times \mathbf{Z} =: \Gamma.$$

An endomorphism ring of an indecomposable projective of the $(\mathbf{Z}\mathcal{S}_p)_{[p]}$ is either isomorphic to Λ_1 or to $\mathbf{Z}$. The Morita multiplicities of the indecomposable projectives in the first case are given by $\binom{p-2}{i}$, $i \in [0, p-2]$ (4.2.8).

We obtain an isomorphism

$$\begin{array}{rcl} (\mathbf{Z}/p^t)^*/\{\pm 1\} &\xrightarrow{\sim}& \Gamma^*_{(p)}/\Lambda^*_{(p)}\Gamma^* = \operatorname{Cl}(\Lambda) \\ x &\longrightarrow& x \times 1. \end{array}$$

In general, $\operatorname{Cl}(\Lambda_1) \simeq C_{(p-1)/2}$ is not annihilated by the Morita multiplicity.

Example D.6.2 Let $p := 2$, let

$$\Lambda := \{x \times y \times z \mid y \equiv_4 z,\ 2x \equiv_8 y + z\} \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} =: \Gamma.$$

Λ is isomorphic to the endomorphism ring of an indecomposable projective of the $(\mathbf{Z}\mathcal{S}_4)_{[2]}$, occurring with Morita multiplicity 2 (S 2.1) and to the endomorphism ring of an indecomposable projective of the $(\mathbf{Z}\mathcal{S}_5)_{[2]}$, occurring with Morita multiplicity 4 (S 2.2.4).
We claim that $\mathrm{Cl}(\Lambda)$ is trivial, i.e. that $\Gamma^*_{(2)} = \Lambda^*_{(2)}\Gamma^*$. But

$$\begin{array}{rcl} 3 \times 1 \times 1 &=& (3 \times -1 \times -1)(1 \times -1 \times -1) \\ 1 \times 3 \times 1 &=& (1 \times 3 \times -1)(1 \times 1 \times -1). \end{array}$$

Example D.6.3 Let $p := 2$, let

$$\Lambda := \{x \times y \times z \times w \mid x - y \equiv_8 z - w \equiv_4 0,\ x \equiv_2 z\}.$$

Λ is isomorphic to the endomorphism ring of an indecomposable projective of the $(\mathbf{Z}\mathcal{S}_4)_{[2]}$, occurring with Morita multiplicity 1 (2.1).
We have an isomorphism

$$\begin{array}{rcl} (\mathbf{Z}/8)^*/\{\pm 1\} & \xrightarrow{\sim} & \Gamma^*_{(p)}/\Lambda^*_{(p)}\Gamma^* = \mathrm{Cl}(\Lambda) \\ x & \longrightarrow & x \times 1 \times 1 \times 1, \end{array}$$

whence $\mathrm{Cl}(\Lambda) \simeq C_2$, which is not annihilated by the Morita multiplicity.
Surjectivity follows using the elements

$$\begin{array}{rcrcrcr} 3 & \times & 3 & \times & 1 & \times & 1 \\ 3 & \times & -1 & \times & 3 & \times & -1 \end{array}$$

of Λ. For to see injectivity, assume that $x \times 1 \times 1 \times 1$ represents the trivial element. Writing it as a product in $\Lambda^*_{(p)}$ and Γ^*, the latter factor has to be of the form $* \times * \times 1 \times 1$ or of the form $* \times * \times -1 \times -1$. Therefore we may conclude that $x \equiv_8 1$ or $x \equiv_8 -1$.

Example D.6.4 Let $p := 2$, let

$$\Lambda := \left\{ a \times b \times c \times d \times e \times f \;\middle|\; \begin{array}{rcl} a & \equiv_2 & e, \\ a + d - 2f & \equiv_{16} & b + c - 2e \equiv_8 0, \\ e - f & \equiv_4 & c - d \equiv_2 0 \end{array} \right\} \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} =: \Gamma.$$

Λ is isomorphic to two of the endomorphism rings of the indecomposable projectives of the $(\mathbf{Z}\mathcal{S}_6)_{[2]}$, both occurring with Morita multiplicity 4 (fAf and gAg in the notation of S 2.3.5).
Note that $(\mathbf{Z}/16)^*/\{\pm 1\}$ is isomorphic to C_4, with generator 3. We **claim** to have an isomorphism

$$\begin{array}{rcl} (\mathbf{Z}/16)^*/\{\pm 1\} & \xrightarrow{\sim} & \Gamma^*_{(2)}/\Lambda^*_{(2)}\Gamma^* = \mathrm{Cl}(\Lambda) \\ x & \longrightarrow & x \times 1 \times 1 \times 1 \times 1 \times 1, \end{array}$$

whence $\mathrm{Cl}(\Lambda) \simeq C_4$, which, however, is annihilated by the Morita multiplicity (cf. D.5.11, D.4.8).
Surjectivity follows using the elements

$$\begin{array}{rcrcrcrcrcr} 3 & \times & -1 & \times & 3 & \times & -1 & \times & 1 & \times & 1 \\ 3 & \times & 3 & \times & -1 & \times & -1 & \times & 1 & \times & 1 \\ -1 & \times & -1 & \times & 3 & \times & 3 & \times & 1 & \times & 1 \\ -1 & \times & 3 & \times & 3 & \times & -1 & \times & 3 & \times & -1 \end{array}$$

of Λ. For to see injectivity we regard the following matrix, whose rows generate Λ over $\mathbf{Z}$,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & 0 & 4 & 2 & 2 & 0 \\ -2 & -2 & 2 & 2 & 0 & 0 \\ 0 & -4 & 4 & 0 & 0 & 0 \\ 8 & 8 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as well as 16 times its inverse,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 4 & 2 & -1 \\ 0 & 0 & 8 & -4 & 0 & 1 \\ 0 & 8 & -8 & -4 & -4 & 2 \\ 16 & -8 & 0 & 4 & 0 & -2 \end{bmatrix}.$$

An element of Γ, regarded as a row vector, is contained in Λ iff the product with the latter matrix is divisible by 16. So we simply have rewritten our ties. In particular, an element of Γ of type $2y \times b \times c \times d \times e \times f$ with $b, c, d, e, f \in \{-2, 0\}$ in Λ is necessarily of one of the following forms

$$\begin{array}{ccccccccccc} 2y & \times & -2 & \times & -2 & \times & -2 & \times & -2 & \times & -2 \\ 2y & \times & -2 & \times & -2 & \times & 0 & \times & -2 & \times & 0 \\ 2y & \times & 0 & \times & 0 & \times & -2 & \times & 0 & \times & -2 \\ 2y & \times & 0 & \times & 0 & \times & 0 & \times & 0 & \times & 0, \end{array}$$

i.e. $y \equiv_8 -1, 0, -1, 0$, respectively. Hence, inserting $2y = \pm x - 1$, the element $x \times 1 \times 1 \times 1 \times 1 \times 1$ is trivial if and only if $\pm x = \pm 1$.

Example D.6.5 Let $p := 3$, let

$$\Lambda := \{a \times b \times c \times d \mid a - d \equiv_9 c - b,\ a \equiv_3 b \equiv_3 c \equiv_3 d\} \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} =: \Gamma.$$

Λ is isomorphic to four of the endomorphism rings of indecomposable projectives of the $(\mathbf{Z}\mathcal{S}_6)_{[3]}$, occurring either with Morita multiplicity 1 or 4 (S 2.3.3).
We **claim** to have an isomorphism

$$\begin{array}{rcl} (\mathbf{Z}/9)^*/\{\pm 1\} & \xrightarrow{\sim} & \Gamma^*_{(3)}/\Lambda^*_{(3)}\Gamma^* = \mathrm{Cl}(\Lambda) \\ x & \longrightarrow & x \times 1 \times 1 \times 1, \end{array}$$

whence $\mathrm{Cl}(\Lambda) \simeq C_3$, which is not annihilated by the Morita multiplicity.
Surjectivity follows using the elements

$$\begin{array}{ccccccc} 1 & \times & 4 & \times & 4 & \times & 1 \\ 4 & \times & 1 & \times & 4 & \times & 1 \end{array}$$

of Λ. Injectivity follows by remarking that $y \times b \times c \times d$ in Λ with $b, c, d \in \{0, -2\}$ implies $y \equiv_9 b = c = d$.

Example D.6.6 Let $p := 3$, let

$$\Lambda := \{a \times b \times c \mid a + c \equiv_9 2b,\ a \equiv_3 b \equiv_3 c\} \subseteq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} =: \Gamma.$$

Λ is isomorphic to the endomorphism ring of an indecomposable projective of the $(\mathbf{Z}\mathcal{S}_6)_{[3]}$, occurring with Morita multiplicity 6 (S 2.3.3).
We **claim** to have an isomorphism

$$\begin{array}{rcl} (\mathbf{Z}/9)^*/\{\pm 1\} & \xrightarrow{\sim} & \Gamma^*_{(3)}/\Lambda^*_{(3)}\Gamma^* = \mathrm{Cl}(\Lambda) \\ x & \longrightarrow & x \times 1 \times 1, \end{array}$$

whence $\mathrm{Cl}(\Lambda) \simeq C_3$, which, however, is annihilated by the Morita multiplicity (cf. D.5.11, D.4.8).
Surjectivity follows using the element

$$-2 \times 4 \times 1$$

of Λ. Injectivity follows by remarking that $y \times b \times c$ in Λ with $b, c \in \{0, -2\}$ implies $y \equiv_9 b = c$.

The list of conventions.

We remind the reader that all conventions we make in (A D) remain valid from the place we state them on to the end of the chapter (A D), in particular, they are valid in the subsequent sections.

(S D.1). By a module over a ring we understand a left module. Finite projective stands for finitely generated projective module. A-proj denotes the category of finite projectives over A. Indecomposable projective stands for finitely generated indecomposable projective module. $\mathrm{ip}(A)$ denotes the set of isomorphism classes of indecomposable projectives over A. We say that Krull-Schmidt holds in A-proj if the decomposition of $P \in A$-proj into indecomposable projectives is unique up to permutation of the summands and up to isomorphism. The unit group of a ring A is denoted by A^*.

(S D.2). Let R be a Dedekind domain with field of fractions K (to which we refer by 'rational') such that $R/\mathfrak{p}$ is finite as a set for each nonzero prime ideal $\mathfrak{p} \subseteq R$. By $\mathfrak{p}, \mathfrak{q}$ we denote nonzero prime ideals of R. Assume K to have finite class number, i.e. assume the set of isomorphism classes of ideals in R to be finite.

An R-order is an R-algebra which is finite projective as an R-module. Let Λ be a full (i.e. rationally equal) R-suborder of a direct product of matrix rings over R, $\Lambda \subset \Gamma := \prod_i (R)_{m_i}$ being strictly included. We fix this embedding throughout. Such an order Λ we call sub split semisimple over R. Γ/Λ is a torsion R-module with annihilator $\mathfrak{a}$ in R.

We abbreviate $K \otimes_R -$ by $K(-)$. A lattice over Λ is a Λ-module that is finite projective over R. A simple Λ-lattice is a Λ-lattice X with KX being a simple $K\Lambda$-module. A pure monomorphism of Λ-lattices has a torsionfree quotient, a full monomorphism has a torsion quotient, a pure epimorphism is surjective.

(S D.3). No further conventions.

(S D.4). Suppose Λ to be a $\mathfrak{p}$-order.

(S D.5). Let R be a principal ideal domain, let $\mathfrak{p} = (p)$. Let Λ be commutative, i.e. let Λ be a full suborder of $\prod_{i=1}^{s} R =: \Gamma$. Let $\mathfrak{a} = \mathfrak{p}^{\xi}$.

(S D.6). Let $R = \mathbf{Z}$.

Appendix E

Radical layers

This is a tentative appendix on the behaviour of the radical of the local orders which occur as endomorphism rings of the indecomposable projectives in a Pierce decomposition of $\mathbf{Z}_{(p)}\mathcal{S}_n$, p a prime dividing $n!$. Suppose given a projective indecomposable lattice P over such a local order surjecting on simple lattices X and Y. The isomorphisms $X/\mathfrak{r}X \xleftarrow{\sim} P/\mathfrak{r}P \xrightarrow{\sim} Y/\mathfrak{r}Y$ imply certain p-ties (E.1.24) between X and Y - intuitively, 'they tie the main diagonal in order not to allow idempotent decompositions'. In searching for a generalization of this assertion we didn't succeed at all, but nevertheless we stumbled over some properties worth recording.

E.1 Recalling some basics

To begin with, we have collected some well-known basic facts from [Row 91] and [Be 91].

E.1.1 Rings

Let A be a left noetherian ring. The Jacobson radical of A, i.e. the intersection of the annihilators of the simple left modules of A, is denoted by $\mathfrak{r}A$, its i-th power is denoted by $\mathfrak{r}^i A := (\mathfrak{r}A)^i$. For a left A-module X we denote $\mathfrak{r}^i X := (\mathfrak{r}^i A)X$. $\mathfrak{r}X$ is called the radical of X.

Remark E.1.1 *Each orthogonal decomposition of 1_A into idempotents of A can be refined to an orthogonal decomposition of 1_A into primitive idempotents.*

Use direct sum decompositions of the corresponding projective left ideals and left noetherianicity of A. Note that a 'family tree' of decompositions of a single idempotent with infinitely many 'generations' would contain an infinite chain, which would yield a properly ascending infinite chain of submodules.

Remark E.1.2 *$\mathfrak{r}A$ is the intersection of the maximal left ideals of A. A ring is called* **local** *if it is the disjoint union of its radical with its unit group. A local ring A has, up to isomorphism, only one simple module, viz. $A/\mathfrak{r}A$, since there is only one maximal left ideal in A.*

Lemma E.1.3 *An element $x \in A$ is called* **left quasiinvertible** *if $1 - x$ is left invertible. An ideal in A is called* **left quasiinvertible** *iff each of its elements is left quasiinvertible. We impose the same definition dropping 'left'. $\mathfrak{r}A$ is the unique maximal quasiinvertible ideal.*

$\mathfrak{r}A$ is left quasiinvertible since no maximal left ideal may contain $1 - x$, $x \in \mathfrak{r}A$, for it does not contain 1.

Any left quasiinvertible ideal $L \subseteq A$ is quasiinvertible. In fact, let $x \in L$ and suppose $y(1-x) = 1$ for some $y \in A$. Since $1 - y \in L$, we obtain $zy = 1$ for some $z \in A$. But $z = zy(1-x) = (1-x)$.

Let $Q \subseteq A$ be a quasiinvertible ideal. If there were a maximal left ideal $M \subseteq A$ not containing Q, then there would be elements $q \in Q$, $m \in M$ such that $q + m = 1$.

Lemma E.1.4 *Let A be a ring, let e be an idempotent in A. We have*

$$e(\mathfrak{r}A)e = \mathfrak{r}(eAe).$$

In particular, $eAe/\mathfrak{r}(eAe) = e(A/\mathfrak{r}A)e$.

To see that $e(\mathfrak{r}A)e \subseteq \mathfrak{r}(eAe)$, we show that the left hand side is quasiinvertible (E.1.3). Consider $x \in \mathfrak{r}A$ and note that $y(1-ex) = 1$ implies $eye(e - exe) = e$.

To see that $\mathfrak{r}A \supseteq \mathfrak{r}(eAe)$, we show that $\mathfrak{r}(eAe)$ annihilates each simple left A-module M. But eM is either a simple eAe-module or zero, since for $em \neq 0$ we have $eAe(em) = e(A(em)) = eM$.

Note that the kernel of the surjection from eAe to $e(A/\mathfrak{r}A)e$ is $e(\mathfrak{r}A)e$.

Proposition E.1.5 (Nakayama's Lemma, proof taken from [Be 91, 1.2.3])
Let A be a ring, let M be a finitely generated left A-module, let $X \subseteq M$ be a submodule. Then $\mathfrak{r}M + X = M$ implies $X = M$.

Passing to M/X, it suffices to show that $\mathfrak{r}M = M$ implies $M = 0$. Write $M = A\langle m_1, \ldots, m_s\rangle$ in a minimal number s of generators. Assume $s \geq 1$ and write $m_s = \sum_{i\in[1,s]} a_i m_i$, $a_i \in \mathfrak{r}A$. This contradicts the invertibility of $1 - a_s$ (E.1.3).

Corollary E.1.6 *Let $A \xrightarrow{\varphi} B$ be a left finite ring morphism, i.e. assume B to be finitely generated as a left module over A via φ, such that in addition $\varphi(\mathfrak{r}A)B$ is a* **left** *ideal in B. Then*

$$\varphi(\mathfrak{r}A) \subseteq \mathfrak{r}B.$$

In particular, let M be a finitely generated B-module. Its radical with respect to A via φ is contained in its radical with respect to B.

We have to show that $\varphi(\mathfrak{r}A)$ is contained in each maximal left ideal M of B. But otherwise

$$M + \varphi(\mathfrak{r}A)B = B$$

holds, since $\varphi(\mathfrak{r}A)B$ is a left ideal in B, which yields $M = B$ (E.1.5).

Remark E.1.7 (dangerous bend) *In general, for a finite ring morphism $A \xrightarrow{\varphi} B$ we may have*

$$\varphi(\mathfrak{r}A) \not\subseteq \mathfrak{r}B.$$

Let

$$A = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b \equiv_2 0,\ a \equiv_2 d\} \overset{\varphi}{\hookrightarrow} (\mathbf{Z}_{(2)})_2 =: B.$$

We **claim** to have

$$\begin{aligned} \mathfrak{r}A &= I := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b \equiv_2 0,\ a \equiv_2 0,\ d \equiv_2 0\}, \\ \mathfrak{r}B &= 2(\mathbf{Z}_{(2)})_2. \end{aligned}$$

$\mathfrak{r}B$ is calculated to be the inverse image of $\mathfrak{r}(B/2) = 0$. A is the disjoint union of its unit group and I.

E.1.2 Algebras

Let k be a field, let A be a finite dimensional k-algebra, let $1_A = \sum e_i$ be an orthogonal decomposition into primitive idempotents. By a module we understand a finitely generated left A-module.

Corollary E.1.8 (to Nakayama's Lemma) $\mathfrak{r}A$ *is the unique maximal nilpotent ideal of* A.

Multiplying A iteratedly with $\mathfrak{r}A$ yields a strictly decreasing sequence (E.1.5), whence $\mathfrak{r}A$ is nilpotent. There exists a unique maximal nilpotent ideal N, since the sum of nilpotent ideals is nilpotent, as one sees elementwise. For a simple module S, equality $NS = S$ is impossible, so that $\mathfrak{r}A = N$.

Proposition E.1.9 (FITTING's Lemma) *The endomorphism ring of an indecomposable module X is local.*

We repeat the argument of (C.2.14).

Apply the Circonference Lemma to the composition $f^n f^n = f^{2n}$, n large, to prove nilpotence of an endomorphism of X which is not an automorphism. Writing down a geometric series, we thus see that either f or $1 - f$ is an automorphism. Via composition, this also holds with an automorphism instead of 1. We conclude that the nonautomorphisms are closed under addition. We are done by (E.1.8) since composition of a nilpotent morphism with an arbitrary morphism cannot be an automorphism, for nilpotent morphisms are neither injective nor surjective.

Corollary E.1.10 *Given a simple module S, there is an indecomposable projective module P, unique up to isomorphism, mapping onto it.*

There is such a P by dropping summands of A. Given P and Q indecomposable projective, mapping surjectively onto S, we may lift to $P \xrightarrow{f} Q$, as well as to $Q \xrightarrow{g} P$. Neither fg nor gf may be nilpotent, so (E.1.9) applies.

Corollary E.1.11 *If Ae_i is not isomorphic to Ae_j, then $e_iAe_j \subseteq \mathfrak{r}A$.*

Let S be a simple module. The existence of an epimorphism from Ae_l onto S is equivalent to the existence of a nonzero element $e_l s$, $s \in S$. Therefore, in case both e_i and e_j do not annihilate S, we derive $Ae_i \simeq Ae_j$ by means of (E.1.10).

Corollary E.1.12 *Assume in addition that A is* **basic,** *i.e. that $Ae_i \simeq Ae_j$ implies $i = j$. Then*

$$\mathfrak{r}A = (\bigoplus_i \mathfrak{r}(e_iAe_i)) \oplus (\bigoplus_{i \neq j} e_iAe_j).$$

Informally, $\mathfrak{r}A$ arises from A by passing to the radicals on the Pierce main diagonal.

This ensues from (E.1.4, E.1.11)

Corollary E.1.13 $e_i(A/\mathfrak{r}A)e_i$ *is a skewfield.*

As a quotient of a local k-algebra (E.1.9, E.1.4), it remains local (E.1.2, E.1.8). Its radical is calculated to be zero by (E.1.4), since $\mathfrak{r}(A/\mathfrak{r}A) = 0$ by correspondence of the maximal left ideals (E.1.2).

Lemma E.1.14 $\mathfrak{r}A$ *is the minimal ideal I in A with respect to the property of having a semisimple quotient, i.e. to A/I being a direct sum of simple modules. $Ae_i/\mathfrak{r}Ae_i$ is simple.*

Let $I \subseteq A$ be an ideal with semisimple quotient. Then A/I is annihilated by $\mathfrak{r}A$, i.e. $\mathfrak{r}A \subseteq I$.

It remains to be shown that $A/\mathfrak{r}A$ is semisimple. We use bars to denote images modulo $\mathfrak{r}A$. It suffices to show that a morphism from Ae_i to $\bar{A}\bar{e}_j$ is an epimorphism or zero, for then $\bar{A}\bar{e}_i$ is shown to be simple. Therefore, it suffices to show that a morphism from $\bar{A}\bar{e}_i$ to $\bar{A}\bar{e}_j$ is an isomorphism or zero. Since $\bar{e}_j\bar{A}\bar{e}_j$ is a skew field (E.1.13), the case $Ae_i \simeq Ae_j$ is done by isomorphic substitution of $\bar{A}\bar{e}_i$ by $\bar{A}\bar{e}_j$. The remaining case is covered by (E.1.11).

Lemma E.1.15 *A module X is said to be* **semisimple** *if it is the direct sum of certain simple submodules. An epimorphic image of a semisimple module is semisimple.*

This ensues from the characterization of a semisimple module as being the sum of all of its simple submodules. In fact, let X have this property and let Y be a maximal semisimple submodule. If Y were properly contained in X, there would be a simple module not entirely contained in Y, whence its intersection with Y would be zero, contradicting the assumed maximality.

Corollary E.1.16 *Given a module X, the radical quotient $X/\mathfrak{r}X$ is semisimple.*

$A/\mathfrak{r}A$ being semisimple (E.1.14), the same is true for $X/\mathfrak{r}X$, writing X as quotient of a finite sum of copies of A (E.1.15).

Corollary E.1.17 *Let P be an indecomposable projective module. $P/\mathfrak{r}P$ is simple.*

Hence, (E.1.10) taken into account, we have a bijective correspondence between the isomorphism classes of indecomposable projective modules and the isomorphism classes of simple modules, given by factoring out the radical.

This ensues from (E.1.14) in view of Krull-Schmidt (C.2.14). Alternatively, $P/\mathfrak{r}P$ is semisimple by (E.1.16), so that it remains to remark that the local endomorphism ring of P surjects onto the endomorphism ring of $P/\mathfrak{r}P$.

E.1.3 Orders

Let R be a discrete valuation ring with maximal ideal (π), residue field $k := R/\pi$ and field of fractions $K := \operatorname{frac} R$. Let Λ be a sub split semisimple R-order, i.e. assume Λ to be fully included into a product of matrix rings $\Gamma := \prod_\lambda (R)_{n_\lambda}$ over R (cf. S D.2). Let $\bar{\Lambda} := \Lambda/\pi$. Let $1_\Lambda = \sum_i e_i$ be an orthogonal decomposition into primitive idempotents. By a module we understand a finitely generated left Λ-module. A lattice is a module which is projective over R.

Lemma E.1.18 *We have*

$$(\mathfrak{r}\Lambda)/(\pi\Lambda) = \mathfrak{r}(\Lambda/\pi\Lambda).$$

From (E.1.6) we take that $\pi\Lambda \subseteq \mathfrak{r}\Lambda$. The result follows by intersection of maximal left ideals.

Lemma E.1.19 *Primitive idempotents of Λ remain primitive modulo π.*

This follows from (C.2.12).

Lemma E.1.20

(i) $e_i\Lambda e_i$ is local for any i.

(ii) We have a bijective correspondence between the isomorphism classes of indecomposable projective left Λ-lattices and the isomorphism classes of simple Λ-modules, given by factoring out the radical.

(iii) Assume for simplicity that Λ is **basic,** *i.e. that $\Lambda e_i \simeq \Lambda e_j$ implies $i = j$. Then*

$$\mathfrak{r}\Lambda = \Big(\bigoplus_i \mathfrak{r}(e_i\Lambda e_i)\Big) \oplus \Big(\bigoplus_{i\neq j} e_i\Lambda e_j\Big).$$

(iv) $\mathfrak{r}\Lambda$ is the unique maximal ideal with respect to the property of having a positive power of it contained in $\pi\Lambda$.

(v) Given a module X, the radical quotient $X/\mathfrak{r}X$ is semisimple.

(i). By (E.1.19, E.1.9), $e_i(\Lambda/\pi)e_i$ is local, whence also $e_i\Lambda e_i$ has a unique maximal left ideal.

(ii). By (E.1.17, E.1.18) it suffices to remark that there is also a bijective correspondence between the indeecomposable projectives over Λ and the indecomposable projectives over Λ/π by factoring out π. This map is well defined by (C.2.15, E.1.19) (here sub split semisimplicity enters). It is surjective by (C.2.14, E.1.19). It is injective by Nakayama's Lemma (E.1.5), applied to show that an isomorphism lifts to an isomorphism.

(iii). By (E.1.18), the radical of Λ is the inverse image under $\Lambda \longrightarrow \Lambda/\pi$ of the radical of Λ/π. By assumption, by (E.1.19) and by the correspondence of projectives remarked in the proof of (ii), Λ/π is basic. Thus (E.1.12) applies.

(iv). Using (E.1.18), this follows from the transcription of the characterization given in (E.1.8).

(v). Using (E.1.18), this follows from (E.1.16).

Corollary E.1.21 *For each Λ-module X there exists a projective Λ-module P surjecting onto it, $P \xrightarrow{f} X$, such that the kernel of f is contained in $\mathfrak{r}P$. Given a second such surjection $P' \xrightarrow{f'} X$, there is an isomorphism $P \xrightarrow[\sim]{u} P'$ with $uf' = f$. P is called the* **projective cover** *of X. The induced morphism*

$$P/\mathfrak{r}P \longrightarrow X/\mathfrak{r}X$$

is an isomorphism.

$X/\mathfrak{r}X$ is semisimple (E.1.20 v), so that we obtain a surjection from the direct sum of indecomposable projectives belonging to its summands, with kernel equal to the radical (E.1.20 ii). This surjection lifts to a surjection onto X by Nakayama's Lemma (E.1.5). Given a second such surjection, we obtain morphisms $P \xrightarrow{u} P'$ with $uf' = f$ and $P' \xrightarrow{v} P$ with $vf = f'$. uv induces an automorphism on $P/\mathfrak{r}P = X/\mathfrak{r}X$, whence it is itself an automorphism by Nakayama's Lemma (E.1.5).

Definition E.1.22 *Let S be a simple module. A module T is said to* **belong to** *S if, for each $i \geq 0$, $\mathfrak{r}^iT/\mathfrak{r}^{i+1}T$ is a finite direct sum of copies of S. A module is called* **neat** *if it belongs to some simple module.*

If T belongs to S, then so does each subquotient of T.

The problem is that one has to cut down a bit to the artinian case in order to apply the Jordan-Hölder Theorem. We know that $T/\mathfrak{r}^iT$ is of finite length since $\pi \in \mathfrak{r}\Lambda$ (E.1.5), thus the composition factors of its subquotients are isomorphic to S.

Suppose given a submodule T' of T. Let $i \geq 0$. We **claim** that for j large we have $\mathfrak{r}^iT' \supseteq T' \cap \mathfrak{r}^jT$. Since $\mathfrak{r}\Lambda$ is nipotent modulo π (E.1.8), we may substitute π^jT for $\mathfrak{r}^jT$ and, of course, π^iT' for $\mathfrak{r}^iT'$ in the assertion, so that a problem on finitely generated R-modules remains. Denoting the torsion resp. torsion free part of T by tT resp. fT, similarly for T', we write the inclusion $T' \hookrightarrow T$ as

$$tT' \oplus fT' \xrightarrow{\left(\begin{smallmatrix}\alpha & 0\\ \beta & \gamma\end{smallmatrix}\right)} tT \oplus fT.$$

Choose j large enough to ensure $\pi^j tT = 0$. Let $x \in tT'$, $y \in fT'$ be given such that $x\alpha + y\beta = 0$ and $y\gamma \in \pi^j fT$. Since γ is injective, we may choose j large enough to obtain $y \in \pi^i fT'$ as well as $y\beta = 0$ (A.1.1). α being injective, this implies $x = 0$. It follows by this claim that $T'/\mathfrak{r}^iT'$ is a quotient of $T'/(T' \cap \mathfrak{r}^jT) = (T' + \mathfrak{r}^jT)/\mathfrak{r}^jT$, which in turn is included in $T/\mathfrak{r}^jT$.

Suppose given a quotient T/T' of T. Let $i \geq 0$. We obtain $(T/T')/\mathfrak{r}^i(T/T') = T/(\mathfrak{r}^iT + T')$.

We attempted to obtain larger modules than the simple ones by considering a smaller ideal than the radical.

Remark E.1.23 (usage unclear) *Assume Λ to be basic (cf. E.1.20 iii). Let*

$$\mathfrak{pr}\Lambda := \Big(\bigoplus_i e_i\Lambda(1-e_i)\Lambda e_i\Big) \oplus \Big(\bigoplus_{i \neq j} e_i\Lambda e_j\Big).$$

be the **preradical** *of* Λ*. It is characterized as the minimal ideal inside* $\mathfrak{r}\Lambda$ *with quotient being a direct product of local rings.*

$\mathfrak{pr}\Lambda$ *is the intersection of the annihilators of the neat modules. Thus the local direct factors of the preradical quotient describe the 'intersections of* Λ-mod *and* $e_i\Lambda e_i$-mod*' in the following sense. The full subcategory of* Λ-mod *consisting of the modules belonging to* $(\Lambda/\mathfrak{r}\Lambda)e_i$ *is just* $e_i\Lambda e_i/e_i\Lambda(1-e_i)\Lambda e_i$-mod.

Since $e_i\Lambda e_i$ is local (E.1.20 i) and since $e_i\Lambda(1-e_i)\Lambda e_i \subseteq \mathfrak{r}(e_i\Lambda e_i)$ (E.1.20 iii), we obtain the quotient

$$\prod_i e_i\Lambda e_i/e_i\Lambda(1-e_i)\Lambda e_i$$

to be in fact a product of local rings and that $\mathfrak{pr}\Lambda \subseteq \mathfrak{r}\Lambda$.

Conversely, let I be an ideal of Λ, contained in $\mathfrak{r}\Lambda$ and having a product of local rings as its quotient Λ/I. $1 = \sum e_i$ remains a decomposition into primitive idempotents of Λ/I since e_i even remains primitive modulo $\mathfrak{r}\Lambda$ (E.1.19). Moreover, the indecomposable projectives $(\Lambda/I)e_i$ are pairwise nonisomorphic since they are even nonisomorphic modulo the radical.

Over the local ring direct factors of Λ/I, given by assumption, Krull-Schmidt holds since it holds for Λ/I as it holds for Λ (C.2.15). Moreover, each such factor is indecomposable as left module over itself, since an orthogonal decomposition $1 = e + f$ into nonzero idempotents e and f would imply e and f to be nonunits, for $xe = 1$ would give $0 = xef = f$. We conclude that there are no nonzero Λ-morphisms beween nonisomorphic indecomposable projective left Λ/I-modules.

We have to show that $e_ixe_j \in I$ for $i \neq j$, $x \in \Lambda$. But the morphism

$$(\Lambda/I)e_i \xrightarrow{(-)e_ixe_j} (\Lambda/I)e_j$$

is forced to be zero.

Let T be a neat module belonging to the simple module $S := (\Lambda/\mathfrak{r}\Lambda)e_i$. We **claim** that $(\mathfrak{pr}\Lambda)T = 0$, thus showing that $\mathfrak{pr}\Lambda$ is contained in the intersection of the annihilators of the neat modules. More precisely, we have to show that e_jxe_l, $j \neq l$, $x \in \Lambda$, annihilates T. It therefore suffices to show that e_j annihilates T for $j \neq i$.

Assume though that $e_jt \neq 0$ for some $t \in T$. The submodule $\langle e_jt\rangle$ of T generated by e_jt belongs to S (E.1.22) so that there is an epimorphism $(\Lambda e_i)^\alpha \longrightarrow \langle e_jt\rangle$ for some $\alpha \geq 0$ (E.1.5). There is, by construction, also an epimorphism from Λe_j to $\langle e_jt\rangle$, sending e_j to e_jt. We obtain two factorizations $\Lambda e_i \xrightarrow{f} (\Lambda e_j)^\alpha$ and $(\Lambda e_j)^\alpha \xrightarrow{g} \Lambda e_i$. Since $\langle e_jt\rangle \neq 0$ we know that $\Lambda e_i \xrightarrow{fg} \Lambda e_i$ is not nilpotent modulo π, **contradicting** (E.1.9, E.1.8) by Krull-Schmidt (C.2.15).

Conversely, in order to show that $\mathfrak{pr}\Lambda$ contains the intersection of the annihilators of the neat modules we need to construct a neat module T which is not annihilated by a given element of Λ which is not contained in $\mathfrak{pr}\Lambda$. In other words, it suffices to show that $A_i := e_i\Lambda e_i/e_i\Lambda(1-e_i)\Lambda e_i$ is neat. But A_i is a local quotient ring of Λ so that it is neat as a left module over itself. Since the radical of A_i as a ring and as a module over Λ coincide, it is neat also as a module over Λ.

Finally, by $(\mathfrak{pr}\Lambda)T = 0$ we conclude that a module that belongs to $(\Lambda/\mathfrak{r}\Lambda)e_i$ is a module over $e_i\Lambda e_i/e_i\Lambda(1-e_i)\Lambda e_i$, and also conversely, since this ring is neat as a Λ-module and belongs to $(\Lambda/\mathfrak{r}\Lambda)e_i$ (cf. E.1.22).

Lemma E.1.24 (ties caused by radical; cf. [P 80/1, (I.27)]) *Let* f *be a primitive idempotent of* Λ*, let* $f = \sum_i f_i$ *an orthogonal decomposition into nonzero idempotents of* Γ*. There is an isomorphism*

$$\Lambda f/\mathfrak{r}\Lambda f \xrightarrow{\sim} \Lambda f_j/\mathfrak{r}\Lambda f_j$$

so that there is also an isomorphism

$$\Lambda f_i/\mathfrak{r}\Lambda f_i \xrightarrow{\sim} \Lambda f_j/\mathfrak{r}\Lambda f_j$$

for all i, j*.*

Choose a k*-linear basis of* $\Lambda f/\mathfrak{r}\Lambda f$*. Map it to a* k*-linear basis of* $\Lambda f_i/\mathfrak{r}\Lambda f_i$ *under this isomorphism. Complement this* k*-linear basis of* $\Lambda f_i/\mathfrak{r}\Lambda f_i$ *to a* k*-linear basis of* $\Lambda f_i/\pi\Lambda f_i$ *and lift it to an* R*-linear basis of* Λf_i *(E.1.5).*

Consider $y \in \Lambda$. Write yf_i in the basis just constructed, dito yf_j. The coefficients of yf_i at those basis elements occurring in the image of the k-linear basis of $\Lambda f/\mathfrak{r}\Lambda f$ in $\Lambda f_i/\mathfrak{r}\Lambda f_i$ are congruent modulo π to those of yf_j at the image in $\Lambda f_j/\mathfrak{r}\Lambda f_j$, for the diagram

$$\begin{array}{ccc} & \Lambda & \\ \swarrow & & \searrow \\ \Lambda f_i/\mathfrak{r}\Lambda f_i & \xrightarrow{\sim} & \Lambda f_j/\mathfrak{r}\Lambda f_j \end{array}$$

commutes by construction. These are the **ties caused by radical.**

The surjection

$$\Lambda f \xrightarrow{(-)f_i} \Lambda f_i$$

induces an isomorphism

$$\Lambda f/\mathfrak{r}\Lambda f \xrightarrow{\sim} \Lambda f_i/\mathfrak{r}\Lambda f_i$$

since the left hand side is simple (E.1.20 ii), the morphism is surjective and the right hand side is nonzero (E.1.5).

Example E.1.25
Let $R = \mathbf{Z}_{(2)}$, let

$$\Lambda := \{x \times y \times z \mid 2x \equiv_8 y + z,\ y \equiv_4 z\} \subseteq R \times R \times R =: \Gamma,$$

Λ being the endomorphism ring of an indecomposable projective $\mathbf{Z}_{(2)}\mathcal{S}_4$-module (S 2.1.1). The radical quotients, as R-modules isomorphic to $R/2$ with Λ-action given by $x \times y \times z$ acting as x, as y resp. as z are isomorphic, yielding $x \equiv_2 y \equiv_2 z$.

Lemma E.1.26 *Let ε be a central idempotent of Γ. Then*

$$(\mathfrak{r}\Lambda)\varepsilon = \mathfrak{r}(\Lambda\varepsilon),$$

$\Lambda\varepsilon$ on the right hand side viewed as a ring, not as a Λ-lattice.

$\subseteq$. Let $M \subseteq \Lambda\varepsilon$ be a maximal left ideal, so that $\Lambda\varepsilon/M$ is simple over $\Lambda\varepsilon$, hence over Λ. Thus it is annihilated by $\mathfrak{r}\Lambda$, i.e. $(\mathfrak{r}\Lambda)\varepsilon \subseteq M$.

$\supseteq$. Let e be a primitive idempotent of Λ. The epimorphism $\Lambda e/(\mathfrak{r}\Lambda)e \longrightarrow \Lambda e\varepsilon/(\mathfrak{r}\Lambda)e\varepsilon$ shows the latter to be zero or simple over Λ (E.1.20 ii), hence over $\Lambda\varepsilon$. Therefore $\mathfrak{r}(\Lambda\varepsilon)e \subseteq (\mathfrak{r}\Lambda)\varepsilon e$.

Corollary E.1.27 *Let ε be a central idempotent of Γ, let P and Q be indecomposable projective Λ-lattices. Then*

$$0 \not\simeq \varepsilon P \simeq \varepsilon Q \implies P \simeq Q.$$

Calculating the radical both in Λ and in $\varepsilon\Lambda$ (E.1.26) we obtain

$$P/\mathfrak{r}P \xrightarrow{\sim} \varepsilon P/\mathfrak{r}\varepsilon P \simeq \varepsilon Q/\mathfrak{r}\varepsilon Q \xleftarrow{\sim} Q/\mathfrak{r}Q.$$

The assertion follows by (E.1.20 ii) or directly by Nakayama's Lemma (E.1.5). This might be considered as the 'reason' for the quasiblock $Q_{(3)}^{(3,2,1)}$ not to be the Gram order induced by the invariant bilinear form (cf. 6.1.29, 6.1.7), namely that its columns 'have to distinguish projectives'.

Lemma E.1.28 (the radical quasiblockwise) *Let $1_\Lambda = \sum \varepsilon_i$ be the orthogonal decomposition into rational central idempotents, which thus lie in Γ. Then*

$$\mathfrak{r}\Lambda = \Lambda \cap \prod_i \mathfrak{r}(\Lambda\varepsilon_i) (\subseteq \prod_i \Lambda\varepsilon_i).$$

The inclusion $\subseteq$ follows by (E.1.26) or by (E.1.6). It remains to be hsown that a positive power of the right hand side is contained in $\pi\Lambda$ (E.1.20 iv). Choose $N \geq 1$ such that $\pi^N \varepsilon_i \in \pi\Lambda$ for all i, choose $M \geq 1$ such that $(\mathfrak{r}(\Lambda\varepsilon_i))^M \subseteq \pi\Lambda\varepsilon_i$ for all i. Then

$$(\Lambda \cap \prod_i \mathfrak{r}(\Lambda\varepsilon_i))^{MN} \subseteq (\prod_i \pi(\Lambda\varepsilon_i))^N \subseteq \pi\Lambda.$$

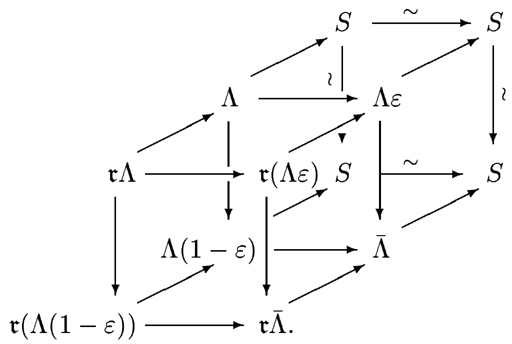
Alternative proof in case Λ is local. It suffices to **claim** that for a central idempotent ε of Γ the radical turns the pullback diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda\varepsilon \\ \downarrow & & \downarrow \\ \Lambda(1-\varepsilon) & \longrightarrow & \bar{\Lambda}, \end{array}$$

where $\bar{\Lambda} = \Lambda\varepsilon / \Lambda \cap \Lambda\varepsilon$, into the pullback diagram

$$\begin{array}{ccc} \mathfrak{r}\Lambda & \longrightarrow & \mathfrak{r}(\Lambda\varepsilon) \\ \downarrow & & \downarrow \\ \mathfrak{r}(\Lambda(1-\varepsilon)) & \longrightarrow & \mathfrak{r}\bar{\Lambda}. \end{array}$$

We may assume $\bar{\Lambda} \neq 0$. Let S be the simple Λ-module and regard the diagram



Example E.1.29 (dangerous bend) *In general, the inclusion*

$$\mathfrak{r}^2\Lambda \subseteq \Lambda \cap \prod_i \mathfrak{r}^2(\Lambda\varepsilon_i),$$

which holds by (E.1.26 or E.1.6), is a strict inclusion.

Let $R := \mathbf{Z}_{(2)}$, $\pi := 2$, and consider

$$\Lambda := \{x \times y \mid x \equiv_4 y\} \subseteq R \times R.$$

By (E.1.28) we obtain

$$\begin{aligned} \mathfrak{r}\Lambda &= \{2x \times 2y \mid 2x \equiv_4 2y\} \subseteq R \times R \\ &= R\langle 2 \times 2,\ 0 \times 4\rangle, \end{aligned}$$

whence

$$\mathfrak{r}^2\Lambda = R\langle 4 \times 4,\ 0 \times 8\rangle \not\ni 0 \times 4.$$

Lemma E.1.30 (the radical of local quasiblocks in easy cases) *Assume in addition Λ to be local and to have a single quasiblock, i.e. assume given a full inclusion of R-orders*

$$\Lambda \subseteq (R)_m =: \Gamma.$$

Let $1_\Lambda = \sum_i f_i$ be an orthogonal decomposition into primitive idempotents of Γ and **assume** *moreover that $f_i \notin f_i\Lambda f_j \Lambda f_i$ for $i \neq j$. Using main diagonal primitive idempotents of Γ, this assumption is satisfied provided there is a single tie either at position ij or at position ji for $i \neq j$. Then*

$$\mathfrak{r}\Lambda = \Lambda \cap \left[(\bigoplus_i \pi f_i \Lambda f_i) \oplus (\bigoplus_{i \neq j} f_i \Lambda f_j) \right] \subseteq \Gamma.$$

Moreover, we could have dropped all but one factor π in this formula without changing the right hand side.

In addition, we obtain

$$k \xrightarrow{\sim} \Lambda/\mathfrak{r}\Lambda.$$

Note that $f_i\Lambda f_i = f_i\Gamma f_i \simeq R$ by primitivity of f_i, because the R-linear generator f_i of the right hand side is contained in the left hand side. In particular, $f_i\Lambda f_i$ is a ring with multiplication given by $(f_i x f_i)(f_i y f_i) = f_i x f_i y f_i$, since the right hand side of this equation in fact is contained in $f_i\Lambda f_i = f_i\Gamma f_i$ (cf. E.1.33).

We **claim** that the right hand side of the equation above is an ideal in Λ. It suffices to show that $f_i x f_j y f_i$ is in $\pi f_i\Lambda f_i$ for $i \neq j$, x and y in Λ. We are reduced to show that $f_i x f_j y f_i$ is a nonunit in $f_i\Lambda f_i$. But otherwise we multiply with its inverse to obtain $f_i \in f_i\Lambda f_j\Lambda f_i$, contrary to our assumption.

We **claim** that the surjection

$$\begin{array}{ccc} \Lambda/\mathfrak{r}\Lambda & \longrightarrow & f_i\Lambda f_i / f_i\mathfrak{r}\Lambda f_i \\ x & \longrightarrow & f_i x f_i \end{array}$$

is an isomorphism of rings for all i. Consider the relation

$$f_i x y f_i - f_i x f_i y f_i = f_i x(1 - f_i) y f_i \in f_i \pi \Lambda f_i \subseteq f_i \mathfrak{r}\Lambda f_i.$$

For $x \in \mathfrak{r}\Lambda$ and $y \in \Lambda$ (resp. vice versa) it shows $f_i\mathfrak{r}\Lambda f_i \subseteq f_i\Lambda f_i$ to be an ideal. For $x, y \in \Lambda$ it shows the map to be a ring morphism. Since $\Lambda/\mathfrak{r}\Lambda$ is a skew field, it is injective.

We derive

$$f_i\mathfrak{r}\Lambda f_i = \mathfrak{r}(f_i\Lambda f_i) = \pi f_i\Lambda f_i,$$

since $f_i\Lambda f_i$ is a local ring in which $f_i\mathfrak{r}\Lambda f_i$ is an ideal that has a skew field as its quotient. It ensues in particular that $\Lambda/\mathfrak{r}\Lambda = k$ (cf. E.1.32).

Now we can set out to prove

$$\mathfrak{r}\Lambda = \Lambda \cap \left[(\bigoplus_i \pi f_i \Lambda f_i) \oplus (\bigoplus_{i \neq j} f_i \Lambda f_j) \right].$$

We **claim** that $\mathfrak{r}\Lambda$ is contained in the right hand side. But this follows from $f_i\mathfrak{r}\Lambda f_i \subseteq \pi f_i\Lambda f_i$ for all i.

We **claim** that the right hand side, with all but one factor π dropped, is contained in $\mathfrak{r}\Lambda$. So suppose given $x \in \Lambda$ such that $f_i x f_i \in \pi f_i\Lambda f_i \subseteq f_i\mathfrak{r}\Lambda f_i$ for some i. By $\Lambda/\mathfrak{r}\Lambda \xrightarrow{\sim} f_i\Lambda f_i/f_i\mathfrak{r}\Lambda f_i$ we conclude that $x \in \mathfrak{r}\Lambda$.

Remark E.1.31 *Let Λ be local. Then*

$$\dim_k \Lambda/\mathfrak{r}\Lambda \leq \min\{n_\lambda \mid \lambda \text{ parametrizing the factors of } \Gamma\}.$$

In particular, in case Λ is commutative, we obtain

$$k \xrightarrow{\sim} \Lambda/\mathfrak{r}\Lambda.$$

This follows from the remark that the column of $(R/\pi)_{n_\lambda}$, regarded as a Λ-module, maps epimorphically onto the simple Λ-module $\Lambda/\mathfrak{r}\Lambda$. Cf. (E.1.32).

Remark E.1.32 *The assumption in (E.1.30) that $f_i \notin f_i\Lambda f_j\Lambda f_i$ for $i \neq j$ cannot be dropped.*

For $u \in R$, let

$$\Lambda_u := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \equiv_\pi d,\ bu \equiv_\pi c\} \subseteq (R)_2 =: \Gamma,$$

which is in fact a suborder, as becomes visible on the set of R-linear generators

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}\}.$$

Note that the ideal $\pi\Gamma \subseteq \Lambda_u$ is contained in $\mathfrak{r}\Lambda_u$ since it is nilpotent modulo π (E.1.20 iv). The ring epimorphism

$$\begin{array}{ccc} k[T] & \longrightarrow & \Lambda_u/\pi\Gamma \\ T & \longrightarrow & \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \end{array}$$

has kernel $(T^2 - u)$. Thus there are several cases to be distinguished.

In **case** u is a square in k^* and $\operatorname{char} k \neq 2$, $\mathfrak{r}\Lambda_u$ coincides with $\pi\Gamma$. $\Lambda_u/\mathfrak{r}\Lambda_u \simeq k \times k$ is not a local ring, so neither is Λ_u, for there are two nonisomorphic simple modules (E.1.2).

In **case** u is a square in k^* and $\operatorname{char} k = 2$, we obtain $\mathfrak{r}(k[T]/(T-\sqrt{u})^2) = (T-\sqrt{u})$, whence $\mathfrak{r}\Lambda_u$ is the ideal generated by $\pi\Gamma$ and $\begin{pmatrix} \sqrt{u} & 1 \\ u & \sqrt{u} \end{pmatrix}$, $\sqrt{u}$ denoting an inverse image in R of the square root of u in k. Therefore $\Lambda_u/\mathfrak{r}\Lambda_u \simeq k$ is simple, whence Λ_u is local (E.1.20 ii, E.1.9). But the subset described in (E.1.30) applied to $1_\Gamma = f_1 + f_2$ with $f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ does **not** give the radical. It is not even an ideal of Λ, since it contains $\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}$ but not $\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}^2$.

In **case** u is not a square in k^*, we obtain $k[T]/(T^2-u)$ to be a quadratic field extension of k, denoted by $k[\sqrt{u}]$ (cf. E.1.31). Thus Λ_u is local with radical $\mathfrak{r}\Lambda_u = \pi\Gamma$, which is likewise **not** in coincidence with the formula given in (E.1.30). Moreover, note that $\mathfrak{r}\Lambda_u/\mathfrak{r}^2\Lambda_u \simeq S \oplus S$, where $S = \Lambda_u/\mathfrak{r}\Lambda_u$ is a column in $\Gamma/\pi\Gamma$.

In **case** $u = 0 \in R$, the radical is in fact calculable by the formula of (E.1.30) to be

$$\mathfrak{r}\Lambda_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, c, d \equiv_\pi 0\} \subseteq (R)_2.$$

Example E.1.33 (dangerous bend) ([1]) Consider

$$\Lambda := \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mid a_{11} \equiv_\pi a_{22} \equiv_\pi a_{33},\ a_{12} \equiv_\pi a_{23} \equiv_\pi a_{31},\ a_{13} \equiv_\pi a_{21} \equiv_\pi a_{32}\} \subseteq (R)_3 =: \Gamma.$$

We claim that Λ is a subring of Γ. Since $\pi\Gamma$ is contained in Λ it suffices, by an extension of the following two elements to an R-linear basis of Λ by 1_Λ and by elements in $\pi\Gamma$, to consider their four products. But

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

are even permutation matrices. Let $e := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We obtain

$$e\Lambda e = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11} \equiv_\pi a_{22}\} \subseteq (R)_2 = e\Gamma e,$$

which is **not** a subring, since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Remark E.1.34 *Assume Λ to be local. Let $f \neq 1$ be an idempotent of Γ. Then*

$$\Lambda \cap \Lambda f \subseteq \mathfrak{r}\Lambda.$$

The quadrangle

[1] S. König pointed out that such an example should exist, contrary to what I had believed. It is modelled on an example H. Weber has shown me.

$$\begin{array}{ccc} \mathfrak{r}\Lambda & \longrightarrow & \mathfrak{r}\Lambda(1-f) \\ \big\downarrow & & \big\downarrow \\ \Lambda & \longrightarrow & \Lambda(1-f) \end{array}$$

is an exact square, as to be seen on the vertical cokernels. The assertion follows by equality of the horizontal kernels.

E.2 Homological inequalities

We derive some inequalities concerning the size of the radical layers in the local case, resulting from a long exact Ext-sequence. To begin with, however, we shall present some elementary facts which give rise to doubts whether the outcome of these considerations will be overly useful.

Keep the conventions from (S E.1.3). Let $\Lambda \subseteq \prod_\lambda (R)_{n_\lambda} =: \Gamma$ be a local sub split semisimple R-order with simple module S and projective module P ($\simeq \Lambda$). Let $D := \mathrm{End}_\Lambda S$. Let X be a left Λ-lattice. Consider the minimal projective resolution

$$\cdots \xrightarrow{d_{i3}} P^{\alpha_{i2}} \xrightarrow{d_{i2}} P^{\alpha_{i1}} \xrightarrow{d_{i1}} P^{\alpha_{i0}} \xrightarrow{d_{i0}} X/\mathfrak{r}^i X,$$

i.e. the projective resolution constructed via projective covers (E.1.21). Let $\omega_{ij}^X := \mathrm{Kern} d_{i,j-1} = \mathrm{Im}\, d_{ij}$. For a Λ-module M we denote by t_M the number defined by

$$M/\mathfrak{r}M \simeq S^{t_M}.$$

Note that by construction we have $\alpha_{ij} = t_{\omega_{ij}^X}$ as well as $\omega_{ij}^X \subseteq \mathfrak{r}P^{\alpha_{i,j-1}}$. If N is a finitely generated torsion R-module, we denote by lN its length in the sense of Jordan-Hölder, i.e. $lN = \sum_{i\geq 0} \dim_k \pi^i N/\pi^{i+1}N$.

For an example in which the inclusion $k \subseteq D$ is strict we refer to (E.1.32).

Remark E.2.1 *The series of isomorphism classes $\mathfrak{r}^i X$ becomes eventually periodic.*

This is a consequence of Jordan-Zassenhaus (D.2.7), using the fact that the inclusion of $\mathfrak{r}^i X$ into X is full. Cf. (6.1.16).

Remark E.2.2 *For $i \geq 0$ we have the upper bound*

$$t_{\mathfrak{r}^i X} \leq \mathrm{rk}_R X / \dim_k S.$$

Note that

$$l(\mathfrak{r}^i X/\mathfrak{r}^{i+1}X) \leq l(\mathfrak{r}^i X/\pi\mathfrak{r}^i X) = \mathrm{rk}_R \mathfrak{r}^i X = \mathrm{rk}_R X.$$

Lemma E.2.3 (stability at the rank) *Assume in addition Λ to be* **commutative,** *i.e. assume $n_\lambda = 1$ for each λ. There is an $I \geq 0$ such that for $i \geq I$ we have*

$$t_{\mathfrak{r}^i X} = \mathrm{rk}_R X.$$

Cf. (E.2.2, E.2.5 c).

Denote $\Gamma X := \Gamma \tilde{\otimes}_\Lambda X$ and note that $\mathrm{rk}_R \Gamma X = \mathrm{rk}_R X$ (B.1.3, B.1.12). The radical of a Γ-lattice Y as a Γ-module, viz. πY, contains the radical of Y as a Λ-lattice by (E.1.6) since Λ is assumed to be commutative. In particular, for $i \geq 0$ we have

$$\mathfrak{r}^i X \subseteq \pi^i \Gamma X.$$

Note that
$$d_{i+1} := l(\pi^{i+1}\Gamma X/\mathfrak{r}^{i+1}X) \le l(\pi^{i+1}\Gamma X/\pi\mathfrak{r}^i X) = l(\pi^i\Gamma X/\mathfrak{r}^i X) =: d_i$$
so that there exists an $I \ge 0$ such that
$$l(\pi^i\Gamma/\mathfrak{r}^i\Lambda) = l(\pi^I\Gamma/\mathfrak{r}^I\Lambda) =: d$$
for $i \ge I$.

The equation
$$\begin{aligned} l(\Gamma X/X) + l(X/\mathfrak{r}^i X) &= l(\Gamma X/\pi^i\Gamma X) + l(\pi^i\Gamma X/\mathfrak{r}^i X) \\ &= i \cdot \mathrm{rk}_R X + d \end{aligned}$$
for $i \ge I$ shows that
$$\mathrm{rk}_R X = \lim_i \frac{1}{i}\sum_{j=1}^{i} l(\mathfrak{r}^{j-1}X/\mathfrak{r}^j X).$$

Subtracting the equations
$$\begin{aligned} l(\pi^i\Gamma X/\mathfrak{r}^i X) + l(\mathfrak{r}^i X/\mathfrak{r}^{i+1}X) &= l(\pi^i\Gamma X/\pi^{i+1}\Gamma X) + l(\pi^{i+1}\Gamma X/\mathfrak{r}^{i+1}X) \\ l(\pi^{i+1}\Gamma X/\mathfrak{r}^{i+1}X) + l(\mathfrak{r}^{i+1}X/\mathfrak{r}^{i+2}X) &= l(\pi^{i+1}\Gamma X/\pi^{i+2}\Gamma X) + l(\pi^{i+2}\Gamma X/\mathfrak{r}^{i+2}X) \end{aligned}$$
yields
$$l(\mathfrak{r}^{i+1}X/\mathfrak{r}^{i+2}X) - l(\mathfrak{r}^i X/\mathfrak{r}^{i+1}X) = d_{i+2} - 2d_{i+1} + d_i$$
which becomes
$$t_{\mathfrak{r}^i X} = l(\mathfrak{r}^i X/\mathfrak{r}^{i+1}X) = l(\mathfrak{r}^{i+1}X/\mathfrak{r}^{i+2}X) = \mathrm{rk}_R X$$
for $i \ge I$, where the left hand side equality holds by (E.1.31).

Corollary E.2.4 *Keep the assumptions and the notation of (E.2.3). For $i \ge I$ we have*
$$\mathfrak{r}^{i+1}X = \pi\mathfrak{r}^i X.$$

Cf. (E.2.1). For a maybe somewhat surprising $\mathfrak{r}^\infty\Lambda$, cf. (E.1.29). The assertion does not hold in case Λ is not commutative, cf. (E.3.4).

Remark E.2.5

(a) I do not know whether the sequence $t_{\mathfrak{r}^i X}$ is ascending. If this was known, the periodicity of (E.2.1) would imply stability of $t_{\mathfrak{r}^i X}$ for large X. Let $1_\Gamma = \sum_\lambda \varepsilon_\lambda$ be the decomposition into central primitive idempotents of Γ. I do not know whether $t_{\mathfrak{r}^i X}$ and $\sum_\lambda t_{\mathfrak{r}^i\varepsilon_\lambda X}$ become equal in the limit.

(b) For Λ commutative local sub split semisimple, we obtain some information on the Hilbert-Samuel polynomial f of Λ, which gives the length of $\Lambda/\mathfrak{r}^i\Lambda$ as Λ-modules for i large [AM 69, 11]. Plugging in $X = \Lambda$ in (E.2.3) we see that f is linear and has leading coefficient $\mathrm{rk}_R\Lambda$. Note that $\Lambda\pi$ is a $\mathfrak{r}\Lambda$-primary ideal (E.1.20 iv), thus the degree of f equals the Krull dimension of Λ equals the minimal number of generators over Λ of an $\mathfrak{r}\Lambda$-primary ideal equals 1, in accordance with [AM 69, 11.14].

(c) We repeat the argument of (E.2.3) in the noncommutative case under an extra assumption. Suppose given an element $\tilde{\pi} \in Z(\Lambda)$ such that there exists a $m \ge 1$ with
$$\tilde{\pi}\Lambda \subseteq \mathfrak{r}^m\Lambda \subseteq \tilde{\pi}\Gamma.$$

For instance, in case Λ is commutative we may take $\tilde{\pi} = \pi$ by (E.1.6). See (E.3.4, E.3.6) for noncommutative examples which fulfill this assumption, and, moreover, for which the assertion of (E.2.3) does not hold.

Denote the valuation of the projection of $\tilde{\pi}$ to $Z((R)_{n_\lambda}) = R$ by σ_λ. Denote by ξ_λ the rank of the projection of ΓX to $(R)_{n_\lambda}X$. So we obtain e.g. for $X = P$ that ξ_λ equals n_λ^2. Note that for $i \ge 0$ we have
$$\mathrm{rk}_{\tilde{\pi}} X := l(\Gamma X/\tilde{\pi}\Gamma X) = l(\tilde{\pi}^i\Gamma X/\tilde{\pi}^{i+1}\Gamma X) = \sum_\lambda \xi_\lambda\sigma_\lambda.$$

The inequality

$$d_{i+1} := l(\tilde{\pi}^{i+1}\Gamma X/\mathfrak{r}^{(i+1)m}X) \leq l(\tilde{\pi}^{i+1}\Gamma X/\tilde{\pi}\mathfrak{r}^{im}X) \leq l(\tilde{\pi}^{i}\Gamma X/\mathfrak{r}^{im}X) =: d_i$$

supplies us with an $I \geq 0$ such that

$$l(\tilde{\pi}^i\Gamma/\mathfrak{r}^{im}\Lambda) = l(\tilde{\pi}^I\Gamma/\mathfrak{r}^{Im}\Lambda) =: d$$

for $i \geq I$.
The equation

$$\begin{aligned} l(\Gamma X/X) + l(X/\mathfrak{r}^{im}X) &= l(\Gamma X/\tilde{\pi}^i\Gamma X) + l(\tilde{\pi}^i\Gamma X/\mathfrak{r}^{im}X) \\ &= i \cdot \mathrm{rk}_{\tilde{\pi}} X + d \end{aligned}$$

for $i \geq I$ shows that

$$\mathrm{rk}_{\tilde{\pi}} X = \lim_i \frac{1}{i} \sum_{j=1}^{i} l(\mathfrak{r}^{(j-1)m}X/\mathfrak{r}^{jm}X).$$

Subtracting the equations

$$\begin{aligned} l(\tilde{\pi}^i\Gamma X/\mathfrak{r}^{im}X) + l(\mathfrak{r}^{im}X/\mathfrak{r}^{(i+1)m}X) &= l(\tilde{\pi}^i\Gamma X/\tilde{\pi}^{i+1}\Gamma X) + l(\tilde{\pi}^{i+1}\Gamma X/\mathfrak{r}^{(i+1)m}X) \\ l(\tilde{\pi}^{i+1}\Gamma X/\mathfrak{r}^{(i+1)m}X) + l(\mathfrak{r}^{(i+1)m}X/\mathfrak{r}^{(i+2)m}X) &= l(\tilde{\pi}^{i+1}\Gamma X/\tilde{\pi}^{i+2}\Gamma X) + l(\tilde{\pi}^{i+2}\Gamma X/\mathfrak{r}^{(i+2)m}X) \end{aligned}$$

yields

$$l(\mathfrak{r}^{im}X/\mathfrak{r}^{(i+1)m}X) = \mathrm{rk}_{\tilde{\pi}} X$$

for $i \geq I$. Moreover, we read off that

$$\mathrm{rk}_{\tilde{\pi}} X \geq l(\mathfrak{r}^{im}X/\mathfrak{r}^{(i+1)m}X)$$

holds for all $i \geq 0$.

Lemma E.2.6 *Let $i, j \geq 1$. We have*

$$\begin{aligned} \dim_D \mathrm{Ext}^j_\Lambda(X/\mathfrak{r}^iX, S) &= t_{\omega^X_{ij}} \\ \dim_D \mathrm{Ext}^j_\Lambda(\mathfrak{r}^iX/\mathfrak{r}^{i+1}X, S) &= t_{\mathfrak{r}^iX} \cdot t_{\omega^P_{1j}} \end{aligned}$$

In fact,

$$\begin{aligned} \mathrm{Ext}^j_\Lambda(X/\mathfrak{r}^iX, S) &= \mathrm{Cokern}\left(\mathrm{Hom}_\Lambda(P^{\alpha_{i,j-1}}, S) \xrightarrow{0} \mathrm{Hom}_\Lambda(\omega^X_{ij}, S)\right) \\ &= \mathrm{Hom}_\Lambda(\omega^X_{ij}, S) \end{aligned}$$

has dimension $t_{\omega^X_{ij}}$ over D, the zero morphism resulting from $\omega^X_{ij} \subseteq \mathfrak{r}P_{i,j-1}$. In particular,

$$\mathrm{Ext}^j_\Lambda(S, S) = t_{\omega^P_{1j}},$$

yielding the second equation.

Definition E.2.7 *A sequence of integers $s_1, s_2, \ldots$ is called* **of Euler characteristic type** *if*

$$\sum_{i \in [1,m]} (-1)^{m-i} s_i \geq 0$$

for all $m \geq 0$.

Proposition E.2.8 (Euler inequalities) *Let $i \geq 2$. The sequence of integers*

$$\begin{array}{lll} & & t_{\mathfrak{r}^{i-1}X}, \\ t_{\omega^X_{i-1,1}}, & t_{\omega^X_{i,1}}, & t_{\mathfrak{r}^{i-1}X} \cdot t_{\omega^P_{11}}, \\ t_{\omega^X_{i-1,2}}, & t_{\omega^X_{i,2}}, & t_{\mathfrak{r}^{i-1}X} \cdot t_{\omega^P_{12}}, \\ t_{\omega^X_{i-1,3}}, & t_{\omega^X_{i,3}}, & t_{\mathfrak{r}^{i-1}X} \cdot t_{\omega^P_{13}}, \\ \ldots & & \end{array}$$

is of Euler characteristic type (E.2.7).

The short exact sequence

$$0 \longrightarrow \mathfrak{r}^{i-1}X/\mathfrak{r}^iX \longrightarrow X/\mathfrak{r}^iX \longrightarrow X/\mathfrak{r}^{i-1}X \longrightarrow 0$$

induces the long exact sequence

$$\begin{array}{ccccccc} \longleftarrow \mathrm{Hom}_\Lambda(\mathfrak{r}^{i-1}X/\mathfrak{r}^iX, S) & \xleftarrow{\;0\;} & \mathrm{Hom}_\Lambda(X/\mathfrak{r}^iX, S) & \xleftarrow{\;\sim\;} & \mathrm{Hom}_\Lambda(X/\mathfrak{r}^{i-1}X, S) & \longleftarrow & 0 \\ \longleftarrow \mathrm{Ext}^1_\Lambda(\mathfrak{r}^{i-1}X/\mathfrak{r}^iX, S) & \longleftarrow & \mathrm{Ext}^1_\Lambda(X/\mathfrak{r}^iX, S) & \longleftarrow & \mathrm{Ext}^1_\Lambda(X/\mathfrak{r}^{i-1}X, S) & \longleftarrow & \\ \longleftarrow \mathrm{Ext}^2_\Lambda(\mathfrak{r}^{i-1}X/\mathfrak{r}^iX, S) & \longleftarrow & \mathrm{Ext}^2_\Lambda(X/\mathfrak{r}^iX, S) & \longleftarrow & \mathrm{Ext}^2_\Lambda(X/\mathfrak{r}^{i-1}X, S) & \longleftarrow & \\ & & & & \ldots & \longleftarrow & \end{array}$$

The inequalities follow by (E.2.6) and by the remark that cutting off this long exact sequence and inserting an image instead, the Euler characteristic of the resulting exact sequence is zero.

Corollary E.2.9 (exponential bound) *For $i \geq 2$ and $X = P$ we obtain*

$$t_{\mathfrak{r}^iP} \leq (t_{\mathfrak{r}P})(t_{\mathfrak{r}^{i-1}P}),$$

thus, in particular,

$$t_{\mathfrak{r}^iP} \leq (t_{\mathfrak{r}P})^i.$$

In view of (E.2.3) in case Λ commutative and of (E.2.1) in the general case, this inequality merely says that the rate of growth is exponentially bounded 'in the start region'.

Remark E.2.10 If $\dim_k S = 1$, e.g. if Λ is commutative (cf. E.1.31), this already follows from the surjection

$$\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda \otimes_k \mathfrak{r}^{i-1}\Lambda/\mathfrak{r}^i\Lambda \longrightarrow \mathfrak{r}^i\Lambda/\mathfrak{r}^{i+1}\Lambda$$

induced by multiplication.

Corollary E.2.11 (regularity is exceptional) *If Λ is commutative, then $t_{\mathfrak{r}P} = 1$ implies $\mathrm{rk}_R P = 1$.*

This follows from (E.2.3, E.2.9). Cf. [AM 69, 11.22].

Remark E.2.12 (growth of $\omega_{1,j}$) *For $j \geq 1$, we have*

$$(*) \qquad (\mathrm{rk}_R\, \omega^P_{1,j+1} + \mathrm{rk}_R\, \omega^P_{1,j})/\mathrm{rk}_R\, P = t_{\omega^P_{1,j}} \leq \mathrm{rk}_R\, \omega^P_{1,j}/\dim_k S,$$

whence

$$(**) \qquad \frac{\mathrm{rk}_R\, \omega^P_{1,j+1}}{\mathrm{rk}_R\, \omega^P_{1,j}} \leq \frac{\mathrm{rk}_R\, P}{\dim_k S} - 1.$$

Moreover, we remark that in case $\mathrm{rk}_R\, P = 2\dim_k S$, we either get $\mathrm{rk}_R\, \omega^P_{1,j} = \mathrm{rk}_R\, P$ for all j, or S has a finite projective resolution.

The equality in $(*)$ is given by construction of the minimal resolution. The inequality in $(*)$ is asserted in (E.2.2). We note that the consequence $\mathrm{rk}_R P \geq \dim_k S$ already follows from $l(P/\pi P) \geq l(P/\mathfrak{r}P)$.

E.3 Examples

We calculate some example for sake of illustration of (S E.1.3, S E.2) as well as in order to examinate some modular morphisms, not necessarily between simple lattices. In particular, we will record the behaviour of such morphisms with respect to the radical layers. The hope was to discover well behaved and predictable morphisms that way, in the spirit of (E.1.24), however, there are none in sight. We restrict our attention to the 'principal building blocks' of $\mathbf{Z}_{(p)}\mathcal{S}_n$, p prime, viz. to the endomorphism rings of the indecomposable projectives. Moreover, we pick some relevant Hom-groups, however, without giving a precise meaning to this attribute. So the following treatment remains tentative, yielding a bunch of experimental material. The first example (E.3.2) is presented rather detailed.

We keep the notation of (S E.1.3, S E.2).

Lemma E.3.1 *Let ε be a central idempotent of Γ, let $I := \Lambda \cap \Lambda(1-\varepsilon)$. The evaluation*

$$\begin{array}{rcl} \mathrm{Hom}_\Lambda(\Lambda\varepsilon, I/\pi^\alpha) & \longrightarrow & \{x \in I/\pi^\alpha \mid Ix = 0\} \\ f & \longrightarrow & \varepsilon f \end{array}$$

is an isomorphism.

The map going from the Hom group to the whole I/π^α is injective. An element $x \in I/\pi^\alpha$ qualifies as an image of ε under a morphism $\Lambda\varepsilon \xrightarrow{f} I/\pi^\alpha$ iff the product of $I = \mathrm{Ann}_\Lambda \varepsilon$ with x vanishes.

Example E.3.2
Let $R := \mathbf{Z}_{(2)}$, $k := \mathbf{F}_2$, let

$$\Lambda := \{x \times y \times z \mid 2x \equiv_8 y + z,\ y \equiv_4 z\} \subseteq R_1 \times R_2 \times R_3 =: \Gamma,$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(2)}\mathcal{S}_4$ (S 2.1.1) as well as over $\mathbf{Z}_{(2)}\mathcal{S}_5$ (S 2.2.4).
By (E.1.28) we obtain

$$\begin{array}{rcl} \Lambda & = & R\langle 1\times 1\times 1,\ 0\times 2\times -2,\ 0\times 0\times 8\rangle \\ \mathfrak{r}\Lambda & = & R\langle 2\times 2\times 2,\ 0\times 2\times -2,\ 0\times 0\times 8\rangle \\ \mathfrak{r}^2\Lambda & = & R\langle 4\times 0\times 0,\ 0\times 4\times 4,\ \ 0\times 0\times 8\rangle \\ \mathfrak{r}^3\Lambda & = & R\langle 8\times 0\times 0,\ 0\times 8\times 8,\ \ 0\times 0\times 16\rangle \\ & = & 2\mathfrak{r}^2\Lambda, \end{array}$$

whence

$$\begin{array}{rcl} \Lambda/\mathfrak{r}\Lambda & = & k\langle 1\times 1\times 1\rangle \\ \mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda & = & k\langle 2\times 2\times 2,\ 0\times 2\times -2\rangle \\ \mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda & = & k\langle 4\times 0\times 0,\ 0\times 4\times 4,\ \ 0\times 0\times 8\rangle. \end{array}$$

We calculate the minimal projective resolution of the simple Λ-module $\Lambda/\mathfrak{r}\Lambda$ (cf. S E.2). The short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega^P_{12} & \longrightarrow & \Lambda \oplus \Lambda & \longrightarrow & \omega^P_{11} & \longrightarrow & 0 \\ & & & & 1 \oplus 0 & \longrightarrow & 2\times 2\times 2 & & \\ & & & & 0 \oplus 1 & \longrightarrow & 0\times 2\times -2 & & \end{array}$$

yields

$$\begin{array}{rcl}\omega_{12}^P & = & \{a\times b\times c\oplus d\times e\times f\in\Lambda\oplus\Lambda \mid 2a\times 2b\times 2c = 0d\times -2e\times 2f\}\\ & \simeq & \{d\times e\times f\in\Lambda \mid 0\times -e\times f\in\Lambda\}\\ & = & \{d\times e\times f\in\Lambda \mid e\equiv_8 f,\ -e\equiv_4 f\}\\ & = & R\langle 2\times 2\times 2,\ 0\times 4\times -4,\ 0\times 0\times 8\rangle,\end{array}$$

whence

$$\omega_{12}^P/\mathfrak{r}\omega_{12}^P = k\langle 2\times 2\times 2,\ 0\times 0\times 8\rangle.$$

The short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{13}^P & \longrightarrow & \Lambda\oplus\Lambda & \longrightarrow & \omega_{12}^P & \longrightarrow & 0\\ & & & & 1\oplus 0 & \longrightarrow & 2\times 2\times 2 & & \\ & & & & 0\oplus 1 & \longrightarrow & 0\times 0\times 8, & & \end{array}$$

yields

$$\begin{array}{rcl}\omega_{13}^P & = & \{a\times b\times c\oplus d\times e\times f\in\Lambda\oplus\Lambda \mid 2a\times 2b\times 2c = 0d\times 0e\times -8f\}\\ & \simeq & \{d\times e\times f\in\Lambda \mid 0\times 0\times -4f\in\Lambda\}\\ & = & \{d\times e\times f\in\Lambda \mid f\equiv_2 0\}\\ & = & R\langle 2\times 2\times 2,\ 0\times 2\times -2,\ 0\times 0\times 8\rangle\\ & = & \omega_{11}^P,\end{array}$$

respecting the bound $(**)$ of (E.2.12), $1\leq 2$.
Let $X := R_1$, let $Y := \{y\times z \mid y\equiv_4 z\}\subseteq R_2\times R_3$. We want to calculate

$$\mathrm{Hom}_\Lambda(X, Y/2^\infty) := \varinjlim_i \mathrm{Hom}_\Lambda(X, Y/2^i)$$

where the injective transition morphisms are induced by

$$Y/2^i \xrightarrow{2(-)} Y/2^{i+1}$$

so that we may as well view this direct limit as a union over this chain of subgroups. A priori we know by the commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_\Lambda(X, Y/2^i) & \longrightarrow & \mathrm{Hom}_\Lambda(X, Y/2^{i+1})\\ \downarrow & & \downarrow\\ \mathrm{Ext}^1_\Lambda(X, Y) & = & \mathrm{Ext}^1_\Lambda(X, Y)\end{array}$$

in which the vertical maps are connectors of long exact Ext-sequences, that this chain stabilizes for $i\geq v_2(\#\mathrm{Ext}^1_\Lambda(X, Y))$, since in this range the injective connectors are also surjective.
Note that the existence of an element

$$f\in\mathrm{Hom}(X, Y/2^i)\backslash\mathrm{Hom}(X, Y/2^{i-1})$$

has the following consequence. Let $y\times z\in Y$ be a representative of $1f\in Y/2^i$, let $a\times b\times c\in\Lambda$. We have $y\times z\notin 2Y$ and, moreover,

$$(ay\times az) - (by\times cz) = (a-b)y\times(a-c)z\in 2^iY$$

Let $\alpha\geq 0$ be minimal such that $2^\alpha\times 0\times 0$ lies in Λ, i.e. $\alpha = 2$. Plugging in $a\times b\times c = 2^\alpha\times 0\times 0$, we obtain for $i\geq\alpha$

$$y\times z\in 2^{i-\alpha}Y,$$

whence $i = \alpha$. We conclude that the chain of subgroups stabilizes at $i = \alpha$. Therefore, actually we have to calculate

$$Y/4\supseteq\mathrm{Hom}_\Lambda(X, Y/4) = \mathrm{Hom}_\Lambda(X, Y/2^\infty).$$

The element $y\times z$ alluded to above determines such a morphism. It yields such a morphism iff

$$(a-b)y\times(a-c)z\in 4Y$$

for all $a \times b \times c$ in a set of R-linear generators of Λ (cf. E.3.1). We obtain the conditions

$$\begin{array}{r} -2y \times 2z \in 4Y \\ 0y \times 8z \in 4Y \end{array}$$

so that

$$\mathrm{Hom}_\Lambda(X, Y/4) \xrightarrow{\sim} R\langle 2 \times -2,\ 0 \times 8\rangle \subseteq Y/8.$$

An element $a \times b \times c$ of Γ acts on Y and commutes with f iff it satisfies the following ties.

$$\begin{array}{rcl} a & \equiv_2 & c \\ 2a & \equiv_8 & b + c. \end{array}$$

An element $a \times b \times c$ of Γ commutes with g iff it satisfies

$$a \equiv_2 c,$$

which is implied by those ties caused by f since g is the composition of f with an endomorphism of $Y/4$. Consider the morphisms induced by

$$\bar{X} := X/4 \xrightarrow{f} Y/4 =: \bar{Y}$$

on the radical layers. The radical layers of $\bar{Y}$ are given by

$$\begin{array}{rcl} \bar{Y}/\mathfrak{r}\bar{Y} & = & k\langle 1 \times 1\rangle \\ \mathfrak{r}\bar{Y}/\mathfrak{r}^2\bar{Y} & = & k\langle 2 \times -2,\ 0 \times 4\rangle \\ \mathfrak{r}^2\bar{Y}/\mathfrak{r}^3\bar{Y} & = & k\langle 4 \times -4\rangle \end{array}$$

so that $\bar{X}/\mathfrak{r}\bar{X}$ injects into $\mathfrak{r}\bar{Y}/\mathfrak{r}^2\bar{Y}$ and $\mathfrak{r}\bar{X}/\mathfrak{r}^2\bar{X}$ injects into $\mathfrak{r}^2\bar{Y}/\mathfrak{r}^3\bar{Y}$. Intuitively, $S = \Lambda/\mathfrak{r}\Lambda$ being the simple module, we obtain

$$\begin{array}{ccccc} \bar{X} & & & \bar{Y} & \\ & & & & \\ & & & S & \\ \boldsymbol{S} & \xrightarrow{f} & \boldsymbol{S} & & S \\ \boldsymbol{S} & & & \boldsymbol{S}. & \end{array}$$

Example E.3.3
Let $R := \mathbf{Z}_{(2)}$, $k := \mathbf{F}_2$, let

$$\Lambda := \{x \times y \times z \times w \mid x \equiv_4 y,\ z \equiv_4 w,\ x \equiv_2 z,\ x - y \equiv_8 z - w\} \subseteq R_1 \times R_2 \times R_3 \times R_4 =: \Gamma,$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(2)}\mathcal{S}_4$ (S 2.1.1).
By (E.1.28) we obtain

$$\begin{array}{rcl} \Lambda & = & R\langle 1 \times 1 \times 1 \times 1,\ \ 0 \times 4 \times 0 \times 4,\ \ 0 \times 0 \times 2 \times 2,\ \ 0 \times 0 \times 0 \times 8\rangle \\ \mathfrak{r}\Lambda & = & R\langle 2 \times 2 \times 0 \times 0,\ \ 0 \times 4 \times 0 \times 4,\ \ 0 \times 0 \times 2 \times 2,\ \ 0 \times 0 \times 0 \times 8\rangle \\ \mathfrak{r}^2\Lambda & = & R\langle 4 \times 4 \times 0 \times 0,\ \ 0 \times 8 \times 0 \times 0,\ \ 0 \times 0 \times 4 \times 4,\ \ 0 \times 0 \times 0 \times 8\rangle \\ \mathfrak{r}^3\Lambda & = & R\langle 8 \times 8 \times 0 \times 0,\ \ 0 \times 16 \times 0 \times 0,\ \ 0 \times 0 \times 8 \times 8,\ \ 0 \times 0 \times 0 \times 16\rangle \\ & = & 2\mathfrak{r}^2\Lambda, \end{array}$$

whence

$$\begin{array}{rcl} \Lambda/\mathfrak{r}\Lambda & = & k\langle 1 \times 1 \times 1 \times 1\rangle \\ \mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda & = & k\langle 2 \times 2 \times 0 \times 0,\ \ 0 \times 4 \times 0 \times 4,\ \ 0 \times 0 \times 2 \times 2\rangle \\ \mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda & = & k\langle 4 \times 4 \times 0 \times 0,\ \ 0 \times 8 \times 0 \times 0,\ \ 0 \times 0 \times 4 \times 4,\ \ 0 \times 0 \times 0 \times 8\rangle \end{array}$$

In particular, both kernel and cokernel of $\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda \longrightarrow \mathfrak{r}\Gamma/\mathfrak{r}^2\Gamma$ have dimension 1.
We calculate the minimal projective resolution of the simple Λ-module. The short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_{12}^P & \longrightarrow & \Lambda \oplus \Lambda \oplus \Lambda & \longrightarrow & \omega_{11}^P & \longrightarrow & 0 \\ & & & & 1 \oplus 0 \oplus 0 & \longrightarrow & 2 \times 2 \times 2 \times 2 & & \\ & & & & 0 \oplus 1 \oplus 0 & \longrightarrow & 0 \times 4 \times 0 \times 4 & & \\ & & & & 0 \oplus 0 \oplus 1 & \longrightarrow & 0 \times 0 \times 2 \times 2 & & \end{array}$$

yields

$$
\begin{array}{rcl}
\omega_{12}^{P} & = & \{a_1 \times b_1 \times c_1 \times d_1 \oplus a_2 \times b_2 \times c_2 \times d_2 \oplus a_3 \times b_3 \times c_3 \times d_3 \in \Lambda \oplus \Lambda \oplus \Lambda \mid \\
& & 2a_1 \times 2b_1 \times 2c_1 \times 2d_1 = 0 \times -4b_2 \times -2c_3 \times -4d_2 - 2d_3\} \\
& \simeq & \{a_2 \times b_2 \times c_2 \times d_2 \oplus a_3 \times b_3 \times c_3 \times d_3 \in \mathfrak{r}\Lambda \oplus \mathfrak{r}\Lambda \mid 0 \times 2b_2 \times c_3 \times 2d_2 + d_3 \in \Lambda\} \\
& = R\langle & \begin{array}{lcl}
2 \times 2 \times 2 \times 2 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 4 \times 0 \times 4 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 2 \times 2 & \oplus & 0 \times 4 \times 0 \times 4, \\
0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 2 \times 2 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 2 \times 2, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8\rangle,
\end{array}
\end{array}
$$

whence

$$
\mathfrak{r}\omega_{12}^{P} \;=\; R\langle \begin{array}{lcl}
4 \times 4 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 8 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 4 \times 4 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 4 \times 4 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 4 \times 4, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8\rangle,
\end{array}
$$

so that

$$
\omega_{12}^{P}/\mathfrak{r}\omega_{12}^{P} \;=\; k\langle \begin{array}{lcl}
2 \times 2 \times 2 \times 2 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 4 \times 0 \times 4 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 2 \times 2 & \oplus & 0 \times 4 \times 0 \times 4, \\
0 \times 0 \times 0 \times 0 & \oplus & 2 \times 2 \times 2 \times 2, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 2 \times 2\rangle.
\end{array}
$$

Therefore,

$$
\begin{array}{rcl}
\omega_{13}^{P} & = & \{a_2 \times b_2 \times c_2 \times d_2 \oplus a_3 \times b_3 \times c_3 \times d_3 \oplus a_5 \times b_5 \times c_5 \times d_5 \in \mathfrak{r}\Lambda \oplus \mathfrak{r}\Lambda \oplus \mathfrak{r}\Lambda \mid \\
& & 2(b_2 - d_2) + (c_3 - d_3) \equiv_8 0,\; 2(b_3 - d_3) + (c_5 - d_5) \equiv_8 0\} \\
& = R\langle & \begin{array}{lclcl}
2 \times 2 \times 2 \times 2 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 4 \times 0 \times 4 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 2 \times 2 & \oplus & 0 \times 4 \times 0 \times 4 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 2 \times 2 \times 2 \times 2 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 2 \times 2 & \oplus & 0 \times 4 \times 0 \times 4, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 2 \times 2 \times 2 \times 2, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 2 \times 2, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8\rangle
\end{array}
\end{array}
$$

and

$$
\mathfrak{r}\omega_{13}^{P} \;=\; R\langle \begin{array}{lclcl}
4 \times 4 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 8 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 4 \times 4 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 4 \times 4 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 4 \times 4 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8 & \oplus & 0 \times 0 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 4 \times 4 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 8 \times 0 \times 0, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 4 \times 4, \\
0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 0 & \oplus & 0 \times 0 \times 0 \times 8\rangle,
\end{array}
$$

so that

$$\begin{array}{rcl}\omega_{13}^P/\mathfrak{r}\omega_{13}^P & = & k\langle\ 2\times 2\times 2\times 2\ \oplus\ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 0\times 0, \\ & & \ 0\times 4\times 0\times 4\ \oplus\ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 0\times 0, \\ & & \ 0\times 0\times 2\times 2\ \oplus\ 0\times 4\times 0\times 4\ \oplus\ 0\times 0\times 0\times 0, \\ & & \ 0\times 0\times 0\times 0\ \oplus\ 2\times 2\times 2\times 2\ \oplus\ 0\times 0\times 0\times 0, \\ & & \ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 2\times 2\ \oplus\ 0\times 4\times 0\times 4, \\ & & \ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 0\times 0\ \oplus\ 2\times 2\times 2\times 2, \\ & & \ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 0\times 0\ \oplus\ 0\times 0\times 2\times 2\rangle. \end{array}$$

Therefore,

$$\begin{array}{rcl}\omega_{14}^P & = & \{a_2\times b_2\times c_2\times d_2\oplus a_3\times b_3\times c_3\times d_3\oplus a_5\times b_5\times c_5\times d_5\oplus a_7\times b_7\times c_7\times d_7\in\mathfrak{r}\Lambda\oplus\mathfrak{r}\Lambda\oplus\mathfrak{r}\Lambda\oplus\mathfrak{r}\Lambda\mid \\ & & 2(b_2-d_2)+(c_3-d_3)\equiv_8 0,\ 2(b_3-d_3)+(c_5-d_5)\equiv_8 0,\ 2(b_5-d_5)+(c_7-d_7)\equiv_8 0\}.\end{array}$$

And so on. We obtain, for $j\geq 1$,

$$\begin{array}{rcl} t_{\omega_{1j}^P} & = & 2j+1 \\ \mathrm{rk}_R\omega_{1j}^P/\mathrm{rk}_R P & = & j,\end{array}$$

respecting the bound $(**)$ of (E.2.12), $(j+1)/j\leq 3$ for $j\geq 1$.
Let $X:=\{x\times y\mid x\equiv_4 y\}\subseteq R_1\times R_2$, let $Y:=\{z\times w\mid z\equiv_4 w\}\subseteq R_3\times R_4$. We want to calculate

$$\mathrm{Hom}_\Lambda(X,Y/2^\infty)=\mathrm{Hom}_\Lambda(X,Y/2)$$

(cf. E.3.2). Let $z\times w$ denote the image of 1×1 under such a morphism, determining it. $z\times w$ qualifies as such an image iff for any $a\times b\times c\times d$ in the annihilator of $1\times 1\in X$ we have $cz\times dw\in 8Y$ (E.3.1). Since this annihilator is generated by $0\times 0\times 2\times 2$ and $0\times 0\times 0\times 8$ over R, this condition translates into

$$\begin{array}{c} 2z\times 2w\in 2Y \\ 0z\times 8w\in 2Y\end{array}$$

so that

$$\mathrm{Hom}_\Lambda(X,Y/2)\xrightarrow{\sim} R\langle 1\times 1,\ 0\times 4\rangle\subseteq Y/2.$$

Let $1\times 1\xrightarrow{f} 1\times 1$. An element $a\times b\times c\times d$ of Γ acts on X and on Y iff

$$\begin{array}{rcl} a & \equiv_4 & b \\ c & \equiv_4 & d.\end{array}$$

In addition, this element commutes with f iff it satisfies the following ties.

$$\begin{array}{rcl} a & \equiv_2 & c \\ b & \equiv_2 & d \\ a-c & \equiv_8 & b-d.\end{array}$$

Note that $X/2\xrightarrow[\sim]{f} Y/2$ is an isomorphism, and so are the induced morphisms on the radical layers.

Example E.3.4
Let $R:=\mathbf{Z}_{(2)}$, $k:=\mathbf{F}_2$, let

$$\begin{array}{rcl}\Lambda & := & \left\{ x\times y\times z\times w\times\left(\begin{smallmatrix} s & t \\ u & v\end{smallmatrix}\right)\ \middle|\ \begin{array}{l} x\equiv_2 y,\ t\equiv_2 0,\ z\equiv_2 v, \\ z-w\equiv_4 t, \\ y-w\equiv_4 2u, \\ x+y+z+w\equiv_8 2s+2v\equiv_4 0\}\end{array}\right\} \\ & \subseteq & R_1\times R_2\times R_3\times R_4\times\left(\begin{smallmatrix} R & R \\ R & R\end{smallmatrix}\right)_5 =: \Gamma,\end{array}$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(2)}\mathcal{S}_5$ (S 2.2.4).

By (E.1.28) we obtain

$$
\begin{aligned}
\Lambda &= R\langle\ 1\times 1\times 1\times 1\times \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \\
&\qquad 0\times 2\times 0\times 2\times \left(\begin{smallmatrix}2&2\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 2\times 2\times \left(\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 4\times \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\rangle \\
\mathfrak{r}\Lambda &= R\langle\ 2\times 2\times 2\times 2\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 2\times 0\times 2\times \left(\begin{smallmatrix}2&2\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 2\times 2\times \left(\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 4\times \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\rangle \\
\mathfrak{r}^2\Lambda &= R\langle\ 4\times 0\times 4\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 4\times 0\times 4\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 4\times 4\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 4\times \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\rangle \\
\mathfrak{r}^3\Lambda &= R\langle\ 8\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 8\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 8\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 8\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\rangle \\
\mathfrak{r}^4\Lambda &= R\langle\ 16\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 16\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 16\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 16\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\rangle \\
\mathfrak{r}^5\Lambda &= R\langle\ 32\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 32\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 32\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 32\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\rangle \\
&\simeq \mathfrak{r}^3\Lambda,
\end{aligned}
$$

whence

$$
\begin{aligned}
\Lambda/\mathfrak{r}\Lambda &= k\langle\ 1\times 1\times 1\times 1\times \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\rangle \\
\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda &= k\langle\ 2\times 2\times 2\times 2\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 2\times 0\times 2\times \left(\begin{smallmatrix}2&2\\0&0\end{smallmatrix}\right), \\
&\qquad 0\times 0\times 2\times 2\times \left(\begin{smallmatrix}2&0\\-1&0\end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$\begin{array}{rcl}
\mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda & = & k\langle\ 4\times 0\times 4\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 4\times 0\times 4\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 4\times 4\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 4\times\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 0\times\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\rangle\\
\mathfrak{r}^3\Lambda/\mathfrak{r}^4\Lambda & = & k\langle\ 8\times 0\times 0\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 8\times 0\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 8\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 8\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 0\times\left(\begin{smallmatrix}0&4\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 0\times\left(\begin{smallmatrix}0&0\\2&0\end{smallmatrix}\right)\rangle\\
\mathfrak{r}^4\Lambda/\mathfrak{r}^5\Lambda & = & k\langle\ 16\times 0\times 0\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 16\times 0\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 16\times 0\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 16\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 0\times\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right),\\
&& 0\times 0\times 0\times 0\times\left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right)\rangle.
\end{array}$$

Note that

$$\tilde{\pi} := 4\times 4\times 4\times 4\times\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\in Z(\Lambda)$$

is an element that satisfies the requirements of (E.2.5 c) with $m = 2$, viz.

$$\tilde{\pi}\Lambda\subseteq\mathfrak{r}^2\Lambda\subseteq\tilde{\pi}\Gamma.$$

The formula for $\mathrm{rk}_{\tilde{\pi}}P = 1\cdot 2+1\cdot 2+1\cdot 2+1\cdot 2+4\cdot 1 = 12$ is in accordance with the observation that $l(\mathfrak{r}^iP/\mathfrak{r}^{i+1}P) = 6$ for i large enough.

Let

$$\begin{array}{rcl}
X := \{\left(\begin{smallmatrix}s&t\\u&v\end{smallmatrix}\right)\mid s\equiv_2 v,\ t\equiv_2 0\} & \subseteq & \left(\begin{smallmatrix}R&R\\R&R\end{smallmatrix}\right)_5\\
Y := \{x\times y\times z\times w\mid x\equiv_2 y\equiv_2 z\equiv_2 w,\ x+y+z+w\equiv_4 0\} & \subseteq & R_1\times R_2\times R_3\times R_4
\end{array}$$

We want to calculate

$$\mathrm{Hom}(X, Y/2^\infty) = \mathrm{Hom}(X, Y/2)\subseteq Y/2$$

(cf. E.3.2). The annihilator of $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\in X$ is generated over Λ by $2\times 2\times 2\times 2\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, so that any element of $Y/2$ qualifies as an image of $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ under a Λ-morphism $X\longrightarrow Y/2$ (E.3.1).

Note that $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\stackrel{f}{\longrightarrow}1\times 1\times 1\times 1$ has the property that each element in $\mathrm{Hom}_\Lambda(X, Y/2)$ can be written as composition of f with an endomorphism of $Y/2$. As we take from the R-linear basis of Λ, f is given by

$$\begin{array}{rcl}
X & \stackrel{f}{\longrightarrow} & Y/2\\
\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) & \longrightarrow & 1\times 1\times 1\times 1\\
\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right) & \longrightarrow & 0\times 0\times 0\times 4\\
\left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right) & \longrightarrow & 0\times 2\times 0\times -2\\
\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) & \longrightarrow & 0\times 0\times -2\times 2.
\end{array}$$

An element $x\times y\times z\times w\times\left(\begin{smallmatrix}s&t\\u&v\end{smallmatrix}\right)$ of Γ acts on X iff it $\left(\begin{smallmatrix}s&t\\u&v\end{smallmatrix}\right)$ is contained in X, it acts on Y iff $x\times y\times z\times w$ is contained in Y. Note that f sends

$$\begin{array}{rcr}
\left(\begin{smallmatrix}s&t\\u&v\end{smallmatrix}\right) & \stackrel{f}{\longrightarrow} & v\cdot(1\times 1\times 1\times 1)\\
& + & (s-v)\cdot(0\times 0\times 0\times 2)\\
& + & t\cdot(0\times 1\times 0\times -1)\\
& + & u\cdot(0\times 0\times -2\times 2).
\end{array}$$

Therefore, in addition, this element commutes with f iff it satisfies the following ties.

$$\begin{array}{rcl}
x-y & \equiv_4 & t\\
x-z & \equiv_4 & 2u\\
x+y+z+w & \equiv_8 & 2s+2v.
\end{array}$$

k-linear bases for the radical layers of $\bar{X} = X/2$ are given by

$$\begin{array}{rcl} \bar{X}/\mathfrak{r}\bar{X} & = & k\langle \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\rangle \\ \mathfrak{r}\bar{X}/\mathfrak{r}^2\bar{X} & = & k\langle \left(\begin{smallmatrix}0&2\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)\rangle \\ \mathfrak{r}^2\bar{X}/\mathfrak{r}^3\bar{X} & = & k\langle \left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)\rangle, \end{array}$$

k-linear bases for the radical layers of $\bar{Y} = Y/2$ are given by

$$\begin{array}{rcl} \bar{Y}/\mathfrak{r}\bar{Y} & = & k\langle 1\times 1\times 1\times 1\rangle \\ \mathfrak{r}\bar{Y}/\mathfrak{r}^2\bar{Y} & = & k\langle 2\times 0\times 0\times -2,\ 0\times 2\times 0\times -2,\ 0\times 0\times -2\times 2\rangle \\ \mathfrak{r}^2\bar{Y}/\mathfrak{r}^3\bar{Y} & = & k\langle 0\times 0\times 0\times 4\rangle. \end{array}$$

Thus, by the description of $X \xrightarrow{f} \bar{Y}$ on an R-linear basis given above, we obtain the following intuitive picture.

$$\begin{array}{ccccc} \bar{X} & & & \bar{Y} & \\ & & & & \\ & \boldsymbol{S} & & \boldsymbol{S} & \\ \boldsymbol{S} & \boldsymbol{S} & \xrightarrow{f} & \boldsymbol{S}\ \boldsymbol{S} & S \\ & \boldsymbol{S} & & \boldsymbol{S} & \end{array}$$

Example E.3.5
Let $R := \mathbf{Z}_{(2)}$, $k := \mathbf{F}_2$, let

$$\Lambda := \left\{ a\times b\times c\times d\times e\times f \;\middle|\; \begin{array}{rcl} c & \equiv_2 & e, \\ b+c-2f & \equiv_{16} & a+d-2e \equiv_8 0, \\ e-f & \equiv_4 & a-b \equiv_2 0 \end{array} \right\} \subseteq R_1\times R_2\times R_3\times R_4\times R_5\times R_6 =: \Gamma.$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(2)}\mathcal{S}_6$ (S 2.3.5).
By (E.1.28) we obtain

$$\begin{array}{rcl} \Lambda & = & R\langle\ 1\times 1\times 1\times 1\times 1\times 1, \\ & & \quad 0\times 2\times 2\times 0\times 0\times 2, \\ & & \quad 0\times 0\times 4\times 4\times 2\times 2, \\ & & \quad 0\times 0\times 0\times 8\times 0\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 4\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 0\times 8\rangle \\ \mathfrak{r}\Lambda & = & R\langle\ 2\times 0\times 0\times 2\times 0\times 2, \\ & & \quad 0\times 2\times 2\times 0\times 0\times 2, \\ & & \quad 0\times 0\times 4\times 4\times 2\times 2, \\ & & \quad 0\times 0\times 0\times 8\times 0\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 4\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 0\times 8\rangle \\ \mathfrak{r}^2\Lambda & = & R\langle\ 4\times 0\times 0\times 4\times 0\times 4, \\ & & \quad 0\times 4\times 4\times 0\times 0\times 4, \\ & & \quad 0\times 0\times 8\times 8\times 0\times 0, \\ & & \quad 0\times 0\times 0\times 8\times 0\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 4\times 4, \\ & & \quad 0\times 0\times 0\times 0\times 0\times 8\rangle \\ \mathfrak{r}^3\Lambda & = & R\langle\ 8\times 0\times 0\times 8\times 0\times 0, \\ & & \quad 0\times 8\times 8\times 0\times 0\times 0, \\ & & \quad 0\times 0\times 16\times 0\times 0\times 0, \\ & & \quad 0\times 0\times 0\times 16\times 0\times 0, \\ & & \quad 0\times 0\times 0\times 0\times 8\times 0, \\ & & \quad 0\times 0\times 0\times 0\times 0\times 8\rangle \\ \mathfrak{r}^4\Lambda & = & \quad 2\mathfrak{r}^3\Lambda, \end{array}$$

whence

$$\begin{array}{rcl} \Lambda/\mathfrak{r}\Lambda & = & k\langle\ 1\times 1\times 1\times 1\times 1\times 1\rangle \\ \mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda & = & k\langle\ 2\times 0\times 0\times 2\times 0\times 2, \\ & & \quad 0\times 2\times 2\times 0\times 0\times 2, \\ & & \quad 0\times 0\times 4\times 4\times 2\times 2\rangle \end{array}$$

$$\begin{array}{rcl}
\mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda & = & k\langle\ 4\times 0\times 0\times 4\times 0\times 4, \\
 & & 0\times 4\times 4\times 0\times 0\times 4, \\
 & & 0\times 0\times 8\times 8\times 0\times 0, \\
 & & 0\times 0\times 0\times 8\times 0\times 4, \\
 & & 0\times 0\times 0\times 0\times 4\times 4\rangle \\
\mathfrak{r}^3\Lambda/\mathfrak{r}^4\Lambda & = & k\langle\ 8\times 0\times 0\times 8\times 0\times 0, \\
 & & 0\times 8\times 8\times 0\times 0\times 0, \\
 & & 0\times 0\times 16\times 0\times 0\times 0, \\
 & & 0\times 0\times 0\times 16\times 0\times 0, \\
 & & 0\times 0\times 0\times 0\times 8\times 0, \\
 & & 0\times 0\times 0\times 0\times 0\times 8\rangle,
\end{array}$$

in accordance with (E.2.3) and respecting the bound of (E.2.9).
Let

$$\begin{array}{rcl}
\{e\times f \mid e\equiv_2 f\} & =: X & \subseteq R_5\times R_6 \\
\{a\times b\times c\times d \mid a+d\equiv_8 b+c\equiv_4 0,\ c\equiv_2 d\} & =: Y & \subseteq R_1\times R_2\times R_3\times R_4 \\
\{a\times b \mid a\equiv_2 b\} & =: Z & \subseteq R_1\times R_2 \\
\{c\times d \mid c\equiv_2 d\} & =: W & \subseteq R_3\times R_4
\end{array}$$

We want to calculate

$$\mathrm{Hom}_\Lambda(W, Z/2^\infty) = \mathrm{Hom}_\Lambda(W, Z/4)$$

(cf. E.3.2). The annihilator of $1\times 1\in W$ is generated over Λ by $4\times 4\times 0\times 0\times 2\times 2$, so that any element $a\times b$ of $Z/4$ qualifies as an image of 1×1 under a Λ-morphism $W\longrightarrow Z/4$ (E.3.1). Thus, any such morphism factors over

$$\begin{array}{rcl}
W & \xrightarrow{h} & Z/4 \\
1\times 1 & \longrightarrow & 1\times 1 \\
0\times 2 & \longrightarrow & 2\times 0
\end{array}$$

An element $a\times b\times c\times d\times e\times f$ of Γ acts on Z iff $a\times b$ is contained in Z, it acts on W iff $c\times d$ is contained in W.

In addition, this element commutes with h iff it satisfies the following ties.

$$a-d\equiv_8 b-c\equiv_4 0.$$

Note that there is a pullback diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & W & & c\times d \\
\downarrow & \ulcorner & \downarrow{\scriptstyle -h} & & \downarrow \\
Z & \longrightarrow & Z/4 & & -d\times -c
\end{array}$$

k-linear bases for the radical layers of $\bar{W} := W/4$ are given by

$$\begin{array}{rcl}
\bar{W}/\mathfrak{r}\bar{W} & = & k\langle 1\times 1\rangle \\
\mathfrak{r}\bar{W}/\mathfrak{r}^2\bar{W} & = & k\langle 2\times 0, 0\times 2\rangle \\
\mathfrak{r}^2\bar{W}/\mathfrak{r}^3\bar{W} & = & k\langle 4\times 0\rangle,
\end{array}$$

similarly $\bar{Z} := Z/4$. Moreover, $\bar{Z}\xrightarrow[\sim]{h}\bar{W}$.
We want to calculate

$$\mathrm{Hom}(X, Y/2^\infty) = \mathrm{Hom}(X, Y/4)$$

(cf. E.3.2). The annihilator of $1\times 1\in X$ is generated over Λ by $2\times -2\times 2\times -2\times 0\times 0$, so that an element $a\times b\times c\times d$ of $Y/4$ qualifies as an image of 1×1 under a Λ-morphism $X\longrightarrow Y/4$ iff

$$a\times -b\times c\times -d\in 2(Y/4).$$

(E.3.1). The condition on $a\times b\times c\times d$ is equivalent to the requirement that

$$a\times b\times c\times d\in\langle 2\times 2\times 2\times 2\rangle\Lambda\subseteq Y/4,$$

so that any such morphism factors over

$$\begin{array}{rcl} X & \xrightarrow{g} & Y/4 \\ 1\times 1 & \longrightarrow & 2\times 2\times 2\times 2 \\ 0\times 2 & \longrightarrow & 0\times 4\times 4\times 0. \end{array}$$

An element $a\times b\times c\times d\times e\times f$ of Γ acts on Y iff $a\times b\times -c\times -d\in Y$, it acts on X iff $e\times f\in X$. **In addition, this element commutes with g iff it satisfies the following ties.**

$$\begin{array}{rcl} b+c-2f & \equiv_{16} & a+d-2e\equiv_8 0, \\ e-f & \equiv_4 & d-c. \end{array}$$

k-linear bases for the radical layers of $\bar{X} := X/4$ are given by

$$\begin{array}{rcl} \bar{X}/\mathfrak{r}\bar{X} & = & k\langle 1\times 1\rangle \\ \mathfrak{r}\bar{X}/\mathfrak{r}^2\bar{X} & = & k\langle 2\times 0, 0\times 2\rangle \\ \mathfrak{r}^2\bar{X}/\mathfrak{r}^3\bar{X} & = & k\langle 0\times 4\rangle, \end{array}$$

k-linear bases for the radical layers of $\bar{Y} := Y/4$ are given by

$$\begin{array}{rcl} \bar{Y}/\mathfrak{r}\bar{Y} & = & k\langle 1\times -1\times 1\times -1\rangle \\ \mathfrak{r}\bar{Y}/\mathfrak{r}^2\bar{Y} & = & k\langle 2\times 2\times 2\times 2,\ 0\times 2\times -2\times 0,\ 0\times 0\times 4\times -4\rangle \\ \mathfrak{r}^2\bar{Y}/\mathfrak{r}^3\bar{Y} & = & k\langle 4\times 0\times 0\times 4,\ 0\times 4\times 4\times 0,\ 0\times 0\times 8\times 0\rangle \\ \mathfrak{r}^2\bar{Y}/\mathfrak{r}^3\bar{Y} & = & k\langle 0\times 8\times 8\times 0\rangle. \end{array}$$

Thus, by the description of $X \xrightarrow{g} \bar{Y}$ on an R-linear basis given above, we obtain the following intuitive picture.

$$\begin{array}{ccccccc} \bar{X} & & & & & \bar{Y} & \\ & & & & & S & \\ \boldsymbol{S} & & & & \boldsymbol{S} & S & S \\ \boldsymbol{S} & \boldsymbol{S} & & \xrightarrow{\quad g \quad} & \boldsymbol{S} & \boldsymbol{S} & S \\ & \boldsymbol{S} & & & & \boldsymbol{S} & \end{array}$$

Example E.3.6
Let $R := \mathbf{Z}_{(2)}$, $k := \mathbf{F}_2$, let

$$\Lambda := \left\{ a\times a'\times b\times b'\times c\times c'\times d\times d'\times \left(\begin{smallmatrix} e & f \\ g & h\end{smallmatrix}\right)\times \left(\begin{smallmatrix} e' & f' \\ g' & h'\end{smallmatrix}\right) \;\middle|\; \begin{array}{rcl} d'-c' & \equiv_4 & b'-a'\equiv_2 0, \\ c' & \equiv_2 & e', \\ f' & \equiv_4 & f', \\ e-h & \equiv_4 & e'-h'\equiv_2 0, \\ c+d+f & \equiv_8 & c'+d'+f', \\ e+e' & \equiv_4 & f'+b'+a'\equiv_2 0, \\ 2f' & \equiv_4 & c'-b', \\ d & \equiv_4 & c'-f', \\ b-3d-f+2g & \equiv_8 & b'-3d'-f'+2g' \\ -3a+b-c-d+2e-2f-2g+2h+2g' & \equiv_{16} & -3a'+b'-c'-d'+2e'-2f'-2g'+2h'+2g\equiv_8 0 \end{array} \right\}$$

$$\subseteq R_1\times R_2\times R_3\times R_4\times R_5\times R_6\times R_7\times R_8\times \left(\begin{smallmatrix} R & R \\ R & R\end{smallmatrix}\right)_9 \times \left(\begin{smallmatrix} R & R \\ R & R\end{smallmatrix}\right)_{10} =: \Gamma.$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(2)}\mathcal{S}_6$ (S 2.3.5). I am not quite content with this system of ties, nevertheless, it is possible to handle it.

By (E.1.28), we obtain

$$\begin{array}{rcl} \Lambda & = & R\langle\ 1\times 1\times 1\times 1\times 1\times 1\times 1\times 1\times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)\times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right), \\ & & 0\times 2\times 2\times 0\times 0\times 2\times 2\times 0\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2\end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0\end{smallmatrix}\right), \\ & & 0\times 0\times 2\times 2\times 0\times 0\times 2\times 2\times \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2\end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2\end{smallmatrix}\right), \\ & & 0\times 0\times 0\times 4\times 0\times 0\times 0\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0\end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 4\end{smallmatrix}\right), \\ & & 0\times 0\times 0\times 0\times 2\times 2\times 2\times 2\times \left(\begin{smallmatrix} 0 & 0 \\ -1 & 2\end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ -1 & -2\end{smallmatrix}\right), \end{array}$$

$$
\begin{aligned}
& 0\times 0\times 0\times 0\times 0\times 4\times 4\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ -2 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 4\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}\Lambda \;=\; R\langle\; & 2\times 0\times 0\times 2\times 2\times 0\times 0\times 2\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 2\times 2\times 0\times 0\times 2\times 2\times 0\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 2\times 2\times 0\times 0\times 2\times 2\times \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 4\times 0\times 0\times 0\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 2\times 2\times 2\times 2\times \left(\begin{smallmatrix} 0 & 0 \\ -1 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ -1 & -2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 4\times 4\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ -2 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 4\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}^2\Lambda \;=\; R\langle\; & 4\times 0\times 0\times 4\times 4\times 0\times 0\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 4\times 4\times 0\times 0\times 4\times 4\times 0\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 4\times 0\times 0\times 0\times 4\times 0\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 4\times 0\times 0\times 0\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 4\times 0\times 0\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ -2 & -4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 4\times 4\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ -2 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 4\times 4\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}^3\Lambda \;=\; R\langle\; & 8\times 0\times 0\times 8\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 8\times 8\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 8\times 0\times 0\times 0\times 8\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 8\times 0\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 8\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 8\times 8\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 8\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 8\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 4 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 8 \\ 0 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 4 & 0 \end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 8 \end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}^4\Lambda \;=\;\; & R\langle\; 16\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 16\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 16\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 16\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 16\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 16\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 16\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 16\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}4&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&4\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right)\rangle\\
\mathfrak{r}^5\Lambda \;=\;\; & R\langle\; 32\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 32\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 32\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 32\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 32\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 32\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 32\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 32\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&8\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\4&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&16\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\8&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right)\rangle\\
\mathfrak{r}^6\Lambda \;=\;\; & R\langle\; 64\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 64\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 64\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 64\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 64\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 64\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 64\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 64\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}8&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&16\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\8&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&8\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}16&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&16\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\8&0\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&16\end{smallmatrix}\right)\rangle\\
\;\simeq\;\; & \mathfrak{r}^4\Lambda,
\end{aligned}
$$

whence

$$
\begin{aligned}
\Lambda/\mathfrak{r}\Lambda \;=\;\; & k\langle\; 1\times 1\times 1\times 1\times 1\times 1\times 1\times 1\times \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\times\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\rangle,\\
\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda \;=\;\; & k\langle\; 2\times 0\times 0\times 2\times 2\times 0\times 0\times 2\times \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\times\left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right),\\
& 0\times 2\times 2\times 0\times 0\times 2\times 2\times 0\times \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right),\\
& 0\times 0\times 2\times 2\times 0\times 0\times 2\times 2\times \left(\begin{smallmatrix}0&2\\0&2\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&2\\0&2\end{smallmatrix}\right),\\
& 0\times 0\times 0\times 0\times 2\times 2\times 2\times 2\times \left(\begin{smallmatrix}0&0\\-1&2\end{smallmatrix}\right)\times\left(\begin{smallmatrix}0&0\\-1&-2\end{smallmatrix}\right)\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda \;=\; k\langle\; & 4\times 0\times 0\times 4\times 4\times 0\times 0\times 4\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}4&0\\0&4\end{pmatrix},\\
& 0\times 4\times 4\times 0\times 0\times 4\times 4\times 0\times \begin{pmatrix}4&0\\0&4\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 4\times 0\times 0\times 0\times 4\times 0\times \begin{pmatrix}0&4\\0&4\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 4\times 0\times 0\times 0\times 4\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&4\\0&4\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 4\times 0\times 0\times 4\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\-2&-4\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 4\times 4\times 0\times \begin{pmatrix}0&0\\-2&4\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 4\times 4\times \begin{pmatrix}0&0\\0&2\end{pmatrix}\times\begin{pmatrix}0&0\\0&2\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}2&0\\0&2\end{pmatrix}\times\begin{pmatrix}2&0\\0&2\end{pmatrix}\rangle\\
\mathfrak{r}^3\Lambda/\mathfrak{r}^4\Lambda \;=\; k\langle\; & 8\times 0\times 0\times 8\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 8\times 8\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 8\times 0\times 0\times 0\times 8\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 8\times 0\times 0\times 0\times 8\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 8\times 0\times 0\times 8\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 8\times 8\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 8\times 0\times \begin{pmatrix}0&0\\0&4\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 8\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&4\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&4\\0&0\end{pmatrix}\times\begin{pmatrix}0&4\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\2&0\end{pmatrix}\times\begin{pmatrix}0&0\\2&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}4&0\\0&4\end{pmatrix}\rangle\\
\mathfrak{r}^4\Lambda/\mathfrak{r}^5\Lambda \;=\; k\langle\; & 16\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 16\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 16\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 16\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 16\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 16\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 16\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 16\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}4&0\\0&0\end{pmatrix}\times\begin{pmatrix}4&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&4\end{pmatrix}\times\begin{pmatrix}0&0\\0&4\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&8\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\4&0\end{pmatrix}\rangle
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{r}^5\Lambda/\mathfrak{r}^6\Lambda \;=\; k\langle\; & 32\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 32\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 32\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 32\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 32\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 32\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 32\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 32\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&8\\0&0\end{pmatrix}\times\begin{pmatrix}0&8\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\4&0\end{pmatrix}\times\begin{pmatrix}0&0\\4&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}8&0\\0&0\end{pmatrix},\\
& 0\times 0\times 0\times 0\times 0\times 0\times 0\times 0\times \begin{pmatrix}0&0\\0&0\end{pmatrix}\times\begin{pmatrix}0&0\\0&8\end{pmatrix}\rangle,
\end{aligned}
$$

respecting the bound of (E.2.9).
Note that

$$\tilde{\pi} := 4\times 4\times 4\times 4\times 4\times 4\times 4\times 4\times \begin{pmatrix}2&0\\0&2\end{pmatrix}\times\begin{pmatrix}2&0\\0&2\end{pmatrix} \in Z(\Lambda)$$

is an element satisfying the requirements of (E.2.5 c) with $m = 2$, viz.

$$\tilde{\pi}\Lambda \subseteq \mathfrak{r}^2\Lambda \subseteq \tilde{\pi}\Gamma.$$

The formula for $\mathrm{rk}_{\tilde{\pi}} P = 8\cdot(1\cdot 2)+2\cdot(4\cdot 1) = 24$ is in accordance with the observation that $l(\mathfrak{r}^i P/\mathfrak{r}^{i+1}P) = 12$ for i large enough.
We drop the consideration of morphisms.

Example E.3.7
Let $R := \mathbf{Z}_{(3)}$, $k := \mathbf{F}_3$, let

$$\Lambda := \left\{ a \times b \times c \times d \;\middle|\; \begin{array}{l} a - b \equiv_9 c - d \equiv_3 0, \\ \quad\; b \equiv_3 c, \end{array} \right\} \subseteq R_1 \times R_2 \times R_3 \times R_4 =: \Gamma.$$

the indices denoting merely an ordering. Λ is isomorphic to four of the endomorphism rings of indecomposable projective lattices over $\mathbf{Z}_{(3)}\mathcal{S}_6$ (S 2.3.3).
By (E.1.28) we obtain

$$\begin{array}{rcl}
\Lambda &=& R\langle\; 1\times 1\times 1\times 1, \\
&& \quad 0\times 3\times 0\times 3, \\
&& \quad 0\times 0\times 3\times 3, \\
&& \quad 0\times 0\times 0\times 9\;\rangle \\
\mathfrak{r}\Lambda &=& R\langle\; 3\times 0\times 3\times 0, \\
&& \quad 0\times 3\times 0\times 3, \\
&& \quad 0\times 0\times 3\times 3, \\
&& \quad 0\times 0\times 0\times 9\;\rangle \\
\mathfrak{r}^2\Lambda &=& R\langle\; 9\times 0\times 0\times 0, \\
&& \quad 0\times 9\times 0\times 0, \\
&& \quad 0\times 0\times 9\times 0, \\
&& \quad 0\times 0\times 0\times 9\;\rangle \\
\mathfrak{r}^3\Lambda &=& R\langle\; 27\times 0\times 0\times 0, \\
&& \quad 0\times 27\times 0\times 0, \\
&& \quad 0\times 0\times 27\times 0, \\
&& \quad 0\times 0\times 0\times 27\;\rangle \\
&=& \quad 3\mathfrak{r}^2\Lambda,
\end{array}$$

whence

$$\begin{array}{rcl}
\Lambda/\mathfrak{r}\Lambda &=& k\langle\; 1\times 1\times 1\times 1\;\rangle \\
\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda &=& k\langle\; 3\times 0\times 3\times 0, \quad 0\times 3\times 0\times 3, \\
&& \quad 0\times 0\times 3\times 3\;\rangle \\
\mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda &=& k\langle\; 9\times 0\times 0\times 0, \quad 0\times 9\times 0\times 0, \\
&& \quad 0\times 0\times 9\times 0, \quad 0\times 0\times 0\times 9\;\rangle,
\end{array}$$

in accordance with (E.2.3) and respecting the bound of (E.2.9).

Example E.3.8
Let $R := \mathbf{Z}_{(3)}$, $k := \mathbf{F}_3$, let

$$\Lambda := \left\{ a \times b \times c \;\middle|\; \begin{array}{rl} b + c \equiv_9 & 2a, \\ a \equiv_3 & b \end{array} \right\} \subseteq R_1 \times R_2 \times R_3 =: \Gamma.$$

the indices denoting merely an ordering. Λ is the endomorphism ring of an indecomposable projective lattice over $\mathbf{Z}_{(3)}\mathcal{S}_6$ (S 2.3.3).
By (E.1.28) we obtain

$$\begin{array}{rcl}
\Lambda &=& R\langle\; 1\times 1\times 1, \quad 0\times 3\times -3, \quad 0\times 0\times 9\;\rangle \\
\mathfrak{r}\Lambda &=& R\langle\; 3\times 3\times 3, \quad 0\times 3\times -3, \quad 0\times 0\times 9\;\rangle \\
\mathfrak{r}^2\Lambda &=& R\langle\; 9\times 0\times 0, \quad 0\times 9\times 0, \quad 0\times 0\times 9\;\rangle \\
\mathfrak{r}^3\Lambda &=& R\langle\; 27\times 0\times 0, \quad 0\times 27\times 0, \quad 0\times 0\times 27\;\rangle \\
&=& \quad 3\mathfrak{r}^2\Lambda,
\end{array}$$

whence

$$\begin{array}{rcl}
\Lambda/\mathfrak{r}\Lambda &=& k\langle\; 1\times 1\times 1\;\rangle \\
\mathfrak{r}\Lambda/\mathfrak{r}^2\Lambda &=& k\langle\; 3\times 3\times 3, \quad 0\times 3\times -3\;\rangle \\
\mathfrak{r}^2\Lambda/\mathfrak{r}^3\Lambda &=& k\langle\; 9\times 0\times 0, \quad 0\times 9\times 0, \quad 0\times 0\times 9\;\rangle,
\end{array}$$

in accordance with (E.2.3) and respecting the bound of (E.2.9).

Appendix F

$\mathbf{Z}\mathcal{S}_7$, quasiblocks

We are concerned with the integral quasiblocks Q^λ of the $\mathcal{S}_7$, λ running over the partitions of 7. For lack of an index formula for quasiblocks (cf. S 1.1.3) we have to ask the reader who wants to check the results, to establish his own algorithm that allows him to conclude by integral linear algebra that the image of the respective ring morphism is surjective. Most of the examples we give are well known [P 80/1, (III, §7)].

We remark that the quasiblocks Q^λ and $Q^{\lambda'}$ are isomorphic, since we may conjugate by the 'Gram matrix' $S^{\lambda,-} \hookrightarrow S^{\lambda,*,-} \simeq S^{\lambda'}$ (6.2.5) to pass from $Q^{\lambda'}$ to Q^λ, and backwards with roles of λ' and λ interchanged.
We will not consider the ties at 7, which are known by (4.2.8).

F.1 The quasiblock $Q^{(2,1,1,1,1,1)}$

Setup

Let $\lambda := (2,1,1,1,1,1)$. A ring morphism $\mathbf{Z}\mathcal{S}_6 \longrightarrow (\mathbf{Z})_6$ having Q^λ as its image is given by

$$(12) \longrightarrow \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.
We shall make use of the possibility to give separate morphisms for the naive localizations at the prime divisors of $n!$, yielding a global surjective morphism

$$\mathbf{Z}\mathcal{S}_7 \longrightarrow Q^\lambda_{[2]} \cap Q^\lambda_{[3]} \cap Q^\lambda_{[5]} \cap Q^\lambda_{[7]} \subseteq (\mathbf{Z})_{n_\lambda},$$

in a constructive, but not explicitely given manner (cf. S 2.2.1).
Moreover, we employ the language of Morita multiplicities (cf. S 2.2.1).

The quasiblock $Q^{(2,1,1,1,1,1)}_{[2]}$

At the prime 2, this quasiblock is the full matrix ring,

$$Q^{(2,1,1,1,1,1)}_{[2]} = (\mathbf{Z})_6.$$

The quasiblock $Q^{(2,1,1,1,1,1)}_{[3]}$

At the prime 3, this quasiblock is the full matrix ring,

$$Q^{(2,1,1,1,1,1)}_{[3]} = (\mathbf{Z})_6.$$

The quasiblock $Q^{(2,1,1,1,1,1)}_{[5]}$

At the prime 5, this quasiblock is the full matrix ring,

$$Q^{(2,1,1,1,1,1)}_{[5]} = (\mathbf{Z})_6.$$

F.2 The quasiblock $Q^{(2,2,1,1,1)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{14}$ having $Q^{(2,2,1,1,1)}$ as its image is given by

$$(12) \longrightarrow \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q^{(2,2,1,1,1)}_{[2]}$

At the prime 2, this quasiblock is the full matrix ring,

$$Q^{(2,2,1,1,1)}_{[2]} = (\mathbf{Z})_{14}.$$

The quasiblock $Q^{(2,2,1,1,1)}_{[3]}$

We conjugate the morphism given in the setup from the left by

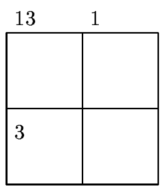
$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
-1&-2&-1&-2&-1&-1&-2&-1&-2&-1&-2&-1&-1&1
\end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix}
-1&0&0&0&1&0&0&0&-1&0&0&0&0&0\\
0&-1&0&0&-1&0&0&0&0&0&0&-1&0&0\\
-1&-2&-2&-2&0&-1&-2&-1&-2&-1&-2&-1&-1&-1\\
0&0&0&-1&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&-1&0&0&1&0&0\\
1&2&1&2&1&1&1&1&3&1&2&1&1&1\\
0&0&0&0&0&0&0&-1&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
-1&-2&-1&-2&-1&-1&-2&-1&-2&-2&-2&0&-1&-1\\
0&0&0&0&0&0&0&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
-1&-2&-1&-2&-1&-1&-2&-1&-2&-1&-2&-1&-2&-1\\
3&6&3&6&3&3&6&3&3&3&6&3&3&2
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
-1&-2&-1&-2&-2&-1&-2&-1&-2&-1&-2&-1&-2&-1\\
1&2&1&2&1&1&2&1&1&1&2&1&2&1\\
-1&-2&-1&-2&-1&-1&-2&-1&-2&-1&-2&-2&-2&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
-1&-2&-1&-2&-1&-1&-2&-1&-2&-1&-2&-1&-2&-1\\
0&-2&-1&-2&-2&-1&-2&-1&-1&-1&-2&-1&-2&-1\\
1&3&1&2&2&1&2&1&2&1&2&2&2&1\\
0&0&1&0&-1&0&0&0&0&0&0&0&-1&0\\
1&2&1&3&2&1&2&1&2&1&2&1&2&1\\
-1&-2&-1&-2&-1&0&-2&-1&-1&-1&-2&-2&-2&-1\\
0&0&0&0&0&0&1&0&-1&0&0&0&1&0\\
-1&-2&-1&-2&-1&-1&-2&0&-1&-1&-2&-1&-2&-1\\
0&0&0&0&0&0&0&0&0&1&0&-1&-1&0\\
0&0&0&0&0&0&0&0&3&0&3&3&0&1
\end{bmatrix}.$$

The image of this morphism, naively localized at 3, i.e. the quasiblock $Q^{(2,2,1,1,1)}_{[3]}$, takes the form



The quasiblock $Q^{(2,2,1,1,1)}_{[5]}$

We conjugate the morphism given in the setup from the left by

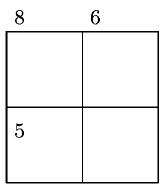
$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
-1&0&0&0&0&-1&1&-1&1&0&0&0&0&0\\
1&-1&1&0&0&1&-1&0&0&1&0&0&0&0\\
1&-1&1&-1&1&0&0&0&0&0&0&0&0&0\\
-1&0&-1&0&0&0&1&0&0&0&-1&1&0&0\\
0&0&0&-1&0&0&0&1&0&0&-1&0&1&0\\
1&-1&0&-1&0&1&0&1&0&0&-1&0&0&1
\end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix} -3 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 5 & -5 & 5 & 5 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 3 & 3 & 0 & -3 \\ -5 & 5 & -5 & 5 & 0 & 0 & 0 & 0 & -1 & 0 & 5 & 1 & 0 & -1 \\ 5 & 0 & 5 & 0 & 5 & 0 & -5 & 0 & 2 & 0 & -2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & -1 \\ -5 & 5 & 0 & 5 & 5 & -5 & 0 & -5 & -3 & 0 & 3 & 3 & 0 & 1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 2 & -2 & 1 & -3 & -2 & 1 & 0 & 2 & 0 & 0 & -1 & 0 & -1 & -1 \\ -2 & 1 & 0 & 2 & 2 & -2 & 1 & -3 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -2 & -3 & 1 & 1 & 2 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 1 & -1 & 2 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \\ 4 & -2 & 1 & -3 & -2 & 2 & -1 & 3 & 1 & 0 & -1 & 0 & -1 & -1 \\ -1 & 3 & 0 & 3 & 3 & -1 & -1 & -2 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ -10 & 10 & -5 & 15 & 10 & -5 & 0 & -10 & -1 & 0 & 5 & 1 & 5 & 4 \\ 10 & -10 & 0 & -15 & -15 & 10 & 0 & 15 & 3 & 0 & -3 & -3 & -6 & -6 \\ 5 & -5 & 0 & -10 & -10 & 5 & 0 & 10 & 1 & 0 & -1 & -1 & -5 & -4 \\ 0 & 5 & 0 & 5 & 5 & 0 & -5 & 0 & 2 & 0 & 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & -5 & -5 & 0 & 0 & 5 & 1 & 1 & -1 & -1 & -4 & 0 \\ 10 & -5 & 5 & -10 & -5 & 5 & -5 & 10 & 3 & 0 & -3 & 1 & -5 & -2 \end{bmatrix}.$$

The image of this morphism, naively localized at 5, i.e. the quasiblock $Q_{[5]}^{(2,2,1,1,1)}$, takes the form



F.3 The quasiblock $Q^{(2,2,2,1)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{14}$ having $Q^{(2,2,2,1)}$ as its image is given by

$$(12) \longrightarrow \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q_{[2]}^{(2,2,2,1)}$

We conjugate the morphism given in the setup from the left by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix} -2 & -2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 4 & 2 & 4 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 4 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 4 & 0 & 2 & 2 & 0 & -1 & 0 & -1 & 2 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 4 & 4 & 4 & 4 & 4 & 4 & 2 & 4 & 2 & 0 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 2 & 0 & 0 & -4 & 0 & -4 & -3 & -3 & -1 & 1 & 1 & -2 & -1 & 0 \\ 0 & 1 & 3 & 6 & 1 & 5 & 5 & 5 & 1 & -1 & 0 & 2 & 2 & 1 \\ 1 & 0 & -1 & -2 & 1 & -2 & -1 & -2 & 0 & 1 & 0 & -1 & -1 & 0 \\ 2 & 0 & 1 & -1 & 1 & -1 & -1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -3 & 1 & -2 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & -2 & 1 & -2 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 2 & 4 & 10 & 0 & 6 & 4 & 4 & 0 & -1 & 0 & 3 & 3 & 1 \\ -6 & -2 & -2 & 4 & -2 & 4 & 2 & 2 & 1 & -2 & -2 & 2 & 1 & -1 \\ -2 & 0 & 0 & 4 & 0 & 4 & 2 & 4 & 1 & -1 & -1 & 2 & 1 & 0 \\ -2 & 0 & -6 & -2 & 0 & -2 & -2 & -2 & 0 & 1 & -2 & 0 & -2 & -1 \\ -4 & 0 & 2 & 6 & -4 & 6 & 4 & 6 & 0 & -3 & -1 & 3 & 3 & 0 \\ -8 & 0 & 0 & 10 & -6 & 8 & 4 & 4 & 0 & -5 & -2 & 6 & 3 & -1 \end{bmatrix}.$$

The image of this morphism, naively localized at 2, i.e. the quasiblock $Q_{[2]}^{(2,2,2,1)}$, takes the form

8	6
2	

The quasiblock $Q_{[3]}^{(2,2,2,1)}$

We conjugate the morphism given in the setup from the left by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -3 & -3 & -3 & 0 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & 0 & -3 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
(1234567) \longrightarrow \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & 1 & -3 & -5 & -6 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & -1 & 4 & 6 & 8 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & -2 & -3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & -3 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -3 & -5 & -6 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & -1 & 2 & 4 & 5 & 6 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 2 & 3 & -1 & 4 & 5 & 6 & 6 \\
0 & -1 & 1 & 2 & 0 & 0 & 0 & -2 & -3 & 1 & -4 & -6 & -6 & -6 \\
0 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 3 & 5 & 6 & -1 & 7 & 9 & 11 & 12 \\
0 & 0 & 0 & 0 & 0 & -1 & -2 & -3 & -4 & 0 & -5 & -6 & -7 & -8
\end{bmatrix}.
$$

The image of this morphism, naively localized at 3, i.e. the quasiblock $Q_{[3]}^{(2,2,2,1)}$, takes the form

13	1
	3

The quasiblock $Q_{[5]}^{(2,2,2,1)}$

We conjugate the morphism given in the setup from the left by

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix}
$$

to obtain the morphism

$$
(12) \longrightarrow \begin{bmatrix}
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & -2 & 1 & 0 & 1 & -1 & 1 & 1 & 0 & -1 & 1 & -1 & -1 & -1 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -2 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -2 & -1 \\
-5 & 5 & -5 & 0 & -5 & 5 & 0 & -5 & 0 & 5 & -5 & 0 & 5 & 4
\end{bmatrix}
$$

$$
(1234567) \longrightarrow \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & -2 & -1 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 2 & 0 & 0 & 3 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 2 & 1 & 0 & 2 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 0 & 2 & 2 & 1 \\
-1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\
5 & -5 & 5 & 0 & 5 & -5 & 0 & -5 & -10 & 5 & -5 & -20 & -15 & -4
\end{bmatrix}.
$$

The image of this morphism, naively localized at 5, i.e. the quasiblock $Q_{[5]}^{(2,2,2,1)}$, takes the form

13	1
5	

F.4 The quasiblock $Q^{(3,1,1,1,1)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{15}$ having $Q^{(3,1,1,1,1)}$ as its image is given by

$$(12) \longrightarrow \begin{bmatrix} -1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\ 0&-1&0&0&-1&0&0&0&0&0&0&1&0&0&0\\ 0&0&-1&0&1&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&-1&0&0&-1&0&0&0\\ 0&0&0&0&0&0&-1&0&1&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&-1&0&0&0&-1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0&0&-1&0&0&0\\ 0&0&1&0&-1&0&0&0&0&0&0&0&0&-1&0\\ 0&0&0&1&1&0&0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&1&0&0&1&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&-1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&-1&0&-1&0\\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&-1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q^{(3,1,1,1,1)}_{[2]}$

We conjugate the morphism given in the setup from the left by

$$\begin{bmatrix} 1&-1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-15\\ 0&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&-1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&-1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&-1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&-1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&-1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1\\ 1&-1&-1&1&-1&1&-1&1&-1&1&-1&1&-1&1&-14 \end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix} 29&0&0&0&-2&-2&-2&-2&-6&-6&-6&-4&-4&0&-30\\ 2&-1&0&0&-2&-2&-2&-2&-2&-2&-2&-1&-1&-2&-2\\ -2&0&-1&0&2&2&2&2&2&2&2&2&2&3&2\\ 2&0&0&-1&-3&-3&-3&-3&-3&-3&-3&-3&-3&-3&-2\\ -4&0&0&0&1&2&2&2&3&3&3&4&4&4&4\\ 2&0&0&0&0&-1&0&0&-2&-2&-2&-3&-3&-2&-2\\ -2&0&0&0&0&0&-1&0&2&2&2&2&2&1&2\\ 4&0&0&0&0&0&0&-1&-3&-3&-3&-3&-3&-3&-4\\ 0&0&0&0&0&0&0&0&1&2&2&1&1&0&0\\ -2&0&0&0&0&0&0&0&0&-1&0&2&2&3&2\\ 2&0&0&0&0&0&0&0&0&0&-1&-3&-3&-3&-2\\ -4&0&0&0&0&0&0&0&0&0&0&1&2&3&4\\ 4&0&0&0&0&0&0&0&0&0&0&0&-1&-3&-4\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 28&0&0&0&-2&-2&-2&-2&-6&-6&-6&-4&-4&0&-29 \end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix} 29&0&2&0&-2&-2&-2&-2&0&-2&0&2&-12&-28&-28\\ 0&0&0&0&0&0&0&0&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 12&-1&-2&-1&-1&0&-1&0&0&1&0&1&0&1&-13\\ -12&0&1&0&-1&-2&-1&-2&-2&-3&-2&-2&-1&-2&13\\ -2&1&0&0&2&2&2&2&2&2&2&1&1&2&2\\ 2&0&1&0&-2&-2&-2&-2&-2&-2&-2&-2&-2&-3&-2\\ 2&0&0&1&2&1&1&1&0&0&0&-1&-1&-1&-2\\ -2&0&0&0&0&1&0&0&2&2&2&3&3&2&2\\ 2&0&0&0&0&0&1&0&-2&-2&-2&-2&-2&-1&-2\\ -4&0&0&0&0&0&0&1&2&1&1&2&2&3&4\\ 2&0&0&0&0&0&0&0&0&1&0&-2&-2&-3&-2\\ 2&0&0&0&0&0&0&0&0&0&1&2&1&0&-2\\ 26&0&2&0&-2&-2&-2&-2&0&-2&0&2&-11&-26&-25 \end{bmatrix}.$$

The image of this morphism, naively localized at 2, i.e. the quasiblock $Q^{(3,1,1,1,1)}_{[2]}$, takes the form

1	14
	2
2	

NB the Specht lattice $S_{(2)}^{(3,1,1,1,1)}$, appearing as a column of $(\mathbf{Z})_{15}$, is not projective over $Q_{(2)}^{(3,1,1,1,1)}$, although $Q_{(2)}^{(3,1,1,1,1)}$ is a Gram order. Cf. also [J 78, 23.10 iii].

The quasiblock $Q_{[3]}^{(3,1,1,1,1)}$

At the prime 3, this quasiblock is the full matrix ring,

$$Q_{[3]}^{(3,1,1,1,1)} = (\mathbf{Z})_{15}.$$

The quasiblock $Q_{[5]}^{(3,1,1,1,1)}$

At the prime 5, this quasiblock is the full matrix ring,

$$Q_{[5]}^{(3,1,1,1,1)} = (\mathbf{Z})_{15}.$$

F.5 The quasiblock $Q^{(3,2,1,1)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{35}$ having $Q^{(3,2,1,1)}$ as its image is given by

$$(12) \longrightarrow \begin{bmatrix}
-1&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&-1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&1&0&-1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&-1&0&0&-1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&-1&0&0&0&0&-1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&-1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0&0&-1&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&-1&0&0&0&1&1&0&0&0&-1&0&0&0&0&0&-1&0&0&0&1&1&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&-1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&-1&0&0&1&-1&-1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-1&-1&0&0&-1&-1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\
-1&0&0&-1&0&0&1&1&1&0&0&0&0&0&0&1&0&0&-1&0&0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0&1\\
0&-1&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&-1&0&0&0&-1&-1&-1\\
0&0&-1&-1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&-1&0&1&1&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&-1&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&-1&0&0&-1&0&0&-1&0&0&0&0&0&1&1&1&0&0&1&0&0&-1&1&1&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0&0&0&-1&0&0&0&0&-1&0&0&0&0&0&1&1&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&-1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&1&0&0&0&0&0&-1&-1&0&0&-1&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&-1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1&-1&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&-1&-1&-1
\end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q_{[2]}^{(3,2,1,1)}$

We conjugate the morphism given in the setup from the left by

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
-18&-6&-6&-10&0&-17&5&0&0&-27&-12&-14&2&-21&-16&0&-20&0&0&3&-14&-17&0&0&3&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
13&3&1&7&0&17&-4&0&0&22&5&9&-1&20&16&0&16&0&0&-3&14&17&0&0&-3&-1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&3&5&3&0&0&0&0&0&5&7&5&-1&0&0&0&4&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-121&-21&29&-37&6&-153&46&0&0&-184&-41&-63&7&-182&-118&0&-136&0&0&63&-103&-137&0&0&27&13&-7&24&0&0&0&-6&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-5&0&-1&-3&0&-7&-1&0&0&-6&-3&-2&0&-9&-12&0&-6&0&0&0&-5&-7&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&-3&7&0&-5&1&-3&-1&-4&-4&0&0&-4&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&0&5&5&-5&0&-2&-5&-1&1&3&1&-1&-4&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
69&23&21&39&0&68&-20&0&0&101&43&49&-8&84&64&0&80&0&0&-12&56&63&0&1&-12&0&4&-4&0&0&0&0&0&0&0\\
-3&0&0&7&-3&-5&-2&0&-1&0&-4&-4&0&-3&-5&1&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
5&-3&-3&3&0&14&2&0&0&7&-1&-1&1&18&24&0&8&0&0&0&10&14&0&0&-2&0&0&0&0&0&0&0&0&0&0\\
10&0&2&-1&0&14&1&0&1&12&2&4&0&14&12&0&12&0&0&0&10&14&0&0&-2&0&0&0&0&0&0&0&0&0&0\\
605&105&-145&185&-30&766&-230&0&0&920&205&315&-35&906&590&0&680&0&0&-315&515&685&0&0&-135&-65&35&-120&0&0&0&30&0&0&0\\
-39&-15&1&-18&0&-70&15&0&-1&-65&-12&-21&3&-70&-52&0&-52&0&0&7&-46&-52&-5&-1&10&4&-4&-1&1&0&0&0&0&0&0\\
5&7&1&7&0&0&0&0&0&-3&-2&-6&0&0&0&0&0&0&0&5&-5&3&2&0&0&0&0&1&-1&1&0&0&0&0&0\\
10&0&-1&13&0&9&3&-7&-1&12&2&-1&0&14&9&0&12&0&0&0&7&14&-1&0&-2&-5&0&0&0&0&1&0&0&0&0\\
5&0&-6&-2&-1&-1&-3&0&0&5&-2&0&0&1&1&0&4&0&0&-7&1&3&0&0&-1&-1&0&-4&0&0&0&1&0&0&0\\
3&1&-1&8&0&0&-3&0&-1&3&1&7&0&0&0&0&-3&0&0&0&0&5&0&-1&0&0&0&3&0&0&0&0&1&0&0\\
-1&0&-5&-6&-7&0&-1&-3&0&-1&-6&-5&0&0&0&0&-3&0&0&-1&0&-7&-5&0&0&0&0&-5&0&0&0&0&0&1&0\\
13&1&4&7&0&7&-2&-5&0&15&3&8&0&14&12&0&12&0&0&0&9&19&-3&-1&-2&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

to obtain the morphism

(12) $\longrightarrow$

−5	0	−7	−6	3	0	−3	0	−11	−1	−21	2	−4	−195	−3	−196	−2	0	−4	−3	−5	0	0	0	1	−2	1	−39	0	0	0	−6	0	0	0
−5	−5	−5	6	5	−5	−4	5	−4	3	−2	3	0	−64	0	−80	1	−5	−3	−5	0	0	1	−1	0	0	1	−16	−1	0	0	0	0	0	0
0	0	−2	−1	0	−4	−8	3	−43	0	−98	−7	−4	−1032	−12	−1060	−4	−3	0	0	−4	1	0	−1	0	−1	0	−211	−1	−1	0	−30	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	−3	−5	−18	−13	−5	−4	−8	−4	−5	−18	−9	−4	−16	−24	0	−8	−5	−7	−1	−6	0	−1	0	0	−1	−2	0	0	0	0	0	0	−1	0
−5	−7	−9	−7	0	−10	−16	0	−51	−5	−120	−8	−8	−1140	−48	−1165	−8	−8	−3	−5	−12	0	0	−2	0	−2	−2	−232	−2	−1	0	−30	0	0	0
25	57	−141	−263	−134	−319	−308	−16	−812	109	−1556	−76	−227	−17871	−176	−18237	14	−156	−412	−243	−14	29	5	−28	4	−2	32	−3630	0	6	0	−546	0	−21	−13
0	0	3	−7	0	0	−1	2	−1	4	4	0	−1	23	0	25	0	0	0	0	0	1	0	0	0	0	0	5	0	0	0	0	0	0	0
−38	−56	129	252	158	262	261	65	741	−94	1434	63	225	16261	156	16520	18	131	353	218	28	−25	2	17	−5	10	−30	3288	−5	−7	0	503	0	20	14
−1	−3	2	2	−3	0	4	0	12	−5	14	−7	5	196	3	196	−2	0	5	3	3	0	0	0	−1	1	−1	39	0	0	0	6	0	0	0
13	8	13	6	−5	9	9	−5	5	9	17	9	1	69	12	80	3	5	8	5	4	0	−1	2	0	1	0	16	1	0	0	0	0	0	0
−5	−3	−4	−3	0	4	8	−3	43	−5	91	1	4	1032	12	1060	0	3	0	0	2	−1	0	1	0	0	0	211	1	1	0	30	0	0	0
9	−4	0	−5	−40	48	37	−57	33	−21	23	−2	−9	742	−20	848	−46	20	48	17	−23	−4	−8	11	0	−10	−4	169	5	1	0	18	0	0	−1
4	3	12	16	16	9	11	8	39	6	99	15	12	869	40	869	14	5	11	8	9	0	1	0	1	2	2	173	0	0	0	25	0	1	0
−3	0	4	0	−3	4	5	8	8	8	26	−1	1	230	6	236	−4	3	0	0	2	1	0	1	−1	0	0	47	0	0	0	6	0	0	1
−208	−100	−1652	−2688	−1448	−2784	−3177	−232	−9696	456	−20230	−1411	−2401	−215593	−4152	−219414	−880	−1403	−3252	−2288	−964	263	−32	−265	−1	−239	123	−43677	−80	0	0	−6389	0	−152	−81
0	−3	−3	−5	−5	0	−3	−8	−3	−8	−16	−4	−3	−39	−24	−25	−6	0	0	0	−6	−1	−1	0	0	−1	−2	−5	0	0	0	0	0	0	0
−3	−1	1	−1	0	0	4	0	43	−3	101	0	4	1060	24	1085	4	2	−5	0	6	0	0	0	0	1	1	216	1	1	0	30	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−7	−8	−3	−2	7	−4	−11	3	−7	−7	−13	−3	−3	−79	−24	−80	−1	0	0	−5	−6	0	0	−2	0	−1	−1	−16	−1	0	0	0	0	1	0
8	−2	−102	−112	−68	−136	−114	−74	−272	−34	−626	−54	−80	−6354	−90	−6470	−6	−78	−146	−90	−17	2	0	−11	4	−3	8	−1288	2	4	0	−190	0	−10	−10
−64	−32	−88	−72	−32	−40	−24	16	−32	−8	−64	−32	−32	−160	−64	−120	−48	−16	−96	−64	−32	0	0	0	0	−13	1	−24	−1	3	0	−4	0	−4	5
−8	24	−48	48	8	0	−32	8	−328	0	−752	−16	−48	−8168	−40	−8360	−32	8	0	−24	−28	0	0	2	0	−13	8	−1664	−1	−4	0	−244	0	1	0
−80	−192	584	1072	568	1264	1208	96	3208	−384	6248	352	896	70848	776	72264	−8	624	1600	960	76	−112	−16	108	−12	13	−117	14384	0	−24	0	2160	0	84	52
−8	−8	−24	32	40	−64	−48	64	−232	16	−504	−24	0	−6024	56	−6304	24	−32	−32	−32	16	8	8	−12	0	3	10	−1255	−8	−4	0	−177	0	0	0
−32	8	200	224	176	224	184	200	528	88	1280	112	160	12568	192	12720	56	136	240	160	52	0	8	11	−8	14	−11	2532	−8	−8	0	380	0	20	20
−48	−40	56	136	128	152	88	40	272	−88	480	24	96	6200	−88	6312	−8	80	216	128	−40	−16	0	2	0	−6	−23	1256	−8	−8	0	208	0	16	8
1024	488	8208	13376	7176	13880	15832	1120	48312	−2312	100720	6992	11952	1074224	20576	1093328	4344	6992	16216	11408	4780	−1316	156	1324	1	1187	−624	217640	400	0	0	31839	0	756	404
152	224	−384	−928	−672	−736	−640	−208	−2296	368	−4528	−264	−688	−51344	−248	−52000	−168	−344	−1112	−672	−60	92	−21	−15	8	−44	95	−10348	32	24	0	−1640	0	−73	−32
−104	−80	−88	−8	104	−112	−192	56	−200	0	−240	24	−80	−2688	−192	−2848	8	−72	−80	−104	−68	7	9	−30	8	−16	15	−568	−16	0	0	−48	0	5	−5
24	−24	88	224	136	152	144	56	176	−64	200	56	152	2392	−8	2328	32	80	240	136	−12	−19	0	3	3	3	−24	463	−13	−13	−1	93	0	16	13
16	40	40	40	−24	104	192	−16	576	24	1264	64	96	13136	392	13440	32	56	40	64	88	−8	0	25	0	13	9	2676	16	8	0	363	0	−4	4
−64	−40	−64	−32	8	−8	8	−8	184	−48	368	−16	8	4712	0	4824	−16	−8	−8	−16	−8	−5	3	0	0	−4	−1	960	2	5	0	144	−1	−1	−1
8	48	40	192	136	40	56	104	56	72	216	104	56	224	312	0	104	40	40	8	84	0	16	0	0	16	26	0	0	0	0	0	0	7	0
−72	−56	40	160	128	192	168	80	368	−72	672	32	136	8080	56	8200	−32	104	240	128	−28	−16	0	12	0	−12	−18	1632	−8	−8	0	256	0	16	15

(1234567) $\longrightarrow$

2	2	6	2	0	0	4	0	43	2	108	5	4	1060	24	1085	8	2	−5	0	8	0	0	0	0	2	1	216	1	1	0	30	0	0	0
3	0	−9	−5	2	0	7	−2	−39	3	−105	0	−3	−1182	−9	−1211	2	2	−4	−7	−13	0	0	1	1	−6	0	−241	0	0	−1	−37	0	0	1
−1	3	−10	−1	−2	−10	−14	−3	−13	−5	−32	−4	−6	−230	−12	−236	−1	−8	−7	−1	0	0	1	−2	0	1	0	−47	0	0	0	−6	−1	−1	−1
0	0	3	0	0	0	0	3	0	4	8	0	0	28	12	25	0	0	0	0	2	1	0	0	0	0	1	5	0	0	0	0	0	0	0
7	0	16	10	1	0	3	2	57	−2	142	6	14	1438	46	1457	8	−2	8	10	22	0	0	−1	0	8	1	290	0	0	1	43	0	0	−1
3	3	−7	5	8	−1	−2	−2	−19	0	−54	3	−3	−584	−10	−603	6	−1	0	0	−10	0	1	−1	1	−4	0	−120	0	0	−1	−18	−1	0	0
309	122	452	116	38	243	340	−122	1602	164	3743	350	251	37129	708	38021	292	122	312	238	234	3	−6	25	6	69	−12	7570	36	23	−5	1041	−15	5	−10
0	0	−5	0	0	−5	−16	−5	−15	−8	−34	−3	−8	−240	−24	−236	−3	−3	0	0	−4	0	0	−2	0	0	−1	−47	0	0	0	−6	−1	0	−1
−298	−128	−521	−124	−31	−262	−359	81	−1857	−178	−4450	−374	−291	−44273	−902	−45321	−301	−134	−324	−262	−340	−3	7	−29	−2	−107	4	−9023	−38	−24	−2	−1254	10	−6	12
−2	−2	−9	4	0	0	−3	−2	−42	−6	−112	−5	−3	−1083	−24	−1110	−8	−2	5	0	−8	−1	0	0	0	−2	−1	−221	−1	−1	0	−30	0	0	0
−3	0	12	−2	−2	0	−9	4	37	1	109	0	1	1204	9	1236	−2	−2	4	7	13	1	0	−1	−1	6	0	246	0	0	1	37	0	0	−1
1	−3	7	1	2	10	14	1	13	1	24	4	6	202	0	211	1	8	7	1	−2	−1	−1	2	0	−1	−1	42	0	0	0	6	1	1	1
−1	7	88	19	−6	23	26	43	312	19	854	37	54	8582	240	8757	17	11	26	34	128	0	−1	4	−4	46	9	1743	2	1	8	256	5	1	−3
−10	−1	−15	−11	−1	0	1	−2	−14	−1	−41	−6	−10	−378	−22	−372	−4	4	−14	−10	−16	0	0	1	0	−7	0	−74	1	1	−1	−13	0	0	1
−6	−7	−19	−13	−7	−19	−33	−8	−67	−14	−160	−16	−17	−1381	−72	−1401	−14	−15	−10	−6	−16	0	0	−4	0	−2	−3	−279	−2	−1	0	−36	−1	−1	−1
2350	983	2975	821	295	1363	1713	−1128	11616	1054	27506	2805	1561	277829	4664	284468	2170	393	2242	1774	1766	10	0	74	80	601	−84	56635	249	168	−28	7890	−171	−11	−149
0	0	7	0	0	1	8	7	25	4	70	5	9	624	39	628	3	1	0	0	16	0	0	1	0	5	2	125	0	0	1	18	1	0	0
0	−3	2	−5	−6	5	14	−2	13	0	18	−1	6	202	0	211	−2	3	0	0	−2	−1	−1	2	0	−1	−1	42	0	0	0	6	1	0	1
0	0	−5	6	0	−4	−7	1	−42	−4	−102	−7	−3	−1055	−12	−1085	−4	−3	0	0	−4	0	0	−1	0	−1	0	−216	−1	−1	0	−30	0	0	0
−5	0	−19	−4	2	0	2	−4	−52	−2	−138	−2	−9	−1430	−29	−1457	−4	2	−8	−10	−20	−1	0	1	1	−8	0	−290	0	0	−1	−43	0	0	1
162	72	292	136	56	206	322	8	892	126	2080	218	212	19394	650	19758	174	126	216	138	191	−3	−4	33	2	47	8	3934	19	9	1	535	3	9	2
32	−32	32	−32	−24	−40	−8	−8	56	32	200	16	16	1616	88	1608	16	−40	−16	−16	20	7	−3	−5	4	5	4	320	−4	1	0	48	0	−3	−1
0	24	8	0	−16	−32	−40	16	−96	40	−160	−16	−24	−1872	−24	−1944	0	−32	−24	0	8	6	4	−4	−3	5	4	−387	−2	−2	0	−53	0	−4	−2
−1232	−480	−1784	−544	−184	−992	−1336	480	−6168	−608	−14336	−1376	−1000	−142336	−2672	−145736	−1136	−488	−1312	−984	−888	−4	24	−96	−24	−267	57	−29016	−135	−83	20	−4000	60	−24	39
8	8	−96	0	0	−40	−64	−32	−440	8	−1096	−40	−80	−11336	−192	−11616	−32	−40	−16	−32	−120	7	4	−7	4	−46	0	−2312	−3	−3	−8	−340	−8	−4	−1
−312	−144	−648	−280	−104	−416	−640	−56	−2032	−256	−4856	−456	−464	−46000	−1480	−46872	−352	−248	−448	−304	−496	6	8	−66	0	−138	−23	−9332	−38	−18	−10	−1290	−10	−18	0
−200	−56	−368	−64	−16	−184	−216	56	−1104	−112	−2648	−224	−184	−26544	−480	−27184	−160	−80	−272	−192	−196	−5	8	−18	0	−67	12	−5412	−19	−11	−2	−758	6	−6	9
−11704	−4904	−14800	−4048	−1464	−6800	−8544	5656	−58000	−5256	−137312	−13992	−7752	−1387312	−23184	−1420536	−10824	−1968	−11144	−8824	−8756	−50	0	−371	−400	−2976	422	−282816	−1248	−844	144	−39392	856	56	742
920	320	1720	400	8	664	720	−304	6192	416	15128	1128	864	154520	2560	158248	832	224	1096	928	1240	13	−24	34	−8	454	−38	31503	101	69	26	4475	−46	10	−71
48	56	−120	−16	24	8	136	−32	−408	48	−1136	8	−32	−13704	80	−14048	72	56	−112	−96	−152	0	0	17	16	−87	17	−2795	10	5	−16	−453	0	−1	15
−256	−88	−408	−128	−32	−256	−288	96	−1216	−80	−2800	−280	−216	−28184	−408	−28952	−200	−144	−328	−224	−196	12	13	−28	0	−73	27	−5764	−25	−12	−2	−806	5	−12	6
−16	−56	64	−8	−48	−24	−16	40	40	−32	160	−56	48	1816	56	1832	−48	−16	8	8	92	−1	−8	−2	−8	42	−4	364	−10	−6	10	74	10	2	1
24	−8	16	80	40	48	48	16	−352	−8	−920	−24	8	−10336	−88	−10616	−8	40	56	16	−84	−1	−5	4	3	−35	−4	−2113	−7	−8	−5	−306	3	8	8
−48	0	−128	−112	−8	16	32	−40	−344	48	−896	−16	−112	−9392	−328	−9472	−40	40	−80	−112	−188	0	0	17	0	−76	−7	−1885	7	7	−11	−295	0	0	10
−240	−104	−472	−96	−48	−296	−392	48	−1592	−168	−3768	−328	−272	−37216	−792	−38112	−248	−176	−328	−232	−280	−1	8	−40	0	−83	5	−7588	−35	−20	−2	−1046	3	−11	6

.

The image of this morphism, naively localized at 2, i.e. the quasiblock $Q_{[2]}^{(3,2,1,1)}$, takes the form

20	1	14
2		
8	4	

An equivalent description of $Q_{(2)}^{(3,2,1,1)}$ is given in [P 80/1, (III.12)].

The quasiblock $Q_{[3]}^{(3,2,1,1)}$

We conjugate the morphism given in the setup from the left by

1	−2	−1	−1	0	−1	0	0	−1	0	−2	0	0	0	0	0	0	0	0	0	0	−2	0	−1	0	0	0	0	0	0	0	−1	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	4	−4	0	0	0	0	0	0	4	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	−4	0	2	−4	0	0	0	0	1	1	0	−2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	−10	−2	11	−4	1	−1	0	0	−3	−8	−7	−2	4	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−1	−2	8	3	−1	7	−3	2	3	4	−1	−5	0	4	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2	−2	−11	11	1	0	0	−3	3	2	−7	−7	−2	−1	0	0	0	0	0	−1	0	0	3	−3	1	0	0	0	0	0	−3	3	0	0	0
−1	−8	4	1	0	1	0	−3	0	1	−1	−4	−1	0	−1	0	0	0	0	0	−1	0	3	0	0	1	0	0	0	0	−3	0	0	0	0
4	1	5	0	−4	1	0	0	−3	2	−1	−5	0	4	−1	0	0	0	0	0	0	−1	0	3	0	0	1	0	0	0	0	−3	0	0	0
2	−7	−5	5	−4	1	0	0	−3	1	−5	−1	−2	4	−1	0	0	0	0	3	−3	−3	2	−3	0	0	0	1	0	0	0	−3	0	0	0
4	−6	6	3	−3	1	0	−3	0	−1	0	−3	0	3	−1	0	0	0	0	−3	−3	−3	−3	2	0	0	0	0	1	0	−3	0	0	0	0
−5	12	−3	3	1	−1	0	0	3	−1	6	3	0	−1	1	0	0	0	0	0	0	6	0	1	0	0	0	0	−1	1	0	3	0	0	0
4	−2	−2	4	−6	7	−3	3	2	−1	−4	−4	−1	3	−1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	−5	−8	8	4	−3	3	3	−3	4	−10	−7	−2	2	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	−18	9	4	−5	14	3	0	−3	−2	0	−9	−1	2	−2	0	0	0	0	3	−3	−6	0	4	0	0	0	0	2	−2	2	0	1	0	0
−15	39	−20	−2	16	−26	−3	−3	6	6	3	17	2	−7	5	0	0	0	0	−6	6	3	0	−1	0	0	0	0	−5	4	−4	2	−2	1	0
−1	9	−4	−1	3	2	−3	−3	−3	1	0	4	1	0	1	0	0	0	0	−3	−2	3	−6	3	0	−1	0	0	0	0	0	3	0	0	1

to obtain the morphism

(12) ⟶

6	18	11	−14	−1	4	6	9	0	5	16	2	6	−2	3	6	0	6	−4	6	0	0	1	0	0	1	−1	0	1	1	0	0	0	0	1
0	−7	4	4	3	−6	3	−6	−1	0	0	−1	−3	0	−3	−3	2	−3	0	−3	1	0	0	0	0	0	0	0	−1	0	1	0	0	0	0
−3	1	−1	2	−6	0	−3	−6	3	−6	2	3	−3	0	−3	−3	0	−3	0	−3	2	0	0	0	0	0	0	0	−1	−1	0	1	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	5	5	12	−2	9	1	−3	4	0	10	−5	−3	1	−6	−3	12	−3	5	−3	−1	−1	0	0	0	0	0	0	−1	0	−1	0	−2	−1	0
0	0	0	0	7	−1	−1	−3	−6	6	−3	−3	−3	2	3	−3	3	−3	3	−3	−2	0	0	0	0	−1	0	0	0	0	0	−1	0	0	−1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−2	−3	−2	−2	−4	1	−3	−4	0	−4	−2	3	0	−2	−4	0	−2	0	−2	0	2	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
−4	−13	1	−5	0	−4	−1	0	−1	−6	−3	4	0	0	−6	0	−7	0	−1	0	−3	0	−1	0	0	0	1	0	0	0	0	0	0	0	0
0	4	−8	1	−3	6	−3	6	2	0	−5	−1	3	0	3	3	−1	3	0	3	0	0	0	0	0	0	0	0	1	0	−1	0	0	0	0
3	1	4	−7	6	0	3	6	−3	6	2	−2	3	0	3	3	0	3	0	3	−3	0	0	0	0	0	0	0	1	1	0	−1	0	0	0
1	7	4	7	−5	−2	3	3	7	−4	9	0	−1	2	1	0	1	0	1	0	0	0	0	0	0	1	0	0	−1	−1	0	0	0	0	1
−3	−1	−7	−15	3	−6	1	3	−5	3	−11	7	3	−4	6	3	−12	3	−4	3	3	2	1	0	0	0	0	0	1	0	1	0	2	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	−4	−2	−5	2	1	0	3	−2	2	−4	0	3	−2	−1	2	−3	3	−2	3	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	3	2	2	−3	−1	3	6	5	−2	5	0	3	0	1	3	−1	2	−2	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	9	3	−1	−4	−1	−3	3	5	−2	8	2	0	1	20	0	−3	0	3	−1	0	−1	0	0	0	0	0	0	0	0	0	0	2	1	0
−6	6	−24	6	−18	12	−12	0	12	−12	−12	0	0	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	−2	−2	2	0	0	0
27	−3	30	51	−12	21	21	27	33	−9	48	−27	15	0	−27	15	36	15	3	15	−9	−8	−3	0	0	5	0	0	0	5	−3	0	−8	−4	5
−27	54	−45	−99	0	−27	−18	0	−36	18	−54	63	9	−18	63	9	−81	9	−27	9	30	15	9	0	0	−2	−3	0	5	−5	6	3	16	8	−2
−72	−54	−54	−45	−18	−36	−63	−81	−27	−45	−81	63	−45	0	−9	−45	−72	−45	0	−45	15	9	0	−1	0	−8	5	0	−7	−13	3	6	10	5	−8
−9	−117	27	18	72	−27	27	−18	−45	27	−36	−18	−9	0	−81	−9	−9	−9	9	−9	−27	0	−5	0	−1	−3	5	0	0	7	3	−7	−8	−4	−3
−27	9	−45	−36	−18	45	−54	−18	−18	−9	−36	18	−9	0	0	−9	−18	−9	9	−9	−3	0	−1	0	0	−8	1	0	4	−3	−7	3	0	0	−7
−54	−18	−54	−81	−18	−63	−18	−54	−27	−27	−81	72	−18	−18	18	−18	−81	−18	−36	−18	42	15	7	0	0	−2	−1	0	−4	−10	10	8	16	8	−2
9	27	−18	−180	54	−36	−9	54	−72	63	−54	54	36	−18	135	36	−108	36	−27	36	−3	9	6	0	0	−5	−1	−1	17	8	6	−6	22	11	−5
−54	54	−90	−81	0	18	−36	−36	−54	27	−99	36	−9	−9	0	−9	−27	−9	−18	−9	30	15	9	0	0	−8	−5	0	4	−9	0	4	6	3	−8
18	−108	18	72	36	−36	27	0	9	0	−18	−36	0	0	−27	0	18	0	9	0	−12	−6	−6	0	0	2	4	0	−2	7	4	−4	−4	−2	2
−27	9	−36	−117	63	−72	−9	−36	−90	63	−99	36	−9	−9	72	−9	−63	−9	−27	−9	24	15	9	0	0	−8	−5	0	5	−2	12	−3	18	9	−8
−9	−36	54	−9	−18	−9	18	−36	0	−36	45	36	−9	−18	−63	−9	−9	−9	−9	−9	6	0	0	0	0	3	2	0	−6	1	3	6	−4	−2	3
−54	198	−144	−99	−36	81	−117	−27	−63	36	−90	45	−27	18	153	−27	−36	−27	18	−27	24	12	11	0	0	−18	−7	0	9	−19	−11	2	17	9	−18
126	−450	324	342	72	−126	234	45	162	−99	234	−144	27	−9	−369	27	153	27	18	27	−87	−39	−33	0	0	35	21	0	−26	39	13	−9	−50	−26	35
9	−81	63	72	90	−45	63	−27	−36	45	9	−54	−27	18	−72	−27	72	−27	27	−27	−27	0	−3	0	0	0	3	0	−9	3	6	−12	−12	−6	−1

$$
(1234567) \longrightarrow
\left[\begin{smallmatrix}
15&6&4&6&16&10&6&13&-3&18&3&-14&4&2&22&3&11&3&9&3&-6&-2&0&0&0&-1&0&0&2&3&-1&-3&0&0&-1\\
2&7&0&-1&-1&2&4&6&5&1&4&2&3&-2&1&3&-1&2&-2&3&2&1&1&0&0&1&0&0&0&0&0&0&0&0&1\\
-12&-20&1&3&7&3&6&-15&-14&0&-7&-4&-9&-1&-27&-6&6&-4&-1&-3&0&1&0&0&0&-1&0&0&-1&-2&1&0&-3&-1&-1\\
-2&-3&-2&-2&-4&1&-3&-3&1&-4&-2&3&0&-2&-4&0&-2&0&-2&0&2&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
-6&-14&-7&-6&5&0&-3&-6&-6&0&-13&4&-3&-2&-8&-3&-6&-2&1&-3&0&1&0&0&0&-1&1&0&0&0&0&0&0&0&-1\\
4&6&-9&-14&-2&-11&-3&15&4&2&-11&7&12&-1&14&9&-16&7&-7&6&1&1&0&0&0&1&0&0&2&2&0&0&3&1&1\\
1&-4&-2&-5&2&1&0&3&-2&2&-4&0&3&-2&-1&3&-2&3&-2&3&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
-7&-5&0&-1&0&8&0&-6&-7&-2&1&-1&-6&0&-5&-3&2&-1&2&0&-1&0&0&0&0&-1&0&0&0&-2&0&0&-1&0&-1\\
2&3&2&2&-3&-1&3&6&5&-2&5&0&3&0&1&3&-1&3&-1&3&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
1&9&8&2&2&-2&3&1&2&2&10&-1&-3&4&5&-2&5&-3&4&-3&-3&0&0&0&0&0&0&0&-1&-1&0&-1&0&0&0\\
-4&-10&-2&-1&-3&-1&-8&-10&-5&-5&-6&1&-3&0&-5&-3&-1&-2&0&-3&0&-1&-1&0&0&-1&0&0&0&0&0&1&0&0&-1\\
14&23&1&-1&-3&-4&-3&19&13&4&9&1&9&3&31&6&-4&4&3&3&-2&-1&0&0&0&1&0&0&1&2&-1&-1&3&1&1\\
-10&-20&5&9&2&5&3&-22&-9&-5&0&-3&-12&-2&-26&-9&10&-7&1&-6&2&0&0&0&0&-1&0&0&-2&-2&1&1&-3&-1&-1\\
7&10&5&1&-3&1&3&9&4&2&9&-4&6&0&6&5&3&5&-3&6&0&-1&0&0&0&1&-1&0&1&1&0&0&0&0&1\\
-1&4&1&4&-2&-1&0&-2&3&-2&4&0&-3&2&1&-3&2&-3&2&-3&0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0\\
13&18&4&7&-1&-2&1&13&12&5&10&-3&6&3&17&3&4&1&4&0&-2&-1&0&0&0&1&0&0&0&2&-1&-1&1&0&1\\
-2&-3&-2&-2&3&0&-4&-7&-6&2&-5&0&-3&0&-1&-3&1&-3&1&-3&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&-1\\
-4&3&8&11&-3&8&0&-12&-1&-5&14&-3&-12&3&2&-8&10&-7&8&-7&-1&-1&0&0&0&-1&0&0&-2&-3&0&0&-1&0&-1\\
2&3&2&2&-3&-1&3&6&5&-2&5&0&2&0&0&2&-2&3&-1&3&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
7&19&6&3&1&1&8&10&5&7&12&-5&3&2&10&3&7&2&0&3&0&0&1&0&0&1&-1&0&0&0&0&-1&0&0&1\\
-24&-48&6&12&24&0&6&-39&-45&6&-18&-18&-24&6&-48&-18&18&-12&6&-12&-8&0&-2&0&0&-4&0&0&-2&-4&2&-2&-6&-2&-4\\
15&33&18&45&-12&39&-24&-27&36&-15&54&3&-33&6&39&-30&33&-30&48&-33&-3&-5&1&0&0&-3&4&0&-8&-3&-5&0&0&0&-3\\
-27&-63&-9&-9&-45&9&9&-27&-18&-36&-27&-9&18&-27&-108&18&0&27&-63&36&27&-3&-2&0&0&3&-8&0&6&1&5&13&-9&-3&3\\
-108&-135&0&-54&27&-45&63&-63&-108&-9&-81&0&-18&-9&-216&0&-18&9&-72&18&6&15&-1&0&0&0&-5&0&0&-15&13&3&-16&-5&0\\
99&144&-9&-36&-90&-63&-9&171&198&-36&54&81&117&-18&117&81&-90&54&-36&45&21&3&3&0&0&23&4&0&4&23&-7&7&21&4&23\\
-18&-108&-36&-36&45&-72&0&-9&-54&27&-126&-9&27&0&-99&9&-27&9&-36&0&-9&3&-7&0&0&-2&0&0&6&13&3&-1&-3&-4&-2\\
-72&-126&9&-9&63&27&54&-99&-135&36&-63&-54&-36&-18&-198&-18&63&-9&-45&9&9&6&2&0&0&-7&-8&0&2&-8&10&1&-24&-8&-7\\
171&288&18&-45&-45&-36&9&261&135&72&99&-18&153&18&315&117&-27&90&-27&90&-9&-12&2&0&0&17&-11&0&23&29&-8&-8&28&9&17\\
-81&-261&-99&-99&36&-54&-27&-63&-144&-9&-234&0&36&-45&-216&27&-81&45&-99&45&15&3&-10&0&0&-6&-3&0&19&13&9&11&-9&-4&-6\\
63&198&36&27&-99&9&-27&81&162&-45&144&54&27&9&144&18&-18&0&18&0&18&0&7&-1&0&14&2&0&-9&-2&-7&5&15&5&14\\
45&90&-45&-72&-63&-45&18&153&54&0&-18&18&117&-18&90&99&-90&90&-90&99&24&3&2&0&0&16&-9&0&20&12&2&5&15&6&16\\
-36&-36&9&-9&18&81&18&-45&-18&0&18&0&-45&-18&-72&-27&36&-18&18&-9&3&3&5&0&0&-6&2&0&-3&-9&-1&1&-12&-3&-6\\
-162&-396&-198&-225&72&-135&-45&-117&-297&9&-432&9&45&-54&-279&45&-162&63&-162&54&12&9&-17&2&-1&-18&-9&0&35&14&17&13&-4&-1&-18\\
288&684&342&324&-117&243&9&162&540&-18&729&63&-90&72&513&-90&243&-144&306&-135&-12&-9&35&-4&2&26&26&0&-65&-16&-35&-19&17&4&27\\
27&117&-36&-63&-36&-45&-45&63&81&-18&18&90&27&0&144&18&-99&0&0&0&12&9&6&0&0&6&6&-1&-1&1&-3&1&24&9&6
\end{smallmatrix}\right].
$$

The image of this morphism, naively localized at 3, i.e. the quasiblock $Q^{(3,2,1,1)}_{[3]}$, takes the form

20	1	1	13
3		3	
3	3		
9	3	3	

An equivalent description of $Q^{(3,2,1,1)}_{(3)}$ is given in [P 80/1, (III.10)].

The quasiblock $Q^{(3,2,1,1)}_{[5]}$

At the prime 5, this quasiblock is the full matrix ring,

$$Q^{(3,2,1,1)}_{[5]} = (\mathbf{Z})_{35}.$$

F.6 The quasiblock $Q^{(3,2,2)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{21}$ having $Q^{(3,2,2)}$ as its image is given by

$$
(12) \longrightarrow
\begin{bmatrix}
-1&0&1&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&-1\\
0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&-1&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&1&0&-1&0&0&0&0&-1&0&0&0&0&1&0\\
0&0&0&0&0&0&-1&-1&0&0&0&0&0&0&0&-1&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&-1&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&0&0&-1&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
$$

$$(1234567) \longrightarrow \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q_{[2]}^{(3,2,2)}$

We conjugate the morphism given in the setup from the left by

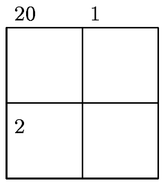
$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1
\end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix}
-2 & -1 & 0 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -2 & -1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 4 & 2 & 4 & 4 & 4 & 4 & 2 & 4 & 4 & 4 & 4 & 2 & 4 & 2 & 2 & 4 & 4 & 4 & 4 & 3
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -2 & -2 & -1 & -2 & 0 & -2 & -1 & -2 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -2 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & -1 & -1 & -2 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 2 & 4 & 4 & 2 & 2 & 0 & 4 & 2 & 1
\end{bmatrix}.$$

The image of this morphism, naively localized at 2, i.e. the quasiblock $Q_{[2]}^{(3,2,2)}$, takes the form



The quasiblock $Q_{[3]}^{(3,2,2)}$

We conjugate the morphism given in the setup from the left by

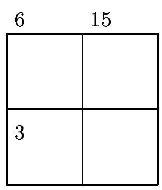
$$\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrr}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0&0&0&0\\
-2&0&0&0&0&0&-1&0&-2&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&-2&0&-1&0&0&-1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&0\\
0&0&0&0&0&0&-1&0&-2&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&-2&0&-1&0&0&-1&0&0&0&0&1&0&0&0&0\\
-1&0&0&0&0&0&-2&0&-1&0&0&-1&0&0&0&0&0&1&0&0&0\\
0&1&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&-1&0&-2&0&0&0&0&0&0&0&0&0&-1&1&0\\
0&0&0&0&0&0&-1&0&-2&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right]$$

to obtain the morphism

$$(12)\longrightarrow\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrr}
-1&0&0&0&-2&-1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&-1\\
-1&-2&0&0&-4&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1\\
-1&0&-2&0&-1&-2&0&0&0&0&0&0&-1&0&0&1&0&0&0&0&-1\\
-1&0&0&-2&-4&-2&0&0&-1&0&0&0&0&0&0&-1&0&0&0&0&-1\\
-1&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&1&0&0&0&0&0\\
-1&0&0&0&-1&-1&0&-1&0&-1&0&0&0&0&0&-1&0&0&0&0&1\\
0&0&0&0&0&0&-1&1&0&0&0&0&-1&0&-2&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
3&0&0&3&6&3&0&0&2&0&0&-1&0&0&-1&1&0&0&0&0&2\\
3&0&0&0&3&0&0&0&0&2&0&-1&0&0&-1&-1&0&0&0&0&1\\
3&0&0&0&3&0&0&1&0&3&-1&-2&1&0&-1&-1&0&-1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
6&0&3&0&6&6&0&2&0&3&0&-1&2&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&-1&-1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
3&0&0&0&3&0&0&1&0&3&0&0&0&0&0&0&0&0&0&0&-1\\
3&0&0&0&3&0&0&2&0&3&0&0&1&0&0&0&-1&-1&0&0&0\\
6&0&3&0&6&6&0&2&0&3&0&-1&1&0&-1&0&0&1&0&0&0\\
0&0&0&0&6&3&0&0&0&0&0&-1&0&0&0&2&0&0&-1&0&1\\
6&3&0&0&9&3&0&1&0&3&0&-1&0&0&-1&-1&0&0&0&2&1\\
3&0&0&0&3&0&0&1&0&3&0&0&0&0&0&-1&0&0&0&0&0
\end{array}\right]$$

$$(1234567)\longrightarrow\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrr}
0&-2&-3&0&-2&-4&0&0&0&0&1&0&-1&0&-1&1&-1&0&0&-1&-1\\
0&-2&-2&0&-2&-4&0&0&0&0&1&0&-1&0&-1&0&-1&0&0&-1&0\\
-2&-3&0&0&-7&-4&-1&1&0&1&-1&0&0&-1&-1&-1&0&0&0&-1&-1\\
0&-1&0&0&-2&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
-1&-1&0&-2&-4&-2&0&0&-1&0&0&-1&0&0&-1&-1&0&0&0&-1&0\\
1&3&0&1&4&2&0&0&1&0&0&1&0&1&1&1&0&0&0&1&0\\
-3&0&6&-3&-6&3&-1&1&-1&1&-3&-1&2&0&0&-2&2&0&-1&0&-1\\
-3&0&0&-6&0&0&0&0&-3&0&0&-2&0&0&0&0&0&0&-2&0&0\\
0&3&3&0&6&6&0&0&0&0&-1&0&1&0&1&-1&1&0&0&1&3\\
3&3&3&6&6&6&0&0&3&0&-1&2&1&0&2&0&1&0&1&2&1\\
3&3&3&3&12&9&1&-1&1&-1&0&1&1&-1&3&0&1&0&0&3&3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&-1\\
0&0&3&0&6&6&1&-2&-1&-2&0&-1&1&-1&1&-1&1&0&0&1&3\\
0&3&3&0&3&6&0&0&0&0&-2&0&2&0&2&0&1&0&0&2&0\\
-3&0&-6&0&0&-6&0&0&0&0&0&0&-2&0&0&2&-1&-1&0&0&0\\
0&-3&0&3&0&0&0&0&1&0&0&1&0&-2&0&0&0&0&0&0&0\\
3&0&0&0&9&6&2&-2&-1&-2&2&-1&0&-1&1&1&0&0&0&1&2\\
3&0&3&0&6&6&1&0&-1&1&0&-1&1&-1&1&-1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1&0&0&0&0&1\\
3&3&9&3&9&15&0&0&1&0&-2&1&3&-1&3&-1&3&1&0&3&1\\
3&3&0&3&12&6&0&0&1&0&0&1&0&0&2&2&0&0&0&3&1
\end{array}\right].$$

The image of this morphism, naively localized at 3, i.e. the quasiblock $Q_{[3]}^{(3,2,2)}$, takes the form



The quasiblock $Q^{(3,2,2)}_{[5]}$

We conjugate the morphism given in the setup from the left by

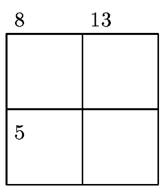
$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-2&-2&-2&2&2&-2&-1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&1&-1&0&-2&0&-1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&1&-1&0&0&-1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
2&-1&-1&1&-2&0&-1&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
2&1&2&2&2&0&0&-1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&-1&1&2&0&-1&-1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
-2&1&-1&-2&-1&2&0&-1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&1&-1&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
1&-2&-1&-2&-1&-2&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&-2&-1&0&2&-2&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
-2&1&2&-1&2&-1&-1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
2&-2&0&2&-1&2&0&-1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
-1&-2&2&-1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix}
2&-5&4&3&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&-1&-1\\
0&-2&0&0&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&-1&1&-2&0&2&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-5&3&0&-5&0&-3&0&0&0&-1&0&0&0&0&-1&0&0&0&0&1&0\\
1&3&-3&1&-1&0&-1&-1&0&0&0&0&0&0&0&-1&0&0&0&0&1\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-5&0&5&5&0&0&-1&1&0&0&0&0&0&-2&0&0&0&0&-1\\
10&0&-5&10&0&5&0&0&0&3&0&0&0&0&1&1&0&0&0&-1&2\\
10&-15&0&10&0&0&0&0&0&1&-1&0&1&0&3&1&0&-1&0&-3&-1\\
5&-15&15&5&0&5&0&0&0&0&0&-1&-1&0&2&0&0&0&0&-2&-5\\
0&-15&10&0&0&5&0&0&0&0&0&0&1&0&2&-1&0&0&0&-2&-4\\
0&-5&5&0&0&10&0&0&0&1&0&0&0&-1&0&-3&0&0&0&-1&-3\\
-10&15&-10&-10&0&-15&0&0&0&-2&0&0&0&0&-3&3&0&0&0&4&4\\
-5&0&0&-5&0&-5&0&0&0&-1&0&0&0&0&-1&2&0&0&0&1&0\\
15&-10&0&15&0&0&0&0&0&2&0&0&0&0&3&1&-1&-1&0&-3&1\\
10&0&0&10&0&10&0&0&0&2&0&0&0&0&2&-2&0&1&0&-2&0\\
0&0&0&0&5&0&0&0&0&1&0&0&0&0&-1&3&0&0&-1&0&1\\
-10&0&10&-10&0&0&0&0&0&-2&0&0&0&0&0&-4&0&0&0&1&-4\\
-5&15&-5&-5&0&-5&0&0&0&-1&0&0&0&0&-2&-1&0&0&0&2&3
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
-1&-4&5&-3&1&0&-3&-2&0&0&1&-1&-1&0&-1&1&-1&0&-1&0&-1\\
0&0&0&2&0&0&-1&0&0&0&0&0&0&-1&0&-1&0&0&0&0&0\\
0&-1&1&0&0&-1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
-3&-1&4&-2&3&0&-1&0&0&0&0&0&0&0&0&0&0&0&-1&0&-1\\
1&2&-2&1&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
-3&3&-3&-2&1&2&2&1&0&0&-1&1&1&0&0&-1&1&0&0&0&0\\
3&-2&0&1&-1&-1&2&1&0&0&0&0&0&1&1&1&0&0&1&0&0\\
0&-1&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&10&-5&0&0&0&0&0&0&0&0&0&0&1&0&-2&0&0&-1&0&2\\
10&0&0&10&-5&-5&0&0&0&0&1&-1&-1&0&1&2&-1&0&2&0&2\\
5&-10&5&5&0&-5&-10&-5&-1&0&3&-3&-3&-1&-1&1&-3&0&-1&0&0\\
-15&-10&20&-15&10&5&-10&-5&0&0&2&-2&-2&-1&-4&0&-2&0&-5&0&-6\\
-5&-5&15&-5&5&-5&-10&-5&0&0&2&-2&-2&-1&-2&3&-2&0&-4&0&-1\\
-5&-5&10&-5&5&-5&-5&0&0&0&0&-1&0&0&0&2&-1&0&-3&0&-2\\
5&15&-20&10&-5&5&10&5&0&0&-4&3&3&-1&2&-5&4&0&4&0&3\\
-5&5&-5&0&0&5&0&0&0&0&-1&1&1&-2&-1&-4&1&0&0&0&0\\
10&-10&0&5&-5&-5&-5&-5&-1&0&3&-3&-3&1&-1&4&-3&0&1&0&1\\
10&0&5&0&-5&0&0&0&1&1&2&-2&-2&2&0&3&-2&0&0&0&1\\
5&10&-10&10&-5&-5&0&0&0&0&0&1&0&-2&0&-1&0&0&1&0&5\\
-20&0&15&-20&10&10&0&0&0&0&0&1&1&2&-2&1&1&1&-4&0&-5\\
5&10&-15&0&-5&-5&10&5&0&0&-2&2&2&3&2&3&2&0&3&1&4
\end{bmatrix}.$$

The image of this morphism, naively localized at 5, i.e. the quasiblock $Q^{(3,2,2)}_{[5]}$, takes the form



F.7 The quasiblock $Q^{(4,1,1,1)}$

Setup

A ring morphism $\mathbf{Z}\mathcal{S}_7 \longrightarrow (\mathbf{Z})_{20}$ having $Q^{(4,1,1,1)}$ as its image is given by

$$(12) \longrightarrow \begin{bmatrix}
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1
\end{bmatrix},$$

as we check via the modified Coxeter relations (S 1.2) and via a comparison of characters.

The quasiblock $Q^{(4,1,1,1)}_{[2]}$

We conjugate the morphism given in the setup from the left by

$$\begin{bmatrix}
1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & -2 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -6 & -6 & -4 & 1 & -2 & -1 & -1 & 2 & 1 & -7 & 2 & 2 & 1 & 1 & -1 & 0 & 0 & 1 & 1 \\
1 & 0 & -2 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -3 & -4 & -4 & 0 & -1 & -1 & -5 & 1 & 0 & -8 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 3 & 1 \\
-1 & 6 & 6 & 4 & -1 & 1 & 1 & 0 & -2 & 0 & 6 & -2 & -2 & -1 & -1 & 1 & 0 & 0 & 0 & -1 \\
0 & -7 & -6 & -4 & -1 & 0 & 0 & -3 & 1 & 0 & -11 & 2 & 2 & 1 & 1 & 0 & 1 & -1 & 2 & 1 \\
3 & -6 & -8 & -5 & 0 & -2 & -1 & -1 & 2 & 0 & -7 & 1 & 2 & 2 & 1 & -1 & 0 & -1 & 1 & 2 \\
-1 & 11 & 12 & 9 & 0 & 3 & 4 & 5 & -4 & -1 & 16 & -3 & -4 & -1 & -2 & 1 & 0 & -1 & -4 & -2 \\
0 & 1 & -1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 0
\end{bmatrix}$$

to obtain the morphism

$$(12) \longrightarrow \begin{bmatrix}
17 & 10 & 30 & 10 & 6 & 42 & 10 & 16 & 21 & 16 & 4 & 14 & 21 & 5 & 12 & 16 & 0 & 4 & 10 & 2 \\
-3 & -5 & -2 & -4 & -1 & -2 & 0 & 1 & 2 & -2 & 0 & 1 & -1 & 0 & 1 & -2 & 0 & 1 & 1 & -1 \\
-1 & 6 & -3 & 4 & -1 & -4 & 2 & 2 & 0 & 5 & 1 & 0 & 2 & 1 & 0 & 5 & 0 & 0 & 0 & 1 \\
-5 & -8 & -10 & -5 & -1 & -16 & -8 & -12 & -14 & -15 & -3 & -10 & -14 & -3 & -7 & -15 & 0 & -4 & -7 & -1 \\
4 & 16 & 2 & 12 & -1 & 4 & 7 & 12 & 6 & 19 & 3 & 6 & 16 & 1 & 4 & 19 & 0 & 2 & 4 & 5 \\
-8 & -12 & -12 & -10 & -2 & -17 & -7 & -11 & -10 & -14 & -3 & -7 & -14 & -3 & -6 & -14 & 0 & -2 & -5 & -3 \\
-10 & -14 & -24 & -10 & -2 & -38 & -18 & -26 & -37 & -33 & -7 & -26 & -31 & -8 & -18 & -33 & 0 & -10 & -18 & 1 \\
4 & -2 & 6 & 2 & 2 & 6 & -3 & -4 & -5 & -8 & 0 & -5 & -4 & -1 & -1 & -8 & 0 & -3 & -3 & 1 \\
16 & 10 & 30 & 10 & 6 & 42 & 10 & 16 & 22 & 16 & 4 & 14 & 21 & 5 & 12 & 16 & 0 & 4 & 10 & 2 \\
24 & 36 & 58 & 18 & 4 & 86 & 48 & 48 & 94 & 81 & 13 & 64 & 65 & 21 & 40 & 82 & 0 & 25 & 43 & -12 \\
10 & 8 & 12 & 8 & 4 & 16 & 0 & 6 & -3 & 2 & 1 & -2 & 7 & 1 & 1 & 2 & 0 & -2 & -1 & 6 \\
8 & 2 & 14 & 4 & 2 & 18 & 5 & 5 & 11 & 6 & 1 & 6 & 7 & 2 & 6 & 6 & 0 & 2 & 5 & -1 \\
-2 & 0 & -2 & -2 & 0 & -2 & 1 & -2 & 3 & 1 & 0 & 2 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & -3 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 3 & 0 & 2 & 1 & 1 & 0 & 3 & 0 & 1 & 1 & -1 \\
16 & 40 & 48 & 20 & -2 & 80 & 58 & 70 & 114 & 106 & 18 & 82 & 86 & 22 & 51 & 106 & 0 & 34 & 56 & -12 \\
-18 & -20 & -46 & -8 & -4 & -66 & -34 & -30 & -70 & -52 & -8 & -46 & -42 & -16 & -29 & -53 & 0 & -18 & -31 & 12 \\
4 & 6 & 28 & -4 & -2 & 46 & 34 & 26 & 80 & 54 & 6 & 56 & 38 & 16 & 31 & 54 & -1 & 24 & 37 & -22 \\
14 & 26 & 46 & 10 & 0 & 74 & 48 & 54 & 102 & 82 & 14 & 72 & 68 & 22 & 45 & 82 & 0 & 29 & 49 & -16 \\
-62 & -76 & -144 & -48 & -12 & -216 & -108 & -124 & -222 & -184 & -34 & -152 & -162 & -46 & -102 & -184 & 0 & -58 & -105 & 24 \\
0 & 0 & -4 & 0 & 0 & -6 & -4 & 0 & -10 & -4 & -2 & -6 & -2 & -2 & -4 & -4 & 0 & -2 & -4 & 5
\end{bmatrix}$$

$$(1234567) \longrightarrow \begin{bmatrix}
-1 & -2 & -3 & 1 & 0 & -3 & -3 & 1 & -2 & -2 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 \\
-3 & -2 & -5 & -1 & 0 & -8 & -5 & -7 & -7 & -6 & -2 & -5 & -7 & 0 & -5 & -6 & 0 & -2 & -4 & 0 \\
-8 & -10 & -11 & -8 & -3 & -16 & -4 & -9 & -7 & -11 & -2 & -6 & -12 & -3 & -4 & -11 & 0 & -2 & -4 & -3 \\
4 & 2 & 6 & 2 & 1 & 8 & 2 & 3 & 2 & 2 & 1 & 1 & 3 & 0 & 2 & 2 & 0 & 0 & 1 & 1 \\
-7 & -9 & -10 & -10 & -4 & -14 & 1 & -6 & -2 & -6 & -1 & -2 & -9 & -3 & -1 & -6 & 0 & 0 & -1 & -5 \\
8 & 11 & 12 & 8 & 3 & 17 & 5 & 9 & 8 & 12 & 2 & 6 & 12 & 3 & 4 & 12 & 0 & 2 & 4 & 3 \\
20 & 4 & 32 & 8 & 6 & 40 & 7 & 7 & 10 & 2 & 3 & 4 & 10 & 1 & 9 & 2 & 0 & -1 & 4 & 2 \\
2 & -2 & 2 & 2 & 0 & 2 & -1 & 0 & -1 & -2 & 0 & -3 & -2 & -1 & 1 & -3 & 0 & -1 & 0 & 0 \\
-6 & -2 & -10 & 0 & -2 & -12 & -4 & -1 & -4 & -1 & 0 & -1 & -2 & 0 & -2 & -1 & 0 & 0 & -1 & 0 \\
-42 & -6 & -74 & -8 & -12 & -96 & -21 & -16 & -36 & -13 & -8 & -15 & -17 & -8 & -23 & -11 & -1 & -1 & -15 & 4 \\
0 & -12 & 4 & -6 & -2 & 4 & 0 & -2 & 1 & -9 & 1 & 0 & -4 & -1 & 3 & -8 & 0 & -1 & 0 & -2 \\
0 & 0 & -2 & 2 & 0 & -2 & -2 & 2 & -2 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\
-8 & -2 & -14 & -2 & -2 & -20 & -5 & -7 & -10 & -7 & -2 & -7 & -9 & -2 & -6 & -7 & 0 & -2 & -5 & 0 \\
2 & 0 & 2 & 0 & 2 & 2 & -2 & -2 & -4 & -4 & -1 & -3 & -2 & -1 & -2 & -4 & 0 & -1 & -2 & 1 \\
-64 & -14 & -110 & -26 & -18 & -146 & -30 & -40 & -56 & -26 & -16 & -28 & -46 & -10 & -41 & -24 & -1 & -3 & -26 & -3 \\
32 & 8 & 58 & 4 & 8 & 78 & 24 & 20 & 40 & 22 & 8 & 20 & 20 & 8 & 23 & 20 & 1 & 5 & 18 & -6 \\
-36 & 10 & -70 & 0 & -6 & -90 & -24 & -16 & -44 & -8 & -10 & -18 & -14 & -6 & -30 & -6 & -1 & -2 & -19 & 10 \\
-66 & -26 & -112 & -28 & -18 & -152 & -42 & -52 & -76 & -50 & -18 & -40 & -58 & -14 & -48 & -46 & -2 & -10 & -37 & 2 \\
116 & 28 & 204 & 34 & 34 & 270 & 62 & 64 & 110 & 52 & 26 & 56 & 78 & 22 & 71 & 49 & 2 & 8 & 48 & -2 \\
18 & 4 & 32 & 4 & 6 & 44 & 8 & 10 & 16 & 8 & 4 & 12 & 16 & 4 & 10 & 9 & 0 & 2 & 7 & 1
\end{bmatrix}.$$

The image of this morphism, naively localized at 2, i.e. the quasiblock $Q^{(4,1,1,1)}_{[2]}$, takes the form

6	8	6
a		
2		
2	2	a

a $x_{11} \equiv_2 x_{33}$

The quasiblock $Q^{(4,1,1,1)}_{[3]}$

At the prime 3, this quasiblock is the full matrix ring,

$$Q^{(4,1,1,1)}_{[3]} = (\mathbf{Z})_{20}.$$

The quasiblock $Q^{(4,1,1,1)}_{[5]}$

At the prime 5, this quasiblock is the full matrix ring,

$$Q^{(4,1,1,1)}_{[5]} = (\mathbf{Z})_{20}.$$

Appendix G

Some centers

We give a list of centers, calculated via the method given in (1.1.11). $Z(\mathbf{Z}\mathcal{S}_9)_{[2]}$ and $Z(\mathbf{Z}\mathcal{S}_{10})_{[2]}$ could not be simplified down to a more or less presentable form yet. Also the examples we present leave something to desire. We have been interested in particular in blocks of weight 2, which are of defect 3 for $p = 2$, and of defect 2 for $p > 2$ [JK 81, 6.2.45]. Cf. [En 90, Th. 11], [Br 88, 1.4.(1)].

$Z(\mathbf{Z}\mathcal{S}_3)$

Let the correspondence of the rational factors to the partitions be

$$\begin{array}{rcl} 1 & : & (1,1,1) \\ 2 & : & (1,1,1)' \\ 3 & : & (2,1). \end{array}$$

We obtain, in abridged notation,

$$\begin{array}{l} Z(\mathbf{Z}\mathcal{S}_3)_{[2]} \xrightarrow{\sim} \{\ x^1 \equiv_2 x^2\ \} \\ Z(\mathbf{Z}\mathcal{S}_3)_{[3]} \xrightarrow{\sim} \{\ x^1 \equiv_3 x^2 \equiv_3 x^3\ \}. \end{array}$$

$Z(\mathbf{Z}\mathcal{S}_4)$

Let the correspondence of the rational factors to the partitions be

$$\begin{array}{rcl} 1 & : & (1,1,1,1) \\ 2 & : & (1,1,1,1)' \\ 3 & : & (2,1,1) \\ 4 & : & (2,1,1)' \\ 5 & : & (2,2). \end{array}$$

We obtain, in abridged notation,

$$Z(\mathbf{Z}\mathcal{S}_4)_{[2]} \xrightarrow{\sim} \{\ x^1 + x^2 \equiv_8 x^3 + x^4 \equiv_8 2x^5,\ x^1 \equiv_4 x^4 \equiv_2 x^7\ \}$$

$$Z(\mathbf{Z}\mathcal{S}_4)_{[3]} \xrightarrow{\sim} \{\ x^1 \equiv_3 x^2 \equiv_3 x^5\ \}.$$

$Z(\mathbf{Z}\mathcal{S}_5)$

Let the correspondence of the rational factors to the partitions be as in (2.2.1), viz.

$$\begin{array}{rcl} 1 &:& (1,1,1,1,1) \\ 2 &:& (1,1,1,1,1)' \\ 3 &:& (2,1,1,1) \\ 4 &:& (2,1,1,1)' \\ 5 &:& (2,2,1) \\ 6 &:& (2,2,1)' \\ 7 &:& (3,1,1). \end{array}$$

We obtain, in abridged notation (cf. 1.1.11),

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_5)_{[2]} &\xrightarrow{\sim} \{\ x^5 + x^6 \equiv_8 x^1 + x^2 \equiv_8 2x^7,\ x^7 \equiv_2 x^1 \equiv_4 x^6, \\ &\qquad x^3 \equiv_2 x^4\} \\ Z(\mathbf{Z}\mathcal{S}_5)_{[3]} &\xrightarrow{\sim} \{\ x^1 \equiv_3 x^4 \equiv_3 x^6, \\ &\qquad x^2 \equiv_3 x^3 \equiv_3 x^5\} \\ Z(\mathbf{Z}\mathcal{S}_5)_{[5]} &\xrightarrow{\sim} \{\ x^1 \equiv_5 x^2 \equiv_5 x^3 \equiv_5 x^4 \equiv_5 x^7\}. \end{aligned}$$

Note that $Z(\mathbf{Z}\mathcal{S}_5)_{[5]}$ is also known by (4.2.8).

$Z(\mathbf{Z}\mathcal{S}_6)$

Let the correspondence of the rational factors to the partitions be as in (2.3.1), viz.

$$\begin{array}{rcl} 1 &:& (1,1,1,1,1,1) \\ 2 &:& (1,1,1,1,1,1)' \\ 3 &:& (2,1,1,1,1) \\ 4 &:& (2,1,1,1,1)' \\ 5 &:& (2,2,1,1) \\ 6 &:& (2,2,1,1)' \\ 7 &:& (2,2,2) \\ 8 &:& (2,2,2)' \\ 9 &:& (3,1,1,1) \\ 10 &:& (3,1,1,1)' \\ 11 &:& (3,2,1). \end{array}$$

We obtain, in abridged notation,

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_6)_{[2]} &\xrightarrow{\sim} \{\ x^3 + x^8 \equiv_8 2x^2,\ x^1 + x^2 \equiv_{16} x^5 + x^6,\ x^1 + x^5 \equiv_8 x^3 + x^7, \\ &\qquad x^3 \equiv_4 x^5,\ x^3 + x^5 \equiv_8 2x^9,\ x^3 + x^4 + x^5 + x^6 \equiv_{16} x^9 + x^{10} \\ &\qquad x^1 \equiv_2 x^3,\ x^3 + x^4 \equiv_{16} x^8 + x^7,\ x^1 + x^7 \equiv_8 2x^{10}\} \\ Z(\mathbf{Z}\mathcal{S}_6)_{[3]} &\xrightarrow{\sim} \{\ x^4 + x^7 \equiv_9 2x^1,\ x^1 + x^2 \equiv_9 2x^{11},\ x^9 + x^{10} \equiv_9 2x^{11}, \\ &\qquad x^3 + x^{10} \equiv_9 2x^7,\ x^4 + x^8 \equiv_9 2x^9, \\ &\qquad x^7 \equiv_3 x^8 \equiv_3 x^9 \equiv_3 x^{11}\} \\ Z(\mathbf{Z}\mathcal{S}_6)_{[5]} &\xrightarrow{\sim} \{\ x^1 \equiv_5 x^2 \equiv_5 x^5 \equiv_5 x^6 \equiv_5 x^{11}\}. \end{aligned}$$

$Z(\mathbf{Z}\mathcal{S}_7)$

Let the correspondence of the rational factors to the partitions be

1 : (7)		9 : (3, 2, 2)	
2 : (6, 1)		10 : (3, 2, 1, 1)	
3 : (5, 2)		11 : (3, 1, 1, 1, 1)	
4 : (5, 1, 1)		12 : (2, 2, 2, 1)	
5 : (4, 3)		13 : (2, 2, 1, 1, 1)	
6 : (4, 2, 1)		14 : (2, 1, 1, 1, 1, 1)	
7 : (4, 1, 1, 1)		15 : (1, 1, 1, 1, 1, 1, 1).	
8 : (3, 3, 1)			

We obtain, in abridged notation,

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_7)_{[2]} \xrightarrow{\sim} \{ \; & x^1 + 2x^3 + x^8 \equiv_{16} x^9 + 2x^{13} + x^{15},\; x^9 + x^{15} \equiv_8 2x^3, \\ & x^4 + x^9 \equiv_8 2x^{13},\; x^3 + x^{13} \equiv_4 x^8 + x^9, \\ & x^4 + x^8 + x^9 + x^{11} \equiv_{16} 2x^3 + 2x^{13},\; x^8 + x^9 \equiv_{16} x^6 + x^{10} \\ & x^8 + x^{10} \equiv_8 2x^9,\; x^6 \equiv_2 x^8 \equiv_2 x^9 \equiv_2 x^{11} \equiv_4 x^{10}, \\ & x^2 + x^{14} \equiv_8 2x^7,\; x^5 + x^{12} \equiv_8 2x^7,\; x^7 \equiv_2 x^{12} \equiv_4 x^{14} \} \end{aligned}$$

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_7)_{[3]} \xrightarrow{\sim} \{ \; & x^1 + x^{15} \equiv_9 2x^7,\; x^3 + x^{12} \equiv_9 2x^{15},\; x^3 + x^6 \equiv_9 2x^5, \\ & x^{10} + x^{13} \equiv_9 2x^{12},\; x^6 + x^{10} \equiv_9 2x^7,\; x^5 \equiv_3 x^7 \equiv_3 x^{10} \equiv_3 x^{12}, \\ & x^2 \equiv_3 x^9 \equiv_3 x^{11}, \\ & x^8 \equiv_3 x^{14} \equiv_3 x^4 \} \end{aligned}$$

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_7)_{[5]} \xrightarrow{\sim} \{ \; & x^2 \equiv_5 x^3 \equiv_5 x^9 \equiv_5 x^{12} \equiv_5 x^{15}, \\ & x^1 \equiv_5 x^5 \equiv_5 x^8 \equiv_5 x^{13} \equiv_5 x^{14} \} \end{aligned}$$

$$Z(\mathbf{Z}\mathcal{S}_7)_{[7]} \xrightarrow{\sim} \{ \; x^1 \equiv_7 x^2 \equiv_7 x^4 \equiv_7 x^7 \equiv_7 x^{11} \equiv_7 x^{14} \equiv_7 x^{15} \}.$$

Note that $Z(\mathbf{Z}\mathcal{S}_7)_{[7]}$ is also known by (4.2.8).

$Z(\mathbf{Z}\mathcal{S}_8)$

Let the correspondence of the rational factors to the partitions be

1 : (8)	12 : (4, 1, 1, 1, 1)
2 : (7, 1)	13 : (3, 3, 2)
3 : (6, 2)	14 : (3, 3, 1, 1)
4 : (6, 1, 1)	15 : (3, 2, 2, 1)
5 : (5, 3)	16 : (3, 2, 1, 1, 1)
6 : (5, 2, 1)	17 : (3, 1, 1, 1, 1, 1)
7 : (5, 1, 1, 1)	18 : (2, 2, 2, 2)
8 : (4, 4)	19 : (2, 2, 2, 1, 1)
9 : (4, 3, 1)	20 : (2, 2, 1, 1, 1, 1)
10 : (4, 2, 2)	21 : (2, 1, 1, 1, 1, 1, 1)
11 : (4, 2, 1, 1)	22 : (1, 1, 1, 1, 1, 1, 1, 1).

We obtain, in abridged notation,

$$Z(\mathbf{Z}\mathcal{S}_8)_{[2]} \xrightarrow{\sim} \{\; x^1 + 2x^8 + x^{17} + 8x^{12} + 8x^{14} \equiv_{128} x^4 + 2x^{18} + x^{22} + 8x^{10} + 8x^{11},$$
$$x^5 + x^7 + x^9 \equiv_{16} x^{11} + x^{15} + x^{19},\; x^9 + 2x^{14} \equiv_{16} x^{15} + 2x^{10},$$
$$x^9 \equiv_8 x^{15},\; x^7 \equiv_8 x^{12},\; x^{11} \equiv_8 x^{13},\; x^7 + 2x^9 + x^{17} \equiv_{64} x^4 + 2x^{15} + x^{12},$$
$$x^4 + 2x^7 + 2x^9 + 2x^{12} + 2x^{13} + 2x^{15} + x^{17} \equiv_{128} 4x^3 + 2x^8 + 2x^{18} + 4x^{20},$$
$$x^4 + x^7 + 4x^8 \equiv_{64} 8x^{11},\; x^9 + x^{11} \equiv_8 2x^{10},\; x^7 + x^{21} \equiv_{32} 2x^{18},$$
$$x^2 + x^{21} + 24x^{11} \equiv_{128} 4x^5 + 4x^{19} + 8x^8 + 8x^{18} + 2x^{13},$$
$$x^7 + 4x^8 + 4x^{12} + x^{21} \equiv_{64} 2x^{18} + 8x^{10},$$
$$2x^5 + x^7 + 2x^8 + x^{12} + 2x^{18} + 4x^{19} \equiv_{128} 6x^9 + 2x^{11} + 6x^{15},$$
$$x^9 + 3x^{11} + 3x^{13} + x^{15} \equiv_{64} 4x^{10} + 4x^{14},\; 2x^9 + x^{13} + x^{18} \equiv_{32} 2x^{12} + x^8 + x^{11},$$
$$x^9 + x^{13} + x^{17} \equiv_{32} x^{11} + x^{12} + x^{15},\; x^7 + x^{17} \equiv_{32} 2x^{15},$$

$$x^6 \equiv_2 x^{16}\}$$

$$Z(\mathbf{Z}\mathcal{S}_8)_{[3]} \xrightarrow{\sim} \{\; x^1 + x^6 \equiv_9 2x^{20},\; x^5 + x^6 \equiv_9 2x^9,\; x^{18} + x^{21} \equiv_9 2x^9,$$
$$x^5 + x^7 \equiv_9 2x^{14},\; x^{18} + x^{20} \equiv_9 2x^{14},$$
$$x^1 \equiv_3 x^5 \equiv_3 x^6 \equiv_3 x^7 \equiv_3 x^9 \equiv_3 x^{14} \equiv_3 x^{18} \equiv_3 x^{20} \equiv_3 x^{21},$$

$$x^{19} + x^{10} \equiv_9 2x^{12},\; x^{19} + x^{16} \equiv_9 2x^{15},\; x^3 + x^8 \equiv_9 2x^{10},$$
$$x^2 + x^8 \equiv_9 2x^{15}, x^2 + x^{12} \equiv_9 2x^{22},$$
$$x^2 \equiv_3 x^3 \equiv_3 x^8 \equiv_3 x^{10} \equiv_3 x^{12} \equiv_3 x^{15} \equiv_3 x^{16} \equiv_3 x^{19} \equiv_3 x^{22},$$

$$x^4 \equiv_3 x^{13} \equiv_3 x^{17}\}$$

$$Z(\mathbf{Z}\mathcal{S}_8)_{[5]} \xrightarrow{\sim} \{\; x^1 \equiv_5 x^8 \equiv_5 x^{14} \equiv_5 x^{16} \equiv_5 x^{17},$$

$$x^4 \equiv_5 x^6 \equiv_5 x^{10} \equiv_5 x^{18} \equiv_5 x^{22},$$

$$x^2 \equiv_5 x^5 \equiv_5 x^{13} \equiv_5 x^{19} \equiv_5 x^{21}\}$$

$$Z(\mathbf{Z}\mathcal{S}_8)_{[7]} \xrightarrow{\sim} \{\; x^1 \equiv_7 x^3 \equiv_7 x^6 \equiv_7 x^{11} \equiv_7 x^{16} \equiv_7 x^{20} \equiv_7 x^{22}\}.$$

$Z(\mathbf{Z}\mathcal{S}_9)$, up to the place 2

The following result has been obtained using MAPLE, in particular, using its routines ismith and ihermite. Let the correspondence of the rational factors to the partitions be

1	: (9)	16	: (4, 2, 2, 1)
2	: (1, 1, 1, 1, 1, 1, 1, 1, 1)	17	: (5, 1, 1, 1, 1)
3	: (2, 1, 1, 1, 1, 1, 1, 1)	18	: (5, 4)
4	: (8, 1)	19	: (2, 2, 2, 2, 1)
5	: (6, 3)	20	: (3, 3, 2, 1)
6	: (2, 2, 2, 1, 1, 1)	21	: (4, 3, 2)
7	: (6, 1, 1, 1)	22	: (3, 3, 1, 1, 1)
8	: (4, 1, 1, 1, 1, 1)	23	: (5, 2, 2)
9	: (5, 3, 1)	24	: (3, 2, 1, 1, 1, 1)
10	: (3, 2, 2, 1, 1)	25	: (6, 2, 1)
11	: (5, 2, 1, 1)	26	: (3, 3, 3)
12	: (4, 2, 1, 1, 1)	27	: (3, 1, 1, 1, 1, 1, 1)
13	: (4, 4, 1)	28	: (7, 1, 1)
14	: (3, 2, 2, 2)	29	: (7, 2)
15	: (4, 3, 1, 1)	30	: (2, 2, 1, 1, 1, 1, 1).

We obtain, in abridged notation,

$$
\begin{array}{ll}
Z(\mathbf{Z}\mathcal{S}_9)_{[2]} \xrightarrow{\sim} & ? \\
Z(\mathbf{Z}\mathcal{S}_9)_{[3]} \xrightarrow{\sim} \{ & x^1+3x^{24} \equiv_{81} x^2+x^{25},\ x^2+3x^{14}+3x^{22}+3x^{23} \equiv_{81} x^{27}+3x^{18}+3x^{19}+3x^{24}, \\
& x^3+3x^{21} \equiv_{81} x^{28}+3x^{19},\ x^4+3x^{20} \equiv_{81} x^{27}+3x^{18}, \\
& x^{28}+3x^5+3x^{27}+9x^{23} \equiv_{81} x^{17}+3x^{18}+3x^{21}+3x^{22}+3x^{25}+3x^{26}, \\
& 4x^{27}+3x^6+6x^{23} \equiv_{81} x^{17}+3x^{19}+3x^{20}+3x^{24}+3x^{26}, \\
& x^7+4x^{27}+5x^{28} \equiv_{81} x^8+3x^{24}+6x^{25},\ x^9+5x^{27}+3x^{20}+3x^{21}+3x^{24} \equiv_{81} 9x^{25}+6x^{26}, \\
& x^{27}+x^{13} \equiv_{81} x^{28}+x^{14},\ x^{27}+2x^{17}+6x^{26} \equiv_{81} 3x^{14}+3x^{22}+3x^{23}, \\
& x^{17}+x^{27}+x^{28} \equiv_{27} 3x^{26},\ x^{18}+2x^{21} \equiv_9 x^{19}+2x^{20}, \\
& x^{19}+x^{21} \equiv_9 2x^{28},\ x^{20}+x^{26} \equiv_9 2x^{21},\ x^{21}+x^{28} \equiv_9 x^{25}+x^{26}, \\
& x^{28}+3x^{22} \equiv_{27} x^{27}+3x^{23},\ x^{22}+x^{23} \equiv_9 x^{20}+x^{25}, \\
& x^{24}+x^{25} \equiv_9 2x^{28},\ x^{25} \equiv_3 x^{26} \equiv_3 x^{28} \equiv_9 x^{27}, \\
& x^{11} \equiv_3 x^{15} \equiv_3 x^{30}, \\
& x^{12} \equiv_3 x^{16} \equiv_3 x^{29}\} \\
Z(\mathbf{Z}\mathcal{S}_9)_{[5]} \xrightarrow{\sim} \{ & x^1 \equiv_5 x^8 \equiv_5 x^{12} \equiv_5 x^{13} \equiv_5 x^{15}, \\
& x^2 \equiv_5 x^7 \equiv_5 x^{11} \equiv_5 x^{14} \equiv_5 x^{16}, \\
& x^3 \equiv_5 x^9 \equiv_5 x^{19} \equiv_5 x^{21} \equiv_5 x^{28}, \\
& x^4 \equiv_5 x^{10} \equiv_5 x^{18} \equiv_5 x^{20} \equiv_5 x^{27}, \\
& x^5 \equiv_5 x^6 \equiv_5 x^{26} \equiv_5 x^{29} \equiv_5 x^{30}\} \\
Z(\mathbf{Z}\mathcal{S}_9)_{[7]} \xrightarrow{\sim} \{ & x^1 \equiv_7 x^3 \equiv_7 x^5 \equiv_7 x^9 \equiv_7 x^{15} \equiv_7 x^{22} \equiv_7 x^{30}, \\
& x^2 \equiv_7 x^4 \equiv_7 x^6 \equiv_7 x^{10} \equiv_7 x^{16} \equiv_7 x^{23} \equiv_7 x^{29}\}.
\end{array}
$$

$Z(\mathbf{Z}\mathcal{S}_{10})$, up to the place 2

The following result has been obtained using MAPLE, in particular, using its routines ismith and ihermite. Let the correspondence of the rational factors to the partitions be

1	:	(10)	22	:	$(4,2,2,2)$
2	:	$(1,1,1,1,1,1,1,1,1,1)$	23	:	$(4,3,2,1)$
3	:	$(2,1,1,1,1,1,1,1,1)$	24	:	$(4,3,3)$
4	:	$(9,1)$	25	:	$(3,3,3,1)$
5	:	$(7,3)$	26	:	$(4,2,2,1,1)$
6	:	$(2,2,2,1,1,1,1)$	27	:	$(5,3,1,1)$
7	:	$(7,1,1,1)$	28	:	$(6,4)$
8	:	$(4,1,1,1,1,1,1)$	29	:	$(2,2,2,2,1,1)$
9	:	$(6,3,1)$	30	:	$(3,3,2,2)$
10	:	$(3,2,2,1,1,1)$	31	:	$(4,4,2)$
11	:	$(6,2,1,1)$	32	:	$(3,3,1,1,1,1)$
12	:	$(4,2,1,1,1,1)$	33	:	$(6,2,2)$
13	:	$(5,5)$	34	:	$(3,2,2,2,1)$
14	:	$(2,2,2,2,2)$	35	:	$(5,4,1)$
15	:	$(5,3,2)$	36	:	$(3,1,1,1,1,1,1,1)$
16	:	$(3,3,2,1,1)$	37	:	$(8,1,1)$
17	:	$(5,2,2,1)$	38	:	$(5,2,1,1,1)$
18	:	$(4,3,1,1,1)$	39	:	$(7,2,1)$
19	:	$(5,1,1,1,1,1)$	40	:	$(3,2,1,1,1,1,1)$
20	:	$(6,1,1,1,1)$	41	:	$(8,2)$
21	:	$(4,4,1,1)$	42	:	$(2,2,1,1,1,1,1,1)$.

We obtain, in abridged notation,

$$Z(\mathbf{Z}\mathcal{S}_{10})_{[2]} \xrightarrow{\sim} \; ?$$

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_{10})_{[3]} \xrightarrow{\sim} \{ \; & x^1 + 3x^{18} \equiv_{81} x^{42} + 3x^5,\; x^1 + 3x^{22} + 3x^{40} \equiv_{81} x^2 + 3x^{21} + 3x^{39}, \\ & 3x^5 + 3x^{17} + 3x^{22} + 3x^{23} + 4x^{39} + 6x^{40} + 2x^{41} \equiv_{81} 24x^{42} \\ & 3x^5 + 3x^{17} + 3x^{22} + 2x^{40} + x^{42} \equiv_{81} 3x^6 + 3x^{18} + 3x^{21} + 2x^{39} + x^{41}, \\ & x^{11} + 6x^{23} + 2x^{39} \equiv_{81} 3x^7 + 3x^{24} + 3x^{25}, \\ & 3x^7 + 6x^{21} + x^{39} + 6x^{41} \equiv_{81} 3x^8 + 6x^{22} + x^{40} + 6x^{42}, \\ & x^{11} + 2x^{38} + 6x^{17} + 6x^{18} + 6x^{21} + 6x^{23} + 3x^{39} + 6x^{41} \equiv_{81} 36x^{42}, \\ & x^{11} + 6x^{21} + 3x^{39} + 3x^{41} \equiv_{81} x^{12} + 6x^{22} + 3x^{38} + 3x^{40}, \\ & 3x^{13} + 3x^{22} + 6x^{24} + 6x^{40} + 2x^{41} \equiv_{81} 3x^{14} + 3x^{21} + 6x^{25} + 6x^{39} + 2x^{42}, \\ & x^{11} + 3x^{17} + 3x^{18} + 3x^{21} + 3x^{23} + x^{42} \equiv_{81} 3x^{14} + 6x^{25} + 2x^{38} + 3x^{39}, \\ & x^{40} + 2x^{42} \equiv_{27} 3x^{17},\; x^{39} + 2x^{41} \equiv_{27} 3x^{18}, \\ & 3x^{21} + x^{41} \equiv_{27} x^{39} + 2x^{40} + x^{42}, \\ & 3x^{22} + x^{39} \equiv_{27} x^{40} + x^{41} + 2x^{42}, \\ & 3x^{23} \equiv_{27} x^{38} + x^{39} + x^{40}, \\ & 3x^{24} \equiv_{27} 2x^{39} + x^{42},\; 3x^{25} \equiv_{27} 2x^{40} + x^{41}, \\ & x^{38} + x^{41} \equiv_{27} 2x^{42},\; x^{39} \equiv_9 x^{40} \equiv_9 x^{41} \equiv_9 x^{42}, \\ & \\ & x^3 + x^{19} \equiv_9 2x^{37},\; x^9 + x^{32} \equiv_9 x^{19} + x^{37}, \\ & x^{15} + x^{19} + 2x^{32} + x^{35} + 4x^{37} \equiv_9 0,\; x^{19} \equiv_3 x^{32} \equiv_3 x^{35} \equiv_3 x^{37}, \\ & x^{29} + x^{32} \equiv_9 2x^{30},\; x^{30} + x^{35} \equiv_9 2x^{37}, \\ & \\ & x^4 + x^{20} \equiv_9 2x^{36},\; x^{10} + x^{33} \equiv_9 x^{20} + x^{36}, \\ & x^{16} + x^{20} + 2x^{33} + x^{34} + 4x^{36} \equiv_9 0,\; x^{20} \equiv_3 x^{33} \equiv_3 x^{34} \equiv_3 x^{36}, \\ & x^{28} + x^{33} \equiv_9 2x^{31},\; x^{31} + x^{34} \equiv_9 2x^{36} \} \end{aligned}$$

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_{10})_{[5]} \xrightarrow{\sim} \{ \; & x^1 + x^{36} \equiv_{25} 2x^{35},\; x^1 + x^2 \equiv_{25} 2x^{38},\; x^1 + x^{37} \equiv_{25} 2x^{30}, \\ & x^4 + x^{31} \equiv_{25} 2x^{36},\; x^7 + x^{36} \equiv_{25} 2x^{30},\; x^{19} + x^{37} \equiv_{25} 2x^{35}, \\ & x^{19} + x^{20} \equiv_{25} 2x^{38},\; x^{13} + x^{34} \equiv_{25} x^{31} + x^{38}, \\ & x^{14} + x^{35} \equiv_{25} x^{30} + x^{38},\; x^{13} + x^{14} \equiv_{25} x^{23} + x^{38}, \\ & x^{26} + x^{35} \equiv_{25} x^{31} + x^{38},\; x^{27} + x^{34} \equiv_{25} x^{30} + x^{38}, \\ & 2x^1 + x^{38} \equiv_{25} 2x^{37} + x^{34}, \\ & \\ & x^9 \equiv_5 x^{24} \equiv_5 x^{29} \equiv_5 x^{39} \equiv_5 x^{42}, \\ & \\ & x^{10} \equiv_5 x^{25} \equiv_5 x^{28} \equiv_5 x^{40} \equiv_5 x^{41} \} \end{aligned}$$

$$\begin{aligned} Z(\mathbf{Z}\mathcal{S}_{10})_{[7]} \xrightarrow{\sim} \{ \; & x^1 \equiv_7 x^{21} \equiv_7 x^{28} \equiv_7 x^{32} \equiv_7 x^{35} \equiv_7 x^{36} \equiv_7 x^{40}, \\ & \\ & x^2 \equiv_7 x^{22} \equiv_7 x^{29} \equiv_7 x^{33} \equiv_7 x^{34} \equiv_7 x^{37} \equiv_7 x^{39}, \\ & \\ & x^3 \equiv_7 x^4 \equiv_7 x^5 \equiv_7 x^6 \equiv_7 x^{15} \equiv_7 x^{16} \equiv_7 x^{23} \}. \end{aligned}$$

Appendix H

References

ALPERIN, J.L.

[A 86] *Local representation theory*, Camb. Univ. Press, Cambridge-New York, 1986.

ATIYAH, M.F., MACDONALD, I.G.

[AM 69] *Introduction to commutative algebra*, Addison-Wesley, Reading, 1969.

BENSON, D.J.

[Be 91] *Representations and Cohomology I*, Camb. Univ. Press, 1991.

BROUÉ, M.

[Br 88] *Blocs, isométries parfaites, catégories dérivées,* C. R. Acad. Sci. Paris, 307, p. 13-18, 1988.

CARTER, R.W., LUSZTIG, G.

[CL 74] *On the Modular Representations of the General Linear and Symmetric Groups*, Math. Z. 136, p. 139-242, 1974.

CARTER, R.W., PAYNE, M.T.J.

[CP 80] *On homomorphisms between Weyl modules and Specht modules*, Math. Proc. Camb. Phil. Soc. 87, p. 419-425, 1980.

CHEVALLEY, C.

[C 36] *L'arithmétique dans les algèbres de matrices*, Exposés Mathématiques 323, Hermann, 1936.

CLIFF, G.H., PLESKEN, W., WEISS, A.

[CPW 87] *Order-Theoretic Properties of the Center of a Block,* Proc. Symp. Pure Math. 47, p. 413-420, 1987.

CURTIS, C.W., REINER, I.

[CR 62] *Representation theory of finite groups and associative algebras*, Interscience, 1962.

[CR 81] *Methods of Representation Theory I,* Pure and App. Math., Wiley, New York, 1981.

DRESS, A.

[D 70] *Vertices of Integral Representations*, Math. Z. 114, p. 159-169, 1970.

ENGUEHARD, M.

[En 90] *Isométries parfaites entre blocs de groupes symétriques,* Astérisque 181/182, p. 157-171, 1990.

ERDMANN, K.

[Er 90] *Blocks of Tame Representation Type and Related Algebras*, SLN 1428, 1990.

ERDMANN, K., MARTIN, S.

[EM 94] *Quiver and Relations for the principal p-Block of* Σ_{2p}, J. Lond. Math. Soc. (2) 49, p. 442-462, 1994.

GABRIEL, P., ROITER, A.V.

[GR 92] *Representations of finite-dimensional algebras*, Encyc. Math. Sci., 73, Algebra VIII, Springer, Berlin, 1992.

GARNIR, H.

[G 50] *Théorie de la répresentation linéaire des groupes symétriques*, Thèse, Mém. Soc. Roy. Sc. Liège, (4), 10, 1950.

HELLER, A.

[H 61] *On group representations over a valuation ring*, Proc. Nat. Am. Soc. 47, p. 1194-1197, 1961.

JAMES, G.D.

[J 78] *The Representation Theory of the Symmetric Groups*, SLN 682, 1978.

JAMES, G.D., KERBER, A.

[JK 81] *The Representation Theory of the Symmetric Group*, Encyc. Math. App. 16, Reading, Mass., 1981.

KAUER, M.

[Ka 98] *Derivierte Äquivalenz von Graphordungen und Graphalgebren*, Thesis, Stuttgart, 1998.

KLESHCHEV, A.

[Kles 98] *Unpublished note*, 1998.

KLEINERT, E.

[Klei 96] *Handling integral p-group rings,* Comm. Alg. 24 (10), p. 3193-3210, 1996.

MITCHELL, B.

[M 65] *Theory of categories*, Academic Press, 1965.

NEBE, G.

[N 98] *The Group Ring of* $\mathrm{SL}_2(p^2)$ *over the p-adic Integers,* J. Alg. 210, p. 593-613, 1998.

PEEL, M. H.

[P 71] *Hook representations of the symmetric groups,* Glasgow Math. J. 12, p. 136-149, 1971.

PLESKEN, W.

[P 80/1] *Gruppenringe über lokalen Dedekindbereichen*, Habilitationsschrift, Aachen, 1980.

[P 80/2] *Projective lattices over group orders as amalgamations of irreducible lattices*, SLN 832, p. 438-448, 1980.

[P 83] *Group Rings of Finite Groups Over p-adic Integers*, SLN 1026, 1983.

ROGGENKAMP, K.W.

[Rog 70] *Lattices over Orders II*, SLN 142, 1970.

[Rog 74] *Darstellungstheorie endlicher Gruppen*, Vorlesungsskript Bielefeld, 1973.

[Rog 80] *Integral representations and structure of finite group rings*, Sém. Math. Sup. 71, Montréal, 1980.

ROWEN, L.H.

[Row 91] *Ring Theory*, Student Edition, Academic Press, 1991.

SERRE, J.P.

[Se 68] *Corps locaux,* Pub. Univ. Nancago, Hermann, Paris, 1968.

[Se 77] *Linear representations of finite groups*, GTM 42, 1977.

SCHUR, I.

[Sch 12] *Über Gruppen linearer Substitutionen mit Koeffizienten aus einem algebraischen Zahlkörper,* Math. Ann. 71, p. 355-367, 1912.

SCOPES, J.

[Sco 91] *Cartan Matrices and Morita Equivalences for Blocks of the Symmetric Groups*, J. Alg. 142, p. 441-455, 1991.

STEINITZ, E.

[St 12] *Rechteckige Systeme und Moduln in algebraischen Zahlkörpern I/II*, Math. Ann. 71, p. 328-354, Math. Ann. 72, p. 297-345, 1912.

SWAN, R.G.

[Sw 62] *Projective modules over group rings and maximal orders*, Ann. Math. 76, 1962.